Correspondance Kervaire $\leftrightarrow$ Milnor about surgery
found in Kervaire Nachlass in February 2009

Abstract

4. Letter Kervaire $\rightarrow$ Milnor dated October 7 1959. 1 page.

Total: 40 pages.
August 22

Dear Kervaire,

Enclosed is a first draft of the lecture I gave in Edinburgh. If you would like to make a joint paper, why don't you work it over, and send it to me at Rorschach. It was supposed to be handed in yesterday, but I don't suppose they were serious about that.

Best regards

John
Dear Michel,

Could you straighten out the references* in the manuscript? I don't have a library here, and it will take a while till I get to work in Princeton. I think the paper is in very good shape otherwise. If you are satisfied you might as well send it on to England. A covering letter to Todd is enclosed.

Is Whitehead's proof that (tangent bundle trivial => normal bundle trivial) readable?

I have forgotten.

As to von Staudt there are two theorems involved, each of which was discovered independently by someone else. The first theorem is found for...
example, in Hardy and Wright. I hope you don't have trouble locating the second (concerning
the numerator of Bn).

Wouldn't it be a good idea to have this mimeographed in Princeton*? It will be a long time before the Congress proceedings come out. I hope that you have some carbon copies. (Otherwise perhaps you could have a photo copy made, to send to Princeton.) Enclosed are copies of two pages I retyped.

Best wishes
John

Fine Hall, Princeton N.J.

* (or in Geneva if you have facilities)
Sept. 23, 1958

Dear Michel,

The manuscript looks fine. The theorem that a 11-manifold $M \times \mathbb{R}^d$ has trivial normal bundle is new to me. In any case there is no point in bringing that in.

As to the references:


[17]: Classification of mappings of an $(n+3)$-dimensional sphere into an $n$-dimensional one...

[13]: Beweis eines Satzes, die Bernoullischen Zahlen betreffend.

Could you also send mimeographed copies to Hirzebruch (Mathematisches Institut der Universität Bonn) and Rohlin (Коломна педагогический институт)? Thanks a lot for having it mimeographed.

Sincerely, John
Dear Milnor,

I need the following statement which should be an easy extension of the surgery theorem you proved in "Differentiable manifolds which are homotopy spheres".

Let $M^n$ be a closed, diff. manifold imbedded in $\mathbb{R}^{n+m}$ with $m$ large. Assume the normal bundle $\nu$ is almost trivial. Let $o(\nu, f)$ be the obstruction to extend some given $\kappa$-section $f$ of $\nu | M - x_0$.

Then surgery in $M^n$ yields a manifold $M'_1$ in $\mathbb{R}^{n+m}$ which is $r$-connected, $r < \frac{3}{2}n$. The normal bundle $\nu_1$ of $M'_1$ is almost trivial and there exists a $\kappa$-section $f'_1$ of $\nu_1 | M'_1 - x_0$ such that $o(\nu, f) = o(\nu_1, f'_1)$. From this $I(M) = I(M'_1)$ is a corollary. Moreover, if $I(M) = 0$, then surgery can make $M'_1$ to be $[\frac{3}{2}n]$-connected, still with existence of $\kappa$-section $f'_1$ of $\nu_1 | M'_1 - x_0$ such that $o(\nu, f) = o(\nu_1, f'_1)$.

1) Do you think the above statement is true?

It would imply that if $n \equiv 1, 2 \pmod{3}$, then $o(\nu, f)$ does not depend on $f$. Can you prove this last statement a priori?

2) If your answer to first question is yes, do you intend to publish a surgery theorem including the statement on the obstructions and the case $r = [\frac{3}{2}n]$?

If there is anything true in the above beyond your statements in the mimeographed notes on homotopy spheres, it would be very useful, I think, to have it in the literature.

I apologize for keeping the manuscript of your paper with Spanier such a long time. I'll make an effort to return it soon.

Very sincerely yours,
Dear Michel,

Unfortunately I do not know how to prove as much as you need. The best I can do is to prove that $\sigma(y, t)$ is independent of $t$, provided that $n = 9 + 10 \pmod{14}$.

1) The assertion that $\sigma(y, t)$ is unchanged by "surgery" can be proved by a slight modification of the argument used in 5.4 of my note D.M. u.a. H.S. Namely it is necessary to work with the Whitney sum (tangent bundle) $\oplus$ (trivial bundle). Do you have an idea for a better proof using the normal bundle? My proof is certainly hard to follow.

2) Suppose that $n = 2k$. Then it is easy to obtain a manifold $M$, which is $(k-1)$-connected.
using surgery. In order to obtain a manifold which is \( k \)-connected it is necessary to assume something further. For \( k \) even the assumption \( I(M) = 0 \) is sufficient, but for \( k \) odd there is an "obstruction" coming from the kernel of

\[
\pi_{k-1}(SO_k) \to \pi_{k-1}(SO_k)
\]

which is usually cyclic of order 2. (Compare 5.11 and 5.12 of my notes.)

However, the assertion that \( \sigma(x, f) \) is independent of \( f \) follows in an easier way if \( n = 2k \) with \( k = 5 \) (mod 8). Given a second cross section \( \tilde{f} \), the only obstruction to a homotopy lies in

\[ H^k(M_1, \pi_{k-1}(SO_k)) = 0. \]

Hence \( \sigma(x, f) = \sigma(y, f) \).

Unfortunately, there is a catch in this argument which I just noticed. Namely, the specific cross-section \( f \) of \( x \) (or of \( x \oplus \text{null} \)) is used in the construction of \( M \).
from $M$; namely it is used in deciding which product structure to give to the normal bundle of a sphere $f(S^1) \subset M$. (See 5.4).

These starting with a different cross-section $f'$ we may arrive at a different $M'$, My ideas ran out at this point.

3) For $n = 2k + 1$ it is again possible to make $M$, $(k-1)$-connected, but it seems very difficult to go any further. (Compare 5.13.) Again it follows that $\sigma(x, t, s)$ is independent of $s$, providing that $k = 4 \pmod{8}$; but again this does not imply anything for $M$.

I am hoping to write a paper on surgery, but haven't started yet.
There is no hurry in returning the Spanish paper. I hope that you are enjoying New York.

Best regards

John [signature]
November 19, 1959

Dear Michel,

Glad to hear that you are still thinking about these problems. Your last letter inspired me to get to work, and I now have a manuscript being typed. I will send you a copy.

Both of your conjectures sound correct. In fact, the second one is contained in my manuscript as part of the proof of the following: $M_1$ can be obtained from $M_2$ by iterated surgery $\iff M_1$ and $M_2$ belong to the same cobordism class. [If $M_1$ and $M_2$ must be closed manifolds of course. Actually I have switched terminology and am using the phrase "$X$-construction" for surgery.]

However, I do not follow your applications of these conjectures. First consider two $k$-spheres in $M^{2k}$ with one "clean" intersection point. Set $A \times B$
$\mathbb{Z}_2$ be the homotopy classes which correspond to their normal bundle. Then replacing these two imbedded spheres by a third, with homotopy class in $\pi_k(M^{2k})$ corresponding to the sum, I claim that the new normal bundle corresponds to the element $a+b+1 \in \mathbb{Z}_2$ (rather than $a+b$ as you claimed). Consider, for example the spheres $S^k \times 0$ and $0 \times S^k$ in $S^k \times S^k$, with $a=b=0$. Then the new sphere which you construct would be isotopic to the diagonal, and therefore have non-trivial normal bundle.

More generally I claim the following. There is a function $\phi: H_k(M^{2k}; \mathbb{Z}_2) \to \mathbb{Z}_2$ defined by $\phi(x) = \begin{cases} 1 & \text{if the normal bundle of an imbedded sphere representing the homotopy class } x \text{ is non-trivial} \\ 0 & \text{if trivial} \end{cases}$. 
This function $\phi$ satisfies the identity
\[ \phi(x+y) = \phi(x) + \phi(y) + \text{Intersection number} \langle x, y \rangle. \]
Thus one obtains a quadratic form over the field $\mathbb{Z}_2$.

Such a form is completely characterized by the middle Betti number, together with its "Caf invariant" which has only two possible values. One can kill $H_k(M; \mathbb{Z})$ if and only if the Caf invariant is trivial. By this method, the proof which I have for these statements are rather involved.

As for the use of Morse theory, didn't Morse make use of the sets $D \leq \text{constant}$ rather than $\phi = \text{constant}$? (where $\phi : M \to \mathbb{R}$). Unfortunately I don't have your thesis with me. The following is the analysis which I had in mind for a $(2k+1)$-manifold. Consider an imbedding $S^k \times D^{k+1} \subset M$ which represents...
a homology class \( \alpha \in H_k(M) \) of order \( r \); 
\( 1 < r < \infty \). Let \( M_0 = M \cup \text{Int}(S^k \times D^{k+1}) \) 
and let \( \lambda, \mu \in H_k(M_0) \) correspond to the 
standard generators of \( H_k(S^k \times S^k) \). Thus \( H_k(M) \) 
is obtained from \( H_k(M_0) \) by adding the relation \( \mu = 0 \). 
Since \( \lambda \to \alpha \) of order \( r \) we have \( r\lambda + 5\mu = 0 \) 
for some \( s \in \mathbb{Z} \). This must be the only relation 
between \( \lambda \) and \( \mu \).

Now performing the "\( X \)-construction" we 
must add the relation \( \lambda = 0 \). Thus the cyclic group 
of order \( r \) is replaced by a group of order \( s \). The 
construction is successful only if \( |s| < r \). 
(The case \( s = 0 \) means that we obtain an infinite 
cyclic group which can be eliminated, as you 
indicated.)

The integer \( s \) itself seems rather hard to
control, however, the residue class of $s$ modulo $r$

is a familiar object: namely the self-linking

number of $\alpha$.

Now consider the extent to which this

picture can be changed by choosing a new trivialization

for the normal bundle of $S \times 0$.

Case 1. $k = 1, 3, 7$. Then $\lambda$ can be replaced by

any $\lambda' = \lambda + i\mu$. Hence $s^*$ can be replaced by

any $s' = s - ir$. Choosing $i$ so that $0 \leq s' < r$

the construction simplifies $\mathcal{H}_c(M)$.

Case 2. $k$ odd $\neq 1, 3, 7$. Then $\lambda$ can be replaced

only by classes of the form $\lambda + 2i\mu$. Hence the best

we can do is to choose $2i$ so that $-r < s' \leq r$.

Thus the construction is successful unless $s \equiv r \pmod{2r}$.

In particular, it is always successful unless
the self linking number

\[ L(\alpha, \beta) = \text{residue of } \frac{1}{\pi} \mod 1 \in \mathbb{Q}/\mathbb{Z} \]

is zero.

If \( L(\alpha, \beta) = 0 \) for all \( \alpha \in H_k(M) \) then the identity

\[ L(\alpha + \beta, \alpha + \beta) = L(\alpha, \alpha) + L(\beta, \beta) + 2L(\alpha, \beta) \]

implies that \( L(\alpha, \beta) = 0 \text{ or } \frac{1}{2} \text{ for all } \alpha, \beta \). This is only possible if \( H_k(M) = \mathbb{Z}_2 + \cdots + \mathbb{Z}_2 \). Thus one can reduce \( M \) to a manifold having only 2-torsion.

What now?

Case 3. \( k \) even. Then \( \lambda \) cannot be changed at all. Do you see some reason to believe that is must be zero? I don't know any examples and don't have any ideas here.

Best regards

John
Nov. 22, 1959

Dear John:

Thanks for correcting my last letter. I believe I can answer your last question, assuming that the $\chi$-construction (explain to me your reason for this terminology, please) is equivalent to passing from one level surface to another with just one non-degenerate critical point inbetween.

Set $r = k+1$, and let $V^{2r}$ be a manifold with boundary $\partial V^{2r} = M' - M$. ($\dim M = \dim M' = 2k+1$.) Let $f : V \rightarrow R$ be differentiable, with just one non-degenerate critical point $0$ of index $r$ in the interior of $V$. Assume $M = f^{-1}(0)$, $M' = f^{-1}(+1)$, $-1 \leq f(x) \leq +1$ for every $x \in V$, and $f(0) = 0$. I am only interested in the case where the element of $H_k(M)$ killed by crossing $0$ is a torsion element, and since $p_k \leq p_k' \leq p_k + 1$, where $p_k = \text{rank } H_k(M; Q)$, $p_k' = \text{rank } H_k(M'; Q)$, it follows that in order to prove that the disturbing element introduced in $H_k(M')$ is of infinite order, it is sufficient to prove that $p_k' \neq p_k$.

The theorem of Morse, concerning $p_k' = p_k$, I was referring to, as contained in his paper: "Homology relations on regular orientable manifolds" Proc. Nat. Acad. Sciences 38 (1952), 247-253. I want to use a refinement of this theorem which runs as follows. (The following is contained in my thesis §9. Sorry I have no more reprints.) Let $\chi^*$ denote the semi-characteristic, then modulo 2:

$$\chi^*(\partial V^{2r}) = \chi(V^{2r}) + \varphi,$$

where $\varphi$ is the rank of the cup-product matrix of $H^r(V^{2r}, \partial V^{2r}; Q)$. (There is a better proof of this formula in "Relative characteristic classes!")

If $r$ is odd, $\varphi$ is congruent 0 modulo 2 because $u \cdot u = 0$ for every $u \in H^r(V, \partial V; Q)$. From the existence of the gradient field of $f$ over $V$, it follows that $\chi(V) = 1$ modulo 2, and since $p_k' = p_k$ for $i < k$, one has $p_k' \neq p_k$. 

100 Bank Street
New York 14, N.Y.
If \( r \) is even, you have reduced the problem to the case where

\[ \text{rk}_k(N) \cong \text{rk}_k(M; \mathbb{Z}_2) \cong \mathbb{Z}_2^+ \cdots \mathbb{Z}_2^+ \]

What I have said before is, I believe, still true, regarding \( p_1, p_1' \) as being rank \( \text{rk}_k(M; \mathbb{Z}_2) \), rank \( \text{rk}_k(M'; \mathbb{Z}_2) \) and replacing "of infinite order" by "non-zero", and \( p_k - 1 \geq p_1' \leq p_k \).

We still have to prove that \( p_1' \neq p_k' \), and this is apparently sufficient. This is equivalent to proving \( \varsigma = 0 \) modulo 2, where \( \varsigma \) is now the rank of the cup-product matrix of \( H^r(V, \partial V; \mathbb{Z}_2) \).

Conjecture: If \( M^{2k+1} \) is a \( n \)-manifold, then \( V \) is also a \( n \)-manifold \( \text{??}\).

If this is true, the statement \( \varsigma = 0 \mod 2 \) follows from \$5$ of my thesis, page 239.

If the conjecture is wrong, I don't know how to prove \( \varsigma = 0 \mod 2 \).

Best regards.
December 15, 1959

Dear Michel,

Your argument sounds good. One thing bothers me: does it only apply to a compact manifold without boundary?

It is known that every compact $\pi_1$-manifold without boundary represents the trivial cobordism class. Hence a series of $X$-constructions can be used to reduce it to a sphere.

The conjecture which you mention is correct and will be included in the paper, which I am still trying to get into shape. If $2p+1 \leq n$ and if the imbedding $f : S^p \times D^{n-p} \rightarrow M^n$ is correctly chosen within its homotopy class, where
M^n is a \(n\)-manifold, then the construction yields a parallelizable \((n+1)\)-dimensional manifold with boundaries \(M^n\) and \(X(M^n)\).

I am afraid that I have no good reason for the terminology "construction". It seemed to be convenient for such notation as \(X(V_f)\) (= the manifold obtained from \(V\) by the \(X\)-construction using the imbedding \(f\)) or "\(X\)-equivalent". It didn't occur to me that it conflicted with the notation for the characteristic or semi-characteristic.

What do you have in mind as application for the argument in your letter? Is it possible to prove that the groups \(\mathbb{H}^{2n-2}(\pi_k)\) (which I defined in DM, a.H.S.) are zero? Is it possible to prove that there exists a homotopy sphere \(M^{8k+1}\) which is not a \(\pi\)-manifold, assuming that the appropriate \(f\)-homomorphism is zero?

Sincerely, John

* \(X\) can be taken as an abbreviation for "Chirurgie"
Dear John:

The argument in my last letter is I think OK for a manifold with boundary provided the boundary is a homotopy sphere. Let $M^{2k+1}$ be the manifold with boundary $\Sigma$, and $M_2$ the mirror image. Perform the constructions on $M = M_1 \cup M_2$, leaving $M_2$ alone. If $\Sigma$ is a homotopy sphere, there will be no "interaction" between the homology of $M_1$ and the homology of $M_2$ in $H_n(M)$.

I did have in mind that $J_{C_{S^g}} = 0$ should imply existence of a $(2s+1)$-homotopy sphere which is not a $n$-manifold. It seems OK now, as well as $\partial^2f(\partial f) = 0$.

There is a series of more or less conjectural statements as follows:

Case I. $\pi_{n+2k}(S^n)$ stable, $S^k$ parallelizable.

For every $a \in \pi_{n+2k}(S^n)$ take $f \in a$ such that $f^{-1}(a) = H^{2k}$ is $(k-1)$-connected. Let $A_1, \ldots, A_q, B_1, \ldots, B_q$ be a "canonical" basis of $H_k(M^{2k}; \mathbb{Z})$. I.e. $A_1 \cdot A_j = B_1 \cdot B_j = 0, A_1 \cdot B_j = \delta_{1j}$. Represent $A_i, B_j$ by imbedded spheres $\alpha_i : S^k \to M^{2k}, \beta_j : S^k \to M^{2k}$. Take fields of normal $k$-frames $\tau_i, \sigma_j$ over $\alpha_i(S^k), \beta_j(S^k)$ respectively. Define
\( \lambda_i \) (resp. \( \mu_j \)) to be the Steenrod-Hopf invariant of \( \mathbb{F}_n \mathbb{F}_n \) \( \mathbb{F}_n \mathbb{F}_n \) (resp. \( \mathbb{F}_n \mathbb{F}_n \)), where \( \mathbb{F}_n \) is the field of normal \( n \)-frames over \( \mathbb{F}_{2^k} \) in \( S^{n+2k} \).

Stern's sequence \( \gamma_k : \pi_k(\text{SO}(k)) \to \pi_k(\text{SO}(k+2)) \to S^2 \to 0 \) is exact if \( S^k \) is parallelizable; it follows that \( \lambda_i, \mu_j \) are well defined modulo 2.

Define \( \pi_{n+2k}(S^n) \to S^2 \) by \( \gamma(a) = \Sigma_1 \lambda_i \mu_j \). For \( k = 1 \), Pontryagin shows that this is indeed well defined, and a homomorphism.

Lemma. If \( \gamma(a) = 0 \), there exists \( f \in a \) such that \( f^{-1}(a) \) is homotopy sphere for some \( a \in S^n \).

Corollary. There exists an exact sequence

\[
0 \to S^{2k} \to \pi_{n+2k}(S^n) \to Z_2 \to 0
\]

for \( k = 1, 3 \) and 7. (n large.)

Corollary. \( S^6 = 0 \). (I don't have Yamanoshita on hand to see what this means for \( S^1 \).)

Case II. \( \pi_{n+2k}(S^n) \) stable, \( k \) odd, \( S^k \) not parallelizable.

For every \( a \in \pi_{n+2k}(S^n) \) pick \( f \in a \) with \( f^{-1}(a) = S^{2k} \) connected. Use your function \( \varphi : H_k(S^{2k}; Z_2) \to Z_2 \) to define

\[
h = \Sigma_1 \varphi(A_1) \varphi(B_1),
\]

where \( A_1, \ldots, A_q, B_1, \ldots, B_q \) is a canonical basis. This expression does not depend on the choice of the basis (provided it is a canonical basis). Is this the Arf invariant?

Do you know whether or not \( h \) is a homotopy invariant \( \pi_{n+2k}(S^n) \to Z_2 \)?

Also, if \( \gamma \) (Case I) is homotopy invariant, it is certainly surjective (it takes value 1 on the composition of a Hopf map with itself). Do you know whether \( h \) is surjective? If \( h \) is homotopy invariant, then

\[
0 \to S^{2k}(\pi) \to \pi_{n+2k}(S^n) \to Z_2 \to 0^{2k+1}(\partial \pi)
\]

is exact.

Case III. \( \frac{S^{2k+1}(\pi)}{\partial \pi} \to \pi_{n+2k+1}(S^n)/3\pi_{2k+1}(SO(n)) \).
Case IV. $e^{hr} = \pi_{n+hr}(s^n)/J_{n+hr}(s^0(n))$.

Best regards,
Dear John:

Enclosed are some more details about the proof of the statements in my last letter in Case I. At the end I have listed the $\mathcal{X}$-theorems which are needed.

As far as Case II is concerned, one should be able to prove that there exists an exact sequence

$$0 \rightarrow \pi_{2k}^2(\pi) \rightarrow \pi_{2k} \rightarrow \mathbb{Z}_2 \rightarrow \pi_{2k-1}^{2k-1}(\pi) \rightarrow \pi_{2k-1}/\text{Im } J \rightarrow 0$$

for $k$ odd and $S^k$ not parallelizable.

The homomorphism $\mathbb{Z}_2 \rightarrow \pi_{2k-1}^{2k-1}(\pi)$ being defined as follows: Let $U, U'$ be two copies of the tubular neighborhood of the diagonal in $S^k \times S^k$. Let $X$ be obtained from the disjoint union $U \cup U'$ by identification of a coordinate neighborhood $R_1^k \times R_2^k$ with its copy $R_1^k \times R_2^k$ under $R_1 \times R_2 \leftrightarrow R_2 \times R_1$. The boundary of $X$ is a homotopy sphere, image of $1 \in \mathbb{Z}_2$ under $\mathbb{Z}_2 \rightarrow \pi_{2k-1}^{2k-1}(\pi)$.

In my opinion, the main problem now would be to decide for which values of $k$ the boundary of $X$ represents the zero $J$-equivalence class.

Best wishes for the new year.
Let $V$ be a finite dimensional vector space over $\mathbb{Z}_2$ with a commutative bilinear product $V \times V \rightarrow \mathbb{Z}_2$ satisfying

1. $x \cdot x = 0$ for every $x \in V$,
2. $a \cdot x = 0$ for every $x \in V$ implies $a = 0$.

It follows that dim $V$ is even; dim $V = 2q$. A basis $a'_1, \ldots, a'_q, b'_1, \ldots, b'_q$ of $V$ is said to be canonical if $a'_i \cdot a'_j = b'_i \cdot b'_j = 0$ and $a'_i \cdot b'_j = \delta_{ij}$ $(1 \leq i, j \leq q)$ There exists at least one canonical basis.

Let $\varphi : V \rightarrow \mathbb{Z}_2$ be a function satisfying

$$\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y$$

**Lemma 1.** Let $a'_1, \ldots, a'_q, b'_1, \ldots, b'_q$ and $a'_1', \ldots, a'_q', b'_1', \ldots, b'_q'$ be two canonical bases of $V$. Then

$$\Gamma = \sum_1^q \varphi(a'_i) \varphi(b'_i) = \sum_1^q \varphi(a'_i') \varphi(b'_i').$$

**Proof.** (Compare L. Pontryagin [1].) One proves that successive transformation of the basis $a'_i, b'_i$ not altering $\sum \varphi(a'_i) \varphi(b'_i)$ bring $a'_i, b'_i$ into $a_i, b_j$. Assume by induction that $a'_k = a_k$ and $b'_k = b_k$ for $r < k \leq q$. Then, $a_r$ is a linear combination of $a'_i, b'_j$ with $i, j \leq r$,

$$a_r = \alpha_1 a'_1 + \ldots + \alpha_r a'_r + \beta_1 b'_1 + \ldots + \beta_r b'_r.$$

One of the coefficients is $\neq 0$. After possible permutation of the indices and interchange of $a$ and $b$, we can assume $\alpha_r = 1$.

Define a new basis $u'_1, \ldots, u'_q, v'_1, \ldots, v'_q$ by

$$u'_i = a'_i + \beta_i b'_i, \quad v'_i = b'_i + \alpha_i b'_i \quad \text{for } 1 \leq i \leq r-1$$

$$u'_r = a'_r, \quad v'_r = b'_r$$

$$u'_k = a'_k, \quad v'_k = b'_k \quad \text{for } r < k \leq q.$$

The new basis is canonical, and
\[ \sum_1^q \varphi(u_1) \cdot \varphi(v_1) = \sum_1^{r-1} \varphi(a_1^i + \beta_i b_1^i) \cdot \varphi(b_1^i + a_1 b_1^i) + \varphi(a_1) \cdot \varphi(b_1^i) + \ldots \]

\[ = \sum_1^q \varphi(a_1^i) \cdot \varphi(b_1^i) + A, \]

where

\[ A = \varphi(b_1^i) \left[ \sum_1^{r-1} (\beta_i \varphi(b_1^i) + a_1 \varphi(a_1^i) + a_1 \beta_i) + (\varphi(a_1) + \varphi(b_1^i)) \right] \]

The expression in brackets is zero because

\[ \varphi(a_1) = \sum_1^{r-1} (\alpha_i \varphi(a_1^i) + \beta_i \varphi(b_1^i) + a_1 \beta_i) + \varphi(a_1) + \beta_i (1 + \varphi(b_1^i)), \]

and

\[ \beta_i \varphi(b_1^i)(1 + \varphi(b_1^i)) = 0. \]

Claim:

\[ b_r = \tau_1 u_1 + \ldots + \tau_r u_r + \sigma_1 v_1 + \ldots + \sigma_{r-1} v_{r-1} + v_r. \]

Indeed, the coefficient of \( v_r \) in the expansion of \( b_r \) is given by

\[ b_r \cdot u_r = b_r \cdot a_r = 1. \]

Interchanging \( u \) and \( v \) and applying the same procedure leads to a new canonical basis \( u_1^i, \ldots, u_q^i, v_1^i, \ldots, v_r^i \) such that

\[ u_k^i = a_k \quad \text{and} \quad v_k^i = b_k \quad \text{for} \quad r \leq k \leq q, \]

and

\[ \sum_1^q \varphi(u_1^i) \cdot \varphi(v_1^i) = \sum_1^q \varphi(a_1^i) \cdot \varphi(b_1^i). \]

Let \( \pi_{2k} \) be the stable homotopy group \( \pi_{n+2k}(s^n), \quad 2k+2 \leq n, \)

and \( \mathfrak{e}^{2k} \) as in J. Milnor [2].

**Theorem 1.**– For \( k = 1, 3, 7 \) there is an exact sequence

\[ 0 \longrightarrow \mathfrak{e}^{2k} \longrightarrow \pi_{2k} \longrightarrow \pi_{2k} \longrightarrow 0. \]

By [2], Corollary 6.8, \( \mathfrak{e}^{2k} \mathfrak{(n)}/\mathfrak{e}^{2k} \mathfrak{(O(n))} \) is naturally isomorphic to a subgroup of \( \pi_{n+2k}(s^n)/J\pi_{n+2k}(SO(n)) \). For \( k = 1, 3 \) or \( 7, \mathfrak{e}^{2k} = \mathfrak{e}^{2k} \mathfrak{(n)} \) and \( \mathfrak{e}^{2k} \mathfrak{(O(n))} = 0 \) by [2], Theorem 5.13.

Since \( \pi_{2k}(SO(n)) = 0 \) for \( k = 1, 3 \) or \( 7, \) we have exactness of

\[ 0 \longrightarrow \mathfrak{e}^{2k} \longrightarrow \pi_{2k} \]
We proceed to the definition of the homomorphism

$$\Gamma : \pi_{2k} \rightarrow \mathbb{Z}_2^*$$

Let $\alpha \in \pi_{n+2k}(S^n)$. Let $f : S^{n+2k} \rightarrow S^n$ be a $C^0$-map representing $\alpha$ and $\mathcal{M}^{2k} = f^{-1}$ (reg. value). If $\alpha$ a field of normal $n$-frames over $\mathcal{M}^{2k}$ such that $\alpha$ is associated with $(\mathcal{M}^{2k}; \mathcal{F}_n)$.

Applying Theorem A, we obtain a $(k-1)$-connected $\pi$-manifold of dimension $2k$ imbedded in $\mathbb{R}^{n+2k}$ and a field of normal $n$-frames over it associated with the same $\alpha$.

I.e. we may assume $\mathcal{M}^{2k}$ to be $(k-1)$-connected. Then $H_k(\mathcal{M}^{2k}; \mathbb{Z})$ is a finitely generated free abelian group. Set $V = H_k(\mathcal{M}^{2k}; \mathbb{Z}_2)$ and define $x \cdot y$ to be the intersection coefficient of $x, y \in V$.

The axioms (1) and (2) of page 01 are satisfied.

Define a function $\varphi : V \rightarrow \mathbb{Z}_2$ as follows: For every $x \in V$ let $X \in H_k(\mathcal{M}^{2k}; \mathbb{Z})$ be such that $X \equiv x$ modulo 2, and let $J_X : S^k \rightarrow \mathcal{M}^{2k}$ be a completely regular immersion representing $X$. The normal bundle (in $\mathcal{M}^{2k}$) of $J_X$ is trivial ($S^k$ is parallelizable). Let $\tau$ be a field of normal $k$-frames. The imbedding of $\mathcal{M}^{2k}$ in $\mathbb{R}^{n+2k}$ induces an immersion of $S^k$ into $\mathbb{R}^{n+2k}$ with a field $\tau \times \mathcal{F}_n$ of normal $(k+n)$-frames. Let $\omega_x$ be the "degree" of the induced map $S^k \rightarrow V_{n+2k}$, $n+k$. Define

$$\varphi(x) = \omega_x + S(J_X) \cdot 1$$

where $S(J_X)$ is the self-intersection coefficient of the imersion $J_X : S^k \rightarrow \mathcal{M}^{2k}$. To be proved:

(a) $\varphi(x)$ does not depend on the choice of $\tau$ (under fixed $X$ and $J_X$);

(b) $\varphi(x)$ does not depend on $J_X$ (under fixed choice of $X$).

Clearly then, $\varphi(x)$ does not depend on the choice of $X$. 
It is easily seen that if $J_x : J_y : S^k \to M^{2k}$ are immersions representing $x$ and $y$ respectively, there exists an immersion $J_{x+y} : S^k \to M^{2k}$ such that
\[ \omega_{x+y} = \omega_x + \omega_y + 1, \]
\[ S(J_{x+y}) = S(J_x) + S(J_y) + x \cdot y. \]

It follows that $\varphi$ satisfies
\[ \varphi(x+y) = \varphi(x) + \varphi(y) + x \cdot y. \]

**Proof of (a).** Let $X \subset H_k(M^{2k},Z)$ and $J_x : S^k \to M^{2k}$ representing $X$ be fixed. Let $\tau, \tau'$ be two fields of normal $k$-frames over $J_x(S^k)$ in $M^{2k}$. There exists a map $\delta : S^k \to SO(k)$ such that $\tau'(u) = \delta(u) \cdot \tau(u)$ for every $u \in S^k$. If $\delta \in \pi_k(SO(k))$ also denotes the homotopy class of $\delta$, and $i^n_\ast : \pi_k(SO(k)) \to \pi_k(SO(n+k))$ is induced by the natural inclusion, then
\[ \omega(\tau') = \omega(\tau) + j_\ast i^n_\ast \delta, \]
where $j_\ast : \pi_k(SO(n+k)) \to \pi_k(V_{n+2k}, n+k)$ is natural.

If $S^k$ is parallelizable, $i^n_\ast \delta$ is divisible by 2. Therefore
\[ \omega(\tau') = \omega(\tau). \]

**Proof of (b).** Let $T_k(M^{2k})$ be the maximal space of the bundle of tangent $k$-frames on $M^{2k}$. We have a diagram
\[ \begin{array}{cccccc}
0 & \to & \pi_k(V_{2k},k) & \to & \pi_k(T_k(M^{2k})) & \to & \pi_k(M^{2k}) & \to & 0 \\
& & \pi_k(V_{n+2k}, n+k) & & \pi_k(M^{2k}) & & \\
\end{array} \]
where the row is exact.

Let $J_0 : S^k \to M^{2k}$ be an immersion with just one self-intersection point, $S(J_0) = 1$, and such that $J_0(S^k)$ is contained in some euclidean neighborhood on $M^{2k}$. (Compare
Proof of (b). Let $T_k(M^{2k})$ be the space of the bundle of tangent $k$-frames on $M^{2k}$. According to M. Hirsch [6] the regular homotopy classes of immersions $S^k \to M^{2k}$ stand in 1-1 correspondence with the $SO(k)$-equivariant homotopy classes of $S^k$-equivariant maps $SO(k+1) \to T_k(M^{2k})$. Since we assume $S^k$ to be parallelizable, this is the same as the homotopy classes of maps $S^k \to T_k(M^{2k})$. The imbedding $f : M^{2k} \to R^{n+2k}$ induces a map $f^* : T_k(M^{2k}) \to V_{n+2k}, n+k$ given by $\tau \mapsto f^*(\tau) \times F_n$. We have a diagram

\[
\begin{array}{cccc}
\pi_k(V_{2k}, k) & i^* & \pi_k(T_k(M^{2k})) & p_* \\
\downarrow f^* & & \downarrow \pi_k(M^{2k}).
\end{array}
\]

Let $J, J' : S^k \to M^{2k}$ be two immersions which are homotopic as maps. Choosing fields of normal $k$-frames $\tau$ and $\tau'$ we obtain liftings $\tilde{J}, \tilde{J}' : S^k \to T_k(M^{2k})$.

Denote the sum of regular homotopy classes of immersions $J, J' : S^k \to M^{2k}$ by $J \cup J'$. This gives a group structure in the set $\pi_k(T_k(M^{2k}))$ of immersions of $S^k$ in $M^{2k}$ which does not coincide with the group structure of $\pi_k(T_k(M^{2k}))$ as homotopy group. Indeed, $J : S^k \to M^{2k}$, the standard imbedding of $S^k$ in some euclidean neighborhood on $M^{2k}$ is the zero of the group of immersions but the corresponding homotopy class in $\pi_k(T_k(M^{2k}))$ is $i_* c$, where $c$ generates $\pi_k(V_{2k}, k)$. On the other hand, the zero homotopy class in $\pi_k(T_k(M^{2k}))$ corresponds to the immersion $J_1 : S^k \to M^{2k}$ with $J_1(S^k)$ contained in some euclidean neighborhood on $M^{2k}$ and precisely one selfintersection point.

Since...
Let \( s_k \) be a fixed field of tangent \( k \)-frames over \( S^k \). With every immersion \( j: S^k \to M^{2k} \) is associated a lifting \( \tilde{j}: S^k \to T_k(M^{2k}) \) given by \( s_k \) and \( j \).

Let \( j_0, j_1: S^k \to M^{2k} \) be respectively a trivial immersion and a Whitney immersion (with precisely one self-intersection point). Define \( \tau(j) = \tilde{j} - \tilde{j}_0 \). If \( j \) is obtained as a sum of \( j' \) and \( j'' \), then \( \tau(j) = \tau(j') + \tau(j'') \).

One has \( f^*(\tau(j)) = \omega_j + 1 \).

Let \( j' \) and \( j'' \) be homotopic (as maps), then \( \tau(j') - \tau(j'') \) is in the kernel of \( p_* \). Since \( \text{Im} \ i_* \) is generated by \( \tau(j_1) \), it follows

\[
\tau(j') = \tau(j'') + a \cdot \tau(j_1) = \tau(j'' + a \cdot j_1)
\]

By M. Hirsch, this means that \( j' \) is regularly homotopic to \( j'' + a \cdot j_1 \). Thus \( S(j') = S(j'' + a \cdot j_1) = S(j'') + a \).

Applying \( f^* \) to the equation \( \tau(j') = \tau(j'') + a \cdot \tau(j_1) \) and using \( f^*(\tau(j_1)) = 1 \), we get

\[
\omega_j + 1 + S(j') = \omega_j + 1 + S(j'') \quad \text{modulo } 2.
\]

Q.E.D.
Since $\Gamma'$ is well defined for a pair $(M^{2k}; F_n)$, where $M^{2k}$ is the disjoint union of $(k-1)$-connected closed manifolds, and clearly additive with respect to the disjoint union of manifolds in $\mathbb{R}^{n+2k}$ with fields of normal $n$-frames, the proof of the homotopy invariance of $\Gamma'$ amounts to proving that $\Gamma'(M^{2k}; F_n) = 0$ if $(M^{2k}; F_n)$ is the restriction of the boundary of some $(M^{2k+1}; F_n)$.

There exists a canonical basis of $H_k(M^{2k}; Z)$ such that $A_1, \ldots, A_q$ is a basis of the kernel of $H_k(M^{2k}) \to H_k(M^{2k+1})$.

By theorem $\text{K}_2$, we can make $W$ to be $(k-1)$-connected without changing the field $F_n$ on the boundary. It follows that $J_x : S^k \to M^{2k}$, immersion representing $x \in [A_1, \ldots, A_q]$ is homotopic to zero in $M^{2k+1}$. Let $A$ be anyone of the classes $A_1, \ldots, A_q$, and $J : S^k \to M^{2k}$ an imbedding representing $A$. (Compare J. Milnor [2], Theorem 5.9.) Let $\tau$ be a field of normal $k$-frames over $J(S^k)$. Since $\varphi(a) = \omega_a + 1$ is a homotopy invariant of the sphere map associated with $J(S^k)$ and $\tau \in F_n$, and since $F_n$ is extended all over $W$, it is sufficient to show that the map $M^{2k} \to S^k$ associated with $J(S^k)$ and $\tau$ can be extended to a map $M^{2k+1} \to S^k$. The only obstruction to such an extension lies in $H^{k+1}(W, M; Z)$. The Poincaré dual in $H_k(W; Z)$ of this obstruction is the image of $A$ under $H_k(M^{2k}; Z) \to H_k(M^{2k+1}; Z)$. It follows that the obstruction is zero. Q.E.D.
If $\alpha, \beta \in \pi_k$ and $h(\alpha), h(\beta)$ is the Steenrod-Hopf invariant of $\alpha, \beta$ respectively. Then $\Gamma(\alpha \circ \beta) = h(\alpha) \cdot h(\beta)$. Therefore is surjective.

Let $\alpha \in \pi_{2k}$ be an element in $\ker \Gamma$. Represent $\alpha$ by a manifold $M^{2k}$ imbedded in $\mathbb{R}^{n+2k}$ with a field of normal $n$-frames $F_n$. We can assume that $M^{2k}$ is $(k-1)$-connected. Since $\Gamma(M^{2k}; F_n) = 0$, there exists a canonical basis $A_1, \ldots, A_q, B_1, \ldots, B_q$ of $H_k(M^{2k}; \mathbb{Z})$ such that $\varphi(A_1) = \varphi(A_2) = \ldots = \varphi(B_q) = 0$.

By Theorem 1.3.2, $(M^{2k}; F_n)$ is homotopic to $(\Sigma^{2k}; G_n)$ where $\Sigma^{2k}$ is a homotopy sphere.
Theorem \(\lambda_1\): Let \(M^d\) be a closed differentiable manifold imbedded in \(R^{d+n}\), where \(n\) is to be large. Let \(F_n\) be a field of normal \(n\)-frames over \(M^d\). There exists \(M'^d\) in \(R^{d+n}\) with a field \(F'_n\) of normal \(n\)-frames such that \(M'^d\) is \([\frac{d-1}{2}]\)-connected and \((M^d; F_n)\) is homotopic to \((M'^d; F'_n)\).

Theorem \(\lambda_2\): If \((M^{d+1}; F_n)\) is a homotopy between \((M'^d; F'_n)\) and \((M'^d; F'^n)\), i.e., \(\mathcal{W} = M^d - M'^d\) and \(F'_n = F_n|_{M'^d}, F'^n = F_n|_{M'^d}\), and if \(M', M''\) are \([\frac{d-1}{2}]\)-connected, then there exists a homotopy \((\mathcal{W}_{d+1}; F_n)\) between \((M'; F'_n)\) and \((M''; F'^n)\) such that \(\mathcal{W}_{d+1}\) is \([\frac{d-1}{2}]\)-connected.

Theorem \(\lambda_3\): Given \((M^{2k}; F_n)\) where \(M^{2k}\) is \((k-1)\)-connected. Then \((M^{2k}; F_n)\) is homotopic to some \((M'; F'_n)\) where \(M'\) is a homotopy sphere iff \(\Gamma(M^{2k}; F_n) = 0\). If \(S^k\) is parallelizable \(\Gamma'\) is defined in the text (page 63). If \(S^k\) is not parallelizable \(\Gamma'\) is as in your letter of Nov. 19.

N.B. to the proof of homotopy invariance of $\Gamma$. (Case I, bottom of page 07.) The map $M^{2k} \to s^k$ defined associated with $J(s^k)$ and $\tau$ can be extended to $W - U \to s^k$, where $U$ is a spherical neighborhood of some point $c$ Int $W$. Thus the map associated with $J(s^k)$ and $\tau \times P_n$ is homotopic to the $n$-th suspension of a map $s^{2k} \to s^k$. The Steenrod-Hopf invariant of such an animal is zero.

Case II.

Definition of $\Gamma$: $\pi_{2k} \to \mathbb{Z}_2$ for $k$ odd, and $s^k$ not parallelizable.

According to M. Hirsch [3], the map $J \to \pi_k(T_k(M^{2k}))$ is bijective, copied from the definition of the Smale invariant is bijective. $(M^{2k}$ unbounded compact manifold, $T_k(M^{2k})$, the space of the bundle of tangent $k$-frames over $M^{2k}$, and $J$ the set of regular homotopy classes of immersions $s^k \to M^{2k}$.)

If $j \in J$, denote by $[j]$ the corresponding element in $\pi_k(T_k(M^{2k}))$. The argument on page 125 of [4] yields

$$[j] = [j'] + [j'']$$

if $j$ is constructed as sum of $j'$ and $j''$. Let $j_1$ be a Whitney immersion.

**Lemma 2.** Let $f : \pi_k(T_k(M^{2k})) \to \mathbb{Z}_2$ be any homomorphism, then there is a function $\varphi : \pi_k(M^{2k}) \to \mathbb{Z}_2$ defined by $\varphi(a) = f([j] + B(j))$, where $j$ is any immersion representing $a$. 
Let \( p : T_k(M^{2k}) \to L_k(M^{2k}) \) be the projection of a \( k \)-frame.

If \( M^{2k} \) is almost parallelizable, there is a map \( f : \pi_k(T_k(M^{2k})) \to \mathbb{Z}_2 \) given by normal bundle. \( f \) is a homomorphism. If \( M^{2k} \) is \((k-1)\)-connected this yields a function \( \varphi : H_k(M^{2k}; \mathbb{Z}_2) \to \mathbb{Z}_2 \) satisfying \( \varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y \).

**Proof of Lemma 2.** Since \( p^*[j] = \text{homotopy class of } j \), where \( p^* : \pi_k(T_k(M^{2k})) \to \pi_k(M^{2k}) \), it follows that if \( j' \) and \( j'' \) are homotopic immersion \( S^k \to M^{2k} \), then

\[
[j'] - [j''] = a[j_1],
\]

for some \( a \in \mathbb{Z}_2 \), where \( j_1 \) is a Whitney immersion. (\( S(j_1) = 1 \) and \( p^*[j_1] = 0 \)). Thus \( j' \) and \( j'' + a j_1 \) are regularly homotopic. Therefore \( S(j') = S(j'') + a \). It follows

\[
f[j'] + S(j') = f[j''] + S(j'').
\]

\( \Gamma \) is thus well defined and additive on pairs \((M^{2k}; F_n)\), where \( M^{2k} \) is a \((k-1)\)-connected unbounded manifold in \( \mathbb{R}^{n+2k} \) and \( F_n \) is a field of normal \( n \)-frames over \( M^{2k} \). To prove the homotopy invariance of \( \Gamma \) it is sufficient to prove that \( \Gamma(M^{2k}; F_n) = 0 \) if \( M^{2k} = \partial W^{2k+1} \) where \( W^{2k+1} \) is a manifold in \( \mathbb{R}^{n+2k+1} \) on which \( F_n \) can be extended as a field of normal \( n \)-frames. It is sufficient to prove \( \varphi(A) = 0 \) for \( A \) in the kernel of \( H_k(M^{2k}; \mathbb{Z}) \to H_k(W^{2k+1}; \mathbb{Z}) \). Let \( j : S^k \to M^{2k} \) be an imbedding representing \( A \). If the normal bundle of \( j \) were nontrivial we would get a map \( f : K^{2k} \to S^k \cup e^{2k} \) (where \( e^{2k} \) is attached by \([i_k, i_k]\))
such that $f_*: H_{2k}^j(S^{2k}; Z) \to H_{2k}^j(S^k \vee e^{2k})$ is an isomorphism.

Again, the extension of $f$ is possible over $W$ except possibly in some spherical neighborhood. The boundary of this neighborhood being $S^{2k}$ we get that the top cycle of $S^k \vee e^{2k}$ is spherical.

I.e. $[1_k, 1_k] = 0$. This contradicts J.P. Adams if $S^k \neq 1, 3, 7$.

(Of course the $J$-construction, theorem $J_2$, has to be used again to make $W(k-1)$-connected and $H^{q+1}(W, M, G) = 0$ for $k < q < 2k$.)

Theorem 2. - For $k \neq 1, 3, 7$ there is an exact sequence

$$0 \to e^{2k}(\pi) \to \pi_{2k} \to z_2 \to e^{2k-1}(\pi) \to \pi_{2k-1}/J \to 0$$

If $\Sigma^{2k-1}$ is a homotopy sphere which bounds a $\pi$-manifold $v^{2k}$, then theorem $J_2$ yields a $v^{2k}$ which is $(k-1)$-connected. Further $J$-construction leaves us either with $v^{2k}$ having the homotopy type of a disk, or $H_k(v^{2k}; Z) \cong Z + Z^2$ with generators represented by imbeddings $j^i: S^k \to v^{2k}$, $j^a: S^k \to v^{2k}$ with $S(j^i, j^a) = 1$ and both normal bundles trivial. If $U$ is a neighborhood of $j^i(S^k) \cup j^a(S^k)$, contractible on $j^i(S^k) \cup j^a(S^k)$, then $U^*$ is a homotopy sphere which is $J$-equivalent to $\Sigma^{2k-1}$. This proves exactness at $e^{2k-1}(\pi)$. 

odd and
March 15, 1960

Dear Michel,

I am still trying to study your last letter, but keep getting sidetracked on other things.

There are two new developments since I wrote last. C.T.C. Wall has written to me indicating that he is also working on these questions, and that he can prove the assertion $\hat{H}^{2k}(\pi_n) = 0$ as well as the assertion $\hat{H}^6 = 0$. He included some details in his letter, but not enough for me to follow. I told him that you had also proved these assertions.

**

O. H. Wallace sent me a copy of a manuscript

* The Loft
  Malton Lane
  Cambridge

** Indiana Univ.
  Bloomington
which should appear in the Canadian Journal in April. This overlaps a great deal with the manuscript which I sent you a few weeks ago. (You probably have received it by now.) However, there is no overlap with what you have done. Wallace uses the term "spherical modification." This does seem better to me than "surgery" or "X-construction." What do you think?

There is an upper Wallace was led to the concept via a forthcoming paper by Aeppli dealing with modifications of algebraic varieties. In any case I plan to publish the manuscript more or less as it stands in the proceedings of the conference on differential geometry which was recently held in Tucson.

I will try to write a more mathematical letter later.

Sincerely, John
May 20, 1960

Dear Michel,

The manuscript which you sent me is very nice. I had tried to prove the existence of a manifold without differentiable structure for a long time, without success.

Smale has announced the same result (in dimensions 8, 12, ...) by a completely different argument. He claims to have proved that, for n \geq 3, 4,

- every $C^\infty$ n-manifold which is a homotopy sphere is homeomorphic to $S^n$ for all $n \geq 3, 4$;
- combinatorially equivalent to $S^n$ for $n$ even.

Using my example of a homotopy 7-sphere which bounds a 3-connected 8-manifold with index 8, it follows that there exists an 8-manifold without differentiable structure.

However, your example is simpler, and
is also sharper in a way. The 10-manifold can be triangulated so that the star of each vertex is a combinatorial cell, whereas this is not known in Smale's examples.

Wall has sent me a mimeographed note proving that \( H^2_3(\mathbb{H}) = 0 \).

Sincerely,

John
Dear Michel,

Unfortunately I haven't gotten too far with our manuscript. The following absurd difficulty came up. It seems to me that the relation of $f$-cobordism as defined is not symmetric. At least for 1-dimensional manifolds there is a definite asymmetry. In higher dimensions I don't really know what happens. In any case some patchwork seems to be needed. There are many possibilities, none of which really appeals to me. (E.g. using $(n+2)$-frames or co-frames in place of $(n+1)$-frames, or dropping the concept of $f$-cobordism completely.)

Perhaps you will have a good idea by the time I get to Berkeley. (Circa July 16)

I have been trying to work on the conjecture that the various exact sequences

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow F_n \rightarrow \cdots \rightarrow \pi_n \rightarrow \cdots$$

I have been trying to work on the conjecture that the various exact sequences

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow F_n \rightarrow \cdots \rightarrow \pi_n \rightarrow \cdots$$
are isomorphic to those of a triple
\[ S^N \cong \text{combinatorial automorphism group of } S^{N-1} \]

The following seems to be a promising candidate for the middle object. Let \( C_{n,k} \) be the c.s.s. group whose \( k \)-simplices are piecewise linear maps
\[(\text{standard } k\text{-simplex}) \times (\text{neighborhood of } 0 \text{ in } \mathbb{R}^N) \to \mathbb{R}^N\]
such that, for each fixed coordinate in the simplex, one obtains a PL-embedding
\[(\text{neighborhood of } 0, 0) \to (\mathbb{R}^N, 0)\]
Two such are to be identified if they coincide over a smaller neighborhood.

Then given any combinatorial manifold one can define a c.s.s. "tangent bundle" acted with \( C_{n,0} \) as structural group.

With best regards

Jack