BERNOULLI NUMBERS, HOMOTOPY GROUPS, AND A THEOREM OF ROHLIN

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A homomorphism $J: \pi_{k-1}(SO_m) \to \pi_{m+k-1}(S^m)$ from the homotopy groups of rotation groups to the homotopy groups of spheres has been defined by H. Hopf and G. W. Whitehead [14]. This homomorphism plays an important role in the study of differentiable manifolds. We will study its relation to one particular problem: the question of possible Pontrjagin numbers of an ‘almost parallelizable’ manifold.

Definition. A connected differentiable manifold $M^k$ with base point $x_0$ is almost parallelizable if $M^k - x_0$ is parallelizable. If $M^k$ is imbedded in a high-dimensional Euclidean space $\mathbb{R}^{n+k}$ ($m \geq k+1$) then this is equivalent to the condition that the normal bundle $\nu$, restricted to $M^k - x_0$, be trivial (compare the argument given by Whitehead [17], or Kervaire [19, §8]).

The following theorem was proved by Rohlin in 1952 (see Rohlin [11, 18], Kervaire [20]).

Theorem (Rohlin). Let $M^k$ be a compact oriented differentiable 4-manifold with Stiefel-Whitney class $w_2$ equal to zero. Then the Pontrjagin number $p_1[M^4]$ is divisible by 48.

Rohlin’s proof may be sketched as follows. It may be assumed that $M^4$ is a connected manifold imbedded in $\mathbb{R}^{n+4}$, $m \geq 5$.

Step 1. It is shown that $M^4$ is almost parallelizable.

Let $f$ be a cross-section of the normal $SO_m$-bundle $\nu$ restricted to $M^k - x_0$. The obstruction to extending $f$ is an element

$$o(\nu, f) \in H^2(M^4; \pi_4(SO_m)) \approx \pi_4(SO_m).$$

Step 2. It is shown that $Jo(\nu, f) = 0$.

Since $J$ carries the infinite cyclic group $\pi_3(SO_m)$ onto the cyclic group $\pi_{m+3}(S^m)$ of order 24, this implies that $o(\nu, f)$ is divisible by 24. Now identify the group $\pi_4(SO_m)$ with the integers.

Step 3. It is shown that the Pontrjagin class $p_1(\nu)$ is equal to $\pm 2o(\nu, f)$.

Since by Whitney duality $p_1(\nu) = -p_1$ (tangent bundle), it follows that $p_1[M^4]$ is divisible by 48.

The first step in this argument does not generalize to higher dimensions. However Step 2, the assertion that $Jo(\nu, f) = 0$, generalizes immediately. In fact we have:

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Lemma 1. Let $a \in \pi_{m-1}(SO_m)$; then $Ja = 0$ if and only if there exists an almost parallelizable manifold $M^k \subset R^{m+k}$ and a cross-section $f$ of the induced normal $SO_m$-bundle $\nu$ over $M^k - x_0$ such that $a = o(\nu, f)$.

Step 2 can be replaced by the following. Identify the group $\pi_{m-1}(SO_m)$, $m > 4n$, with the integers (compare Bott [22]). Define $a_n$ to be equal to 2 for $n$ odd and 1 for $n$ even.

Lemma 2. Let $f$ be a stable $SO_m$-bundle over a complex $K$ (dim $K < m$), and let $f$ be a cross-section of $\xi$ restricted to the skeleton $K^{m-1}$. Then the obstruction class $o(f, \xi) \in H^{m-1}(K; \pi_{m-1}(SO_m))$ is related to the Pontrjagin class $p_1(\xi)$ by the identity $p_1(\xi) = \pm a_{m-1} \cdot (2n-1)! \cdot o(\xi, f)$.

Combining Lemmas 1 and 2, we obtain the following theorem.

Define $f_n$ as the order of the finite cyclic group $J_n \pi_{m-1}(SO_m)$ in the stable range $m > 4n$.

Theorem 1. The Pontrjagin number $p_1[M^{4n}]$ of an almost parallelizable 4n-manifold is divisible by $j_n a_n (2n - 1)!$.

(For $n = 1$, this gives Rohlin’s assertion, since $j_1 = 24, a_1 = 2$.)

Proof. This follows since $o(\nu, f)$ must be divisible by $j_n$.

Conversely:

Theorem 2. There exists an almost parallelizable manifold $M^{4n}$ with $p_1[M^{4n}] = j_n a_n (2n - 1)!$.

The proof is clear.

Proof of Lemma 1. Given an imbedding $i: V^{k-1} \to R^{m+k-1}$ of a compact differentiable manifold $V^{k-1}$ into Euclidean space, and given a cross-section $f$ of the normal $SO_m$-bundle over $V^{k-1}$, a well-known procedure due to Thom associates with $i$ and $f$ a sphere mapping $\phi: S^{m+k-1} \to S^m$ (compare Kervaire [20], p. 223).

The map $\phi$ is homotopic to zero if and only if there exists a bounded manifold $Q^k$ with boundary $V^{k-1}$ imbedded in $R^{m+k}$ on one side of $R^{m+k-1}$ such that:

(i) the restriction to $V^{k-1}$ of the imbedding of $Q^k$ is the given imbedding of $V^{k-1}$ in $R^{m+k-1}$;

(ii) $Q^k$ meets $R^{m+k-1}$ orthogonally so that the restriction to $V^{k-1}$ of the normal bundle of $Q^k$ is the normal bundle of $V^{k-1}$ in $R^{m+k-1}$; and

(iii) the cross-section $f$ can be extended throughout $Q^k$ as a cross-section $f$ of the normal $SO_m$-bundle.

These facts follow from Thom [13], ch. I, § 2 and Lemmas IV, 5, IV, 5'.

To obtain Lemma 1 above, take $V^{k-1} = S^{k-1}$ and take $i(S^{k-1})$ to be the unit sphere in $R^{k-1}$. Since the normal $\omega$-plane at each point of $i(S^{k-1})$ in $R^{m+k-1}$ admits a natural basis (consisting of the radius vector followed by the vectors of a basis for $R^{m+k-1}$), the cross-section $f$
provides a mapping \( \alpha: S^{k-1} \to \text{SO}_m \). Let \( \alpha \in \pi_{n+1}(\text{SO}_m) \) be its homotopy class. It is easily seen (compare Kervaire\(^\text{[5]}\), §1.8) that the map \( \phi: S^{m+k-1} \to S^k \) associated with \( \alpha \) and \( \beta \) represents \( \alpha \chi \) up to sign.

If \( Jx = 0 \) then there exists a bounded manifold \( Q^k \subset R^{m+k} \) satisfying conditions (i), (ii) and (iii). Let \( M^k = R^{m+k} \) denote the unbounded manifold obtained from \( Q^k \) by adjoining a \( k \)-dimensional hemisphere, which lies on the other side of \( R^{m+k-1} \) and has the same boundary \( i(S^{k-1}) \). Since the normal bundle \( v \) restricted to \( Q^k \) has a cross-section \( f' \), it follows that \( M^k \) is almost parallelizable. Clearly the obstruction class \( \varphi(v, f') \) is equal to \( \alpha \).

Conversely, let \( M^k \) be a manifold imbedded in \( R^{m+k} \) and let \( f \) be a cross-section of the normal bundle \( v \) restricted to \( M^k - x_0 \). After modifying this imbedding by a diffeomorphism of \( R^{m+k} \) we may assume that some neighborhood of \( x_0 \) in \( M^k \) is a hemisphere lying on one side of the hyperplane \( R^{m+k-1} \), and that the rest of \( M^k \) lies on the other side. Removing this neighborhood we obtain a bounded manifold \( Q^k \subset R^{m+k} \) just as above, having the unit sphere \( S^{k-1} \subset R^k \subset R^{m+k-1} \) as boundary. The cross-section \( f \) restricted to \( S^{k-1} \) gives rise to a map \( \alpha: S^{k-1} \to \text{SO}_m \) which represents the homotopy class \( \varphi(v, f) \). The argument above shows that \( Jf(v, f) = 0 \); which completes the proof of Lemma 1.

Remark. Lemma 1 could also be proved using the interpretation of \( J \) given in Milnor\(^\text{[6]}\).

Proof of Lemma 2. (Compare Kervaire\(^\text{[5]}\).) The \( \text{SO}_m \)-bundle \( \xi \) induces a \( \text{U}_m \)-bundle \( \xi' \) and hence a \( \text{U}_m \big|_{\text{U}_m} \)-bundle \( \xi' \). Similarly, the partial cross-section \( f \) induces partial cross-sections \( f' \) and \( f'' \). By definition the obstruction class \( \varphi(p, \xi', f') \) is equal to the Chern class \( c_2(p, \xi', f') \) and hence to the Pontrjagin class \( p_4 \circ \varphi \). Therefore \( p_4(\xi) \) equals \( q_4 \circ h_4 \circ \varphi \), where

\[ h: \pi_{n+1}(\text{SO}_m) \to \pi_{n+1}(\text{U}_m) \quad \text{and} \quad q: \pi_{n+1}(\text{U}_m) \to \pi_{n+1}(\text{U}_{m-1}) \]

are the natural homomorphisms and \( h_4, q_4 \) are the homomorphisms in the cohomology of \( K \) induced by the coefficient homomorphism \( h, q \).

Using the following computations of Bott\(^\text{[3]}\):

\[ \pi_{n+1}(\text{U}_m) \cong Z, \quad \pi_{n+1}(\text{U}_m/\text{SO}_m) \cong Z_{2^n}, \quad \pi_{n+1}(\text{SO}_m) = 0, \quad \pi_{n+1}(U_{m-1}) \cong Z_{2^{m-1}} \]

it follows that \( h \) carries a generator into \( 2^n \) times a generator. Similarly, using the fact that

\[ \pi_{n+1}(U_{m-1}) \cong Z_{2^{m-1}}, \quad \pi_{n+1}(U_{m-1}) = 0, \quad \pi_{n+1}(U_{m-1}) = 0, \]

it follows that \( q \) carries a generator into \( (2^n - 1) \times 2^m \) times a generator. Therefore \( p_4(\xi) = \pm 2^n(2^n - 1)! \alpha(\xi, f) \). This completes the proof of Lemma 2.
Combining Lemma 3 with Theorem 4, we see that the stable homotopy groups of spheres contain elements of arbitrary finite order. In fact:

**Corollary.** If $2n$ is a multiple of the Euler $f$ function $F(r)$, then the stable group $\pi_{m+1}(S^n)$ contains an element of order $r$.

**REFERENCES**


[10] Milnor, J. W. On the cobordism ring $\Omega^*$ and a complex analogue. (In preparation.)


**ON THE FOURTEENTH PROBLEM OF HILBERT**

By Masayoshi Nagata

The purpose of the present paper is to show that the answer to the 14th problem of Hilbert is negative, even in the following restricted case, which may be called the original 14th problem of Hilbert:

Let $G$ be a subgroup of the full linear group of the polynomial ring in indeterminates $x_1, \ldots, x_n$ over a field $k$, and let $\sigma$ be the set of elements of $k[x_1, \ldots, x_n]$ which are invariant under $G$. Is $\sigma$ finitely generated?

Our construction of a counter-example is independent of the characteristic of $k$, and $k$ and $n$ can be the field of complex numbers.

1. The construction of a counter-example

Let $\{a_{ij}\} \ (i = 1, 2, 3; \ j = 1, 2, \ldots, 16)$ be algebraically independent elements over the prime field $\mathbb{F}$ of arbitrary characteristic, and let $k$ be a field containing the $a_{ij}$. Let $V$ be the vector space of dimension 16 over $k$ and let $V^\ast$ be the set of vectors in $V$ which are orthogonal to the vectors $(a_{1i}, a_{2i}, \ldots, a_{16i}) \ (i = 1, 2, 3)$. ($V^\ast$ is a subspace of dimension 13.)

Let $x_1, \ldots, x_{16}, t_1, \ldots, t_{16}$ be algebraically independent elements over $k$ and let $G$ be the set of linear transformations $\sigma$ such that (i) $\sigma(t_i) = t_i$ for any $i$ and (ii) $\sigma(x_i) = x_i + b_i t_i$ with $(b_1, \ldots, b_{16}) \in V^\ast$. Then:

The set of elements of $k[x_1, \ldots, x_{16}, t_1, \ldots, t_{16}]$ which are invariant under $G$ is not finitely generated.

2. A lemma on plane curves

In order to prove the example, we need the following lemma on plane curves:

Fundamental lemma. Let $P_1, \ldots, P_d$ be independent generic points of the projective plane $\mathbb{P}$ over the prime field $\mathbb{F}$. For any curve $C$ of degree $d$, the sum of the multiplicities of $P_i$ on $C$ is less than $4d$.

Proof. Assume that there exists a curve $C$ of degree $d$ such that $\Sigma m_i \geq 4d$, where $m_i$ is the multiplicity of $P_i$ on $C$. Since the $P_i$ are independent generic points, the $P_i$ can be specialized to any permutation of the $P_i$ and therefore we see that there exists a curve of degree $d'$ such that the multiplicity of the $P_i$ is equal to $m$ for every $i$ and $d' < 4m$. Therefore it is sufficient to prove the following lemma (which is equivalent to the fundamental lemma):