Brown-Kervaire Invariants

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Stephan Klaus
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Contents

Introduction

M.Kervaire constructed in [29] a $\mathbb{Z}/2$-valued bordism invariant for closed $(4m+2)$-dimensional framed manifolds. For such a manifold M, the invariant $K(M)$ is defined by taking the Arf invariant of a quadratic refinement for the intersection pairing on $H^{2m+1}(M;\mathbb{Z}/2)$. Here, the quadratic refinement can be constructed geometrically from the framing of the manifold M.

In [17], E.H.Brown generalized this invariant to other manifolds using a homotopy-theoretical translation for the quadratic refinement. For closed $2n$-dimensional manifolds M with ξ-structure, where $\xi : B \to BO$ is a given fibration, Brown obtained again bordism invariants

$$K_h : \Omega_{2n}^{\xi} \longrightarrow \mathbb{Z}/8.$$

The main differences to the invariant K of Kervaire consist in three points: Firstly, the construction of K_h works iff the universal Wu class $v_{n+1}(\xi)$ vanishes. Secondly, the construction gives in general a family of invariants as one has to choose a parameter h in a universal parameter space Q_{2n}^{ξ}. Thirdly, the invariants live in $\mathbb{Z}/8$, because one has to deal with $\mathbb{Z}/4$-valued quadratic refinements as the self-intersection on H^nM for ξ-manifolds is in general non-trivial.

Our main subject of this thesis is the examination of these Brown-Kervaire invariants for *Spin*-manifolds, which we do in part I. We give now an abstract on the main theorem:

For $(8m+2)$-dimensional closed *Spin*-manifolds, one has on the one hand the finite set of Brown-Kervaire invariants, and on the other hand the k-invariant defined by S.Ochanine. Both are $\mathbb{Z}/2$-valued invariants of *Spin*-bordism, where the first are defined cohomologically as the Arf-invariant of certain quadratic refinements of the intersection form on $H^{4m+1}(M;\mathbb{Z}/2)$, and the second can be defined as a KO-characteristic number which by the Real Family Index Theorem has an analytic interpretation as the mod 2 index of a twisted Dirac-operator. Ochanine showed that these invariants agree on the class of *Spin*-manifolds, for which all Stiefel-Whitney numbers containing an odd-dimensional Stiefel-Whitney class vanish. Moreover, it is not difficult to construct two different Brown-Kervaire invariants in dimension 34.

We show in 7.1.1 that the Ochanine k-invariant is in fact a Brown-Kervaire invariant; in particular, it vanishes if $H^{4m+1}(M;\mathbb{Z}/2) = 0$, and is an invariant of the *Spin*-homotopy type. This result is in analogy to the Hirzebruch Signature Theorem and can be considered as a $\mathbb{Z}/2$-valued cohomological index theorem for the above operator. The proof uses the integral elliptic homology of M.Kreck and S.Stolz, which in particular characterizes invariants with a multiplicativity property in $\mathbb{H}P^2$-bundles with structure group $PSp(3)$; and the theory of L.Kristensen about secondary cohomology operations, which gives a Cartan formula necessary for the computation of certain secondary operations in these $\mathbb{H}P^2$-bundles.

In part II of this thesis, we consider also Brown-Kervaire invariants for other manifolds. In particular, we examine the two classes of oriented manifolds and of $BO\langle 8\rangle$-manifolds. The first case of oriented manifolds, being easier than *Spin*-manifolds, gives an example where all Brown-Kervaire invariants can be explicitely determined. In fact, the main work

consisting in the computation of the oriented bordism ring and the action of the Pontrjagin squaring operation on H^*BSO was already done. In the second case of $BO\langle 8\rangle$-manifolds, being more difficult than *Spin*-manifolds, we restrict us to consider the analogue problem of multiplicativity of Brown-Kervaire invariants for $\mathbb{C}ayP^2$-bundles with structure group F_4.

It follows a more detailed survey on the ten sections and the appendix in this thesis, with emphasis on the new results:

(1) In the first section, we recall the definition of Brown-Kervaire invariants, list some directly following properties, and come then to the more difficult problems of a product formula and of the connection to the Adams spectral sequence. Almost all results can be found (at least implicitely) in the literature, and our main work consisted in giving a unified representation. The simple corollary 1.2.5 of the addition formula seems to be new and will be applied in section 2, and in 1.2.3 we indicate a proof for a statement in [22] on the parameter set Q^{ξ}_{2n}. We also give more details how to apply the product formula of Brown to show that $K_h(M^{8m} \times \overline{S^1} \times \overline{S^1}) \equiv sign(M^{8m}) \bmod 2$ in the case of *Spin*-manifolds (1.3.16), which was stated in [48]. The connection of Brown-Kervaire invariants to the Adams spectral sequence examined in [22] is only included for the sake of completeness; we have seen no good application to our main subject of *Spin*-manifolds (see 1.4.9).

(2) In the second section and beginning with part I, we give the basic information on Brown-Kervaire invariants of *Spin*-manifolds. We start with the known results on the structure of the *Spin*-bordism ring due to Anderson, Brown and Peterson, and give in 2.1.12 an explicit formula for the Poincaré series of $Tors(\Omega^{Spin}_*)$, following the ideas in [5]. As an illustration, we computed tables for Ω^{Spin}_* up to dimension 48. Then we come to the Brown-Kervaire invariants

$$K_h : \Omega^{Spin}_{8m+2} \longrightarrow \mathbb{Z}/2$$

and the results of Ochanine on generalized Kervaire invariants. By modification of an example of Ochanine, we prove in 2.2.7 the existence of at least two different Brown-Kervaire invariants for Ω^{Spin}_{34}. As a next point, we recall the construction of certain Brown-Kervaire invariants by unstable secondary cohomology operations going back to a Theorem of Brown and Peterson. Thus we call these invariants *Brown-Peterson-Kervaire invariants.* At last, we consider the addition formula for these invariants leading to *generalized Wu classes.* We show how to compute these classes universally in 2.4.4 and give a table up to dimension 17. Using this computation, we prove in 2.4.6 that there are at most two different Brown-Peterson-Kervaire invariants for Ω^{Spin}_{34}.

(3) In the third section, we introduce the Ochanine k-invariant, the Ochanine genus κ, and the Ochanine elliptic genus β:

$$\begin{array}{rlcl} k: & \Omega^{Spin}_{8m+2} & \longrightarrow & \mathbb{Z}/2, \\ \kappa: & \Omega^{Spin}_{*} & \longrightarrow & KO_* \otimes \mathbb{Z}/2, \\ \beta: & \Omega^{Spin}_{*} & \longrightarrow & KO_*[[q]]. \end{array}$$

We recall their known properties, for example multiplicativity in certain fibre bundles. For β, multiplicativity (3.2.10) was proved by Keck and Stolz in [32] (using the Bott-Taubes Rigidity Theorem), which gives the multiplicativity of k as a corollary in 3.2.11. At last we consider the problem to express k as a KO-characteristic number, which has then by the

Real Family Index Theorem an analytic interpretation as the mod 2 index of a twisted Dirac-operator. Such expressions were first obtained by Ochanine [48] and Rubinsztein [55] with certain multiplicative series for KO-numbers, in analogy to the Hirzebruch formalism. We give here a more explicit representation (3.3.3, 3.3.5) by direct computation of the highest coefficient in the polynomial decomposition of β into the basic integral modular forms, using an inverse transformation formula for power series (3.3.2).

(4) In section 4, we start with the examination of the relationship between the Ochanine k-invariant and the Brown-Kervaire invariants K_h. We attack this problem by *integral elliptic homology* of Kreck and Stolz [32], where fibre bundles over $Spin$-manifolds with fibre $\mathbb{H}P^2$ and structure group $PSp(3)$ play a fundamental role. First we recall a Theorem of Stolz [60] on the cohomological structure of the universal $PSp(3)$-$\mathbb{H}P^2$-bundle, as we need in section 5 and section 7 that $Sq^1 x = 0$ for the universal Leray-Hirsch generator x (4.1.3). Then we obtain in 4.2.7 a characterization of the Ochanine k-invariant, which follows from the determination of the coefficients of integral elliptic homology by Kreck and Stolz. In particular, k is characterized by multiplicativity in $PSp(3)$-$\mathbb{H}P^2$-bundles and $k(M^{8m} \times \overline{S^1} \times \overline{S^1}) \equiv sign(M^{8m})$ mod 2. In order to check the multiplicativity for Brown-Peterson-Kervaire invariants, one has to compute the Brown-Peterson secondary cohomology operation ϕ on the total space of a $PSp(3)$-$\mathbb{H}P^2$-bundle, which reduces by the quadratic sub-Lagrangian lemma for the Arf invariant to the computation of a product formula for $\phi(xp^*y)$, with y a middle-dimensional cohomology class of the basis manifold.

(5) Section 5 is a first attempt to decide this multiplicativity problem by computing the product formula for $\phi(xp^*y)$ with standard homotopy theory. Using the glueing approach for secondary operations, we prove in 5.2.2 a very general product formula for secondary operations (which could also be of interest for other problems). In 5.3.3, we apply this method to our case of $PSp(3)$-$\mathbb{H}P^2$-bundles, but the result contains a deviation $\epsilon = \phi'(v_{\phi,\phi'})$ to multiplicativity, which is obtained by glueing together certain homotopies. Because we were not able to compute this term by homotopy theory as an explicit linear combination of known invariants, this formula has only very restricted application to our problem. At least, we can prove in 5.3.4 multiplicativity under certain assumptions, for example for $Sp(3)$-$\mathbb{H}P^2$-bundles over almost complex manifolds. Moreover, we analyze in 5.3.6 the dependence of the critical term on the choice of the homotopies. Unfortunately, the freedom in choosing the homotopies is not large enough to garantee a priori the vanishing of $v_{\phi,\phi'}$.

(6) In this section we give a survey to another approach to secondary cohomology operations which is due to Kristensen. Using cochain operations to represent cohomology operations, Kristensen obtained sum and product formulas in a series of papers [33], [34] and [35], see in particular [37] for a short survey on his product formula. Cochain operations seem to provide the algebraic analogue of homotopies. They form an infinite-dimensional graded $\mathbb{Z}/2$-vector space $\mathcal{O}^*$ carrying a non-linear composition operation and a differential Δ, and Kristensen proved that the homology $H(\mathcal{O}^*;\Delta)$ is naturally isomorphic to the Steenrod algebra A^*. For some of them there exist explicit and manageable expressions in terms of $\cup_i$-products, which explains the success of Kristensen's method. Moreover, Kristensen computed the complete product formula (including the critical primary term ϵ) for the secondary cohomology operations associated to special series of relations, see 6.3.5 and 6.3.8.

(7) Here, we prove the Main Theorem 7.1.1: The Ochanine k-invariant is in fact a Brown-Kervaire invariant. Using Kristensen's method of cochain operations, we can compute the critical term ϵ in the product formula for $\phi(xp^*y)$ by a combination of the special cochain operations (=homotopies) due to Kristensen, showing that the primary operation ϵ obtained by 'glueing homotopies together' does not vanish for general spaces, but contributes $v_{\phi,\phi'} = 0$ in our case of $PSp(3)$-$\mathbb{H}P^2$-bundles of 1-connected *Spin*-manifolds. As an application of the Main Theorem, k vanishes if $H^{4m+1}(M;\mathbb{Z}/2) = 0$ (7.2.1), and is an invariant of the *Spin*-homotopy type (7.2.2). This suggests the problem to determine all KO-characteristic numbers which are invariants of the *Spin*-homotopy type (7.2.4).

(8) In section 8, we begin with part II of this thesis and consider also other manifolds than *Spin*-manifolds. First we recall very shortly the two extreme cases for Brown-Kervaire invariants: The most general case of Wu-bordism, and the most special case of framed bordism. Then we compute in 8.2.1 the vanishing dimensions for the universal Wu class in the bordism of immersions, generalizing (and proving) a statement in [22]. At last we consider bordism theories with $w_i(\xi) = 0$ for all $i < k$ with some fixed $k \in \mathbb{N}$, where we can assume that k is a power of two. In this case, we compute the vanishing dimensions of the universal Wu class in 8.3.2, which applies for example to $BO\langle 2^r\rangle$-manifolds. We obtain series $2n = 2^{s+1}m + (2^s - 2)$ of dimensions $2n$ (with $s \leq r$), where Brown-Kervaire invariants exist, starting with the classical Kervaire invariant in the critical dimension $2^s - 2$ (see 8.3.6). At last, we generalize the construction of Brown-Kervaire invariants for *Spin*-manifolds by the secondary cohomology operations of Brown and Peterson to the bordism theories $\Omega_*^{\langle 2^r\rangle}$. Using the *canonical decomposition* for Sq^{n+1} (8.3.6), we construct in 8.3.7 again *Brown-Peterson-Kervaire invariants*; the main problem consists here in proving that the quadratic forms are defined on the whole middle-dimensional cohomology.

(9) The subject of section 9 are orientable manifolds, where Brown-Kervaire invariants can be defined in dimensions $4m$. Here, there exists a canonical quadratic refinement given by the Pontrjagin square, being an unstable *primary* cohomology operation $\wp : H^*(\ ;\mathbb{Z}/2) \to H^{2*}(\ ;\mathbb{Z}/4)$. We list the properties of $\wp$ and recall the Theorem of Morita [45], which says that the generalized Arf invariant of $\wp$ is given by the signature modulo 8. By the addition formula, the problem to determine all Brown-Kervaire invariants is reduced to the computation of $\wp$ on $H^*(BSO;\mathbb{Z}/2)$ and to the corresponding $\mathbb{Z}/4$-valued characteristic numbers for oriented bordism. We recall the relevant facts from the literature, i.e. the computation of the oriented bordism ring, the cohomology of BSO with integer coefficients, and the action of $\wp$ on Stiefel-Whitney classes.

(10) In the last section, we consider the Brown-Kervaire invariants for $BO\langle 8\rangle$-manifolds in some more detail. First, we give some background information on $\Omega_*^{\langle 8\rangle}$. In 8.3.4, we have showed that the interesting dimensions are $2n = 16m + 6$. The multiplicativity problem for $PSp(3)$-$\mathbb{H}P^2$-bundles over *Spin*-manifolds suggests an analogous problem for F_4-$\mathbb{C}ayP^2$-bundles over $BO\langle 8\rangle$-manifolds. Thus we recall some facts on $\mathbb{C}ayP^2$ and the cohomology of $BSpin(9)$ and BF_4, giving in 10.2.5 the Leray-Hirsch Theorem for F_4-$\mathbb{C}ayP^2$-bundles and the total Steenrod square on the universal Leray-Hirsch generator. We consider only certain F_4-$\mathbb{C}ayP^2$-bundles over $BO\langle 8\rangle$-manifolds as the tangent bundle along the fibres of the universal F_4-$\mathbb{C}ayP^2$-bundle has no $BO\langle 8\rangle$-structure (see 10.2.6). Then we consider the multiplicativity problem of Brown-Peterson-Kervaire invariants for these bundles, which leads

by the quadratic sub-Lagrangian lemma again to the computation of $\phi(xp^*y)$. We compute in 10.3.4 the product formula for $\phi(xp^*y)$ by our homotopy-theoretical method of section 5. As a corollary, we obtain in 10.3.5 multiplicativity for almost complex basis manifolds, for example. Then, we give some remarks on the application of Kristensen's product formula, and conclude with some conjectures concerning the generalization of our Main Theorem to the case of $BO\langle 8\rangle$-manifolds.

(A) We included an appendix on quadratic forms and generalized quadratic forms defined on $\mathbb{Z}/2$-vector spaces, which seems to be not well-presented in the standard algebraic books on quadratic forms. First, we consider the classification of symmetric inner products which is given by dimension and type. Then, we define and classify non-degenerate quadratic forms (values in $\mathbb{Z}/2$), where the Arf invariant comes in. Because $\mathbb{Z}/2$-valued quadratic forms can only be defined for even type inner products, one generalizes them to $\mathbb{Z}/4$-valued quadratic forms which exist also for odd type inner products. But then, the generalization of the Arf invariant takes values in $\mathbb{Z}/8$. We prove a corresponding classification result A.3.7, which seems to be not in the literature. In all three categories of symmetric / quadratic / generalized quadratic inner product spaces, we consider also the stable classification and the Witt ring classification. Furthermore, we prove in A.2.18, A.3.18 a sub-Lagrangian lemma which we need for *Spin*- and $BO\langle 8\rangle$-manifolds (only the $\mathbb{Z}/2$-case). At least in the $\mathbb{Z}/2$-case, this is well-known, but we did not find a good reference for it.

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Conventions

In the following we will use some conventions and abbreviations in order to simplify the notation:

- All manifolds we consider are smooth and compact. All maps between manifolds, and all fibre bundles of manifolds are smooth.
- We set $K_n := K(\mathbb{Z}/2, n)$, an Eilenberg-MacLane space in dimension n with $\mathbb{Z}/2$-coefficients. If we take other coefficients than $\mathbb{Z}/2$, then they are included in the notation.
- We set $H^n X := H^n(X; \mathbb{Z}/2)$, singular cohomology of X with $\mathbb{Z}/2$-coefficients. In the same way, $H_n X := H_n(X; \mathbb{Z}/2)$. If we take other coefficients than $\mathbb{Z}/2$, then they are included in the notation. We often use the identification $H^n X = [X, K_n]$ given by $f^*\iota_n \leftrightarrow f$, where $[\ ,\ ]$ denotes the set of homotopy classes of maps between spaces, and $\iota_n \in H^n K_n$ is the fundamental class, which corresponds to the identity map. Sometimes, we take the direct product of the $H^n X$, which we denote by $H^\bullet X$ (for example, the total Stiefel-Whitney class w lives in $H^\bullet BO$).
- H denotes the Eilenberg-MacLane spectrum with $\mathbb{Z}/2$-coefficients. Other coefficients A are denoted by HA. For spectra, $[\ ,\ ]$ means the abelian group of homotopy classes of maps between spectra.
- A^* denotes the Steenrod algebra for $\mathbb{Z}/2$-coefficients, and A^n the homogenous subspace of degree n. The direct product of the A^n is denoted by $A^\bullet$ (for example, the total squaring operation Sq lives in $A^\bullet$). The canonical anti-automorphism of A^* (and of $A^\bullet$) is denoted as usual by χ.
- S^0 denotes the sphere spectrum; thus $S^0 \wedge X$ is the suspension spectrum of a space X.
- K_h denotes a Brown-Kervaire invariant with parameter h.
- $i_n^{mn} : \mathbb{Z}/n \to \mathbb{Z}/mn$ denotes the canonical injection $i_n^{mn}(1) := m$. We consider in particular i_2^4, i_2^8 and i_4^8.
- In the appendix, IPS stands for an *inner product space*. We consider symmetric, quadratic, and generalized quadratic IPSs over the field $\mathbb{Z}/2$.

1 Brown-Kervaire Invariants

The classical Kervaire invariant can be defined in the following way: Given a closed $2n$-dimensional framed manifold M with n odd, one constructs with the framing a quadratic refinement $q : H^nM \to \mathbb{Z}/2$ of the cup-pairing $\cup : H^nM \times H^nM \to \mathbb{Z}/2$. The Kervaire invariant $K(M) \in \mathbb{Z}/2$ of M is then defined as the Arf invariant $\mathrm{Arf}(q)$ of the quadratic form q. In fact, this gives a bordism invariant $K : \Omega_{2n}^{\mathrm{fr}} \longrightarrow \mathbb{Z}/2$.

In [17], Brown generalized this construction to closed $2n$-dimensional manifolds (for all n) with a given ξ-structure, where $\xi : B \to BO$ is a fibration. But here one gets three difficulties:

1. A universal construction of the quadratic form for closed $2n$-dimensional ξ-manifolds works only for such ξ which satisfy a certain condition, namely the vanishing of the Wu class $v_{n+1}(\xi) = 0$.
2. In this case one further has to choose an element h in a universal parameter set Q_{2n}^{ξ} in order to do the construction. This parameter set is an affine space with vector space isomorphic to H^nB, thus there is in general no natural choice for the parameter h.
3. The cup-pairing on a ξ-manifold is in general not even (there can exist elements $x \in H^nM$ with self-intersection $x^2[M] \neq 0$). So one has to work with $\mathbb{Z}/4$-valued quadratic forms and gets a $\mathbb{Z}/8$-valued Arf invariant.

Like the classical Kervaire invariant, Brown's generalization gives a bordism invariant

$$K : Q_{2n}^{\xi} \times \Omega_{2n}^{\xi} \to \mathbb{Z}/8.$$

In this section, we recall Brown's construction, list some properties of his invariant, and give then a short survey on further results like multiplicative properties and the connection to the Adams spectral sequence. More details about Brown-Kervaire Invariants and most of the following properties can be found in [17], [16], [18], [22]. The definition and some properties of Arf invariants of $\mathbb{Z}/2$- and $\mathbb{Z}/4$-valued quadratic forms can be found in the appendix.

1.1 The Construction

1.1.1 In order to describe Brown's construction, we first recall the definition of ξ-structures on manifolds, see [63] (we consider here only differentiable manifolds, although Brown's definition also works for Poincaré complexes). Let $\xi : B \to BO$ be a given fibration (with B connected and of the homotopy type of a CW-complex). Roughly, a ξ-structure on a manifold M is a lift $\tilde{\nu} : M \to B$ of the stable normal bundle $\nu : M \to BO$ over ξ. More precisely, one has to choose an embedding $M \hookrightarrow \mathbb{R}^N$ (respectively $M \hookrightarrow \mathbb{R}^+ \times \mathbb{R}^{N-1}$ such that ∂M meets $0 \times \mathbb{R}^{N-1}$ transversally, if $\partial M \neq \emptyset$). We denote by $\nu : M \to BO$ the corresponding Gauss map of the stable normal bundle. A *homotopy lift* $\tilde{\nu} : M \to B$ of ν is defined as an equivalence class of lifts of ν over ξ, where 'equivalent' means 'homotopic

over ν'. If N is large enough, two embeddings are isotopic, with different isotopies being themselves isotopic. As an isotopy between two embeddings gives a homotopy between their normal Gauss maps, we get (by lifting this homotopy) a bijection between the two sets of homotopy lifts, which is independent of the choice of the isotopy. Then a *ξ-structure* on M is defined to be an equivalence class of homotopy lifts under these bijections. It can be represented by a particular lift $\tilde{\nu} : M \to B$. As we will consider only $\tilde{\nu}$ and not ν, we will in the following write ν for $\tilde{\nu}$ in order to simplify the notation.

1.1.2 If $\xi : B \to BO$ is a principal fibration with fibre an H-space G, then the set of ξ-structures on M (if non-empty) is acted transitively and effectively on by the group $[M, G]$. For example, we have a tower $BO\langle 8\rangle \to BSpin \to BSO \to BO$ of principal fibrations with classifying maps w_1, w_2 and $\frac{p_1}{2}$:

$$\begin{array}{ccccc}
K(\mathbb{Z},3) & \longrightarrow & BO\langle 8\rangle & & \\
 & & \downarrow & & \\
K(\mathbb{Z}/2,1) & \longrightarrow & BSpin & \xrightarrow{\frac{p_1}{2}} & K(\mathbb{Z},4) \\
 & & \downarrow & & \\
K(\mathbb{Z}/2,0) & \longrightarrow & BSO & \xrightarrow{w_2} & K(\mathbb{Z}/2,2) \\
 & & \downarrow & & \\
 & & BO & \xrightarrow{w_1} & K(\mathbb{Z}/2,1)
\end{array}$$

which shows that

$$\begin{array}{ccc}
\{\text{orientations on } M\} & \approx & \begin{cases} H^0M & \text{if } w_1(M)=0 \\ \emptyset & \text{otherwise} \end{cases} \\
\left\{\begin{array}{l}\text{Spin-structures on } M \\ \text{over a fixed orientation}\end{array}\right\} & \approx & \begin{cases} H^1M & \text{if } w_2(M)=0 \\ \emptyset & \text{otherwise} \end{cases} \\
\left\{\begin{array}{l}BO\langle 8\rangle\text{-structures on } M \\ \text{over a fixed } Spin\text{-structure}\end{array}\right\} & \approx & \begin{cases} H^3(M,\mathbb{Z}) & \text{if } \frac{p_1}{2}(M)=0 \\ \emptyset & \text{otherwise.} \end{cases}
\end{array}$$

1.1.3 Now we come to the definition of Brown-Kervaire invariants. These turn out to be invariants

$$K : Q^{\xi}_{2n} \times \Omega^{\xi}_{2n} \to \mathbb{Z}/8$$

of the bordism theory Ω^{ξ}_* of ξ-manifolds, which is associated to the fibration $\xi : B \to BO$, and by the Pontrjagin-Thom construction is given as $\pi_* M\xi$ with $M\xi$ the Thom spectrum of the stable bundle ξ (see [63]). Here Q^{ξ}_{2n} denotes the *set of parameters*

$$Q^{\xi}_{2n} := \{h \in Hom(\pi_{2n}(M\xi \wedge K_n), \mathbb{Z}/4) \mid h(\lambda_\xi) = 2\},$$

where $K_n := K(\mathbb{Z}/2, n)$, and where the stable map $\lambda_\xi : S^{2n} \to S^0 \wedge K_n \to M\xi \wedge K_n$ is induced by the non-trivial map in $\pi^{\mathrm{st}}_{2n}(K_n) = \mathbb{Z}/2$ ([17]). For a $2n$-dimensional ξ-manifold M^{2n} Brown defined $K_h(M^{2n})$, which we call the *Brown-Kervaire invariant* of M^{2n} *with parameter* h, as the $\mathbb{Z}/8$-valued Arf-invariant of the $\mathbb{Z}/4$-valued quadratic form

$$q_h : H^n M^{2n} = [M^{2n}_+, K_n] \xrightarrow{\Sigma^0} [S^0 \wedge M^{2n}_+, S^0 \wedge K_n] \xrightarrow{S} [S^{2n}, M\nu \wedge K_n] \xrightarrow{\nu_*} [S^{2n}, M\xi \wedge K_n] \xrightarrow{h} \mathbb{Z}/4.$$

Here the first map Σ^0 is the stabilization map, which associates to each unstable homotopy class between spaces its stable homotopy class between the suspension spectra. This is the

non-linear part in the composition. The second map S is the S-duality isomorphism for the suspension spectrum $S^0 \wedge M^{2n}_+$ and the Thom spectrum of the stable normal bundle $M\nu$; for any two spectra X, Y it is defined with the Thom map $t : \Sigma^{2n} S^0 \to M\nu$ and the diagonal $\Delta : M\nu \to M\nu \wedge M^{2n}_+$ as

$$S : [M^{2n}_+ \wedge X, Y] \to [\Sigma^{2n} X, M\nu \wedge Y], \qquad S(f) := (1_{M\nu} \wedge f)((\Delta t) \wedge 1_X)$$

and is an isomorphism ([64]). The third map ν_* is induced by the ξ-structure $\nu : M^{2n} \to B$ on M^{2n}, which gives a corresponding map between the Thom spectra $M\nu$ and $M\xi$. Now we can write down the definition of the Brown-Kervaire invariant K_h with parameter h, and a further invariant $q_{x,h}$, with $x \in H^n B$:

$$\begin{array}{rcll} K_h(M^{2n}) & := & \widetilde{\mathrm{Arf}}(q_h : H^n M^{2n} \to \mathbb{Z}/2) & \in \mathbb{Z}/8, \\ q_{x,h}(M^{2n}) & := & q_h(\nu^* x) & \in \mathbb{Z}/4. \end{array}$$

We refer to the appendix for the definition and properties of the *Arf invariant* Arf and its generalization $\widetilde{\mathrm{Arf}}$. To prove that the map q_h is a $\mathbb{Z}/4$-valued quadratic refinement of the $\mathbb{Z}/2$-intersection pairing $H^n M^{2n} \times H^n M^{2n} \to \mathbb{Z}/2$, i.e.

$$q_h(x + y) = q_h(x) + q_h(y) + i^4_2(xy),$$

Brown showed that:

1.1.4 Theorem: (Brown [16], [17]) *(i) The Postnikov tower up to dimension* $2n$ *of the suspension spectrum* $S^0 \wedge K_n$ *is given by*

$$\begin{array}{ccccc} & & S^0 \wedge K_n & & \\ & & \downarrow_j & & \\ \Sigma^{2n} H & \xrightarrow{i} & E & & \\ & & \downarrow_p & & \\ & & \Sigma^n H & \xrightarrow{Sq^{n+1}} & \Sigma^{2n+1} H \end{array}$$

where H *is the Eilenberg-MacLane spectrum for* $\mathbb{Z}/2$*-coefficients,* E *is the fibre of the map* $Sq^{n+1} \in [\Sigma^n H, \Sigma^{2n+1} H] = A^{n+1}$*, and* $pj \in [S^0 \wedge K_n, \Sigma^n H] = H^n K_n$ *is the fundamental class.*
(ii) For a $2n$*-dimensional CW-complex* X^{2n}*, one has a short exact sequence*

$$0 \longrightarrow H^{2n} X^{2n} \xrightarrow{j_*^{-1} i_*} [S^0 \wedge X^{2n}, S^0 \wedge K_n] \xrightarrow{p_* j_*} H^n X^{2n} \longrightarrow 0,$$

where $j_* : [S^0 \wedge X^{2n}, S^0 \wedge K_n] \to [S^0 \wedge X^{2n}, E]$ *is an isomorphism and the stabilization map* $\Sigma^0 : H^n X^{2n} = [X^{2n}, K_n] \to [S^0 \wedge X^{2n}, S^0 \wedge K_n]$ *satisfies*

$$\Sigma^0(x + y) = \Sigma^0(x) + \Sigma^0(y) + j_*^{-1} i_*(xy).$$

1.1.5 We recall from [14] the definition of the *total Wu class*

$$v(\xi) = 1 + v_1(\xi) + v_2(\xi) + \ldots \in H^\bullet B := \prod_i H^i B$$

of the stable bundle $\xi : B \to BO$: First, we have for the total Stiefel-Whitney class $w(\xi) = \xi^* w \in H^\bullet B$ by a result of Thom (see [44]) that

$$w(\xi)U_\xi = Sq\, U_\xi,$$

where $Sq := 1+Sq^1+Sq^2+\ldots \in A^\bullet := \prod_i A^i$ is the total squaring operation, $U_\xi \in H^0 M\xi$ the stable Thom class, and $x \in H^k B \mapsto xU_\xi \in H^k M\xi$ denotes the stable Thom isomorphism. Now, for any closed manifold N its *total Wu class* $v_N \in H^* N$ is defined by Poincaré duality as the 'eigenvalue of Sq', i.e.

$$\langle v_N x, [N]\rangle = \langle Sq(x), [N]\rangle \qquad \text{for all } x \in H^* N,$$

where $[N] \in H_{dim(N)} N$ denotes the fundamental class and $\langle\ ,\ \rangle$ the Kronecker pairing. If N has ξ-structure $\nu : N \to B$, it holds by a result of Wu ([44]) that

$$v_N = \nu^* Sq^{-1}(w(\xi)^{-1}),$$

where $Sq^{-1} \in A^\bullet$ denotes the multiplicative inverse of the element Sq in the non-abelian group of units of $A^\bullet$. We remark that we have to take in the formula $w(\xi)^{-1}$ instead of $w(\xi)$ as we consider *normal* structures. The map $Sq^{-1} : H^*(\ ) \to H^*(\ )$ gives an automorphism of the cohomology ring (as Sq does); moreover, it holds $Sq^{-1} = \chi(Sq)$ with $\chi : A^* \to A^*$ denoting the *canonical anti-automorphism* of the Steenrod algebra, in particular $\chi(\alpha\beta) = \chi(\beta)\chi(\alpha)$ (these two properties characterize χ). Thus one defines the total Wu class of ξ by

$$v(\xi) := \chi(Sq)(w(\xi)^{-1}) = (\chi(Sq)(w(\xi)))^{-1},$$

which gives $v_N = \nu^* v(\xi)$. Furthermore,

$$v(\xi)U_\xi = \chi(Sq)U_\xi,$$

because it holds $U_\xi = \chi(Sq)Sq\, U_\xi = \chi(Sq)(w(\xi)U_\xi) = \chi(Sq)(w(\xi)) \cdot \chi(Sq)U_\xi$.

1.1.6 Theorem (Brown [16], [17]): *(i) For all $h \in Q^\xi_{2n}$, the map q_h is a quadratic refinement of the cup pairing. The parameter set Q^ξ_{2n} is non-empty iff the Wu class $v_{n+1}(\xi)$ vanishes. In this case, it is an affine space for the vector space $H^n B$, and one gets for the corresponding quadratic forms on a ξ-manifold M^{2n}:*

$$q_{x+h}(y) = i_2^4\langle \nu^*(x)y, [M^{2n}]\rangle + q_h(y) \qquad \textit{for } x \in H^n B,\ y \in H^n M^{2n},\ h \in Q^\xi_{2n}.$$

(ii) The Brown-Kervaire invariant K_h and the invariant $q_{x,h}$ are bordism invariants

$$\begin{array}{rrcl} K: & Q^\xi_{2n} \times \Omega^\xi_{2n} & \longrightarrow & \mathbb{Z}/8, \\ q: & H^n B \times Q^\xi_{2n} \times \Omega^\xi_{2n} & \longrightarrow & \mathbb{Z}/4. \end{array}$$

Proof: Since not all statements are formulated in this way in the references, we give a short proof: By Brown's theorem 1.1.4, we get for a ξ-manifold M of dimension $2n$ a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{2n}M & \xrightarrow{i_*} & [S^0 \wedge M_+, S^0 \wedge K_n] & \xrightarrow{p_*} & H^n M & \longrightarrow & 0 \\ & & \downarrow S & & \downarrow S & & \downarrow S & & \\ 0 & \longrightarrow & H_0 M\nu & \xrightarrow{i_*} & \pi_{2n}(M\nu \wedge K_n) & \xrightarrow{p_*} & H_n M\nu & \longrightarrow & 0 \\ & & \downarrow \nu_* & & \downarrow \nu_* & & \downarrow \nu_* & & \\ H_{n+1}M\xi & \xrightarrow{Sq_*^{n+1}} & H_0 M\xi & \xrightarrow{i_*} & \pi_{2n}(M\xi \wedge K_n) & \xrightarrow{p_*} & H_n M\xi & \longrightarrow & 0 \end{array}$$

For $h \in Q^\xi_{2n}$, 1.1.4 (ii) shows that $q_h = h\nu_* S\Sigma^0$ is a quadratic refinement of the cup pairing. As $\lambda_\xi = i_* u_\xi$ where $u_\xi \in H_0 M\xi$ is the generator (dual to the Thom class U_ξ), the condition $h(\lambda_\xi) = 2$ for the parameter h can be satisfied iff u_ξ is not in the image of the map Sq^{n+1}_*. Dualizing this and using a result of Thom that the dual of Sq^{n+1}_* is given by the map $\chi(Sq^{n+1}) : H^0 M\xi \to H^{n+1} M\xi$, one sees that this is equivalent to the vanishing of $\chi(Sq^{n+1})(U_\xi) = v_{n+1}(\xi)U_\xi$. Obviously, Q^ξ_{2n} is an affine space for the vector space $\{h \in Hom(\pi_{2n}(M\xi \wedge K_n), \mathbb{Z}/4) | h(\lambda_\xi) = 0\}$ which by the third exact row and the Thom isomorphism is isomorphic to $Hom(H_n M\xi, \mathbb{Z}/4) \cong H^n B$. The operation of $H^n B$ on Q^ξ_{2n} is explicitely given by

$$(x + h)(a) := i^4_2 \langle xU_\xi, p_*(a)\rangle + h(a)$$

for $x \in H^n B$, $h \in Q^\xi_{2n}$ and $a \in \pi_{2n}(M\xi \wedge K_n)$; this shows (i). For (ii), the first statement is proved in [17] (a zero-bordism gives by Poincaré-Lefschetz duality a Lagrangian of q_h), and the second follows by anticipating the first addition formula in 1.2.4, which is a consequence of the corresponding algebraic fact A.3.12. ∎

1.2 Some Properties

We list some properties of the Brown-Kervaire invariant K and the invariant q which follow directly from the definition. Almost all properties can be found in [17] or [22], with the exception of the naturality lemma 1.2.1 (which is trivial), of the last two addition formulas in 1.2.4, and of its corollary 1.2.5 (which all directly follow from the corresponding algebraic facts A.3.12, A.3.13). The simple result 1.2.5 will be used in 2.2.7 to prove the existence of two different Brown-Kervaire invariants for *Spin*-manifolds in dimension 34. Moreover, we give a proof for a non-trivial unproved statement in [22], see 1.2.3.

First we have naturality in the following sense: Let $B \xrightarrow{a} B' \xrightarrow{\xi'} BO$ be a factorization of ξ and denote the induced maps by $H^n B \xleftarrow{a^*} H^n B'$ and

$$\Omega^\xi_{2n} \xrightarrow{a_*} \Omega^{\xi'}_{2n}, \qquad Q^\xi_{2n} \xleftarrow{a^*} Q^{\xi'}_{2n}.$$

1.2.1 Lemma (Naturality): *Let $M \in \Omega^\xi_{2n}$, $h' \in Q^{\xi'}_{2n}$ and $x' \in H^n B'$. Then:*

$$K_{a^*h'}(M) = K_{h'}(a_* M) \quad \textit{and} \quad q_{a^*x',a^*h'}(M) = q_{x',h'}(a_*M).$$

With the classification result A.3.7 in the appendix, we can give a simple criterion when our invariants are $\mathbb{Z}/2$-valued:

1.2.2 Lemma: *Let $v_{n+1}(\xi) = 0$. If the middle dimensional Wu class also vanishes,*

$$v_n(\xi) = 0,$$

then the pairing is even and $q_h : H^n M^{2n} \to \mathbb{Z}/4$ takes values in $\mathbb{Z}/2 \subset \mathbb{Z}/4$; in this case $K_h(M^{2n})$ is the ordinary $\mathbb{Z}/2$-valued Arf-invariant of the quadratic form q_h. In particular, K and q take then values in $\mathbb{Z}/2 \subset \mathbb{Z}/8$, respectively $\mathbb{Z}/2 \subset \mathbb{Z}/4$.

1.2.3 *Remark:* In fact, a stronger result holds, which is stated in [22] but without proof:

Let $v_{n+1}(\xi) = 0$. The exact sequence $0 \to \mathbb{Z}/2 \xrightarrow{i} \pi_{2n}(M\xi \wedge K_n) \xrightarrow{p} H_n B \to 0$ is split iff $v_n(\xi) = 0$.

Actually, this can be proved with a result of Kristensen in [36], p.139, which says the following:

For the cohomology theory E^* associated to the 2-stage Postnikov-system of spectra

$$E \longrightarrow H \xrightarrow{k} \Sigma^d H$$

with k-invariant $k \in A^d = [H, \Sigma^d H]$, the extension in the long exact sequence

$$\ldots \longrightarrow H^{*-1}X \xrightarrow{k} H^{*+d-1}X \longrightarrow E^*X \longrightarrow H^*X \xrightarrow{k} H^{*+d}X \longrightarrow \ldots$$

for any space or spectrum X is detected by $\kappa(k) \in A^{d-1}$.

Here, $\kappa : A^* \to A^{*-1}$ is the Kristensen derivation in the Steenrod algebra (see [33]); and 'detection' means, that $\tilde{x} \in E^*X$ mapping to $x \in ker(k : H^*X \to H^{*+d}X)$ has order 2 if $\kappa(k)(x) \in im(k : H^{*-1}X \to H^{*+d-1}X)$, and order 4 otherwise.

Now, the statement above follows with $k = Sq^{n+1}$ from $\kappa(Sq^{n+1}) = Sq^n$ (see [33]). ■

By definition, the maps K and q are linear in the last variable $M \in \Omega_{2n}^{\xi}$, but not in the other variables h and x, where the dependence is given as follows:

1.2.4 Lemma (Addition formula): Let $M \in \Omega_{2n}^{\xi}$, $h \in Q_{2n}^{\xi}$ and $x, y \in H^nB$. Then we have:

$$\begin{array}{rclcl} K_{x+h}(M) & = & K_h(M) & - & i_4^8 q_{x,h}(M) \\ q_{x+y,h}(M) & = & q_{x,h}(M) & + & q_{y,h}(M) \quad + \quad i_2^4 xy[M] \\ q_{x,y+h}(M) & = & q_{x,h}(M) & + & i_2^4 xy[M]. \end{array}$$

The first formula follows directly from the addition formula A.3.12 in the appendix, and the other two by the definition of $q_{x,h}$ and the quadratic property of $q_h : H^nM \to \mathbb{Z}/4$. We have shortly denoted the H^*B-characteristic number $\langle \nu^*(z), [M] \rangle \in \mathbb{Z}/2$ for $z \in H^{2n}B$ by $z[M]$. As in the appendix (A.3.13), the first two formulas have an important consequence:

1.2.5 Corollary: *Let $M \in \Omega_{2n}^{\xi}$, $h \in Q_{2n}^{\xi}$ and $x, y \in H^nB$. If $xy[M] \neq 0$, then at least one of the three Brown-Kervaire invariants K_{x+h}, K_{y+h}, or K_{x+y+h} is different from K_h.*

In particular, it is to some extend possible to measure the non-triviality of Brown-Kervaire invariants by characteristic numbers of the form $xy[\,]$.

Finally, we have the following formulas for the mod 2 and mod 4 reductions of our invariants K and q, which can be expressed by characteristic classes (see A.3.10):

1.2.6 Lemma (Reductions mod 2 and mod 4): *We have:*

$$\begin{array}{rclcll} K_h(M^{2n}) & \equiv & q_{v_n,h}(M^{2n}) & & & \mathrm{mod} 4 \\ q_{x,h}(M^{2n}) & \equiv & x^2[M^{2n}] & \equiv & v_n x[M^{2n}] & \mathrm{mod} 2 \\ K_h(M^{2n}) & \equiv & v_n^2[M^{2n}] & \equiv & \mathrm{rank}(H^nM^{2n}) & \mathrm{mod} 2. \end{array}$$

We remark also the well-known connection of $\mathrm{rank}(H^nM^{2n})$ to the Euler characteristic and the signature of M^{2n}, which is given by $\mathrm{rank}(H^nM^{2n}) \equiv \chi(M^{2n})$ mod 2; and for M^{2n} oriented, $\chi(M^{2n}) \equiv sign(M^{2n})$ mod 2 (in particular, $\equiv 0$ for n odd).

1.3 The Product Formula of Brown

In [18], Brown considered the problem to compute his invariants K_h on a product $M \times N$ of two manifolds. In order to solve this problem, he introduced a new bordism theory, which we call *strong Wu-bordism*. In difference to *Wu-bordism*, which is the universal bordism theory for his invariant K_h and defined by manifolds with a '*trivialization of the Wu class* v_{n+1}', strong Wu-bordism is defined by simultanous '*trivializations of all Wu classes* v_i, $i \geq n+1$' and is not further universal for K_h.

As his product formula lives in this bordism theory, one has to be careful to get applications of his results for other bordism theories. In particular, we will need for *Spin*-bordism the property that $K_h(M^{8m} \times \overline{S^1} \times \overline{S^1}) \equiv sign(M^{8m}) \bmod 2$ for all Brown-Kervaire invariants of $(8m+2)$-dimensional *Spin*-manifolds (2.2.3). This is stated in [48] by referring to [18] without further comments. We now give a survey on the product formula of Brown in [18] and show in more detail, how to obtain this and similar results from his formula.

1.3.1 First we recall that Brown-Kervaire invariants exist on Ω_{2n}^{ξ} iff $v_{n+1}(\xi) = 0$. This restriction comes in, because we want to construct a *universally defined* quadratic form on $H^n M^{2n}$ for all closed ξ-manifolds M^{2n}.

We can also do this *individually*: For each closed manifold M^{2n} we have

$$v_{n+1}(\nu_M) = v_{n+1}(M) = 0$$

with $\nu_M : M^{2n} \to BO$ the stable normal bundle. Thus we always have Brown-Kervaire invariants K_h in dimension $2n$ for the bordism theory $\Omega_{2n}^{\nu_M}$ associated to the normal bundle; actually, the parameter set $Q_{2n}^{\nu_M}$ can here be identified in a natural way with the set of all $\mathbb{Z}/4$-quadratic refinements on $H^n M^{2n}$ ([17], see also 1.1.4 (ii), A.3.1).

But this does not imply the existence of Brown-Kervaire invariants on unoriented bordism Ω_{2n}^{O}. In fact, the condition $v_{n+1} = 0$ is here never satisfied because $v_{n+1} \in H^*BO$ is non-zero for all n (consider $\mathbb{R}P^{2(n+1)}$).

The universal bordism theory which satisfies the condition $v_{n+1}(\xi) = 0$ is called *Wu-*$(n+1)$*-bordism* $\Omega_*^{\langle v_{n+1}\rangle}$. It is constructed from BO just by killing v_{n+1}, i.e. we take

$$\xi := p_{\langle v_{n+1}\rangle} : BO\langle v_{n+1}\rangle \longrightarrow BO$$

as the pullback of the path fibration $PK_{n+1} \to K_{n+1}$ by a map $v_{n+1} : BO \to K_{n+1}$ realizing the universal Wu class $v_{n+1} \in H^{n+1}BO$. We denote a $p_{\langle v_{n+1}\rangle}$-structure on M also as a *Wu-*$(n+1)$*-structure*, which should be considered as a trivialization of v_{n+1}. Then for any bordism theory Ω_*^{ξ} with $v_{n+1}(\xi) = 0$, there exists a lift of $\xi : B \to BO$ to $BO\langle v_{n+1}\rangle$, and the Brown-Kervaire invariants on Ω_{2n}^{ξ} factor over those of $\Omega_{2n}^{\langle v_{n+1}\rangle}$ (see also 8.1.1).

Now, for two closed manifolds M^{2m} and N^{2n} with a Wu-$(m+1)$-, respectively a Wu-$(n+1)$-structure, there seems to exist no natural way to define a Wu-$(m+n+1)$-structure on the product $M^{2m} \times N^{2n}$. But this can be done - after certain universal choices, again - if one strengthens the definition of a Wu-$(n+1)$-structure.

1.3.2 Thus, Brown defined in [18] a stronger version of Wu-$(n+1)$-bordism, which we therefore call *strong Wu-$(n+1)$-bordism* $\Omega_*^{\langle v_i \,|\, i>n\rangle}$. Here, we kill not only v_{n+1}, but all v_i with $i > n$; i.e. we take the pullback

$$p_{\langle v_i \,|\, i>n\rangle} : BO\langle v_i \,|\, i > n\rangle \longrightarrow BO$$

of the path fibration $\prod_{i>n} PK_i \to \prod_{i>n} K_i$ by a map $(v_i)_{i>n} : BO \to \prod_{i>n} K_i$ realizing all Wu classes $v_i \in H^i BO$, $i > n$. If M^d is a closed manifold of dimension d, we have $v_i(M^d) = 0$ for all $i > \frac{d}{2}$ showing that M^d has always *strong Wu-$(n+1)$-structures* for $n+1 > \frac{d}{2}$, which should be considered as a choice of trivializations for all v_i with $i > \frac{d}{2}$. Since $BO\langle v_i \,|\, i > n\rangle \to BO$ is a principal fibration, the set of these structures on M^d is an affine space with associated vector space

$$[M^d, \prod_{i>n} \Omega K_i] = \bigoplus_{i=n}^{d} H^i M^d.$$

If $n = [\frac{d}{2}]$ (integer part of $\frac{d}{2}$), we speak shortly of a *strong Wu-structure* on M^d.

1.3.3 As an example, the sphere S^d has exactly two strong Wu-$(n+1)$-structures for $0 \leq n \leq d$, and one otherwise. The standard trivialization of the stable normal bundle ν_{S^d} (for the Standard embedding $S^d \subset \mathbb{R}^{d+1}$) gives a stable bundle map $V : \nu_{S^d} \to \gamma_{BO}$ over the constant map $S^d \to BO$. Now, let $g : S^d \to BO\langle v_i \,|\, i > \frac{d}{2}\rangle$ be induced from the generator of $\pi_d K_d = \mathbb{Z}/2$ in the fibre $\prod_{i>\frac{d}{2}} \Omega K_i$. Then V and g define a strong Wu-structure $U : S^d \to BO\langle v_i \,|\, i > \frac{d}{2}\rangle$. According to [18], we call

$$\overline{S^d} := (S^d, U)$$

the *d-sphere with the non-trivial strong Wu-structure.*

1.3.4 As the total Wu class $v \in H^\bullet BO$ is multiplicative,

$$\oplus^* v = v \otimes v,$$

with $\oplus : BO \times BO \to BO$ denoting the Whitney sum, we can $\oplus$ lift to a map

$$\mu : BO\langle v_i \,|\, i > m\rangle \times BO\langle v_i \,|\, i > n\rangle \longrightarrow BO\langle v_i \,|\, i > m+n\rangle,$$

defining a product for the associated bordism theories which we also denote by μ:

$$\mu : \Omega_a^{\langle v_i \,|\, i>m\rangle} \otimes \Omega_b^{\langle v_i \,|\, i>n\rangle} \longrightarrow \Omega_{a+b}^{\langle v_i \,|\, i>m+n\rangle}.$$

This is the first step in the direction to the product formula; but we remark, that the lift μ of $\oplus$ is not unique, which later will play a role.

1.3.5 Next, Brown generalized his construction of the quadratic form $q_h : H^n M^{2n} \to \mathbb{Z}/4$ for a closed $2n$-dimensional manifold with a Wu-$(n+1)$-structure to a series of maps

$$_{(h)}q_i^d : H^i M^d \longrightarrow \begin{cases} \mathbb{Z}/4, & \text{for } d = 2i \\ \mathbb{Z}/2, & \text{otherwise} \end{cases}$$

for closed d-dimensional manifolds with a strong Wu-structure. Here the index (h) indicates, that these maps also depend on the choice of certain universal parameters (remark: q_i^d is in

[18] denoted by ϕ_i^d). For the construction, we recall that q_h was defined as $q_h = h\theta$, where $\theta : H^n M^{2n} \to \pi_{2n}(M\xi \wedge K_n)$ denotes the map (1.1.3)

$$\theta : H^n M^{2n} = [M_+^{2n}, K_n] \xrightarrow{\Sigma^0} [S^0 \wedge M_+^{2n}, S^0 \wedge K_n] \xrightarrow{S} [S^{2n}, M\nu \wedge K_n] \xrightarrow{\nu_*} [S^{2n}, M\xi \wedge K_n].$$

Now, Brown considers for M^d as above, with V being its strong Wu-structure, the map

$$\theta : H^i M^d = [M_+^d, K_i] \xrightarrow{\Sigma^0} [S^0 \wedge M_+^d, S^0 \wedge K_i] \xrightarrow{S} [S^d, M\nu \wedge K_i] \xrightarrow{V_*} [S^d, MO^{\langle v_i \,|\, i > \frac{d}{2} \rangle} \wedge K_i].$$

According to [18], we abbreviate the last term by

$$G_i^d := \pi_d(MO^{\langle v_i \,|\, i > \frac{d}{2} \rangle} \wedge K_i),$$

and remark, that the cup-product $\cup : K_i \times K_j \to K_{i+j}$ and the product μ define a product $\mu_* : G_i^m \otimes G_j^n \to G_{i+j}^{m+n}$. The maps $q_i^d = {}_{(h)}q_i^d$ above are then defined by

$$q_i^d := h_i^d \theta,$$

where one has to choose homomorphisms ('*universal parameters*')

$$h_i^d : G_i^d \longrightarrow \begin{cases} \mathbb{Z}/4, & \text{for } d = 2i \\ \mathbb{Z}/2, & \text{otherwise.} \end{cases}$$

We recall, that h_n^{2n} is just a universal parameter $h = h_n^{2n} \in Q_{2n}^{\langle v_i \,|\, i > n \rangle}$ for a Brown-Kervaire invariant on $\Omega_{2n}^{\langle v_i \,|\, i > n \rangle}$, iff the condition $h(\lambda_n) = 2$ is satisfied ($\lambda_n \in G_n^{2n} = \mathbb{Z}/2$ the generator), since then $q_h = h\theta$ is quadratic. The following theorem of Brown generalizes this to all q_i^d:

1.3.6 Theorem (Brown [18]): *There exists a choice of liftings μ and parameters h_i^d, such that*

$$\begin{aligned} q_i^d(x) &= \begin{cases} (Sq\ x)[M^d] & \text{if } i > \frac{d}{2} \\ (Sq\ x)[M^d] \bmod 2 & \text{if } i = \frac{d}{2} \end{cases} \\ q_i^d(x+y) &= \begin{cases} q_i^d(x) + q_i^d(y) & \text{if } i < \frac{d}{2} \\ q_i^d(x) + q_i^d(y) + i_2^4\ xy[M^d] & \text{if } i = \frac{d}{2} \end{cases} \end{aligned}$$

for all closed d-dimensional manifolds M^d with a strong Wu-structure, and $x, y \in H^i M^d$. Furthermore, define the ring Λ by

$$\Lambda := \frac{\mathbb{Z}/4[t, \alpha_k \mid k \in \mathbb{N}]}{\alpha_0 - 2,\ 2t,\ 2\alpha_k,\ \alpha_k\alpha_l,\ t\alpha_k - \alpha_{k-1}}$$

and a map $q : H^* M^d \to \Lambda$ by

$$q(x) := q_{d/2}^d(x) + \sum_{i > \frac{d}{2}} q_i^d(x) t^{2i-d} + \sum_{i < \frac{d}{2}} q_i^d(x) \alpha_{d-2i}.$$

Then we have for all $x \otimes y \in H^*(M \times N)$ that

$$q(x \otimes y) = q(x)q(y),$$

where we take on $M \times N$ the product of the two strong Wu-structeres on M, N defined by μ. Moreover, each choice of the parameters h_n^{2n} with $h_n^{2n}(\lambda_n) = 2$ for all $n \in \mathbb{N}$ can be uniquely extended to a choice of all (μ, h_i^d), giving these equations for the functions q_i^d.

1.3.7 *Remarks ([18]):*
(i) Of course, the term $q^d_{d/2}(x)$ is defined to be zero for d odd.

(ii) The ring Λ is obtained from $G := \bigoplus_{d,i} G^d_i$ by dividing out the largest ideal possible without losing the quadratic property of q. In fact, the main work in [18] consisted in analyzing G and the product μ_*.

(iii) The statement in 1.3.6 about the choices of (μ, h^d_i) in dependence on h^{2n}_n can be found in [18] on p.299 and p.307.

1.3.8 In the last step, Brown introduced numerical invariants $\sigma_k(M^d)$, $k \in \mathbb{N}$, of the function $q : H^*M^d \to \Lambda$, where $\sigma_0(M^d) \in \mathbb{Z}/8$, and $\sigma_k(M^d) \in \mathbb{Z}/2$ for $k \geq 1$:

$$\sigma_0(M^d) \;:=\; \begin{cases} \widetilde{\mathrm{Arf}}(q^{2n}_n), & \text{for } d = 2n \\ 0, & \text{otherwise} \end{cases}$$

$$\sigma_k(M^d) \;:=\; \mathrm{Arf}(Q_k : V^k(M^d) \to \mathbb{Z}/2) \quad \text{for } k \geq 1,$$

with Q_k and $V^k(M^d)$ defined by

$$V^k(M^d) \;:=\; \begin{cases} H^iM^d \oplus H^{d-i}M^d, & \text{if } d-k = 2i \\ 0, & \text{if } d-k \text{ is odd} \end{cases}$$

$$Q_k(x,y) \;:=\; q^d_k(x) + (Sq\ y)[M^d] + xy[M^d],$$

which turns out to be a non-degenerate quadratic form $Q_k : V^k(M^d) \to \mathbb{Z}/2$.

In particular, $\sigma_0(M^{2n})$ is just the Brown-Kervaire invariant $K_h : \Omega^{\langle v_i \,|\, i>n\rangle}_{2n} \to \mathbb{Z}/8$ with parameter $h = h^{2n}_n$. Using 1.3.6, Brown proved the following product formula:

1.3.9 Theorem (Brown [18]): *For all $n \in \mathbb{N}$, let h^{2n}_n be chosen with $h^{2n}_n(\lambda_n) = 2$, and let (μ, h^d_i) be the liftings and parameters giving the unique extension of the h^{2n}_n in the sense of 1.3.6. Let M^m and N^n be two closed manifolds with strong Wu-structures, and take on $M^m \times N^n$ the product strong Wu-structure defined by μ. Then the invariants σ_k, $k \in \mathbb{N}$, (defined by the h^d_i) satisfy:*

$$\sigma_0(M^m \times N^n) \;=\; \sigma_0(M^m)\sigma_0(N^n) + i^8_2\Big(\textstyle\sum_{k\geq 1} \sigma_k(M^m)\sigma_k(N^n)\Big),$$

$$\sigma_k(M^m \times N^n) \;=\; \sigma_0(M^m)\sigma_k(N^n) + \sigma_k(M^m)\sigma_0(N^n).$$

1.3.10 *Remarks:*
(i) The product formula above can also expressed more elegantly in a ring A ([18]) by

$$\Sigma(M^m \times N^n) = \Sigma(M^m)\Sigma(N^n),$$

where the ring A and the invariant $\Sigma(M) \in A$ are defined by

$$A := \frac{\mathbb{Z}/8[a_k \mid k \geq 1]}{2a_k, a_k^2 - 4, a_k a_l \ (k \neq l)}, \qquad \Sigma(M) := \sigma_0(M) + \sum_{k\geq 1} \sigma_k(M) a_k.$$

(ii) In particular, $\sigma_0(M^m \times N^n)$ is in general not the same as $\sigma_0(M^m)\sigma_0(N^n)$; and the Brown-Kervaire invariants K_h *behave in general not multiplicative*. As an example, we have $\sigma_0(\overline{S^n}) = 0$ since there is no middle dimensional cohomology, but we will see in 1.3.12 that

$$\sigma_0(\overline{S^n} \times \overline{S^n}) = 4 \in \mathbb{Z}/8.$$

(iii) In [18], a strong Wu-structure on M^d is called *preferred*, if $q : H^*M^d \to \Lambda$ vanishes on H^iM^d for all $i < \frac{d}{2}$. In this case it holds that $\sigma_k(M^d) = 0$ for all $k \geq 1$. There always exist preferred strong Wu-stuctures on each M^d, which differ by the trivializations of $v_s(M^d)$ where $s := [\frac{d}{2}] + 1$, i.e. they form an affine space with associated vector space H^sM^d. For example, the preferred strong Wu-structure on S^d is the trivial one. The product of preferred Wu-structures is again preferred, and it holds then $\sigma_0(M^m \times N^n) = \sigma_0(M^m)\sigma_0(N^n)$.

(iv) Furthermore, all constructions and the results 1.3.6 and 1.3.9 are not only valid for strong Wu-structures. In fact, Brown in [18] carried out the construction of $BO\langle v_i \,|\, i > n\rangle$, θ, q_i^d and σ_k not only for BO, but for any classifying space B with respect to the only restriction, that Whitney sums $\oplus : B \times B \to B$ exist in order to construct the liftings

$$\mu : B\langle v_i \,|\, i > m\rangle \times B\langle v_i \,|\, i > m\rangle \to B\langle v_i \,|\, i > m + n\rangle.$$

For example, we can also take BSO, $BSpin$, BPL, ... instead of BO ([18], p.296). We call then a lift

$$V : M^d \longrightarrow B\langle v_i \,|\, i > \tfrac{d}{2}\rangle$$

a *strong B-Wu-structure* on M^d. In the case of oriented bundles (in the usual sense, for example BSO), Brown obtained additionally to 1.3.6 and 1.3.9 using a Theorem of Morita ([45]) the following:

1.3.11 Theorem (Brown [18]): *If the bundles with B-structure are oriented and hence M^d has an orientation class $[M^d] \in H_d(M^d;\mathbb{Z})$ via its B-structure, then we can for all $d = 4m$ choose h_{2m}^{4m} such that $q_{2m}^{4m}(x) = \wp(x)[M^{4m}]$; here $\wp$ denotes the Pontrjagin square. It holds then*

$$\sigma_0(M^{4m}) \equiv sign(M^{4m}) \bmod 8.$$

We refer to section 9 for the definition and properties of the Pontjagin square and the Theorem of Morita. Brown showed also the following results using the definition of σ_k and 1.3.9:

1.3.12 Theorem (Brown [18]):
(1) If $k > d$ or $d - k$ odd, then it holds $\sigma_k(M^d) = 0$.

(2) If $d - k = 2i$ and $k \geq 1$, then it holds $\sigma_k(M^d) = q_i^d(v_i(M^d))$.

(3) If $k \geq 1$, then it holds $i_2^8(\sigma_k(M^d)) = \sigma_0(\overline{S^k} \times M^d)$.

(4) It holds $\sigma_k(\overline{S^k}) = 1$, $\sigma_0(\overline{S^k} \times \overline{S^k}) = 4$, and $\sigma_k(\overline{S^k} \times \mathbb{R}P^{2n}) = 1$.

(5) If the Wu classes $v_i(M)$ vanish for $i > 0$, and if the Λ-valued quadratic form on M satisfies $q_{(M)}(1) = 0 \in \Lambda$, then it holds $\sigma_0(M \times N) = \sigma_0(M)\chi(N)$.

Now we come to results which are not in [18]. With the product formula 1.3.9 and 1.3.12, we obtain:

1.3.13 Proposition:
(1) The invariants σ_k, $k \in \mathbb{N}$ are for each choice of (μ, h_i^d) as in 1.3.6 invariants of strong B-Wu-bordism, i.e.

$$\sigma_k : \Omega_{2n}^{B\langle v_i \,|\, i>n\rangle} \longrightarrow \begin{cases} \mathbb{Z}/8 & \text{for } k = 0 \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

If we set $Wu(B)_* := \bigoplus_{n\in\mathbb{N}} \Omega_{2n}^{B\langle v_i \,|\, i>n\rangle}$, then Σ in 1.3.10 (i) gives a ring-homomorphism

$$\Sigma : Wu(B)_* \longrightarrow A.$$

(2) For all M and N with strong B-Wu-structures and $k \geq 1$, we have

$$\begin{array}{rcl} \sigma_k(N \times N) & = & 0 \\ \sigma_0(M \times N \times N) & = & \sigma_0(M)\sigma_0(N \times N). \end{array}$$

In particular, $\sigma_0(M \times \overline{S^n} \times \overline{S^n}) \in \{0, 4\}$; and if B-bundles are oriented and M^{4m} is of dimension $4m$, we get with the choice of h_{2m}^{4m} as in 1.3.11 that

$$\frac{1}{4}\sigma_0(M^{4m} \times \overline{S^n} \times \overline{S^n}) \equiv sign(M^{4m}) \bmod 2.$$

Proof: (1): We already have seen that σ_0 is a bordism invariant, as it is the Brown-Kervaire invariant with parameter h_n^{2n}. The bordism invariance of σ_k, $k \geq 1$ follows then by 1.3.12 (3). The multiplicativity of $\Sigma : Wu(B)_* \to A$ is just the product formula. (2): The vanishing of $\sigma_k(N \times N)$, $k \geq 1$ follows from the product formula, which gives then also $\sigma_0(M \times N \times N)$. For $\sigma_0(M \times \overline{S^n} \times \overline{S^n})$, we use $\sigma_0(\overline{S^n} \times \overline{S^n}) = 4$ (1.3.12) and $\sigma_0(M) \equiv sign(M) \bmod 8$ (1.3.11). (Another proof for this can also be given with 1.3.12 (5).) ■

1.3.14 At last, we consider the initial problem to get product formulas for Brown-Kervaire invariants in a bordism theory Ω_*^ξ associated to $\xi : B \to BO$. Even if B has Whitney sums and $v_{n+1}(\xi) = 0$, we cannot expect to have a lift $\Omega_*^\xi \to \Omega_*^{B\langle v_i \,|\, i>n\rangle}$ to strong B-Wu-bordism, since in general it does not hold that $v_i(\xi) = 0$ for all $i \geq n+2$. (For the 'classical' bordism theories in [63], this holds only in the case of framed bordism Ω_*^{fr}.)

We only have a map in the other direction

$$p_*^n : \Omega_*^{B\langle v_i \,|\, i>n\rangle} \longrightarrow \Omega_*^B = \Omega_*^\xi,$$

which is induced by the projection map $p^n : B\langle v_i \,|\, i > n\rangle \to B$ and is an *epimorphism* in dimensions $* \leq 2n$. This holds, because any ξ-manifold M in this dimensions has a strong B-Wu-structure over its ξ-structure as $v_i(M) = 0$ for all $i > \frac{1}{2}dim(M)$. Under the assumption

$v_{n+1}(\xi) = 0$, there exist a Brown-Kervaire invariant K_h for Ω^B_{2n} with parameter $h \in Q^B_{2n}$, and by naturality 1.2.1, $K_{p^{n*}h}$ is a Brown-Kervaire invariant for $\Omega^{B\langle v_i \,|\, i>n\rangle}_{2n}$ with parameter $p^{n*}h$.

Suppose now that we have $v_{m+1}(\xi) = v_{n+1}(\xi) = v_{m+n+1}(\xi) = 0$, and choose parameters $h_m \in Q^\xi_{2m}$, $h_n \in Q^\xi_{2n}$, and $h_{m+n} \in Q^\xi_{2(m+n)}$ (if $m = n$, we choose the parameter h_m only one times, of course). According to 1.3.6, we can extend this to a choice of (μ, h^d_i) with $h^{2a}_a = p^{a*}h_a$ for $a \in \{m, n, m+n\}$, such that the product formula 1.3.9 holds in strong B-Wu-bordism. By definition, we have

$$K_{h_a} p^a_* = \sigma_0 : \Omega^{B\langle v_i \,|\, i>a\rangle}_{2a} \longrightarrow \mathbb{Z}/8$$

for $a \in \{m, n, m+n\}$. Now, let $M^{2m}, N^{2n} \in \Omega^\xi_*$ and choose strong B-Wu-structures U, V on M, N lying over their ξ-structures. Then 1.3.9 gives

$$K_{h_{m+n}}(M^{2m} \times N^{2n}) = \sigma_0((M^{2m}, U) \times (N^{2n}, V)) =$$

$$= K_{h_m}(M^{2m}) K_{h_n}(N^{2n}) + i^8_2\Big(\sum_{k\geq 1} \sigma_k(M^{2m}, U)\sigma_k(N^{2n}, V)\Big).$$

1.3.15 As an application, we consider the case $B = BSpin$ of *Spin*-bordism, where we will see in 2.2.1 that $v_{4m+2}(BSpin) = 0$. Hence there are Brown-Kervaire invariants in dimension $8m+2$, which moreover are $\mathbb{Z}/2$-valued as also $v_{4m+1}(BSpin) = 0$:

$$K_h : \Omega^{Spin}_{8m+2} \longrightarrow \mathbb{Z}/2 \subset \mathbb{Z}/8.$$

Furthermore, the non-trivial strong Wu-structure of $\overline{S^1}$ is nothing but the non-trivial *Spin*-structure, and we obtain with 1.3.13 the property stated in [48]:

1.3.16 Proposition: For all $h \in Q^{Spin}_{8m+2}$ and $M^{8m} \in \Omega^{Spin}_{8m+2}$, it holds

$$K_h(M^{8m} \times \overline{S^1} \times \overline{S^1}) \equiv sign(M^{8m}) \bmod 2.$$

1.4 Connection to the Adams Spectral Sequence

We consider now the connections of Brown-Kervaire invariants to the Adams spectral sequence, which was firstly examined by Browder [14] in the case of framed bordism, and then generalized to other bordism theories by Cohen, Jones and Mahowald [22].

This generalization was in [22] applied to the bordism theories of codimension-1 immersions, respectively to oriented codimension-2 immersions, proving certain results for the stable homotopy groups of spheres.

In our case of *Spin*-manifolds, I have seen no good application; so I included this subsection only for the sake of completeness (see 1.4.9).

1.4.1 We recall the *Adams spectral sequence* (always with $\mathbb{Z}/2$-coefficients) for the bordism theory Ω^ξ_*, which comes in by the Pontrjagin-Thom isomorphism $\Omega^\xi_* \cong \pi_* M\xi$ (given by $M^d \mapsto t_{M^d}$, where $t_{M^d} : \Sigma^d S^0 \to M\xi$ denotes the *stable Thom map* of M^d):

$$Ext^{s,t}_{A^*}(H^*M\xi, \mathbb{Z}/2) = E_2^{s,t} \Longrightarrow E_\infty^{s,t} = Gr_s({}_{(2)}\pi_{t-s}M\xi).$$

In particular, we get on Ω^ξ_* the *Adams filtration* of $\pi_*M\xi$, which we denote by $AF(\,)$. Thus, we have $AF(M) \geq s$ iff all functional higher order cohomology operations, of order $< s$ and defined with $t_M \in \pi_*M\xi$, vanish on H^*M (see [1], [40]). Hence $AF(M) \geq 0$ for all $M \in \Omega^\xi_*$, and $AF(M) \geq 1$ iff all H^*B-characteristic numbers vanish for M, i.e M lies in the kernel of the *stable Hurewicz homomorphism*

$$h : \pi_*M\xi \to H_*M\xi.$$

1.4.2 As a corollary of the addition formulas in 1.2.4, we have for $AF(M) \geq 1$ that

$$K_{x+h}(M) - K_h(M) = i_4^8(q_{x,h}(M))$$

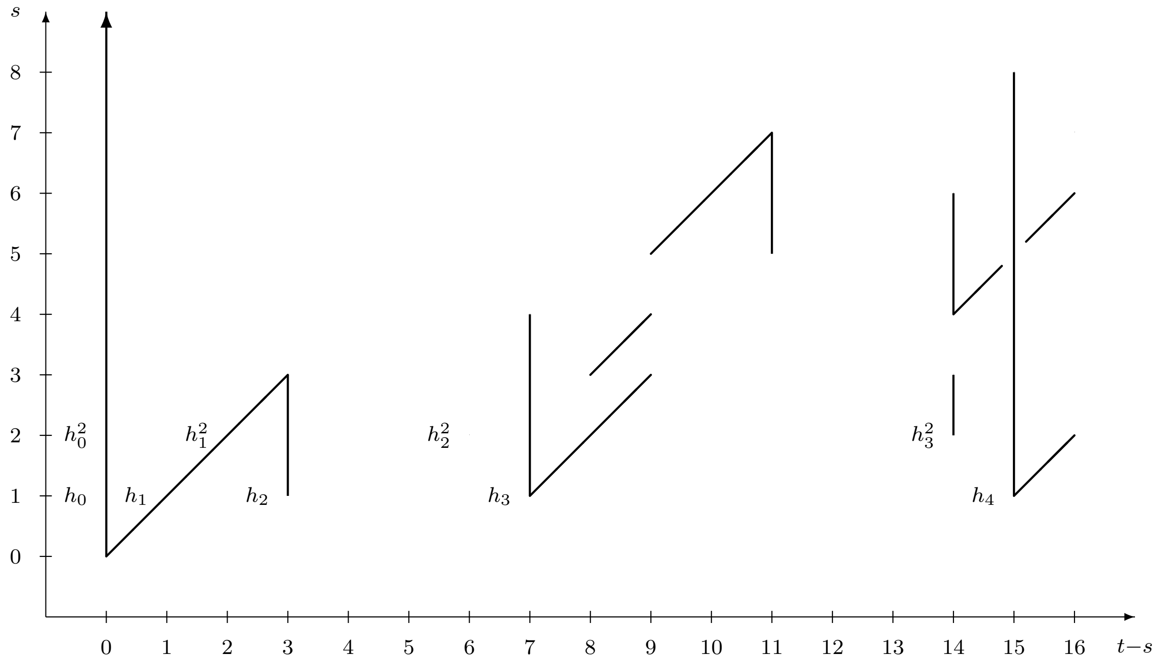
is independent of h and depends linearly on x, because all characteristic numbers $xy[M]$ vanish. Moreover, by 1.2.2, we have $K_h(M) \in \mathbb{Z}/2 \subset \mathbb{Z}/8$ and $q_{x,h}(M) \in \mathbb{Z}/2 \subset \mathbb{Z}/4$.

The connection of Brown-Kervaire invariants to the Adams spectral sequence was examined by Cohen, Jones, and Mahowald in [22]; their results generalize the methods of Browder ([14]) in his study of the Kervaire invariant for framed bordism $\Omega^{\text{fr}}_* = \pi^{\text{st}}_*$.

1.4.3 We recall the main result in [14], where Browder showed, that the Kervaire invariant

$$K : \Omega^{\text{fr}}_{4m+2} \longrightarrow \mathbb{Z}/2$$

vanishes for $4m+2 \neq 2^r - 2$, and is non-trivial for $4m+2 = 2^r - 2$ iff the element $h^2_{r-1} \in Ext^{2,2^r}_{A^*}(\mathbb{Z}/2, \mathbb{Z}/2)$ in the Adams spectral sequence for π^{st}_* survives to E_∞:



Here, $h_i \in Ext_{A^*}^{1,2^i}(\mathbb{Z}/2, \mathbb{Z}/2)$ corresponds to the indecomposable generator Sq^{2^i} of the Steenrod-algebra. According to Adams [1], h_i survives to E_∞ iff there exists a map

$$S^{2^{i+1}-1} \longrightarrow S^{2^i}$$

of Hopf invariant one. Moreover, this is the case iff $i = 0, 1, 2, 3$, where such maps are given by the Hopf fibrations

$$\cdot 2 : S^1 \longrightarrow S^1, \quad \eta : S^3 \longrightarrow S^2, \quad \nu : S^7 \longrightarrow S^4, \quad \sigma : S^{15} \longrightarrow S^8$$

constructed with the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{C}ay$. The analogous problem for the elements h_{r-1}^2 is just the Kervaire invariant one problem. It is conjectured, that all h_{r-1}^2 survive to E_∞. This is known to be true for $r-1 = 0, 1, ..., 6$, but unknown for $r-1 \geq 7$.

Browder showed this by factoring the Kervaire invariant over Wu-bordism (1.3.1, 8.1.1), where it is in the Adams spectral sequence represented by certain functional secondary cohomology operations, hence detecting only elements of Adams filtration ≤ 2.

S. Stolz pointed out to me the well-known fact, that this implies by universality of Wu-bordism the vanishing of Brown-Kervaire invariants for manifolds of Adams filtration ≥ 3:

1.4.4 Theorem (see [14]): *Let $v_{n+1}(\xi) = 0$, and $M \in \Omega_{2n}^\xi$ with $AF(M) \geq 3$. Then $K_h(M)$ and $q_{x,h}(M)$ vanish for all $h \in Q_{2n}^\xi$ and $x \in H^{2n}B$.*

1.4.5 Now we come to the main technical results in [22], generalizing the results of Browder: Consider for $n \in \mathbb{N}$ the following cohomology operation φ^n, where X denotes any space or spectrum:

$$\varphi^n := \sum_{i=0}^{n} Sq^{n+1-i} : \bigoplus_{i=0}^{n} H^{*+i}X \longrightarrow H^{*+n+1}X.$$

We recall from [46], that for a map $f : Y \to X$, one can define the *functional primary cohomology operation*

$$\varphi^n_f : ker(\varphi^n, f^*) \longrightarrow coker(\varphi^n, f^*),$$

where

$$\begin{array}{lcl} ker(\varphi^n, f^*) & := & \{x := (x_0, ..., x_n) \in \bigoplus_{i=0}^n H^{*+i}X \mid \varphi^n(x) = 0 \text{ and all } f^*x_i = 0\}, \\ coker(\varphi^n, f^*) & := & H^{*+n}Y \big/ _{(\varphi^n(\bigoplus_{i=0}^n H^{*-1+i}Y) + Im(f^* : H^{*+n}X \to H^{*+n}Y))}. \end{array}$$

Furthermore, we define for $x \in H^nB$ the element $\tilde{x} \in \bigoplus_{i=0}^n H^{n+i}M\xi$ by

$$\tilde{x} := (xU_\xi \ , \ xv_1(\xi)U_\xi \ , \ \dots \ , \ xv_n(\xi)U_\xi),$$

where the multiplication with U_ξ denotes the stable Thom isomorphism $H^nB \cong H^nM\xi$, of course. Then we have:

1.4.6 Theorem (Cohen, Jones, Mahowald [22]): *Let $v_{n+1}(\xi) = 0$, $h \in Q^\xi_{2n}$, and $M \in \Omega^\xi_{2n}$.*

(1) For all $x \in H^nB$, it holds $\varphi^n(\tilde{x}) = 0$.

(2) If $AF(M) \geq 1$, then the functional operation $\varphi^n_{t_M}(\tilde{x}) \in \mathbb{Z}/2$ is well-defined, and it holds

$$q_{x,h} = i^4_2 \left(\varphi^n_{t_M}(\tilde{x})\right).$$

In particular, we recover 1.4.2, saying that for $AF(M) \geq 1$, $q_{x,h}(M)$ is $\mathbb{Z}/2$-valued and independent on the choice of h.

1.4.7 Corollary: *Let $v_{n+1}(\xi) = 0$, and $M \in \Omega^\xi_{2n}$ with $AF(M) \geq 2$. As we can express $q_{x,h}$ by the functional primary operation $\varphi^n_{t_M}(\tilde{x})$, it vanishes for all $x \in H^nB$ and $h \in Q^\xi_{2n}$. Thus by 1.2.4, $K_h(M) \in \mathbb{Z}/2 \subset \mathbb{Z}/8$ is independent on the choice of $h \in Q^\xi_{2n}$.*

Now, we consider K_h. Let $\widetilde{H}$ be the cofibre of the map $S^0 \to H$ of spectra corresponding to the generator of $H^0S^0 = \mathbb{Z}/2$, and let ∂ be the boundary in the long exact sequence

$$\dots \longrightarrow \pi_*M\xi \longrightarrow \pi_*(H \wedge M\xi) \longrightarrow \pi_*(\widetilde{H} \wedge M\xi) \xrightarrow{\partial} \pi_{*-1}M\xi \longrightarrow \dots$$

induced by the cofibration $M\xi = S^0 \wedge M\xi \to H \wedge M\xi \to \widetilde{H} \wedge M\xi$.

Furthermore, let $z^n_i \in H^{n+1+i}(\widetilde{H} \wedge M\xi)$ be the element

$$z^n_i := \sum_{s+t=i,\ p+q=n+1} \tbinom{t+q}{q} \chi(Sq^{t+q}) \otimes v_s(\xi)v_p(\xi)U_\xi,$$

and $z^n(\xi) := (z^n_o, \dots, z^n_n) \in \bigoplus_{i=0}^n H^{n+1+i}(\widetilde{H} \wedge M\xi)$. Here, we have identified $H^*\widetilde{H}$ with the augmentation ideal $\overline{A}^* := \bigoplus_{i\geq 1} A^i$ of the Steenrod algebra A^*.

1.4.8 Theorem (Cohen, Jones, Mahowald [22]): *Let $v_{n+1}(\xi) = 0$, $h \in Q^{\xi}_{2n}$, and $M \in \Omega^{\xi}_{2n}$.*

(1) In $H^*(\widetilde{H} \wedge M\xi)$, it holds $\varphi^n(z^n(\xi)) = 0$.

(2) If $AF(M) \geq 2$, then $K_h(M) \neq 0$ iff there exists an element $\beta \in \pi_{2n+1}(\widetilde{H} \wedge M\xi)$ with

$$\partial(\beta) = t_M \quad \text{and} \quad \varphi^n_\beta(z^n(\xi)) \neq 0.$$

1.4.9 Actually, these results of Cohen, Jones, Mahowald have found no application in this thesis, because for our main subject of *Spin*-manifolds, one has in the interesting dimensions $8m+2$ that either $AF(M) = 0$ or $AF(M) \geq 2$, see 2.1.16. But $AF(M) = 0$ is the difficult case, where 1.4.6 and 1.4.8 cannot be applied; and for $AF(M) \geq 2$, 1.4.8 could be applied; but in this case Ochanine obtained a sharper result (see 2.2.4).

Part I:

Brown-Kervaire Invariants of *Spin*-Manifolds

2 *Spin*-Manifolds

In this section, we start considering Brown-Kervaire invariants of *Spin*-manifolds. We first give some background information on *Spin*-bordism, which was computed by Anderson, Brown and Peterson [6]. Then we consider Brown-Kervaire invariants and generalized Kervaire invariants, which were defined by Ochanine [48]. As a new result, we show in 2.2.7 the existence of at least two different Brown-Kervaire invariants in dimension 34. Then we recall the construction of Brown and Peterson [20], [21] of some Brown-Kervaire invariants by certain secondary cohomology operations, which we therefore call Brown-Peterson-Kervaire invariants. At last, the addition formula of these invariants leads us to generalized Wu classes.

2.1 *Spin*-Bordism

We now describe the results of Anderson, Brown and Peterson on the *Spin*-bordism ring. As references, we used [6], [5], [4] and the book of Stong [63].

2.1.1 First, we need H^*BSpin and H^*MSpin as modules over the Steenrod algebra A^*. We recall the principal fibration

$$B\mathbb{Z}/2 \longrightarrow BSpin \xrightarrow{p} BSO$$

with classifying map $w_2 : BSO \to K_2$, obtained by applying the B-functor to the central extension $\mathbb{Z}/2 \to Spin \to SO$ and $B\mathbb{Z}/2 = \mathbb{R}P^\infty = K_1 \simeq \Omega K_2$. From this it follows at once that p induces an isomorphism in cohomology with $\mathbb{Z}[\frac{1}{2}]$-coefficients, which gives then also

$$\Omega_*^{Spin} \otimes \mathbb{Z}[\frac{1}{2}] \cong \Omega_*^{SO} \otimes \mathbb{Z}[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}][M^4, M^8, ...].$$

For $\mathbb{Z}/2$-coefficients, we apply the Serre spectral sequence to $BSpin \to BSO \to K_2$. With $H^*K_2 = \mathbb{Z}/2[Sq^I\iota_2 \,|\, I = (2^{r-1}, .., 2, 1), r \geq 0]$, one gets (see also [63]):

2.1.2 Proposition (Thomas [66]): *The map $p^* : H^*BSO \to H^*BSpin$ is surjective with kernel the ideal generated by $Sq^I w_2$, where $I = (2^{r-1}, .., 2, 1)$ and $r \geq 0$ (with $Sq^I := 1$ for $r = 0$). In particular, there is an isomorphism of algebras*

$$\begin{aligned} H^*BSpin &\cong \mathbb{Z}/2[w_k \,|\, k \neq 1, 2^r + 1, \, r \geq 0] \\ &= \mathbb{Z}/2[w_k \,|\, \alpha(k-1) \geq 2] \\ &= \mathbb{Z}/2[w_4, w_6, w_7, w_8, w_{10}, ...], \end{aligned}$$

where $\alpha : \mathbb{N} \to \mathbb{N}$ counts the number of one's in the dyadic expansion of a natural number.

2.1.3 The statement about the multiplicative structure in 2.1.2 follows from the Wu formula [44]

$$Sq^k w_n = \sum_{i=0}^{k} \binom{k-n}{i} w_{k-i} w_{n+i},$$

which gives by induction on r in $I = (2^{r-1}, .., 2, 1)$ that $Sq^I w_2 = w_{2^r+1}$+decomposables, since $\binom{2^r+1}{2^r} \equiv 1 \bmod 2$. Thus $w_{2^r+1} \in H^*BSpin$ is in general not zero, but only decomposable. For example,

$$w_2 = w_3 = w_5 = w_9 = 0, \qquad \text{but} \qquad w_{17} = w_{13}w_4 + w_{11}w_6 + w_{10}w_7.$$

The A^*-module structure of H^*BSpin is then given by the Wu formula together with these decompositions of w_{2^r+1}. By the Thom isomorphism $H^*MSpin = H^*BSpin \cdot U_{Spin}$ and $Sq^k U_{Spin} = w_k U_{Spin}$, the A^*-module structure of H^*MSpin is now explicitely computable and an application of the Adams spectral sequence for low-dimensional computations is possible.

2.1.4 In order to have a better understanding of the A^*-module structure, Anderson, Brown and Peterson examined certain KO-characteristic numbers of Spin manifolds. We recall the KO-orientation of Spin manifolds given by the Dirac operator $MSpin \to KO$ (see [8], [64]) and denote the KO-orientation class of $M \in \Omega_n^{Spin}$ by $[M]_{KO} \in KO_n(M)$. For an element $x \in KO^m(BSpin)$ we can define a (normal) *KO-characteristic number*

$$x[M] := \langle \nu_M^* x, [M]_{KO} \rangle \in KO_{n-m}$$

by the Kronecker pairing $KO^m(X) \otimes KO_n(X) \to KO^{m-n} = KO_{n-m}$, which gives bordism invariants

$$KO^m(BSpin) \otimes \Omega_n^{Spin} \to KO_{n-m}.$$

The bordism invariance follows by $\langle \nu_M^* x, [M] \rangle = \langle i^* \nu_W^* x, \partial[W, M] \rangle = 0$ for $M = \partial W \overset{i}{\hookrightarrow} W$. For $m - n = 0 \bmod 4$ it is possible to express the integer-valued invariant $x[M]$ by rational Pontrjagin numbers: The *Pontrjagin character* $ph(x) := ch(x \otimes \mathbb{C})$ is defined by the Chern character after complexification,

$$ph : KO^m(X) \to H^{4*+m}(X; \mathbb{Q}),$$

which for X a point is injective in dimensions $0, 4 \bmod 8$, with image $\mathbb{Z}$ in the first case, and $2\mathbb{Z}$ in the second case. Then there is the *Topological Riemann-Roch Theorem*:

2.1.5 Theorem (see [63]): *Let M be a closed Spin-manifold and $x \in \widetilde{KO}^* M$, then*

$$ph\langle x, [M]_{KO} \rangle = \langle \hat{A}(M) ph(x), [M] \rangle,$$

where $\hat{A}(M)$ denotes the $\hat{A}$-genus of M, and the right side is the Kronecker pairing in cohomology.

2.1.6 Now we introduce *KO-Pontrjagin classes* $\pi_k(\xi) \in KO^0(X)$ ([4]) of an n-dimensional real vector bundle ξ over a space X, which are defined by

$$\pi_u(\xi) = \Lambda_t(\xi - n),$$

where $\Lambda_t(\xi) := \sum_{k=0}^\infty t^k \Lambda^k(\xi)$ is the *total exterior power*, and $\pi_u(\xi) := \sum_{k=0}^\infty u^k \pi_k(\xi)$ with $u := \frac{t}{(1+t)^2}$. This equation is to be understood in the formal power ring $KO(X)[[t]]$ and makes also sense for a virtual bundle ξ. Explicitely, we have $(1+t)^n \sum_{k=0}^\infty \frac{t^k}{(1+t)^{2k}} \pi_k(\xi) = \sum_{k=0}^\infty t^k \Lambda^k(\xi)$.

2.1.7 *Remarks:*
(i) Let $\overline{\Lambda^k} := \Lambda^k(\xi - n)$, with $n := dim(\xi)$, be the reduced exterior power, then the first of these classes are given by (computation with 'Mathematica'):

$$\begin{array}{rcl}
\pi_0(\xi) & = & 1, \\
\pi_1(\xi) & = & \overline{\Lambda^1}, \\
\pi_2(\xi) & = & \overline{\Lambda^2} + 2\overline{\Lambda^1}, \\
\pi_3(\xi) & = & \overline{\Lambda^3} + 4\overline{\Lambda^2} + 5\overline{\Lambda^1}, \\
\pi_4(\xi) & = & \overline{\Lambda^4} + 6\overline{\Lambda^3} + 14\overline{\Lambda^2} + 14\overline{\Lambda^1}, \\
\pi_5(\xi) & = & \overline{\Lambda^5} + 8\overline{\Lambda^4} + 27\overline{\Lambda^3} + 48\overline{\Lambda^2} + 42\overline{\Lambda^1}.
\end{array}$$

(ii) Since $\pi_k(\xi)$ is again a virtual vector bundle, we can consider the natural transformation $\xi \mapsto \pi_k(\xi)$ as being an element $\pi_k \in KO^0(BO)$. We denote the corresponding elements in $KO^0(BSO)$ and $KO^0(BSpin)$ also by π_k. Then the following theorem of Anderson [3] holds (see [4]):

$$KO^*(BSO) \cong KO^*[[\pi_k | k \geq 1]].$$

2.1.8 Furthermore, one defines (normal) *KO-Pontrjagin numbers* $\pi_I[M^n]$ of a closed *Spin*-manifold M^n for any sequence $I = (i_1, .., i_r)$ of integers by

$$\pi_I[M^n] := \langle \pi_{i_1}(\nu_{M^n}) \cdot ... \cdot \pi_{i_r}(\nu_{M^n}), [M^n]_{KO} \rangle \qquad \in KO_n.$$

We will consider only sequences with $i_1 \geq ... \geq i_r > 1$, which we call *special sequences*, and set $|I| := i_1 + ... + i_r$. As the $\pi_I[M^n]$ are invariants of *Spin*-bordism, we can also view the KO-Pontrjagin numbers as maps $\pi_I : MSpin \to KO$ from the Thom spectrum $MSpin$ to the real K-theory spectrum KO. Studying these maps, Anderson, Brown and Peterson proved:

2.1.9 Theorem (Anderson, Brown, Peterson [6]): *The map $\pi_I : MSpin \to KO$ can be lifted to the $(4|I|-1)$-connected covering spectrum $KO\langle 4|I|\rangle$ if $|I|$ is even, and to the $(4|I|-3)$-connected covering spectrum $KO\langle 4|I|-2\rangle$ if $|I|$ is odd. There exist liftings $\widetilde{\pi}_I$ of π_I and homogeneous elements $z_i \in H^*MSpin = [MSpin, \Sigma^* H]$, $i \in Z$, such that the map*

$$(\widetilde{\pi}_I, z_i) : MSpin \longrightarrow (\bigvee_{|I| even} KO\langle 4|I|\rangle) \vee (\bigvee_{|I| odd} KO\langle 4|I|-2\rangle) \vee (\bigvee_{i \in Z} \Sigma^{|z_i|} H)$$

gives a 2-primary homotopy equivalence of spectra. Since Ω_*^{Spin} has no odd torsion, one gets an additive isomorphism

$$\Omega_*^{Spin} \cong (X_* \otimes KO\langle 0\rangle_*) \oplus (Y_* \otimes \Sigma^{-2} KO\langle 2\rangle_*) \oplus Z_*,$$

where $X_* = \bigoplus \mathbb{Z} x_I$ is the graded free abelian group generated by elements x_I of degree $4|I|$ for all special sequences I with $|I|$ even, $Y_* = \bigoplus \mathbb{Z} y_I$ is the graded free abelian group generated by elements y_I of degree $4|I|-2$ for all special sequences I with $|I|$ odd, and $Z_* = \bigoplus \mathbb{Z}/2 z_i$ is the graded $\mathbb{Z}/2$-vector space generated by the z_i, $i \in Z$ of degree $|z_i|$. Since the z_i can be expressed by Stiefel-Whitney numbers, two closed Spin-manifolds are Spin-cobordant, iff they have the same KO-Pontrjagin numbers and Stiefel-Whitney numbers.

2.1.10 *Remarks:*

(i) The coefficients of the (-1)-connected KO-theory $KO\langle 0\rangle$ and of the shifted 1-connected KO-theory $\Sigma^{-2}KO\langle 2\rangle$ are given by the real Bott periodicity:

$n:$	$n<0:$	$n\geq 0$, mod8 :							
		$n\equiv 0$	$n\equiv 1$	$n\equiv 2$	$n\equiv 3$	$n\equiv 4$	$n\equiv 5$	$n\equiv 6$	$n\equiv 7$
$KO\langle 0\rangle_n:$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0
$\Sigma^{-2}KO\langle 2\rangle_n:$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$

(ii) The graded free abelian group X_* is concentrated in dimensions $0 \bmod 8$, and Y_* is concentrated in dimensions $2 \bmod 8$. In fact, the first two summands $P_* := (X_*\otimes KO\langle 0\rangle_*)\oplus(Y_*\otimes\Sigma^{-2}KO\langle 2\rangle_*)$ of Ω_*^{Spin} in Theorem 2.1.10 give the following contribution:

$n:$	$8m$	$8m+1$	$8m+2$	$8m+3$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
$P_n:$	$(\mathbb{Z})^{\wp(2m)}$	$(\mathbb{Z}/2)^{\wp(2m)}$	$(\mathbb{Z}/2)^{\wp(2m+1)}$	0	$(\mathbb{Z})^{\wp(2m+1)}$	0	0	0

Here, $\wp(k)$ denotes the number of partitions of $k\in\mathbb{N}$, which has the generating function $\sum_{k\geq 0}\wp(k)t^k=\prod_{n\geq 1}(1-t^n)^{-1}$ (see [26], chapter 8).

2.1.11 The graded $\mathbb{Z}/2$-vector space Z_* in theorem 2.1.10 is not explicitely given. Using [5], it can be computed as follows, giving thus the degrees of all $z_i\in H^*MSpin$, $i\in Z$ (but not the actual classes!):

The *Poincaré-Hilbert series* $p_V(t)\in\mathbb{Z}[[t]]$ of a graded $\mathbb{Z}/2$-vector space V is defined by

$$p_V(t):=\sum_{i=0}^{\infty} dim_{\mathbb{Z}/2}(V_i)t^i.$$

Let $p_{MSpin}(t)$, $p_A(t)$, $p_{A_1}(t)$, $p_J(t)$, $p_X(t)$, $p_Y(t)$, and $p_Z(t)$ be the Poincaré-Hilbert series of the graded $\mathbb{Z}/2$-vector spaces

$$\begin{array}{l} H^*MSpin,\\ A^*,\\ A_1^* \;=\; A^*/(Sq^1A^*+Sq^2A^*)\;\cong\; H^*KO\langle 0\rangle,\\ J^* \;:=\; A^*/(Sq^3A^*)\;\cong\; H^*\Sigma^{-2}KO\langle 2\rangle\;\; \text{(the 'Joker')},\\ X_*\otimes\mathbb{Z}/2,\\ Y_*\otimes\mathbb{Z}/2,\\ Z_*, \end{array}$$

respectively. One knows that ([5], [46]):

$$\begin{array}{lcl} p_{MSpin}(t) &=& \prod_{n\neq 1,2^r+1}(1-t^n)^{-1}\\ p_A(t) &=& \prod_{n=2^r-1,r\geq 1}(1-t^n)^{-1}\\ p_{A_1}(t) &=& (1-t^4)^{-1}(1-t^6)^{-1}\prod_{n=2^r-1,r\geq 3}(1-t^n)^{-1}\\ p_J(t) &=& (1+t+t^2+t^3+t^4)(1-t^4)^{-1}(1-t^6)^{-1}\prod_{n=2^r-1,r\geq 3}(1-t^n)^{-1} \end{array}$$

According to [5], one then gets Z_* by an 'inductive computation' using

$$H^*MSpin\cong(X_*\otimes H^*KO\langle 0\rangle)\oplus(Y_*\otimes H^*\Sigma^{-2}KO\langle 2\rangle)\oplus(Z_*\otimes A^*),$$

which holds, because $(\tilde{\pi}_I, z_i)$ gives a 2-primary homotopy equivalence. But in fact, a very explicit formula for $p_Z(t)$ can be given: A special sequence I with $|I| = n$ is just a partition of n into numbers $i_s \geq 2$, showing that the number $a(n)$ of these partitions I can be computed by

$$\sum_{k \geq 0} a(k) t^k = \prod_{n \geq 2} (1 - t^n)^{-1}.$$

With the definition of X_* and Y_* one easily gets

$$\begin{aligned} p_X(t) &= \tfrac{1}{2}(\textstyle\prod_{k\geq 2}(1 - t^{4k})^{-1} + \prod_{k \geq 2}(1 - (-1)^k t^{4k})^{-1}) \\ p_Y(t) &= \tfrac{1}{2t^2}(\textstyle\prod_{k\geq 2}(1 - t^{4k})^{-1} - \prod_{k \geq 2}(1 - (-1)^k t^{4k})^{-1}). \end{aligned}$$

Thus $p_Z(t)$ is given by $p_Z(t) = p_A(t)^{-1}\Big(p_{MSpin}(t) - p_X(t)p_{A'}(t) - p_Y(t)p_{A''}(t)\Big)$, hence:

2.1.12 Proposition: *The Hilbert series $p_Z(t)$ giving the Eilenberg-MacLane part in $MSpin$ is given by*

$$p_Z(t) =$$

$$(1+t+t^2+t^3)^{-1}(1+t^3)^{-1}\left(\prod_{n \neq 2^r \pm 1, n \geq 8} (1-t^n)^{-1} - p_X(t) - (1+t+t^2+t^3+t^4)p_Y(t)\right).$$

For the convenience of the reader, we now give P_* and Z_* below dimension 48, which also gives $\Omega_*^{Spin} \cong P_* \oplus Z_*$. We use the short notation $\infty^a 2^b := (\mathbb{Z})^a \oplus (\mathbb{Z}/2)^b$. This table was computed from the above formulas using a 'Mathematica' program:

$n:$	0	1	2	3	4	5	6	7
$P_n:$	∞	2	2	0	∞	0	0	0
$Z_n:$	0	0	0	0	0	0	0	0
$n:$	8	9	10	11	12	13	14	15
$P_n:$	∞^2	2^2	2^3	0	∞^3	0	0	0
$Z_n:$	0	0	0	0	0	0	0	0
$n:$	16	17	18	19	20	21	22	23
$P_n:$	∞^5	2^5	2^7	0	∞^7	0	0	0
$Z_n:$	0	0	0	0	2	0	2	0
$n:$	24	25	26	27	28	29	30	31
$P_n:$	∞^{11}	2^{11}	2^{15}	0	∞^{15}	0	0	0
$Z_n:$	0	0	0	0	2^2	2	2^3	0
$n:$	32	33	34	35	36	37	38	39
$P_n:$	∞^{22}	2^{22}	2^{30}	0	∞^{30}	0	0	0
$Z_n:$	2	2	2	0	2^6	2^2	2^7	2
$n:$	40	41	42	43	44	45	46	47
$P_n:$	∞^{42}	2^{42}	2^{56}	0	∞^{56}	0	0	0
$Z_n:$	2^4	2^3	2^4	2^2	2^{14}	2^6	2^{17}	2^4

2.1.13 The result 2.1.10 gives only the additive structure of Ω_*^{Spin}, whereas its full multiplicative structure is not yet known. This is caused by the Stiefel-Whitney numbers z_i and their (unknown) relations to KO-Pontrjagin numbers. But at least, KO-Pontrjagin classes behave multiplicatively,

$$\pi_n(x \otimes y) = \sum_{i+j=n} \pi_i(x)\pi_j(y),$$

which was used in [6] (see also [63]) to prove the following theorem:

2.1.14 Theorem (Anderson, Brown, Peterson [6]): *Let $I_* \subset \Omega_*^{Spin}$ be the ideal generated by all Spin-manifolds with vanishing KO-Pontrjagin numbers. Then $I_* \cong Z_*$, and Ω_*^{Spin}/I_* is the subring of R_* generated by α, x_{8i}, $x_{8i+4}x_{8j+4}$, $2x_{8i+4}$ and y_{8i+2}, where*

$$R_* := \frac{\mathbb{Z}[\alpha,\ x_{4i},\ y_{8i+2} \mid i \geq 1]}{2\alpha,\ 2y_{8i+2},\ \alpha^3,\ y_{8i+2}y_{8j+2},\ \alpha y_{8i+2},\ \alpha^2 x_{8i+4} - x_4 y_{8i+2},\ x_{8i+4}y_{8j+2} - x_{8j+4}y_{8i+2}}.$$

Furthermore, the relations between (integral) Pontrjagin numbers, and between Stiefel-Whitney numbers have been computed for *Spin*-bordism ([63]):

2.1.15 Theorem [63]:

(i) All relations between integral Pontrjagin numbers are given by the Topological Riemann-Roch Theorem 2.1.5:

$$\langle \hat{A}(M^{8m}) ph(\nu_M^* x), [M^{8m}] \rangle \in \mathbb{Z} \qquad \textit{and} \qquad \langle \hat{A}(M^{8m+4}) ph(\nu_M^* x), [M^{8m+4}] \rangle \in 2\mathbb{Z},$$

where $ph(\nu_M^* x)$ is expressed as a rational Pontragin class for all KO-Pontrjagin classes x as explained in [63].

(ii) All relations between Stiefel-Whitney classes are given by the vanishing of w_1, w_2, together with the the Wu formula 2.1.3, which can be expressed by

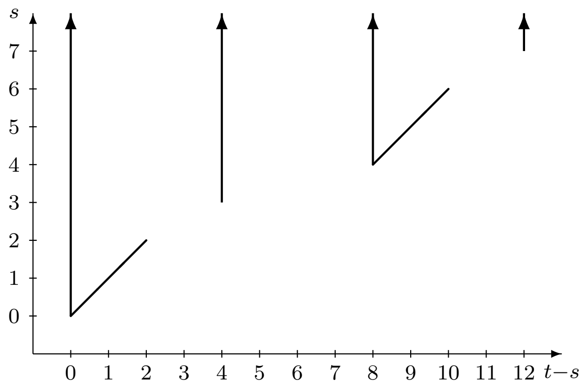
$$\langle \nu_M^*(Sq^i a - v_i a), [M] \rangle = 0 \quad \textit{for all } a \in H^*BO.$$

We remark, that the relations between KO-Pontrjagin numbers (mod 2) and Stiefel-Whitney numbers of *Spin*-manifolds seem not to be known; otherwise, one could reconstruct the multiplicative structure of Ω_*^{Spin} by 2.1.9.

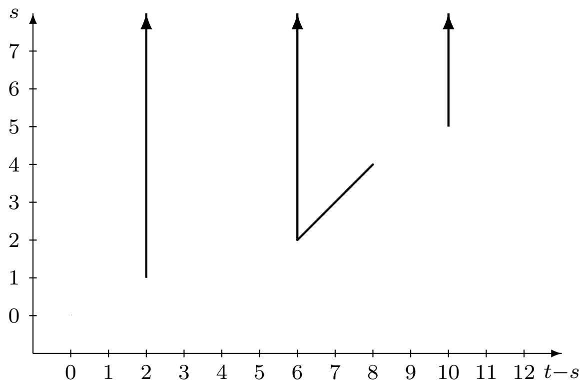
At last, we consider the Adams spectral sequence for *Spin*-bordism:

2.1.16 Theorem ([54]): *The E_2-term for the Adams spectral sequence with $\mathbb{Z}/2$-coefficients, converging to $\Omega_*^{Spin} = \pi_* MSpin$, decomposes according to 2.1.9 as a direct sum, where three types of summands occur:*

(i) $Ext_2^{s,t}(H^*KO\langle 0\rangle, \mathbb{Z}/2)$, which is $(8,4)$-periodic in $(t-s, s)$:



(ii) $Ext_2^{s,t}(\Sigma^{-2}H^*KO\langle 2\rangle, \mathbb{Z}/2)$, which is also $(8,4)$-periodic in $(t-s, s)$:



(iii) $Ext_2^{s,t}(A^*, \mathbb{Z}/2) = \mathbb{Z}/2$, concentrated in bidegree $(0,0)$.

These summands are shifted to the geometric dimensions $t-s = 8m$ in the first case, to $t-s = 8m+2$ in the second case, and to $t-s = |z_i|$ in the third case. The Adams spectral sequence collapses, $E_2 = E_\infty$. Thus, the image of the stable Hurewicz homomorphism,

$$h_* := im(h : \Omega_*^{Spin} \to H_* MSpin),$$

as a graded vector space is isomorphic to

$$(X_* \otimes \mathbb{Z}/2) \oplus (Y_* \otimes \mathbb{Z}/2) \oplus Z_*.$$

2.1.17 A computation gives for $k_* := (X_* \oplus Y_*) \otimes \mathbb{Z}/2$ and h_* the following values below dimension 48:

$n:$	0	1	2	3	4	5	6	7
$k_n:$	2	0	0	0	0	0	0	0
$h_n:$	2	0	0	0	0	0	0	0

$n:$	8	9	10	11	12	13	14	15
$k_n:$	2	0	2	0	0	0	0	0
$h_n:$	2	0	2	0	0	0	0	0

$n:$	16	17	18	19	20	21	22	23
$k_n:$	2^2	0	2^2	0	0	0	0	0
$h_n:$	2^2	0	2^2	0	2	0	2	0

$n:$	24	25	26	27	28	29	30	31
$k_n:$	2^4	0	2^4	0	0	0	0	0
$h_n:$	2^4	0	2^4	0	2^2	2	2^3	0

$n:$	32	33	34	35	36	37	38	39
$k_n:$	2^7	0	2^8	0	0	0	0	0
$h_n:$	2^9	2	2^9	0	2^6	2^2	2^7	2

$n:$	40	41	42	43	44	45	46	47
$k_n:$	2^{12}	0	2^{14}	0	0	0	0	0
$h_n:$	2^{16}	2^3	2^{18}	2^2	2^{14}	2^6	2^{17}	2^4

2.2 Brown-Kervaire Invariants and Generalized Kervaire Invariants

Now we ask when our existence condition $v_{n+1}(\xi) = 0$ for Brown-Kervaire invariants is satisfied for *Spin*-manifolds. This surely is a well-known fact:

2.2.1 Lemma: *For Spin-bordism $\xi : BSpin \to BO$, the total Wu class has the form*

$$v(\xi) = 1 + v_4 + v_8 + v_{12} + \dots$$

with $v_{4k} \neq 0$ for all k. Thus there exist Brown-Kervaire invariants for Spin-manifolds exactly in dimensions $2n$ with $n+1 \neq 4k$, which gives the two cases $2n = 4m$ and $2n = 8m+2$. The invariants are $\mathbb{Z}/2$-valued in the second case.

Proof: Since in [48] only a slightly weaker statement was shown ($v_{4m+2}M^{8m+2} = 0$), we give here the proof following the same argument: By the Adem relations $Sq^1Sq^{2k} = Sq^{2k+1}$ and $Sq^2Sq^{4k} = Sq^{4k+2} + Sq^{4k+1}Sq^1$ we get $\chi(Sq^{2k+1}) = \chi(Sq^{2k})Sq^1$ and $\chi(Sq^{4k+2}) = \chi(Sq^{4k})Sq^2 + Sq^1\chi(Sq^{4k})Sq^1$, showing that the condition $v_{n+1}(\xi) = 0$ is satisfied for $n = 2m$ and $n = 4m+1$. The statement for v_{4k} follows by consideration of the *Spin*-manifold $\mathbb{H}P^{2k}$. ■

2.2.2 In the first case, the Pontrjagin square $\wp$ gives a canonical $h_\wp \in Q_{4m}^{Spin}$ and then $K_\wp$ is equal to the signature mod 8 by a theorem of Morita [45]. In fact, this is already true for oriented manifolds, which we will consider in section 9.

The second case is more complicated; here one also has $v_n(\xi) = 0$ and thus gets by 1.2.2 $\mathbb{Z}/2$-valued invariants which we also denote by

$$\begin{array}{rrcl} K: & Q_{8m+2}^{Spin} \times \Omega_{8m+2}^{Spin} & \longrightarrow & \mathbb{Z}/2, \\ q: \; H^{4m+1}BSpin \times & Q_{8m+2}^{Spin} \times \Omega_{8m+2}^{Spin} & \longrightarrow & \mathbb{Z}/2. \end{array}$$

The parameter set Q_{8m+2}^{Spin} grows according to $|Q_{8m+2}^{Spin}| = |H^{4m+1}BSpin|$, and $H^*BSpin = \mathbb{Z}/2[w_k | k \neq 1, 2^s+1]$. This gives for the first values $|Q_2^{Spin}| = |Q_{10}^{Spin}| = |Q_{18}^{Spin}| = 1$, $|Q_{26}^{Spin}| = 4$, and $|Q_{34}^{Spin}| = 16$, where a $\mathbb{Z}/2$-basis of $H^{17}BSpin$ is given by

$$w_{13}w_4, \quad w_{11}w_6, \quad w_{10}w_7, \quad w_7w_6w_4.$$

2.2.3 In fact, Ochanine proved in [48] that in dimensions 2, 10, 18 and 26, the Brown-Kervaire invariants agree with his invariant k (defined in 3.1.3 in the next section). Actually, he proved this for the larger class of homomorphisms $K : \Omega_{8m+2}^{Spin} \to \mathbb{Z}/2$ with the properties

$$\begin{array}{lrcl} (1) & K(M^{8m} \times \bar{S}^1 \times \bar{S}^1) & \equiv & sign(M^{8m}) \bmod 2 \\ (2) & H^{4m+1}M^{8m+2} = 0 & \Rightarrow & K(M^{8m+2}) = 0, \end{array}$$

where $\bar{S}^1$ denotes the circle with the non-trivial *Spin*-structure. Ochanine called these invariants *generalized Kervaire invariants* because for Brown-Kervaire invariants, (2) is obviously satisfied, and (1) follows from the results in [18], see 1.3.16.

Ochanine got this result by proving the following theorem:

2.2.4 Theorem (Ochanine [48]): *Let $C_* \le \Omega_*^{Spin}$ be the subring of all bordism classes of manifolds, for which all Stiefel-Whitney numbers containing an odd dimensional Stiefel-Whitney class vanish. Then all generalized Kervaire invariants coincide on C_{8m+2} with the Ochanine invariant k. Furthermore, for two generalized Kervaire invariants K_1 and K_2, the difference $K_1 - K_2$ is a $\mathbb{Z}/2$-characteristic number $x[\,]$, where $x \in H^{8m+2}BSpin$ lies in the ideal generated by the w_{2i+1} with i not a power of 2.*

2.2.5 *Remarks:*
(i) The statement in 2.2.3 is then a consequence of $C_* = \Omega_*^{Spin}$ for $* = 2, 10, 18, 26$.

(ii) According to [63], the subgroup C_* can also be characterized by

$$\begin{array}{rcl} C_* & = & \{M \in \Omega_*^{Spin} \,|\, M \text{ is unoriented cobordant to an almost complex manifold}\} \\ & = & \{M \in \Omega_*^{Spin} \,|\, M \text{ is unoriented cobordant to a square}\}. \end{array}$$

In [48], there is also an example of two different generalized Kervaire invariants in dimension 34: Let M^{10} and M^{24} be closed *Spin*-manifolds with the only (tangential) non-zero Stiefel-Whitney numbers

$$w_6w_4[M^{10}], w_8^2w_4^2[M^{24}], w_7^2w_6w_4[M^{24}], w_6^4[M^{24}], w_6^2w_4^2[M^{24}], w_4^6[M^{24}]$$

(M^{10} and M^{24} exist by [41] and [6]). Define $M^{34} := M^{10} \times M^{24}$ and let $K : \Omega_{34}^{Spin} \to \mathbb{Z}/2$ be a generalized Kervaire invariant.

2.2.6 Proposition (Ochanine [48]): *$K + w_{12}w_8w_7^2[\ \ ]$ is also a generalized Kervaire invariant which is different from K because of $w_{12}w_8w_7^2[M^{34}] \neq 0$.*

A modification of this example shows also the existence of two different Brown-Kervaire invariants in dimension 34: Let $x := w_{13}w_4$ and $y := w_{10}w_7$, then we have:

2.2.7 Proposition: *At least one of the Brown-Kervaire invariants K_{x+h}, K_{y+h} or K_{x+y+h} is different from K_h for every $h \in Q_{34}^{Spin}$.*

Proof: A short computation shows $xy[M^{34}] \neq 0$ and hence the proposition, by Corollary 1.2.5 of the addition formula. ∎

At this point I would like to thank again Rainer Jung for his help in writing a computer program to compute all relations between Stiefel-Whitney numbers of 34-dimensional *Spin*-manifolds, according to theorem 2.1.15(ii). I also thank the Max-Planck Institut für Mathematik in Bonn for supporting computer time. As a result of this computation, the Stiefel-Whitney number $xy[\ \ ]$ turned out to be non-trivial. Later I observed, that Ochanine's *Spin*-manifold M^{34} above serves as an example to prove this directly.

2.3 Brown-Peterson-Kervaire Invariants

We come now to the construction of certain Brown-Kervaire invariants for *Spin*-manifolds by unstable secondary cohomology operations, see [15]. We will use this method also in 8.3.7 for $BO\langle 2^r\rangle$-manifolds. The starting point is the following Theorem of Brown and Peterson:

2.3.1 Theorem: (Brown, Peterson [20]) *Let $Sq^{n+1} = \sum_{i=1}^{s} \alpha_i\beta_i$ be a decomposition in the Steenrod algebra A^*, with $\alpha_i \in A^{n_i}$, $\beta_i \in A^{m_i}$, and $m_i + n_i = n+1$, $m_i, n_i > 0$. Let ϕ be a secondary cohomology operation which is associated to this decomposition,*

$$\phi : ker(\beta) \longrightarrow coker(\alpha),$$

$$\begin{array}{llll} \alpha := & \sum_{i=1}^{s} \alpha_i : & \bigoplus_{i=1}^{s} H^{2n-n_i}X & \longrightarrow & H^{2n}X, \\ \beta := & (\beta_i)_{i=1..s} : & H^nX & \longrightarrow & \bigoplus_{i=1}^{s} H^{n+m_i}X. \end{array}$$

Then ϕ is unstable, and satisfies for all x, y in the range of definition $ker(\beta)$

$$\phi(x+y) = \phi(x) + \phi(y) + xy.$$

This equation is valid modulo the indeterminacy, i.e. in $coker(\alpha)$.

2.3.2 *Remarks:*
(i) A decomposition of Sq^{n+1} in A^* exists iff $n+1$ is not a power of 2. This follows from the Adem relations, see [46] and 8.3.3:

$$Sq^aSq^b = \sum_{j=0}^{[\frac{a}{2}]} \tbinom{b-1-j}{a-2j} Sq^{a+b-j}Sq^j.$$

(ii) As usual, the operations ϕ are constructed as cohomology classes of a generalized 2-stage Postnikov system, see [46]:

Let $K := \prod_{i=1}^{s} K_{n+m_i}$ and $\Omega K \overset{j}{\hookrightarrow} E \overset{\pi}{\to} K_n$ be the pullback of the path fibration $\Omega K \hookrightarrow PK \to K$ by a map $\beta : K_n \to K$ realizing the β_i. Because $\tau(\alpha\iota_{\Omega K}) = \alpha\beta\iota_n = Sq^{n+1}\iota_n = 0$, with τ the transgression in the Serre spectral sequence for $\Omega K \to E \to K_n$, there exists a $\phi \in H^{2n}E$ with $\alpha\iota_{\Omega K} = j^*\phi$. These operations ϕ are unstable, because $\alpha\beta\iota_k = 0$ (a 'relation') holds only in dimensions $k \leq n$.

(iii) Alternatively, the operations ϕ are also given by a glueing construction using a zero-homotopy for the composite map $\alpha\beta$, or by Kristensen's method of cochain operations. We will come in section 5 and section 6 to these other points of view.

(iv) The construction shows that two operations ϕ, ϕ' differ by an element $\pi^*\epsilon \in H^{2n}E$, where $\epsilon \in H^{2n}K_n$ can take an arbitrary value. As we are in characteristic 2, this group is in the stable range as well: By a theorem of Serre one has

$$H^*K_n = \mathbb{Z}/2[Sq^I\iota_n | I \text{ admissible}, exc(I) < n],$$

showing that the map $A^k \to H^{n+k}K_n$, $\gamma \mapsto \gamma\iota_n$ is an isomorphism for $k < n$ (see [46]). But in fact, a little more is true in our case of $\mathbb{Z}/2$-coefficients: $H^{2n}K_n$ has a $\mathbb{Z}/2$-basis consisting of the $Sq^I\iota_n$ with I admissible, $|I| = n$ and $exc(I) < n$, and also ι_n^2. But $\iota_n^2 = Sq^n\iota_n$, and because of $exc(I) \leq |I|$ (with equality iff $I = (n)$), we have also an isomorphism $A^n \cong H^{2n}K_n$ given by $Sq^I \mapsto Sq^I\iota_n$. We remark that the corresponding statement is not true in characteristic > 2. Thus two operations ϕ, ϕ' differ by a stable primary cohomology operation $\epsilon \in A^n$.

(v) Theorem 2.3.1 is a special case of a Theorem of Kristensen [33] which computes the deviation from linearity of more general secondary cohomology operations.

2.3.3 Now we consider for $m \geq 1$ the decomposition

$$Sq^{4m+2} = Sq^2Sq^{4m} + Sq^1Sq^{4m}Sq^1$$

which follows from the Adem relations $Sq^1Sq^{2k} = Sq^{2k+1}$ and $Sq^2Sq^{4m} = Sq^{4m+2} + Sq^{4m+1}Sq^1$. For $m = 0$, we get no decomposition, but the relation $Sq^1Sq^1 = 0$. By Theorem 2.3.1 of Brown and Peterson, we have for $m \geq 1$ associated secondary cohomology operations $\phi : ker(\beta) \to coker(\alpha)$ with range of definition and indeterminacy given by

$$\begin{array}{rrlcl} \alpha := & Sq^2 + Sq^1 : & H^{8m}X \oplus H^{8m+1}X & \longrightarrow & H^{8m+2}X, \\ \beta := & (Sq^{4m}, Sq^{4m}Sq^1) : & H^{4m+1}X & \longrightarrow & H^{8m+1}X \oplus H^{8m+2}X. \end{array}$$

We call them *Brown-Peterson secondary cohomology operations*. Two of them differ by a primary operation $\epsilon : H^{4m+1}X \longrightarrow H^{8m+2}X$ which is given by an arbitrary element $\epsilon \in A^{4m+1}$ in the Steenrod algebra.

If X is a *Spin*-manifold M of dimension $2n = 8m+2$, we have $coker(\alpha) = H^{2n}M$ because of $Sq^1H^{2n-1}M = v_1H^{2n-1}M = 0$ and $Sq^2H^{2n-2}M = v_2H^{2n-2}M = 0$. Furthermore, $ker(\beta) = ker(Sq^{4m} : H^nM \to H^{2n-1}M)$ because the map $Sq^{4m}Sq^1 : H^nM \to H^{2n}$ is zero: This

follows from $Sq^{4m}Sq^1 = Sq^2Sq^{4m-1} + Sq^1Sq^{4m}$ together with $Sq^2Sq^{4m-1}x = v_2Sq^{4m-1}x = 0$ and $Sq^1Sq^{4m}x = v_1Sq^{4m}x = 0$. We have shown:

2.3.4 Proposition: (Brown [15]) *Let M be a connected Spin manifold of dimension $2n = 8m+2$ with $H^1M = 0$ and ϕ be a Brown-Peterson secondary cohomology operation. Then ϕ gives a quadratic refinement $\phi : H^nM \to H^{2n}M$ of the intersection pairing in $\mathbb{Z}/2$-cohomology.*

2.3.5 *Remarks:*
(i) In [15] and [21] the authors consider the slightly different decomposition

$$Sq^{4m+2} = Sq^2Sq^{4m} + Sq^1Sq^2Sq^{4m-1}$$

which by $Sq^{4m}Sq^1 = Sq^2Sq^{4m-1} + Sq^1Sq^{4m}$ and $Sq^1Sq^1 = 0$ is equivalent to our decomposition. We prefer to use our decomposition because the computations in the proof of the Main Theorem 7.1.1 with Kristensen's theory have then fewer terms. Also, our decomposition seems to be the more natural candidate because it is obtained by applying the Adem relations only two times. Only the proof of $ker(\beta) = H^nM$ requires the above relation for $Sq^{4m}Sq^1$.

(ii) In fact, the two sets of secondary operations which belong to these two decompositions are the same; this follows from a general Theorem of Kristensen in [33], p.77.

2.3.6 Now one can define an Brown-Kervaire invariant of *Spin*-bordism in dimension $8m+2$ with $m \geq 1$ as follows: Take any Brown-Peterson secondary cohomology operation ϕ. For a bordism class in Ω^{Spin}_{8m+2}, take a 1-connected representing manifold M, which is possible by surgery as $BSpin$ is 3-connected. Define

$$K_\phi(M) \in \mathbb{Z}/2$$

to be the Arf invariant of the quadratic refinement $H^nM \to \mathbb{Z}/2$, $x \mapsto \phi(x)[M]$. As in the proof for Brown-Kervaire invariants, this gives a bordism invariant because a zero-bordism (which by surgery can also be assumed to be 1-connected) produces a Lagrangian of ϕ in H^nM. Alternatively, we can construct a secondary operation for spectra (analogously to 2.3.2(ii)) associated to a (fixed) Brown-Peterson secondary cohomology operation ϕ, using that the composition

$$MSpin \wedge K_{4m+1} \xrightarrow{1\wedge\beta} MSpin \wedge (K_{8m+1} \times K_{8m+2}) \xrightarrow{1\wedge\alpha} MSpin \wedge K_{8m+3}$$

is zero in $[MSpin \wedge K_{4m+1}, MSpin \wedge K_{8m+3}]$. Applied to the sphere spectrum $\Sigma^{2n}S^0$, this secondary operation gives a homomorphism

$$\pi_{2n}(MSpin \wedge K_{4m+1}) = [\Sigma^{2n}S^0, MSpin \wedge K_{4m+1}] \longrightarrow \mathbb{Z}/2 = [\Sigma^{2n}S^0, \Sigma^{-1}(MSpin \wedge K_{8m+3})],$$

which is just an element $h \in Q^{Spin}_{8m+2}$, producing a quadratic form $q_h : H^{4m+1}M \to \mathbb{Z}/2$ for each closed $(8m+2)$-dimensional *Spin*-manifold M. This also shows that K_ϕ is actually a Brown-Kervaire invariant with parameter h. We call the invariants K_ϕ constructed by this method *Brown-Peterson-Kervaire invariants.*

2.3.7 We used the fact, that each secondary cohomology operation for spaces associated to a zero-homotopic map $K \xrightarrow{\beta} K' \xrightarrow{\alpha} K''$ between Eilenberg-MacLane spaces has also a stable version on the level of spectra, which is associated to $E \wedge K \xrightarrow{1\wedge\beta} E \wedge K' \xrightarrow{1\wedge\alpha} E \wedge K''$, where E is a CW-spectrum. This can be seen immediately with the glueing construction for secondary cohomology operations (see 5.1.3), where the operation is constructed by choosing a zero-homotopy $H : CK \to K''$ for the composite map $\alpha\beta : K \to K''$. (Here, C denotes the cone. In section 5, we work with $H : K \to PK''$, where P denotes the path space; this is equivalent as C, P are adjoined functors.) Now, H defines a *function* $1 \wedge H : E \wedge CK \to E \wedge K''$ of CW-spectra, which is a stronger notion than a map (see [64]). Thus, for two zero-homotopies H_1, H_2 of maps, the glueing construction 5.1.3 is not only defined on the level of spaces, but also for spectra: $1 \wedge H_1$ and $1 \wedge H_2$ are functions which coincide on the bottom of the cone, and hence can be glued to give a function $1 \wedge (H_1 - H_2) : E \wedge SK \to E \wedge K''$.

2.4 Generalized Wu Classes

2.4.1 We have seen in 1.2.4 that the difference of two Brown-Kervaire invariants K_h and $K_{h'}$ is measured by the invariant $q_{x,h}$, where $x \in H^nB$ gives the difference of h and h'. We consider here the same problem for Brown-Peterson-Kervaire invariants of *Spin* manifolds in dimension $2n = 8m + 2$. As two Brown-Peterson secondary cohomology operations ϕ, ϕ' differ by $\epsilon \in A^{4m+1}$, the addition formula 1.2.4 shows that the Arf invariants $K_\phi(M)$ and $K_{\phi'}(M)$ differ by

$$K_\phi(M) - K_{\phi'}(M) = \phi(v_\epsilon)[M].$$

Here, $v_\epsilon \in H^nM$ denotes the *generalized Wu class* of $\epsilon \in A^n$, which corresponds by Poincaré duality to the homomorphism $H^nM \to \mathbb{Z}/2$, $x \mapsto \epsilon(x)[M]$. Thus v_ϵ is defined by the property $\epsilon(x)[M] = v_\epsilon x[M]$ for all $x \in H^nM$.

2.4.2 The definition of generalized Wu classes works for arbitrary closed manifolds. Given a manifold M of dimension d, one obtains for each $k \in \mathbb{N}$ a $\mathbb{Z}/2$-linear map $A^k \to H^kM$, $\alpha \mapsto v_\alpha$, with the defining property $\alpha(x) = v_\alpha x$ for all $x \in H^{d-k}M$. (Here we can drop the evaluation on the fundamental class $[M]$ since this is in the top dimension an isomorphism to $\mathbb{Z}/2$ for connected manifolds; for non-connected manifolds the same argument applies to each component.) For example, $Sq^k \in A^k$ gives the usual Wu class v_k, with

$$v := v_{Sq} = 1 + v_1 + v_2 + ... \in H^*M$$

the *total Wu class* of M. By the Wu formula we have $Sq(v) = w(T_M) = w(\nu_M)^{-1}$, hence the v_k can be expressed as polynomials in Stiefel-Whitney classes and are thus characteristic classes, $v_k \in H^kBO$. In particular,

$$v = v_{Sq} = Sq^{-1}(w^{-1}) \in H^\bullet BO$$

gives the expression of the *universal total Wu class* v as a normal characteristic class by the Stiefel-Whitney classes.

2.4.3 We prove the analogous statement for the generalized Wu classes, $v_\alpha \in H^*BO$, and give a recursive formula for the corresponding universal map $A^* \to H^*BO$. We clearly have

for $x \in H^*M$ and $\alpha, \beta \in A^*$

$$v_{\alpha+\beta} = v_\alpha + v_\beta \quad \text{and} \quad v_\alpha \beta(x)[M] = \alpha(\beta(x))[M] = v_{(\alpha\beta)} x[M],$$

but there seems to be no general expression for $v_{(\alpha\beta)}$ in terms of v_α and v_β. For $x, y \in H^*M$ and $\alpha \in A^*$ we have furthermore

$$xSq(y)[M] = Sq(Sq^{-1}(x))Sq(y)[M] = Sq(Sq^{-1}(x)y)[M] = vSq^{-1}(x)y[M],$$

and in particular $v_{(\alpha Sq)} y[M] = v_\alpha Sq(y)[M] = Sq^{-1}(v_\alpha) v y[M]$, showing that $v_{(\alpha Sq)} = Sq^{-1}(v_\alpha) v$. Since $Sq^{-1} = \chi(Sq)$, we have proved:

2.4.4 Lemma: *The generalized Wu classes can be recursively computed (after the lenght of the monomial α) by*

$$v_{\alpha Sq^n} = \sum_{i=0}^{n} v_i \chi(Sq^{n-i})(v_\alpha),$$

which shows also that they are universally defined as characteristic classes $v_\alpha \in H^*BO$.

2.4.5 For $I = (i_1, ..., i_r)$, we define $Sq^I := Sq^{i_1} ... Sq^{i_r}$ and $v_I := v_{Sq^I} \in H^*BO$. Because the map $A^* \to H^*BO$ is linear, we have only to consider admissible monomials I since they give a basis of A^*. Furthermore, as $v_I x[M] = v_{i_1} Sq^{(i_2,...,i_r)}(x)[M]$, and $v_{i_1} = 0$ for $i_1 \not\equiv 0 \bmod 4$ for *Spin*-manifolds, we have only to consider admissible monomials with the first number $i_1 \equiv 0 \bmod 4$. This gives by a lengthy computation with 2.4.4 all generalized Wu classes up to dimension 17 in the case of *Spin*-manifolds (in terms of normal Stiefel-Whitney classes):

dim:	v_I :		polynomial in w_i :	dim:	v_I :		polynomial in w_i :
4 :	v_4	=	w_4	14 :	$v_{12,2}$	=	$w_{10}w_4 + w_8w_6$
5 :	$v_{4,1}$	=	0				$+w_7^2 + w_4^2 w_6$
6 :	$v_{4,2}$	=	w_6		$v_{8,4,2}$	=	w_{14}
7 :	$v_{4,2,1}$	=	w_7	15 :	$v_{12,3}$	=	0
8 :	v_8	=	w_8		$v_{12,2,1}$	=	$w_{11}w_4 + w_8w_7 + w_4^2 w_7$
9 :	$v_{8,1}$	=	0		$v_{8,4,2,1}$	=	w_{15}
10 :	$v_{8,2}$	=	w_{10}	16 :	v_{16}	=	$w_{16} + w_{12}w_4$
11 :	$v_{8,3}$	=	0				$+w_{10}w_6 + w_8 w_4^2$
	$v_{8,2,1}$	=	w_{11}		$v_{12,4}$	=	$w_{12}w_4 + w_{10}w_6 + w_8w_4^2$
12 :	v_{12}	=	$w_8w_4 + w_6^2 + w_4^3$		$v_{12,3,1}$	=	0
	$v_{8,4}$	=	w_{12}	17 :	$v_{16,1}$	=	0
	$v_{8,3,1}$	=	0		$v_{12,5}$	=	0
13 :	$v_{12,1}$	=	0		$v_{12,4,1}$	=	$w_{13}w_4 + w_{11}w_6 + w_{10}w_7$
	$v_{8,4,1}$	=	$w_{13} + w_7w_6$				

From this computation, we get the following result:

2.4.6 Proposition: *In dimension 34, all characteristic numbers $v_\alpha v_\beta[\,]$ for Ω_{34}^{Spin} vanish, where $v_\alpha, v_\beta \in H^{17}BSpin$ are the generalized Wu classes associated to $\alpha, \beta \in A^{17}$. There are at most 2 different Brown-Peterson-Kervaire invariants $K_\phi : \Omega_{34}^{Spin} \to \mathbb{Z}/2$.*

Proof: According to the above computation, the subspace in $H^{17}BSpin$ spanned by all v_α is 1-dimensional with generator $v_{12,4,1}$, thus the only non-trivial class of the form $v_\alpha v_\beta$ is given by $v_{12,4,1}^2$. But the associated characteristic number vanishes, because this is a square of an odd-dimensional class:

$$x^2[\,] = (Sq^{17}(x))[\,] = (Sq^1 Sq^{16}(x))[\,] = (v_1 Sq^{16}(x))[\,] = 0$$

as $v_1 = 0$ in H^*BSpin. By the addition formula $q_{x,y+h} = q_{x,h} + xy[\ ]$ (1.2.4), the *Spin*-bordism invariant $q_{v_\alpha,\phi}(M)$ $(:= \langle \phi(\nu_M^* v_\alpha), [M]\rangle)$ is independent on the choice of the Brown-Peterson secondary cohomology operation ϕ. Moreover, $q_{v_{12,4,1};\phi}(\ )$ is the only candidate for a non-trivial invariant. By $K_{x+h} - K_h = q_{x,h}$ (1.2.4), all Brown-Peterson-Kervaire invariants on Ω_{34}^{Spin} are given by the two invariants K_ϕ and $K_{v_{12,4,1}+\phi}$ (with ϕ any Brown-Peterson operation). ∎

2.4.7 *Remark:* It seems to be difficult to compute the invariant $q_{v_{12,4,1};\phi} : \Omega_{34}^{Spin} \to \mathbb{Z}/2$ measuring the difference between K_ϕ and $K_{v_{12,4,1}+\phi}$. It is not possible to compute this invariant universally by applying the secondary operation ϕ directly to $v_{12,4,1} \in H^*BSpin$, because $v_{12,4,1}$ does not lie in the range of definition $ker(Sq^{16}, Sq^{16}Sq^1) \subset H^{17}BSpin$ of ϕ.

3 The Ochanine k-Invariant

We introduce here the Ochanine k-invariant, which is a homomorphism $k : \Omega^{Spin}_{8m+2} \to \mathbb{Z}/2$ and can be considered as a signature defect of $(8m+2)$-dimensional closed *Spin*-manifolds. In this definition, Ochanine's signature theorem for $(8m+4)$-dimensional closed *Spin*-manifolds plays a crucial role.

Then we recall that k is also given by the Ochanine elliptic genus $\beta : \Omega^{Spin}_* \to KO_*[[q]]$, which is an important KO-theoretic invariant, refining for *Spin*-manifolds the rational universal elliptic genus $\phi : \Omega^{SO}_* \to \mathbb{Q}[\delta, \epsilon] \hookrightarrow \mathbb{Q}[[q]]$. Hence k is a KO-characteristic number.

By the real family index theorem of Atiyah and Singer, k can also be computed as the mod 2 index of a twisted Dirac operator. As a new result, we give an explicit formula for the virtual bundles needed for the twistings.

3.1 The Ochanine Signature Theorem and k-Invariant

First we recall the Theorem of Ochanine on the signature of $(8m+4)$-dimensional closed *Spin*-manifolds:

3.1.1 Theorem (Ochanine [48]): *The signature of an* $(8m+4)$*-dimensional closed Spin-manifold is always divisible by* 16.

3.1.2 *Remarks:*
(i) The products $K^4 \times (\mathbb{H}P^2)^m$, where K^4 is the *Kummer surface* (a 4-dimensional closed *Spin*-manifold with $sign(K^4) = -16$), and where $\mathbb{H}P^2$ is the *quaternion projective plane* (an 8-dimensional closed *Spin*-manifold with $sign(\mathbb{H}P^2) = 1$), show that 3.1.1 gives the best possible result. Of course, in dimensions $8m$ the signature can take any value since $sign((\mathbb{H}P^2)^m) = 1$.

(ii) A $(4k)$-dimensional closed oriented manifold M with $v_{2k} = 0$ has even $\mathbb{Z}$-valued intersection form and hence $sign(M) \equiv 0 \bmod 8$. By 2.2.1, this is the case for $(8m+4)$-dimensional closed *Spin*-manifolds. Theorem 3.1.1 strengthens this result.

(iii) The original proof [48] of 3.1.1 worked with SU-bordism. Another proof using the Ochanine elliptic genus β was given by Landweber, see [39]. Here, the strengthening from 8 to 16 comes in by the Pontrjagin character (2.1.4), which maps $KO_{8m+4} \cong \mathbb{Z}$ injectively onto $2\mathbb{Z} \subset \mathbb{Q}$.

3.1.3 In [48], Ochanine defined a homomorphism

$$k : \Omega^{Spin}_{8m+2} \longrightarrow \mathbb{Z}/2$$

in the following way: Let $\overline{S^1}$ be the circle with the non-trivial *Spin*-structure, which gives the generator α of $\Omega^{Spin}_1 = \mathbb{Z}/2$. For an $(8m+2)$-dimensional closed *Spin*-manifold M^{8m+2}, the

product $M^{8m+2} \times \overline{S^1}$ has vanishing KO-Pontrjagin numbers (by dimension) and vanishing Stiefel-Whitney numbers (because of $\overline{S^1}$). Thus by Theorem 2.1.9 of Anderson, Brown and Peterson, there exists an $(8m+4)$-dimensional *Spin*-manifold W^{8m+4} with boundary

$$\partial W^{8m+4} = M^{8m+2} \times \overline{S^1}.$$

Furthermore, there is a 2-dimensional *Spin*-manifold P^2 with boundary $\partial P^2 = \overline{S^1} \sqcup \overline{S^1}$, hence one can construct the $(8m+4)$-dimensional closed *Spin*-manifold

$$X^{8m+4} := (W^{8m+4} \sqcup W^{8m+4}) \cup -(M^{8m+2} \times P^2),$$

because the two *Spin*-manifolds on the right have the same boundary. Now one gets by the Novikov additivity theorem for the signature [47] and Ochanine's theorem 3.1.1 that

$$2sign(W^{8m+4}) = sign(X^{8m+4}) \equiv 0 \bmod 16,$$

which shows that $sign(W^{8m+4}) \equiv 0 \bmod 8$. Then Ochanine defined

$$k(M^{8m+2}) := \frac{sign(W^{8m+4})}{8} \bmod 2 \quad \in \mathbb{Z}/2,$$

which does not depend on the choice of W^{8m+4}, and moreover depends only on the *Spin*-bordism class of M^{8m+2}: Firstly, one has for another W' by 3.1.1 $sign(W \cup_M -W') \equiv 0 \bmod 16$, hence $sign(W) \equiv sign(W') \bmod 16$. Secondly, for $M^{8m+2} = \partial V^{8m+3}$ one can choose $W^{8m+4} := V^{8m+3} \times \overline{S^1}$ which has $sign(W^{8m+4}) = 0$, hence $k(M^{8m+2}) = 0$. We call this invariant k of *Spin*-bordism the *Ochanine k-invariant.*

Now we come to the multiplicative properties of the Ochanine k-invariant. Ochanine shows in [48] that $k(\overline{S^1} \times \overline{S^1}) = 1$, in particular there exists a 4-dimensional *Spin*-manifold W^4 with $\partial W^4 = (\overline{S^1})^3$ and $\frac{1}{8}sign(W^4) \equiv 1 \bmod 2$ (we remark, that there exists an explicit example for W^4 by a certain elliptic surface). With the definition of k one obtains:

3.1.4 Theorem (Ochanine [48], [49]): *For an 8m-dimensional closed Spin-manifold M^{8m}, one has*

$$k(M^{8m} \times \overline{S^1} \times \overline{S^1}) \equiv sign(M^{8m}) \bmod 8.$$

3.1.5 Moreover, Ochanine defined in [48] a homomorphism $\kappa : \Omega_*^{Spin} \to KO_* \otimes \mathbb{Z}/2$ by

$$\kappa(M^n) := \begin{cases} sign(M^n) \cdot \mu^m & \otimes 1 \quad \text{for} \quad n = 8m \\ k(M^n \times \overline{S^1}) \cdot \eta\mu^m & \otimes 1 \quad \text{for} \quad n = 8m+1 \\ k(M^n) \cdot \eta^2\mu^m & \otimes 1 \quad \text{for} \quad n = 8m+2 \\ \frac{1}{16}sign(M^n) \cdot \omega\mu^m & \otimes 1 \quad \text{for} \quad n = 8m+4 \\ 0 & \qquad \text{otherwise} \end{cases}$$

where the generators $\eta, \omega, \mu \in KO_*$ are described in 3.2.1, and

$$KO_* \otimes \mathbb{Z}/2 = \frac{\mathbb{Z}/2[\eta, \omega]}{\eta^3, \eta\omega, \omega^2} \otimes \mathbb{Z}/2[\mu, \mu^{-1}].$$

The multiplicative properties of k can then be summarized by:

3.1.6 Theorem (Ochanine [48]): *The map κ is a ring-homomorphism.*

3.2 The Ochanine Elliptic Genus β

3.2.1 Ochanine gave in [49] another construction of k in terms of KO-characteristic numbers. We recall the coefficients of KO-theory

$$KO_* = \frac{\mathbb{Z}[\eta, \omega, \mu, \mu^{-1}]}{2\eta,\ \eta^3,\ \eta\omega,\ \omega^2 - 4\mu},$$

n:	$8m$	$8m+1$	$8m+2$	$8m+3$	$8m+4$	$8m+5$	$8m+6$	$8m+7$
KO_n:	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0
generator:	μ^m	$\eta\mu^m$	$\eta^2\mu^m$	0	$\omega\mu^m$	0	0	0

where $\eta \in KO_1$, $\omega \in KO_4$ and $\mu \in KO_8$ are given by the normalized Hopf bundles $\gamma_{\mathbb{R}P^1} - 1$, $\gamma_{\mathbb{H}P^1} - 4$, and $\gamma_{\mathbb{C}ayP^1} - 8$ (viewed as real vector bundles) over the real, quaternion, and Cayley projective lines $\mathbb{R}P^1 = S^1$, $\mathbb{H}P^1 = S^4$, and $\mathbb{C}ayP^1 = S^8$.

3.2.2 Now we define according to [48] for a real vector bundle $E \to X$

$$\Theta_q(E) := \bigotimes_{n\geq 1}(\Lambda_{-q^{2n-1}}(E) \otimes S_{q^{2n}}(E)) \qquad \in KO^0(X)[[q]]$$

where $\Lambda_u(E) = \sum_{k\geq 0} u^k \Lambda^k(E)$ and $S_u(E) = \sum_{k\geq 0} u^k S^k(E)$ are the total exterior respectively symmetrical powers of E. For the trivial line bundle we have in particular

$$\theta(q) := \Theta_q(1) = \prod_{n\geq 1} \frac{(1-q^{2n-1})}{(1-q^{2n})} = 1 - q + q^2 - 2q^3 + 3q^4 - 4q^5 + 5q^6 - 7q^7 + 10q^8 \pm \ldots$$

which we also view as an element in $\mathbb{Z}[[q]]$. We remark, that this power series can also be represented as a generating function of certain partitions, which follows from a lemma in the theory of elliptic modular functions, see [26], chapter 8: There are four basic power series

$$\begin{array}{lcl} Q_0 & := & \prod_{n=1}^{\infty}(1-q^{2n}), \\ Q_1 & := & \prod_{n=1}^{\infty}(1+q^{2n}), \\ Q_2 & := & \prod_{n=1}^{\infty}(1+q^{2n-1}), \\ Q_3 & := & \prod_{n=1}^{\infty}(1-q^{2n-1}), \end{array}$$

and we have the strange identity

$$Q_1 Q_2 Q_3 = 1,$$

which is proved by multiplication with Q_0. This gives $\theta(q) = \frac{Q_3}{Q_0} = \frac{1}{Q_0 Q_1 Q_2}$, so

$$\theta(-q) = \prod_{n=1}^{\infty}(1-q^{4n})^{-1}(1-q^{2n-1})^{-1}$$

is the generating function for the number $a(k)$ of partitions of $k \in \mathbb{N}$ into natural numbers, which are divisible by four or odd; and $\theta(q) = \sum_{k=0}^{\infty}(-1)^k a(k) q^k$.

Because $\Lambda_u(E)$ and $S_u(E)$ are multiplicative in E, we also have $\Theta_q(E \oplus F) = \Theta_q(E)\Theta_q(F)$ and can thus extend Θ_q to

$$\Theta_q : KO^0(X) \to KO^0(X)[[q]].$$

For an n-dimensional closed *Spin*-manifold M^n, Ochanine defined

$$\beta(M^n) \quad := \quad \langle \Theta_q(TM^n - n), [M^n]_{KO} \rangle \quad = \quad \theta(q)^{-n} \langle \Theta_q(TM^n), [M^n]_{KO} \rangle \quad \in KO_n[[q]]$$

where $[M^n]_{KO} \in KO_n(M^n)$ denotes the Atiyah-Bott-Shapiro orientation of M^n and $\langle \, , \, \rangle : KO^m(X) \otimes KO_n(X) \to KO_{n-m}$ is the Kronecker pairing. Since β is multiplicative, this gives a genus for *Spin*-bordism, the *Ochanine elliptic genus*

$$\beta : \Omega_n^{Spin} \to KO_n[[q]].$$

If we write $\beta(M) = \sum_{k=0}^{\infty} \beta_k(M) q^k$, then we have $\beta_k(M) = \langle b_k(M), [M]_{KO} \rangle \in KO_n$, where b_k is given by the KO-Pontrjagin classes π_i of 2.1.6 as follows (see [49]):

$$\begin{aligned}
b_0 &= 1 \\
b_1 &= -\pi_1 \\
b_2 &= \pi_2 - \pi_1 \\
b_3 &= -\pi_3 + 4\pi_2 - \pi_1^2 - 4\pi_1 \\
b_k &= (-1)^k \pi_k + \text{ polynomial in } \pi_1, ..., \pi_{k-1}
\end{aligned}$$

The coefficient $\langle 1, [M^n]_{KO} \rangle \in KO_n$ of q^0 is called the *Atiyah α-invariant*, which is a genus

$$\alpha : \Omega_n^{Spin} \to KO_n.$$

3.2.3 *Remark:* Ochanine proves in [49] that the Pontrjagin character of β gives the q-expansion at the cusp ∞ (in the notation of [25], the cusp 0) of the universal elliptic genus (see [39])

$$\phi : \Omega_*^{SO} \to \mathbb{Z}[\frac{1}{2}][\delta, \epsilon],$$

where $\delta = \phi(\mathbb{C}P^2)$, $\epsilon = \phi(\mathbb{H}P^2)$. Thus β extends ϕ for *Spin*-manifolds to the dimensions $8m+1$ and $8m+2$. See [39] and [25] for the definition and properties of ϕ.

Furthermore, β takes values in the ring of modular forms over KO_*, more precisely:

3.2.4 Theorem (Ochanine [49]): *The image of β is the subring generated by*

$$\eta, \omega\delta_0, \mu\delta_0^2 \text{ and } \mu\epsilon$$

of the ring $M^{\Gamma_0(2)}(KO_*) = \frac{KO_*[\delta_0, \epsilon]}{\eta(\delta_0 - 1)} \subset KO_*[[q]]$ of modular forms over KO_*, where $\delta_0 \in \mathbb{Z}[[q]]$ and $\epsilon \in \mathbb{Z}[[q]]$ are given by the formal power series

$$\delta_0 = -8\delta = 1 + 24 \sum_{n \geq 1} (\sum_{d|n, d \text{ odd}} d) q^n, \qquad \epsilon = \sum_{n \geq 1} (\sum_{d|n, n/d \text{ odd}} d^3) q^n.$$

In particular, we have for $n = 8m + r + 4s$ with $r = 0, 1, 2$ and $s = 0, 1$ that

$$\beta(M^n) = \Big(a_0(M^n)\delta_0^{2m+s} + a_1(M^n)\delta_0^{2m+s-2}\epsilon + \ldots + a_{m-1}(M^n)\delta_0^{2+s}\epsilon^{m-1} + a_m(M^n)\delta_0^s\epsilon^m \Big) \eta^r \omega^s \mu^m,$$

with uniquely defined homomorphisms $a_i \cdot \eta^r\omega^s\mu^m : \Omega_n^{Spin} \to KO_n$*, i.e.* $a_i(M) \in \mathbb{Z}$ *for* $n \equiv 0 \bmod 4$ *and* $a_i(M) \in \mathbb{Z}/2$ *for* $n \equiv 1, 2 \bmod 8$.

(We remark, that a_i in [49] denotes our $a_i \cdot \eta^r\omega^s\mu^m$.)

3.2.5 In dimensions $8m+1$ and $8m+2$, δ_0 and ϵ can be replaced by their reductions mod 2. These are given by

$$\begin{array}{rcl} \delta_0 & \equiv & 1 \mod 2 \\ \epsilon & \equiv & \bar{\epsilon} \mod 2, \end{array}$$

where (see [49])

$$\bar{\epsilon} := \sum_{k \geq 0} q^{(2k+1)^2} = q + q^9 + q^{25} + q^{49} + \ldots \qquad \in \mathbb{Z}/2[[q]].$$

3.2.6 The lowest coefficient a_0 in the expansion of 3.2.4 is given again by the Atiyah α-invariant, because the power series of ϵ and $\bar{\epsilon}$ have vanishing constant term. The highest coefficient $a_m(M^n)$, with $m = [\frac{n}{8}]$ the integer part of $\frac{n}{8}$, was in [49] determined as:

3.2.7 Theorem (Ochanine [49]): *For any closed Spin-manifold* M^n *of dimension* $n = 8m + r + 4s$ *with* $r = 0, 1, 2;\ s = 0, 1$*, the highest coefficient* $a_m(M^n)$ *in 3.2.4 is given by*

$$a_m(M^n) \cdot \eta^r\omega^s\mu^m \otimes 1 = \kappa(M^n),$$

where this equation holds in $KO_n \otimes \mathbb{Z}/2$.

In particular, $a_m(M^{8m+2})$ is just the Ochanine k-invariant,

$$a_m(M^{8m+2}) = k(M^{8m+2}).$$

3.2.8 *Remarks:*
(i) A geometric interpretation of the other coefficients a_i, $1 \leq i \leq m-1$, seems not to be known.

(ii) From the proof of 3.2.7 in [49], it follows that the signature is obtained from β by replacing δ_0 by -8, ϵ by 1, ω by 2, and μ by 1. Thus:

$$\begin{array}{rcl} sign(M^{8m}) & = & \Big(8^{2m}a_0(M^{8m}) - 8^{2m-2}a_1(M^{8m}) \pm \ldots + a_m\Big), \\ sign(M^{8m+4}) & = & -16\Big(8^{2m}a_0(M^{8m}) - 8^{2m-2}a_1(M^{8m}) \pm \ldots + a_m\Big), \end{array}$$

which indicates also Ochanine's Signature Theorem 3.1.1. In fact, the proof given by Landweber [39] works in this way.

3.2.9 At last, we come to the multiplicative properties of β and k in certain fibre bundles. The Bott-Taubes Rigidity Theorem [12] implies the multiplicativity (see [57]) of the universal elliptic genus $\phi : \Omega_*^{SO} \to \mathbb{Z}[\frac{1}{2}][\delta, \epsilon]$ for fibre bundles $F \to E \to B$ of closed *Spin*-manifolds with a compact connected Lie group as structure group, i.e.

$$\phi(E) = \phi(F)\phi(B).$$

Using this result, Kreck and Stolz proved:

3.2.10 Theorem (Kreck, Stolz [32]): *Let G be a compact connected Lie group acting on a closed Spin-manifold F of dimension k preserving the Spin-structure. Assume that $k = 0, 3 \bmod 4$ or $G = S^1$. Then for any fibre bundle $p : E \to B$ over a closed Spin-manifold B with fibre F and structure group G, we have*

$$\beta(E) = \beta(F)\beta(B).$$

This gives the following corollary, which we need in the next section:

3.2.11 Corollary: *Let G be a compact connected Lie group acting on a closed Spin-manifold F of dimension $0 \bmod 8$ preserving the Spin-structure. Then for any fibre bundle $p : E \to B$ over a closed Spin-manifold B of dimension $2 \bmod 8$ with fibre F and structure group G, we have*

$$k(E) = sign(F)k(B).$$

Proof: Using 3.2.4, we get from 3.2.10 that $a_{m+l}(E) = a_l(F)a_m(B)$, where $l := \frac{1}{8}dim(F)$ and $m := [\frac{1}{8}dim(B)]$ (hence $m + l = [\frac{1}{8}dim(E)]$). But then the statement follows from 3.2.7. ■

3.3 Analytic Interpretation

3.3.1 As Ochanine remarked in [49], with 3.2.7 it is possible to give an expression of k in terms of KO-characteristic numbers: Let $q(\epsilon) \in \mathbb{Z}[[\epsilon]]$ be any formal power series whose $\mathbb{Z}/2$-reduction is the inverse power series of $\bar{\epsilon} \in \mathbb{Z}/2[[q]]$, then a_m is the coefficient of $\bar{\epsilon}^m$ in the polynomial $\beta_{q(\epsilon)} \in KO_{8m+2}[\bar{\epsilon}]$ which is obtained from the formal power series $\beta \in KO_{8m+2}[[q]]$ by inserting $q(\epsilon)$ for q.

Together with the Real Family Index Theorem in dimension $8m + 2$, this shows that k has an analytical interpretation as the mod 2 index of a twisted Dirac operator.

We remark, that this is not the only way to express k by KO-Pontrjagin numbers, since Ochanine obtained such expressions also in [48] by considering certain multiplicative series for KO-numbers, in analogy to the Hirzebruch formalism for genera of oriented bordism (see [44]). But these series are more complicated and not uniquely given (in contrast to the L-series of Hirzebruch giving the signature). In [55], Rubinsztein used this representation of k together with the Real Family Index Theorem, to represent k as a mod 2 index of a twisted Dirac operator; but his formulas are not very explicit.

3.3.2 Actually, one can do the transformation in 3.3.1 from β to the coefficients a_m very explicitly: We recall that we get the coefficient of x^m in a power series $c(x) = \sum_{k=0}^{\infty} c_m x^m$ by the Residuum Theorem as

$$c_m = \frac{1}{2\pi i} \oint \frac{dx}{x^{m+1}} c(x).$$

Thus we get (suppressing $\eta^2\mu^m$)

$$a_m \equiv \frac{1}{2\pi i}\oint \frac{d\epsilon}{\epsilon^{m+1}}\beta_{q(\epsilon)} \equiv \frac{1}{2\pi i}\oint \frac{dq}{q^{m+1}}\left(\frac{d\epsilon}{dq}\right)\frac{q^{m+1}}{\epsilon(q)^{m+1}}\beta_q \mod 2$$

$$\equiv \frac{1}{2\pi i}\oint \frac{dq}{q^{m+1}}(\frac{\epsilon}{q})^{-m}\beta_q \mod 2,$$

because we have $q\frac{d\epsilon}{dq} \equiv \epsilon \bmod 2$ by $\bar{\epsilon} = \sum_{k\geq 0} q^{(2k+1)^2} = q + q^9 + q^{25} + \dots$. Furthermore, $\epsilon/q \equiv f(q^8) \equiv f(q)^8 \bmod 2$, where

$$f(q) := \sum_{n\geq 1} q^{\binom{n}{2}} = 1 + q + q^3 + q^6 + \ldots \in \mathbb{Z}[q]].$$

Thus we have shown:

3.3.3 Proposition: *The Ochanine k-invariant in dimension $8m+2$ is given by the coefficient of q^m in*

$$f(q)^{-8m}\cdot\beta = \langle f(q)^{-8m}\theta(q)^{-8m-2}\Theta_q(TM), [M]_{KO}\rangle \quad \in KO_{8m+2}[[q]].$$

We use this result together with the Real Family Index Theorem to get an analytical interpretation of k as the mod 2 index of an explicitly given twisted Dirac operator. We recall:

3.3.4 Theorem (Atiyah, Singer [7]): *Let M be an $(8m+2)$-dimensional closed Spin-manifold and $E \in KO^0(M)$. Define $e \in \mathbb{Z}/2$ by $\langle E, [M]_{KO}\rangle = e\eta^2\mu^m \in KO_{8m+2}$, then*

$$e \equiv dim_{\mathbb{C}} ker(D_E) \bmod 2,$$

where D_E is the Dirac operator of M twisted by the virtual bundle E.

In particular, with 3.3.3 we have shown:

3.3.5 Corollary: *Let M be an $(8m+2)$-dimensional closed Spin-manifold and $E_m \in KO^0(M)$ be the virtual bundle, which is the coefficient of q^m in*

$$f(q)^{-8m}\theta(q)^{-8m-2}\Theta_q(TM) \in KO^0(M)[[q]].$$

Then the Ochanine k-invariant of M is equal to the mod 2 index of the Dirac operator twisted by the virtual bundle E_m,

$$k(M) \equiv dim_{\mathbb{C}} ker(D_{E_m}) \bmod 2.$$

4 $\mathbb{H}P^2$-Bundles and Integral Elliptic Homology

It seems to be difficult to compare k directly with the Brown-Kervaire invariants K_h for *Spin*-bordism. Whereas k can be given as a very concrete KO-characteristic number, the Brown-Kervaire invariants, constructed with quadratic forms on the middle-dimensional $\mathbb{Z}/2$-cohomology, form an affine space and thus seem to have no distinguished element.

Fortunately, it is possible to characterize k by a multiplicativity property for certain $\mathbb{H}P^2$-bundles. Thus k is a Brown-Kervaire invariant iff we can find one with this multiplicativity property. This result is a corollary of a Theorem of Kreck and Stolz [32] showing that the kernel of the Ochanine elliptic genus β consists of all *Spin*-bordism classes of $\mathbb{H}P^2$-bundles with structure group $PSp(3)$ over zero-bordant *Spin*-manifolds.

We consider now these $\mathbb{H}P^2$-bundles, recall the results of Kreck and Stolz which are relevant to our problem, and at last examine the behaviour of Brown-Peterson secondary cohomology operations in $\mathbb{H}P^2$-bundles. The problem, if there exists a multiplicative Brown-Peterson-Kervaire invariant (which would then be equal to k), will be reduced to a product formula for the secondary operation involved.

4.1 $\mathbb{H}P^2$-Bundles

4.1.1 According to Stolz [60], we consider fibre bundles $p : N^{k+8} \to M^k$ of closed *Spin*-manifolds N^{k+8}, M^k with fibre the *quaternionic projective plane* $\mathbb{H}P^2$ and structure group the *projective symplectic group* $PSp(3)$. We shortly call these bundles $PSp(3)$-$\mathbb{H}P^2$*-bundles.*

We recall, that the quaternionic projective plane can be constructed by several equivalent methods. For example, one can take the Hopf map $\nu : S^7 \to S^4$ and form $\mathbb{H}P^2 := S^4 \cup_\nu D^8$. This gives the integral cohomology ring as $H^*(\mathbb{H}P^2;\mathbb{Z}) = \mathbb{Z}[x]/x^3$ with $|x| = 4$, and hence the total tangential Stiefel-Whitney class as $w(\mathbb{H}P^2) = 1 + x + x^2$ (x denotes also its mod 2 reduction). Alternatively, one can take the quotient of $S^{11} \subset \mathbb{H}^3$ by the (free) action of $S^3 \subset \mathbb{H}$, the quaternions of length 1. With $S^3 = Sp(1)$ and $S^{11} = Sp(3)/Sp(2)$, this shows that we can consider $\mathbb{H}P^2$ also as

$$\mathbb{H}P^2 = Sp(3)/(Sp(2) \times Sp(1)) = PSp(3)/P(Sp(2) \times Sp(1)).$$

Here, $G := PSp(3) = Sp(3)/Z(Sp(3))$ is the projective symplectic group, which is a compact connected Lie group of dimension 21 and $\pi_1(PSp(3)) = \mathbb{Z}/2$ since $Z(Sp(3)) = \{\pm 1\} = \mathbb{Z}/2$; furthermore, the isotropy group $H := P(Sp(2) \times Sp(1)) = (Sp(2) \times Sp(1))/Z(Sp(3))$ is also compact connected, of dimension 13 and $\pi_1(H) = \mathbb{Z}/2$. The group G acts (in contrast to $Sp(3)$) effectively on $\mathbb{H}P^2$. In fact, giving $\mathbb{H}P^2$ the standard Riemannian metric, G is just the group of isometries on $\mathbb{H}P^2$. Finally, we remark that the total tangential Pontrjagin class was in [11] computed as $p(\mathbb{H}P^2) = 1 + 2x + 7x^2$.

Now, bundles as above are the pullback of the *universal* $PSp(3)$-$\mathbb{H}P^2$*-bundle*

$$\mathbb{H}P^2 = G/H \to BH \to BG$$

by a classifying map $f: M^k \to BG$. Stolz, using a result of Kono [30], computed in [60] the cohomology rings in this universal bundle as modules over the Steenrod algebra:

4.1.2 Theorem (Stolz [60]): *There exist elements $t_2, t_3, t_8, t_{12} \in H^*BG$ and $u_2, u_3, u_4, u_8 \in H^*BH$, the index denoting the degree, such that the cohomology rings of these classifying spaces are the polynomial rings*

$$\begin{aligned} H^*BG &= \mathbb{Z}/2[t_2, t_3, t_8, t_{12}] \\ H^*BH &= \mathbb{Z}/2[u_2, u_3, u_4, u_8]. \end{aligned}$$

Furthermore, the induced map of the projection $p = Bi : BH \to BG$ satisfies

$$p^*(t_2) = u_2, \quad p^*(t_3) = u_3, \quad p^*(t_8) = u_4^2 + u_8, \quad p^*(t_{12}) = u_4 u_8,$$

and the operation of the Steenrod algebra is given by

$$\begin{aligned} Sq(t_2) &= t_2 + t_3 + t_2^2 \\ Sq(t_3) &= t_3(1 + t_2 + t_3) \\ Sq(t_8) &= t_8 + t_2^2 t_8 + t_{12} + t_3^2 t_8 + t_2 t_{12} + \\ &\quad + t_3 t_{12} + t_8^2 \\ Sq(t_{12}) &= t_{12}(1 + t_2 + t_3 + t_2^2 + t_2^2 t_3 + \\ &\quad + t_2 t_3^2 + t_8 + t_3^3 + t_2 t_8 + t_3 t_8 + t_{12}) \end{aligned} \qquad \begin{aligned} Sq(u_2) &= u_2 + u_3 + u_2^2 \\ Sq(u_3) &= u_3(1 + u_2 + u_3) \\ Sq(u_4) &= u_4(1 + u_2 + u_3 + u_4) \\ Sq(u_8) &= u_8(1 + u_2^2 + u_4 + u_3^2 + \\ &\quad + u_2 u_4 + u_3 u_4 + + u_8). \end{aligned}$$

The total Stiefel-Whitney class of the tangent bundle along the fibres T^Δ is

$$w(T^\Delta) = 1 + (u_2^2 + u_4) + (u_2 u_4 + u_3^2) + u_3 u_4 + u_8,$$

and the Serre spectral sequence of the universal bundle collapses, showing that H^*BH is a free H^*BG-module on the basis $\{1, w_4(T^\Delta), w_4^2(T^\Delta)\}$. In particular, the Leray-Hirsch Theorem holds for all $\mathbb{H}P^2$-bundles $\mathbb{H}P^2 \to E \to B$ with structure group $PSp(3)$, where we get the basis of H^*E as a H^*B-module by pulling back the universal basis with the map $\tilde{f} : E \to BH$ associated to the classifying map $f : B \to BG$.

Proof: In [60], only the action of Sq^1 and Sq^2 on H^*BG is given, although the proof works also for H^*BH and the other Steenrod squares. Since we need in section 5 the form of $Sq(w_4(T^\Delta))$, we indicate a proof following the argument of Stolz in [60]. We cite the following inclusions of Lie groups

$$(\mathbb{Z}/2)^4 \longrightarrow P(Sp(1)^3) \longrightarrow H \longrightarrow G,$$

which induce monomorphisms for the cohomology rings of classifying spaces

$$H^*B((\mathbb{Z}/2)^4) \longleftarrow H^*BP(Sp(1)^3) \longleftarrow H^*BH \longleftarrow H^*BG.$$

With $H^*B((\mathbb{Z}/2)^4) = \mathbb{Z}/2[x_1, x_1', y_1, y_1']$, Stolz proved that (identifying the polynomial generators above with their monomorphic images):

$$\begin{aligned} t_2 &= x_1^2 + x_1 x_1' + x_1'^2 \\ t_3 &= x_1^2 x_1' + x_1 x_1'^2 \\ t_8 &= s_4^2 + s_4 s_4' + s_4'^2 \\ t_{12} &= s_4^2 s_4' + s_4 s_4'^2 \end{aligned} \qquad \begin{aligned} u_2 &= t_2 \\ u_3 &= t_3 \\ u_4 &= s_4 + s_4' \\ u_8 &= s_4 s_4', \end{aligned}$$

where $s_4, s'_4 \in H^4B((\mathbb{Z}/2)^4)$ are defined by

$$\begin{array}{rcl} s_4 & := & y_1^4 + y_1^2 t_2 + y_1 t_3 \\ s'_4 & := & y_1'^4 + y_1'^2 t_2 + y'_1 t_3. \end{array}$$

A straightforward computation in $H^*B((\mathbb{Z}/2)^4)$ gives then the result. ■

4.1.3 Corollary: *The total Steenrod square applied to the universal Leray-Hirsch generator $x := w_4(T^\Delta) = u_2^2 + u_4 \in H^4BH$ is given by*

$$Sq(x) = x + (u_3^2 + u_2u_4) + u_3u_4 + x^2.$$

In particular, $Sq^1(x) = 0$.

4.2 Integral Elliptic Homology

This theory of Kreck and Stolz [32] is a refinement at the prime two of the elliptic homology theory of Landweber, Ravenel and Stong (see [39]) and has a very geometric definition in terms of $PSp(3)$-$\mathbb{H}P^2$-bundles. Here we need only the coefficients of this theory.

4.2.1 On the level of bordism the construction of $PSp(3)$-$\mathbb{H}P^2$-bundles by classifying maps, respectively the forgetting of the classifying map, can be expressed by homomorphisms (see [60])

$$\begin{array}{rlcll} \Psi: & \Omega_k^{Spin}(BPSp(3)) & \longrightarrow & \Omega_{k+8}^{Spin}, & [M,f] \mapsto [N = f^*E] \\ \pi: & \Omega_k^{Spin}(BPSp(3)) & \longrightarrow & \Omega_k^{Spin}, & [M,f] \mapsto [M], \end{array}$$

and we set

$$\begin{array}{ll} T_* := im\Psi & = \{\text{total spaces of } PSp(3)\text{-}\mathbb{H}P^2\text{-bundles in } \Omega_*^{Spin}\}, \\ \tilde{T}_* := \Psi(ker\pi) & = \{\text{total spaces of } PSp(3)\text{-}\mathbb{H}P^2\text{-bundles with zero-bordant base in } \Omega_*^{Spin}\}. \end{array}$$

4.2.2 *Remarks:*
(i) In the construction of Ψ as a transfer map in the sense of Boardman (see [60]), one uses that the tangent bundle along the fibres T^Δ of the universal $PSp(3)$-$\mathbb{H}P^2$-bundle has a *Spin*-structure. Thus for any classifying map $f: M \to BPSp(3)$, the pullback f^*E is again a *Spin*-manifold.

(ii) These definitions can also be made with an auxiliary space X, i.e. one can also define $T_*(X)$ and $\tilde{T}_*(X)$.

Now, Stolz proves in [60] the following deep result, which was the key step in his proof of the Gromov-Lawson conjecture:

4.2.3 Theorem (Stolz [60]): *The subgroup $T_* \subset \Omega_*^{Spin}$ is equal to the kernel of the Atiyah α-invariant,*

$$ker(\alpha) = T_*.$$

In [32], this was used by Kreck and Stolz for the computation of $ker(\beta)$ and $\Omega_*^{Spin}/ker(\beta)$:

4.2.4 Theorem (Kreck, Stolz [32]): *For the Ochanine elliptic genus $\beta : \Omega_*^{Spin} \to KO_*[[q]]$, we have*

$$\begin{array}{lcl} ker(\beta) & = & \tilde{T}_* \\ \Omega_*^{Spin}/ker(\beta) & = & \dfrac{\mathbb{Z}[s,k,b,h]}{2s,\ s^3,\ sk,\ ,k^2-4(b+64h)}, \end{array}$$

where $s := [\bar{S}^1]$, $k := [K^4]$, $b := [B^8]$ and $h := [\mathbb{H}P^2]$ denote the Spin-bordism classes of the non-trivial circle, the Kummer-surface, the Bott-manifold and the quaternion plane. In fact, $\Omega_*^{Spin}/\tilde{T}_*$ is isomorphic to the subalgebra $S_* \subset \Omega_*^{Spin}$ generated by s, k, b, h, and there is an additive splitting $\Omega_*^{Spin} \cong S_* \oplus \tilde{T}_*$.

4.2.5 Remarks:
(i) With $\Omega_*^{Spin}/\tilde{T}_* \cong im(\beta)$, we get again theorem 3.2.4 of Ochanine.

(ii) According to [32], one defines $ell_*(X)$ for any space X by

$$ell_*(X) := \Omega_*^{Spin}(X)/\tilde{T}_*(X).$$

We cite from [32] that this does not give a homology theory; but one obtains homology theories by the following two constructions (we will not use this result in this thesis):

4.2.6 Theorem (Kreck, Stolz [32]):
(1) $ell_*(X) \otimes \mathbb{Z}_{(2)}$ is a multiplicative homology theory.

(2) For any element $v \in ell_q$ of positive degree, one gets a multiplicative homology theory $El_*^v(X) := ell_*(X)[v^{-1}]$, which agrees with the (Franke-)Landweber-Ravenel-Stong elliptic homology (see [39], [32]) $Ell_*^v(X) := \Omega_*^{SO}(X) \otimes_{\Omega^{SO}} \mathbb{Z}[\frac{1}{2}][\delta,\epsilon][\phi(v)^{-1}]$ after inverting 2.

For $v = h = \mathbb{H}P^2$, one gets

$$El_n^h(X) = \bigoplus_{k\in\mathbb{Z}} \Omega_{n+8k}^{Spin}(X)/\sim,$$

where $\sim$ identifies for every G-$\mathbb{H}P^2$-bundle $p : E \to B$ the class $[f : B \to X] \in \Omega_*^{Spin}(X)$ with $[fp : E \to X] \in \Omega_{*+8}^{Spin}(X)$, i.e. total spaces of G-$\mathbb{H}P^2$-bundles are identified with their base ([32]).

The following corollary of 4.2.4 and 3.2.11 gives a characterization of the Ochanine k-invariant within all homomorphisms $\Omega_{8m+2}^{Spin} \to \mathbb{Z}/2$:

4.2.7 Corollary: *Let $K : \Omega_{8m+2}^{Spin} \to \mathbb{Z}/2$ be a homomorphism with:*

(i) *K vanishes on $\tilde{T}_{8m+2}$.*
(ii) *For all $N \in \Omega_{8m}^{Spin}$, it holds $K(N \times \overline{S^1} \times \overline{S^1}) \equiv sign(N) \bmod 2$.*

Then K is equal to the Ochanine k-invariant,

$$K = k.$$

Proof: By (i), K factors over ell_{8m+2} which is a $\mathbb{Z}/2$-vector space with basis $s^2h^m, s^2h^{m-1}b, \ldots, s^2b^m$. By (ii), we have $K(s^2h^ib^j) \equiv sign(h^ib^j) \bmod 2$. But the same applies to k by 3.2.11 and 3.1.4. ■

4.2.8 *Remark:* Condition (i) in 4.2.7 is equivalent to :

(i') There exists a homomorphism $K' : \Omega^{Spin}_{8m-6} \to \mathbb{Z}/2$, such that $K(N) = K'(M)$ for all G-$\mathbb{H}P^2$-bundles $p : N \to M$ of closed Spin-manifolds N, M of dimension $8m+2$, $8m-6$ respectively.

This holds, because (i') means exactly that $K \circ \psi = K' \circ \pi$ (with ψ and π as in 4.2.1), which is equivalent to (i) by the definition of $\tilde{T}_*$. ■

4.3 Secondary Operations and $\mathbb{H}P^2$-Bundles

4.3.1 In order to apply the above result 4.2.7, we want to compute Brown-Peterson-Kervaire invariants of $\mathbb{H}P^2$-bundles. Let

$$p : N^{8m+10} \to M^{8m+2}$$

be a G-$\mathbb{H}P^2$-bundle of closed *Spin*-manifolds classified by $f : M^{8m+2} \to BG$. Since BG is 1-connected, we can by surgery assume that M^{8m+2} is 1-connected without changing the bordism class $[M^{8m+2}, f] \in \Omega^{Spin}_{8m+2}(BG)$. Then N^{8m+10} is also 1-connected. As the Leray-Hirsch Theorem applies to G-$\mathbb{H}P^2$-bundles (4.1.2), we have

$$H^{4m+5}N^{8m+10} = V_- \oplus V_0 \oplus V_+,$$

where

$$\begin{array}{rcr} V_- & := & p^*H^{4m+5}M^{8m+2} \\ V_0 & := & xp^*H^{4m+1}M^{8m+2} \\ V_+ & := & x^2p^*H^{4m-3}M^{8m+2}. \end{array}$$

Here, $x \in H^4N^{8m+10}$ denotes the pullback of the universal Leray-Hirsch generator, which we called also $x = w_4(T^\Delta) \in H^4BH$, to H^4N^{8m+10}, see 4.1.2.

4.3.2 Now assume that

$$\phi : (ker\beta)^{4m+5} \to (coker\alpha)^{8m+10}$$

is a Brown-Peterson secondary cohomology operation (see 2.3.3), giving a Brown-Peterson-Kervaire invariant K_ϕ in dimension $8m+10$ (2.3.6). For N^{8m+10}, we get $ker\beta = H^{4m+5}N^{8m+10}$ and $coker\alpha = H^{8m+10}N^{8m+10} = \mathbb{Z}/2$, and for $a = p^*a' \in V_-$ we have by naturality

$$\phi(a) = p^*\phi(a') \quad \in p^*H^{8m+10}M^{8m+2} = 0.$$

On the two other summands $V_0 = xp^*H^{4m+1}M^{8m+2}$ and $V_+ = x^2p^*H^{4m-3}M^{8m+2}$ the operation ϕ does not vanish in general; in fact, it can be very complicated on V_+. But with the quadratic sub-Lagrangian lemma A.2.18 we obtain:

4.3.3 Proposition: *Let $q_\phi : H^{4m+5}N^{8m+10} \to \mathbb{Z}/2$ be the quadratic form belonging to our Brown-Peterson-Kervaire invariant K_ϕ, defined by $q_\phi(z) = \langle \phi(z), [N^{8m+10}]\rangle$. Then we have*

$$K_\phi(N^{8m+10}) = Arf(q_\phi) = Arf(q_\phi|_{V_0}).$$

Proof: The multiplikative structure of H^*N^{8m+10} gives by the Leray-Hirsch theorem that $V_-^\perp = V_- \oplus V_0$. Thus $V_0 = V_-^\perp/V_-$, and we can apply A.2.18. ∎

This proposition tells us, that we have to understand the Brown-Peterson secondary cohomology operation ϕ only on the middle summand V_0. The optimal case would be, that there exists another appropriate Brown-Peterson operation

$$\phi' : (ker\beta^{4m+1}) \to (coker\alpha)^{8m+2}$$

(giving a Brown-Peterson-Kervaire invariant $K_{\phi'}$ in dimension $8m+2$), such that $\phi(xp^*a') = x^2p^*\phi'(a')$ on V_0. In this case, we get

$$q_\phi(xp^*a') = \langle x^2p^*\phi'(a'), [N]\rangle = \langle \phi'(a'), [M]\rangle = q_{\phi'}(a') \in \mathbb{Z}/2,$$

hence $K_\phi(N) = \mathrm{Arf}(q_\phi|_{V_0}) = K_{\phi'}(M)$. With 2.2.3 and 4.2.7 (see also remark 4.2.8), one obtains then $K_\phi = k$.

In the next three sections we will attack this problem, which consists in computing a Cartan formula for Brown-Peterson secondary cohomology operations. We remark, that this problem of computing Cartan formulas for secondary operations (with smallest possible undeterminacy) seems to be very hard. Even in the stable case, Adams showed in [1] the existence of a Cartan formula by universal models but could not give the actual form of the terms in the product expansion. Later Adem [2] proved that $\phi(xy) = \phi(x)y + x\phi(y)$ for certain stable operations ϕ; the simple form of the Cartan formula in this case comes from the 'very small' range of definition of these operations, i.e. the condition $\beta(x) = 0$ to define $\phi(x)$ is so restrictive, that all primary operations $\epsilon(x)$ vanish. In particular, the relation $\alpha\beta = 0$ here fixes the operation ϕ.

As we have seen in 2.4.5, this is definitely not true in our case of Brown-Peterson operations ϕ, where the condition $\beta(x) = 0$ on the middle-dimensional cohomology of the *Spin*-manifolds we consider is not restrictive at all (2.3.4). Fortunately, Kristensen gave a method to compute a Cartan formula for secondary cohomology operations, which we review in section 6 and apply to our problem in section 7. But first, we will see in the next section how far we come by standard homotopy theory.

5 A Homotopy Theoretical Product Formula

The previous section shows that we need a product formula for the unstable secondary cohomology operation ϕ applied to $x \cdot y$ with $x \in H^4 N^{8m+10}$ and $y := p^* a$, $a \in H^{4m+1} M^{8m+2}$. In this section we prove by standard homotopy theory a product formula for secondary operations of a very general type. This gives a first result for $\phi(xy)$, but it contains a primary term which is obtained by glueing together certain homotopies. Because we were not able to compute this term by homotopy theory as an explicit linear combination of known invariants, this formula has only very restricted application to our problem.

On the other side, we hope that our formula has some interest in its own, because it seems to show more directly than Kristensen's theory (for which we give a survey in the next section) how a product formula arises and where the hard part of the problem lies (namely, in the primary operation which comes from glueing homotopies). Also, our method is not restricted to cohomology with $\mathbb{Z}/p$-coefficients, as is the theory of Kristensen.

We point to a little difference in the notation of 'relations' in this section, which disagrees with the rest of this thesis: We introduce here relations as zero-homotopic compositions

$$A \xrightarrow{a} B \xrightarrow{b} C,$$

i.e. $ba \simeq 0$, and the associated secondary operations go then from '$ker(a)$' to '$coker(b)$'. In contrast to this, relations in the Steenrod algebra (and similar for cochain operations in the next section) are denoted by $\sum \alpha_i \beta_i = 0$, giving secondary cohomology operations going from $ker(\beta)$ to $coker(\alpha)$. The first notation seems to fit for commutative diagrams of spaces, whereas the second notation fits for 'algebraic' equations. We hope, that these two different notations will not cause any confusion.

5.1 Secondary Operations

5.1.1 We recall the definition of secondary operations in the language of homotopy theory, see [46] and [59]. Let A and B be spaces (we work in the category of pointed compactly generated spaces) and consider the homotopy functors $[\ \ , A]$ and $[\ \ , B]$. By the Yoneda lemma, the *primary operations* from $[\ \ , A]$ to $[\ \ , B]$ are given by the elements in $[A, B]$. Now fix a map $a : A \to B$ and consider the pullback $\pi_a : a^*PB \to A$ of the path fibration $\pi_B : PB \to B$, which comes with the canonical map $\tilde{a} : a^*PB \to PB$ over a. We remark that by the exponential law, a zero-homotopy for a fixed map $X \to B$ is the same as a lift $X \to PB$. Similarly, for a fixed map $x : X \to A$ its lifts $\tilde{x} : X \to a^*PB$ correspond (by the pullback property) bijectively to the zero-homotopies h $(:= \tilde{a}\tilde{x})$ of the composite map ax:

$$\begin{tikzcd} & a^*PB \arrow[r, "\tilde{a}"] \arrow[d, "\pi_a"] & PB \arrow[d, "\pi_B"] \\ X \arrow[ur, "\tilde{x}"] \arrow[urr, "h"] \arrow[r, "x"] & A \arrow[r, "a"] & B. \end{tikzcd}$$

5.1.2 Now, a *secondary operation* Φ from $[\ \ ,A]$ to $[\ \ ,D]$ is given by $\phi \in [a^*PB, D]$ where D is a further space; then one defines for any space X:

$$\Phi : [X,A] \longrightarrow \mathcal{P}[X,D]$$

$$\Phi([x]) := \{\phi[\tilde{x}] \mid \tilde{x} : X \to a^*PB \text{ is a lift of } x : X \to A\},$$

where $\mathcal{P}$ denotes the power set (set of all subsets) and $[x]$ the homotopy class of a map $x : X \to A$. By the above remarks we see that $\Phi([x]) = \emptyset$ iff $[a][x] \neq 0$. We say that Φ has

$$ker[a] := \{[x] \in [X,A] \mid [a][x] = 0\}$$

as range of definition. The naturality of Φ is expressed by $\Phi(f^*[x]) \subset f^*\Phi([x])$ for a map $f : Y \to X$ (here $f^*[x] = [xf]$ and similar for $f^*\Phi([x])$).

5.1.3 Now we show how secondary operations arise from a relation between primary operations. Suppose we have a *relation* (a,b), i.e. two maps $A \xrightarrow{a} B \xrightarrow{b} C$ where the composite map ba is zero-homotopic. Fixing a zero-homotopy of ba, which is the same as a lift $H : A \to PC$, we get a secondary operation

$$\phi := H\pi_a - (Pb)\,\tilde{a} : a^*PB \to \Omega C$$

by glueing together the two maps $H\pi_a : a^*PB \to PC$ and $(Pb)\tilde{a} : a^*PB \to PC$,

$$\begin{tikzcd}
 & \Omega B \arrow[r, equal] \arrow[d] & \Omega B \arrow[r, "\Omega b"] \arrow[d] & \Omega C \arrow[d] \\
 & a^*PB \arrow[r, "\tilde{a}"] \arrow[d, "\pi_a"'] \arrow[urr, "\phi"'] & PB \arrow[r, "Pb"] \arrow[d, "\pi_B"'] & PC \arrow[d, "\pi_C"'] \\
X \arrow[r, "x"] \arrow[ur, "\tilde{x}"] \arrow[urr, "h"] & A \arrow[r, "a"] \arrow[urr, "H"] & B \arrow[r, "b"] & C.
\end{tikzcd}$$

Here, *glueing* means composition with the map

$$- : \pi_C^*PC = \{(w_1,w_2) \in PC \times PC \mid \pi_C w_1 = \pi_C w_2\} \longrightarrow \Omega C$$

$$w_1 - w_2 := \left(t \mapsto \begin{cases} w_1(2t), & t \in [0,1/2] \\ w_2(1-2t), & t \in [1/2,1] \end{cases} \right).$$

5.1.4 We analyse the undeterminacy of the secondary operation Φ in this case: Since a lift $\tilde{x}$ of x is the same as a zero-homotopy $h : X \to PB$ of ba, we see that the subset $\Phi([x]) = \{[\phi\tilde{x}]\}$ of the group $[X, \Omega C]$ is obtained by glueing together the two maps $Hx : X \to PC$ and $(Pb)h : X \to PC$ for all zero-homotopies h of ax. As the path fibration and its induced fibrations are principal fibrations, different h differ (up to homotopy) by adding maps $X \to \Omega B$ with

$$+ : \Omega B \times PB \longrightarrow PB$$
$$w_1 + w_2 := \left(t \mapsto \begin{cases} w_1(2t), & t \in [0,1/2] \\ w_2(2t-1), & t \in [1/2,1] \end{cases} \right),$$

thus $\Phi([x])$ is in fact a right coset of the subgroup $(\Omega b)_*[X, \Omega B] \leq [X, \Omega C]$. In particular, if ΩC is a homotopy commutative H-space, the group $[X, \Omega C]$ is abelian and Φ can be considered to take values in the factor group $coker[\Omega b] := [X, \Omega C]/(\Omega b)_*[X, \Omega B]$,

$$\Phi : ker[a] \longrightarrow coker[\Omega b].$$

5.1.5 A different choice of the zero-homotopy H for the relation (a, b) differs (up to homotopy) by adding a map $d : A \to \Omega C$, which gives then the difference between the associated secondary operations:

$$\phi_{d+H} = d\pi_a + \phi_H$$
$$\Phi_{d+H}([x]) = d[x] + \Phi_H([x]),$$

where the last "+" is of course the product in the group $[X, \Omega C]$ which is non-commutative in general.

5.1.6 Remark: This shows that if one does not specify a choice of H for the relation (a, b), one gets only an element in the double coset

$$\langle x, a, b\rangle := \bigcup_H \Phi_H([x]) \in x^*[A, \Omega C]\backslash[X, \Omega C]/(\Omega b)_*[X, \Omega B],$$

which is called the *secondary composition* (or *Toda bracket*) of x, a and b (see [59]).

5.2 A Product Formula

5.2.1 Now we come to a product formula for secondary operations. By this we mean an expansion of $\mu^*\Phi$ where $\mu : E \wedge A' \to A$ is a given map. We suppose further that we have a relation (a', b'), and maps $\nu : E \wedge B' \to B$ and $\eta : E \wedge C' \to C$, such that the following diagram is commutative up to homotopy:

$$\begin{tikzcd}
A \arrow[r, "a"] & B \arrow[r, "b"] & C \\
E \wedge A' \arrow[u, "\mu"] \arrow[r, "1\wedge a'"] & E \wedge B' \arrow[u, "\nu"] \arrow[r, "1\wedge b'"] & E \wedge C' \arrow[u, "\eta"]
\end{tikzcd}$$

We will in the applications 5.3.1 and 10.3.3 see how such a diagram follows from the relation (a, b) and product formulas for the primary operations a, b with respect to μ.

5.2.2 Theorem: *Let (a, b) and (a', b') be relations and μ, ν and η be maps such that the above diagram is commutative up to basepoint-preserving homotopies. Choose zero-homotopies $H : A \to PC$ and $H' : A' \to PC'$ for the relations and pointed homotopies*

$$H^L : I \times (E \wedge A') \to B \qquad H^R : I \times (E \wedge B') \to C$$

for the left and the right square of the diagram, where $H_0^L = \nu(1 \wedge a')$, $H_1^L = a\mu$, $H_0^R = \eta(1 \wedge b')$ and $H_1^R = b\nu$. Then the following two maps from $E \wedge a'^*PB'$ to ΩC,

$$\phi_H \tilde{\mu} \quad \text{and} \quad \epsilon_{H,H',H^L,H^R}(1 \wedge \pi_{a'}) + (\Omega_{(2)}\eta)\phi_{H'},$$

are homotopic. Here, $\tilde{\mu}$ is a lift of μ constructed by H^L, $\Omega_{(2)}$ denotes partial looping in the second variable, and ϵ_{H,H',H^L,H^R} is obtained by glueing together the homotopies H, H', H^L, H^R. In particular, if $(x, y) : X \to E \wedge A'$ is a map with $a'y \simeq 0$, then the secondary operations $\Phi_{H'}([y])$ and $\Phi_H([\mu(x, y)])$ are defined, and it holds

$$\Phi_H([\mu(x, y)]) = [\epsilon_{H,H',H^L,H^R}]([x], [y]) + [\Omega_{(2)}\eta]([x], \Phi_{H'}([y])).$$

Proof: First we show the existence of pointed maps $\tilde{\mu}$, $P_{(2)}\nu$ and $P_{(2)}\eta$ and homotopies $\tilde{H}^L : I \times (E \wedge a'^*PB') \to PB$ and $P_{(2)}H^R : I \times (E \wedge PB') \to PC$ in the following diagram

$$\begin{tikzcd}
 & a^*PB \arrow[rr, "\tilde{a}"] \arrow[dd, "\pi_a"] \arrow[dr, phantom, "\tilde{H}^L \Uparrow"] & & PB \arrow[rr, "Pb"] \arrow[dd, "\pi_B"] \arrow[dr, phantom, "P_{(2)}H^R \Uparrow"] & & PC \arrow[dd, "\pi_C"] \\
E \wedge a'^*PB' \arrow[ur, "\tilde{\mu}"] \arrow[rr, "1\wedge\tilde{a}'"] \arrow[dd, "1\wedge\pi_{a'}"] & & E \wedge PB' \arrow[ur, "P_{(2)}\nu"] \arrow[rr, "1\wedge Pb'"] \arrow[dd, "1\wedge\pi_{B'}"] & & E \wedge PC' \arrow[ur, "P_{(2)}\eta"] \arrow[dd, "1\wedge\pi_{C'}"] & \\
 & A \arrow[rr, "a"] \arrow[dr, phantom, "H^L \Uparrow"] & & B \arrow[rr, "b"] \arrow[dr, phantom, "H^R \Uparrow"] & & C \\
E \wedge A' \arrow[ur, "\mu"] \arrow[rr, "1\wedge a'"] & & E \wedge B' \arrow[ur, "\nu"] \arrow[rr, "1\wedge b'"] & & E \wedge C' \arrow[ur, "\eta"] &
\end{tikzcd}$$

such that

$$\begin{array}{llcll}
(1) & \pi_a\tilde{\mu} & = & \mu(1 \wedge \pi_{a'}) & \\
(2) & \tilde{a}\tilde{\mu} & \simeq & (P_{(2)}\nu)(1 \wedge \tilde{a}') & \text{by } \tilde{H}^L \\
(3) & \pi_B(P_{(2)}\nu) & = & \nu(1 \wedge \pi_{B'}) & \\
(4) & \pi_C(P_{(2)}\eta) & = & \eta(1 \wedge \pi_{C'}) & \\
(5) & (Pb)(P_{(2)}\nu) & \simeq & (P_{(2)}\eta)(1 \wedge Pb') & \text{by } P_{(2)}H^R.
\end{array}$$

The maps $P_{(2)}\nu$, $P_{(2)}\eta$ and the homotopy $P_{(2)}H^R$ are constructed by applying the *partial path space functor $P_{(2)}$ in the second variable* which associates to a map $f : X \wedge Y \to Z$ the map $P_{(2)}f : X \wedge PY \to PZ$ defined by $P_{(2)}f : (x, (t \mapsto y_t)) \mapsto (t \mapsto f(x, y_t))$. In the same way one defines $(P_{(2)}H^R)_s := P_{(2)}(H_s^R)$ for the homotopy $H^R = (H_s^R)_{s \in I}$. A straightforward calculation shows that these maps are well-defined and satisfy (3), (4) and (5).

Now we want to construct the map $\tilde{\mu}$ and the homotopy $\tilde{H}^L$. Because of (1), we want $\tilde{\mu}$ to be a lift of $\mu(1 \wedge \pi_{a'})$, thus we look for a zero-homotopy of the map $a\mu(1 \wedge \pi_{a'})$. But this map is — by the homotopy$(H_s^L(1 \wedge \pi_{a'}))_{s \in I}$ — homotopic to the map

$$\nu(1 \wedge a')(1 \wedge \pi_{a'}) = \nu(1 \wedge \pi_{B'})(1 \wedge \tilde{a'}) = \pi_B(P_{(2)}\nu)(1 \wedge \tilde{a'}),$$

which has the canonical zero-homotopy $(P_{(2)}\nu)(1 \wedge \tilde{a'})$. Thus $\tilde{\mu}$ is constructed by glueing together this zero-homotopy $(P_{(2)}\nu)(1 \wedge \tilde{a'})$ of $\nu(1 \wedge a')(1 \wedge \pi_{a'})$ with the homotopy $H_s^L(1 \wedge \pi_{a'})$ between $\nu(1 \wedge a')(1 \wedge \pi_{a'})$ and $a\mu(1 \wedge \pi_{a'})$ to give a zero-homotopy of the map $a\mu(1 \wedge \pi_{a'})$. In fact, the intermediate steps of this glueing process define then also the homotopy $\tilde{H}^L$ with the property (2). In order to give an explicit expression, we recall that an element of $E \wedge a'^* PB'$ is given by $(e, (u, (t \mapsto v_t)))$ where $e \in E$, $u \in A'$ and $(t \mapsto v_t) \in PB'$ with $a'(u) = v_1$, modulo the identifications of the smash product. Then $\tilde{\mu}$ and $\tilde{H}^L$ are given by

$$\begin{array}{llcl}
\tilde{\mu}: & E \wedge a'^* PB' & \longrightarrow & a^* PB \\
\tilde{\mu}: & (e, (u, (t \mapsto v_t))) & \mapsto & (\mu(e, u), (t \mapsto \begin{cases} \nu(e, v_{2t}) & , t \in [0, 1/2] \\ H^L_{2t-1}(e, u) & , t \in [1/2, 1] \end{cases}))
\end{array}$$

$$\begin{array}{llcl}
\tilde{H}^L_s: & E \wedge a'^* PB' & \longrightarrow & PB, \qquad s \in I \\
\tilde{H}^L_s: & (e, (u, (t \mapsto v_t))) & \mapsto & (t \mapsto \begin{cases} \nu(e, v_{2t/(2-s)}) & , t \in [0, 1-\frac{s}{2}] \\ H^L_{2(t-1)+s}(e, u) & , t \in [1-\frac{s}{2}, 1] \end{cases})
\end{array}$$

Tedious but straightforward computations show that these maps are again well-defined and satisfy (1) and (2), and also $\pi_B \tilde{H}_s^L = H_s^L(1 \wedge \pi_{a'})$.

As next, we want to compute $\mu^*\Phi$ which we represent by the map

$$\phi\tilde{\mu} : E \wedge a'^* PB' \to \Omega C,$$

because by (1) a lift $\tilde{x} : X \to E \wedge a'^* PB'$ of a map $x : X \to E \wedge A'$ gives a lift $\tilde{\mu}\tilde{x} : X \to a^* PB$ of $\mu x : X \to A$. We have $\phi\tilde{\mu} = H\pi_a\tilde{\mu} - (Pb)\tilde{a}\tilde{\mu}$, where the both maps $H\pi_a\tilde{\mu}$ and $(Pb)\tilde{a}\tilde{\mu}$ lie over the map $ba\pi_a\tilde{\mu} = ba\mu(1 \wedge \pi_{a'})$. Our strategy is to pull these both maps from the back face of our diagram to the front face with the help of the homotopies H^L and H^R (respectively $\tilde{H}^L$, $P_{(2)}H^R$). In fact we will define homotopies F and G with

$$\begin{array}{lll}
F : I \times (E \wedge A') \to PC, & F_0 = H\mu, & \\
G : I \times (E \wedge a'^* PB') \to PC, & G_0 = (Pb)\tilde{a}\tilde{\mu}, & G_1 = (P_{(2)}\eta)(1 \wedge (Pb')\tilde{a}')
\end{array}$$

and F_1 is obtained by glueing together the homotopies H, H^L and H^R. Then we get

$$\phi\tilde{\mu} \quad = \quad F_0(1 \wedge \pi_{a'}) - G_0 \quad \simeq \quad F_1(1 \wedge \pi_{a'}) - G_1,$$

and in the last step we will compare the right side with

$$(\Omega_{(2)}\eta)(1 \wedge \phi').$$

We have to be a little careful because we take the "difference" by the map $- : p_C^* PC \to \Omega C$ which can only be applied if the two maps to PC lie over the same map, i.e. the range of definition of "$-$" is $p_C^* PC$ and not $PC \times PC$. In particular, the homotopies F and G have to satisfy

$$\pi_C F_s(1 \wedge \pi_{a'}) = \pi_C G_s \qquad \text{for all } s.$$

Now we write down the explicit definitions of F and G. We start with G which is obtained by glueing together the homotopies $(Pb)\tilde{H}^L$ and $(P_{(2)}H^R)(1\wedge\tilde{a'})$:

$$G_s := \begin{cases} (Pb)(\tilde{H}^L_{1-2s}) & \text{for } s\in[0,\frac{1}{2}],\\ (P_{(2)}H^R)_{2-2s}(1\wedge\tilde{a'}) & \text{for } s\in[\frac{1}{2},1]. \end{cases}$$

(We often switch here between $map(I\times X,Y)$ and $map(X,map(I,Y))$ using the exponential law.) For the definition of $F=(F_s : E\wedge A'\to PC)_{s\in I}$, we glue together the homotopies $(H_t\mu)_{t\in I}$, $(bH^L_t)t\in I$ and $(H^R_t(1\wedge a'))_{t\in I}$ and obtain a homotopy $(\hat{H}_t)_{t\in[0,3]}$, which gives F_s by running from $\hat{H}_0=0$ to $\hat{H}_{1+2s}$:

$$\hat{H} : [0,3]\times(E\wedge A')\longrightarrow PC,$$

$$\hat{H}_t := \begin{cases} H_t\mu, & \text{for } t\in[0,1]\\ bH^L_{2-t}, & \text{for } t\in[1,2]\\ H^R_{3-t}(1\wedge a'), & \text{for } t\in[2,3] \end{cases}, \qquad F_s := (t\mapsto H_{(1+2s)t}).$$

It is straightforward to verify all properties of F and G we wanted to have. Thus we get by 'adding zero':

$$\begin{aligned} \phi\tilde{\mu} \quad &\simeq \quad F_1(1\wedge\pi_{a'}) - (P_{(2)}\eta)(1\wedge(Pb')\tilde{a'})\\ \simeq \Big(F_1(1\wedge\pi_{a'}) - (P_{(2)}\eta)&(1\wedge H'\pi_{a'})\Big) + \Big((P_{(2)}\eta)(1\wedge H'\pi_{a'}) - (P_{(2)}\eta)(1\wedge(Pb')\tilde{a'})\Big). \end{aligned}$$

It is clear how this process of 'adding zero' works and that it does not change the homotopy class, but again this is tedious to write down: Given maps $r^1,r^2,r^3 : X\to PY$ over the same map $X\to Y$, we get a homotopy $(R_s)_{s\in I}$ from $R_0=r^1-r^3$ to $R_1=r^1-r^2+r^2-r^3$ by

$$R_s := \left(t\mapsto \begin{cases} r^1_{a_1(s,t)} & \text{for } t\in[0,\frac{1}{2}-\frac{1}{4}s]\\ r^2_{a_2(s,t)} & \text{for } t\in[\frac{1}{2}-\frac{1}{4}s,\frac{1}{2}]\\ r^2_{a'_2(s,t)} & \text{for } t\in[\frac{1}{2},\frac{1}{2}+\frac{1}{4}s]\\ r^3_{a_3(s,t)} & \text{for } t\in[\frac{1}{2}+\frac{1}{4}s,1] \end{cases} \right)$$

where the functions a_1,a_2,a'_2,a_3 can be characterized by linearity in t and their values at the 'boundaries':

$$\begin{array}{lcl@{\qquad}ll} a_1(s,0) & = & 0 & a_1(s,\frac{1}{2}-\frac{1}{4}s) & =1\\ a_2(s,\frac{1}{2}-\frac{1}{4}s) & = & 1 & a_2(s,\frac{1}{2}) & =1-s\\ a'_2(s,\frac{1}{2}) & = & 1-s & a'_2(s,\frac{1}{2}+\frac{1}{4}s) & =1\\ a_3(s,\frac{1}{2}+\frac{1}{4}s) & = & 1 & a_3(s,1) & =0. \end{array}$$

(By a similar method, we got the expressions for $\tilde{\mu}$ and $\tilde{H}^L$ above.)

Going back to our computation, the left difference r^1-r^2 is (up to reparametrization in t) given by $\epsilon_{H,H',H^L,H^R}(1\wedge\pi_{a'})$, where

$$\epsilon_{H,H',H^L,H^R} : E\wedge A'\longrightarrow \Omega C,$$

$$(\epsilon_{H,H',H^L,H^R})_t := \begin{cases} H_{4t}\mu, & \text{for } t\in[0,\frac{1}{4}]\\ bH^L_{2-4t}, & \text{for } t\in[\frac{1}{4},\frac{1}{2}]\\ H^R_{3-4t}(1\wedge a'), & \text{for } t\in[\frac{1}{2},\frac{3}{4}]\\ (P_{(2)}\eta)(1\wedge H'_{4-4t}), & \text{for } t\in[\frac{3}{4},1] \end{cases}$$

For the right difference $r^2 - r^3$, we remark that the map $P_{(2)}\eta : E \wedge PC' \to PC$ was for fixed parameter $t \in I$ just given by η; thus we obtain

$$\begin{aligned}(P_{(2)}\eta)(1 \wedge H'\pi_{a'}) - (P_{(2)}\eta)(1 \wedge Pb'\tilde{a'}) &= (\Omega_{(2)}\eta)(1 \wedge (H'\pi_{a'} - Pb'\tilde{a'})) = \\ &= (\Omega_{(2)}\eta)(1 \wedge \phi'),\end{aligned}$$

where $\Omega_{(2)}$ denotes the *partial loop space functor in the second variable*, which associates to a map $f : X \wedge Y \to Z$ the map $\Omega_{(2)}f : X \wedge \Omega Y \to \Omega Z$ defined by

$$\Omega_{(2)}f : (x, (t \mapsto y_t)) \mapsto (t \mapsto f(x, y_t)).$$

This ends the proof. ■

5.3 Application to $\mathbb{H}P^2$-Bundles

5.3.1 We want to apply the product formula 5.2.2 to a Brown-Peterson-Kervaire invariant ϕ on an $PSp(3)$-$\mathbb{H}P^2$-bundle $p : N^{8m+10} \to M^{8m+2}$ of closed 1-connected *Spin*-manifolds with structure group $PSp(3)$. According to 4.3.3, we have to compute ϕ on the middle dimensional summand $xp^*H^{4m+1}M^{8m+2}$. Thus we consider the following diagram giving the initial data 5.2.1 for the product formula 5.2.2:

$$\begin{array}{ccccccc} & & K_{4m+5} & \xrightarrow{a} & K_{8m+9}\times K_{8m+10} & \xrightarrow{b} & K_{8m+11} \\ & & \uparrow{\scriptstyle\mu} & & \uparrow{\scriptstyle\nu} & & \uparrow{\scriptstyle\eta} \\ N^{8m+10} & \xrightarrow{(\bar{f},p^*y)} & E \wedge K_{4m+1} & \xrightarrow{1\wedge a'} & E \wedge B' & \xrightarrow{1\wedge b'} & E \wedge C', \end{array}$$

where $E := BP(Sp(2) \times Sp(1))$ is the total space of the universal $PSp(3)$-$\mathbb{H}P^2$-bundle, $f : M^{8m+2} \to BPSp(3)$ denotes the classifying map of the bundle $N^{8m+10} \to M^{8m+2}$ with associated bundle map $\bar{f} : N^{8m+10} \to E$, and $\mu := x \cup \iota_{4m+1}$ with $x : E \to K_4$ representating the universal Leray-Hirsch generator $x \in H^4E$ (we denote also the pullback $\bar{f}^*x \in H^4N^{8m+10}$ by x and hope that this will not lead to any confusion). Furthermore, $a := (Sq^{4m+4}, Sq^{4m+4}Sq^1)$ and $b := Sq^2 + Sq^1$ give the relation for Brown-Peterson secondary cohomology operations in dimension $8m + 10$. Here and in the following, $+$ means the H-space structure of K_i, ι_i denotes the identity (=fundamental class) on K_i, and $\cup : K_i \wedge K_j \to K_{i+j}$ is a map realizing the cup product in cohomology.

In order to define the remaining part of the diagram, we apply the Cartan formula to $a\mu$:

$$\begin{array}{rcll} Sq^{4m+4}(x\cdot\iota_{4m+1}) &=& x^2 Sq^{4m}\iota_{4m+1} & + \; Sq^3x\cdot Sq^{4m+1}\iota_{4m+1}, \\ Sq^{4m+4}Sq^1(x\cdot\iota_{4m+1}) &=& x^2 Sq^{4m}Sq^1\iota_{4m+1} & + \; Sq^3x\cdot Sq^{4m+1}Sq^1\iota_{4m+1} \\ & & & + \; Sq^2x\cdot Sq^{4m+2}Sq^1\iota_{4m+1}. \end{array}$$

Here we have used that $Sq\,x = x + Sq^2x + Sq^3x + x^2$ (4.1.3) and $Sq^k y = 0$ for $k > 4m + 1$. Thus we set

$$\begin{array}{rcl} B' &:=& (K_{8m+1} \times K_{8m+2}) \times (K'_{8m+2} \times K_{8m+3} \times K_{8m+4}), \\ a' &:=& ((Sq^{4m}, Sq^{4m+1}), (Sq^{4m}Sq^1, Sq^{4m+1}Sq^1, Sq^{4m+2}Sq^1)), \\ \nu &:=& (x^2 \cup \iota_{8m+1} + Sq^3x \cup \iota_{8m+2}) \times (x^2 \cup \iota'_{8m+2} + Sq^3x \cup \iota_{8m+3} + Sq^2x \cup \iota_{8m+4}). \end{array}$$

Of course, x^2, Sq^3x, Sq^2x mean here representing maps from E to K_8, K_7, K_6. Because there are two factors K_{8m+2} in the definition of B', we denote the second by K'_{8m+2} and its fundamental class by ι'_{8m+2}. This defines the left square of the diagram which by construction is commutative up to homotopy. For the right square we apply again the Cartan formula, but this time to $b\nu$:

$$Sq^2((x^2\iota_{8m+1} + Sq^3x\cdot\iota_{8m+2}) + Sq^1(x^2\iota'_{8m+2} + Sq^3x\cdot\iota_{8m+3} + Sq^2x\cdot\iota_{8m+4}) =$$

$$x^2Sq^2\iota_{8m+1} + Sq^3x\cdot Sq^2\iota_{8m+2} + x^2Sq^1\iota'_{8m+2} + Sq^3x\cdot Sq^1\iota_{8m+3} + Sq^3x\cdot\iota_{8m+4} + Sq^2x\cdot Sq^1\iota_{8m+4}.$$

Here we used also some Adem relations, like $Sq^2Sq^3 = Sq^5 + Sq^4Sq^1$. Thus we set

$$\begin{array}{lcl} C' & := & K_{8m+3} \times K_{8m+4} \times K_{8m+5}, \\ b' & := & (Sq^2\iota_{8m+1} + Sq^1\iota'_{8m+2},\ Sq^2\iota_{8m+2} + Sq^1\iota_{8m+3} + \iota_{8m+4},\ Sq^1\iota_{8m+4}), \\ \eta & := & x^2 \cup \iota_{8m+3} + Sq^3x \cup \iota_{8m+4} + Sq^2x \cup \iota_{8m+5}, \end{array}$$

defining the right square of our diagram which is homotopy commutative by construction. Now we can prove:

5.3.2 Theorem: *Under the above assumptions we have the following equation living in $H^{8m+10}N^{8m+10} \cong \mathbb{Z}/2$:*

$$\Phi(xp^*y) = x^2p^*\Big(\Phi'(y) + \epsilon(y)\Big),$$

where $x \in H^4N^{8m+10}$ is the Leray-Hirsch generator, $y \in H^{4m+1}M^{8m+2}$, Φ and Φ' are the Brown-Peterson secondary cohomology operations on $H^{4m+5}N^{8m+10}$ and $H^{4m+1}M^{8m+2}$ constructed with the homotopies H and H', and ϵ comes from the primary operation constructed by glueing together the homotopies H, H', H^L, and H^R. Furthermore, ϵ is given as a linear combination of $PSp(3)$-bundle characteristic classes and stable primary cohomology operations:

$$\epsilon(y) = \sum_{i=0}^{4m+1} f^*(u_i)\alpha_i(y),$$

with $u_i \in H^iBPSp(3)$, $\alpha_i \in A^{4m+1-i}$, and $f : M^{8m+2} \to BPSp(3)$ is the classifying map of the $\mathbb{H}P^2$-bundle $p : N^{8m+10} \to M^{8m+2}$.

Proof: By 5.2.2, we obtain

$$\Phi(xp^*y) = \Phi_H(\mu(\bar{f}, p^*y)) = \epsilon_{H,H',H^L,H^R}(\bar{f}, p^*y) + (\Omega_{(2)}\eta)(\bar{f}, \Phi_{H'}(p^*y)),$$

where Φ is the Brown-Peterson secondary cohomology operation in dimension $8m+10$ constructed with the zero-homotopy H. We have to compute the right terms in the formula.

The secondary operation $\Phi_{H'}$ is *not* a Brown-Peterson secondary cohomology operation, since by the application of the Cartan and Adem formulas, a' and b' have become more complicated. According to $C' = K_{8m+3} \times K_{8m+4} \times K_{8m+5}$, the operation $\Phi_{H'}$ splits into three secondary cohomology operations Φ_1, Φ_2, Φ_3, associated to the unstable relations

$$\begin{array}{llcccl} (1) & K_{4m+1} & \xrightarrow{(Sq^{4m}, Sq^{4m}Sq^1)} & K_{8m+1} \times K'_{8m+2} & \xrightarrow{Sq^2+Sq^1} & K_{8m+3} \\ (2) & K_{4m+1} & \xrightarrow{(Sq^{4m+1}, Sq^{4m+1}Sq^1, Sq^{4m+2}Sq^1)} & K_{8m+2} \times K_{8m+3} \times K_{8m+4} & \xrightarrow{Sq^2+Sq^1+\iota} & K_{8m+4} \\ (3) & K_{4m+1} & \xrightarrow{Sq^{4m+2}Sq^1} & K_{8m+4} & \xrightarrow{Sq^1} & K_{8m+5}. \end{array}$$

But Φ_1 is just a Brown-Peterson secondary cohomology operation in dimension $8m+2$, and the two other operations Φ_2, Φ_3 take values in dimension $8m+3$, $8m+4$ respectively. Applied to the class p^*y, $y \in H^{4m+1}M^{8m+2}$, we see by naturality the vanishing of $\Phi_2(p^*y)$ and $\Phi_3(p^*y)$.

Now, we come to the map $\Omega_{(2)}\eta$, which can be considered as a derivative of the product operation η after the second variable. We use the following fact on $\Omega_{(2)}$ applied to the cup-product $\cup : K_i \wedge K_j \to K_{i+j}$:

With the natural homotopy equivalence $\Omega K_l \simeq K_{l-1}$, the map $\Omega_{(2)}\cup : K_i \wedge \Omega K_j \to \Omega K_{i+j}$ is again given by the cup-product $\cup : K_i \wedge K_{j-1} \to K_{i+j-1}$.

In fact, this is part of the ring-spectrum property of the Eilenberg-MacLane spectrum $H = H\mathbb{Z}/2$, and can be found in [63] (there, it is formulated using suspensions, giving by the exponential law the above formulation). We remark, that the corresponding fact for the total loop space functor is not true: It holds $\Omega\cup \simeq 0$, which is just the vanishing of all cup products in the suspension of a space ([64]). This shows that

$$\eta : E \wedge (K_{8m+3} \times K_{8m+4} \times K_{8m+5}) \quad \longrightarrow \quad K_{8m+11}$$
$$\eta = x^2 \cup \iota_{8m+3} + Sq^3x \cup \iota_{8m+4} + Sq^2x \cup \iota_{8m+5}$$

gives

$$\Omega_{(2)}\eta : E \wedge (K_{8m+2} \times K_{8m+3} \times K_{8m+4}) \quad \longrightarrow \quad K_{8m+10}$$
$$\Omega_{(2)}\eta = x^2 \cup \iota_{8m+2} + Sq^3x \cup \iota_{8m+3} + Sq^2x \cup \iota_{8m+4},$$

since $\Omega K_l \simeq K_{l-1}$ is compatible with the H-space structures defining the addition in cohomology. In particular, we have computed ($x = \bar{f}^*x$)

$$(\Omega_{(2)}\eta)(\bar{f}, \Phi_{H'}(p^*y)) = x^2 p^* \Phi_{H'}(y).$$

At last, we consider the primary term

$$\epsilon_{H,H',H^L,H^R} \;\in\; [E \wedge K_{4m+1}, \Omega K_{8m+11}] = H^{8m+10}(E \wedge K_{4m+1}) = \bigoplus_{i+j=8m+10} (H^iE \otimes H^jK_{4m+1}),$$

hence $\epsilon_{H,H',H^L,H^R} = \sum_{i+j=8m+10} Z_i \otimes \alpha_j$ with $Z_i \in H^iE$, $\alpha_j \in H^jK_{4m+1}$ and

$$\epsilon(x,y) := \epsilon_{H,H',H^L,H^R}(\bar{f}, p^*y) = \sum_{i+j=8m+10} \bar{f}^*(Z_i)\; p^*\alpha_j(y) \quad \in H^{8m+10}N^{8m+10} = \mathbb{Z}/2.$$

Using that H^*E is a free $H^*BPSp(3)$-module via p^* on the generators 1, x, x^2 (4.1.2), we have $Z_i = x^2p^*(z_{i-8}) + xp^*(z'_{i-4}) + p^*(z''_i)$ with $z_l, z'_l, z''_l \in H^lBPSp(3)$. Since $\epsilon(x,y)$ lives in the top-dimension of N^{8m+10}, only the terms with x^2 can survive:

$$\epsilon(x,y) = \sum_{l+j=8m+2} x^2p^*f^*(z_l)\; p^*\alpha_j(y).$$

But then, $\alpha_j \in H^jK_{4m+1} = A^{j-(4m+1)}$ is in the stable range as $j \leq 8m+2$, see 2.3.2 (iv). If we shift the index j by $4m+1$ and call l again i, we have shown that

$$\epsilon(x,y) = x^2p^*\Big(\sum_{i+j=4m+1} f^*(z_i)\; \alpha_j(y)\Big) \qquad \text{with } z_i \in H^iBPSp(3),\ \alpha_j \in A^j. \quad \blacksquare$$

5.3.3 Corollary: *Let $N^{8m+10} \to M^{8m+2}$ be a $PSp(3)$-$\mathbb{H}P^2$-bundle of closed 1-connected Spin-manifolds and choose Brown-Peterson-Kervaire invariants K_ϕ and $K_{\phi'}$ in dimension $8m+10$ and $8m+2$, respectively. Then*

$$K_\phi(N^{8m+10}) = K_{\phi'}(M^{8m+2}) + \langle \phi'(v_{\phi,\phi'}), [M^{8m+2}] \rangle,$$

where $v_{\phi,\phi'} \in H^{4m+1}M^{8m+2}$ corresponds by Poincaré duality to the linear form

$$\epsilon^* : H^{4m+1}M^{8m+2} \to \mathbb{Z}/2, \qquad y \mapsto \langle \epsilon(y), [M^{8m+2}] \rangle.$$

Moreover, $v_{\phi,\phi'}$ is given by the pullback

$$v_{\phi,\phi'} = \sum_s f^*(\tilde{z}_s)\nu_M^*(\tilde{w}_s),$$

where $f : M^{8m+2} \to BPSp(3)$ is the classifying map of the bundle, $\nu_M : M^{8m+2} \to BSpin$ is the Spin-structure, and

$$\sum_s \tilde{z}_s \otimes \tilde{w}_s \quad \in H^{4m+1}(BPSp(3) \times BSpin)$$

is a universal class depending only on the choice of the Brown-Peterson-Kervaire invariants K_ϕ, $K_{\phi'}$.

Proof: The first statement follows by application of the sub-lagrangian lemma 4.3.3, the product formula 5.2.2, and the addition formula 1.2.4:

$$\begin{array}{rcl} K_\phi(N) & := & \mathrm{Arf}(\phi : H^{4m+5}N \to \mathbb{Z}/2) \\ & = & \mathrm{Arf}(\phi : xp^*H^{4m+1}M \to \mathbb{Z}/2) \\ & = & \mathrm{Arf}(y \mapsto \langle \phi'(y) + \epsilon(y), [M] \rangle) \\ & = & \mathrm{Arf}(y \mapsto \langle \phi'(y), [M] \rangle) + \langle \phi'(PD(\epsilon^*)), [M] \rangle \quad =: \quad K_{\phi'}(M) + \langle \phi'(v_{\phi,\phi'}), [M] \rangle. \end{array}$$

For the second statement, we have to show that the class $v_{\phi,\phi'} \in H^{4m+1}M$ corresponding by Poincaré duality to the linear form

$$y \mapsto \epsilon^*(y) = \langle \sum_{i+j=4m+1} f^*(z_i)\alpha_j(y), [M] \rangle$$

comes from a universal element in $H^{4m+1}(BPSp(3) \times BSpin)$. One can prove this by a method similar to the computation of generalized Wu classes in 2.4.3:

Let M^{2d} be a closed connected manifold of dimension $2d$, $z \in H^iM^{2d}$, and $Sq^I \in A^{d-i}$ with $I = (n, I')$, i.e. the multiindex I starts with n. Then we have for any $y \in H^dM^{2d}$ using the Cartan formula:

$$\langle zSq^I(y), [M] \rangle = \langle zSq^nSq^{I'}(y), [M] \rangle =$$
$$= \langle Sq^n(zSq^{I'}(y)) - \sum_{i=0}^{n-1} Sq^{n-i}(z)Sq^iSq^{I'}(y), [M] \rangle.$$

This shows with $\langle Sq^n(zSq^{I'}(y)), [M] \rangle = \langle v_n zSq^{I'}(y), [M] \rangle$ by a double induction on the length of the monomial I and the first index n of I, that

$$\langle zSq^I(y), [M] \rangle = \langle z'y, [M] \rangle,$$

where $z' \in H^d M^{2d}$ is of the form $z' = \sum_s \tilde{w}_s \alpha_s(z)$ with certain polynomials $\tilde{w}_s$ in Stiefel-Whitney classes and elements $\alpha_s \in A^*$.

This ends the proof of the second statement. ■

5.3.4 Corollary: *Let $N^{8m+10} \to M^{8m+2}$ be a $PSp(3)$-$\mathbb{H}P^2$-bundle of closed 1-connected Spin-manifolds. If the $PSp(3)$-bundle characteristic class $f^*t_3 \in H^3 M^{8m+2}$ and all odd-dimensional Stiefel-Whitney classes of M^{8m+2} vanish, then it holds*

$$K_\phi(N^{8m+10}) = K_{\phi'}(M^{8m+2})$$

for all choices of Brown-Peterson-Kervaire invariants K_ϕ and $K_{\phi'}$ in dimension $8m+10$ and $8m+2$, respectively.

Proof: This follows from $v_{\phi,\phi'} = \sum_s f^*(\tilde{z}_s)\nu_M^*(\tilde{w}_s)$ since by the assumptions the subrings $f^*(H^*BPSp(3)) \subset H^*M^{8m+2}$ and $\nu_M^*(H^*BSpin) \subset H^*M^{8m+2}$ are concentrated in even dimensions, whereas $v_{\phi,\phi'}$ lives in odd dimension $4m+1$. ■

5.3.5 *Remarks:*
(i) We recall from 4.1.2 that $H^*BPSp(3) = \mathbb{Z}/2[t_2, t_3, t_8, t_{12}]$ and $t_3 = Sq^1 t_2$. In particular, the first condition in 5.3.4 is satisfied for $f^*t_2 = 0$, which is equivalent to the existence of a lifting from the structure group $PSp(3)$ of the fibre bundle to the full symplectic group $Sp(3)$. In analogy to the case of *Spin*-structures on oriented manifolds, this can be seen by the principal fibration

$$B\mathbb{Z}/2 \longrightarrow BSp(3) \longrightarrow BPSp(3)$$

which must have $t_2 : BPSp(3) \to K_2$ as classifying map because $BSp(3)$ is 3-connected.

(ii) In fact, this result also follows from theorem 2.2.4 of Ochanine: By the assumptions it is not difficult to see that M^{8m+2}, N^{8m+10} belong to $C_* \subset \Omega_*^{Spin}$, where all Brown-Kervaire invariants coincide with the Ochanine k-invariant, being multiplicative (3.2.11). A more interesting application of 5.2.2 will be given in 10.3.5 for $BO\langle 8\rangle$-manifolds, showing that an analogous result holds. But there, Ochanine's methods [48] in the proof of 2.2.4 seem not to work out for $BO\langle 8\rangle$-manifolds.

5.3.6 At last, we consider the dependence of the critical term $x^2p^*\epsilon(y) = x^2p^*(v_{\phi,\phi'}y)$ in 5.3.3 on the choice of the Brown-Peterson-Kervaire invariants K_ϕ and $K_{\phi'}$, i.e. on the homotopies H and H' (and also H^L, H^R). Looking at the definition of ϵ_{H,H',H^L,H^R} in the proof of 5.2.2, we obtain the following table:

homotopy:	change:	contribution to $[\epsilon_{H,H',H^L,H^R}]$:
H	$[A, \Omega C]$	$\mu^*[A, \Omega C]$
H'	$[A', \Omega C']$	$(\Omega_{(2)}\nu)_*(1 \wedge [A', \Omega C'])$
H^L	$[E \wedge A', \Omega B]$	$(\Omega b)_*[E \wedge A', \Omega B]$
H^R	$[E \wedge B', \Omega C]$	$(1 \wedge a')^*[E \wedge B', \Omega C]$

Strictly speaking, this is only true if ΩC is homotopy commutative, as we do not want to specify the way of glueing the contributions to $[\epsilon_{H,H',H^L,H^R}]$. In our applications, C is an Eilenberg-MacLane space and we have no problems with commutativity. We see, that the

choice of H^L and H^R for the product formula does not play any role, as the contributions by changing H^L vanish in $coker[\Omega b]$ and those of changing H^R vanish on $ker[a']$.

In our case 5.3.3, we see that $\epsilon_{H,H'}(x,y) = x^2p^*(\sum f^*(z_i)\alpha_j(y))$ depends on the choice of H and H' as follows:

$$\begin{array}{lcllcl} \epsilon_{H+\delta,H'}(x,y) & = & x^2p^*(\sum f^*(z_i)\alpha_j(y)) + \delta(xp^*y) & \text{for } \delta \in [K_{4m+5}, K_{8m+10}] & = & A^{4m+5} \\ \epsilon_{H,H'+\delta'}(x,y) & = & x^2p^*(\sum f^*(z_i)\alpha_j(y) + \delta'(y)) & \text{for } \delta' \in [K_{4m+1}, K_{8m+2}] & = & A^{4m+1} \end{array}$$

Thus we get $v_{\phi,\phi'+\delta'} = v_{\phi,\phi'} + v_{\delta'}$, with $v_{\delta'} \in H^{4m+1}M^{8m+2}$ the generalized Wu class associated to δ', see 2.4.2. The dependence of $v_{\phi+\delta,\phi'}$ on δ is more complicated: One has to expand $\delta(xp^*y)$ as

$$\delta(xp^*y) = x^2p^*\delta_0(y) + xp^*\delta_1(y) + p^*\delta_2(y)$$

which is possible by the Leray-Hirsch theorem, where $\delta_i \in H^*BPSp(3) \otimes A^*$. Then one proceeds as in the proof of 5.3.3 to represent δ_0 by multiplication with an element $u_{\delta_0} \in H^*(BPSp(3) \times BSpin)$ and obtains $v_{\phi+\delta,\phi'} = v_{\phi,\phi'} + u_{\delta_0}$.

The problem of the existence of Brown-Peterson-Kervaire invariants behaving multiplicatively for $PSp(3)$-$\mathbb{H}P^2$-bundles is by 5.3.3 reduced to the question, if there exists a choice of homotopies H and H', such that $\langle \phi'(v_{\phi,\phi'}), [M^{8m+2}] \rangle$ vanishes for all these bundles. Thus it is natural to ask, if there exists a choice of H, H' such that $v_{\phi,\phi'}$ itself is universally zero, i.e. in $H^{4m+1}(BPSp(3) \times BSpin)$. Unfortunately, the freedom of choosing H, H' is in general not large enough to get all elements in $H^{4m+1}(BPSp(3) \times BSpin)$ as $v_{\delta'} + u_{\delta_0}$, which would be sufficient for this (I examined this by lenghty computations in dimension 34).

6 The Product Formula of Kristensen

By the previous section there is a deviation $\phi'(v_{\phi,\phi'})[\ ]$ to multiplicativity with $v_{\phi,\phi'} \in H^*(BPSp(3) \times BSpin)$, which is obtained by glueing together the homotopies H, H', H^L and H^R. There exists a multiplicative Brown-Peterson secondary cohomology operation (i.e., $\phi(xp^*y) = x^2p^*\phi'(y)$), if there is a choice of H, H' such that $\phi'(v_{\phi,\phi'})[\,]$ vanishes. It seems to be very hard (if not impossible) to decide by standard homotopy theory, if the affine set of all possible $v_{\phi,\phi'}$ contains 0, because homotopies are so flabby, and also the spaces and maps (and the 'reasons' why they are homotopic) involved have no simple structure.

In this section we give a survey to another approach to secondary cohomology operations which is due to Kristensen. Using cochain operations to represent cohomology operations, Kristensen obtained sum and product formulas in a series of papers [33], [34] and [35], see in particular [37] for a short survey on his product formula. Cochain operations seem to provide the algebraic analogue of homotopies. They form an infinite dimensional $\mathbb{Z}/2$-vector space carrying a non-linear composition operation, and for some of them there exist explicit and manageable expressions in terms of $\cup_i$-products, which explains the success of Kristensen's method.

With this method of cochain operations, we will finally compute $\phi(xp^*y)$ in section 7 using special cochain operations (=homotopies) due to Kristensen, in fact showing that the primary operation ϵ obtained by 'glueing them together' does not vanish for general spaces, but contributes $v_{\phi,\phi'} = 0$ in our case of $PSp(3)$-$\mathbb{H}P^2$-bundles of 1-connected $Spin$-manifolds.

6.1 Cochain Operations

6.1.1 Kristensen worked in the category of simplicial sets which is no restriction because its homotopy category is equivalent to the homotopy category of topological spaces, and used cochain operations to represent secondary cohomology operations. A *cochain operation* $a = (a_k)_{k\in\mathbb{N}}$ of degree $n \in \mathbb{N}$ is a series of natural transformations $a_k : C^k(\quad) \to C^{k+n}(\quad)$ of the normalized cochain functor for simplicial sets (coefficients are always $\mathbb{Z}/2$). The a_k need neither to be linear nor to commute with the coboundary $\delta : C^k(\quad) \to C^{k+1}(\quad)$. We denote the graded $\mathbb{Z}/2$-vector space of these cochain operations by $\mathcal{O}^*$; as an example, the coboundary δ itself is a cochain operation in $\mathcal{O}^1$. As we use *normalized* cochains (i.e. $c \in C^k(X) = Hom(C_k(X), \mathbb{Z}/2)$ vanishes on all degenerate simplices of X), we have $a(0) = 0$ for all $a \in \mathcal{O}^*$ which follows by naturality from the vanishing of the normalized cochains of the simplicial point. Kristensen defined a differential Δ of degree 1 in $\mathcal{O}^*$ by $(\Delta a)_k := \delta a_k + a_{k+1}\delta$ (here, we use $a(0) = 0$), and showed the following:

6.1.2 Theorem (Kristensen [33]): *Let $a \in \mathcal{O}^n$ with $\Delta a = 0$ and define a cohomology operation $[a]$ of degree n in each dimension k by $[a]([x]) := [a(x)]$ for all $x \in C^kX$ with $\delta x = 0$, then $[a]$ is well-defined and stable. This gives an isomorphism*

$$H(\mathcal{O}^*, \Delta) = A^*.$$

6.1.3 This isomorphism is also compatible with composition, but in contrast to the Steenrod algebra A^*, the cochain operations $\mathcal{O}^*$ do not built an algebra because in general its elements consist of non-linear mappings and the composition is thus not right distributive. For example, by using a system of cup-i products (see [46], [33])

$$\cup_i : C^n X \times C^m X \longrightarrow C^{n+m-i} X,$$

one defines cochain operations $sq^i \in \mathcal{O}^i$ as

$$(sq^i)_k(x) := x \cup_{k-i} x + x \cup_{k-i+1} \delta x, \quad x \in C^k X,$$

which give the Steenrod squares $Sq^i = [sq^i]$. While the Sq^i are linear they are induced from quadratic maps sq^i.

6.1.4 Kristensen proved also an r-variable version of the above Theorem

$$H(\mathcal{O}^{*(r)}, \Delta^{(r)}) = \bigoplus_r A^*,$$

where a cochain operation a of degree n in r variables is a series of natural transformations $a_k : C^k(\ \ ) \times .. \times C^k(\ \ ) \to C^{k+n}(\ \ )$ and the differential $\Delta^{(r)}$ is defined by $(\Delta^{(r)} a)(x_1, .., x_r) := \delta a(x_1, .., x_r) + a(\delta x_1, .., \delta x_r)$. As an application, for each $a \in \mathcal{O}^n$ with $\Delta a = 0$ there exists an r-variable cochain operation $d_a \in \mathcal{O}^{n-1\,(r)}$ with

$$(\Delta^{(r)} d_a)(x_1, .., x_r) = a(\sum_{i=1}^r x_i) - \sum_{i=1}^r a(x_i),$$

because the left side measures the deviation of a from linearity which vanishes in $\bigoplus_r A^n$ since $[a] \in A^n$ is linear. It is also possible to normalize d_a in the sense that $d_a(x_1, .., x_r) = 0$ if all but one x_i vanish. Moreover, a similar formula exists for arbitrary $a \in \mathcal{O}^n$, which can be seen in the following way (see [31]): The cochain operation $\Delta a \in \mathcal{O}^{n+1}$ is a cycle, so there exists $d_{\Delta a} \in \mathcal{O}^{n\,(r)}$ as above. Applying $\Delta^{(r)}$ shows that the cochain operation $(x_1, .., x_r) \mapsto a(\sum_{i=1}^r x_i) + \sum_{i=1}^r a(x_i) + d_{\Delta a}(x_1, .., x_r)$ is a cycle in $\mathcal{O}^{n\,(r)}$, hence giving a cohomology operation $(\alpha_1, .., \alpha_r) \in \bigoplus_r A^*$. Inserting zero for all $x_1, .., x_r$ with the exception of x_i shows that $\alpha_i = 0$. Thus, there exists a cochain operation $d_a \in \mathcal{O}^{n-1\,(r)}$ with

$$(\Delta^{(r)} d_a)(x_1, .., x_r) = a(\sum_{i=1}^r x_i) + \sum_{i=1}^r a(x_i) + d_{\Delta a}(x_1, .., x_r).$$

For example, we obtain for cocycles $x_1, .., x_r$ the sum formula

$$a(\sum_{i=1}^r x_i) = \sum_{i=1}^r a(x_i) + d_{\Delta a}(x_1, .., x_r) + \delta d_a(x_1, .., x_r).$$

6.2 Secondary Cohomology Operations

6.2.1 We come now to the representation of secondary cohomology operations by cochain operations, see [33]. We start with a relation $\sum_{i=1}^{s} \alpha_i\beta_i = \gamma$ of degree n in the Steenrod algebra, where $\alpha_i \in A^{n_i}$, $\beta_i \in A^{m_i}$ with $n_i + m_i = n$ for $i = 1, .., s$, and $\gamma \in A^n$. If we write α_i, β_i and γ as sums of admissible monomials in the Sq^k, then the corresponding expressions with Sq^k replaced by sq^k are representing cochain operations a_i, b_i and c. The cochain operation $r := \sum_{i=1}^{s} a_i b_i + c \in \mathcal{O}^n$ has the property $\Delta r = 0$ and $[r] = 0$, thus there exists a cochain operation $R \in \mathcal{O}^{n-1}$ with $\Delta R = r$. Now, let $[x] \in H^k X$ be in the kernel of all the β_i, and $k < excess(\gamma)$. Since $[b_i(x)] = 0$ there are $w_i \in C^{k+m_i-1}X$ with $\delta w_i = b_i(x)$, and furthermore $c(x) = 0$ by the definition of the excess and of the sq^i. Consider

$$\phi(x) := R(x) + \sum_{i=1}^{s} a_i(w_i),$$

then $\phi(x) \in C^{k+n-1}X$ and a short computation gives $\delta\phi(x) = 0$. Kristensen shows that choosing other w_i' with $\delta w_i' = b_i(x)$ or another $x' \in [x]$ changes the cohomology class $[\phi(x)] \in H^{k+n-1}X$ by elements in $\sum_{i=1}^{s} im(\alpha_i : H^{k+m_i-1}X \to H^{k+n-1}X)$. Thus for $k < excess(\gamma)$ we have defined a secondary cohomology operation

$$\phi : ker(H^k X \overset{(\beta_1,..,\beta_s)}{\longrightarrow} \bigoplus_{i=1}^{s} H^{k+m_i}X) \longrightarrow coker(\bigoplus_{i=1}^{s} H^{k+m_i-1}X \overset{\alpha_1+..+\alpha_s}{\longrightarrow} H^{k+n-1}X),$$

which is stable if γ vanishes. Furthermore, a different choice of R' with $\Delta R' = r$ is given by $e := R' - R \in \mathcal{O}^{n-1}$ with $\Delta e = 0$, and then one has $\phi' - \phi = [e] \in A^{n-1}$ for the corresponding secondary cohomology operations. In the following we say that ϕ is associated to the 'relation'

$$\rho := \sum_{i=1}^{s} \alpha_i \otimes \beta_i \in A^* \otimes A^*$$

and is defined in dimensions $k < excess(\mu(\rho))$, where $\mu : A^* \otimes A^* \to A^*$ denotes the product in the Steenrod algebra. These operations are equivalent to those constructed in the topological category from $\sum_{i=1}^{s} \alpha_i\beta_i = \gamma$ ([33]).

6.2.2 We remark that in the case of $|x| < excess(\beta_i)$ for all $i = 1..s$ we have $b_i(x) = 0$ and can thus make the canonical choice $w_i = 0$. But also r vanishes then in this dimension and R can be chosen with $R(x) = 0$ (one can easily see this in the topological category by choosing the zero map between the appropriate Eilenberg-MacLane spaces as a representative of β_i). In particular, one has then $\phi([x]) = 0$. See also the Theorem on p.76 in [33] for a sharper statement.

6.3 The Product Formula of Kristensen

6.3.1 Now we want to compute a product formula for the operation ϕ. The product formula for a stable primary cohomology operation is given by the coproduct $\psi : A^* \to A^* \otimes A^*$ in the Steenrod algebra, and for relations we have the coproduct

$$\psi^{(2)} := (1 \otimes t \otimes 1)(\psi \otimes \psi) : A^* \otimes A^* \to A^* \otimes A^* \otimes A^* \otimes A^*$$

with the Hopf algebra property $(\mu \otimes \mu)\psi^{(2)} = \psi\mu$. Suppose now that we have

$$\psi^{(2)}\rho = \sum_{n\in N} \rho'_n \otimes \epsilon''_n + \sum_{m\in M} \epsilon'_m \otimes \rho''_m$$

with $\rho'_n, \rho''_m, \epsilon'_m, \epsilon''_n \in A^* \otimes A^*$, where we regard the ρ'_n, ρ''_m as relations. This decomposition is designed for the case that $0 = \beta_i([x][y]) = \sum_{j\in B_i} \beta'_{ij}([x])\beta''_{ij}([y])$ holds true because in each summand at least one factor is zero, which Kristensen calls the *complementary case*; his method works only under this condition (see [34] and [35]).

6.3.2 A first conjecture would be that (on the common domain of definition and modulo the total indeterminacy) one has then $\phi([x][y]) = \sum_{n\in N} \phi'_n([x])\delta''_n([y]) + \sum_{m\in M} \delta'_m([x])\phi''_m([y])$ with secondary cohomology operations ϕ'_n, ϕ''_m associated to ρ'_n, ρ''_m, and $\delta''_n := \mu(\epsilon''_n)$, $\delta'_m := \mu(\epsilon'_m)$. But the situation is more complicated because a relation gives in general more than one secondary operation (which differ by stable primary operations), so this equation can be only true if one adds at the left side $\epsilon([x] \otimes [y])$ with a certain primary cohomology operation $\epsilon \in A^* \otimes A^*$, whose computation was the main problem in the product formula of Kristensen for ϕ.

6.3.3 For the computation of $\phi([x][y]) = [R(xy) + \sum_{i=1}^{s} b_i(w_i)]$ we need two parts: Firstly, an expansion of $R(xy)$ (with xy meaning the cup product of cochains), and secondly, cochains w_i with $\delta w_i = b_i(xy)$ which are given in terms of the complementarity condition.

The second problem leads to *cochain operations* $\mathcal{Q}^*$ *of the second kind*; these are series $G = (G_{i,j})_{i,j\in\mathbb{N}}$ of natural transformations $G_{i,j} : C^i(\quad) \times C^j(\quad) \to C^{i+j+n}(\quad)$, and one has a differential $\nabla : \mathcal{Q}^* \to \mathcal{Q}^{*+1}$ by $(\nabla G)(x,y) := \delta G(x,y) + G(\delta x, y) + G(x, \delta y)$. Kristensen proves in [34] that

$$H(\mathcal{Q}^*, \nabla) = A^* \otimes A^*.$$

As an application of this Theorem, let $\alpha \in A^n$ and the terms in the coproduct $\psi\alpha = \sum \alpha'_k \otimes \alpha''_k$ be represented by cochain operations a, a'_k and a''_k. Then there exists a cochain transformation $T_a \in \mathcal{Q}^{n-1}$ of the second kind measuring the deviation from the Cartan formula on the cochain level,

$$\nabla T_a(x,y) = a(xy) + \sum a'_k(x)a''_k(y) + d_a(\delta x\, y, x\delta y) + |x| d_a(x\delta y, x\delta y).$$

In particular, we obtain for cocycles x, y that

$$a(xy) = \sum a'_k(x)a''_k(y) + \delta T_a(x,y).$$

For the proof, one computes that ∇ of the right side is zero (in order to get this, one has to include the linearity defects d_a) and that it represents $\alpha([x][y]) + \sum \alpha'_k([x])\alpha''_k([y]) = 0$. Now

we can construct our w_i with $\delta w_i = b_i(xy)$ as

$$w_i := \sum_{j\in B_i'} w_{ij}' b_{ij}''(y) + \sum_{j\in B_i''} b_{ij}'(x) w_{ij}'' + T_{b_i}(x,y),$$

where $\delta w_{ij}' = b_{ij}'(x)$ for $j \in B_i'$, $\delta w_{ij}'' = b_{ij}''(y)$ for $j \in B_i''$, and $B_i = B_i' \sqcup B_i''$.

6.3.4 Attacking the first problem, Kristensen defines the following cochain operation $A \in \mathcal{Q}^{n-1}$ of the second kind:

$$A(x,y) := R(x\cdot y) + C_R(x,y) + T_r(x,y).$$

Here, $C_R(x,y) := \sum_{n\in N} R_n'(x) d_n''(y) + \sum_{m\in M} d_m'(x) R_m''(y)$ is the 'Cartan-term', where the cochain operations $\Delta R = r$, $\Delta R_n' = r_n'$ and $\Delta R_m'' = r_m''$ represent in $\mathcal{O}^*$ the relations $\rho, \rho_n', \rho_m'' \in A^*\otimes A^*$ and d_n'', d_m' represent $\delta_n'', \delta_m' \in A^*$. The cochain operation $T_r \in \mathcal{Q}^{n-1}$ is characterized as T_a above by the property

$$\nabla T_r(x,y) = r(xy) + \sum_{n\in N} r_n'(x) d_m''(y) + \sum_{m\in M} d_m'(x) r_m''(y) + d_r(\delta x\, y, x\delta y) + |x| d_r(x\delta y, x\delta y),$$

hence measures the Cartan defect of the relation r on the cochain level. Using the decomposition $r = \sum_{i=1}^s a_i b_i + c$, it can be related to the cochain operations T_{a_i}, T_{b_i} and T_c by lengthy formulas which can be found in [31], [35], or the appendix of [37] (we do not use them here). Kristensen shows in [34] that

$$\nabla A(x,y) = R(\delta x\, y + x\delta y) + R(\delta x\, y) + R(x\delta y) + d_r(\delta x\, y, x\delta y) + |x| d_r(x\delta y, x\delta y) =$$

$$= \Delta d_R(\delta x\, y, x\delta y) + |x| d_r(x\delta y, x\delta y);$$

thus for cocycles x, y, one obtains also a cocycle $A(x,y)$. In particular, one gets a primary operation

$$\epsilon \in A^*\otimes A^*, \qquad \epsilon([x]\otimes[y]) := [A(x,y)],$$

and Kristensen proves

6.3.5 Theorem (Kristensen [34], [37]): *Under the complementarity assumptions on the cohomology classes $[x]$, $[y]$ and $\psi^{(2)}\rho = \sum_{n\in N} \rho_n' \otimes \epsilon_n'' + \sum_{m\in M} \epsilon_m' \otimes \rho_m''$, we have (on the common domain of definition and modulo the total indeterminacy)*

$$\phi([x][y]) = \sum_{n\in N} \phi_n'([x]) \delta_n''([y]) + \sum_{m\in M} \delta_m'([x]) \phi_m''([y]) + \epsilon([x]\otimes[y])$$

with secondary cohomology operations ϕ, ϕ_n', ϕ_m'' associated to the relations ρ, ρ_n', ρ_m''; with $\delta_n'' := \mu(\epsilon_n'')$, $\delta_m' := \mu(\epsilon_m')$; and with $\epsilon \in A^*\otimes A^*$ constructed as above.

6.3.6 *Remark:* We used for $A(x,y)$ the definition in [37]. In [34], there was also included a term $D_R \in \mathcal{Q}^{n-1}$ in the definition of $A(x,y)$ (which is there called $E(x,y)$). There, D_R has the property that

$$\nabla D_R(x,y) = R(\delta x\, y + x\delta y) + R(\delta x\, y) + R(x\delta y) + d_r(\delta x\, y, x\delta y) + |x| d_r(x\delta y, x\delta y),$$

giving just $\nabla E = \nabla(A + D_R) = 0$ and showing by $H(\mathcal{Q}^*, \nabla) = A^*\otimes A^*$ directly that E represents a primary operation.

6.3.7 In the application of this formula, one has the problem that the term ϵ is not effectively computed by the other data. This problem was later solved by Kristensen, see [35] and [37]. In particular, he gave an explicit formula for the following triple series of relations, which are linear combinations of the Adem relations:

$$\rho^k_{ab} := \sum_{j\in\mathbb{Z}}(\tbinom{b-1-j}{k+b-a-2j} + \tbinom{b-1-j}{j+b-a})Sq^{k-j}\otimes Sq^j, \qquad k,a,b\in\mathbb{Z}.$$

Here we use the conventions $Sq^k = 0$ for $k<0$ and $\binom{n}{k} = \frac{n(n-1)..(n-k+1)}{k!}$ for $k\geq 0$, $\binom{n}{k}=0$ for $k<0$. Applying the Cartan formula and reindexing gives a decomposition

$$\psi^{(2)}\rho^k_{ab} = \sum_{i,j\in\mathbb{Z}} \rho^{k-i-j}_{a-2i,b-i}\otimes(Sq^j\otimes Sq^i) + \sum_{i,j\in\mathbb{Z}}(Sq^j\otimes Sq^i)\otimes\rho^{k-i-j}_{a-j,b-i}.$$

6.3.8 Theorem (Kristensen [35], [37]): *There exists a choice of cochain operations R^k_{ab} for the relations ρ^k_{ab} such that the primary term ϵ in the product formula for the associated secondary cohomology operations is given by*

$$\epsilon^k_{ab} = (Sq^1\otimes(Sq^2Sq^1+Sq^3))\cdot\psi\left(\sum_{j\in\mathbb{Z}}(\tbinom{b-1-j}{k+b-a-2j}+\tbinom{b-1-j}{j+b-a})(Sq^{k-j-3}Sq^{j-2}+Sq^{k-j-2}Sq^{j-3})\right).$$

The cochain operations R^k_{ab} are unique up to addition of boundaries of the Kristensen differential Δ.

We only mention that his proof uses (see [37]):

- The Eilenberg-MacLane complex $K(\mathbb{Z}/2,1)$, which is a simplicial $\mathbb{Z}/2$-vector space with zero differential in its normalized cochain complex, and has thus the only non-zero cochain $u^n\in C^nK(\mathbb{Z}/2,1)$ in each dimension where u is the fundamental cocycle. Then one can relate the action of $A^k_{ab}\in\mathcal{Q}^{k-1}$ on (u^n,u^m) to the action of R^k_{ab} and T^k_{ab}.
- The cobar resolution $\bar{A}^{\otimes n}$ of the Steenrod algebra, which has homology $\Lambda(Q_0,Q_1,...)$ where in particular $Q_0 = Sq^1$ and $Q_1 = Sq^2Sq^1+Sq^3$. Then A^k_{ab} gives an element in $\bar{A}^{\otimes 2}$ whose boundary in $\bar{A}^{\otimes 3}$ can again be expressed in terms of A^k_{ab}.
- Special systems of cochain operations for the T_r with good combinatorial properties, whose existence was proved in [31]. See also [35] and the appendix of [37].

7 The Ochanine k-Invariant is a Brown-Kervaire Invariant

7.1 Proof of the Main Theorem

We prove now the main result:

7.1.1 Theorem: *In each dimension $8m+2$, there exists a Brown-Peterson-Kervaire invariant K_ϕ which is equal to the Ochanine k-invariant.*

Proof: By 4.2.7 and 4.2.8, we have to show that for each $m \in \mathbb{N}$, there exists a Brown-Peterson-Kervaire invariant $K_{\phi_{m+1}}$ in dimension $8m+10$ and an invariant K' in dimension $8m+2$ which satisfy the multiplicativity property (i') in 4.2.8 (actually we will show that K' is also a Brown-Peterson-Kervaire invariant K_{ϕ_m}). By 4.3.3, we have for each $PSp(3)$-$\mathbb{H}P^2$-bundle of closed 1-connected *Spin*-manifolds $p : N^{8m+10} \to M^{8m+2}$, that $K_{\phi_{m+1}}(N^{8m+10}) = \mathrm{Arf}(q_0)$ where $q_0 : H^{4m+1}M^{8m+2} \to \mathbb{Z}/2$ denotes $q_0(y') := \phi_{m+1}(x \cdot p^*y')[N^{8m+10}]$. We compute now the product formula for $\phi_{m+1}(xy)$, $y := p^*y'$, by Kristensen's theory, where ϕ_{m+1} is associated to

$$\rho_{m+1} := Sq^2 \otimes Sq^{4m+4} + Sq^1 \otimes Sq^{4m+4}Sq^1.$$

We first have to check the complementarity conditions for xy. The summands in

$$Sq^{4m+4}(xy) = \sum_{i=0..4m+4} Sq^i x \cdot Sq^{4m+4-i} y$$

and

$$Sq^{4m+4}Sq^1(xy) = \sum_{i=0..4m+4} Sq^i Sq^1 x \cdot Sq^{4m+4-i} y + \sum_{i=0..4m+4} Sq^i x \cdot Sq^{4m+4-i} Sq^1 y$$

are all zero, in detail:

- For $i = 0, 1, 2$ in the first and the second sum, and $i = 0, 1$ in the third sum, because then the dimension $|y| = 4m+1$ is smaller than the excess of the operation acting on y.
- For $i = 5, .., 4m+4$ in the first and the third sum, and $i = 6, .., 4m+4$ in the second sum, because then the dimension $|x| = 4$ is smaller than the excess of the operation acting on x.
- For $i = 3, 4$ in the first sum because then $Sq^{4m+1}y = p^*Sq^1(Sq^{4m}y') = 0$ since M^{8m+2} is *Spin*, respectively $Sq^{4m}y = p^*Sq^{4m}y' = 0$ since M^{8m+2} is 1-connected.
- For $i = 3, 4, 5$ in the second sum because then $Sq^i Sq^1 x = 0$; here we use that $x \in H^4 N^{8m+10}$ is the pullback of the universal Leray-Hirsch generator $x \in H^4 E$ which satisfies $Sq^1 x = 0$ (see 4.1.3).

- For $i = 2,3,4$ in the third sum because then $Sq^{4m+2}Sq^1y \in p^*H^{8m+4}M^{8m+2} = 0$ and $Sq^{4m+1}Sq^1y \in p^*H^{8m+3}M^{8m+2} = 0$, respectively $Sq^{4m}Sq^1y = p^*Sq^{4m}Sq^1y' = 0$ since M^{8m+2} is $Spin$.

According to these facts we choose our splitting of

$$\psi^{(2)}\rho_{m+1} = \sum_{\substack{i=0,..,4m+4\\ j=0,1,2}} \sigma^1_{ji} + \sum_{\substack{i=0,..,4m+4\\ j=0,1}} \sigma^2_{ji} + \sum_{\substack{i=0,..,4m+4\\ j=0,1}} \sigma^3_{ji}$$

with

$$\begin{array}{lll} \sigma^1_{ji} := (Sq^j \otimes Sq^i) & \otimes & (Sq^{2-j} \otimes Sq^{4m+4-i}) \\ \sigma^2_{ji} := (Sq^j \otimes Sq^iSq^1) & \otimes & (Sq^{1-j} \otimes Sq^{4m+4-i}) \\ \sigma^3_{ji} := (Sq^j \otimes Sq^i) & \otimes & (Sq^{1-j} \otimes Sq^{4m+4-i}Sq^1) \end{array}$$

in the following way:

$$\psi^{(2)}\rho_{m+1} = (\sum_{\substack{i,j\\ i\neq 3,4}} \sigma^1_{ji} + \sum_{\substack{i,j\\ i\neq 3,4,5}} \sigma^2_{ji} + \sum_{\substack{i,j\\ i\neq 2,3,4}} \sigma^3_{ji}) + (\sum_{\substack{i,j\\ i=3,4}} \sigma^1_{ji} + \sum_{\substack{i,j\\ i=3,4,5}} \sigma^2_{ji} + \sum_{\substack{i,j\\ i=2,3,4}} \sigma^3_{ji}),$$

where we denote the first bracket by Σ_1 and the second by Σ_2. Now the summands of Σ_1, which we consider as $\rho' \otimes \epsilon''$ or $\epsilon' \otimes \rho''$ according to that x or y gives the 'reason' for being zero, contribute all with $\phi'(x)\delta''(y) = 0$ or $\delta'(x)\phi''(y) = 0$ to the sum formula for $\phi(xy)$. This holds because the kernel condition for ϕ' respectively ϕ'' is satisfied by the fact that the excess is larger than the dimension (see 6.2.2). We say that Σ_1 consists of *trivial terms*. In contrast to this, the 18 summands in Σ_2 do not vanish by this reason; we call them *critical terms*. We show now that in our situation 16 of these terms vanish, with the remaining two terms giving exactly $x^2 \cdot p^*\phi_m(y)$.

We remark that in the case where the secondary operation of the one side of a term is defined and the primary operation of the other side of the term vanishes, the whole term (including its undeterminacy) vanishes. This applies to the terms σ^1_{13}, σ^1_{23}, σ^1_{14}, σ^1_{24}, σ^3_{13}, σ^3_{14}, which we view as $\epsilon' \otimes \rho''$; and to σ^2_{03}, σ^2_{13}, σ^2_{04}, σ^2_{14}, σ^2_{05}, which we view as $\rho' \otimes \epsilon''$. Furthermore, if a term $\epsilon' \otimes \rho''$ has the property that the degree of the relation satisfies $|\rho''| > 4m+2$, one gets $\phi''(y) \subset p^*\phi''(y') = 0$ because $\phi''(y') \subset H^{4m+|\rho''|}M^{8m+2} = 0$. This applies to σ^1_{03}, σ^3_{02}, σ^3_{12} and σ^3_{03}.

It remain the terms σ^2_{15}, σ^1_{04} and σ^3_{04}. We consider first

$$\sigma^2_{15} = (Sq^1 \otimes Sq^5Sq^1) \otimes (Sq^0 \otimes Sq^{4m-1}) =: \rho' \otimes \epsilon''.$$

The associated secondary operation ϕ' has the property $0 \in \phi'(x)$, which can be seen by $\phi'(\ ) \subset \phi(Sq^1(\ ))$ and $Sq^1x = 0$, where ϕ is associated to the relation $Sq^1Sq^5 = 0$. Moreover, the undeterminacy of $\phi'(x)$ is $im(Sq^1)$; thus we get for $z \in \phi'(x)$ that

$$z \cdot Sq^{4m-1}y = z \cdot Sq^1Sq^{4m-2}y = Sq^1(z \cdot Sq^{4m-2}y) = v_1(z \cdot Sq^{4m-2}y) = 0$$

showing that the term $\phi'(x)\delta''(y)$ (including its undeterminacy) vanishes. The sum of the last two terms can be factorized as

$$\sigma^1_{04} + \sigma^3_{04} = (Sq^0 \otimes Sq^4) \otimes (Sq^2 \otimes Sq^{4m} + Sq^1 \otimes Sq^{4m}Sq^1) = (1 \otimes Sq^4) \otimes \rho_m.$$

With the product formula 6.3.5 of Kristensen, we have proved

$$\phi_{m+1}(xy) = Sq^4(x)\phi_m(y) + \epsilon(x \otimes y) = x^2 \cdot p^*\phi_m(y') + \epsilon(x \otimes y),$$

which we independently obtained by our product formula in 5.3.2.

Now, we have to compute the primary term ϵ, which comes from the cochain operation $A_R(x,y) = R(x \cdot y) + C_R(x,y) + T_r(x,y)$ in 6.3.4, where r and R have to be replaced by $r_{m+1} := sq^{4m+6} + sq^2 sq^{4m+4} + sq^1 sq^{4m+4} sq^1$ and $\Delta R_{m+1} = r_{m+1}$. We note that the Kristensen relations $\rho^{a+b}_{2b,b}$ in 6.3.7 are nothing but the Adem relations written as

$$\rho^{a+b}_{2b,b} = Sq^a \otimes Sq^b + \sum_{j\in\mathbb{Z}} \tbinom{b-1-j}{a-2j} Sq^{a+b-j} \otimes Sq^j$$

and the corresponding primary terms $\epsilon^{a+b}_{2b,b}$ in 6.3.8 are ($Q_0 := Sq^1$, $Q_1 := Sq^2Sq^1 + Sq^3$)

$$\epsilon^{a+b}_{2b,b} = (Q_0 \otimes Q_1)\cdot\psi\left(Sq^{a-3}Sq^{b-2} + Sq^{a-2}Sq^{b-3} + \sum_{j\in\mathbb{Z}} \tbinom{b-1-j}{a-2j} (Sq^{a+b-j-3}Sq^{j-2} + Sq^{a+b-j-2}Sq^{j-3}) \right).$$

We need in particular the following relations and primary terms:

$$\begin{array}{lll} \dot\rho_n & := \rho^{2n+1}_{4n,2n} & = Sq^1 \otimes Sq^{2n} + Sq^{2n+1} \otimes 1, \\ \dot\epsilon_n & := \epsilon^{2n+1}_{4n,2n} & = (Q_0 \otimes Q_1)\psi(0) \\ & & = 0 \end{array}$$

$$\begin{array}{lll} \ddot\rho_m & := \rho^{4m+2}_{8m,4m} & = Sq^2 \otimes Sq^{4m} + Sq^{4m+2} \otimes 1 + Sq^{4m+1} \otimes Sq^1, \\ \ddot\epsilon_m & := \epsilon^{4m+2}_{8m,4m} & = (Q_0 \otimes Q_1)\psi(Sq^{4m-3}) \\ & & = \sum_{i=0}^{4m-3} Sq^1 Sq^i \otimes (Sq^2Sq^1 + Sq^3)Sq^{4m-3-i} \end{array}$$

and denote the special cochain operations $R^{a+b}_{2b,b}$ of Kristensen in 6.3.8 by

$$\begin{array}{llll} \dot R_n & := R^{2n+1}_{4n,2n} & , \Delta\dot R_n = \dot r_n & := sq^1 sq^{2n} + sq^{2n+1} \\ \ddot R_m & := R^{4m+2}_{8m,4m} & , \Delta\ddot R_m = \ddot r_m & := sq^2 sq^{4m} + sq^{4m+2} + sq^{4m+1} sq^1. \end{array}$$

Since our cochain operation r_{m+1} decomposes as $r_{m+1} = \ddot r_{m+1} + \dot r_{2m+2} sq^1$, we construct R_{m+1} as the linear combination

$$R_{m+1} := \ddot R_{m+1} + \dot R_{2m+2} sq^1$$

with the special system of Kristensen's cochain operations above. This works because of

$$\Delta(\ddot R_{m+1} + \dot R_{2m+2} sq^1) = \delta \ddot R_{m+1} + \delta \dot R_{2m+2} sq^1 + \ddot R_{m+1}\delta + \dot R_{2m+2} sq^1 \delta =$$

$$= \Delta(\ddot R_{m+1}) + \Delta(\dot R_{2m+2}) sq^1 = r_{m+1},$$

where we have used $\delta sq^1 - sq^1\delta = \Delta(sq^1) = 0$ and also the *left* distributivity of the cochain operation sq^1.

We claim now that this decomposition of r_{m+1} gives the following decomposition of our primary term $A_{R_{m+1}}(x,y)$ $(:= R_{m+1}(x \cdot y) + C_{R_{m+1}}(x,y) + T_{r_{m+1}}(x,y))$, if x, y are cocycles:

$$A_{R_{m+1}}(x,y) = A_{\ddot R_{m+1}}(x,y) + A_{\dot R_{2m+2}}(sq^1 x, y) + A_{\dot R_{2m+2}}(x, sq^1 y) + \delta(...).$$

Here and in the following, $\delta(...)$ denotes a coboundary which we do not want to write down explicitely in order to make the formulas more readable. In the end of the computation, we take the cohomology class of $A_{R_{m+1}}(x,y)$, so these terms $\delta(...)$ contribute zero.

To prove this decomposition of $A_{R_{m+1}}(x,y)$, we first consider $R_{m+1}(x\cdot y) = \ddot{R}_{m+1}(x\cdot y) + \dot{R}_{2m+2}sq^1(x\cdot y)$. By 6.3.3, we have $sq^1(xy) = (sq^1x)y + x(sq^1y) + \delta T_{sq^1}(x,y)$ for cocycles x, y, thus we get $\dot{R}_{2m+2}sq^1(x\cdot y) = \dot{R}_{2m+2}(sq^1x\cdot y) + \dot{R}_{2m+2}(x\cdot sq^1y) + \dot{R}_{2m+2}\delta T_{sq^1}(x,y) + d_{\Delta\dot{R}_{2m+2}}(sq^1x\cdot y, x\cdot sq^1y, \delta T_{sq^1}(x,y)) + \delta(...)$ according to 6.1.4. Moreover, $\dot{R}_{2m+2}\delta T_{sq^1}(x,y) = \dot{r}_{2m+2}T_{sq^1}(x,y) + \delta(...)$ by the definition of $\dot{R}_{2m+2}$. This shows

$$R_{m+1}(x\cdot y) = \ddot{R}_{m+1}(x\cdot y) + \dot{R}_{2m+2}(sq^1x\cdot y) + \dot{R}_{2m+2}(x\cdot sq^1y) +$$

$$+\dot{r}_{2m+2}T_{sq^1}(x,y) + d_{\dot{r}_{2m+2}}((sq^1x)y, x(sq^1y), \delta T_{sq^1}(x,y)) + \delta(...).$$

Next, we consider the Cartan term $C_{R_{m+1}}$. We obtained $\psi^{(2)}\rho_{m+1}$ in the beginning of the proof by applying the Cartan formula (i.e. the codiagonal ψ) to each Steenrod square Sq^k in $\rho_{m+1} = Sq^2\otimes Sq^{4m} + Sq^1\otimes Sq^{4m}Sq^1$. But in the same way, $\psi^{(2)}\rho^k_{a,b}$ in 6.3.8 is obtained. In particular, with our decomposition $r_{m+1} = \ddot{r}_{m+1} + \dot{r}_{2m+2}sq^1$ and with $\psi(Sq^1) = Sq^1\otimes 1 + 1\otimes Sq^1$ we obtain for the Cartan terms that

$$C_{R_{m+1}}(x,y) = C_{\ddot{R}_{m+1}}(x,y) + C_{\dot{R}_{2m+2}}(sq^1x,y) + C_{\dot{R}_{2m+2}}(x,sq^1y),$$

if we choose for the summands in $C_{R_{m+1}}(x,y)$ just the cochain operations which come from the Kristensen cochain cochain operations $R^k_{a,b}$ used in the definition of the right side $C_{\ddot{R}_{m+1}}(x,y) + C_{\dot{R}_{2m+2}}(sq^1x,y) + C_{\dot{R}_{2m+2}}(x,sq^1y)$. This fixes the 'zero-homotopies' which we used to define the secondary operations belonging to the summands σ^1_{ij}, σ^2_{ij}, σ^3_{ij} of $\psi^{(2)}\rho_{m+1}$; we have seen that these summands all vanish with the exception of $\sigma^1_{04}+\sigma^3_{04} = (1\otimes Sq^4)\otimes\rho_m$, where we have to use the special operation $R_m := \ddot{R}_m + \dot{R}_{2m}sq^1$, again.

At last, we consider the term $T_{r_{m+1}}$ measuring the Cartan defect of r_{m+1}. It is straightforward to see that $T_{a+b} = T_a + T_b$, so it remains to expand $T_{\dot{r}_{2m+2}sq^1}$. By the Cartan formula, we have $(Sq^1Sq^{4m+4} + Sq^{4m+5})(xy) = \sum\alpha'_i(x)\alpha''_i(y)$, where we do not need to make α'_i, α''_i explicit. Hence for cocycles x, y, it follows $\dot{r}_{2m+2}(xy) = \sum r'_i(x)r''_i(y) + \delta T_{\dot{r}_{2m+2}}(x,y)$ with r'_i, r''_i corresponding to α'_i, α''_i (see 6.3.3). Using 6.1.4, we get for cocycles x, y:

$$\begin{array}{rcl}
\dot{r}_{2m+2}sq^1(xy) & = & \dot{r}_{2m+2}((sq^1x)y + x(sq^1y) + \delta T_{sq^1}(x,y)) \\
& = & \dot{r}_{2m+2}((sq^1x)y) + \dot{r}_{2m+2}(x(sq^1y)) + \dot{r}_{2m+2}(\delta T_{sq^1}(x,y)) \\
& & +\delta d_{\dot{r}_{2m+2}}((sq^1x)y, x(sq^1y), \delta T_{sq^1}(x,y)) \\
& = & \sum r'_isq^1(x)r''_i(y) + \delta T_{\dot{r}_{2m+2}}(sq^1x,y) + \sum r'_i(x)r''_isq^1(y) + \delta T_{\dot{r}_{2m+2}}(x,sq^1y) \\
& & +\delta\dot{r}_{2m+2}T_{sq^1}(x,y) + \delta d_{\dot{r}_{2m+2}}((sq^1x)y, x(sq^1y), \delta T_{sq^1}(x,y))
\end{array}$$

showing that

$$T_{r_{m+1}}(x,y) = T_{\ddot{r}_{m+1}}(x,y) + T_{\dot{r}_{2m+2}}(sq^1x,y) + T_{\dot{r}_{2m+2}}(x,sq^1y) +$$

$$+\dot{r}_{2m+2}T_{sq^1}(x,y) + d_{\dot{r}_{2m+2}}((sq^1x)y, x(sq^1y), \delta T_{sq^1}(x,y)) + \delta(...).$$

This proves our claim for $A_{R_{m+1}}$ modulo coboundaries by taking the sum of the three expansions of $R_{m+1}(xy)$, $C_{R_{m+1}}(x,y)$ and $T_{r_{m+1}}(x,y)$. Hence, for cocycles x, y our primary term $\epsilon(x \otimes y)$ is given by the following combination of the Kristensen primary terms:

$$\epsilon(x \otimes y) = \ddot{\epsilon}_{m+1}(x \otimes y) + \dot{\epsilon}_{m+1}(Sq^1 x \otimes y) + \dot{\epsilon}_{m+1}(x \otimes Sq^1 y) =$$

$$= \sum_{i=0}^{4m+1} Sq^1 Sq^i x \cdot (Sq^2 Sq^1 + Sq^3) Sq^{4m+1-i} y.$$

Applied to our case of *Spin*-manifolds, the only term which can give a contribution has to contain the factor $x^2 = Sq^4 x$ since we are in the top dimension $8m+10$, but this term does not show up in the sum because of $Sq^1 Sq^i = 0$ for i odd. Summarizing our computation we have shown that

$$\phi_{m+1}(xp^* y) = x^2 p^* \phi_m(y')$$

where the secondary operations ϕ_{m+1} and ϕ_m are constructed by using Kristensen's special system of cochain operations. Now the proof is finished since

$$K_{m+1}(N^{8m+10}) = \mathrm{Arf}(\phi_{m+1}) = \mathrm{Arf}(y' \mapsto \phi_{m+1}(xp^* y')[N^{8m+10}])$$

$$= \mathrm{Arf}(y' \mapsto \phi_m(y')[M^{8m+2}]) = K_m(M^{8m+2}). \quad \blacksquare$$

7.2 Applications

As Brown-Kervaire invariants are defined by quadratic forms on the middle dimensional cohomology, we get from 7.1.1:

7.2.1 Corollary: *The Ochanine k-invariant vanishes for* $H^{4m+1}M^{8m+2} = 0$.

Ochanine showed in [48] that an orientation preserving homotopy equivalence between two closed oriented manifolds with $w_2 = 0$ gives a natural bijection between the both sets of *Spin*-structures on the two manifolds. In particular, one defines a *Spin-homotopy equivalence* between two *Spin*-manifolds as an orientation preserving homotopy equivalence which maps the *Spin*-structure of the one to that of the other. Furthermore, Ochanine showed that generalized Kervaire invariants are invariants of the *Spin*-homotopy type.

7.2.2 Corollary: *The Ochanine k-invariant is an invariant of the Spin-homotopy type.*

In [48] , Ochanine defined an extension $\kappa : \Omega_*^{Spin} \to KO_* \otimes \mathbb{Z}/2$ of k using also the signature (see 3.1.5) and showed that κ is a ring homorphism; this summarizes the multiplicative properties of k. Now, the signature is an invariant of the oriented homotopy type, and the definition of *Spin*-homotopy equivalence is compatible with products.

7.2.3 Corollary: κ *is an invariant of the Spin-homotopy type.*

In contrast to this, the Atiyah α-invariant (and thus also the Ochanine β-invariant) is *not* an invariant of the *Spin*-homotopy type, because it detects some exotic spheres in dimension 9 which have clearly the *Spin*-homotopy type of the standard sphere.

According to a Theorem of Kahn ([28]) for closed oriented manifolds, the rational multiples of the signature are the only rational characteristic numbers which are invariants of the oriented homotopy type. One can ask the analogous question for KO-characteristic numbers of closed $Spin$-manifolds, and we have seen that the signature $sign(M^{4m})$ and the Ochanine k-invariants $k(M^{8m+1} \times \overline{S^1})$ and $k(M^{8m+2})$ are invariants of the $Spin$-homotopy type.

7.2.4 Problem: *Determine the KO-characteristic numbers for Spin-manifolds which are invariants of the Spin-homotopy type.*

We remark, that by the Theorem of Anderson, Brown and Peterson 2.1.9, the KO-Pontrjagin number $\pi_I[\,]$ detects in $MSpin$ the summand $\Sigma^{4|I|}KO\langle 0\rangle$ for $|I|$ even, and $\Sigma^{4|I|-2}KO\langle 2\rangle$ for $|I|$ odd. As these start in dimension $4|I|$ respectively $4|I|-2$ in Adams filtration 0 (2.1.16), the homomorphisms

$$\begin{array}{llcllll} \pi_I \bmod 2: & \Omega^{Spin}_{4|I|} & \longrightarrow & KO_{4|I|} \otimes \mathbb{Z}/2 & \cong & \mathbb{Z}/2 & \text{for } |I| \text{ even} \\ \pi_I: & \Omega^{Spin}_{4|I|-2} & \longrightarrow & KO_{4|I|-2} & \cong & \mathbb{Z}/2 & \text{for } |I| \text{ odd} \end{array}$$

can also be a expressed by $\mathbb{Z}/2$-cohomology characteristic numbers, i.e. by Stiefel-Whitney numbers. In particular, these homomorphisms are invariants of the unoriented homotopy type as well.

Part II:

Other Manifolds

8 Examples of Brown-Kervaire Invariants

In this section we want to give some examples of fibrations $\xi : B \to BO$ and dimensions $2n$ with $v_{n+1}(\xi) = 0$, thus providing Brown-Kervaire invariants for the associated bordism theories Ω^{ξ}_{2n}. We first recall two extreme cases, namely Wu bordism and framed bordism. Then we consider fibrations $\xi : B \to BO$ where the Stiefel-Whitney classes have an upper bound (i.e., $w_i(\xi) = 0$ for $i > k$), for example the bordism theories of immersions $\Omega^{O(k)}_*$ and $\Omega^{SO(k)}_*$. At last we consider the 'dual' case where the Stiefel Whitney classes have a lower bound ($w_i(\xi) = 0$ for $i < k$), for example the bordism theories $\pi_* MO\langle k\rangle = \Omega^{\langle k\rangle}_*$ associated to the $(k-1)$-connected covers $BO\langle k\rangle \to BO$.

In the first two cases, we restrict us to look for the vanishing dimensions of the Wu classes, but do not examine the Brown-Kervaire invariants further. Only for the third case, we generalize the construction of Brown-Kervaire invariants for *Spin*-manifolds by secondary cohomology operations to the bordism theories $\Omega^{\langle 2^r\rangle}_*$, introducing again Brown-Peterson-Kervaire invariants. The case of $\Omega^{\langle 8\rangle}_*$ will be considered in some more detail in section 10.

8.1 Wu Bordism and Framed Bordism

8.1.1 Wu Bordism $\Omega^{\langle v_{n+1}\rangle}_*$: This is the most general bordism theory for Brown-Kervaire invariants in dimension $2n$. Here, the fibration is given by

$$\pi_{\langle v_{n+1}\rangle} : BO\langle v_{n+1}\rangle \longrightarrow BO,$$

where $BO\langle v_{n+1}\rangle := v^*_{n+1} PK_{n+1}$ is defined as the pullback of the path fibration $PK_{n+1} \to K_{n+1}$ by a map $v_{n+1} : BO \to K_{n+1}$ realising the universal Wu class v_{n+1}. In dimension $2n$ there exist Brown-Kervaire invariants $K : Q^{\langle v_{n+1}\rangle}_{2n} \times \Omega^{\langle v_{n+1}\rangle}_{2n} \to \mathbb{Z}/8$ as $v_{n+1}(\pi_{\langle v_{n+1}\rangle}) = 0$ by construction. Since $v_{n+1}(M) = 0$ for any closed $2n$-dimensional manifold, we have always $\pi_{\langle v_{n+1}\rangle}$-structures on M which we call *Wu-$(n+1)$-structures*. The fibre homotopy classes of these structures form an affine space with associated vector space $H^n M = [M, \Omega K_{n+1}]$. Because in general $v_n^2[M] \neq 0$ (example: $\mathbb{R}P^{2n}$), the invariant K does not restrict to a $\mathbb{Z}/2$-valued invariant, $\mathbb{Z}/2 \subset \mathbb{Z}/8$ (see 1.2.6). Given another $\xi : B \to BO$ with $v_{n+1}(\xi) = 0$, there exists a lift $\tilde{\xi} : B \to BO\langle v_{n+1}\rangle$ of ξ which implies by naturality 1.2.1 that $K_{\tilde{\xi}^* h}(M) = K_h(\tilde{\xi}_* M)$ for all $M \in \Omega^{\xi}_{2n}$ and $h \in Q^{\langle v_{n+1}\rangle}_{2n}$. The fibre homotopy classes of these lifts $\tilde{\xi}$ form an affine space with $H^n B$ as associated vector space; using this one shows that each Brown-Kervaire invariant for Ω^{ξ}_{2n} comes from any fixed Brown-Kervaire invariant for $\Omega^{\langle v_{n+1}\rangle}_{2n}$ by a suitable lift $\tilde{\xi}$.

8.1.2 Framed Bordism Ω^{fr}_*: This is the most special bordism theory for Brown-Kervaire invariants. Here, the fibration is given by the universal bundle

$$p^{\mathrm{fr}} : EO \longrightarrow BO,$$

and a p^{fr}-structure on M is just a *framing* of the stable normal bundle. Thus M has a p^{fr}-structure iff ν_M is stably trivial, $\nu_M = 0 \in KO^0(M)$. The fibre homotopy classes of

framings on ν_M form an affine space with $KO^{-1}(M) = [M, O]$ as associated vector space. According to $H^n EO = 0$, the parameter set Q_{2n}^{fr} consists of the single element $h = i_2^4 : \pi_{2n}^{\mathrm{st}}(K_n) = \mathbb{Z}/2 \to \mathbb{Z}/4$, showing that there exists exactly one Brown-Kervaire invariant K with values in $\mathbb{Z}/2$, which is nothing but the classical Kervaire invariant $K(M)$ (for n odd). We remark that by the Pontrjagin-Thom construction, framed bordism Ω_*^{fr} is just given by the stable homotopy groups of spheres, i.e. $\Omega_*^{\mathrm{fr}} = \pi_*^{\mathrm{fr}}$. Given $\xi : B \to BO$, because of $EO \simeq *$ there is a lift $\tilde{p} : EO \to B$ of p^{fr}, and by naturality we have $K(M) = K_h(\tilde{p}_* M)$ for all $M \in \Omega_{2n}^{\mathrm{fr}}$ and $h \in Q_{2n}^{\xi}$. We remark, that $K(M^{4m}) = 0$ for all $M^{4m} \in \Omega_{4m}^{\mathrm{fr}}$ which follows from the Theorem of Morita [45] (see section 9), as $K(M^{4m}) \equiv sign(M^{4m}) \bmod 8$, but $sign(M^{4m}) = 0$. Furthermore, the Main Theorem of Browder in [14] says, that the classical Kervaire invariant K vanishes in all dimensions $2n \neq 2^r - 2$; and in the critical dimensions $2n = 2^r - 2$, K is non-trivial iff the element $h_{r-1}^2 \in Ext_{A^*}^{2,2^r}(\mathbb{Z}/2, \mathbb{Z}/2)$ in the E_2-term of the Adams spectral sequence for the stable homotopy groups of spheres survives to E_∞. This is known to hold true for $2n = 2, 6, 14, 30$, and 62, but is not known in general.

8.2 Bordism of Immersions

We consider now the case where $w_i(\xi) = 0$ for all $i > k$ for some $k \in \mathbb{N}$. Examples of this case are provided by the non-stable classifying spaces

$$BO(k) \hookrightarrow BO \quad \text{and} \quad BSO(k) \hookrightarrow BO.$$

According to [24] (see [22], [63]), the associated bordism theories $\Omega_n^{O(k)}$ and $\Omega_n^{SO(k)}$ give the bordism groups of codimension-k immersions of (oriented) n-dimensional manifolds in Euclidean space. The Brown-Kervaire invariants in the case of $\Omega_{2n}^{O(1)}$ and $\Omega_{2n}^{SO(2)}$ are studied in [22], where the authors state (without proof) that $v_{n+1}(\xi) = 0$ for $n \neq 2^t - 2$ in the first bordism theory and for $n \neq 2^t - 3$ in the second bordism theory. We generalize this now to arbitrary k:

8.2.1 Proposition: *Let $\xi : B \to BO$ be a fibration with $w_i(\xi) = 0$ for all $i > k$, for some fixed $k \in \mathbb{N}$. For $m \in \mathbb{N}$, let $\alpha(m) \in \mathbb{N}$ be the number of one's in the dyadic expansion of m. Then $v_m(\xi) = 0$ for all $m \in \mathbb{N}$ with $\alpha(m+k) > k$. In particular, there exist Brown-Kervaire invariants for Ω_{2n}^{ξ} if $\alpha(n+k+1) > k$, and these are $\mathbb{Z}/2$-valued for $\alpha(n+k) > k$.*

Proof: We have to compute the total Wu class $v(\xi) \in H^\bullet B$, which is given by

$$v(\xi) = \chi(Sq)(w(\xi)^{-1}) = (\chi(Sq)w(\xi))^{-1},$$

where $w = 1 + w_1 + w_2 + ... + w_k$ is the total normal Stiefel-Whitney class. By the splitting principle, we can assume $w(\xi) = \prod_{i=1}^k (1 + x_i)$, so

$$v(\xi) = \prod_{i=1}^{k} (1 + \sum_{j=0}^{\infty} x_i^{2^j})^{-1}$$

since $\chi(Sq)(1+x) = 1 + x + x^2 + x^4 + x^8 + ... = 1 + \sum_{j=0}^{\infty} x^{2^j}$ for a 1-dimensional cohomology class x; this follows from $Sq(x^{2^j}) = Sq(x)^{2^j} = x^{2^j} + x^{2^{j+1}}$, showing that Sq applied to the

series gives $1 + x$. Thus we have to compute the multiplicative inverse of the power series

$$f(x) := 1 + \sum_{j=0}^{\infty} x^{2^j} \qquad \in \mathbb{Z}/2[[x]].$$

Because over the field $\mathbb{Z}/2$ the square of any polynomial or power series is given by doubling all exponents of x, we have

$$f(x)^2 = 1 + \sum_{j=0}^{\infty} x^{2^j+1} = 1 + x^2 + x^4 + x^8 + ... = f(x) - x,$$

which gives $f(x)(1 + f(x)) = x$ and thus

$$f(x)^{-1} = \frac{1 + f(x)}{x} = \sum_{j=0}^{\infty} x^{2^j-1} = 1 + x + x^3 + x^7 + x^{15} + ... \qquad \in \mathbb{Z}/2[[x]].$$

This shows that the total Wu class is given by the product

$$v(\xi) = \prod_{i=1}^{k} (1 + x_i + x_i^3 + x_i^7 + x_i^{15} + ...),$$

producing zero terms $v_m(\xi)$ in each dimension m not of the form $\sum_{i=1}^{k}(2^{s_i} - 1)$, with $s_i \in \mathbb{N}$. Thus $m + k \neq \sum_{i=1}^{k} 2^{s_i}$, which means that $m + k$ must have more than k one's in its dyadic expansion. ■

8.2.2 *Remarks:*
(i) For $B = BU(k) \to BO(2k) \to BO$, a similar computation applied to

$$v(\xi) = \chi(Sq)((1 + w_2 + w_4 + ... + w_{2k})^{-1}) = (\prod_{i=1}^{k} \chi(Sq)(1 + x_i^2))^{-1}$$

shows that $v_m = 0$ for $m \neq \sum_{i=1}^{k} 2(2^{s_i} - 1)$, which is equivalent to m odd, or m even with $\alpha(\frac{m}{2} + k) > k$.

(ii) In the two cases $BO(1)$ and $BU(1) = BSO(2)$, one obtaines the conditions above, $n \neq 2^t - 2$ respectively $n \neq 2^t - 3$.

(iii) According to Brown [17], p.376, there are exactly two Brown-Kervaire invariants $K : \Omega_2^{BO(1)} \to \mathbb{Z}/8$, which are isomorphisms, and one is the negative of the other.

8.3 $BO\langle n\rangle$-Manifolds and Brown-Peterson-Kervaire Invariants

At last we consider the case that $w_i(\xi) = 0$ for all $i < k$ with some fixed $k \in \mathbb{N}$. There are some fibrations are of this type, for example the $(k-1)$-connected cover

$$\pi\langle k\rangle : BO\langle k\rangle \to BO,$$

but also the space $\pi\langle w_i\rangle_{i<k} : BO\langle w_i\rangle_{i<k} \to BO$ obtained by pulling back the path fibration over $\prod_{i<k} K_i$ with the map $(w_i)_{i<k} : BO \to \prod_{i<k} K_i$. This includes

$$BSO \simeq BO\langle 2\rangle \simeq BO\langle w_1\rangle$$

$$BSpin \simeq BO\langle 4\rangle \simeq BO\langle w_1, w_2\rangle$$

but we have only a lift $BO\langle k\rangle \to BO\langle w_i\rangle_{i<k}$ for $k \geq 4$. We remark that bundles of this type are also given by BU ($k = 2$), BSU and BSp ($k = 4$) and their $(k-1)$-connected covers. It is well-known ([44]) that we can assume in the condition $w_i(\xi) = 0$ for $i < k$ without loss of generality that k is a power of two:

8.3.1 Lemma: *Let $\xi : B \to BO$ be a bundle with $w_{2^i}(\xi) = 0$ for all $i = 0, ..., r-1$, $r \geq 1$. Then $w_i(\xi) = 0$ for all $i = 1, ..., 2^r - 1$.*

Proof: By induction on r. Thus we have to show that $w_i = 0$ for $1 \leq i \leq 2^{r-1}$ implies that $w_i = 0$ also for $1 \leq i \leq 2^r - 1$. We apply the Wu formula $Sq^k w_n = \sum_{i=0}^k \binom{k-n}{i} w_{k-i} w_{n+i}$ to the case of $n = 2^{r-1}$ and $1 \leq k \leq 2^{r-1} - 1$, and obtain with $\binom{k-2^{r-1}}{k} \equiv \binom{2^{r-1}-1}{k} \equiv 1 \bmod 2$ that

$$w_{2^{r-1}+k} = Sq^k w_{2^{r-1}} - \sum_{i=0}^{k-1} \binom{k-2^{r-1}}{i} w_{k-i} w_{2^{r-1}+i} = 0.$$

■

This shows that instead of $BO\langle w_i\rangle_{i<k}$ we can also consider the 'smaller' space $BO\langle w_{2^j}\rangle_{j<r}$. We remark that there exists a lift $BO\langle 2^r\rangle \to BO\langle w_{2^j}\rangle_{j<r}$. The following theorem is a generalisation of 2.2.1, where the situation is described for Spin manifolds:

8.3.2 Proposition: *Let $\xi : B \to BO$ be a bundle with $w_{2^i}(\xi) = 0$ for all $i = 0, ..., r-1$. Then the total Wu class of ξ has the form*

$$v(\xi) = \sum_{i\geq 0} v_{2^r i}(\xi) = 1 + v_{2^r} + v_{2^r 2} + v_{2^r 3} + \dots .$$

In particular, there exist Brown-Kervaire invariants on Ω^ξ_{2n} for $n \neq 2^r i - 1$, which are $\mathbb{Z}/2$-valued for $n \neq 2^r i$.

In the proof we will need the following result of Stong:

8.3.3 Proposition (Stong [62]): *Let $A_s := \langle Sq^{2^i}\rangle_{i=0...,s}$ be the Hopf sub-algebra of the Steenrod algebra A generated by $Sq^1, ..., Sq^{2^s}$ and $\bar{A}_s$ be the augmentation ideal. Then Sq^k is contained in the right ideal $\bar{A}_s A$ of A, where s is the 2-exponent of k: $k = 2^s(2m+1)$. We have thus a decomposition in the Steenrod algebra A^**

$$Sq^k = \sum_{i=0}^s Sq^{2^i} \beta_i, \qquad \beta_i \in A^{k-2^i}.$$

Proof of 8.3.3: (See [62]) By the Adem formula $Sq^a Sq^b = \sum_{c=0}^{[\frac{a}{2}]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$ we have

$$Sq^{2^s} Sq^{2^{s+1}m} = Sq^{2^{s+1}m+2^s} + \sum_{j=0}^{s-1} Sq^{2^{s+1}m+2^s-2^j} Sq^{2^j}$$

since $\binom{2^{s+1}m-c-1}{2^s-2c} \equiv \binom{(2^{s+1}-1)-c}{(2^s-1)+c} \equiv 1 \bmod 2$ iff $c = 0$ or $c = 2^j$. Thus

$$Sq^k = Sq^{2^s} Sq^{2^{s+1}m} + \sum_{j=0}^{s-1} Sq^{2^j l_j} Sq^{2^j},$$

where $l_j := 2^{s-j}(2m+1) - 1$ is odd, showing that $2^j l_j$ has 2-exponent $j < s$. Hence the result follows by induction on the 2-exponent s of k, which gives the decompositions of the $Sq^{2^j l_j}$. ■

Proof of 8.3.2: By induction on r. Thus we know already that $v(\xi) = 1 + v_{2^r} + v_{2^r 2} + v_{2^r 3} + ...$ and have to show that $v_{2^{r-1}(2m+1)} = 0$. We apply 8.3.3 to $k := 2^{r-1}(2m+1)$, which has 2-exponent $s := r-1$, and get

$$v_k(\xi)U_\xi = \chi(Sq^k)U_\xi = \sum_{i=0}^{s} \chi(\beta_i)\chi(Sq^{2^i})U_\xi$$

But $\chi(Sq^{2^i})U_\xi = v_{2^i}(\xi)U_\xi = 0$, since by the Wu formula the vanishing of the Stiefel-Whitney classes $w_i(\xi) = 0$ for all $i = 1, ..., 2^s$ implies the vanishing of the corresponding Wu classes. ■

In particular, Brown-Kervaire invariants exist for Ω_*^{SO} in dimensions $4m$, for Ω_*^{Spin} in dimensions $4m$ and $8m+2$, and for $\Omega_*^{\langle 8 \rangle}$ in dimensions $4m$, $8m+2$ and $16m+6$. In fact, we have a little more:

8.3.4 Proposition: *For $B = BSO$, $BSpin$ and $BO\langle 8\rangle$, all Brown-Kervaire invariants are given by*

$$\begin{array}{lcccccc} K^{SO} & : & Q_{4m}^{SO} & \times & \Omega_{4m}^{SO} & \to & \mathbb{Z}/8 \\ K^{Spin} & : & Q_{8m+2}^{Spin} & \times & \Omega_{8m+2}^{Spin} & \to & \mathbb{Z}/2 \\ K^{\langle 8\rangle} & : & Q_{16m+6}^{\langle 8\rangle} & \times & \Omega_{16m+6}^{\langle 8\rangle} & \to & \mathbb{Z}/2 \end{array}$$

and their compositions with the induced maps of $BO\langle 8\rangle \to BSpin \to BSO$.

Proof: First we have to show, that there are no other dimensions, where Brown-Kervaire invariants exist. In other words, the universal Wu classes in

$$\begin{array}{lcccccccccc} v_{SO} & = & 1 & + & v_2 & + & v_4 & + & v_6 & + & ... \\ v_{Spin} & = & 1 & + & v_4 & + & v_8 & + & v_{12} & + & ... \\ v_{BO\langle 8\rangle} & = & 1 & + & v_8 & + & v_{16} & + & v_{24} & + & ... \end{array}$$

are non-zero. This follows by the consideration of the manifolds $(\mathbb{C}P^2)^m \in \Omega_{4m}^{SO}$, $(\mathbb{H}P^2)^m \in \Omega_{8m}^{Spin}$ and $(\mathbb{C}ayP^2)^m \in \Omega_{16m}^{\langle 8\rangle}$: If their middle-dimensional Wu classes vanish, then the intersection form will be even and the signature will be divisible by 8. But they have all signature equal to 1.

Furthermore, by naturality 1.2.1 the only thing to show is the surjectivity of the induced maps for the parameter spaces

$$Q_{4m}^{SO} \longrightarrow Q_{4m}^{Spin} \qquad \text{and} \qquad Q_{8m+2}^{Spin} \longrightarrow Q_{8m+2}^{\langle 8\rangle}.$$

But this follows with 1.1.6 from the surjectivity of the induced maps in cohomology $H^{4m}BSO \to H^{4m}BSpin$ and $H^{8m+2}BSpin \to H^{8m+2}BO\langle 8\rangle$, see 2.1.2 and 10.1.1. ■

8.3.5 *Remark:* The corresponding statement in 8.3.4 for $BO\langle 2^r\rangle$, $r \geq 4$, is not true because the mappings in cohomology are then not further surjective ([63]).

8.3.6 At last, we construct the analoges of Brown-Peterson-Kervaire invariants of *Spin*-manifolds, but this time for $BO\langle 2^r\rangle$-manifolds. By 8.3.2, there exist Brown-Kervaire invariants on $\Omega_{2n}^{\langle 2^r\rangle}$ if $n+1$ has 2-exponent smaller than r, i.e. for

$$n+1 \quad \in \quad \{2m+1, 2(2m+1), \ldots, 2^{r-1}(2m+1) \,|\, m \in \mathbb{N}\},$$

which gives

$$\begin{aligned} 2n \quad &\in \quad \{2^{i+1}m + (2^i - 2) \,|\, m \in \mathbb{N}, 1 \leq i \leq r\} = \\ &= \{4m, 8m+2, 16m+6, \ldots, 2^{r+1}m + (2^r - 2) \,|\, m \in \mathbb{N}\}. \end{aligned}$$

Furthermore, by 8.3.2 these invariants are all $\mathbb{Z}/2$-valued, with the exception of the first case $2n = 4m$. We are mainly interested in the case $2n = 2^{r+1} + (2^r - 2)$, which comes up as a new case by going from $BO\langle 2^{r-1}\rangle$ to $BO\langle 2^r\rangle$. For the other cases, one has already invariants on $\Omega_*^{\langle 2^i\rangle}$ with $i < r$, which give invariants on $\Omega_{2n}^{\langle 2^r\rangle}$ by naturality.

It is interesting, that the series $\{2^{r+1}m + (2^r - 2)\}$ starts with the critical dimension $2n = 2^r - 2$ of the Kervaire invariant problem 8.1.2. In fact, the natural map $\Omega_*^{\mathrm{fr}} \to \Omega_*^{\langle 2^r\rangle}$ is an isomorphism in dimensions $* \leq 2^r - 2$, because then a $BO\langle 2^r\rangle$-structure on a manifold is nothing but a framing of the stable normal bundle ($* \leq 2^r - 1$), and the same for the bordisms ($* \leq 2^r - 2$). In particular, there exists exactly one Brown-Kervaire invariant on $\Omega_{2^r-2}^{\langle 2^r\rangle}$, which is just the classical Kervaire invariant in this critical dimension.

Now, in order to apply the Theorem of Brown and Peterson 2.3.1, we need a decomposition of Sq^{n+1}, where $n+1 = 2^r m + 2^{r-1}$. In fact, the proof of 8.3.3 gives for $m \geq 1$ an explicit decomposition of Sq^{n+1}, which we call the *canonical decomposition.* It is constructed with the Adem relation

$$Sq^{2^r m + 2^{r-1}} = Sq^{2^{r-1}} Sq^{2^r m} + \sum_{j=1}^{r-1} Sq^{2^j m_j + 2^{j-1}} Sq^{2^{j-1}}$$

and its recursive application to the elements $Sq^{2^j m_j + 2^{j-1}}$, where $m_j := 2^{r-j}m + 2^{r-j-1} - 1$. For example,

$$\begin{aligned} Sq^{2m+1} \quad &= \quad Sq^1(Sq^{2m}), \\ Sq^{4m+2} \quad &= \quad Sq^2(Sq^{4m}) + Sq^1(Sq^{4m}Sq^1), \\ Sq^{8m+4} \quad &= \quad Sq^4(Sq^{8m}) + Sq^2(Sq^{8m}Sq^2) + Sq^1(Sq^{8m+2}Sq^1 + Sq^{8m}Sq^1Sq^2), \\ Sq^{16m+8} \quad &= \quad Sq^8(Sq^{16m}) + Sq^4(Sq^{16m}Sq^4) + Sq^2(Sq^{16m+4}Sq^2 + Sq^{16m}Sq^2Sq^4) \\ &\quad + Sq^1(Sq^{16m+6}Sq^1 + Sq^{16m+4}Sq^1Sq^2 + Sq^{16m+2}Sq^1Sq^4 + Sq^{16m}Sq^1Sq^2Sq^4). \end{aligned}$$

The parentheses ( ) indicate the $\beta_i \in A$. For $m = 0$ one obtains no decomposition of $Sq^{n+1} = Sq^{2^{r-1}}$ (which is impossible, as $Sq^{2^{r-1}} \in A^*$ is indecomposable), but an interesting relation in A^*. We remark that it is also possible to apply the expansion further on the

factors $Sq^{2^r m+j}$ in the β_i, to get a decompositions of the form $Sq^{n+1} = \sum \beta_i' Sq^{2^r m} \beta_i''$ with $\beta_i', \beta_i'' \in \bar{A}_s$, for example

$$Sq^{8m+4} = Sq^4 Sq^{8m} + Sq^2 Sq^{8m} Sq^2 + Sq^1 Sq^2 Sq^{8m} Sq^1 + Sq^1 Sq^{8m} Sq^1 Sq^2.$$

These can be used to construct a decomposition of any element in A^* in terms of the undecomposable elements Sq^{2^s}, but we will not consider this further.

We generalize now 2.3.6 to construct certain Brown-Kervaire invariants for $BO\langle 2^r\rangle$-manifolds (with $r \geq 2$) by Brown-Peterson secondary cohomology operations, living in dimensions $2n = 2^{r+1}m + (2^r - 2)$ with $m \geq 1$. We call these homomorphisms $\Omega_{2n}^{\langle 2^r\rangle} \to \mathbb{Z}/2$ again *Brown-Peterson-Kervaire invariants.*

8.3.7 Theorem: *For $n+1 = 2^r m + 2^{r-1}$ with $m \geq 1$ and $r \geq 2$, let $Sq^{n+1} = \sum_{i=0}^{r-1} \alpha_i \beta_i$ be the canonical decomposition, where $\alpha_i := Sq^{2^i}$ and $\beta_i \in A^{n+1-2^i}$. Let ϕ be an associated Brown-Peterson secondary cohomology operation (2.3.1)*

$$\phi : ker(\beta) \longrightarrow coker(\alpha),$$

$$\begin{array}{rrlcl} \alpha := & \sum_{i=0}^{r-1} Sq^i : & \bigoplus_{i=0}^{r-1} H^{n-2^i} X & \longrightarrow & H^{2n} X, \\ \beta := & (\beta_0, ..., \beta_{r-1}) : & H^n X & \longrightarrow & \bigoplus_{i=0}^{r-1} H^{2n+1-2^i} X. \end{array}$$

Then one gets a $\mathbb{Z}/2$-valued Brown-Kervaire invariant in dimension $2n = 2^{r+1}m + (2^r - 2)$

$$K_\phi : \Omega_{2n}^{\langle 2^r\rangle} \longrightarrow \mathbb{Z}/2$$

by $K_\phi(M^{2n}) := Arf(q_\phi : H^n M^{2n} \to \mathbb{Z}/2)$, where $q_\phi(x) := \langle \phi(x), [M^{2n}]\rangle$, and M^{2n} is taken as a closed $(2^{r-1}-1)$-connected $BO\langle 2^r\rangle$-manifold in its bordism class.

Proof: First, we remark that for $d \geq 2^{r+1}$, each bordism class in $\Omega_d^{\langle 2^r\rangle}$ can by surgery be represented by a closed $(2^r - 1)$-connected $BO\langle 2^r\rangle$-manifold. Moreover, a $BO\langle 2^r\rangle$-bordism between two such representatives can also be chosen $(2^r - 1)$-connected. In particular, this applies to $d = 2n$ for $m \geq 1$. In the proof of the theorem, we will only need to choose M^{2n} $(2^{r-1} - 1)$-connected; in fact, $H^i M^{2n} = 0$ for $i = 1, .., 2^{r-1} - 1$ is enough.

Now, we will show that K_ϕ is well-defined on $H^n M^{2n}$, i.e. $coker(\alpha) = H^{2n} M^{2n} \cong \mathbb{Z}/2$ and $ker(\beta) = H^n M^{2n}$. The first statement follows simply by the vanishing of the Wu classes $v_1 = v_2 = ... = v_{2^{-1}} = 0$.

The second statement is more complicated: By the connectivity assumption on M^{2n} and Poincaré duality, the maps $\beta_1, ..., \beta_{r-1}$ are zero on $H^n M^{2n}$ because they go to the codimensions $2^i - 1$, for $i = 1, ..., r-1$. But this does not apply to β_0 (in fact, one has to show that the generalized Wu class 2.4.3 associated to β_0 vanishes in $H^* BO\langle 2^r\rangle$). Thus, we must know β_0 more explicit; and the following formula in proposition 8.3.8 is illustrated by the explicit decompositions in 8.3.6:

$$\begin{array}{lcl} 2n = 2m+1 & : & \beta_0 = Sq^{2m}, \\ 2n = 4m+2 & : & \beta_0 = Sq^{4m} Sq^1, \\ 2n = 8m+4 & : & \beta_0 = Sq^{8m+2} Sq^1 + Sq^{8m} Sq^1 Sq^2, \\ 2n = 16m+8 & : & \beta_0 = Sq^{16m+6} Sq^1 + Sq^{16m+4} Sq^1 Sq^2 + Sq^{16m+2} Sq^1 Sq^4 + Sq^{16m} Sq^1 Sq^2 Sq^4. \end{array}$$

8.3.8 Proposition: *Let* $Sq^k = \sum_{j=0}^{s} Sq^{2^j}\beta_j^{(k)}$ *be the canonical decomposition of* Sq^k*, where* s *denotes the* 2*-exponent of* k*. Then the term* $\beta_0^{(k)} \in A^{k-1}$ *is for* k *even given by*

$$\beta_0^{(k)} = \sum_{\substack{\{i_2, i_3, .., i_t\} \subseteq \{2,4,8,..,2^{s-1}\} \\ \text{with } i_2 < i_3 < ... < i_t}} Sq^{(k-2)-(i_2+i_3+..+i_t)} Sq^1 Sq^{i_2} ... Sq^{i_t},$$

and for k *odd, one has* $\beta_0^{(k)} = Sq^{k-1}$.

Proof of 8.3.8: If k is odd ($s=0$), the statement follows from $Sq^k = Sq^1 Sq^{k-1}$. If k is even ($s \geq 1$), we make induction on the 2-exponent of k. In order to do this, we recall from 8.3.3 and 8.3.6, that we obtain the canonical decomposition of Sq^k by inserting in

$$Sq^k = Sq^{2^s} Sq^{k-2^s} + \sum_{j=0}^{s-1} Sq^{k-2^j} Sq^{2^j}$$

the canonical decomposition of the Sq^{k-2^j}. As $k-2^j$ has 2-exponent j, these are given by $Sq^{k-2^j} = \sum_{l=0}^{j} Sq^{2^l}\beta_l^{(k-2^j)}$, and we obtain $Sq^k = Sq^{2^s}Sq^{k-2^s} + \sum_{j=0}^{s-1}\sum_{l=0}^{j} Sq^{2^l}\beta_l^{(k-2^j)} Sq^{2^j}$. In particular, we get

$$\beta_0^{(k)} = \sum_{j=0}^{s-1} \beta_0^{(k-2^j)} Sq^{2^j}.$$

Now, for $s=1$ we have $k = 4m+2$ and $Sq^k = Sq^2 Sq^{4m} + Sq^1 Sq^{4m} Sq^1$, showing the statement since $\beta_0^{(k)} = Sq^{4m}Sq^1$. For $s \geq 2$, we get by the induction hypothesis

$$\beta_0^{(k)} = Sq^{k-2}Sq^1 + \sum_{j=1}^{s-1} \left(\sum_{\substack{\{i_2,..,i_t\} \subseteq \{2,..,2^{j-1}\} \\ \text{with } i_2 < ... < i_t}} Sq^{(k-2^j-2)-(i_2+..+i_t)} Sq^1 Sq^{i_2} ... Sq^{i_t} \right) Sq^{2^j},$$

which by $i_{t+1} := 2^j$ gives just the statement, because then $(k-2^j-2)-(i_2+..+i_t) = (k-2)-(i_2+..+i_t+i_{t+1})$ and $i_t < i_{t+1}$. This gives the proof of proposition 8.3.8. ■

We continue with the proof of 8.3.7:
As a corollary of 8.3.8, β_0 in the canonical decomposition of $n+1 = 2^{r-1}(2m+1)$ gives

$$ker(\beta_0) = ker(Sq^{2^r m} Sq^1 Sq^2 Sq^4 \dots Sq^{2^{r-2}} : H^n M \longrightarrow H^{2n} M),$$

because all proper subsets $\{i_2, i_3, ..., i_t\} \subset \{2,4,8,...,2^{r-2}\}$ in 8.3.8 contribute summands with first factor $Sq^{2^r m + u}$, where $u := (2^{r-1}-2)-(i_2+i_3+...+i_t) > 0$. But these factors can again be decomposed with 8.3.3 and thus lie in $\bar{A}^*_{(r-1)}A^*$, showing that the whole summand vanishes on $H^n M$ as $v_1 = ... = v_{2^{r-1}} = 0$ in H^*M.

Moreover, also $\gamma := Sq^{2^r m} Sq^1 Sq^2 Sq^4 \dots Sq^{2^{r-2}}$ vanishes on $H^n M$, which follows as for *Spin*-manifolds by the Adem relations $Sq^2 Sq^{4k-1} = Sq^{4k+1} + Sq^{4k}Sq^1$ and $Sq^1 Sq^{2k} = Sq^{2k+1}$: This gives $Sq^{2^r m}Sq^1 = Sq^1 Sq^{2^r m} + Sq^2 Sq^{2^r m - 1}$, hence

$$\gamma = Sq^1 Sq^{2^r m} Sq^2 Sq^4 \dots Sq^{2^{r-2}} + Sq^2 Sq^{2^r m-1} Sq^2 Sq^4 \dots Sq^{2^{r-2}},$$

showing with $v_1 = v_2 = 0$ the vanishing of γ on $H^n M$. Thus, we have shown that $ker(\beta) = H^n M^{2n}$.

After this effort, the remaining part of 8.3.7 is simple to prove. The bordism invariance of K_ϕ follows as for *Spin*-manifolds 2.3.6, because a $BO\langle 2^r\rangle$-zero-bordism, which we can assume to be $2^r - 1$-connected, gives by Poincaré-Lefschetz duality a Lagrangian of q_ϕ, implying the vanishing of K_ϕ. The fact that K_ϕ is a Brown-Kervaire invariant follows as in 2.3.6 by considering a secondary operation of spectra belonging to the zero-homotopic composition

$$MO\langle 2^r\rangle \wedge K_n \xrightarrow{1\wedge\beta} MO\langle 2^r\rangle \wedge K \xrightarrow{1\wedge\alpha} MO\langle 2^r\rangle \wedge K_{2n+1},$$

where $K := \times_{i=0}^{r-1} K_{2n+1-2^i}$. We take in the construction of this stable version of ϕ the stabilization of the zero-homotopy used in the construction of ϕ (see 2.3.7). This defines then the parameter h_ϕ of ϕ by application to the sphere spectrum $\Sigma^{2n}S^0$:

$$\pi_{2n}(MO\langle 2^r\rangle \wedge K_n) = [\Sigma^{2n}S^0, MO\langle 2^r\rangle \wedge K_n] \longrightarrow \mathbb{Z}/2 = [\Sigma^{2n}S^0, \Sigma^{-1}(MO\langle 2^r\rangle \wedge K_{2n+1})].$$

(In particular, we get again the bordism invariance.) This gives the proof of theorem 8.3.7. ■

8.3.9 *Remarks:* Our construction of Brown-Peterson-Kervaire invariants for $\Omega_*^{\langle 2^r\rangle}$ does not work in two cases:

(i) For $m = 0$, which is the critical dimension $2n = 2^r - 2$ of the classical Kervaire invariant. The reason is, that there exists no decomposition of $Sq^{2^{r-1}}$ in A^*. But according to Adams [1], one has for $r - 1 \geq 4$ a decomposition of $Sq^{2^{r-1}}$ in certain stable secondary cohomology operations. In [20], Brown and Peterson state (without detailed proof), that this decomposition gives then a tertiary cohomology operation ϕ, defined in dimension n with values in dimension $2n$, which behaves also quadratic with respect to the cup product (and hence is unstable).

For a closed $2n$-dimensional stably framed manifold M, which by framed surgery can be assumed to $(n-1)$-connected, the operation ϕ has then $H^n M$ as range of definition and takes values in $H^{2n}M \cong \mathbb{Z}/2$, because the stable secondary cohomology operations do not produce a proper kernel respectively cokernel. This follows from the connectedness of M, and because there is also no critical 'Wu class' contribution by operations going from dimension n to dimension $2n$ (as all Stiefel-Whitney classes of stably framed manifolds vanish). Thus ϕ gives a quadratic refinement of the intersection pairing on $H^n M$, and $\mathrm{Arf}(\phi)$ is just the classical Kervaire invariant $K : \Omega^{\mathrm{fr}}_{2^r-2} \to \mathbb{Z}/2$.

(ii) For $r = 1$, which is oriented bordism in dimension $2n = 4m$. The construction does not work here, because $ker(\beta_0) = \ker(Sq^{2m})$ is in general not $H^n M$ (equality holds iff the middle dimensional Wu class of M vanishes). In the next section, we will construct Brown-Kervaire invariants on Ω_{4m}^{SO} not by secondary cohomology operations, but by the Pontrjagin square, which is an unstable primary cohomology operation.

9 Oriented Manifolds

In this section we consider orientable fibrations $\xi : B \to BO$ (i.e., $w_1(\xi) = 0$), for example $B = BSO$. In particular, ξ-manifolds are also orientable. On the one hand, there exist then Brown-Kervaire invariants $K : Q^{\xi}_{4m} \times \Omega^{\xi}_{4m} \longrightarrow \mathbb{Z}/8$ by proposition 8.3.2.

On the other hand, after fixing a orientation $o : M\xi \to H\mathbb{Z}$ for ξ, one gets for each closed d-dimensional ξ-manifold M a orientation class $[M]_o \in H_d(M;\mathbb{Z})$ and can thus define the signature homomorphism $sign : \Omega^{\xi}_{4m} \longrightarrow \mathbb{Z}$.

Here, we want to compare these bordism invariants. This is possible, because there is in this case a distinguished Brown-Kervaire invariant: The Pontrjagin squaring operation $\wp : H^*(\quad;\mathbb{Z}/2) \to H^{2*}(\quad;\mathbb{Z}/4)$, together with the orientation, gives a canonical element $h_{\wp,o} \in Q^{\xi}_{4m}$ which is characterized by $q_{\wp,o}(x) = \langle \wp(x), [M]_o \rangle \in \mathbb{Z}/4$ for all $x \in H^{2m}M$. By a theorem of Morita, we have then $K_{\wp,o}(M) \equiv sign(M)$ mod 8.

With the addition formula 1.2.4, we can then express all Brown-Kervaire invariants for Ω^{ξ}_{4m} as $i^4_2\langle \nu^*\wp(x), [M]_o\rangle + sign(M)$ mod 8, with $x \in H^{2m}B$. The problem reduces thus to the computation of $\wp$ on $H^{2m}B$ and the corresponding $\mathbb{Z}/4$-characteristic numbers for ξ-bordism. For oriented bordism Ω^{SO}_{4m}, the action of $\wp$ on $H^{2m}BSO$ follows from known properties of $\wp$, and the characteristic numbers are also known, as Ω^{SO}_* was completely determined.

9.1 The Pontrjagin Square and the Theorem of Morita

9.1.1 The Pontrjagin square (see [51], [46]) is an unstable primary cohomology operation

$$\wp : H^*(\quad;\mathbb{Z}/2) \to H^{2*}(\quad;\mathbb{Z}/4)$$

which is defined on the level of cochains by

$$\wp(x) := x \cup_0 x + x \cup_1 \delta x,$$

where $x \in C^k(X;\mathbb{Z})$ is an integral cochain, and the construction of integral cup-i products $\cup_0$, $\cup_1$ can be found for example in [46]. The coboundary formula for the $\cup_i$-products shows that for x a cocycle mod 2, $\wp(x)$ gives a cocycle mod 4, whose cohomology class depends only on the cohomology class of x. We give now some properties of the Pontrjagin square, combining results in [13] and [70]:

9.1.2 Theorem: *The Pontrjagin square is uniquely characterized by the following properties:*

$$\begin{array}{lrcl} (1) & r^2_4\wp(x) & = & x^2 \\ (2) & \wp(r^2_4 y) & = & y^2 \\ (3) & \sigma^{-1}\wp(\sigma x) & = & i^4_2(xSq^1x) \end{array}$$

where $x \in H^kX$ and $y \in H^k(X;\mathbb{Z}/4)$, $\sigma : H^*(X;A) \to H^{*+1}(SX;A)$ is the suspension isomorphism, and $i^4_2 : H^*(\quad;\mathbb{Z}/2) \to H^*(\quad;\mathbb{Z}/4)$ as well as $r^2_4 : H^*(\quad;\mathbb{Z}/4) \to H^*(\quad;\mathbb{Z}/2)$

denote the coefficient homomorphisms induced by $0 \to \mathbb{Z}/2 \stackrel{i_2^4}{\to} \mathbb{Z}/4 \stackrel{r_4^2}{\to} \mathbb{Z}/2 \to 0$. *It has the following further properties:*

$$(4) \quad \wp(x+x') \;=\; \wp(x)+\wp(x')+\begin{cases} 0 & \text{for } k \text{ odd} \\ i_2^4(xx') & \text{for } k \text{ even} \end{cases}$$

$$(5) \quad \wp(xy) \;=\; \wp(x)\wp(y)+i_2^4((xSq^1x)(Sq^{l-1}y)+(Sq^{k-1}x)(ySq^1y))$$

where $x, x' \in H^kX$ *and* $y \in H^lX$.

9.1.3 *Remarks:*
(i) The cohomology operation $\wp' : H^*(\quad:\mathbb{Z}/2) \to H^{2*+1}(\quad;\mathbb{Z}/4)$ with

$$\wp'(x) := i_2^4(xSq^1x)$$

is also called the *Postnikov square* (see [13]). Thus in 9.1.2, (3) says that the suspension of $\wp$ is $\wp'$, and (5) can be formulated as $\wp(xy) = \wp(x)\wp(y) + \wp'(x)(Sq^{l-1}y) + (Sq^{k-1}x)\wp'(y)$.

(ii) In [70], (4) is formulated as $\wp(x+x') = \wp(x)+\wp(x')+(1+(-1)^{|x|})(xx')$, where '2' has here the meaning of i_2^4.

9.1.4 Thus one obtains for a $4m$-dimensional closed oriented manifold M^{4m} a $\mathbb{Z}/4$-valued quadratic refinement $q : H^{2m}M^{4m} \to \mathbb{Z}/4$ of the $\mathbb{Z}/2$-intersection pairing by

$$q(x) := \langle \wp(x), [M^{4m}] \rangle,$$

where $[M] \in H_{4m}(M;\mathbb{Z}/4)$ denotes the orientation class and $\langle\ ,\ \rangle$ is the Kronecker pairing in cohomology with $\mathbb{Z}/4$-coefficients. It is now natural to consider the invariant $\widetilde{\mathrm{Arf}}(q) \in \mathbb{Z}/8$ of M^{4m}. Brown in [17] conjectured the following theorem, which was proved by Morita:

9.1.5 Theorem (Morita [45]): $\widetilde{\mathrm{Arf}}(q) \equiv sign(M^{4m}) \bmod 8$.

9.1.6 In fact, $\widetilde{\mathrm{Arf}}(q)$ is a Brown-Kervaire invariant (this is implicitely stated by Brown in [17]). The corresponding parameter $h_\wp \in Q_{4m}^{SO}$ is given by the map

$$\pi_{4m}(MSO \wedge K_{2m}) \xrightarrow{o\wedge\wp} \pi_{4m}(H\mathbb{Z} \wedge K(\mathbb{Z}/4, 4m)) = H_{4m}(K(\mathbb{Z}/4,4m);\mathbb{Z}) = \mathbb{Z}/4,$$

where $o : MSO \to H\mathbb{Z}$ denotes the orientation. More general, let $\xi : B \to BO$ any orientable fibration (i.e., $w_1(\xi) = 0$), and choose an orientation $o : B \to H\mathbb{Z}$ (there are only two choices as B is always asumed to be connected). This defines a lift $f : B \to BSO$ of ξ, and one obtains by the naturality 1.2.1 a Brown-Kervaire invariant

$$K_{\wp,o}(M^{4m}) := \widetilde{\mathrm{Arf}}(x \mapsto \langle \wp(x), [M^{4m}]_o \rangle)$$

for Ω_{4m}^ξ with parameter $h_{\wp,0} := f^*h_\wp$. It also has the property $K_{\wp,o}(M) \equiv sign(M) \bmod 8$.

9.1.7 Now we consider the remaining Brown-Kervaire invariants for Ω_{4m}^ξ with ξ and o as above. As Q_{2n}^ξ is an affine space with H^nB the associated vector space, the parameters

$h \in Q^{\xi}_{4m}$ are given by $h = x + h_{\wp,o}$ with $x \in H^{2m}B$. Applying the addition formula 1.2.4 gives

$$\begin{aligned} K_h &= K_{\wp,o}(M) - i_4^8\langle \wp(\nu^* x), [M]_o\rangle \\ &\equiv sign(M) - i_4^8\langle \nu^*\wp(x), [M]_o\rangle \bmod 8. \end{aligned}$$

In particular, in this case the determination of all Brown-Kervaire invariants is reduced to

1. the signature homomorphism $sign : \Omega^{\xi}_{4m} \to \mathbb{Z}$,
2. the computation of $\wp : H^{2m}B \to H^{4m}(B; \mathbb{Z}/4)$,
3. the determination of all $\mathbb{Z}/4$-valued characteristic numbers of the form $\langle \nu^*\wp(x), [M]_o\rangle$.

9.2 Oriented Bordism

We want to show how this determination works for oriented bordism Ω^{SO}_*. First we recall the cohomology of BSO with $\mathbb{Z}/2$- and $\mathbb{Z}[\frac{1}{2}]$-coefficients:

9.2.1 Proposition ([44]): *The cohomology rings of BSO with $\mathbb{Z}/2$- and $\mathbb{Z}[\frac{1}{2}]$-coefficients is given as follows:*

$$\begin{aligned} H^*BSO &= \mathbb{Z}/2[w_k \,|\, k \geq 2], \\ &= \mathbb{Z}/2[w_2, w_3, w_4, w_5, ...], \\ H^*(BSO; \mathbb{Z}[\tfrac{1}{2}]) &= \mathbb{Z}[\tfrac{1}{2}][p_1, p_2, p_3, ...], \end{aligned}$$

where $w_n \in H^nBSO$ and $p_n \in H^{4n}(BSO; \mathbb{Z}[\frac{1}{2}])$ are the Stiefel-Whitney respectively Pontrjagin classes of the universal bundle.

9.2.2 *Remarks:*
(i) The A^*-module structure of H^*BSO is given by the Wu formula 2.1.3 together with $w_1 = 0$. In particular, $Sq^1 w_{2n} = w_{2n+1}$ and $Sq^1 w_{2n+1} = 0$. Using this together with the rational cohomology, Borel showed in [10] that the torsion in $H^*(BSO; \mathbb{Z})$ has order two. This gives the additive structure of $H^*(BSO; \mathbb{Z})$ as follows:

Let $p_{T^*BSO}(t)$ be the Poincaré series of the torsion part $Tors(H^*(BSO; \mathbb{Z}))$, and $p_{H^*BSO}(t) = \prod_{n \geq 2}(1 - t^n)^{-1}$, $p_{F^*BSO}(t) = \prod_{n \geq 1}(1 - t^{4n})^{-1}$ be the Poincaré series of H^*BSO and the free part $(H^*(BSO; \mathbb{Z})/Tors(H^*(BSO; \mathbb{Z}))) \otimes \mathbb{Z}/2$, respectively. By the universal coefficient theorem, we have $H^nBSO \cong (H^n(BSO; \mathbb{Z}) \otimes \mathbb{Z}/2) \oplus Tors(H^{n+1}(BSO; \mathbb{Z}))$, hence $p_{H^*BSO}(t) = p_{F^*BSO}(t) + (1 + t^{-1})p_{T^*BSO}(t)$, giving

$$\begin{aligned} p_{T^*BSO}(t) &= t(1+t)^{-1}\left(\prod_{n \geq 2}(1 - t^n)^{-1} - \prod_{n \geq 1}(1 - t^{4n})^{-1}\right) \\ &= 1 + t^3 + t^5 + t^6 + 3t^7 + t^8 + \dots . \end{aligned}$$

Thus, we obtain the following table:

$n:$	0	1	2	3	4	5	6	7	8	...
$H^n(BSO;\mathbb{Z}):$	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}^2 \oplus \mathbb{Z}/2$	...

(ii) We have $H^*BSO = \mathbb{Z}/2[w_k \,|\, \alpha(k-1) \geq 1]$, where $\alpha : \mathbb{N} \to \mathbb{N}$ counts the number of one's in the dyadic development of an integer. One can compare this with 2.1.2 and 10.1.1, but also look at remark 10.1.2.

Now we recall the computation of $\Omega_*^{SO} = \pi_* MSO$ by Thom, Milnor, Wall, and others (see [63]):

9.2.3 Theorem ([63]): *The following results hold for the oriented bordism ring:*

(1) One has $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, ...]$.

(2) One has $\Omega_*^{SO}/Tors(\Omega_*^{SO}) = \mathbb{Z}[M^4, M^8, M^{12}, ...]$*, where an oriented closed manifold* M^{4m} *can be taken as a polynomial generator iff the characteristic number* $s_m[M^{4m}] \in \mathbb{Z}$ *(see [63]) satisfies*

$$s_m[M^{4m}] = \begin{cases} \pm 1, & \textit{if } 2m+1 \textit{ is not a prime power} \\ \pm p, & \textit{if } 2m+1 = p^s \textit{ with } p \textit{ a prime} \end{cases}$$

(3) $Tors(\Omega_*^{SO})$ *is a* $\mathbb{Z}/2$*-algebra. Let* $W_* := \mathbb{Z}/2[x_j, (x_{2^s})^2 \,|\, j \neq 2^s, 2^s - 1]$ *and* $\partial : W_* \to W_*$ *be the derivation given by*

$$\partial x_{2m-1} := 0, \quad \partial x_{2m} := x_{2m-1}, \quad \partial (x_{2^s})^2 := 0.$$

Then there is an isomorphism $Tors(\Omega_*^{SO}) \cong im(\partial)$.

(4) Let I *be a partition of* $|I| =: m$ *and let* $p_I : MSO \to \Sigma^{4m} H\mathbb{Z}$ *be the map corresponding to the Pontrjagin number* $p_I[\ ] : \Omega_{4m}^{SO} \to \mathbb{Z}$*. Then there exist homogeneous elements* $t_i \in H^*MSO = [MSO, \Sigma^* H]$*,* $i \in T$*, such that the map*

$$(\pi_I, t_i) : MSO \longrightarrow (\bigvee_I \Sigma^{|I|} H\mathbb{Z}) \vee (\bigvee_{i \in T} \Sigma^{|t_i|} H)$$

gives a 2*-primary homotopy equivalence of spectra. Since* Ω_*^{SO} *has no odd torsion, one gets an additive isomorphism*

$$\Omega_*^{SO} \cong F_* \oplus T_*,$$

where $F_* \cong \Omega_*^{SO}/Tors(\Omega_*^{SO})$ *is the graded free abelean group generated by elements* p_I *of order* $4|I|$ *for all partitions* I*, and* $T_* \cong Tors(\Omega_*^{SO})$ *is the graded* $\mathbb{Z}/2$*-vector space generated by the* t_i*,* $i \in T$ *of order* $|t_i|$*. Since the* t_i *can be expressed by Stiefel-Whitney numbers, two oriented manifolds are oriented cobordant, iff they have the same Pontrjagin numbers and Stiefel-Whitney numbers.*

In fact, Wall constructed explicit manifolds representing the generators in (2) and (3), see [68]. Also, all relations between Pontrjagin numbers and Stiefel-Whitney numbers were determined ([63], [68]).

9.2.4 *Remark:* Analogeously to Ω_*^{Spin} (see 2.1.11), it is possible to compute all degrees $|t_i|$ by considering the Poincaré-Hilbert series in the equation

$$H^*MSO \cong (F_* \otimes H^*(H\mathbb{Z})) \oplus (T_* \otimes A^*).$$

Let $p_{MSO}(t)$, $p_{A_0}(t)$, $p_F(t)$, and $p_T(t)$ be the Hilbert series of the graded $\mathbb{Z}/2$-vector spaces H^*MSO, $H^*(H\mathbb{Z}) \cong A^*/(A^*Sq^1) =: A_0$, $F_* \otimes \mathbb{Z}/2$, and T_*, respectively. One knows that

$$\begin{array}{lcl} p_{MSO}(t) & = & \prod_{n\geq 2}(1-t^n)^{-1} \\ p_A(t) & = & \prod_{n=2^r-1, r\geq 1}(1-t^n)^{-1} \\ p_{A_0}(t) & = & (1-t^2)^{-1}\prod_{n=2^r-1, r\geq 2}(1-t^n)^{-1} \\ p_F(t) & = & \prod_{n\geq 1}(1-t^{4n})^{-1}, \end{array}$$

since $H^*MSO \cong H^*BSO = \mathbb{Z}/2[w_i \,|\, i \geq 2]$; for p_{A_0} see [46]. Furthermore, $p_F(t)$ follows from $F_* \cong \mathbb{Z}[p_i \,|\, i \in \mathbb{N}]$. This gives $p_T(t) = p_A(t)^{-1}(p_{MSO}(t) - p_F(t)p_{A_1}(t))$, hence:

9.2.5 Proposition: *The Poincaré-Hilbert series of the torsion part in Ω_*^{SO} is given by*

$$p_T(t) = (1+t)^{-1}\prod_{n\geq 1}(1-t^{4n})^{-1}\left(\Big(\prod_{n\neq 2^r-1, n\neq 0 \bmod 4, n\geq 5}(1-t^n)^{-1}\Big) - 1\right).$$

9.3 Determination of all Brown-Kervaire Invariants

In order to compute the Brown-Kervaire invariants for Ω_*^{SO}, we first want to understand $H^*(BSO;\mathbb{Z}/4)$ and then consider the Pontragin square $\wp$ on H^*BSO. Let

$$\ldots \xrightarrow{\beta} H^*(BSO;\mathbb{Z}) \xrightarrow{\cdot 2} H^*(BSO;\mathbb{Z}/2) \xrightarrow{r_0^2} H^*BSO \xrightarrow{\beta} H^{*+1}(BSO;\mathbb{Z}) \longrightarrow \ldots$$

be the Bockstein long exact sequence associated to the exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{r_0^2} \mathbb{Z}/2 \longrightarrow 0.$$

It is well-known that $r_0^2\beta = Sq^1$. Now, Brown determined $H^*(BSO;\mathbb{Z})$ as follows:

9.3.1 Theorem (Brown [19]): *Let $p_k := (-1)^k c_{2k}(\gamma_{SO}\otimes\mathbb{C}) \in H^*(BSO;\mathbb{Z})$, $k \in \mathbb{N}$, be the integral Pontragin classes of the stable universal bundle γ_{SO}. Then there is ring isomorphism:*

$$H^*(BSO;\mathbb{Z}) = \frac{\mathbb{Z}[p_1, p_2, p_3, \ldots, \beta(w_{2I}) \,|\, 0 < i_1 < i_2 < \ldots < i_s,\ s \in \mathbb{N}]}{2\beta(w_{2I}),\ \beta(w_{2I})\beta(w_{2J}) - \sum_{k\in I}\beta(w_{2k})p_{K'}\beta(w_{K''})},$$

where $w_{2I} := w_{2i_1}\ldots w_{2i_s}$ and $p_I := p_{i_1}\ldots p_{i_s}$ for $I = (i_1,..,i_s)$; and in the sum, one has $K' := (I-k)\cap J$ and $K'' := ((I-k)\cup J) - K'$. Furthermore, $r_0^2(p_k) = w_{2k}^2$, and under Whitney sum $\oplus : BSO \times BSO \to BSO$ one has:

$$\begin{array}{lcl} \oplus^*(w_k) & = & \sum_{i=0}^k w_i \otimes w_{k-i}, \\ \oplus^*(p_k) & = & \sum_{i=0}^{2k} u_i \otimes u_{2k-i}, \end{array}$$

where $u_{2i} := p_i$ and $u_{2i+1} := (\beta w_{2i})^2 + p_i\beta w_1$.

9.3.2 For $H^*(BSO;\mathbb{Z}/4)$, we consider the following commutative diagram of Bockstein long exact sequences (the maps are self-explaining):

$$
\begin{array}{ccccccccc}
 & & & & \dots & & \dots & & \\
 & & & & \downarrow{\scriptstyle p_0^2} & & \downarrow{\scriptstyle p_4^2} & & \\
 & & & & H^{n-1}BSO & = & H^{n-1}BSO & & \\
 & & & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle Sq^1} & & \\
\dots \xrightarrow{\beta} & H^n(BSO;\mathbb{Z}) & \xrightarrow{\cdot 2} & & H^n(BSO;\mathbb{Z}) & \xrightarrow{p_0^2} & H^nBSO & \xrightarrow{\beta} & H^{n+1}(BSO;\mathbb{Z}) \xrightarrow{\cdot 2} \dots \\
 & \| & & & \downarrow{\scriptstyle \cdot 2} & & \downarrow{\scriptstyle i_2^4} & & \| \\
\dots \xrightarrow{\beta'} & H^n(BSO;\mathbb{Z}) & \xrightarrow{\cdot 4} & & H^n(BSO;\mathbb{Z}) & \xrightarrow{p_0^4} & H^n(BSO;\mathbb{Z}/4) & \xrightarrow{\beta'} & H^{n+1}(BSO;\mathbb{Z}) \xrightarrow{\cdot 4} \dots \\
 & & & & \downarrow{\scriptstyle p_0^2} & & \downarrow{\scriptstyle p_4^2} & & \\
 & & & & H^nBSO & = & H^nBSO & & \\
 & & & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle Sq^1} & & \\
 & & & & \dots & & \dots & &
\end{array}
$$

We denote the torsion subgroup of $H^n(BSO;\mathbb{Z})$ by $T^n := Tors(H^n(BSO;\mathbb{Z}))$; by theorem 9.3.1 of Brown we have that T^n is the ideal generated by the elements $\beta(w_{2I})$ of order two. Now, the second row in the above diagram shows that $H^n(BSO;\mathbb{Z}/4)$ is an extension of $ker(\cdot 4) = T^{n+1}$ with $coker(\cdot 4) = H^n(BSO;\mathbb{Z}) \otimes \mathbb{Z}/4$ (which by the universal coefficient theorem is trivial). As T^{n+1} is also equal to $ker(\cdot 2)$, it follows from the diagram that $H^n(BSO;\mathbb{Z}/4)$ is generated by $p_0^4(H^n(BSO;\mathbb{Z}))$ and $i_2^4(H^nBSO)$. In particular, all $H^*(BSO;\mathbb{Z}/4)$-characteristic numbers for Ω_*^{SO} are given by Pontrjagin numbers reduced mod 4 and Stiefel-Whitney numbers embedded in $\mathbb{Z}/4$ by i_2^4.

At last we come to the description of the action of $\wp$ on H^*BSO. By the product formula in 9.1.2, we only have to know $\wp$ on the Stiefel-Whitney classes. This was determined by Wu in the (more general) unoriented case of H^*BO:

9.3.3 Theorem (Wu [70]): *The Pontrjagin square $\wp : H^*BO \to H^{2*}(BO;\mathbb{Z}/4)$ takes on the Stiefel-Whitney classes the values*

$$
\begin{array}{lcl}
\wp(w_{2n+1}) & = & r_0^4\beta(Sq^{2n}w_{2n+1}) + i_2^4(w_1 Sq^{2n}w_{2n+1}), \\
\wp(w_{2n}) & = & r_0^4(p_n) + r_0^4\beta(w_{2n-1}w_{2n}) + i_2^4(w_1 Sq^{2n-1}w_{2n} + \sum_{i=0}^{n-1} w_{2i}w_{4n-2i}).
\end{array}
$$

Of course, this gives the action of $\wp$ on H^*BSO by naturality and the fact, that $w_1 = 0$.

9.3.4 We consider dimension 4 as an example: Since $H^2BSO = \mathbb{Z}/2 \cdot w_2$, there are at most two Brown-Kervaire invariants on Ω_4^{SO}. In fact, one has the two Brown-Kervaire invariants

$$\begin{array}{rclcccl} K_{\wp} & \equiv & sign(\ ) \bmod 8 & : & \Omega_4^{SO} & \longrightarrow & \mathbb{Z}/8, \\ K_{-\wp} & \equiv & -sign(\ ) \bmod 8 & : & \Omega_4^{SO} & \longrightarrow & \mathbb{Z}/8. \end{array}$$

This follows also from $\wp(w_2) = r_0^4(p_1) + i_2^4(w_4)$, giving by the addition formula 1.2.4 that

$$K_{w_2+\wp} \equiv sign(\ ) - (2p_1[\ ] + i_2^8(w_4[\ ])) \bmod 8.$$

But this takes the value $-1 \bmod 8$ on the generator $\mathbb{C}P^2 \in \Omega_4^{SO} \cong \mathbb{Z}$, so $K_{w_2+\wp} = K_{-\wp}$.

10 $BO\langle 8\rangle$-Manifolds

The multiplicativity problem 4.2.8 for Brown-Peterson-Kervaire invariants of $\mathbb{H}P^2$-bundles of *Spin*-manifolds, which led us to the Main Theorem 7.1.1, suggests a corresponding question for $\mathbb{C}ayP^2$-bundles of $BO\langle 8\rangle$-manifolds. Here, $\mathbb{C}ayP^2$ is the Cayley projective plane, which has a representation as a homogeneous space $F_4/Spin(9)$, and is a 16-dimensional $BO\langle 8\rangle$-manifold with signature 1. As we are here in $BO\langle 8\rangle$-bordism, we consider Brown-Peterson-Kervaire invariants in dimension $16m+6$

$$K_\phi : \Omega^{\langle 8\rangle}_{16m+6} \longrightarrow \mathbb{Z}/2,$$

and ask for the existence of invariants K_ϕ, $K_{\phi'}$ such that $K_\phi(N^{16m+22}) = K_{\phi'}(M^{16m+6})$ for fibre bundles $N^{16m+22} \to M^{16m+6}$ of closed $BO\langle 8\rangle$-manifolds with fibre $\mathbb{C}ayP^2$ and structure group F_4. We study this problem at first with our product formula 5.2.2, and then give some remarks on the application of Kristensen's product formula 6.3.5. We conclude with some remarks and conjectures concerning the generalization of our Main Theorem 7.1.1 to the case of $BO\langle 8\rangle$-manifolds.

10.1 $BO\langle 8\rangle$-Bordism

We first give a survey on some known results on $BO\langle 8\rangle$-bordism $\Omega^{\langle 8\rangle}_*$. In contrast to *Spin*-bordism, there is no known general formula for the additive structure of $\Omega^{\langle 8\rangle}_*$, which seems to be caused by three reasons (see [23]):

1. The structure of $H^*MO\langle 8\rangle$ as an A^*-module is very complicated, with no known general decomposition into smaller parts (although it is explicitly given in 10.1.1).
2. There are non-trivial differentials in the Adams spectral sequence for $\Omega^{\langle 8\rangle}_* = \pi_* MO\langle 8\rangle$.
3. There is also 3-primary torsion in $\Omega^{\langle 8\rangle}_*$, with the same problems as in (1) and (2).

Here, we consider only the 2-primary part of $BO\langle 8\rangle$-bordism. We recall the principal fibration $K(\mathbb{Z},3) \to BO\langle 8\rangle \to BSpin$ with classifying map $\frac{p_1}{2} : BSpin \to K(\mathbb{Z},4)$. Using that $H^*K(\mathbb{Z},3) = \mathbb{Z}/2[Sq^I\iota_3 \,|\, I = (i_1,..,i_r),\ I \text{ admissible},\ i_r > 1,\ excess(I) < 3]$, Stong computed the structure of $H^*BO\langle 8\rangle$:

10.1.1 Theorem (Stong [61], see also [23]): *The map $H^*BSpin \to H^*BO\langle 8\rangle$ is surjective, and there is an isomorphism as an algebra*

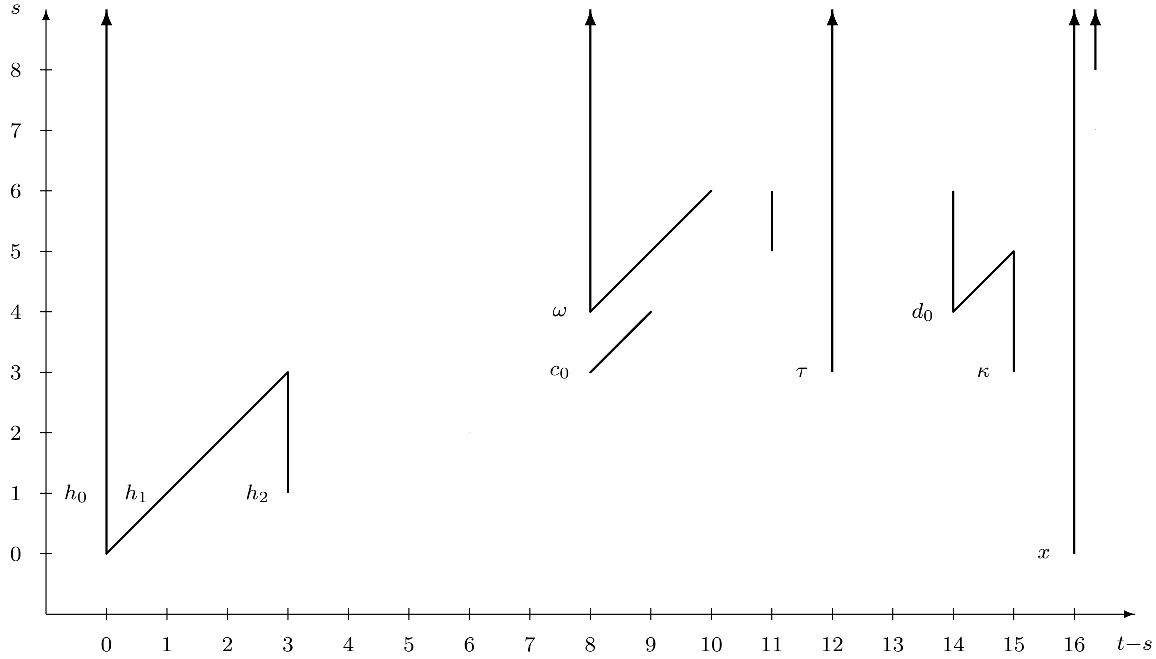
$$\begin{aligned} H^*BO\langle 8\rangle &\cong \mathbb{Z}/2[w_k \,|\, k \neq 1, 2^r+1, 2^s+2^r+1, \ \ \textit{where } s > r \geq 0] \\ &= \mathbb{Z}/2[w_k \,|\, \alpha(k-1) \geq 3] \\ &= \mathbb{Z}/2[w_8, w_{12}, w_{14}, w_{15}, w_{16}, w_{20}, ...], \end{aligned}$$

where $\alpha : \mathbb{N} \to \mathbb{N}$ counts the number of one's in the dyadic expansion of a natural number.

10.1.2 *Remark:* In fact, Stong computed in [61] the $\mathbb{Z}/2$-cohomology of all $(n-1)$-connected covers $BO\langle n\rangle$ of BO. But beyond $BO\langle 2\rangle = BSO$, $BO\langle 4\rangle = BSpin$ and $BO\langle 8\rangle$, the structure of $H^*BO\langle n\rangle$ becomes more complicated and is not only given by Stiefel-Whitney classes. In particular, the projection maps to BO are then not further surjective in cohomology.

10.1.3 By the Thom isomorphism $H^*MO\langle 8\rangle = H^*BO\langle 8\rangle \cdot U_{\langle 8\rangle}$, we can now compute the A^*-module structure of $H^*MO\langle 8\rangle$ up to any dimension. But in contrast to *Spin*-bordism (2.1.9), no general decomposition into smaller sub-modules is known. At least, one knows that $H^*MO\langle 8\rangle$ is an extended A_2^*-module, where A_2^* is the Hopf sub-algebra $A_2^* = \langle Sq^1, Sq^2, Sq^4\rangle$ of the Steenrod algebra A^* (see [50]). The computation of the Adams spectral sequence for $\Omega_*^{\langle 8\rangle}$ in low dimensions was first done by Giambalvo:

10.1.4 Theorem (Giambalvo [23]): *The E_2-term of the Adams spectral sequence for $\Omega_*^{\langle 8\rangle}$ has for $t-s \leq 16$ nine multiplicative generators $h_0, h_1, h_2, c_0, \omega, \tau, d_0,$ and x:*



In the diagram, the multiplication with h_0 and h_1 is indicated by the vertical and diagonal lines (the multiplicative relations between the other generators can be found in [23]). All differentials in this range are determined by $d_2(\kappa) = h_0 d_0$, $d_2(\tau) = h_2\omega$, and $d_r(x) = 0$ for all $r \geq 2$. The 2-primary part of $\Omega_*^{\langle 8\rangle}$ is given by:

$n:$	0	1	2	3	4	5	6	7	8
${}_{(2)}\Omega_n^{\langle 8\rangle}:$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$

$n:$	9	10	11	12	13	14	15	16
${}_{(2)}\Omega_n^{\langle 8\rangle}:$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z})^2$

10.1.5 *Remark:*
(i) In [23], there was shown that $\Omega_*^{\langle 8\rangle}$ contains also 3-torsion, but no p-torsion for $p \geq 5$. All 3-torsion in the range $t - s \leq 16$ is given by summands $\mathbb{Z}/3$ in dimensions 3, 10, and 13.

(ii) The natural map $\Omega_*^{\mathrm{fr}} \to \Omega_*^{\langle 8\rangle}$ is an isomorphism in dimension $* \leq 6$, see 8.3.6. In particular, the generators of $\Omega_k^{\langle 8\rangle}$ for $k = 1, 2, 3, 6$ are given by $\overline{S^1}$, $(\overline{S^1})^2$, $\overline{S^3}$, and $(\overline{S^3})^2$. Here, $\overline{S^1}$ and $\overline{S^3}$ denote the spheres S^1 and S^3 with the non-trivial framings, which are obtained by twisting the trivial framings with the generators in $\pi_1 O = \mathbb{Z}/2$ and $\pi_3 O = \mathbb{Z}$.

10.2 Cayley Projective Plane Bundles

10.2.1 In dimension 16 there exists an interesting closed $BO\langle 8\rangle$-manifold, the Cayley projective plane $\mathbb{C}ayP^2$ (see [11], [69]). It is constructed as $\mathbb{C}ayP^2 := S^8 \cup_\sigma D^{16}$, where $\sigma : S^{15} \to S^8$ is the Hopf map given by the division in the Cayley octaves $\mathbb{C}ay$. Thus $\mathbb{C}ayP^2$ has cohomology ring

$$H^*(\mathbb{C}ayP^2; \mathbb{Z}) = \mathbb{Z}[x]/x^3,$$

where $x \in H^8(\mathbb{C}ayP^2; \mathbb{Z})$ is the canonical 8-dimensional generator coming from S^8. In particular, $sign(\mathbb{C}ayP^2) = 1$. $\mathbb{C}ayP^2$ has also a representation as a homogeneous space (see [11], [69])

$$\mathbb{C}ayP^2 \cong F_4/Spin(9),$$

where F_4 denotes the compact 1-connected Lie group with exceptional Lie algebra F_4, and $Spin(9) \subset F_4$ is the standard inclusion. The tangential Stiefel-Whitney classes of $\mathbb{C}ayP^2$ are clearly given by

$$w(T_{\mathbb{C}ayP^2}) = 1 + x + x^2;$$

and the tangential Pontrjagin classes of $\mathbb{C}ayP^2$ were computed in [11] as

$$p(T_{\mathbb{C}ayP^2}) = 1 + 6x + 39x^2,$$

with x denoting also the reduction mod 2 of x.

10.2.2 Now we consider fibre bundles $E \to B$ with fibre $\mathbb{C}ayP^2$ and structure group F_4, which we shortly call *F_4-$\mathbb{C}ayP^2$-bundles.* There is the universal F_4-$\mathbb{C}ayP^2$-bundle

$$\mathbb{C}ayP^2 \longrightarrow BSpin(9) \longrightarrow BF_4,$$

and we have the following results (compare with $PSp(3)$-$\mathbb{H}P^2$-bundles in 4.1.2):

10.2.3 Theorem (Quillen ([52]): *The cohomology ring of $BSpin(9)$ is given by the polynomial ring*

$$H^*BSpin(9) = \mathbb{Z}/2[w_4, w_6, w_7, w_8, \Delta_{16}],$$

where w_i is the restriction of the universal Stiefel-Whitney class, and Δ_{16} is the Stiefel-Whitney class $w_{16}(\Delta_{Spin(9)})$ of the Spin-representation $\Delta_{Spin(9)} : Spin(9) \to O(16)$.

10.2.4 Theorem (Borel [9], see also [67]): *The cohomology ring of BF_4 is given as the polynomial ring*

$$H^*BF_4 = \mathbb{Z}/2[x_4, Sq^2x_4, Sq^3x_4, x_{16}, Sq^8x_{16}],$$

where x_4, x_{16} denote polynomial generators of dimension 4 and 16 respectively.

10.2.5 Corollary: *Via the induced map of the inclusion, $H^*BSpin(9)$ is a free H^*BF_4-module on generators 1, x, x^2, with $x \in H^8BSpin(9)$ denoting the universal Leray-Hirsch generator mapping to $x \in H^8\mathbb{C}ayP^2$. The elements $x_4, Sq^2x_4, Sq^3x_4 \in H^*BF_4$ are mapped to $w_4, w_6, w_7 \in H^*BSpin(9)$. We have*

$$Sq\,x = x + Sq^4x + Sq^6x + Sq^7x + x^2,$$

and the Leray-Hirsch theorem holds for any F_4-$\mathbb{C}ayP^2$-bundle.

Proof: First we show, that the Serre spectral sequence of $\mathbb{C}ayP^2 \to BSpin(9) \to BF_4$ collapses. We recall the following well-known fact ([40]):

Assume that cohomology is taken with coefficients in a field $\mathbb{K}$. Let $F \to E \to B$ be a fibration (F connected, B 1-connected), where we assume the cohomology rings H^*F, H^*E, H^*B to be locally finite dimensional. Then the Poincaré series $p(\ ) := \sum_{n\geq 0} t^n dim_{\mathbb{K}} H^n(\ ) \in \mathbb{Z}[[t]]$ are defined and it holds $p(E) \leq p(F)p(B)$. Moreover, $p(E) = p(F)p(B)$ iff the Serre spectral sequence collapses.

Now, the above claim follows with 10.2.3 and 10.2.4, since

$$\begin{aligned}\frac{p(E)}{p(B)} &= \frac{(1-t^4)^{-1}(1-t^6)^{-1}(1-t^7)^{-1}(1-t^8)^{-1}(1-t^{16})^{-1}}{(1-t^4)^{-1}(1-t^6)^{-1}(1-t^7)^{-1}(1-t^{16})^{-1}(1-t^{24})^{-1}} = \\ &= \frac{1-t^{24}}{1-t^8} = 1 + t^8 + t^{16} = p(F).\end{aligned}$$

In particular, $H^*BSpin(9)$ is a free H^*BF_4-module on $1, x, x^2$, where either $x = w_8$ or $x = w_8 + w_4^2$. In both cases, one computes by the Wu formula and with $w_1 = w_2 = 0$ that $Sq^1x = Sq^2x = Sq^3x = Sq^5x = 0$. Moreover, x_4 is mapped to w_4, and one has $w_6 = Sq^2w_4$, $w_7 = Sq^3w_4$ in $H^*BSpin(9)$. As the Leray-Hirsch theorem holds for the universal bundle, it holds for all bundles. ■

10.2.6 There is an important difference between F_4-$\mathbb{C}ayP^2$-bundles over $BO\langle 8\rangle$-manifolds and $PSp(3)$-$\mathbb{H}P^2$-bundles over $Spin$-manifolds: Whereas the total spaces of the latter are again $Spin$-manifolds, the corresponding property for F_4-$\mathbb{C}ayP^2$-bundles over $BO\langle 8\rangle$-manifolds is in general not true. The reason is that the tangent bundle along the fibres T^Δ of the universal F_4-$\mathbb{C}ayP^2$-bundles $\mathbb{C}ayP^2 \to BSpin(9) \to BF_4$ has no $BO\langle 8\rangle$-structure (but it has a unique $Spin$-structure as $H^iBSpin(9) = 0$ for $i = 1, 2, 3$). To see this, we compute the first Pontrjagin class of T^Δ according to Borel and Hirzebruch [11] as follows: The complementary roots of $i : Spin(9) \hookrightarrow F_4$ are the 16 roots $\frac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4)$, where x_i denote the standard linear forms on the Lie algebra of $SO(n)$. The total Pontrjagin class $p(T^\Delta) \in H^*(BSpin(9); \mathbb{Q})$ is then given by their product

$$\frac{1}{4}\prod(\pm x_1 \pm x_2 \pm x_3 \pm x_4),$$

in particular $p_1(T^\Delta) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) =: \hat{x} \in H^4(BSpin(9); \mathbb{Q})$, which is also invariant under the Weyl group of F_4 and hence lives in $H^4(BF_4; \mathbb{Q}) = \mathbb{Q}$ as well.

Assume now that we have an F_4-$\mathbb{C}ayP^2$-bundles over a manifold M with classifying map $f : M \to BF_4$ and total space N:

$$\begin{array}{ccc} N & \xrightarrow{\tilde{f}} & BSpin(9) \\ \downarrow_\pi & & \downarrow_{Bi} \\ M & \xrightarrow{f} & BF_4. \end{array}$$

Then it holds $TN = \pi^* TM \oplus \tilde{f}^* T^\Delta$ and hence for the first (tangential) Pontrjagin class

$$p_1(N) = \pi^*(p_1(M) + f^* \hat{x}).$$

In the case that M is a 3-connected $BO\langle 8\rangle$-manifold, we have that $H^4(M; \mathbb{Z})$ is free and $\pi^* : H^4(M; \mathbb{Z}) \to H^4(N; \mathbb{Z})$ is an isomorphism. Thus, N is then also a $BO\langle 8\rangle$-manifold iff $f^* \bar{x} = 0 \in H^4(M; \mathbb{Z})$, where $\bar{x} \in H^4(BF_4; \mathbb{Z}) = \mathbb{Z}$ denotes the generator.

10.2.7 *Remark:* In a similar formulation as for $PSp(3)$-$\mathbb{H}P^2$-bundles over $Spin$-manifolds in 4.2.1, we do not have a transfer map from $\Omega_*^{\langle 8\rangle}(BF_4)$ to $\Omega_{*+16}^{\langle 8\rangle}$, but a transfer map

$$\Omega_*^{\langle 8\rangle}(BF_4\langle \bar{x}\rangle) \longrightarrow \Omega_{*+16}^{\langle 8\rangle}$$

where the fibration $BF_4\langle \bar{x}\rangle \longrightarrow BF_4$ is obtained from BF_4 by killing $\bar{x}$, i.e. by pulling back the path fibration $PK(\mathbb{Z}, 4) \to K(\mathbb{Z}, 4)$ with a map $\bar{x} : BF_4 \to K(\mathbb{Z}, 4)$ realizing $\bar{x}$.

10.3 Application of the Homotopy-theoretical Product Formula

10.3.1 We consider now Brown-Peterson-Kervaire invariants

$$K_\phi : \Omega_{16m+6}^{\langle 8\rangle} \longrightarrow \mathbb{Z}/2,$$

which were defined in 8.3.7 using the relation

$$Sq^{8m+4} = Sq^4 \beta_2 + Sq^2 \beta_1 + Sq^1 \beta_0,$$

$$\beta_2 := Sq^{8m}, \qquad \beta_1 := Sq^{8m} Sq^2, \qquad \beta_0 := Sq^{8m+2} Sq^1 + Sq^{8m} Sq^3$$

by $K_\phi(M^{16m+6}) := \mathrm{Arf}(z \in H^{8m+3} M^{16m+6} \mapsto \phi(z)[M^{16m+6}] \in \mathbb{Z}/2)$. Here, M^{16m+6} is a closed 3-connected $BO\langle 8\rangle$-manifold of dimension $16m+6$ and ϕ is a Brown-Peterson secondary cohomology operation associated to the above relation. (Remark: In the proof of 8.3.7, it was enough to assume $H^1 M = H^2 M = H^3 M = 0$ in order to define K_ϕ. In $\Omega_*^{\langle 8\rangle}$ $(* > 6)$, this is up to bordism equivalent to our stronger assumption which is more convenient to write down.)

10.3.2 For a F_4-$\mathbb{C}ayP^2$-bundle $p : N^{16m+22} \to M^{16m+6}$ of such manifolds, we want to compare $K_\phi(N)$ and $K_{\phi'}(M)$ for Brown-Peterson-Kervaire invariants K_ϕ, $K_{\phi'}$ in the corresponding dimensions. As in 4.3.3 for $PSp(3)$-$\mathbb{H}P^2$-bundles of $Spin$-manifolds, we get by the sub Lagrangian lemma that

$$K_\phi(N^{16m+22}) = \mathrm{Arf}\Big(y \in H^{8m+3}M^{16m+6} \mapsto \phi(xp^*y)[N^{16m+22}]\Big),$$

because the Leray-Hirsch theorem 10.2.5 is satisfied. In particular, we again have reduced the multiplicativity problem to the computation of a Cartan formula for $\phi(xp^*y)$

10.3.3 First, we use our product formula 5.2.2 and proceed analogously to the proof of 5.3.3, obmitting some details of the computation. Because we have to apply K_ϕ in dimension $16m+22$ to the product xp^*y, we obtain μ and the upper half of the diagram 5.2.1:

$$\begin{array}{ccccccc}
 & & K_{8m+11} & \xrightarrow{a} & \begin{pmatrix} K_{16m+19} \\ \times K_{16m+21} \\ \times K_{16m+22} \end{pmatrix} & \xrightarrow{b} & K_{16m+23} \\
 & & \uparrow{\scriptstyle\mu} & & \uparrow{\scriptstyle\nu} & & \uparrow{\scriptstyle\eta} \\
N^{16m+22} & \xrightarrow{(\bar f, p^*y)} & E \wedge K_{8m+3} & \xrightarrow{1\wedge a'} & E \wedge B' & \xrightarrow{1\wedge b'} & E \wedge C',
\end{array}$$

Here, $E := BSpin(9)$ is the total space of the universal F_4-$\mathbb{C}ayP^2$-bundle, $a := (\beta_2, \beta_1, \beta_0)$, $b := Sq^4 + Sq^2 + Sq^1$, $\mu := x \cup \iota_{8m+3}$ is the multiplication with the Leray-Hirsch generator $x : E \to K_8$, $\bar f : N^{16m+22} \to E$ is the bundle map over the classifying map $f : M^{16m+6} \to BF_4$, and $y : M^{16m+6} \to K_{8m+3}$ is a middle dimensional cohomology class.

The lower half of the diagram follows by application of the Cartan and Adem formulas. Using $Sq\, x = x + Sq^4x + Sq^6x + Sq^7x + x^2$, the computation gives:

$$B' = \begin{pmatrix} & (K_{16m+3} & \times & K_{16m+4} & \times & K_{16m+5}) & & \\ \times & (K'_{16m+5} & \times & K'_{16m+6} & \times & K'_{16m+7} & \times & K'_{16m+9}) \\ \times & (K''_{16m+6} & \times & K''_{16m+7} & \times & K''_{16m+8} & \times & K''_{16m+10}) \end{pmatrix}$$

$$a' = \begin{pmatrix} (Sq^{8m} & , Sq^{8m+1} & , Sq^{8m+2}), & \\ (Sq^{8m}Sq^2 & , Sq^{8m+1}Sq^2 & , Sq^{8m+2}Sq^2 & , Sq^{8m+4}Sq^2), \\ (Sq^{8m+2}Sq^1 + Sq^{8m}Sq^3, & Sq^{8m+3}Sq^1 + Sq^{8m+1}Sq^3, & Sq^{8m+4}Sq^1 + Sq^{8m+2}Sq^3, & Sq^{8m+4}Sq^3) \end{pmatrix}$$

$$\nu = \begin{pmatrix} & (x^2 \cup \iota_{16m+3} & + & Sq^7x \cup \iota_{16m+4} & + & Sq^6x \cup \iota_{16m+5}) & & \\ \times & (x^2 \cup \iota'_{16m+5} & + & Sq^7x \cup \iota'_{16m+6} & + & Sq^6x \cup \iota'_{16m+7} & + & Sq^4x \cup \iota'_{16m+9} \\ \times & (x^2 \cup \iota''_{16m+6} & + & Sq^7x \cup \iota''_{16m+7} & + & Sq^6x \cup \iota''_{16m+8} & + & Sq^4x \cup \iota''_{16m+10}) \end{pmatrix}$$

$$C' = K_{16m+7} \times K_{16m+8} \times K_{16m+9} \times K_{16m+11}$$

$$b' = \begin{pmatrix} (Sq^4\iota_{16m+3} + Sq^2\iota'_{16m+5} + Sq^1\iota''_{16m+6})\times \\ (Sq^4\iota_{16m+4} + Sq^3\iota_{16m+5} + Sq^2\iota'_{16m+6} + Sq^1\iota'_{16m+7} + Sq^1\iota''_{16m+7} + \iota''_{16m+8})\times \\ (Sq^4\iota_{16m+5} + Sq^2\iota'_{16m+7} + Sq^1\iota''_{16m+8} + \iota'_{16m+9})\times \\ (Sq^2\iota'_{16m+9} + Sq^1\iota''_{16m+10}) \end{pmatrix}$$

$$\eta = x^2 \cup \iota_{16m+7} + Sq^7x \cup \iota_{16m+8} + Sq^6x \cup \iota_{16m+9} + Sq^4x \cup \iota_{16m+11}$$

Now we choose homotopies H,H',H^L,H^R and get with our product formula 5.2.2 that

$$\phi_H(xp^*y) = \epsilon_{H,H',H^L,H^R}(\bar{f}, p^*y) + (\Omega_{(2)}\eta)(\bar{f}, \phi_{H'}(p^*y)).$$

According to the above decomposition of C', the secondary operation $\phi_{H'}$ splits into four unstable secondary cohomology operations ϕ_1, ϕ_2, ϕ_3, ϕ_4, where ϕ_1 is just a Brown-Peterson secondary cohomology operation in dimension $16m+6$. The other operations ϕ_2, ϕ_3, ϕ_4 take values in dimension $16m+7$, $16m+8$, $16m+10$, respectively, hence vanish on $H^{8m+3}M^{16m+6}$. With the same arguments as in the proof of 5.3.2, one shows that

$$\Omega_{(2)}\eta = x^2 \cup \iota_{16m+6} + Sq^7x \cup \iota_{16m+7} + Sq^6x \cup \iota_{16m+8} + Sq^4x \cup \iota_{16m+10}$$

and

$$\epsilon_{H,H',H^L,H^R}(\bar{f}, p^*y) = x^2p^*\left(\sum_{i+j=8m+3} f^*(z_i)\ \alpha_j(y)\right) =: x^2p^*\epsilon(y)$$

where $f : M^{16m+6} \to BF_4$ denotes the classifying map of the F_4-$\mathbb{C}ayP^2$-bundle, and $z_i \in H^iBF_4$, $\alpha_j \in A^j$ depend only on the choice of the homotopies H,H',H^L,H^R and not on the specific bundle. Furthermore, as in the proof of 5.3.3 one can represent the map $y \mapsto \epsilon(y)$ by multiplication with the pullback $v_{\phi,\phi'} \in H^{8m+3}M^{16m+6}$ of a universal class in $H^{8m+3}(BF_4 \times BO\langle 8\rangle)$. Proceeding then as in the proof of 5.3.3, we have shown:

10.3.4 Theorem: *Let $N^{16m+22} \to M^{16m+6}$ be a F_4-$\mathbb{C}ayP^2$-bundle of closed 3-connected $BO\langle 8\rangle$-manifolds and K_ϕ, $K_{\phi'}$ be two Brown-Peterson-Kervaire invariants in dimension $16m+22$, $16m+6$ respectively. Then*

$$K_\phi(N^{16m+22}) = K_{\phi'}(M^{16m+6}) + \langle \phi'(v_{\phi,\phi'}), [M^{16m+6}]\rangle,$$

with

$$v_{\phi,\phi'} = \sum_s f^*(\tilde{z}_s)\nu_M^*(\tilde{w}_s), \qquad \sum_s \tilde{z}_s \otimes \tilde{w}_s \in H^{8m+3}(BF_4 \times BO\langle 8\rangle),$$

where $f : M^{16m+6} \to BF_4$ is the classifying map of the bundle, $\nu_M : M^{16m+6} \to BO\langle 8\rangle$ is the $BO\langle 8\rangle$-structure, and $\sum_s \tilde{z}_s \otimes \tilde{w}_s$ is a universal class depending only on the choice of the Brown-Peterson-Kervaire invariants K_ϕ, $K_{\phi'}$.

As for *Spin*-manifolds in 5.3.4, we get from this result the following corollary:

10.3.5 Corollary: *If all odd-dimensional Stiefel-Whitney classes of M^{16m+6} vanish, then it holds*

$$K_\phi(N^{16m+22}) = K_{\phi'}(M^{16m+6})$$

for all choices of Brown-Peterson-Kervaire invariants K_ϕ, $K_{\phi'}$. In particular, this is true for an almost complex $BO\langle 8\rangle$-manifold M^{16m+6}.

Proof: From $f^*\bar{x} = 0$ (10.2.6), it follows that also the F_4-bundle characteristic class $f^*x_4 \in H^4M^{16m+6}$ vanishes since x is the mod 2 reduction of $\bar{x}$. As H^*BF_4 is the polynomial ring in x_4, Sq^2x_4, Sq^3x_4, x_{16}, Sq^8x_{16}, the subrings $f^*(H^*BF_4) \subset H^*M^{16m+6}$ and $\nu_M^*(H^*BO\langle 8\rangle) \subset H^*M^{16m+6}$ are then concentrated in even dimensions, whereas $v_{\phi,\phi'}$ lives in odd dimension $8m+3$. ■

This suggests that an analogous result as Ochanine's result 2.2.4 for *Spin*-manifolds could be true for $BO\langle 8\rangle$-manifolds. As Ochanine's proof in [48] works with the image of *Spin*-bordism in oriented bordism, which is known from the results 2.1.9 of Anderson, Brown and Peterson on the structure of the *Spin*-bordism ring, there seems to be at this time no chance to generalize his proof the case of $BO\langle 8\rangle$-manifolds because the corresponding facts for $\Omega_*^{\langle 8\rangle}$ are not known.

10.4 Application of Kristensen's Product Formula

10.4.1 Now, we give some remarks on the application of Kristensen's machinery (section 6) to our problem to compute $\phi(xy)$. This turns out to be a lot of work and we do not give the full computation (we hope to give this in a forthcoming paper). We first have to check the complementarity conditions 6.3.1 on $\beta_{m+1}(xy)$. With

$$\beta_m = (Sq^{8m}, Sq^{8m}Sq^2, Sq^{8m}Sq^3 + Sq^{8m+2}Sq^1)$$

(see 8.3.7), we obtain by the Cartan formula that

$$\beta_{m+1}(xy) =$$

$$\begin{pmatrix} \sum_{i=0}^{8m+8}\left(Sq^ixSq^{8m+8-i}y\right), \\ \sum_{i=0}^{8m+8}\left(Sq^iSq^2xSq^{8m+8-i}y + Sq^iSq^1xSq^{8m+8-i}Sq^1y + Sq^ixSq^{8m+8-i}Sq^2y\right), \\ \sum_{i=0}^{8m+8}\left(Sq^iSq^3xSq^{8m+8-i}y + Sq^iSq^2xSq^{8m+8-i}Sq^1y + Sq^iSq^1xSq^{8m+8-i}Sq^2y+\right. \\ \left.+Sq^ixSq^{8m+8-i}Sq^3y\right) + \sum_{i=0}^{8m+10-i}\left(Sq^iSq^1xSq^{8m+10-i}y + Sq^ixSq^{8m+10-i}Sq^1y\right) \end{pmatrix}.$$

The vanishing of each summand can be seen by an analogous discussion as in the *Spin*-case; it also follows at the same time we discuss the terms of the codiagonal of our relation

$$\rho_{m+1} := Sq^4 \otimes Sq^{8m} + Sq^2 \otimes Sq^{8m}Sq^2 + Sq^1 \otimes (Sq^{8m}Sq^3 + Sq^{8m+2}Sq^1),$$

which is given by applying the Cartan formula a further time:

$$\psi^{(2)}\rho_m =$$

$$\sum_{j=0}^{4}\sum_{i=0}^{8m+8}\sigma_{ji}^1 + \sum_{j=0}^{2}\sum_{i=0}^{8m+8}(\sigma_{ji}^2+\sigma_{ji}^3+\sigma_{ji}^4) + \sum_{j=0}^{1}\left(\sum_{i=0}^{8m+8}(\sigma_{ji}^5+\sigma_{ji}^6+\sigma_{ji}^7+\sigma_{ji}^8) + \sum_{i=0}^{8m+10}(\sigma_{ji}^9+\sigma_{ji}^{10})\right).$$

Here, we have set:

$$
\begin{array}{llllllll}
\sigma^1_{ji} & := & (Sq^j & \otimes & Sq^i) & \otimes & (Sq^{4-j} & \otimes & Sq^{8m+8-i}) \\
\sigma^2_{ji} & := & (Sq^j & \otimes & Sq^i Sq^2) & \otimes & (Sq^{2-j} & \otimes & Sq^{8m+8-i}) \\
\sigma^3_{ji} & := & (Sq^j & \otimes & Sq^i Sq^1) & \otimes & (Sq^{2-j} & \otimes & Sq^{8m+8-i} Sq^1) \\
\sigma^4_{ji} & := & (Sq^j & \otimes & Sq^i) & \otimes & (Sq^{2-j} & \otimes & Sq^{8m+8-i} Sq^2) \\
\sigma^5_{ji} & := & (Sq^j & \otimes & Sq^i Sq^3) & \otimes & (Sq^{1-j} & \otimes & Sq^{8m+8-i}) \\
\sigma^6_{ji} & := & (Sq^j & \otimes & Sq^i Sq^2) & \otimes & (Sq^{1-j} & \otimes & Sq^{8m+8-i} Sq^1) \\
\sigma^7_{ji} & := & (Sq^j & \otimes & Sq^i Sq^1) & \otimes & (Sq^{1-j} & \otimes & Sq^{8m+8-i} Sq^2) \\
\sigma^8_{ji} & := & (Sq^j & \otimes & Sq^i) & \otimes & (Sq^{1-j} & \otimes & Sq^{8m+8-i} Sq^3) \\
\sigma^9_{ji} & := & (Sq^j & \otimes & Sq^i Sq^1) & \otimes & (Sq^{1-j} & \otimes & Sq^{8m+8-i}) \\
\sigma^{10}_{ji} & := & (Sq^j & \otimes & Sq^i) & \otimes & (Sq^{1-j} & \otimes & Sq^{8m+8-i} Sq^1)
\end{array}
$$

As in the case of *Spin*-manifolds, most of the terms vanish by *trivial reasons*, i.e. because the kernel condition for the secondary operation in a summand of the product expansion is satisfied by the fact that the excess is larger than the dimension. This applies to σ^1_{ji} for $i > 8$ or $i < 5$, to σ^2_{ji} for $i > 10$ or $i < 5$, to σ^3_{ji} for $i > 9$ or $i < 4$, to σ^4_{ji} for $i > 8$ or $i < 3$, to σ^5_{ji} for $i > 11$ or $i < 5$, to σ^6_{ji} for $i > 10$ or $i < 4$, to σ^7_{ji} for $i > 9$ or $i < 3$, to σ^8_{ji} for $i > 8$ or $i < 2$, to σ^9_{ji} for $i > 9$ or $i < 7$, and to σ^{10}_{ji} for $i > 8$ or $i < 6$, because one gets then $exc(\beta') > |x| = 8$, respectively $exc(\beta'') > |y| = 8m + 3$. But there remain 142 *critical terms* which have to be considered in detail. In particular, the following 4 critical terms contribute

$$\sigma^1_{08} + \sigma^4_{08} + \sigma^8_{08} + \sigma^{10}_{08} = (1 \otimes Sq^8) \otimes \rho_m,$$

which gives just $x^2\phi_m(y)$ as a summand in Kristensen's product formula. In analogy to the *Spin*-case and by our homotopy-theoretical result 10.3.4, one expects that the other 138 critical summands contribute 0 to the product formula.

In the last step, one has to compute the primary term in Kristensen's product formula. In order to do this, we use again the special system of cochain operations 6.3.7 due to Kristensen to construct our secondary operations. Here, we need not only $\dot{R}_m$ and $\ddot{R}_m$ (see the proof of the Main Theorem 7.1.1), but also (see 6.3.8)

$$
\begin{array}{lll}
\hat{\rho}_m & := \rho^{8m+4}_{16m,8m} & = Sq^4 \otimes Sq^{8m} + Sq^{8m+4} \otimes 1 + Sq^{8m+2} \otimes Sq^2 + Sq^{8m+3} \otimes Sq^1, \\
\hat{r}_m & := r^{8m+4}_{16m,8m} & = sq^4 sq^{8m} + sq^{8m+4} + sq^{8m+2} sq^2 + sq^{8m+3} sq^1, \\
\hat{\epsilon}_m & := \epsilon^{8m+4}_{16m,8m} & = (Q_0 \otimes Q_1)\psi(Sq^1 Sq^{8m-2} + Sq^2 Sq^{8m-3} + Sq^{8m-1})
\end{array}
$$

and denote the special cochain operations $R^{8m+4}_{16m,8m}$ in 6.3.8 with $\Delta R^{8m+4}_{16m,8m} = r^{8m+4}_{16m,8m}$ by $\hat{R}_m$. Now, our relation is on the cochain level represented by

$$r_m := sq^{8m+4} + sq^4 sq^{8m} + sq^2 sq^{8m} sq^2 + sq^1 sq^{8m} sq^1 sq^2 + sq^1 sq^{8m+2} sq^1$$

(here, we have to be a little careful as $sq^1(sq^{8m}sq^3 + sq^{8m+2}sq^1) \neq sq^1 sq^{8m} sq^1 sq^2 + sq^1 sq^{8m+2} sq^1$) which we can decompose by

$$r_m = \hat{r}_m + \ddot{r}_{2m} sq^2 + \dot{r}_{4m} sq^1 sq^2 + \dot{r}_{4m+1} sq^1.$$

Thus we can choose

$$R_m := \hat{R}_m + \ddot{R}_{2m} sq^2 + \dot{R}_{4m} sq^1 sq^2 + \dot{R}_{4m+1} sq^1,$$

and proceed then with similar computations as in the *Spin*-case to show that the primary term $A_{R_{m+1}}(x,y)$ with this choice of 'zero-homotopies' is given by the following linear combination of primary terms for the special cochain operations of Kristensen:

$$A_{R_{m+1}}(x,y) = A_{\hat{R}_{m+1}}(x,y) + A_{\ddot{R}_{2m+2}}(sq^2x,y) + A_{\ddot{R}_{2m+2}}(sq^1x,sq^1y) + A_{\ddot{R}_{2m+2}}(x,sq^2y)+$$

$$+A_{\dot{R}_{4m+4}}(sq^1sq^2x,y) + A_{\dot{R}_{4m+4}}(sq^2x,sq^1y) + A_{\dot{R}_{4m+4}}(sq^1sq^1x,sq^1y) + A_{\dot{R}_{4m+4}}(sq^1x,sq^1sq^1y)+$$
$$+A_{\dot{R}_{4m+4}}(sq^1x,sq^2y) + A_{\dot{R}_{4m+4}}(x,sq^1sq^2y) + A_{\dot{R}_{4m+5}}(sq^1x,y) + A_{\dot{R}_{4m+5}}(x,sq^1y) + \delta(...).$$

Hence, for cocycles x, y our primary term $\epsilon(x\otimes y)$ is given by the following combination of the Kristensen primary terms ($Q_0 := Sq^1$, $Q_1 := Sq^2Sq^1 + Sq^3$):

$$\epsilon_{R_{m+1}}(x,y) = \hat{\epsilon}_{m+1}(x,y) + \ddot{\epsilon}_{2m+2}(Sq^2x,y) + \ddot{\epsilon}_{2m+2}(Sq^1x,Sq^1y) + \ddot{\epsilon}_{2m+2}(x,Sq^2y)+$$

$$+\dot{\epsilon}_{4m+4}(Sq^3x,y) + \dot{\epsilon}_{4m+4}(Sq^2x,Sq^1y) + \dot{\epsilon}_{4m+4}(Sq^1x,Sq^2y) + \dot{\epsilon}_{4m+4}(x,Sq^3y)+$$
$$+\dot{\epsilon}_{4m+5}(Sq^1x,y) + \dot{\epsilon}_{4m+5}(x,Sq^1y) =$$
$$= \mu(Q_0\otimes Q_1)\Big(\psi(Sq^1Sq^{8m+6} + Sq^2Sq^{8m+5} + Sq^{8m+7}) + \psi(Sq^{8m+5})\psi(Sq^2)\Big)(x\otimes y) =$$
$$= \mu\Big((Q_0\otimes Q_1)\psi(Sq^{8m+6}Sq^1 + Sq^{8m+5}Sq^2)\Big)(x\otimes y).$$

Applied to our case of F_4-$\mathbb{C}ayP^2$-bundles, using $Sq\,x = x + Sq^4x + Sq^6x + Sq^7x + x^2$ we obtain

$$\epsilon_{R_{m+1}}(x,y) = \sum_{i+j=8m+6} Q_0Sq^ix\cdot Q_1Sq^jSq^1y + \sum_{i+j=8m+5} Q_0Sq^ix\cdot Q_1Sq^jSq^2y.$$

But this vanishes because the only term which can give a contribution has to contain the factor $x^2 = Sq^8x$ since we are in the top dimension $16m+22$, and this term does not show up in the two sums. Summarizing the results one gets

$$\phi_{m+1}(xp^*y) = x^2p^*\phi_m(y')$$

where the secondary operations ϕ_{m+1} and ϕ_m are constructed by using Kristensen's special system of cochain operations.

In particular, this implies the existence of Brown-Peterson-Kervaire invariants which behave multiplicative in F_4-$\mathbb{C}ayP^2$-bundles $p : N^{16m+22} \to M^{16m+6}$ of closed 3-connected $BO\langle 8\rangle$-manifolds. As we did not carry out all computational details, we do not claim this here as a theorem. We hope to give the full computation in a forthcoming paper.

10.5 Concluding Remarks

The definition of the Ochanine k-invariant $k : \Omega_{8m+2}^{Spin} \to \mathbb{Z}/2$ in 3.1.3 and the Main Theorem 7.1.1 suggest the question, if one can define in the same way a k-invariant for $(16m+6)$-dimensional closed $BO\langle 8\rangle$-manifold; and if k is then a Brown-Kervaire invariant, again. We conclude with some remarks and conjectures concerning this (open) question.

As a first step in this direction one has to determine the image of the signature homomorphism for closed $(16m+8)$-dimensional $BO\langle 8\rangle$-manifolds M^{16m+8}. If we let $s_m \in \mathbb{N}$ be the minimal non-trivial signature,

$$s_m := \left| \frac{\mathbb{Z}}{sign(\Omega_{16m+8}^{\langle 8\rangle})} \right|,$$

then it holds that s_{m+1} divides s_m for all m because we can multiply M^{16m+8} with the Cayley projective plane $\mathbb{C}ayP^2$ which has signature 1. As the universal Wu class $v_{8m+4} \in H^{8m+4}BO\langle 8\rangle$ vanishes (8.3.2), the $\mathbb{Z}$-valued intersection form of M^{16m+8} is even and the signature thus always divisible by 8, in particular $8|s_m$. Furthermore, an 8-dimensional closed $BO\langle 8\rangle$-manifold is almost parallelizable and thus

$$s_0 = 2^5 \cdot 7 = 224$$

by the results of Kervaire and Milnor in [43]. In analogy to the Ochanine signature Theorem 3.1.1, one can make the conjecture (I) that this gives then also the value for all s_m.

The second step consists in analyzing the multiplication with $\overline{S^1}$ on $\Omega_*^{\langle 8\rangle}$. One needs in particular

$$S_{16m+6} := ker(\Omega_{16m+6}^{\langle 8\rangle} \xrightarrow{\cdot \overline{S^1}} \Omega_{16m+7}^{\langle 8\rangle})$$

and can make the conjecture (II) that $S_{16m+6} = \Omega_{16m+6}^{\langle 8\rangle}$ (in analogy to the *Spin*-case). Then we define *the k-invariant for $BO\langle 8\rangle$-bordism* by

$$k : S_{16m+6} \longrightarrow \mathbb{Z}/2$$

$$k(M^{16m+6}) := \frac{sign(W^{16m+8})}{\frac{1}{2}s_m} \bmod 2,$$

where $\partial W^{16m+8} = M^{16m+6} \times \overline{S^1}$, which is well-defined by the same argumentation as in the *Spin*-case (see 3.1.3).

Now, in order to compare k with the Brown-Kervaire invariants, one can try again to get a characterization by multiplicativity properties. The product formula of Brown 1.3.13 (see also the proof of 1.3.16) shows that for Brown-Kervaire invariants

$$K_h : \Omega_{16m+6}^{\langle 8\rangle} \longrightarrow \mathbb{Z}/2$$

we have

$$K_h(M^{16m} \times \overline{S^3} \times \overline{S^3}) \equiv sign(M^{16m}) \bmod 2.$$

For the corresponding formula with K_h replaced by k, we remark that $\overline{S^3} \times \overline{S^3} \times \overline{S^1} = \partial W^8$ because of $\Omega_7^{\langle 8\rangle} = 0$. In particular, $\overline{S^3} \times \overline{S^3} \in S_6 = \Omega_6^{\langle 8\rangle}$, and one can make the conjecture

(III) that $k(\overline{S^3} \times \overline{S^3}) = 1$ (i.e., $sign(W^8) \equiv 112 \bmod 224$. This seems to be more easy to check than (I) or (II).) Furthermore, $M^{16m} \times \overline{S^3} \times \overline{S^3} \in S_{16m+6}$ for all $M^{16m} \in \Omega_{16m}^{\langle 8 \rangle}$ because of $M^{16m} \times \overline{S^3} \times \overline{S^3} \times \overline{S^1} = \partial(M^{16m} \times W^8)$, which shows that

$$k(M^{16m} \times \overline{S^3} \times \overline{S^3}) \equiv \lambda_m sign(M^{16m}) \bmod 2$$

with $\lambda_m := \frac{sign(W^8)}{s_m/2} = k(\overline{S^3} \times \overline{S^3}) \cdot \frac{s_0}{s_m}$.

Thus, if (I), (II) and (III) hold true, we have the half (ii) of a speculative analogous theorem as the characterization 4.2.7 of the Ochanine k-invariant. But the remaining part of

1. checking the analogous multiplicativity property (i) as in 4.2.7 with F_4-$\mathbb{C}ayP^2$-bundles,
2. and analyzing if (i) and (ii) characterize k (by computing $\Omega_*^{\langle 8 \rangle}$ modulo F_4-$\mathbb{C}ayP^2$-bundles),

seems to be considerably more difficult than (I), (II) and (III) and depends on a better understanding of $\Omega_*^{\langle 8 \rangle}$.

Appendix:

Quadratic Forms on

$\mathbb{Z}/2$-Vector Spaces

We give here a survey on symmetric inner products and non-degenerate quadratic forms on finite-dimensional $\mathbb{Z}/2$-vector spaces. This characteristic 2 case differs strongly from the well-presented theory of quadratic forms in characteristic $\neq 2$ (see for example [38], [56]), but unfortunately is excluded in these and many other algebra books on quadratic forms. The material here is taken from several sources: [17], [42], and the appendix in [22] (but see also remark A.1.6). We included the totally elementary proofs of all statements.

First, we consider the classification of symmetric inner products which is given by dimension and type. Then, we define and classify non-degenerate quadratic forms (values in $\mathbb{Z}/2$), where the Arf invariant comes in. Because $\mathbb{Z}/2$-valued quadratic forms can only be defined for even type inner products, one generalizes them to $\mathbb{Z}/4$-valued quadratic forms which exist also for odd type inner products. But then, the generalization of the Arf invariant takes values in $\mathbb{Z}/8$. We prove a corresponding classification result A.3.7, which seems to be not in the literature. In all three categories of symmetric / quadratic / generalized quadratic IPSs (where we here and in the following set *IPS* := inner product space), we consider also the stable classification and the Witt ring classification. Furthermore, we prove in A.2.18, A.3.18 a sub-Lagrangian lemma which we need for *Spin*- and $BO\langle 8\rangle$-manifolds (only the $\mathbb{Z}/2$-case). At least in the $\mathbb{Z}/2$-case, this is well-known (see [53]), but we did not find a good reference for it.

We end with the remark that the category of generalized ($= \mathbb{Z}/4$-valued) quadratic forms behaves different from the category of the usual ($= \mathbb{Z}/2$-valued) quadratic forms in the following points:

1. All symmetric bilinear forms have generalized quadratic refinements.
2. In the definition of the tensor product of generalized quadratic forms, we do not need the cumbersome factor 2 as for $\mathbb{Z}/2$-valued forms.
3. The Witt cancellation theorem is not true for generalized quadratic IPSs, but holds in the case of $\mathbb{Z}/2$-valued forms.
4. The generalized Arf invariant ($\in \mathbb{Z}/8$) is multiplicative, in contrast to the $\mathbb{Z}/2$-valued one.
5. The ring of stable isomorphism classes, and the Witt ring of generalized quadratic IPSs both have a unit element, in contrast to the case of $\mathbb{Z}/2$-valued forms.

(Of course, 2., 4. and 5. are related to each other.)

A.1 Symmetric Inner Products

A.1.1 Let V be a $\mathbb{Z}/2$-vector space of finite dimension $d := dim_{\mathbb{Z}/2}V$. A *symmetric inner product* μ on V is a non-degenerate symmetric bilinear form

$$\mu : V \times V \longrightarrow \mathbb{Z}/2.$$

The pair (V, μ) is called a *symmetric IPS*. We set $\mu(x, y) =: xy$ and denote the adjoined linear isomorphism by $\mu' : V \to V^*$, where $\mu'(x) : y \mapsto xy$. The notion of isomorphism of symmetric inner products is obvious; orthogonal sum and tensor product are defined by $(x \oplus x')(y \oplus y') := xy + x'y'$ and $(x \otimes x')(y \otimes y') := (xy) \cdot (x'y')$, and are again non-degenerate.

A.1.2 Because we are in characteristic 2, the *squaring map* $Sq : V \to \mathbb{Z}/2$, $Sq(x) := x^2$ is linear and thus given by

$$Sq(x) = vx,$$

with $v := \mu'^{-1}(Sq) \in V$ the *Wu class* of the symmetric inner product μ. We call (V, μ) of *even type* (or *symplectic*) if $v = 0$, and of *odd type* otherwise. The type of the direct sum and of the tensor product are given by $v_{V\oplus W} = v_V \oplus v_W$ and $v_{V\otimes W} = v_V \otimes v_W$.

A.1.3 The two fundamental examples of symmetric IPSs are the *standard space* (1) and the *hyperbolic plane* H:

$$\begin{array}{lcl} (1) & := & (\mathbb{Z}/2, (1)) \\ H & := & ((\mathbb{Z}/2)^2, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})), \end{array}$$

where the inner products are given by the matrices (1) and $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ with respect to the canonical basis. There are isomorphisms

$$\begin{array}{lcl} H \oplus (1) & \cong & (1) \oplus (1) \oplus (1) \\ H \otimes H & \cong & H \oplus H \end{array}$$

where the first isomorphism is given by $e_1 \mapsto f_1 + f_3$, $e_2 \mapsto f_2 + f_3$, $e \mapsto f_1 + f_2 + f_3$, with (e_1, e_2, e) and (f_1, f_2, f_3) the canonical bases of $H \oplus (1)$ and $(1) \oplus (1) \oplus (1)$. This gives an example that the Witt Cancellation Theorem for IPSs is not true in characteristic 2, because $H \ncong (1) \oplus (1)$, as H is even, but $(1) \oplus (1)$ is odd. Both isomorphisms follow also from the classification result:

A.1.4 Proposition: (Classification of symmetric IPSs) *The isomorphism class of a symmetric IPS (V, μ) is given by the dimension d and the type:*

$$\begin{array}{lcll} (V, \mu) & \cong & \bigoplus_{d'} H \quad \textit{with } d = 2d' & \textit{for } \mu \textit{ of even type} \\ (V, \mu) & \cong & \bigoplus_{d} (1) & \textit{for } \mu \textit{ of odd type.} \end{array}$$

Proof: We first recall the *orthogonal decomposition lemma* [42] (valid for symmetrical or skew-symmetrical inner products over any commutative ring): *If $W \leq V$ is a submodule such that the restriction μ_W of μ to W is non-degenerate, then* (with $W^\perp$ the *orthogonal complement*):

$$(V, \mu) \cong (W, \mu_W) \oplus (W^\perp, \mu_{W^\perp}).$$

In fact, if $x \in W \cap W^\perp$, then $xw = 0$ for all $w \in W$ and thus $x = 0$; and if $x \in V$, then the restriction of the linear form $\mu'(x)$ to W is represented by multiplication with a $w_x \in W$, giving the decomposition $x = w_x + (x - w_x)$ with $x - w_x \in W^\perp$.

Now we prove A.1.4 by induction on the dimension: (i) If μ is even, let $x, y \in V - 0$ such that $xy \neq 0$. Then $H \cong \langle x, y\rangle$ splits off from (V, μ) with the complement also of even type. (ii) If μ is odd, there is an $x \in V - 0$ with $x^2 \neq 0$, and $(1) \cong \langle x\rangle$ splits off from (V, μ). If $\langle x\rangle^\perp$ is of even type, we proceed by (i) and use $H \oplus (1) \cong (1) \oplus (1) \oplus (1)$ in the end ■

A.1.5 In particular, for an even type symmetric IPS the dimension is always even, $d = 2d'$, and there exists a basis $(x_i, y_i)_{i=1..d'}$ of V such that $x_i x_j = y_i y_j = 0$ and $x_i y_j = \delta_{ij}$, which is called a *symplectic basis* of (V, μ).

A.1.6 *Remark:* In [22], this result is expressed in a misleading fashion by using a third symmetric IPS $F := ((\mathbb{Z}/2)^2, (\begin{smallmatrix}0&1\\1&1\end{smallmatrix}))$ and intoducing in the odd type case a further distinction between the cases $v^2 = 0$ ('case II') and $v^2 \neq 0$ ('case III'). In fact, $F \cong (1) \oplus (1)$ by the isomorphism $e_1 \mapsto f_1 + f_2$, $e_2 \mapsto f_2$, where (e_1, e_2) and (f_1, f_2) are the canonical bases of F and $(1) \oplus (1)$; and the invariant $v^2 \in \mathbb{Z}/2$ of (V, μ) can be read off by the next lemma:

A.1.7 Lemma: *For a symmetric IPS (V, μ), one has $v^2 \equiv dim_{\mathbb{Z}/2} V$* mod 2.

Proof: If (V, μ) is of even type this is true since the dimension is even. If (V, μ) is of odd type, we can assume $(V, \mu) = \bigoplus_d (1)$ where the Wu class is given by $v = (1, 1, .., 1)$. But then $v^2 = 1 + .. + 1 \equiv d \bmod 2$. ■

A.1.8 Now we consider two stable classification problems. We define $KS(\mathbb{Z}/2)$ as the unreduced *Grothendieck ring of stable isomorphism classes* of symmetric IPSs. Then we get from $H \oplus (1) \cong (1) \oplus (1) \oplus (1)$ and the unstable classification A.1.4:

A.1.9 Proposition: *There is a ring-isomorphism $KS(\mathbb{Z}/2) \cong \mathbb{Z}$ induced by the dimension, and (1) is a generator.*

A.1.10 A symmetric IPS is defined to be *split* if there exists a *Lagrangian*; that is a subspace $L < V$ of half the dimension which is self-orthogonal, $L = L^\perp$ (as we consider vector spaces, all sub-modules are direct summands). For example, H is split with $L = \langle e_1\rangle$, but also $(1) \oplus (1)$ is split with $L = \langle e_1 + e_2\rangle$ since we are in characteristic two. The *symmetric Witt ring* $W(\mathbb{Z}/2)$ is defined as the set of *Witt classes* of symmetric IPSs, where the Witt class of a space is obtained by stabilization with split spaces (see[42]). Again we get from $H \oplus (1) \cong (1) \oplus (1) \oplus (1)$ and the unstable classification A.1.4:

A.1.11 Proposition: *There is a ring-isomorphism $W(\mathbb{Z}/2) \cong \mathbb{Z}/2$ given by the dimension mod 2, and (1) is a generator.*

A.1.12 A *sub-Lagrangian* is a subspace $S \leq V$ which is self-orthogonal, $S \cdot S = 0$. Thus the orthogonal complement $S^\perp$ contains S, and one defines

$$V_- := S, \quad V_0 := S^\perp/S, \quad V_+ := V/S^\perp.$$

Now, we choose linear splittings $V_0 \to S^\perp \subseteq V$ and $V_+ \to V$ of the projection maps, and obtain a linear isomorphism

$$V \cong V_- \oplus V_0 \oplus V_+.$$

Pulling the inner product μ back to the sum $V_- \oplus V_0 \oplus V_+$ gives then bilinear pairings

$$\begin{array}{c|ccc} & V_- & V_0 & V_+ \\ \hline V_- & 0 & 0 & \mu_\pm \\ V_0 & 0 & \mu_0 & \mu' \\ V_+ & \mu_\pm & \mu' & \mu'' \end{array}$$

where

$$\begin{array}{llllll} \mu_0 : & V_0 \otimes V_0 & \to & \mathbb{Z}/2, & \mu_0(v+S, w+S) & = & \mu(v,w), \\ \mu_\pm : & V_+ \otimes V_- & \to & \mathbb{Z}/2, & \mu_\pm(v+S^\perp, s) & = & \mu(v,s) \end{array}$$

are independent of the chosen splittings and moreover non-degenerate, as μ is non-degenerate. In particular, (V_0, μ_0) is again a symmetric IPS.

With $dim_{\mathbb{Z}/2}V_0 = dim_{\mathbb{Z}/2}S^\perp - dim_{\mathbb{Z}/2}S = dim_{\mathbb{Z}/2}V - 2dim_{\mathbb{Z}/2}S$, we have shown:

A.1.13 Lemma: (Symmetric sub-Lagrangian lemma) *The symmetric IPSs (V, μ) and (V_0, μ_0) give the same element in the symmetric Witt group $W(\mathbb{Z}/2)$.*

A.2 Quadratic Forms

A.2.1 Now we come to *quadratic forms* on V. Because we are in characteristic 2, these cannot be constructed by bilinear forms, but are defined to be maps $q : V \to \mathbb{Z}/2$ such that the symmetric map $\mu_q : V \times V \to \mathbb{Z}/2$,

$$\mu_q(x, y) := q(x + y) - q(x) - q(y)$$

is bilinear. If μ_q is non-degenerate, then q is also called *non-degenerate*, (V, q) is called a *quadratic IPS*, and (V, μ_q) is called the *associated symmetric IPS*. The notion of isomorphism for quadratic forms is obvious; and the orthogonal sum is defined by $q_{V\oplus W}(v\oplus w) := q_V(v) + q_W(w)$ which gives $\mu_{q_{V\oplus W}} = \mu_{q_V} \oplus \mu_{q_W}$ and remains non-degenerate if q_V, q_W are.

A.2.2 Lemma: *For a quadratic form (V, q), the associated symmetric bilinear form μ_q has always even type.*

Proof: We have $q(0) = \mu_q(0, 0) = 0$ and thus $\mu_q(x, x) = q(2x) - 2q(x) = q(0) - 0 = 0$ for all $x \in V$. ■

A.2.3 If μ_q is given, it is enough to know q on a basis $(e_1, .., e_d)$ of V because one sees by induction $q(\sum_i x_i) = \sum_i q(x_i) + \sum_{i<j} x_i x_j$. In fact, we will see in A.2.20 that the values $q(e_i)$, $i = 1, ..., d$, can be chosen arbitrarily to give a quadratic form q. Thus, for fixed μ of even type there are exactly 2^d quadratic forms with associated (V, μ).

A.2.4 In the definition of the tensor product of quadratic forms (on R-modules for some ring R) one has to be careful, see [42]: First, one defines the tensor product of a symmetric

bilinear form (V, μ) with a quadratic form (V', q') to be the quadratic form $(V \otimes V', q)$ uniquely defined by the two equations

$$\begin{array}{lcl} q(x \otimes x') & := & \mu(x,x)q'(x'), \\ \mu_q(x \otimes x', y \otimes y') & := & \mu(x,y)\mu_{q'}(x',y'). \end{array}$$

Note the isomorphism $(1) \otimes (V,q) \cong (V,q)$. Then one defines the tensor product of two quadratic forms (V,q) and (V',q') to be the tensor product of (V, μ_q) and (V', q'). Thus $(V \otimes V', q \otimes q')$ is uniquely defined by the two equations

$$\begin{array}{lcl} (q \otimes q')(x \otimes x') & := & 2q(x)q'(x'), \\ \mu_{q \otimes q'}(x \otimes x', y \otimes y') & := & \mu_q(x,y)\mu_{q'}(x',y'). \end{array}$$

because $\mu(x,x) = 2q(x)$. In particular, the tensor product of two non-degenerate forms remains non-degenerate. In our case $R = \mathbb{Z}/2$, the quadratic form $q \otimes q'$ vanishes on pure tensors because of the cumbersome factor 2 in the first equation, and depends only on the associated symmetric products.

A.2.5 There are two fundamental examples q_0 and q_1 of non-degenerate quadratic forms, which have the hyperbolic plane H as associated symmetric inner product and are given by

$$\begin{array}{lcll} q_0 & : & (\mathbb{Z}/2)^2 \to \mathbb{Z}/2, & q_0(e_1) = q_0(e_2) = 0 \\ q_1 & : & (\mathbb{Z}/2)^2 \to \mathbb{Z}/2, & q_1(e_1) = q_1(e_2) = 1. \end{array}$$

There is an isomorphism $q_1 \oplus q_1 \cong q_0 \oplus q_0$ which is given by the automorphism $\alpha \in GL(\mathbb{Z}/2, 4)$ defined by $e_i \mapsto (e_1 + e_2 + e_3 + e_4) - e_i$, $i = 1..4$, where e_i is the canonical basis of $(\mathbb{Z}/2)^4$. As we need α also in the next section about generalized quadratic forms, we list some properties: In matrix notation, α is given by

$$\alpha = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and satisfies $\alpha^T = \alpha$, $\alpha^2 = 1$ (showing that α is an isometry of $(1) \oplus (1) \oplus (1) \oplus (1)$), and $\alpha\beta\alpha = \beta$ where $\beta := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (showing that α is also an isometry of $H \oplus H$).

Furthermore we remark that the other two quadratic forms on H (defined by $q_0(e_1) = 0$, $q_0(e_2) = 1$ respectively $q_0(e_1) = 1$, $q_0(e_2) = 0$) are both isomorphic to q_0 (by $e_1 \mapsto e_1$, $e_2 \mapsto e_1 + e_2$ respectively $e_1 \mapsto e_1 + e_2$, $e_2 \mapsto e_2$).

A.2.6 This gives the classification of a quadratic form (V,q): The associated (V, μ) decomposes as $\bigoplus_{d'} H$, thus q decomposes also into the orthogonal sum of d' many quadratic forms on H which have to be isomorphic to q_0 or q_1. Because of $q_1 \oplus q_1 \cong q_0 \oplus q_0$, we see that q is isomorphic to $\bigoplus_{d'} q_0$ or $q_1 \oplus \bigoplus_{d'-1} q_0$. In order to show that these two quadratic forms are not isomorphic, a new invariant has to be introduced:

A.2.7 Define the *Arf invariant* $\mathrm{Arf}(q) \in \mathbb{Z}/2$ of a quadratic IPS (V,q) by

$$(-1)^{\mathrm{Arf}(q)} := sgn \sum_{x \in V} (-1)^{q(x)}.$$

We denote the sum by $S(q)$ and remark that always $S(q) \neq 0$, thus $\mathrm{Arf}(q)$ is well-defined. In fact, $S(q)$ obviously depends only on the isomorphism class of (V, q) and has the property $S(q \oplus q') = S(q)S(q')$, which implies the additivity of the Arf invariant under $\oplus$. As $S(q_0) = 2$ and $S(q_1) = -2$, we see that $S(q) = \pm 2^{d'}$, with '+' for $q \cong \bigoplus_{d'} q_0$ and '$-$' for $q \cong q_1 \oplus \bigoplus_{d'-1} q_0$. Thus we have shown:

A.2.8 Proposition: (Classification of quadratic IPSs) *The isomorphism class of a quadratic IPS (V, q) is given by the dimension $d = 2d'$ and the Arf invariant:*

$$\begin{array}{lcll} (V,q) & \cong & \bigoplus_{d'} q_0 & \text{for Arf(q)=0} \\ (V,q) & \cong & q_1 \oplus \bigoplus_{d'-1} q_0 & \text{for Arf(q)=1.} \end{array}$$

A.2.9 *Remarks:*
(i) As $S(q)$ counts the number of x with $q(x) = 0$ minus that with $q(x) = 1$, the Arf invariant is one iff the majority of vectors x has $q(x) = 1$ (so it is also called the 'democratic invariant'). This number of $x \in V$ with $q(x) = 1$ is given by $\frac{1}{2}(|V| - S(q))$, which is $2^{2d'-1} - 2^{d'-1}$ for $\mathrm{Arf}(q) = 0$ and $2^{2d'-1} + 2^{d'-1}$ for $\mathrm{Arf}(q) = 1$.

(ii) In contrast to the case of symmetric IPSs in characteristic 2, the Witt Cancellation Theorem remains true for quadratic IPSs ([42]). The classification result A.2.8 above illustrates this fact.

A.2.10 Lemma: *Let $(x_i, y_i)_{i=1..d'}$ be a symplectic basis of (V, μ). Then*

$$\mathrm{Arf}(q) = \sum_{i=1..d'} q(x_i)q(y_i).$$

Proof: As the right side is also additive under $\oplus$, one has only to check equality for q_0 and q_1. ■

A.2.11 The multiplicative properties of Arf follow from the isomorphisms

$$q_0 \otimes q_0 \cong q_0 \otimes q_1 \cong q_1 \otimes q_1 \cong q_0 \oplus q_0,$$

which hold because the tensor product of quadratic forms in the case $R = \mathbb{Z}/2$ depends only on the underlying symmetric inner products ($= H$). Thus we have $\mathrm{Arf}(q \otimes q') = 0$ for all quadratic inner products q, q'; i.e. the Arf invariant is not multiplicative with respect to the canonical product in $\mathbb{Z}/2$. See also remark A.2.16.

A.2.12 As for symmetric inner products, we consider now two stable classification problems. We define $KQ(\mathbb{Z}/2)$ to be the unreduced *Grothendieck group of stable isomorphism classes* of quadratic IPSs. Then we get from $q_1 \oplus q_1 \cong q_0 \oplus q_0$ and the unstable classification A.2.8:

A.2.13 Proposition: *There is an additive isomorphism $KQ(\mathbb{Z}/2) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ induced by half the dimension and the Arf invariant. A generator of the first summand is q_0, and of the second summand is $q_1 - q_0$. The dimension is multiplicative, but not the Arf invariant which shows that the torsion part $\mathbb{Z}/2$ has the trivial product structure.*

A.2.14 A quadratic IPS (V, q) is defined to be *split* if there exists a *Lagrangian*; that is a subspace $L < V$ of half the dimension which satisfies $q(L) = 0$. In particular, L is then also a Lagrangian of the associated symmetric IPS (V, μ_q), and the Arf invariant of (V, q) vanishes. For example, q_0 is split with $L = \langle e_1 \rangle$, but q_1 is not split. The *quadratic Witt ring* $WQ(\mathbb{Z}/2)$ is defined as the set of *Witt classes* of quadratic IPSs, where the Witt class of a space is obtained by stabilization with split spaces (see[42]). Again we get from $q_1 \oplus q_1 \cong q_0 \oplus q_0$ and the unstable classification A.2.8:

A.2.15 Proposition: *There is an additive isomorphism $WQ(\mathbb{Z}/2) \cong \mathbb{Z}/2$ induced by the Arf invariant, and q_1 gives a generator. The multiplicative structure is the trivial one, since $q_1 \otimes q_1$ has vanishing Witt class.*

A.2.16 *Remark:* Associating (V, μ_q) to (V, q) gives a map $WQ(R) \to W(R)$ which is zero in our case $R = \mathbb{Z}/2$ because here (V, μ_q) always has a symplectic basis (alternatively, this follows also by the different multiplicative structures on $\mathbb{Z}/2$).

A.2.17 A *sub-Lagrangian* of (V, q) is a subspace $S \leq V$ which satisfies $q(S) = 0$. In particular, S is also a sub-Lagrangian of (V, μ_q). One defines a quadratic form q_0 on $V_0 = S^\perp/S$ by $q_0(x+S) := q(x)$. This is well-defined, and non-degenerate as its associated bilinear form μ_0 is. Then we have:

A.2.18 Lemma: (Quadratic sub-Lagrangian lemma) *The quadratic IPSs (V, q) and (V_0, q_0) give the same element in the quadratic Witt group $WQ(\mathbb{Z}/2)$.*

Proof: By the orthogonal decomposition lemma (see the proof of A.1.4) and A.1.12, (V_0, μ_0) splits as direct summand from (V, μ), hence the same is true for (V_0, q_0) and (V, q). But $V_- = S$ is a Lagrangian in the orthogonal complement of (V_0, q_0), which gives the proof. ■

1.2.19 *Remark:* Another proof using the definition of the Arf invariant as the 'democratic invariant' (A.2.9) follows as a special case from A.3.18.

A.2.20 If one has two non-degenerate quadratic forms q, q' (on the same V) such that the associated symmetric inner products coincide, then the difference map $\delta := q' - q : V \to \mathbb{Z}/2$ is a linear form since $\delta(x+y) - \delta(x) - \delta(y) = \mu_{q'}(x, y) - \mu_q(x, y) = 0$. This shows that the set $Q(V, \mu)$ of quadratic forms with fixed associated symmetric inner product μ, which we also call the set of *quadratic refinements* of μ, is an affine space with associated vector space the dual space V^*. By the isomorphism $\mu' : V \to V^*$, we see that $Q(V, \mu) = \{q + \mu'(x) | x \in V\}$ for any fixed $q \in Q(V, \mu)$. Then we have an addition formula for the Arf invariant:

A.2.21 Lemma: (Addition formula) *The Arf invariant of the quadratic form $q + \mu'(x)$, $x \in V$ is given by*

$$\mathrm{Arf}(q + \mu'(x)) = \mathrm{Arf}(q) + q(x).$$

Proof: This holds for quadratic forms on H and follows then in the general case by additivity. Alternatively, this follows from $S(q + \mu'(x)) = \sum_{y \in V} (-1)^{q(y)+xy} = (-1)^{-q(x)} \sum_{y' \in V} (-1)^{q(y')}$ with $y' := x + y$ by $q(y) + xy = q(x + y) - q(x)$. ■

A.2.22 Corollary: *Let $x, y \in V$ with $xy \neq 0$. Then the three numbers $q(x), q(y), q(x+y) \in \mathbb{Z}/2$ cannot be all zero, thus at least one of the three quadratic forms $q + \mu'(x)$, $q + \mu'(y)$, $q + \mu'(x+y)$ has a different Arf invariant than q.*

A.2.23 At last we remark that a quadratic refinement $q : V \to \mathbb{Z}/2$ of μ can also be considered as a homorphism $h : V^\mu \to \mathbb{Z}/2$ with the property $h(\lambda) = 1$, and vice versa:

$$Q(V, \mu) \approx \{h \in Hom(V^\mu, \mathbb{Z}/2) | h(\lambda) = 1\}.$$

Here the abelian group V^μ is defined as $\mathbb{Z}/2 \times V$ with the addition $(a, v) + (a', v') := (a + a' + vv', v + v')$, and $\lambda := (1, 0)$. This is nothing but the abelian extension of V by $\mathbb{Z}/2$ associated to the 2-cocycle $\mu : V \times V \to \mathbb{Z}/2$:

$$0 \to \mathbb{Z}/2 \xrightarrow{j_1} V^\mu \xrightarrow{p_2} V \to 0.$$

The other canonical injection j_2 respectively projection p_1 are non-linear in general. One obtains the bijection above by

$$q \mapsto h := qp_2 + p_1, \qquad h \mapsto q := hj_2.$$

A.3 Generalized Quadratic Forms

A.3.1 We have seen that quadratic refinements q of (V, μ) exist iff μ has even type. In order to generalize quadratic forms to the odd type case, we consider again the abelian extension V^μ of (V, μ). For all types, its elements of order 2 form the subgroup $\mathbb{Z}/2 \times ker(Sq)$. Thus, there exist in V^μ elements of order 4 iff μ has odd type; these have then the form (ϵ, x) with $\epsilon \in \mathbb{Z}/2$ and $x^2 \neq 0$, and satisfy $2(\epsilon, x) = (1, 0) = \lambda$. In this case, there exists no homomorphism $h : V^\mu \to \mathbb{Z}/2$ with $h(\lambda) = 1$ (which would be equivalent to a quadratic refinement $q = hj_2$ of μ). But of course there exist homomorphisms $h : V^\mu \to \mathbb{Z}/4$ with $h(\lambda) = 2$; and these correspond by $q = hj_2$ and $h = qp_2 + i_2^4 p_1$ bijectively to $\mathbb{Z}/4$-*valued quadratic refinements* $q : V \to \mathbb{Z}/4$ of μ,

$$q(x + y) = q(x) + q(y) + i_2^4 xy,$$

which we call *generalized quadratic forms*, and (V, q) a *generalized quadratic IPS* if μ is non-degenerate (here and in the following, $i_n^{mn} : \mathbb{Z}/n \hookrightarrow \mathbb{Z}/mn$ denotes the canonical monomorphism given by $1 \mapsto m$). The type of μ is also called the *type of* q. If we denote the set of generalized quadratic refinements of μ by $GQ(V, \mu)$, then the above correspondence shows that

$$GQ(V, \mu) \approx \{h \in Hom(V^\mu, \mathbb{Z}/4) | h(\lambda) = 2\}.$$

We remark that V^μ contains no elements of order higher than 4, thus every homorphism $V^\mu \to \mathbb{Z}/2^s$ factors through $\mathbb{Z}/4$ which therefore gives the correct generalization of quadratic forms.

A.3.2 Again we have $q(0) = 0$ by the quadratic property, and $2q(x) = q(2x) - i_2^4 x^2 = i_2^4 x^2$ shows that $q(x) \in \{1, 3\}$ iff $x^2 \neq 0$ and $q(x) \in \{0, 2\}$ iff $x^2 = 0$. Thus, the even type $\mathbb{Z}/4$-valued forms come exactly from the $\mathbb{Z}/2$-valued forms by the imbedding $i_2^4 : \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$.

A.3.3 The notion of isomorphism and direct sum is defined as for $\mathbb{Z}/2$-valued quadratic forms; furthermore one defines $(V, -q)$ by $(-q)(x) := -q(x)$. But in contrast to A.2.4, one can define the tensor product of the generalized quadratic forms (V, q) and (V', q') without the cumbersome factor 2 as

$$\begin{array}{rcl} (q \otimes q')(x \otimes x') & := & q(x)q'(x'), \\ \mu_{q\otimes q'}(x \otimes x', y \otimes y') & := & \mu_q(x, y)\mu_{q'}(x', y'). \end{array}$$

Of course, the product $q(x)q'(x')$ is the canonical one in $\mathbb{Z}/4$, and the factor 2 is absorbed by $i_2^4(a)i_2^4(b) = 2i_2^4(ab)$. Again, the tensor product of two non-degenerate forms remains non-degenerate; and for $q_1 = i_2^4 q_1'$, $q_2 = i_2^4 q_2'$ of even type, this gives our old definition $(q_1' \otimes q_2')(x_1 \otimes x_2) = 2q_1'(x_1)q_2'(x_2)$.

A.3.4 Besides the two examples $i_2^4 q_0 : H \to \mathbb{Z}/4$ and $i_2^4 q_1 : H \to \mathbb{Z}/4$, we have a further fundamental generalized quadratic IPS γ:

$$\gamma : (1) \longrightarrow \mathbb{Z}/4, \qquad \gamma(e) := 1.$$

As (1) is of odd type, γ and $-\gamma$ are the only generalized quadratic forms on (1). There are isomorphisms

$$\begin{array}{ccccccccc} i_2^4 q_0 & \oplus & \gamma & \cong & -\gamma & \oplus & \gamma & \oplus & \gamma \\ i_2^4 q_0 & \oplus & -\gamma & \cong & -\gamma & \oplus & -\gamma & \oplus & \gamma \\ i_2^4 q_1 & \oplus & \gamma & \cong & -\gamma & \oplus & -\gamma & \oplus & -\gamma \\ i_2^4 q_1 & \oplus & -\gamma & \cong & \gamma & \oplus & \gamma & \oplus & \gamma \end{array}$$

which follow straightforward using the explicit isomorphism $H \oplus (1) \cong (1) \oplus (1) \oplus (1)$ or also the classification A.3.7 below. Thus the Witt Cancellation Theorem is not true for generalized quadratic product spaces, because for example $i_2^4 q_0 \not\cong -\gamma \oplus \gamma$. Additionally, there is an isomorphism

$$\bigoplus_4 \gamma \cong \bigoplus_4 (-\gamma),$$

which is given by the transformation $\alpha \in GL(\mathbb{Z}/2, 4)$ (we use this isomorphism in the proof of the classification result below).

A.3.5 As in the $\mathbb{Z}/2$-valued case, we can attack now the classification problem for generalized quadratic IPSs (V, q): For even type we have seen that $q = i_2^4 q'$ with $q' \cong \bigoplus_{d'} q_0$ or $q' \cong q_1 \oplus \bigoplus_{d'-1} q_0$. For odd type, the decomposition $\mu_q \cong \bigoplus_d (1)$ shows that q decomposes into the orthogonal sum of d many generalized quadratic forms on (1), which have to be γ or $-\gamma$. Because of $\bigoplus_4 \gamma \cong \bigoplus_4 (-\gamma)$, the form q is isomorphic to $\bigoplus_a (-\gamma) \oplus \bigoplus_{d-a} \gamma$ with $0 \leq a < 4$. It remains to show that these 4 generalized quadratic forms are not isomorphic.

A.3.6 We define the *generalized Arf invariant* $\widetilde{\mathrm{Arf}}(q) \in \mathbb{Z}/8$ of a generalized quadratic IPS (V, q) according to [17] by the Gauß sum

$$(\epsilon_8)^{\widetilde{\mathrm{Arf}}(q)} := s\Big(\sum_{x \in V} i^{q(x)}\Big).$$

Here $i \in \mathbb{C}$ and $\epsilon_8 := \frac{1+i}{\sqrt{2}} \in \mathbb{C}$ are the fourth respectively eighth primitive roots of unity and the function $s : \mathbb{C} - 0 \to S^1 \subset \mathbb{C}$ is defined by $s(z) := \frac{z}{|z|}$. We have to show that $\widetilde{\mathrm{Arf}}$

is well-defined (for a different proof, see [17]): We set again $S(q) := \sum_{x \in V} i^{q(x)} \in \mathbb{C}$, which is well-defined because of $i^4 = 1$ and obviously depends only on the isomorphism class of (V, q). Furthermore, $S(q)$ has the properties $S(q_1 \oplus q_2) = S(q_1)S(q_2)$, $S(-q) = \overline{S(q)}$, and takes the values $S(i_2^4 q_0) = 2$, $S(i_2^4 q_1) = -2$, and $S(\gamma) = 1 + i = \sqrt{2}\epsilon_8$. Thus $|S(q)| = \sqrt{2}^d$ and $\widetilde{\mathrm{Arf}}(q)$ is well-defined, with $\widetilde{\mathrm{Arf}}(q_1 \oplus q_2) = \widetilde{\mathrm{Arf}}(q_1) + \widetilde{\mathrm{Arf}}(q_2)$, $\widetilde{\mathrm{Arf}}(-q) = -\widetilde{\mathrm{Arf}}(q)$. On the fundamental spaces it takes the values $\widetilde{\mathrm{Arf}}(i_2^4 q_0) = 0$, $\widetilde{\mathrm{Arf}}(i_2^4 q_1) = 4$, and $\widetilde{\mathrm{Arf}}(\gamma) = 1$. We have shown:

A.3.7 Proposition: (Classification of generalized quadratic IPSs) *The isomorphism class of a generalized quadratic IPS (V, q) is given by its dimension d, its type, and its generalized Arf invariant $\widetilde{Arf}(q) \in \mathbb{Z}/8$.*
One has for even type:

$$\begin{array}{rcll} (V,q) & \cong & i_2^4(\bigoplus_{d'} q_0) & \text{for } \widetilde{\mathrm{Arf}}(q) = 0, \\ (V,q) & \cong & i_2^4(q_1 \oplus \bigoplus_{d'-1} q_0) & \text{for } \widetilde{\mathrm{Arf}}(q) = 4, \end{array}$$

and for odd type:

$$\begin{array}{rclll} (V,q) & \cong & \bigoplus_d \gamma & \text{for } \widetilde{\mathrm{Arf}}(q) \equiv d & \mathrm{mod} 8, \\ (V,q) & \cong & -\gamma \oplus \bigoplus_{d-1} \gamma & \text{for } \widetilde{\mathrm{Arf}}(q) \equiv d-2 & \mathrm{mod} 8, \\ (V,q) & \cong & -\gamma \oplus -\gamma \oplus \bigoplus_{d-2} \gamma & \text{for } \widetilde{\mathrm{Arf}}(q) \equiv d-4 & \mathrm{mod} 8, \\ (V,q) & \cong & -\gamma \oplus -\gamma \oplus -\gamma \oplus \bigoplus_{d-3} \gamma & \text{for } \widetilde{\mathrm{Arf}}(q) \equiv d-6 & \mathrm{mod} 8. \end{array}$$

Other combinations of dimension, type and $\widetilde{Arf}$ do not occur. In particular, $\widetilde{\mathrm{Arf}} \equiv d \bmod 2$ *for all types, and* $\widetilde{\mathrm{Arf}} \equiv 0 \bmod 4$ *for even type.*

Now we come to several properties of the generalized Arf invariant (which give corresponding results for Brown-Kervaire invariants in section 1.2).

A.3.8 Lemma: (Multiplicativity) *For generalized quadratic IPSs (V, q) and (V', q'), we have $\widetilde{Arf}(q \otimes q') = \widetilde{Arf}(q)\widetilde{Arf}(q')$ with the canonical product in $\mathbb{Z}/8$.*

Proof: See [17]. Another proof is given by the classification A.3.7 and the fact that $\widetilde{\mathrm{Arf}}$ is multiplicative if we take for q, q' the fundamental spaces $i_2^4 q_0$, $i_2^4 q_1$ and γ. ∎

A.3.9 Proposition: (Connection with $\mathbb{Z}/2$-valued and $\mathbb{Z}$-valued forms)
(i) Let (V, q) be a ($\mathbb{Z}/2$-valued) quadratic IPS. Then

$$\widetilde{Arf}(i_2^4 q) = i_2^8 Arf(q).$$

(ii) Let $\mu : F \times F \to \mathbb{Z}$ be a symmetric unimodular bilinear form on a finitely generated free abelian group F. Let $V := F \otimes \mathbb{Z}/2$ and define $q : V \to \mathbb{Z}/4$ by $q(x + 2F) := \mu(x, x) \bmod 4$. Then q is well-defined and (V, q) is a generalized quadratic IPS with

$$\widetilde{\mathrm{Arf}}(q) \equiv sign(\mu) \bmod 8.$$

Proof: See [17]: (i) follows by the definition, and for (ii) one uses that the Grothendieck group $KS(\mathbb{Z})$ of symmetric unimodular bilinear forms over $\mathbb{Z}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ by rank and signature (see [58]). ∎

A.3.10 Lemma: (Reductions mod 4 and mod 2) *Let (V, q) be a generalized quadratic IPS, and $v \in V$ be the Wu class of μ_q. Then:*

$$\begin{array}{lllll} \widetilde{\mathrm{Arf}}(q) & \equiv & q(v) & & \mathrm{mod} 4 \\ q(x) & \equiv & x^2 & = \quad vx & \mathrm{mod} 2 \\ \widetilde{\mathrm{Arf}}(q) & \equiv & v^2 & \equiv \quad dim_{\mathbb{Z}/2} V & \mathrm{mod} 2. \end{array}$$

Proof: The results $v^2 \equiv dim_{\mathbb{Z}/2} V$ mod 2 and $q(x) \equiv x^2 = vx$ mod 2 already have shown to be true. Thus we only must show the first statement. For even type q this is true because of $q = i_2^4 q'$ and $v = 0$. For odd type q, we can assume that $q = \bigoplus_a (-\gamma) \oplus \bigoplus_{d-a} \gamma$ where $v = (1, ..., 1)$, thus $\widetilde{\mathrm{Arf}}(q) = d - 2a$ mod $8 \equiv q(v)$ mod 4. ∎

A.3.11 Now we consider again the set $GQ(V, \mu)$ of generalized quadratic refinements on a symmetric IPS (V, μ). As for $\mathbb{Z}/2$-valued quadratic refinements, one sees that this set is an affine space with associated vector space the dual space V^*. We can replace the dual space by V using the adjoined isomorphism μ', and get for any $q \in GQ(V, \mu)$:

$$GQ(V, \mu) = \{q + i_2^4 \mu'(x) | x \in V\}.$$

Now, the following generalization of A.2.21 holds, where additionally a 'minus' comes in:

A.3.12 Lemma: (Addition formula) *The generalized Arf invariant of the generalized quadratic form $q + i_2^4 \mu'(x)$, $x \in V$ is given by*

$$\widetilde{\mathrm{Arf}}(q + i_2^4 \mu'(x)) = \widetilde{\mathrm{Arf}}(q) - i_4^8 q(x).$$

Proof: This holds for generalized quadratic forms on (1) and H and follows then in the general case by additivity. Alternatively, this follows from $S(q + i_2^4 \mu'(x)) = \sum_{y \in V} i^{q(y) + i_2^4 xy} = i^{-q(x)} \sum_{y' \in V} i^{q(y')}$ with $y' := x + y$ (see [17]). ∎

A.3.13 Corollary: *Let $x, y \in V$ with $xy \neq 0$. Then the three numbers $q(x), q(y), q(x+y) \in \mathbb{Z}/4$ cannot be all zero, thus at least one of the three generalized quadratic forms $q + i_2^4 \mu'(x)$, $q + i_2^4 \mu'(y)$, $q + i_2^4 \mu'(x + y)$ has a different Arf invariant than q.*

A.3.14 We end with considering the stable classification (in the usual and the Witt sense) and the sub-Lagrangian Lemma. As for $\mathbb{Z}/2$-valued forms, $KGQ(\mathbb{Z}/2)$ denotes the *Grothendieck ring of stable isomorphism classes* of generalized quadratic IPSs (V, q). A space (V, q) is *split* if there exists a *Lagrangian*, i.e. $L \leq V$ of half the dimension with $q(L) = 0$. For example, $i_2^4 q_0$ is split but not $i_2^4 q_1$ and γ. The *generalized quadratic Witt ring* $WGQ(\mathbb{Z}/2)$ is defined by stabilization with split spaces. A *sub-Lagrangian* $S \leq V$ satisfies $q(S) = 0$. Again, this defines a well-defined and non-degenerate quadratic form q_0 on $V_0 = S^\perp / S$ by $q_0(x + S) := q(x)$. From the classification we get:

A.3.15 Proposition: *Dimension and generalized Arf invariant give a multiplicative isomorphism*

$$KGQ(\mathbb{Z}/2) \cong \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z}/8 \mid a \equiv b \bmod 2\}.$$

A.3.16 Proposition: *The generalized Arf invariant gives a multiplicative isomorphism*

$$WGQ(\mathbb{Z}/2) \cong \mathbb{Z}/8.$$

A.3.17 *Remark:* The maps $KQ(\mathbb{Z}/2) \to KGQ(\mathbb{Z}/2)$ and $WQ(\mathbb{Z}/2) \to WGQ(\mathbb{Z}/2)$ defined by associating $(V, i_2^4 q)$ to a quadratic IPS (V, q) are given by the formula $\widetilde{\mathrm{Arf}}(i_2^4 q) = i_2^8 \mathrm{Arf}(q)$. This fits also with the unusual multiplicative behaviour of Arf since $4 \cdot 4 = 16 \equiv 0 \bmod 8$.

A.3.18 Lemma: (Generalized quadratic sub-Lagrangian lemma) *The generalized quadratic IPSs (V, q) and (V_0, q_0) give the same element in the generalized quadratic Witt group $WGQ(\mathbb{Z}/2)$.*

Proof: We prove this using the definition A.3.6 of $\widetilde{\mathrm{Arf}}$ by a Gauß sum:

$$\widetilde{\mathrm{Arf}}(q) = \frac{8}{2\pi i} \ln(S(q)) \ \in \mathbb{Z}/8, \qquad \text{with } S(q) = s(\sum_{x \in V} i^{q(x)}) \ \ \in S^1 \subset \mathbb{C}.$$

According to A.1.12, the associated symmetric IPS (V, μ_q) decomposes as

$$\begin{array}{c|ccc} & V_- & V_0 & V_+ \\ \hline V_- & 0 & 0 & \mu_\pm \\ V_0 & 0 & \mu_0 & \mu' \\ V_+ & \mu_\pm & \mu' & \mu'' \end{array} .$$

Together with $q(v_-) = 0$, this gives $q(v_- + v_0 + v_+) = q(v_0) + q(v_+) + i_2^4(v_- v_+) + i_2^4(v_0 v_+)$, showing that

$$\sum_{x \in V} i^{q(x)} = \sum_{v_- \in V_-, v_0 \in V_0, v_+ \in V_+} i^{q(v_- + v_0 + v_+)}$$

$$= \sum_{v_0 \in V_0} i^{q(v_0)} \left(\sum_{v_+ \in V_+} i^{q(v_+) + i_2^4(v_0 v_+)} \left(\sum_{v_- \in V_-} i^{i_2^4(v_- v_+)} \right) \right).$$

But $\sum_{v_- \in V_-} i^{i_2^4(v_- v_+)} = \sum_{v_- \in V_-} (-1)^{v_- v_+}$ is 0 for $v_+ \neq 0$, since the kernel of the linear form $v_- \mapsto v_- v_+$ on V_- has codimension 1, hence half of the elements of V_- are mapped to $+1$, and the other half is mapped to -1. For $v_+ = 0$, one has $\sum_{v_- \in V_-} i^{i_2^4(v_- v_+)} = |V_-|$. So

$$\sum_{v \in V} i^{q(v)} = \sum_{v_0 \in V_0} i^{q(v_0)} \left(i^{q(0)+0} \cdot |V_-| \right)$$

$$= \left(\sum_{v_0 \in V_0} i^{q(v_0)} \right) \cdot |V_-|,$$

which gives $S(q) = S(q_0)$, ending the proof. ■

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Lebenslauf

Stephan Klaus, geboren am 5.10.1963 in Saarbrücken

1968:	Einschulung in die Knappenroth-Volksschule Saarbrücken
1972:	Besuch des Otto Hahn-Gymnasiums in Saarbrücken
6.82	Abitur
10.82:	Beginn des Diplom-Studiums in Mathematik und Physik an der Johannes Gutenberg-Universität Mainz
10.84:	Vordiplom in Mathematik
7.85:	Vordiplom in Physik
2.89:	Diplom in Mathematik, Gesamtnote: sehr gut Betreuer: Prof. Dr. Matthias Kreck Thema der Diplomarbeit: "Einfach-zusammenhängende kompakte homogene Räume bis zur Dimension 9"
5.89-9.90:	Zivildienst beim Arbeiter-Samariter-Bund in Bonn
10.90-9.92:	Physikstudium an der Universität in Bonn
1.92:	Doktorand im Fachbereich Mathematik der Universität Mainz Betreuer: Prof. Dr. Matthias Kreck Thema der Dissertation: "Brown-Kervaire Invariants"
10.92:	Einstellung als wissenschaftlicher Mitarbeiter im Fachbereich Mathematik der Universität Mainz