FILTRATION OF THE CLASSICAL KNOT CONCORDANCE GROUP AND
CASSON-GORDON INVARIANTS

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Abstract. It is known that if any prime power branched cyclic cover of a knot in \( S^3 \) is a homology sphere, then the knot has vanishing Casson-Gordon invariants. We construct infinitely many examples of (topologically) non-slice knots in \( S^3 \) whose prime power branched cyclic covers are homology spheres. We show that these knots generate an infinite rank subgroup of \( \mathcal{F}_{(1.0)}/\mathcal{F}_{(1.5)} \) for which Casson-Gordon invariants vanish in Cochran-Orr-Teichner’s filtration of the classical knot concordance group. As a corollary, it follows that Casson-Gordon invariants are not a complete set of obstructions to a second layer of Whitney disks.

1. Introduction

A knot in the 3-sphere is (topologically) slice if it bounds a locally flat 2-disk in the 4-ball. Two knots are said to be (topologically) concordant if the connected sum of one and the mirror image of the other with reversed orientation is slice. (Equivalently, there is a locally flat embedding of an annulus \( S^1 \times [0,1] \) into \( S^3 \times [0,1] \) whose restrictions to the boundary components give the knots.) This concordance relation is an equivalence relation, and the concordance classes form an abelian group \( 
\mathcal{C} \), the classical knot concordance group, under the connected sum operation. In \( \mathcal{C} \), the identity element is the class of slice knots.

In [COT], Cochran, Orr, and Teichner (henceforth COT) define a geometric filtration of the classical knot concordance group \( \mathcal{C} \)

\[
0 \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_{n} \subset \cdots \subset \mathcal{F}_{1.5} \subset \mathcal{F}_{1.0} \subset \mathcal{F}_{0.5} \subset \mathcal{F}_{0} \subset \mathcal{C}
\]

where \( \mathcal{F}_{m} \) is the set of \( (m) \)-solvable knots. (See Definition 3.4.) They show that \( (1.5) \)-solvable knots have vanishing Casson-Gordon invariants and that \( \mathcal{F}_{(2.0)}/\mathcal{F}_{(2.5)} \neq 0 \), thus giving the first examples of knots with vanishing Casson-Gordon invariants which are not (topologically) slice. (Refer to [CG] for Casson-Gordon invariants.) In [COT2], they extend their results to show \( \mathcal{F}_{(2.0)}/\mathcal{F}_{(2.5)} \) has infinite rank. We improve their results further and prove:

Theorem 1.1 (Main Theorem). In the above filtration, \( \mathcal{F}_{(1.0)}/\mathcal{F}_{(1.5)} \) has an infinite rank subgroup of knots for which Casson-Gordon invariants vanish.

Theorem 1.1 implies that Casson-Gordon invariants are not a complete set of obstructions to \( (1.5) \)-solvability. By contrast to the above result, the examples of [COT] are \( (2.0) \)-solvable.

To show Casson-Gordon invariants of our examples vanish, we use the following theorem of Livingston.

Theorem 1.2. ([Liv, Theorem 0.5]) A knot \( K \) has a prime power branched cyclic cover with nontrivial homology if and only if its Alexander polynomial has a nontrivial factor that is not an \( n \)-cyclotomic polynomial with \( n \) divisible by three distinct primes.

The group of examples in Theorem 1.1 have a spanning set of knots with a fixed Seifert form and Alexander polynomial. The shared Alexander polynomial of these generators is \( (\Phi_{30})^2 \),

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the square of the 30-cyclotomic polynomial. By Theorem 1.2, these generators have prime power branched cyclic covers which are homology spheres. (One might compare this to the fact that if every finite branched cyclic cover of a knot is a homology sphere, then its Alexander polynomial is 1, hence the knot is topologically slice by Freedman’s work [F].) It follows from the definition of Casson-Gordon invariants that Casson-Gordon invariants vanish for these knots. (See Proposition 6.4.) In fact, since any prime power branched cyclic cover is a homology sphere all the concordance invariants known prior to Cochran-Orr-Teichner’s $L^{(2)}$-signature invariants, such as Gilmer’s extension of Casson-Gordon invariants ([G]), Kirk and Livingston’s twisted Alexander invariants ([KL]), and Letsche’s invariants ([Le]), vanish for these knots.

Theorem 1.1 has a significant geometric consequence. Freedman’s disk embedding theorem ([F]), together with the Cappell-Shaneson homology surgery approach ([CS]) to classifying knot concordance group, suggest that the Casson-Gordon invariants obstruct the construction of a second layer of Whitney disks for a Cappell-Shaneson surgery kernel of an algebraically slice knot. That this is so was shown in [COT1, Section 8 and 9]. Indeed, in [COT1], they showed that a knot is (1.5)-solvable if and only if for zero surgery on the knot in $S^3$, there exists an $H_1$-bordism which contains a spherical Lagrangian admitting a Whitney tower of height (1.5). (See [COT1, Theorem 8.4] and Section 3 in this paper). Since (1.5)-solvable knots have vanishing Casson-Gordon invariants, it follows that Casson-Gordon invariants obstruct a Whitney tower of height (1.5) in the above sense. Precise definitions of a Whitney tower and other terminologies are given in [COT1, Section 8] and are reviewed in Section 3 in this paper.

We briefly discuss Whitney towers here. In 4-manifolds, Whitney disks may no longer be embedded, but may themselves have intersections, which might or might not occur in algebraically cancelling pairs. If these intersections occur in algebraically cancelling pairs, one can construct immersed Whitney disks for these cancelling pairs of points in the usual manner. Very roughly speaking, a Whitney tower is obtained by iterating this procedure. We have the following corollary of Theorem 1.1.

**Corollary 1.3.** There is an algebraically slice knot with vanishing Casson-Gordon invariants such that zero surgery on the knot in $S^3$ does not bound an $H_1$-bordism which contains a spherical Lagrangian admitting a Whitney tower of height (1.5).

**Proof.** It follows from Theorem 1.1 and [COT1, Theorem 8.4].

Corollary 1.3 says that Casson-Gordon invariants are not a complete set of obstructions to a second layer of Whitney disks.

To find the knots generating the subgroup in Theorem 1.1, we follow the method of COT. We begin by constructing a ribbon knot with the rational Alexander module $\mathbb{Q}[t, t^{-1}]/(\Phi_{30}(t))^2$. In particular, its Alexander polynomial is $(\Phi_{30}(t))^2$. Henceforth we refer to this ribbon knot as the seed knot to the examples of Theorem 1.1. (See Remark 2.3. See [K] for the definition of a ribbon knot. In particular, a ribbon knot is a slice knot.) We modify this seed knot using a family of Arf invariant zero knots in a way described in [COT2, Section 3] and reviewed in Section 4 in this paper. The resulting knots are shown to have the same Seifert form with the seed knot, so their prime power branched cyclic covers are also homology spheres by Theorem 1.2. Another important fact, which will be used significantly in this paper, is that $\mathbb{Q}[t, t^{-1}]/(\Phi_{30}(t))^2$ has a unique nontrivial proper submodule. (See the proofs of Lemma 5.3, Proposition 5.4 and Theorem 1.4.)
This paper is organized in the following manner. In Section 2, we construct a ribbon knot whose rational Alexander module is cyclic of order \((\Phi_{30}(t))^2\), i.e., \(\mathbb{Q}[t, t^{-1}]/(\Phi_{30}(t))^2\). This will be our seed knot. In Section 3, we explain the definition and properties of the Cochran-Orr-Teichner filtration of the classical knot concordance group and its relation to Whitney towers. In Section 4, we discuss how to construct a family of \((n)\)-solvable knots from a given ribbon knot using a certain Arf invariant zero knot. This method, applied to the ribbon knot mentioned above, will be used to construct the generators of the desired subgroup. In Section 5, \(L^{(2)}\)-signatures and their properties are reviewed. Finally, in Section 6, we provide the construction of a set of generators of the subgroup in Theorem 1.1 and the proof of Theorem 1.1.

Remark 1.4. For any \(n \in \mathbb{N}\) which is divisible by at least three distinct primes, we can also find an infinite rank subgroup of \(F_{(1,0)}/F_{(1,5)}\) that has generators with the Alexander polynomial \((\Phi_n(t))^2\) (the square of the \(n\)-cyclotomic polynomial). The only difference in the proof will be finding a seed knot with the rational Alexander module \(\mathbb{Q}[t, t^{-1}]/(\Phi_n(t))^2\) as we do for \(n = 30\) in Section 2.

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2. Construction of the Seed Knot

In this section, we construct our seed knot. That is, we will construct a knot which is a ribbon knot and has the rational Alexander module \(\mathbb{Q}[t, t^{-1}]/(\Phi_{30}(t))^2\). Then by Theorem 1.2 of C. Livingston, the seed knot will have prime power branched cyclic covers that are homology spheres.

First, we find a Seifert matrix whose Alexander polynomial is \(\Phi_{30}(t)\). Recall that \(\Phi_{30}(t) = t^8 + t^7 - t^5 - t^4 - t^3 + t + 1\). This can be done by applying Levine’s arguments in [L, 14 on page 236] which originated from Seifert [S]. But we will need a Seifert surface that is a boundary connected sum of disks with two bands, and it’s not clear how to find such a Seifert surface from the resulting Seifert matrix. So we modify Levine’s arguments a little. The final matrix is

\[
A = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & -9 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -9 & 1 & 0 & 1 & 26 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 26 & 1 & 24 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

That is, \(\det(A^T - tA) = \Phi_{30}(t)\). One can easily construct a knot, say \(K_1\), and its Seifert surface whose Seifert matrix with respect to a certain choice of basis is the matrix \(A\). The Seifert surface is obtained in the usual way as the boundary connected sum of disks with two bands under proper twists and intertwining among bands. Figure 1 is a part of \(K_1\) and its Seifert surface. The rectangles containing integers symbolize full twists between the two strands which pass vertically through the rectangles. Thus the rectangle labelled \(+24\) symbolizes 24 right-handed full twists. Let \(u_i, 1 \leq i \leq 8\), be the simple closed curves on the Seifert surface each of which goes once around a band. With proper orientations, \(\{u_i\}_{1 \leq i \leq 8}\) is a basis with respect to which the Seifert matrix is the matrix \(A\). It is known that \(A^T - tA\) is a presentation
matrix of the rational Alexander module of $K_1$. By column and row operations on $A^T - tA$ over $\mathbb{Q}[t, t^{-1}]$-coefficients, we determine that the rational Alexander module of $K_1$ is isomorphic to $\mathbb{Q}[t, t^{-1}]/\Phi_{30}(t)$, whose only generator is represented by a dual of $u_8$. (A dual of $u_8$ is a simple closed curve in the complement of the Seifert surface such that it has linking number one with $u_8$ and no linking with the other $u_i$’s.)

Let $K_2 = K_1 \# (-K_1)$, the connected sum of $K_1$ and its inverse. Then $K_2$ is a ribbon knot. (See, for instance, [K, Proposition 5.10 p.83].) Its rational Alexander module is $\mathbb{Q}[t, t^{-1}]/\Phi_{30}(t) \oplus \mathbb{Q}[t, t^{-1}]/\Phi_{30}(t)$, and its Seifert surface is obtained as the boundary connected sum of the Seifert surface of $K_1$ and that of $-K_1$ which is the mirror image of the Seifert surface of $K_1$. See Figure 2 below. Let $M_2$ denote zero surgery on $K_2$ in $S^3$. The rational Alexander module of $K_2$, $H_1(M_2; \mathbb{Q}[t, t^{-1}])$, is generated by $v_i, 1 \leq i \leq 16$, where for $1 \leq i \leq 8$, $v_i = u_i$, and for $9 \leq i \leq 16$, $v_i$ is the mirror image of $-u_{(17-i)}$. With this choice of basis, the Seifert matrix of $K_2$ is the matrix $B = (b_{ij}), 1 \leq i, j \leq 16$, defined by

$$b_{ij} = \begin{cases} a_{ij} : 1 \leq i, j \leq 8 \\ -a_{(17-i)(17-j)} : 9 \leq i, j \leq 16 \\ 0 : \text{otherwise} \end{cases}$$

Figure 1.

Figure 2.
Even though \( K_2 \) is a ribbon knot, its rational Alexander module is generated by two elements. In particular, it’s not cyclic. So we modify \( K_2 \) a little more. Choose an unknot \( \alpha \) around the Seifert surface of \( K_2 \) as in Figure 2. After +1 surgery on \( \alpha \), \( K_2 \) will be modified to a new knot, say \( K_s \), in \( S^3 \) since the resulting ambient manifold obtained by +1 surgery on an unknot in \( S^3 \) is homeomorphic with \( S^3 \). A part of \( K_s \) is illustrated in Figure 3. Let \( w_i \), \( 1 \leq i \leq 16 \), denote the image of \( v_i \) under the surgery. \( \{w_i\}_{1 \leq i \leq 16} \) is a basis of the Seifert form of \( K_s \). The Seifert matrix with respect to this basis is obtained by changing the matrix \( B \) such that only \( b_{ij} \) with \( 7 \leq i,j \leq 10 \) are changed from

\[
\begin{pmatrix}
24 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & -1 & -24
\end{pmatrix}
\]

to

\[
\begin{pmatrix}
24 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & -1 & -24
\end{pmatrix}
\]

Denote the resulting matrix by \( C \). Let \( M_s \) denote zero surgery on \( K_s \) in \( S^3 \).

**Figure 3.**

**Proposition 2.1.** The rational Alexander module of \( K_s \) is cyclic of order \((\Phi_{30}(t))^2\), i.e., \( H_1(M_s; \mathbb{Q}[t, t^{-1}]) \cong \mathbb{Q}[t, t^{-1}]/(\Phi_{30}(t))^2\).

**Proof.** \( C^T - tC \) is a presentation matrix of \( H_1(M_s; \mathbb{Q}[t, t^{-1}]) \). By column and row operations on \( C^T - tC \) over \( \mathbb{Q}[t, t^{-1}] \)-coefficients, we find out that \( H_1(M_s; \mathbb{Q}[t, t^{-1}]) \cong \mathbb{Q}[t, t^{-1}]/(\Phi_{30}(t))^2 \) whose only generator is a dual of \( w_9 \).

**Proposition 2.2.** \( K_s \) is a ribbon knot.

**Proof.** One can construct a ribbon disk for the ribbon knot \( K_2 \) using the method in [K, Proposition 5.10]. In particular, a ribbon disk can be obtained such that \( \alpha \) is disjoint from the ribbon disk and the spanning disk of \( \alpha \) has no intersection with the singularities of the ribbon disk. After +1 surgery along \( \alpha \), the image of the ribbon disk of \( K_2 \) would be a ribbon disk of \( K_s \).

**Remark 2.3.** By Proposition 2.1 and 2.2, \( K_s \) has the rational Alexander module which is cyclic of order \((\Phi_{30}(t))^2\) and it is a ribbon knot. So \( K_s \) is our desired seed knot.

3. **Filtering the knot concordance group and Whitney towers**

This section and the next two sections are brief expositions of some of the work in [COT1] and [COT2]. These sections contain no new results but serve to clarify ideas and make this paper more self-contained.
In [COT1], Cochran, Orr, and Teichner established a geometric filtration of the knot concordance group $\mathcal{C}$

$$0 \subset \cdots \subset \mathcal{F}_{(n,5)} \subset \mathcal{F}_{(n)} \subset \cdots \subset \mathcal{F}_{(1,5)} \subset \mathcal{F}_{(1,0)} \subset \mathcal{F}_{(0,5)} \subset \mathcal{F}_{(0)} \subset \mathcal{C}$$

where $\mathcal{F}_{(n)}$ is the subgroup of $(n)$-solvable knots for $n \in \{0, 0.5, 1.0, 1.5, \cdots \}$. The precise definition of this filtration and Whitney towers, and their relations will be discussed in this section.

Let $G^{(i)}$ denote the $i$-th derived subgroup of a group $G$, inductively defined by $G^{(0)} \equiv G$ and $G^{(i+1)} \equiv [G^{(i)}, G^{(i)}]$. For a CW-complex $W$, we denote the regular covering of $W$ corresponding to the subgroup $\pi_1(W)^{(n)}$ by $W^{(n)}$. If $W$ is a spin 4-manifold, then we have the usual intersection form

$$\lambda_n : H_2(W^{(n)}) \times H_2(W^{(n)}) \rightarrow \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}]$$

A more detailed description of $\lambda_n$ and the self-intersection invariant $\mu_n$ can be found in [W, Chapter 5] and [COT1, Section 7]. In particular, $\lambda_0$ is the ordinary intersection form on $H_2(W)$.

**Definition 3.1.** An $H_1$-bordism is a 4-dimensional spin manifold $W$ with boundary $M$ such that the inclusion map induces an isomorphism $H_1(M) \xrightarrow{\cong} H_1(W)$.

An $(n)$-surface is a generic immersion of a closed oriented surface $F$, say $f : F \to X$, such that $f_*(\pi_1(F)) \leq \pi_1(X)^{(n)}$.

**Definition 3.2.** Let $W$ be an $H_1$-bordism such that $\lambda_0$ is a hyperbolic form on $H_2(W)$.

1. A Lagrangian for $\lambda_0$ is a direct summand of $H_2(W)$ of half rank on which $\lambda_0$ vanishes.
2. An $(n)$-Lagrangian is a submodule of $H_2(W^{(n)})$ on which $\lambda_n$ and $\mu_n$ vanish and which maps onto a Lagrangian of $\lambda_0$ on $H_2(W)$.
3. A spherical Lagrangian is a submodule of $\pi_2(W)$ on which $\lambda_n, \mu_n$ ($n \geq 0$) vanish and which maps onto a Lagrangian of $\lambda_0$.
4. For $k \leq n$, $(k)$-duals of an $(n)$-Lagrangian generated by $(n)$-surfaces $\ell_1, \cdots, \ell_g$ are $(k)$-surfaces $d_1, \cdots, d_g$ such that $H_2(W)$ has rank $2g$ and

$$\lambda_k(\ell_i, d_j) = \delta_{i,j}.$$

Before giving the definition of $(n)$-solvability, we discuss Whitney towers. Let $W$ be a 4-manifold with boundary $M$ and $\gamma$ be a framed circle in $M$. A Whitney disk is an immersed disk $\Delta$ in $W$ which bounds $\gamma$ and such that the unique framing on the normal bundle of $\Delta$ restricts to the given framing on $\gamma$. $\gamma$ is called its Whitney circle.

**Definition 3.3.**

1. A Whitney tower of height $(0)$ is a collection $C_0$ of 2-spheres $S_i \subset W^4$.
2. For $n \in \mathbb{N}$, a Whitney tower of height $(n)$ on $C_0$ is a sequence $C_j = \{\Delta_{j,k}\}_{k=1}^{j}$ of framed immersed Whitney disks $\Delta_{j,k}$ in general position such that for $j = 2, \cdots, n$, the collection $C_j$ pairs up all $C_{j-1}$-(self)-intersections and has interiors disjoint from $C_1, \ldots, C_{j-1}$.
3. For $n \in \mathbb{N}$, a Whitney tower of height $(n.5)$ on $C_0$ is a sequence $C_j = \{\Delta_{j,k}\}_{k=1}^{j+1}$ of framed immersed Whitney disks such that $C_1, \ldots, C_n$ consist of a Whitney tower of height $n$ on $C_0$ and $C_{n+1}$ pairs up all $C_n$-(self)-intersections and has interiors disjoint from $C_1, \ldots, C_{n-1}$ (but $C_{n+1}$ is allowed to intersect the previous collection $C_n$).

Refer to [COT1, Section 7] for more details about Whitney towers.
Definition 3.4. A 3-manifold $M$ is $(n)$-solvable (resp. $(n.5)$-solvable) if there is an $H_1$-bordism $W$ which contains an $(n)$-Lagrangian (resp. $(n+1)$-Lagrangian) with $(n)$-duals. If $M$ is zero surgery on a knot or a link then the corresponding knot or link is called $(n)$-solvable (resp. $(n.5)$-solvable).

In Definition 3.4, $M$ is said to be $(n)$-solvable via $(n.5)$-solvable via) $W$, and $W$ is called an $(n)$-solution (resp. $(n.5)$-solution) for $M$.

Theorem 3.5. ([COT1, Theorem 8.4, 8.8]) Let $M$ be a closed oriented 3-manifold and $n \in \{0, 0.5, 1.0, 1.5, \cdots\}$. Then $M$ is $(n)$-solvable if and only if there is an $H_1$-bordism which contains a spherical Lagrangian admitting a Whitney tower of height $(n)$.

Remark 3.6. The exterior of a slice disk is an $(n)$-solution for the slice knot (and for its zero surgery $M$) for all $n$.

4. Constructing $(n)$-solvable knots

In this section, we obtain an $(n)$-solvable knot by modifying a given ribbon knot $K$. For this purpose, we make use of a grafting construction, which produces a satellite knot of $K$. For further details on this construction and more general cases, the reader should consult [COT2 Section 3].

Simply speaking, seize a collection of parallel strands of $K$ in one hand and tie these into a knot, say $J$. More precisely, choose a circle, say $\eta$, in $S^3 \setminus K$ which bounds an embedded disk in $S^3$. Now cut open $K$ along this disk and tie all the strands passing through this disk into $J$, or more exactly, through a tubular neighborhood of $J$ with 0-framing. Then the resulting ambient manifold is still homeomorphic with $S^3$, and under this identification, we obtain a new knot $K'$ which is the image of $K$. We denote the resulting knot $K'$ by $K(J, \eta)$. Moreover, this construction has another very useful description. $K(J, \eta)$ is obtained by taking the union of the exterior of $\eta$ and that of $J$ along the boundary in such a way that the resulting ambient manifold is homeomorphic with $S^3$.

Making use of the above construction, we get the following proposition due to COT. We outline a proof here for completeness and to establish notation for what follows. $M$ (resp. $M_J$) denotes zero surgery on $K$ (resp. $J$) in $S^3$. Note that a knot is $(0)$-solvable if and only if it has Arf invariant zero. ([COT1, Remark 8.2].)

Proposition 4.1. If $\eta \in \pi_1(M)^{(n)}$ and $J$ has Arf invariant zero, then $K(J, \eta)$ is $(n)$-solvable.

Proof. This is a special case of [COT2 Proposition 3.1]. Let $W$ be the exterior of a ribbon disk for $K$ in $B^4$. (Note that $W$ may be viewed as an $(n)$-solution.) Let $W_J$ be the $(0)$-solution for $J$ such that a canonical epimorphism $\pi_1(M_J) \rightarrow Z$ extends to $\pi_1(W_J)$. By doing surgery on elements in $\pi_1(W_J)^{(1)}$, we can assume that $\pi_1(W_J) \cong Z$. Let $\mu_J$ denote the meridian of a tubular neighborhood of $J$ and let $\ell_J$ be the 0-framed longitude. Then $\partial W_J = M_J = E_J \cup (S^1 \times D^2)$ where $S^1 \times \{\ast\}$ is $\mu_J$, and $\{\ast\} \times \partial D^2$ is $\ell_J$. Let $W'$ be the 4-manifold obtained from $W_J$ and $W$ by identifying the solid torus $S^1 \times D^2 \subset \partial W_J$ with $\eta \times D^2 \subset \partial W$. Observe that $\partial W' = M'$, zero surgery on $K' = K(J, \eta)$. Then $W'$ is an $(n)$-solution for $K'$. See [COT2, Proposition 3.1] for more details.
5. Detecting \((n)\)-solvability using \(L^{(2)}\)-signatures

A group \(\Gamma\) is called poly-torsion-free-abelian (PTFA) if it admits a normal series \(\langle 1\rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = \Gamma\) such that the factors \(G_{i+1}/G_i\) are torsion-free abelian. If \(\Gamma\) is PTFA, then the group ring \(\mathbb{Q}\Gamma\) is a right Ore domain, hence \(\mathbb{Q}\Gamma\) embeds in its classical right ring of quotients \(K\Gamma\).

(\textit{COT}, Proposition 2.5.) Let \(M\) be an oriented closed 3-manifold. Suppose \(\phi : \pi_1(M) \to \Gamma\) is a homomorphism where \(\Gamma\) is a PTFA group and suppose there are an oriented compact 4-manifold \(W\) bounded by \(M\) and a homomorphism \(\psi : \pi_1(W) \to \Gamma\) which extends \(\phi\), \(i.e.,\) \((M, \phi) = \partial(W, \psi)\). Then the (reduced) \(L^{(2)}\)-signature or von Neumann \(\rho\)-invariant \(\rho(M, \phi) \in \mathbb{R}\) is defined to be \(\rho(M, \phi) = \sigma^{(2)}_\Gamma(W, \psi) - \sigma_0(W)\) where \(\sigma^{(2)}_\Gamma\) is the \(L^{(2)}\)-signature of the intersection form on \(H_2(W; \mathbb{Z}_\Gamma)\) and \(\sigma_0\) is the ordinary signature. We refer the reader to \(\textit{COT},\) Section 5 for more discussion of \(L^{(2)}\)-signatures. The following theorem, due to COT, gives an obstruction for a knot being \((n.5)\)-solvable.

**Theorem 5.1.** (\textit{COT}, Theorem 4.2) Suppose \(\Gamma\) is an \((n)\)-solvable group and \(M\) is \((n)\)-solvable. If \(\phi : \pi_1(M) \to \Gamma\) extends over some \((n.5)\)-solution \(W\) for \(M\), then \(\rho(M, \phi) = 0\).

**Corollary 5.2.** If \(K\) is a slice knot and \(\phi\) extends over the exterior of a slice disk, then \(\rho(M, \phi) = 0\) for any PTFA group \(\Gamma\) where \(M\) is zero surgery on \(K\) in \(S^3\).

As to calculating \(\rho\)-invariants, if \(\Gamma = \mathbb{Z}\) and \(\phi\) is not trivial, then \(\rho(M, \phi)\) is easily calculated as a certain integral over \(S^3\). (See \textit{COT}, Property 2.4.) In particular, if \(K\) has Arf invariant zero (\(i.e.,\) \(K\) is \((0)\)-solvable), then we can assign a real value \(\rho(K)\) to \(K\) that is “canonically” induced from \(\rho\)-invariants as follows. Let \(M\) be zero surgery on \(K\) in \(S^3\). Choose a \((0)\)-solution \(W\) of \(M\) such that a canonical epimorphism \(\phi : \pi_1(M) \to \mathbb{Z}\) extends to \(\psi : \pi_1(W) \to \mathbb{Z}\) and \(\pi_1(W) \cong \mathbb{Z}\) as we did in the proof of Proposition \(\text{[1,]}\). Then we can calculate \(\rho(M, \phi)\) via \((W, \psi)\) and we define \(\rho(K)\) to be these “canonical” real numbers. These “canonical” real numbers will play an important role in our work. (See Proposition \(\text{[1.3]}\) and the paragraph preceding Proposition \(\text{[1.4]}\).

Now we investigate how \(\rho\)-invariants change under the grafting construction described in Section \(\text{[1]}\). Though there is no general additive property of \(\rho\)-invariants, if the representations of the fundamental groups of the relevant manifolds are matched up nicely under the grafting construction, we can derive an additive property. In particular, to prove the main theorem, we only need to look into \(\rho\)-invariants of \(K(J, \eta)\) where \(K\) is a ribbon knot and \(J\) has \(n\)-invariant zero.

Suppose \(K\) is a ribbon knot and \(J\) has \(n\)-invariant zero. Let \(W\) be the exterior of a ribbon disk for \(K\). \(W, J, M, \eta\) are defined as in Proposition \(\text{[1]}\). Suppose we are given homomorphisms \(\phi : \pi_1(M) \to \Gamma\) and \(\phi_J : \pi_1(M_J) \to \Gamma\) such that \(\phi([\eta]) = \phi_J([\mu_J])\) where \(\Gamma\) is a PTFA group. Then \(\phi\) and \(\phi_J\) produce a unique homomorphism \(\phi' : \pi_1(M') \to \Gamma\) (For this, observe that \(M' = (M \setminus \eta \times D^2) \cup \bigcup_{S^1 \times S^1} E_J\) where \(M \setminus (\eta \times D^2) \subset M\) and \(E_J \subset M_J\)). Use Van Kampen Theorem noticing that for \(\{*\} \times \partial D^2 \subset \eta \times D^2\), \(\phi((\{*\} \times \partial D^2)) = \phi_J((\ell_J) = 0)\). Then we have the following proposition due to COT.

**Proposition 5.3.** Suppose \(\phi'\) extends to \(\psi' : \pi_1(W') \to \Gamma\). Then \(\rho(M', \phi') = \rho(J)\) if \(\phi(\eta) \neq 1\), and \(\rho(M', \phi') = 0\) if \(\phi(\eta) = 1\).

**Proof.** By \(\textit{COT},\) Proposition 3.2, \(\rho(M', \phi') = \rho(M, \phi) + \rho(M_J, \phi_J)\). \(\rho(M, \phi) = 0\) by Corollary \(\text{[5.2]}\). Since \(\eta\) generates \(\pi_1(W_J) \cong \mathbb{Z}\), \(\rho(M_J, \phi_J) = \rho(J)\) if \(\phi(\eta) \neq 1\) and \(\rho(M_J, \phi_J) = 0\) if \(\phi(\eta) = 1\) by Property 2.3 and 2.5 in \(\textit{COT}\).

\[\square\]
6. Proof of main theorem

We begin this section by briefly reviewing some very useful machinery for the proof of the main theorem. This originates from [COT], Section 2, 3, and 4, so for all the detailed arguments and more generalized facts, the readers should consult [COT].

**Definition 6.1.** The family of rationally universal groups \( \{ \Gamma^U_n \} \) is defined inductively by \( \Gamma^U_0 = \mathbb{Z} \), \( R^U_0 = \mathbb{Q}[t, t^{-1}] \) and for \( n \geq 0 \), setting

\[
S_n = \mathbb{Q}[\Gamma^U_n, \Gamma^U_{n+1}] - \{0\}, \quad R^U_n = (Q\Gamma^U_n)S^{-1}_n
\]

and

\[
\Gamma^U_{n+1} = K_n/R^U_n \times \Gamma^U_n.
\]

Here \( K_n \) is the right ring of quotients of \( \mathbb{Q}[\Gamma^U_n] \).

It is shown in [COT], Proposition 2.5] that \( \mathbb{Q}[\Gamma^U_n] \) is an Ore domain, i.e., \( \mathbb{Q}[\Gamma^U_n] \) has a right ring of quotients. (Note that inductively \( \Gamma^U_n \) is PTFA.) The semi-direct product is defined via the left multiplication of \( \Gamma^U_n \) on \( K_n/R^U_n \). One can show that \( \Gamma^U_n \) is an \( (n) \)-solvable group for all \( n \geq 0 \). Observe that \( K_0 = \mathbb{Q}(t) \) and \( \Gamma^U_1 = \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}] \cong \mathbb{Z} \).

Suppose \( M \) is a closed 3-manifold with \( \beta_1(M) = 1 \) and we have a homomorphism \( \phi_0 : \pi_1(M) \rightarrow \Gamma^U_1 \). Then we can define the rational Alexander module \( A_0(M) \equiv H_1(M; R^U_0) \) and the (non-singular) Blanchfield form \( B\ell_0 : A_0(M) \times A_0(M) \rightarrow K_0/R^U_0 \). Then,

\[
A_0(M) \equiv H_1(M; K_0/R^U_0) \cong H^2(M; R^U_0) \cong H^1(M; K_0/R^U_0).
\]

and there is a bijection \( f : H^1(M; K_0/R^U_0) \rightarrow \text{Rep}^*_{\Gamma^U_0} (\pi_1(M), \Gamma^U_1) \)

(\text{Rep}^*_{\Gamma^U_0} (\pi_1(M), \Gamma^U_1) \) is defined to be the representations from \( \pi_1(M) \) to \( \Gamma^U_1 \) which agree with \( \phi_0 \) after composing with the projection \( \Gamma^U_1 \rightarrow \Gamma^U_0 \) modulo \( K_0/R^U_0 \)-conjugations.\) So any choice \( x_0 \in A_0(M) \) will (together with \( \phi_0 \)) induce \( \phi_1 : \pi_1(M) \rightarrow \Gamma^U_1 \). We refer to this as the **character induced by \( x_0 \).** Now we obtain some very useful facts which are summarized in the following remark.

**Remark 6.2.** (\text{[COT] Theorem 3.5, 3.6, and 4.4}) Suppose \( M = \partial W \) is a compact 3-manifold with \( \beta_1(M) = 1 \) and \( \phi_0 : \pi_1(M) \rightarrow \Gamma^U_0 \) is given.

(i) The isomorphism \( H_1(M; R^U_0) \cong H^1(M; K_0/R^U_0) \) with \( f \) gives a natural bijection \( \hat{f} : A_0(M) \rightarrow \text{Rep}^*_{\Gamma^U_0} (\pi_1(M), \Gamma^U_1) \).

(ii) If \( x \in A_0(M) \), then the character induced by \( x \) is given by \( y \mapsto B\ell_0(x, y) \).

(iii) Assume that the non-trivial map \( \phi_0 : \pi_1(M) \rightarrow \Gamma^U_0 \) extends to a map \( \psi_0 : \pi_1(W) \rightarrow \Gamma^U_0 \) and that \( \phi_1 \) is a representative of a class in \( \text{Rep}^*_{\Gamma^U_0} (\pi_1(M), \Gamma^U_1) \) corresponding to \( x \in H_1(M; R^U_0) \). Let

\[
P_0 = \ker\{ j_* : H_1(M; R^U_0) \rightarrow H_1(W; R^U_0) \}\}

Then if \( M \) is \( (1) \)-solvable via \( W \), then \( \phi_1 \) extends to \( \pi_1(W) \) if and only if \( x \in P_0 \).

(iv) Suppose \( M \) is \( (1) \)-solvable via \( W \) and \( \phi_0 \) is a non-trivial coefficient system that extends to \( \pi_1(W) \). Then the Blanchfield form \( B\ell_0 \) is hyperbolic, and in fact the kernel of \( j_* : H_1(M; R^U_0) \rightarrow H_1(W; R^U_0) \) is self-annihilating. (i.e., \( \ker j_* = (\ker j_*)^\perp \).)
From Theorem 5.1 and Remark 6.2, to prove that a knot $K$ is not $(1.5)$-solvable, basically we need to investigate the representations of the fundamental group induced from all self-annihilating submodules of $A_0(M) \equiv H_1(M; \mathbb{Q}[t, t^{-1}])$ where $M$ is zero surgery on $K$ in $S^3$. But in case $A_0(M)$ has a unique proper submodule, we have the following useful lemma.

**Lemma 6.3.** Suppose $M$ is $(1)$-solvable and $A_0(M)$ has a unique proper submodule $P$. If there exists $p \in P$ such that $\rho(M, \phi) \neq 0$ for $\phi : \pi_1(M) \rightarrow \Gamma^U_1$ induced from $p$, then $M$ is not $(1.5)$-solvable.

**Proof.** Suppose $M$ is $(1.5)$-solvable via $W$. Let $A_0(W) \equiv H_1(W; \mathbb{Q}[t, t^{-1}])$. Since $W$ is also a $(1)$-solution of $M$, the kernel of the inclusion-induced map $i_\ast : A_0(M) \rightarrow A_0(W)$ is self-annihilating with respect to $B_{\ell_0}$ by Remark 6.2 (iv). Since $A_0(M)$ has a unique proper submodule $P$, $\text{Ker } i_\ast = P$. By Remark 6.2 (i) and (iii) , $\phi : \pi_1(M) \rightarrow \Gamma^U_1$ induced from $p(\in P)$ extends to $\pi_1(W)$. Then since $W$ is assumed to be a $(1.5)$-solution of $M$, by Theorem 5.1, $\rho(M, \phi) = 0$. This leads us to a contradiction. \hfill \square

Through this section, $K_s$ denotes our seed ribbon knot which was constructed in Section 2 and $\eta$ is the designated circle in the complement of the Seifert surface of $K_s$ in $S^3$ as in Figure 4. Notice that $\eta$ is a dual of $w_9$, so it represents the homology class which generates the rational Alexander module of $K_s$.

![Figure 4.](image)

By [COT2, Proposition 2.6], there are infinitely many Arf invariant zero knots $J_i (i \in \mathbb{N})$ such that $\{\rho(J_i)\}_{i \in \mathbb{N}}$ is linearly independent over integers. In particular, $\rho(J_i) \neq 0$. Let $K_i \equiv K_s(J_i, \eta)$ be the family of knots resulting from the grafting construction as described in Section 4. In the following propositions, we exploit important properties of $K_i$.

**Proposition 6.4.** $K_i (i \in \mathbb{N})$ are $(1)$-solvable but not $(1.5)$-solvable.

**Proof.** $\eta$ lifts to a closed circle in the infinite cyclic cover of $S^3 \setminus K_s$, hence $\eta \in \pi_1(M)^{(1)}$ where $M$ is zero surgery on $K_s$ in $S^3$. Now it is clear from Proposition 1.1 that $K_i$ are $(1)$-solvable.

We need to show that $K_i$ are not $(1.5)$-solvable. Fix $i$. Let $W'$ denote the $(1)$-solution for $K_i$ formed as in the proof of Proposition 1.1 and let $M'$ denote zero surgery on $K_i$ in $S^3$. Recall that $\Gamma^U_0 = \mathbb{Z}$. Let $\pi_1(M') \rightarrow \Gamma^U_0$ be the canonical epimorphism which extends uniquely to an epimorphism $\pi_1(W')$. Looking into the grafting construction more closely, one can see that $K_i$ has the same Seifert form as that of $K_s$, so the rational Alexander module $A_0(M')$ is
Proof. Because $K_i$ and $K_j$ have the same Seifert form, they have the same Alexander polynomial which is $(\Phi_{30}(t))^2$. By Theorem 1.2, any prime power branched cyclic cover of $K_i$ is a homology sphere. Hence all Casson-Gordon invariants vanish on $K_i$ by [Lit, Corollary B2].

Now we are ready to prove the main theorem.

Proof of Theorem 1.1. First, we show that no non-trivial linear combination of $K_i, j \in \mathbb{N}$ is (1.5)-solvable. We follow [COT2]. Refer to the proof of [COT2, Theorem 4.1]. The only crucial difference in this proof is that we deal with (1.5)-solubility instead of (2.5)-solubility, so we use the second order invariants instead of the third order invariants. Since the proof will follow almost the same course of COT’s proof for [COT2, Theorem 4.1], some details will be omitted. For convenience, we follow the notations used in [COT2].

Suppose that a non-trivial linear combination $\sum_{i=1}^{m} n_i K_i, n_i' \neq 0$, is (1.5)-solvable. We may assume all $n_i' > 0$ by replacing $K_i$ by $-K_i$ if $n_i' < 0$ and $n_i' > 1$ if $m = 1$. Let $M_i$ denote $M_{K_i}$, and note that $-M_i = M_{-K_i}$. Let $M_0$ denote 0-surgery on $\sum_{i=1}^{m} n_i K_i$. Let $W_0$ be a (1.5)-solution of $M_0$. Let $W_i$ ($i > 0$) denote the specific (1)-solution for $M_i$ constructed as in Proposition 1.1 with the exterior of a ribbon disk for $K_i$. Let $n_1 = n_1' - 1$ and $n_i = n_i'$ if $i > 1$. Let $W$ be the union of $W_0, C$ ($C$ is defined in the next paragraph), and all the copies of $W_i$ ($i = 1, 2, \ldots, m$) where there are $n_i$ copies of $W_i$ below $C$. Refer to Figure 5 below. Later we will show that $W$ is a (1)-solution of $M_1$.

The 4-manifold $C$ is a standard cobordism between 0-surgery on $\#_{i=1}^{m} n_i' K_i$ and the disjoint union of 0-surgeries on the summands of $\#_{i=1}^{m} n_i' K_i$. Briefly, start with a collar on the disjoint

![Diagram](image-url)
union of 0-surgeries, and add 1-handles to get a connected 4-manifold whose upper boundary is given by surgery on the link consisting of the split union of $K_i$’s, each with 0-framing. Next add 0-framed 2-handles to get a 0-surgery on a connected sum of $K_i$’s on the upper boundary. See [COT2, Theorem 4.1] for more details, and note that $C$ has a handlebody decomposition, relative to $\bigsqcup_{i=1}^m n_i M_i$, consisting of $(\sum_{i=1}^m |n_i|)\,1$-handles and the same number of 2-handles. Moreover, $J_1(C;\mathbb{Z}) \cong \mathbb{Z}$ and the inclusion from any of its boundary components induces an isomorphism on $H_1$. One can also see that $H_2(C) \cong H_2(\bigsqcup_{i=1}^m n_i M_i)$.

We prove that $W$ is a (1)-solution of $M_1$. Since the inclusion-induced homomorphisms $H_1(M_i) \to H_1(W)$ are isomorphisms for $i \geq 0$, the inclusion-induced $H_1(M_1) \to H_1(W)$ is also an isomorphism. For $i \geq 0$, $H_2(M_i) \to H_2(W)$ is the zero map since the boundary map $H_3(W_i, M_i) \to H_2(M_i)$, the dual map of the inclusion induced $H^1(W_i) \to H^1(M_i)$, is an isomorphism. Using this and Mayer-Vietoris sequence we can prove that $H_2(W) \cong H_2(W_0) \bigoplus_{i=1}^m n_i H_2(W_i)$. Now if one looks carefully at (1)-Lagrangians and their duals for $W_0$ and the $W_i$’s, one can see that they form (1)-Lagrangian and its dual for $W$. So $W$ is a (1)-solution for $M_1$.

We repeat the argument in Proposition \[5.4.\] Let $\pi_1(M_1) \to \Gamma_1^U$ be the canonical epimorphism which extends uniquely to an epimorphism $\pi_1(W)$. Recall that the rational Alexander module $A_0(M_1) = H_1(M_1;\mathbb{Q}[t, t^{-1}])$ is isomorphic to $\mathbb{Q}[t, t^{-1}]/(\Phi_{20}(t))^2$. Let $A_0(W) = H_1(W;\mathbb{Q}[t, t^{-1}])$. By Remark \[5.2.(iv),\] since $W$ is a (1)-solution for $M_1$, the kernel of the inclusion-induced map $j_* : A_0(M_1) \to A_0(W)$ is self-annihilating with respect to the Blanchfield form $B\ell_0$. Since $A_0(M_1)$ has a unique proper submodule, say $P_0$, the latter is this kernel. Choose a non-zero $p_0 \in P_0$, inducing $\phi_1 : \pi_1(M_1) \to \Gamma_1^U$ by Remark \[5.2.(i).\] By Remark \[5.2.(iii),\] $\phi_1$ extends to $\psi_1 : \pi_1(W) \to \Gamma_1^U$. Therefore $\rho(M_1, \phi_1)$ can be computed using $(W, \psi_1)$.

We compute $\rho(M_1, \phi_1)$ using $(W, \psi_1)$. Let $\phi_{i,j}$ denote the restriction of $\psi_1$ to the $j$th copy of $\pi_1(M_i)$, $1 \leq j \leq n_i$. Let $\phi_0$ denote the restriction of $\psi_1$ to $\pi_1(M_0)$. Let $K_1$ denote the classical right ring of quotients of $\mathbb{Z}\Gamma_1^U$. $H_4(M_i; K_1) = 0$ for $i \geq 0$ ([COT1, Propositions 2.9 and 2.11]), so a Mayer-Vietoris sequence shows that $H_2(W; K_1) \cong H_2(W_0; K_1) \oplus H_2(C; K_1) \oplus H_2(W_1; K_1) \oplus \cdots \oplus H_2(W_m; K_1)$ where $W_i$ occurs $n_i$ times. Here the coefficient systems on $W_i$ ($i \geq 0$) and $C$ are induced by inclusions into $W$. By [COT2, Lemma 4.2] $H_2(C; K_1) = 0$. And the intersection form on $H_2(W; K_1)$ splits along the direct sum. From [COT1, Section 5], $\sigma^{(2)}_{\Gamma_1^U}$ can be viewed as a homomorphism from the Witt group of non-singular hermitian forms on finitely generated $K_1$ modules, so we have

$$\rho(M_1, \phi_1) = \rho(M_0, \phi_0) + \sum_{i=1}^m \sum_{j=1}^{n_i} \rho(-M_i, \phi_{i,j})$$

Here $\rho(M_0, \phi_0) = 0$ by Theorem \[5.1\] because $\phi_0$ extends to (1.5)-solution $W_0$. By Proposition \[5.3\] $\rho(-M_i, \phi_{i,j}) = -\rho(M_i, \phi_{i,j}) = -\rho(J_i)$ or 0. So we deduce that

$$\rho(M_1, \phi_1) + \sum_{i=1}^m c_i \rho(J_i) = 0$$

for some non-negative constants $c_i$’s.

Now as in Proposition \[5.4\], pick $p_0 \in P_0$ such that $\phi_1(\eta) \neq 1$. Note that $P_0$ is equal to the kernel of the inclusion-induced map $i_* : A_0(M_1) \to A_0(W_1)$. Then by Proposition \[5.3\], $\rho(M_1, \phi_1) = \rho(J_1)$, so we have
which contradicts that \( \{ \rho(J_i) \}_{i \in \mathbb{N}} \) is linearly independent over integers.

Now it remains to show that Casson-Gordon invariants vanish on the subgroup generated by \( K_i \). Recall that every prime power branched cyclic cover of \( K_i \) is a homology 3-sphere. One can show that the connected sum of two homology 3-spheres is a homology 3-sphere. Since a finite branched cyclic cover of the connected sum of two knots over \( S^3 \) is homeomorphic with the connected sum of the finite branched cyclic covers of the knots, the assertion follows from [Lit, Corollary B2].

References


