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ORTHOGONAL POLYNOMIALS AND CONTINUED FRACTIONS

From Euler's Point of View

Sergey Khrushchev

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Orthogonal Polynomials and Continued Fractions

Continued fractions, studied since the time of Ancient Greece, only became a powerful tool in the eighteenth century, in the hands of the great mathematician Euler. This book tells how Euler introduced the idea of orthogonal polynomials and combined the two subjects, and how Brouncker's formula of 1655 can be derived from Euler's efforts in special functions and orthogonal polynomials. The most interesting applications of this work are discussed, including Markoff's theorem on the Lagrange spectrum, Abel's theorem on integration in finite terms, Chebyshev's theory of orthogonal polynomials and very recent advances in orthogonal polynomials on the unit circle. As continued fractions become more important again, in part due to their use in finding algorithms in approximation theory, this timely book revives the approach of Wallis, Brouncker and Euler and illustrates the continuing significance of their influence. A translation of Euler's famous paper "Continued fractions, observations", is included as an appendix.

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of Galina Kreidtner (8/10/1920–23/12/2000)

Contents

	Preface		*page* ix
1	**Continued fractions: real numbers**		1
	1.1	Historical background	1
	1.2	Euler's theory of continued fractions	11
	1.3	Rational approximations	17
	1.4	Jean Bernoulli sequences	36
	1.5	Markoff sequences	49
2	**Continued fractions: algebra**		71
	2.1	Euler's algorithm	71
	2.2	Lagrange's theorem	81
	2.3	Pell's equation	84
	2.4	Equivalent irrationals	92
	2.5	Markoff's theory	98
3	**Continued fractions: analysis**		123
	3.1	Convergence: elementary methods	123
	3.2	Contribution of Brouncker and Wallis	131
	3.3	Brouncker's method and the gamma function	146
4	**Continued fractions: Euler**		158
	4.1	Partial sums	158
	4.2	Euler's version of Brouncker's method	163
	4.3	An extension of Wallis' formula	169
	4.4	Wallis' formula for sinusoidal spirals	174
	4.5	An extension of Brouncker's formula	177
	4.6	On the formation of continued fractions	180
	4.7	Euler's differential method	183
	4.8	Laplace transform of hyperbolic secant	191
	4.9	Stieltjes' continued fractions	194
	4.10	Continued fraction of hyperbolic cotangent	199
	4.11	Riccati's equation	206

5	**Continued fractions: Euler's influence**		**228**
	5.1	Bauer–Muir–Perron theory	229
	5.2	From Euler to Scott–Wall	232
	5.3	The irrationality of π	238
	5.4	The parabola theorem	240
6	***P*-fractions**		**247**
	6.1	Laurent series	247
	6.2	Convergents	253
	6.3	Quadratic irrationals	258
	6.4	Hypergeometric series	272
	6.5	Stieltjes' theory	285
7	**Orthogonal polynomials**		**296**
	7.1	Euler's problem	296
	7.2	Quadrature formulas	298
	7.3	Sturm's method	303
	7.4	Chebyshev's approach to orthogonal polynomials	310
	7.5	Examples of orthogonal polynomials	315
8	**Orthogonal polynomials on the unit circle**		**322**
	8.1	Orthogonal polynomials and continued fractions	322
	8.2	The Gram–Schmidt algorithm	336
	8.3	Szegő's alternative	346
	8.4	Erdös measures	356
	8.5	The continuum of Schur parameters	360
	8.6	Rakhmanov measures	364
	8.7	Convergence of Schur's algorithm on $\mathbb{T}$	368
	8.8	Nevai's class	371
	8.9	Inner functions and singular measures	380
	8.10	Schur functions of smooth measures	388
	8.11	Periodic measures	390
	Appendix	*Continued fractions, observations L. Euler (1739)*	426
	References		466
	Index		475

Preface

$$s^2 = \left\{ (s-1) + \frac{1^2}{2(s-1)} {}_{+} \frac{3^2}{2(s-1)} {}_{+} \frac{5^2}{2(s-1)} {}_{+} \cdots \right\}$$

$$\times \left\{ (s+1) + \frac{1^2}{2(s+1)} {}_{+} \frac{3^2}{2(s+1)} {}_{+} \frac{5^2}{2(s+1)} {}_{+} \cdots \right\} \qquad (1)$$

This book has emerged as a result of my attempts to understand the theory of orthogonal polynomials. I became acquainted with this theory by studying the excellent book by Geronimus (1958). However, the fundamental reasons for its beauty and difficulty remained unclear to me. From time to time I returned to this question but the first real progress occurred only in the autumn of 1987, when I visited the Mittag-Leffler Institute right at the beginning of the foundation of the Euler International Mathematical Institute in St Petersburg (the EIMI). A simple proof of Geronimus' theorem on the parameters of orthogonal polynomials had been found (Khrushchev 1993). The paper specified an important relationship of orthogonal polynomials on the unit circle to Schur's algorithm. Later this very paper was the starting point for Khrushchev (2001). It took about eight years to complete the EIMI project, which occupied all my time, leaving no chance to continue this research.

I returned to the subject matter of this book again only in the summer of 1998 in Almaty, where I was able to get back to mathematics with the assistance of mountains and the National and Central Scientific libraries of Kazakhstan. Both these libraries had a complete collection of Euler's books translated into Russian as well as a lot of other wonderful old Russian mathematical literature such as the Russian translation of Szegő's *Orthogonal Polynomials* (1975) by Geronimus, complete with his careful and comprehensive comments. All this was very helpful for my paper Khrushchev (2001), but the question of what is the driving mechanism for, and how it explains the mystery of, orthogonal polynomials remained open. It was clear to me that most likely this mechanism is continued fractions. And my impression that something very important had disappeared from the modern theory was supported by Chebyshev's and Markoff's contributions to the subject area as well as by the following remark of Szegő in 1939: “Despite the close relationship between continued fractions and the problem of moments, and notwithstanding recent important advances in the latter subject, continued fractions have been gradually abandoned as a starting point for the theory of orthogonal polynomials”.

The study of the book by W. B. Jones and W. J. Thron (1980), which I found in Pushkin's library in Almaty, indicated that perhaps the right answer to my question could be found in Euler's research on continued fractions. That not all Euler's papers on continued fractions had been carefully studied was already mentioned by Khovanskii (1957). There are two great papers of Euler in this field: (1744) and (1750b). The brief summary of Euler (1744) published as the eighteenth chapter of Euler (1748) is usually also mentioned. English translations of Euler (1744, 1748) are available. As for Euler (1750b), its first English translation is given as an appendix to this book. I thank Alexander Aptekarev (Institute of Applied Mathematics, Moscow) for arranging a translation from Latin to Russian. Then using my understanding of Euler (1750b), its Russian translation and Latin–Russian dictionaries, I translated it into English. Therefore it is not as professional as the translation of Euler (1744) but will, I hope, be acceptable. The most important facts on continued fractions from Euler (1744, 1748) are presented in the first chapter here.

I saw the Latin version of Euler (1750b) only in January 2003 when, at the kind invitation of Barry Simon, I visited Caltech, California, to lecture from a preliminary version of the present book. At that time I knew nothing of the project of the Euler Archive run by the Euler Society (www.eulersociety.org), which possibly then was only under construction. Lecturing in front of Simon's group in Caltech strongly reminded me of the golden years in St Petersburg in the 1970s. There are of course some differences because of the location. For instance, they do not have late evening tea and instead take lunch before the seminar where you can enjoy, if you are brave enough, hot Mexican pepper.

Even a very brief inspection of Euler (1750b) shows that it was motivated by the remarkable formula (1) discovered in March 1655 by Brouncker, the first President of the Royal Society of London. The proof of this formula was included in section 191 of Wallis (1656). I did not have access to this striking book at that period but from historical literature in Russian, for instance from Kramer (1961), I discovered that the presentation in this particular part is just impossible to understand. This was indirectly confirmed by Euler (1750b), who, in spite of the fact that Wallis' *Arithmetica Infinitorum* was a permanent feature of his desk, complained that Brouncker's proof was seemingly irreparably lost.

Nonetheless, in the summer of 2004 in the mountains of Almaty I came to the conclusion that possibly this can be done very easily if suitably transformed partial Wallis products are written as continued fractions. I couldn't find the required transformation and was about to give up but suddenly help arrived from the Amazon bookstore. It has a very good knowledge system which makes proposals based on the captured interests of their customers. Amazon's email claimed that the English translation of *Arithmetica Infinitorum* by Jacqueline Stedall (2004) was available from Oxford.

When I arrived back in Ankara the book awaited me in the post office. I opened Wallis' comment on section 191 and saw the following: "*The Noble Gentleman noticed that two consecutive odd numbers, if multiplied together, form a product which is the square of the intermediate even number minus one... He asked, therefore, by what ratio the factors must be increased to form a product, not those squares minus one, but equal to the squares themselves*". When I read this I could immediately understand how Brouncker proved (1). It took some time to complete the calculations and this proof is now available in Chapter 3. Wallis' previously unclear remarks are now used to confirm that the proof presented is exactly that discovered by Brouncker.

A few words explaining why (1) is so important. It is the functional equation $b(s-1)b(s+1) = s^2$, reminding us on the one hand of an elementary formula $(s-1)(s+1) = s^2 - 1$ from algebra and, on the other hand, of the functional equation for Euler's gamma function $\Gamma(x+1) = x\Gamma(x)$. In fact these two functions are related by the Ramanujan formula (see Theorem 3.25). Another mystery is that Ramanujan's formula in turn is an easy consequence of Brouncker's theory... Combining the Ramanujan formula with Chebyshev's arguments presented in Section 7.4, one easily obtains that the polynomials written explicitly in Wallis (1656, §191) are orthogonal with respect to the weight

$$d\mu = \frac{1}{8\pi^3}\left|\Gamma\left(\frac{1+it}{4}\right)\right|^4 dt\,.$$

In (1977), J. A. Wilson, following some ideas of R. Askey on the gamma function, introduced a new class of orthogonal polynomials depending on a number of independent parameters. An impressive property of Wilson's polynomial family is that almost all the so-called classical orthogonal polynomials are placed on its boundary. An inquiry into Andrews, Askey and Roy (1999) shows that, on the contrary, Brouncker's polynomials are placed at the very center, corresponding to the choice $a=0$, $b=1/2$, $c=d=1/4$. Thus Brouncker's formula in 1655 already listed important orthogonal polynomials, though not in a direct form. But neither was the Universe in its first few minutes similar to the present world. In addition to special functions, Brouncker's formula stimulated, or it is better to say could stimulate, developments in two other important directions.

The first is the moment problem considered by Stieltjes (1895). One can easily notice a remarkable similarity of Brouncker's arguments to those of Stieltjes. The second is the solution of Pell's equation obtained by Brouncker as his answer to the challenge of Fermat. It looks as if Fermat carefully studied Wallis' book. Still, I have never heard that he ever mentioned §191 in his letters. Instead Fermat proposed to outstanding British mathematicians a problem which they could solve by the method of Brouncker presented in this very paragraph. And indeed Brouncker solved Fermat's problem by applying a part of the argument he used to answer Wallis' question. After that Wallis developed his own method. This is considered in more detail in Chapter 2.

It is just unbelievable that such a partial, on first glance, result obtained in 1655 encapsulated a considerable part of the further development of algebra and analysis. True, this was a result on the quadrature problem obtained with continued fractions . . .

From the critical analysis of Brouncker's proof of (1), two interesting properties of continued fractions can be observed. If some continued fractions give a development of one part of mathematics then it is quite possible that similar progress can be made with other continued fractions in another part. In most cases the arguments could be simplified, as for instance Wallis did for Pell's equation, but at the cost of losing some substantial relationships, regarding which Euler was such a great master. I assume that the right explanation of this phenomenon lies in approximation theory. Any continued fraction is nothing other than an algorithm whose elementary steps are simple Möbius transforms. Therefore, adjusting these parameters in an appropriate way at each step, one can significantly change the original result. The art is to make this choice properly so that a new result can at least be stated.

In 1880 A. A. Markoff completed his Master's thesis at St Petersburg University, which was devoted to the theory of binary quadratic forms of positive determinant. I strongly believe that this was the best work of Markoff's whole mathematical career. It appears to have determined his later significant papers, in particular those in probability theory. It was not just one more application of continued fractions. Rather it was an incredibly beautiful demonstration of what can be done with their proper use. Therefore, although I was forced to sacrifice Stieltjes' theory of moments to a great extent in consequence, I have included this important theory in Chapter 2. The theory of moments is well presented in a number of books (Akhiezer 1961, Shohat and Tamarkin 1943 and Stieltjes 1895), whereas Markoff's original approach to this problem is not. In addition Markoff's theory has some relations to my own research (Khrushchev 2001a, b, 2002).

The key to both is Lagrange's formula (1.50). The Lagrange function $\mu(\xi)$ is defined for irrational ξ as the supremum of $c > 0$ such that

$$\left|\frac{p}{q} - \xi\right| < \frac{1}{cq^2}$$

has infinitely many solutions in the integers p, q, $q > 0$. So, the greater $\mu(\xi)$ is, the better can ξ be approximated by rational numbers. The range of μ is called the Lagrange spectrum. Markoff proved that, on the one hand, for $\mu(\xi) < 3$ the Lagrange spectrum is discrete and any ξ with $\mu(\xi) < 3$ is a quadratic irrational that is equivalent to the continued fraction

$$\xi(\theta, \delta) = \frac{1}{r_1} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_3} + \cdots + \frac{1}{r_n} + \frac{1}{r_n} + \cdots, \tag{2}$$

with $\delta = 0$ and a rational $0 < \theta \leqslant 1$. Here

$$r_n = [(n+1)\theta + \delta] - [n\theta + \delta]$$

is a Jean Bernoulli sequence,[1] which Jean Bernoulli introduced in his treatise on astronomy (1772). On the other hand, $\mu(\xi(\theta, \delta)) = 3$ for irrational $\theta \in (0, 1)$. Hence there are transcendental numbers with $\mu(\xi) = 3$. Moreover, they may be represented by regular continued fractions (2), which are simply expressed via Jean Bernoulli sequences. These are the worst transcendental numbers from the point of view of rational approximation, as follows from Markoff's main result, in contrast with Liouville's constant

$$\begin{aligned} L &= \sum_{n=0}^{\infty} 10^{-n!} \\ &= \frac{1}{9} + \frac{1}{11} + \frac{1}{99} + \frac{1}{1} + \frac{1}{10} + \frac{1}{9} + \frac{1}{999\,999\,999\,999} + \frac{1}{1} + \cdots. \end{aligned}$$

In Khrushchev (2001, 2002), Lagrange's formula, see Theorem 8.67, is applied not to regular continued fractions but to Wall continued fractions, which are nothing other than a form of the classical Schur algorithm. In the case of numbers one usually considers either their decimal representations or regular continued fraction expansions, and in this case there are three closely related objects. The first is the continuum $\mathfrak{P}(\mathbb{T})$ of all probability Borel measures on $\mathbb{T}$. The second and the third are the continuums of analytic functions F^σ with positive real part in the unit disc $\mathbb{D}$:

$$F^\sigma(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta) = \frac{1 + zf^\sigma}{1 - zf^\sigma} \tag{3}$$

and contractive analytic functions f^σ in $\mathbb{D}$. Any such f^σ expands into a Wall continued fraction

$$f(z) = a_0 + \frac{(1 - |a_0|^2)z}{\bar{a}_0 z} + \frac{1}{a_1} + \frac{(1 - |a_1|^2)z}{\bar{a}_1 z} + \cdots, \tag{4}$$

where, by Geronimus' theorem, which I mentioned right at the start, $\{a_n\}_{n\geqslant 0}$ are on the one hand the Verblunsky parameters of σ and on the other hand the Schur parameters of f^σ. The even convergents to (4) are contractive rational functions A_n/B_n, which by Schur's theorem (1917) converge to f^σ uniformly on compact subsets of $\mathbb{D}$. The substitution of f^σ in (3) with A_n/B_n results in rational functions Ψ_n^*/Φ_n^*, where $\{\Phi_n\}_{n\geqslant 0}$ are monic orthogonal polynomials in $L^2(d\sigma)$. By Lagrange's formula, asymptotic

[1] So-called by A. A. Markoff, who discovered important properties of these sequences.

properties of the normalized orthogonal polynomials $\{\varphi_n\}_{n\geqslant 0}$, $|\varphi|^2 d\sigma \in \mathfrak{P}(\mathbb{T})$, can be studied from the point of view of approximation on $\mathbb{T}$ either of f^σ by A_n/B_n or F^σ by Ψ_n^*/Φ_n^*. For instance, it turns out that Szegő measures, i.e. measures with finite entropy

$$\int_{\mathbb{T}} \log \sigma' dm > -\infty \,, \tag{5}$$

where m is the Lebesgue measure on $\mathbb{T}$, are exactly the measures such that $A_n/B_n \to f$ in L^1, where the distance between values of A_n/B_n and f is measured in the Poincaré metric of the non-euclidean geometry of $\mathbb{D}$; see Theorem 8.56.

A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is called a *Rakhmanov* measure if

$$*\text{-}\lim_n |\varphi_n|^2 d\sigma = dm$$

in the $*$-weak topology of $\mathfrak{P}(\mathbb{T})$. A measure is a Rakhmanov measure if and only if the Máté–Nevai condition

$$\lim_n a_n a_{n+k} = 0 \;\text{ for }\; k \geqslant 1 \,,$$

for the Verblunsky parameters $\{a_n\}_{n\geqslant 0}$ is satisfied (Theorem 8.73). Moreover $A_n/B_n \rightrightarrows f^\sigma$ in measure on $\mathbb{T}$ for any Rakhmanov measure σ. With this theorem to hand we can prove that $A_n/B_n \rightrightarrows f^\sigma$ in measure on $\mathbb{T}$ if and only if either σ is singular or $\lim_n a_n = 0$; see Theorem 8.78. The last two results have an important practical application. Let σ be any Szegő measure. By Geronimus, theorem (5) is equivalent to

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty \,.$$

Let us now modify the sequence $\{a_n\}_{n\geqslant 0}$ on an arbitrary sparse subset Λ of integers, which in addition starts far from $n = 0$. We replace a_n with $10^{-1000!}$ if $n \in \Lambda$. The sequence obtained, $\{a_n^*\}_{n\geqslant 0}$, is a Máté–Nevai sequence. Hence the measure σ^* with Verblunsky parameters $\{a_n^*\}_{n\geqslant 0}$ is a Rakhmanov measure, implying that $A_n/B_n \rightrightarrows f^{\sigma^*}$ on $\mathbb{T}$. Since $\lim_n a_n^* \neq 0$, we obtain that σ^* is singular. It is impossible to distinguish in practice the Szegő measure σ from the singular measure σ^* just by observing their first Verblunsky parameters.

We also construct in Chapter 8 examples of extremely transcendental σ, such that the sequence $\{|\varphi_n|^2 d\sigma\}_{n\geqslant 0}$ is dense in $\mathfrak{P}(\mathbb{T})$. Elements of the theory of periodic measures are also considered at this point.

Although the analogy described above is rather remote, in my opinion it is beautiful and justifies the inclusion of Markoff's results in Chapter 2. At many points I follow the Russian version of Markoff's thesis. However, there is an important difference. From Lagrange's formula Markoff quite naturally arrives at a combinatorial description of Jean Bernoulli sequences. In this book the properties of Jean Bernoulli sequences are first studied in detail in Chapter 1. Then these results are applied in Chapter 2, which makes the basic ideas of Markoff look more natural. With Jean Bernoulli sequences and the formulas for Jean Bernoulli and Markoff periods one can easily calculate numerically as many points of the Lagrange spectrum as necessary.

Markoff's theory completes the algebraic part of this book. The analytic part begins in Chapter 3 and is followed in Chapter 4 by Euler's research. To a great extent this chapter covers Euler (1750b) but with the difference that Brouncker's method recovered in Chapter 3 is applied. The method can be extended from the unit circle considered by Wallis to the class of sinusoidal spirals introduced into mathematics in 1718 by another great British mathematician, Colin Maclaurin. Chapter 4 covers the forgotten Euler differential method of summation of some continued fractions of hypergeometric type. It turns out that, for instance, approximately half the continued fractions discovered later by Stieltjes can be easily summed up by this method of Euler. However, the differential method does have some limitations. Attempting to overcome them Euler arrived at the beautiful theory of Riccati equations. In (1933) Sanielevici presented Euler's method in a very general form. Later Khovanskii (1958) using Sanielevici's results developed as continued fractions many elementary functions. Still, I think that Euler's method as stated by Euler makes everything more clear. In this part I filled some gaps in the proofs while trying not to violate Euler's arguments. The central result here is the continued fraction for the hyperbolic cotangent. I collect in Chapter 5 some results which were or could be directly or indirectly influenced by Euler's formulas. Such an approach sheds new light on the subject.

Chapter 6 presents results either obtained by Wallis interpolation or by a direct transfer from the regular continued fractions of number theory to polynomial continued fractions, i.e. to P-fractions. Euler's results on hypergeometric functions play a significant role here. Another interesting topic of this chapter is the periodicity of P-fractions. As before, the first results were obtained by Euler. Using continued fractions of the radicals of quadratic polynomials as guidance, Euler found his now well-known substitutions for integration. This was extended by Abel in one of his first papers, which incidentally preceded his discoveries in elliptic functions. I include in this chapter a beautiful result of Chebyshev on integration in finite terms.

Chapter 7 indicates how Euler's ideas eventually led to the discovery of orthogonal polynomials. Finally, I present in Chapter 8 my own research on the convergence of Schur's algorithm.

A few words on the title. It varied several times but since the essential part of the book is related to Euler I believe that finally I made a good choice. Moreover, in 2007 the tercentenary of Euler was celebrated.

Following Euler, I have split the book into small numbered "paragraphs" (subsections). This was an old tradition in mathematics, now almost forgotten. It makes the book easier to read. The difference from what Euler did is that almost all the "paragraphs" also have titles.

This book is not a complete account of what has been done in orthogonal polynomials or in continued fractions. In orthogonal polynomials Szegő's book (1939) is still important. There are also two important contributions made by Nevai (1979, 1986), and another two books by Saff and Totik (1997) and by Stahl and Totik (1992). As to orthogonal polynomials on the unit circle there is the recent and exhaustive work in two volumes of Barry Simon (2005). More on continued fractions can be found in Jones and Thron (1980), Khinchin (1935), Khovanskii (1958), Perron (1954, 1957) and Wall (1943).

Most parts of this book require only some knowledge of calculus and an undergraduate course on algebra. In Chapters 6–8 elementary facts from complex analysis are used occasionally. In Chapter 8 in addition it is important to know basic facts on Hardy spaces. There are two relatively new and very well written books on Hardy spaces: Garnett (1982) and Koosis (1998).

I wish to thank a number of people and organizations supporting me in one or another way during the work on this project. First of all I express deep gratitude to my aunt Galina Kreidtner. She was not a mathematician, she was an architect and artist. Nonetheless she enthusiastically and helpfully discussed with me the idea of this book in Almaty. She died at the age of 80 in 2000 and so I lost one of the best friends in my life. I dedicate this book to her memory.

Another very good friend, Paul Nevai from the Ohio State University, made a right choice in favor of orthogonal polynomials at the very beginning of his mathematical career in St Petersburg University, where we were fellow students. His support was also extremely valuable and sincere.

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I express my sincere and deep gratitude to Barry Simon (Caltech, Pasadena) who provided very important personal support for this book.

The book turned out to be so much influenced by Wallis' *Arithmetica Infinitorum* (1656) that I cannot avoid a temptation to finish this preface with a citation from its very end:

> *There remains this: we beseech the skilled in these things, that what we thought worth showing, they will think worth openly receiving, and whatever it hides, worth imparting more properly by themselves to the wider mathematical community.*

PRAISE BE TO GOD

Almaty–Ankara (1998–2007)

1

Continued fractions: real numbers

1.1 Historical background

1 Euclidean algorithm. Any pair $x_0 > x_1$ of positive integers generates a decreasing sequence $x_0 > x_1 > x_2 > \cdots$ in the set $\mathbb{N}$ of all positive integers:

$$\begin{aligned} x_0 &= b_0 x_1 + x_2, \\ x_1 &= b_1 x_2 + x_3, \\ x_2 &= b_2 x_3 + x_4, \\ &\vdots \\ x_{n-2} &= b_{n-2} x_{n-1} + x_n, \\ x_{n-1} &= b_{n-1} x_n, \end{aligned} \tag{1.1}$$

with $b_j \in \mathbb{N},\ j = 0, 1, \ldots$ Since any decreasing sequence in $\mathbb{N}$ is finite, there exists $n \in \mathbb{N}$ such that for $x_{n-1} = b_{n-1} x_n$ the algorithm stops at this line.

Reading the equations in (1.1) from the top to $x_{n-2} = b_{n-2} x_{n-1} + x_n$, which precedes the last equation $x_{n-1} = b_{n-1} x_n$, we obtain that any common divisor of x_0 and x_1 divides x_n. Reading the same equations from the bottom to the top, we obtain that x_n is a common divisor of x_0 and x_1. Hence x_n is the greatest common divisor $d = (x_0, x_1)$ for x_0 and x_1. This is the standard form of the Euclidean algorithm, which provides a foundation for multiplicative number theory.

To explain the role played by the coefficients b_k in (1.1) we will consider (1.1) as a system of linear algebraic equations with integer coefficients $b_0, b_1, b_2, \ldots$ Eliminating the unknowns x_k from (1.1) we obtain

$$\frac{x_{k-1}}{x_k} = b_{k-1} + \frac{1}{x_k / x_{k+1}}, \qquad k = 1, 2, \ldots,$$

which obviously yields the development of x_0/x_1 into a finite *regular continued fraction*

$$\frac{x_0}{x_1} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots + \cfrac{1}{b_n}}}$$

To save space, Rogers (1907) proposed that the following notation could be used, in which the continued fraction is written in line form:

$$\frac{x_0}{x_1} = b_0 + \frac{1}{b_1} {}_{+} \frac{1}{b_2} {}_{+ \cdots +} \frac{1}{b_{n-1}}. \tag{1.2}$$

This shows that any rational number equals the value of a regular continued fraction (1.2), where b_0 is an integer ($b_0 \in \mathbb{Z}$) and $b_1, b_2, \ldots, b_{n-1}$ are positive integers. The advantage of such a representation compared with popular decimal or dyadic representations is that it is universal and does not reflect particular properties of the base. *Thus the continuum* $\mathbb{R}$ *of real numbers can be parameterized by a sequence of integer parameters* $\{b_k\}_{k \geqslant 0}$ restricted to $b_0 \in \mathbb{Z}$ and $b_k \in \mathbb{N}$ if $k \geqslant 1$.

2 Hippasus of Metapontum. The algebraic construction of continued fractions discussed above originates in one important problem of geometry solved by the Pythagorean Hippasus of Metapontum in the fifth century BC. By the way, this problem is related to the notion of orthogonality; namely, given $AB \perp AD$, $x_1 = |AB| = |AD|$, prove that BD, $|BD| = x_0$ and AD have no common unit of measurement.

Hippasus' geometrical construction is remarkably similar to the construction of continued fractions (see Fig. 1.1). First, $x_0 > x_1 > x_2 = |ED|$, where E is defined so

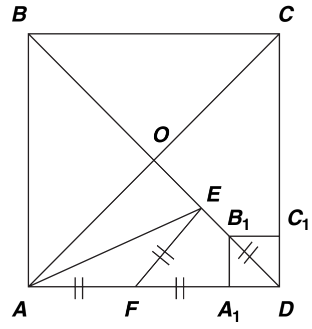

Fig. 1.1. Hippasus' construction for $x_1 = 2x_2 + x_3$.

that $|AB| = |BE|$. Computations with the angles in ΔABE, ΔAEF and ΔFED show that $|AF| = |FE| = |ED|$. Hence

$$x_0 = x_1 + x_2;$$

$$x_1 = 2x_2 + x_3, \quad |A_1D| = x_3 < x_2.$$

Observing that $\Delta ABD \sim \Delta EFD$, we have $x_2 = 2x_3 + x_4$. The construction can now be run by induction and it will never stop (notice that A_n never equals D). The result is that x_0/x_1 can be represented by an infinite continued fraction:

$$\frac{x_0}{x_1} = 1 + \frac{1}{2} {}_{+} \frac{1}{2} {}_{+} \frac{1}{2} {}_{+} \frac{1}{2} {}_{+ \cdots} . \tag{1.3}$$

Since rational numbers are values of finite regular continued fractions and the development into a regular continued fraction is unique, this, by the way, shows that $\sqrt{2} = |BD|/|AD|$ is an irrational number.

3 Bombelli's method. In *L'algebra* R. Bombelli (1572) considered a method of computation of square roots $\sqrt{N}$, where N is a positive integer which is not a perfect square. Let a be the greatest positive integer satisfying $a^2 < N$. Then $N = a^2 + r$ with $r > 0$ and

$$\sqrt{a^2 + r} = a + x \Leftrightarrow x = \frac{r}{2a + x},$$

implying that

$$\sqrt{N} = a + \frac{r}{2a} {}_{+} \frac{r}{2a} {}_{+} \frac{r}{2a} {}_{+ \cdots} .$$

In particular for $N = 13$ we obtain

$$\sqrt{13} = 3 + \frac{4}{6} {}_{+} \frac{4}{6} {}_{+} \frac{4}{6} {}_{+ \cdots} = 3 + \frac{2}{3} {}_{+} \frac{1}{3} {}_{+} \frac{1}{3} {}_{+} \frac{1}{3} {}_{+ \cdots} .$$

4 Ascending continued fractions. It is easy to see that any finite regular continued fraction represents a rational number. To find the rational number corresponding to the continued fraction

$$2 + \frac{1}{3} {}_{+} \frac{1}{2} {}_{+} \frac{1}{1} {}_{+} \frac{1}{4} {}_{+} \frac{1}{2} {}_{+} \frac{1}{3}$$

we rewrite it, starting from the right-hand side of the above expression, in the form of an *ascending continued fraction*

$$\cfrac{1}{\cfrac{1}{\cfrac{1}{3} + 2} + 4} + \cdots ,$$

which in six steps of elementary arithmetic operations results in 825/359.

5 Huygens' method. The theory of regular continued fractions originates in the practical problem of the approximation in the lowest terms of rational numbers with large numerators and denominators by rational numbers with much smaller ones. The first such problem was considered systematically by Huygens (1698). In this book Huygens studied a planetarium problem. For a planetarium to work accurately one should arrange the gear ratio to be

$$\frac{77\,708\,431}{2\,640\,858}.$$

Since it was impossible to arrange this ratio in practice, Huygens developed the ratio into the continued fraction

$$29+\frac{1}{2}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{5}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{4}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{6}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{10}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{3},$$

and studied the successive approximations

$$\begin{aligned}
\frac{P_{-1}}{Q_{-1}} &= \frac{1}{0} = +\infty,\\
\frac{P_0}{Q_0} &= \frac{29}{1} = 29,\\
\frac{P_1}{Q_1} &= 29+\frac{1}{2} = \frac{59}{2} = 29.5,\\
\frac{P_2}{Q_2} &= 29+\frac{1}{2}\,{}_{+}\,\frac{1}{2} = \frac{147}{5} = 29.4,\\
\frac{P_3}{Q_3} &= 29+\frac{1}{2}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{1} = \frac{206}{7} = 29.428\,571\,43\ldots,\\
\frac{P_4}{Q_4} &= 29+\frac{1}{2}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{5} = \frac{1177}{40} = 29.425,\\
\frac{P_5}{Q_5} &= 29+\frac{1}{2}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{5}\,{}_{+}\,\frac{1}{1} = \frac{1383}{47} = 29.425\,531\,91\ldots,\\
\frac{P_6}{Q_6} &= 29+\frac{1}{2}\,{}_{+}\,\frac{1}{2}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{5}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{4} = \frac{6709}{228} = 29.425\,438\,60\ldots
\end{aligned} \tag{1.4}$$

A simple analysis of (1.2) shows that the value of the continued fraction lies between its consecutive *convergents* P_k/Q_k. Therefore to estimate the approximation error one can simply find the following differences:

$$\frac{59}{2}-\frac{29}{1}=\frac{1}{2\times 1},\ \frac{59}{2}-\frac{147}{5}=\frac{1}{2\times 5},$$
$$\frac{206}{7}-\frac{147}{5}=\frac{1}{7\times 147},\ \frac{206}{7}-\frac{1177}{40}=\frac{1}{7\times 40},\tag{1.5}$$
$$\frac{1383}{47}-\frac{1177}{40}=\frac{1}{47\times 40},\ \frac{1383}{47}-\frac{6709}{228}=\frac{1}{47\times 228}=\frac{1}{10\,716}.$$

The fact that all these differences are *aliquot* fractions, i.e. fractions with unit numerators and integer denominators, cannot be accidental. Basically it is this fact which yields a good rational approximation, 6709/228, to Huygens' fraction.

6 Continued fractions and the Gregorian calendar. Following Euler (1748), we consider an application of continued fractions to the calendar problem.

Problem 1.1 *Precise astronomical observations show that one year lasts*

$$365^d 5^h 48^m 55^s.$$

Find a calendar that will not accumulate a noticeable error for a long interval of time.

The assumption that one year lasts 365 days leads to an error of 5 hours per year. The error accumulates fairly fast and in 100 years results in a noticeable shift of the seasons. If we assume that one year lasts 366 days the disagreement with the seasons will be observed much earlier.

To solve this problem we first express the duration of one year in days:

$$1 \text{ year} = 365+\frac{5}{24}+\frac{48}{60}\times\frac{1}{24}+\frac{55}{60}\times\frac{1}{60}\times\frac{1}{24}\text{ days} = 365+\frac{20\,935}{86\,400}\text{ days}.$$

This is itself, of course, an approximate duration but the error is so small that it will not be noticeable for more than 10 000 years.

To find a good approximation to 20 935/86 400 we develop this rational number into a regular continued fraction. It is clear that the numbers 20 935 and 86 400 are both divisible by 5, so that

$$\frac{20\ 935}{86\ 400}=\frac{4187}{17\ 280}.$$

One can easily prove that the last fraction is in the lowest terms. Indeed $20\,935 = 5\times 53\times 79$, whereas $86400 = 2^7\times 3^3\times 5^2$, which implies that 5 is the greatest common divisor.

We have

$$\frac{4187}{17\,280} = \frac{1}{4} + \frac{532}{4187} = \frac{1}{4} + \frac{1}{7} + \frac{463}{532} = \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{69}{463}$$

$$= \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{1}{6} + \frac{49}{69} = \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{20}{49}$$

$$= \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{2}{9}$$

$$= \frac{1}{4} + \frac{1}{7} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{2}.$$

The convergents to this continued fraction can be arranged into the following table:

$$\begin{array}{cccccccccc} 0 & 4 & 7 & 1 & 6 & 1 & 2 & 2 & \cdots \\ \frac{1}{0} & \frac{0}{1}^{l} & \frac{1}{4}^{g} & \frac{7}{29}^{l} & \frac{8}{33}^{g} & \frac{55}{227}^{l} & \frac{63}{260}^{g} & \frac{181}{747}^{l} & \cdots \end{array} \tag{1.6}$$

The first row of this table contains the partial denominators of the continued fractions. The second row consists of the corresponding convergents, shifted to the right by 1. This convenient notation is due to Euler and is explained below, in the paragraph before Theorem 1.5. The index l means that the convergent is less than the value of the continued fraction. The index g means that the convergent is greater than this value. It follows that every four years contribute a little bit less than 1 extra day. This gives rise to the Julian Calendar, which adds one extra day (29 February) every leap year (i.e. each year that is divisible by 4).

It should also be clear from (1.6) that every 33 years contribute a little bit less than eight days. Since

$$100 = 3 \times 33 + 1,$$

we obtain that every $400 = 4 \times 3 \times 33 + 4$ years contribute a little bit less than $4 \times 3 \times 8 + 1 = 97$ extra days. To arrange a convenient compensation, the Gregorian Calendar converts $3 = 100 - 97$ leap years within the range of every 400 years into ordinary years. Thus 1700, 1800, 1900 were ordinary years (to remove the three extra days contributed by the Julian Calendar). However, 1600 and 2000 were leap years. Since

$$\frac{97}{400} - \frac{4187}{17\,280} = 0.000\,197\,456\ldots,$$

the Gregorian Calendar contributes about two extra days every 10 000 years.

The Gregorian Calendar was introduced in 1582 by Pope Gregory. By that time the difference between the two calendars was already 10 days. The new calendar was introduced on 5 October 1582 and, to compensate the difference of 10 days, the day of 5 October 1582 was announced to be 15 October 1582 (Kiselev 1915, §97). It is interesting to notice that although the Gregorian calendar, as shown above, is closely related to continued fractions it was one of the contributors to the field at this time, Wallis, who advised the British authorities to reject it in Great Britain (Zeuthen 1903).

7 The well-tempered clavier. Here is an impressive application of continued fractions to music. The Weber–Fechner law states that the response of human beings to physical phenomena obeys a logarithmic law (see Maor 1994, pp. 111–12). This ability of human beings makes them less sensitive to the changes of the outside world by converting outside impulses with exponential growth into a linear response scale and so reduces our reaction to the most significant ones. In particular, our ear registers not the direct frequency ratio of two sounds but its logarithm. The main problem in music is to arrange a system of sounds which will create an impression of harmony under this logarithmic law of response. In practice this means that the frequencies in a musical scale should correspond to a linear set of logarithmic responses, i.e. these responses should divide up the logarithmic image of the scale into a number of equal parts. If a string of length l creates a sound of frequency $\omega = 512$ Hz then a string of length $l/2$ doubles the frequency to 2ω. The logarithmic base a is then chosen so as to normalize the following number to unity:

$$\log_a (2\omega : \omega) = \log_a 2 = 1,$$

which implies that $a = 2$. The ratio $2\omega : \omega = 2$ determines an interval $(\omega, 2\omega)$, called an *octave*. The ratio $3\omega/2 : \omega$ corresponding to half the interval $(\omega, 2\omega)$ (the frequency $3\omega/2$ is generated by a string of length $2l/3$) is called a *perfect fifth*; the ear hears this ratio as

$$\log_2 \left(\tfrac{3}{2}\omega : \omega\right) = \log_2 3 - 1.$$

Our ear hears a perfect fifth best, and therefore one should divide the logarithmic image of an octave into a number of equal parts in such a way that the above logarithmic image of a perfect fifth is well approximated. It can be shown that

$$\log_2 3 - 1 = 0.584\,962\,500\,721\ldots = \frac{1}{1}\,\frac{}{+}\,\frac{1}{1}\,\frac{}{+}\,\frac{1}{2}\,\frac{}{+}\,\frac{1}{2}\,\frac{}{+}\,\frac{1}{3}\,\frac{}{+}\,\frac{1}{1}\,\frac{}{+}\,\frac{1}{5}\,\frac{}{+}\,\frac{1}{2}\,\frac{}{+}\,\frac{1}{23}\,\frac{}{+}\cdots$$

The convergents to the continued fraction of $\log_2 3 - 1$ form the series

$$1, \quad \frac{1}{2}, \quad \frac{3}{5}, \quad \frac{7}{12}, \quad \frac{24}{41}, \quad \ldots \tag{1.7}$$

and represent successive ways of dividing up the image of the octave. The approximations 1 and 1/2 are too inexact. The approximation 3/5 is used in eastern music. The approximation 7/12 is the best. It divides the octave into 12 semitones and seven such semitones correspond to a fifth. To study what happens at other frequencies we observe that the distance between two notes measured as the ratio of their frequencies is called an interval. If the interval between two notes is a ratio of small integers these two notes are called consonant. Otherwise they are called dissonant.[1] There are seven

[1] An original theory of sound classification by the "degree of pleasure" was developed by Euler in his monograph (1739).

intervals that are commonly considered as consonant (they had appeared already in Descartes' table; see Brouncker 1653, p. 13):

$2/1$ (octave)	$5/4$ (major third)
$3/2$ (perfect fifth)	$6/5$ (minor third)
$4/3$ (perfect fourth)	$5/3$ (major sixth)
	$8/5$ (minor sixth)

The analysis of these numbers shows that they form the sequence

$$1 < \frac{6}{5} < \frac{5}{4} < \frac{4}{3} < \frac{3}{2} < \frac{8}{5} < \frac{5}{3} < 2 \tag{1.8}$$

satisfying the relations

$$\frac{5}{3} \times \frac{6}{5} = \frac{5}{4} \times \frac{8}{5} = \frac{3}{2} \times \frac{4}{3} = 2, \qquad \frac{5}{4} \times \frac{6}{5} = \frac{3}{2}.$$

This implies that the binary logarithms of these intervals are linear combinations of 1, $\log_2 3/2$ and $\log_2 5/4$ with coefficients in $\{0, 1, -1\}$. Hence the error in the approximation by a uniform scale is completely determined by the errors for $\log_2 3/2$ and $\log_2 5/4$ and cannot exceed the maximum of the two. Now

$$\log_2\left(\frac{5}{4}\right) = 0.321\,928\,094\,887\,362\,347\,87\ldots = \frac{1}{3} \,{}_{+}\, \frac{1}{9} \,{}_{+}\, \frac{1}{2} \,{}_{+}\, \frac{1}{2} \,{}_{+}\, \frac{1}{4} \,{}_{+}\, \frac{1}{6} \,{}_{+}\cdots$$

shows that $1/3 = 4/12$ is a convergent to $\log_2 5/4$. This guarantees that the equal-temperament system of 12 uniform semitones gives a good rational approximation to the two basic intervals 3/2 and 5/4, and hence to all seven consonant intervals.

See Dunne and McConnell (1999) for a more detailed discussion. Excellent comments on this topic can also be found in the appendix by V. G. Boltyanskii to the Russian translation of Klein (1932).

8 Quadrature of the unit circle. The unit circle $\mathbb{T}$ is the boundary of the unit disc $\mathbb{D}$ centered at zero. The area of $\mathbb{D}$ is denoted by π. Having been introduced by W. Jones in (1706), the notation π became standard only after Euler published his monograph (1748). According to Cajori (1916, p. 32), in introducing π Jones was probably motivated by Oughtred's notation π/δ for the ratio of the circumference and the diameter of a circle; Oughtred was Wallis' teacher.

Theorem 1.2 (Archimedes) *The length of* $\mathbb{T}$ *is* 2π.

Proof Let us inscribe a regular n-polygon P_n in $\mathbb{T}$. Then its sides of equal length l_n make small triangles with the origin of area $l_n(1-o(1))/2$. Hence the area of P_n is $n \times l_n(1-o(1))/2$. It approaches π as $n \to +\infty$, whereas nl_n approaches the length of $\mathbb{T}$, which proves the theorem. $\square$

Problem 1.3 (The quadrature problem) *Find a good rational approximation to the length of* $\mathbb{T}$.

The quadrature (squaring) of the circle was one of the most difficult ancient mathematical problems. Attempts to solve it resulted in significant progress in mathematical analysis and especially in the theory of continued fractions. The meaning of the problem has been changing in mathematics with time. In Archimedes' time the practical side of the problem was to construct with ruler and compass the side of a square having area π. The theoretical side of the problem was to either prove that $\pi \in \mathbb{Q}$ or at least find a good rational approximation to π. Now with Wolfram's Mathematica program[2] everybody can find thousands of digits of π:

$$\pi = 3.141\,592\,653\,589\,793\,238\,462\,643\,383\,279\ldots,$$

but originally the calculation of the correct decimal places of π was a difficult problem. The first important contribution to the quadrature problem was made by Archimedes, who developed the method of inscribed and superscribed regular n-polygons with $n = 6,\ 12,\ 24,\ 48,\ 96, \ldots, 3 \times 2^k$. For instance, consideration of a regular hexagon inscribed in a circle shows that $3 < \pi$.

Archimedes method looks especially beautiful in the form of Gregory (1667); see O'Connor and Robertson (2004). Let a_k be the semiperimeter of a regular n-polygon ($n = 3 \times 2^k$) inscribed in $\mathbb{T}$. In Fig. 1.2 its side is AC ($k = 1$), $AP = AC/2$ and AB is half the side of a superscribed regular n-polygon with semiperimeter b_k. It follows that $\angle AOB = \pi/n$. Since ΔOAB is similar to ΔOAP, we obtain that $|AB| = \tan \pi/n$ and $|AP| = \sin \pi/n$. Hence

$$a_k = n \sin \frac{\pi}{n} < \pi < b_k = n \tan \frac{\pi}{n}.$$

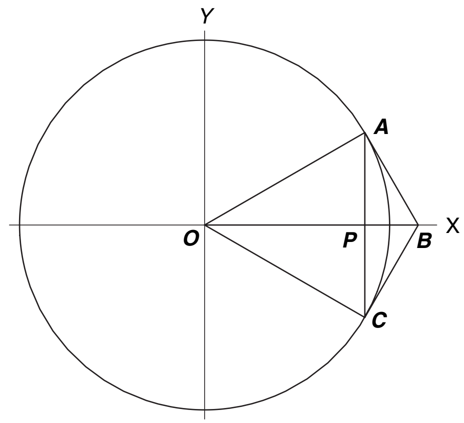

Fig. 1.2. Archimedes' construction to find an approximation to π.

[2] www.wolfram.research.com.

Clearly $a_1 = 3$, $b_1 = 2\sqrt{3} = 3.464101615\ldots$ Obvious trigonometry,

$$\frac{1}{\tan\theta} + \frac{1}{\sin\theta} = \frac{1}{\tan(\theta/2)}, \qquad 2\tan\frac{\theta}{2}\sin\theta = \left(2\sin\frac{\theta}{2}\right)^2,$$

results in the recurrence relations

$$b_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \qquad a_{n+1} = \sqrt{a_n b_{n+1}}.$$

Archimedes obtained his inequalities $3.1410 < \pi < 3.1427$ by calculating a_5 and b_5. His method was considerably improved by Huygens (see Rudio 1892).

The first algebraic algorithm for the calculation of an arbitrary number of places of π was proposed by Brouncker in 1656;[3] see §**63** in Section 3.2. Although Brouncker's simple calculations remained unnoticed,[4] the more complicated calculations of Huygens resulted in significant progress in the rational approximation of π. In particular, it was shown that

$$\pi = 3 + \frac{1}{7} \,\frac{}{+}\, \frac{1}{15} \,\frac{}{+}\, \frac{1}{1} \,\frac{}{+}\, \frac{1}{292} \,\frac{}{+}\, \frac{1}{1} \,\frac{}{+}\, \frac{1}{1} \,\frac{}{+}\, \frac{1}{1} \,\frac{}{+}\, \frac{1}{2} \,\frac{}{+}\, \frac{1}{1} \,\frac{}{+}\, \frac{1}{3} \,\frac{}{+}\, \frac{1}{1} \,\frac{}{+}\, \frac{1}{14} \,\frac{}{+}\, \frac{1}{2} \,\frac{}{+}\, \frac{1}{1} \,\frac{}{+}\, \cdots.$$

In a similar way to (1.6), the convergents to the continued fraction of π can be arranged in the following table:

$$\begin{array}{ccccccc} 3 & 7 & 15 & 1 & 292 & 1 & \ldots \\ \frac{1}{0}^{g} & \frac{3}{1}^{l} & \frac{22}{7}^{g} & \frac{333}{106}^{l} & \frac{355}{113}^{g} & \frac{103\,993}{33\,102}^{l} & \ldots \end{array} \tag{1.9}$$

The approximation 355/113 is called Metius' approximation. The error in Metius' approximation is less than

$$\begin{aligned} 0 < \frac{355}{113} - \pi < \frac{355}{113} - \frac{103\,993}{33\,102} &= \frac{11\,751\,210 - 11\,751\,209}{113 \times 33\,102} \\ &= \frac{1}{113 \times 33\,102} = \frac{1}{3\,740\,526} \\ &= 0.000\,000\,267\ldots \end{aligned}$$

One can easily check with Wolfram's Mathematica program that there is another dramatic jump in the series of moderately small values of partial denominators for π. It happens at $n = 431$: $b_{431} = 20\,776$, whereas $b_{430} = 4$ and $b_{432} = 1$.

By Corollary 1.16 the continued fraction for π is finite if and only if $\pi \in \mathbb{Q}$. If π were a rational number then the quadrature of the circle would have a positive solution. Indeed, using ruler and compass one can easily construct any rational number a on the number axis. Then $\sqrt{a}$ is the length of the diagonal of the square with side a. Using a continued fraction for the cotangent of an angle, discovered by Euler, Lambert in (1761) proved that $\pi \notin \mathbb{Q}$. Later Legendre gave a simpler proof. We discuss this in more detail later in §**113** at the start of Section 5.3. The quadrature problem in the most general form was solved in 1882 by Lindemann, who showed that π does not satisfy any algebraic equation with integer coefficients; for

[3] According to a letter from Wallis to Digby sent on 6 June 1657.
[4] See §**63**, noting that Huygens was informed about Brouncker's calculations at his request.

a proof, see for instance LeVeque (1996, §9.7). On the contrary, the length of any segment that can be constructed with ruler and compass must satisfy just such an equation.

9 Euler's example (1744). In the following example, Euler computed the convergents of Hippasus' continued fraction (1.3) each decreased by unity. For further purposes we list the convergents to $\sqrt{2}$ itself:

$$\begin{array}{cccccccccc} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \dots \\ \frac{1}{0} & \frac{1}{1} & \frac{3}{2} & \frac{7}{5} & \frac{17}{12} & \frac{41}{29} & \frac{99}{70} & \frac{239}{169} & \dots \end{array} \tag{1.10}$$

It follows from (1.3) that the even convergents are smaller than the odd convergents. Therefore $239/169 < \sqrt{2} < 99/70$ and the error in representing $\sqrt{2}$ by the fifth convergent 99/70 cannot exceed

$$\frac{99}{70} - \frac{239}{169} = \frac{169 \times 99 - 70 \times 239}{70 \times 169} = \frac{16\,731 - 16\,730}{11\,830} < 10^{-4}.$$

Computations now show that

$$\sqrt{2} = 1.414\,213\,562\,37\dots,$$

$$1 + \frac{29}{70} = 1.414\,285\,714\,28\dots$$

The identity $169 \times 99 - 70 \times 239 = 1$ is explained by Euler's theory, discussed below.

1.2 Euler's theory of continued fractions

10 The Euler–Wallis formulas. Replacing the 1's multiplying the x_{j+1} on the right-hand side of (1.1) by *nonzero coefficients* a_j and letting the number of equations be infinite, we obtain

$$\begin{aligned} x_0 &= b_0 x_1 + a_1 x_2, \\ x_1 &= b_1 x_2 + a_2 x_3, \\ x_2 &= b_2 x_3 + a_3 x_4, \\ &\vdots \end{aligned} \tag{1.11}$$

Eliminating the unknowns x_k, we get a general continued fraction:

$$b_0 + \frac{a_1}{b_1} {\atop +} \frac{a_2}{b_2} {\atop +\cdots} = b_0 + \mathop{\mathbf{K}}_{k=1}^{\infty} \left(\frac{a_k}{b_k} \right). \tag{1.12}$$

The numbers a_k are called the *kth partial numerators*, and the b_k are called the *kth partial denominators,* of (1.12).

We will consider (1.12) just as an algorithm for obtaining rational approximants. More precisely, for every positive integer n we can stop the process in (1.12) at the term a_n/b_n and perform all algebraic operations without cancellations. Then

$$\frac{P_n}{Q_n} \equiv b_0 + \mathop{\mathbf{K}}_{k=1}^{n}\left(\frac{a_k}{b_k}\right) \tag{1.13}$$

is called the *nth convergent* to the continued fraction (1.12). By (1.16) below, P_n and Q_n cannot both vanish, so one can always assign a value, finite or infinite, to (1.13). This is the reason for the requirement $a_k \neq 0$.

Theorem 1.4 (Brounker: Euler 1748 and Wallis 1656) *Let*

$$\xi = b_0 + \mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right) = b_0 + \frac{a_1}{b_1}{}_{+\cdots+}\frac{a_n}{b_n}{}_{+}\frac{a_{n+1}}{\xi_{n+1}} \tag{1.14}$$

be a formal continued fraction with convergents $\{P_n/Q_n\}_{n\geqslant 1}$. *Let*

$$P_{-1} = 1, \qquad P_0 = b_0,$$
$$Q_{-1} = 0, \qquad Q_0 = 1.$$

Then $\{P_n\}_{n\geqslant 0}$ *and* $\{Q_n\}_{n\geqslant 0}$ *satisfy the Euler–Wallis formulas*

$$\begin{aligned} P_n &= b_n P_{n-1} + a_n P_{n-2}, \\ Q_n &= b_n Q_{n-1} + a_n Q_{n-2}, \end{aligned} \tag{1.15}$$

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} a_1 \cdots a_n, \tag{1.16}$$

$$\xi = \frac{\xi_{n+1} P_n + a_{n+1} P_{n-1}}{\xi_{n+1} Q_n + a_{n+1} Q_{n-1}}. \tag{1.17}$$

Proof We have $P_0/Q_0 = b_0$, $b_0 + a_1/b_1 = (b_0 b_1 + a_1)/b_1 = P_1/Q_1$, which by definition implies that

$$P_1 = b_0 b_1 + a_1 = b_1 P_0 + a_1 P_{-1}, \qquad Q_1 = b_1 = b_1 Q_0 + a_1 Q_{-1}.$$

In other words (1.15) holds for $n = 1$. The proof is now completed by induction. Assuming that (1.15) holds for a given n for any formal continued fraction, we will prove that it holds for $n+1$. If $b_{n+1} = 0$, then

$$\cdots{}_{+}\frac{a_{n-1}}{b_{n-1}}{}_{+}\frac{a_n}{b_n}{}_{+}\frac{a_{n+1}}{b_{n+1}}{}_{+} = \cdots{}_{+}\frac{a_{n-1}}{b_{n-1}},$$

since $a_{n+1}/b_{n+1} = \infty$. Hence $P_{n+1} = a_{n+1}P_{n-1}$ and $Q_{n+1} = a_{n+1}Q_{n-1}$, in complete correspondence with (1.15). If $b_{n+1} \neq 0$ then, observing that

$$\frac{a_n}{b_n + a_{n+1}/b_{n+1}} = \frac{a_n b_{n+1}}{b_n b_{n+1} + a_{n+1}},$$

we put $a'_n = a_n b_{n+1}$, $b'_n = b_n b_{n+1} + a_{n+1}$ and consider an auxiliary finite continued fraction with n terms:

$$b_0 + \frac{a_1}{b_1} {}_{+} \frac{a_2}{b_2} {}_{+} \cdots {}_{+} \frac{a_{n-1}}{b_{n-1}} {}_{+} \frac{a'_n}{b'_n}.$$

By the definition of P_{n+1} and induction on the auxiliary continued fraction, we obtain

$$\begin{aligned} P_{n+1} = P'_n &= b'_n P_{n-1} + a'_n P_{n-2} \\ &= b_n b_{n+1} P_{n-1} + a_{n+1} P_{n-1} + b_{n+1} a_n P_{n-2} \\ &= b_{n+1}(b_n P_{n-1} + a_n P_{n-2}) + a_{n+1} P_{n-1} \\ &= b_{n+1} P_n + a_{n+1} P_{n-1}. \end{aligned}$$

This implies (1.15) for P_{n+1}. Similarly, (1.15) holds for Q_{n+1}.

To prove (1.16) we observe that $P_1 Q_0 - P_0 Q_1 = a_1$. Assuming that (1.16) holds for n, we apply (1.15) to show that

$$\begin{aligned} P_{n+1} Q_n - P_n Q_{n+1} &= (b_{n+1} P_n + a_{n+1} P_{n-1}) Q_n \\ &\quad - P_n (b_{n+1} Q_n + a_{n+1} Q_{n-1}) \\ &= - a_{n+1} (P_n Q_{n-1} - P_{n-1} Q_n), \end{aligned}$$

which completes the proof of (1.16). To prove (1.17) we put $a'_{n+1} = a_{n+1}$, $b'_{n+1} = \xi_{n+1}$ and notice that by (1.15)

$$P'_{n+1} = (\xi_{n+1}) P_n + a_{n+1} P_{n-1},$$
$$Q'_{n+1} = (\xi_{n+1}) Q_n + a_{n+1} Q_{n-1},$$

which proves (1.17) since $\xi = P'_{n+1}/Q'_{n+1}$. □

The story of these formulas goes back to Brouncker, when in March 1655 on a request of Wallis he found a beautiful continued fraction for the quadrature problem; see §**59** in Section 3.2. The important formula (1.17) is due to Euler. It was Euler who systematically applied these formulas and moreover stated them explicitly (1748). The choice of the first convergents 1/0 and $b_0/1$ is also due to Euler and was motivated by Brouncker's theorem 1.7; see (1.21) below. Theorem 1.4 is very useful in converting the convergents of continued fractions into fractions in their lowest terms. Compared with the straightforward method of ascendant continued fractions presented in §**4**, formulas (1.15) simplify the calculations considerably. As we have seen, Euler found (see Euler 1744) a convenient way to arrange the calculations of convergents in tables; see (1.6), (1.9) and (1.10). Euler's idea is well demonstrated by (1.9):

$$\frac{333 + 292 \times 355}{106 + 292 \times 113} = \frac{103\,993}{33\,102}.$$

Theorem 1.5 *The numerators P_n and denominators Q_n of (1.13) satisfy*

$$\begin{aligned}\frac{P_n}{P_{n-1}} &= b_n + \frac{a_n}{b_{n-1}}\,{}_{+}\,\frac{a_{n-1}}{b_{n-2}}\,{}_{+\cdots+}\,\frac{a_1}{b_0},\\ \frac{Q_n}{Q_{n-1}} &= b_n + \frac{a_n}{b_{n-1}}\,{}_{+}\,\frac{a_{n-1}}{b_{n-2}}\,{}_{+\cdots+}\,\frac{a_2}{b_1}.\end{aligned} \tag{1.18}$$

Proof Apply (1.15) to the left-hand sides of (1.18) iteratively. □

Corollary 1.6 *A sequence of positive integers $\{b_n\}_{n\geqslant 0}$ satisfies, for $k \geqslant 2$,*

$$\{b_1,\, b_2,\, \ldots\, b_k\} = \{b_k,\, b_{k-1},\, \ldots\, b_1\} \tag{1.19}$$

if and only if $P_k = Q_{k-1}$, where $\{P_n/Q_n\}_{n\geqslant 0}$ are convergents to $\underset{n\geqslant 1}{\mathbf{K}}(1/b_n)$.

Proof By (1.18) and by the uniqueness of representation by regular continued fractions,

$$\frac{Q_{k-1}}{Q_k} = \frac{1}{b_k}\,{}_{+}\,\frac{1}{b_{k-1}}\,{}_{+\cdots}\,\frac{1}{b_1}$$

is the convergent P_k/Q_k if and only if (1.19) holds. □

The interlacing property of convergents for continued fractions with positive terms was first discovered by Brouncker; see §**60** in Section 3.2.

Theorem 1.7 (Brouncker 1655) *Let $b_0 + \mathbf{K}_{k=1}^{\infty}(a_k/b_k)$ be a formal continued fraction with positive terms. Then*

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}a_1\cdots a_n}{Q_nQ_{n-1}}, \qquad n = 1, 2, \ldots, \tag{1.20}$$

$$\frac{P_0}{Q_0} < \cdots < \frac{P_{2k}}{Q_{2k}} < \cdots < \frac{P_{2k+1}}{Q_{2k+1}} < \cdots < \frac{P_1}{Q_1} < \frac{P_{-1}}{Q_{-1}} = +\infty. \tag{1.21}$$

Proof To obtain (1.20) we divide both sides of (1.16) by Q_nQ_{n-1}. Adding formulas (1.20) for consecutive values of n, we obtain

$$\begin{aligned}\frac{P_n}{Q_n} - \frac{P_{n-2}}{Q_{n-2}} &= \frac{(-1)^{n-1}a_1\cdots a_n}{Q_nQ_{n-1}} + \frac{(-1)^{n-2}a_1\cdots a_{n-1}}{Q_{n-1}Q_{n-2}}\\ &= \frac{(-1)^n a_1\cdots a_{n-1}}{Q_{n-1}}\left(\frac{1}{Q_{n-2}} - \frac{a_n}{Q_n}\right)\\ &= \frac{(-1)^n a_1\cdots a_{n-1}}{Q_{n-1}}\,\frac{Q_n - a_nQ_{n-2}}{Q_{n-2}Q_n} = \frac{(-1)^n a_1\cdots a_{n-1}b_n}{Q_nQ_{n-2}},\end{aligned} \tag{1.22}$$

since $Q_n - a_nQ_{n-2} = b_nQ_{n-1}$ by (1.15). It follows that the even convergents increase whereas the odd convergents decrease. Putting $n = 2k+1$ in (1.20) results in

$$\frac{P_{2k+1}}{Q_{2k+1}} - \frac{P_{2k}}{Q_{2k}} = \frac{1}{Q_{2k+1}Q_{2k}},$$

which implies that the $(2k+1)$th convergent is always greater than the $2k$th convergent and hence than any even convergent. $\square$

11 The Euler–Mindingen formulas (Perron 1954). According to the Euler–Wallis formulas (1.15)–(1.17), the numerators P_n and denominators Q_n of the convergents P_n/Q_n to a general continued fraction (1.12) are polynomials in $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$. We are going to obtain formulas for these polynomials. Let

$$A_n = b_0 b_1 \cdots b_n \left(1 + \frac{a_1}{b_0 b_1}\right)\left(1 + \frac{a_2}{b_1 b_2}\right) \cdots \left(1 + \frac{a_n}{b_{n-1} b_n}\right). \tag{1.23}$$

Multiplying out the factors shows that A_n contains terms of "integer" type,

$$b_0 b_1 \cdots b_n \frac{a_1}{b_0 b_1} = b_2 \cdots b_n a_1, \qquad b_0 b_1 \cdots b_n \frac{a_1}{b_0 b_1}\frac{a_3}{b_2 b_3} = b_4 \cdots b_n a_1 a_3,$$

and also terms of "fractional" type,

$$b_0 b_1 \cdots b_n \frac{a_1}{b_0 b_1}\frac{a_2}{b_1 b_2} = \frac{b_3 \cdots b_n}{b_1} a_1 a_2.$$

Let us split A_n into a sum $[\![A_n]\!]$ of the "integer" terms and a sum $\{\{A_n\}\}$ of the "fractional" terms:

$$A_n = [\![A_n]\!] + \{\{A_n\}\}.$$

Theorem 1.8 (Euler) *For every integer n,*

$$P_n = [\![A_n]\!]. \tag{1.24}$$

Proof By definition $A_0 = b_0 = P_0$ and $A_1 = b_0 b_1 + a_1 = P_1$. Hence (1.24) holds for $n = 0$ and $n = 1$ and the proof will be complete if we can establish the Euler–Wallis formula for $[\![A_n]\!]$:

$$[\![A_n]\!] = b_n [\![A_{n-1}]\!] + a_n [\![A_{n-2}]\!].$$

By the definition of A_n we have

$$A_n = b_n \left(1 + \frac{a_n}{b_{n-1} b_n}\right) A_{n-1} = b_n A_{n-1} + \frac{a_n}{b_{n-1}} A_{n-1}. \tag{1.25}$$

Let us write this formula again but with n replaced by $n-1$:

$$A_{n-1} = b_{n-1} A_{n-2} + \frac{a_{n-1}}{b_{n-2}} A_{n-2}.$$

Substituting this formula for A_{n-1} into the last term of (1.25), we get

$$A_n = b_n A_{n-1} + a_n A_{n-2} + \frac{a_n a_{n-1}}{b_{n-1} b_{n-2}} A_{n-2}. \tag{1.26}$$

Now A_{n-1} depends neither on b_n nor on a_n, which implies that $[\![b_n A_{n-1}]\!] = b_n A_{n-1}$. We also have $[\![a_n A_{n-2}]\!] = a_n A_{n-2}$, since a_n does not enter the denominators of A_{n-2}.

Finally, since A_{n-2} does not depend on b_{n-1}, the last term in (1.26) is obviously fractional. □

Formulas (1.23) and (1.24) imply an explicit formula for P_n:

$$P_n = b_0 b_1 \cdots b_n \left(1 + \sum_{i}^{0,n-1} \frac{a_{i+1}}{b_i b_{i+1}} + \sum_{i<k}^{0,n-2} \frac{a_{i+1}}{b_i b_{i+1}} \frac{a_{k+2}}{b_{k+1} b_{k+2}} + \sum_{i<k<l}^{0,n-3} \frac{a_{i+1}}{b_i b_{i+1}} \frac{a_{k+2}}{b_{k+1} b_{k+2}} \frac{a_{l+3}}{b_{l+2} b_{l+3}} + \cdots \right) \tag{1.27}$$

Similarly, we define

$$B_n = b_1 b_2 \cdots b_n \left(1 + \frac{a_2}{b_1 b_2}\right)\left(1 + \frac{a_3}{b_2 b_3}\right) \cdots \left(1 + \frac{a_n}{b_{n-1} b_n}\right),$$

and obtain that

$$Q_n = [\![B_n]\!] = b_1 b_2 \cdots b_n \left(1 + \sum_{i}^{1,n-1} \frac{a_{i+1}}{b_i b_{i+1}} + \sum_{i<k}^{1,n-2} \frac{a_{i+1}}{b_i b_{i+1}} \frac{a_{k+2}}{b_{k+1} b_{k+2}} + \sum_{i<k<l}^{1,n-3} \frac{a_{i+1}}{b_i b_{i+1}} \frac{a_{k+2}}{b_{k+1} b_{k+2}} \frac{a_{l+3}}{b_{l+2} b_{l+3}} + \ldots \right) \tag{1.28}$$

12 Continuants. The Euler–Wallis formulas (1.15)–(1.17) can be considered as a system of linear equations in the unknowns P_k, $k = 0, \ldots, n$:

$$\begin{array}{rrrrrcr}
-P_0 & & & & & = & -b_0, \\
b_1 P_0 & -P_1 & & & & = & -a_1, \\
a_2 P_0 & +b_2 P_1 & -P_2 & & & = & 0, \\
 & a_3 P_1 & +b_3 P_2 & -P_3 & & = & 0, \\
 & & & & & \vdots & \\
 & & a_n P_{n-2} & +b_n P_{n-1} & -P_n & = & 0.
\end{array}$$

Applying Cramer's rule and moving the last column to the first place, we obtain the formula

$$P_n = \begin{vmatrix}
b_0 & -1 & & & & & \\
a_1 & b_1 & -1 & & & & \\
 & a_2 & b_2 & -1 & & & \\
 & & & \vdots & & & \\
 & & & & a_{n-1} & b_{n-1} & -1 \\
 & & & & & a_n & b_n
\end{vmatrix}$$

This determinant is called a *continuant*. Since the formulas for Q_k in this form are shifted by one, we obtain a continuant formula for Q_n:

$$Q_n = \begin{vmatrix} b_1 & -1 & & & & & \\ a_2 & b_2 & -1 & & & & \\ & a_3 & b_3 & -1 & & & \\ & & & \vdots & & & \\ & & & & a_{n-1} & b_{n-1} & -1 \\ & & & & & a_n & b_n \end{vmatrix}$$

Applying the cofactor expansion of P_n with respect to the first row and differentiating in b_0, we obtain the interesting formula

$$Q_n = \frac{\partial P_n}{\partial b_0}. \tag{1.29}$$

Applying the cofactor expansion to the last rows in the above formulas for P_n and Q_n, we obtain (1.15).

1.3 Rational approximations

13 Algorithm of regular fractions. Every irrational ξ can be easily developed into a regular continued fraction with the help of the following algorithm. For $\xi \in \mathbb{R}$ let $[\xi]$ be the greatest integer satisfying $n \leqslant \xi$ and $0 \leqslant \xi - [\xi] = \{\!\{\xi\}\!\} < 1$ be the fractional part of ξ. The function $x \to [x]$ is called the *integer part* or the *floor* function. The latter term originates in the form of the graph of $\xi \to [\xi]$ resembling a flight of stairs. The following simple formulas are useful in operations involving $[\xi]$:

$$\xi = [\xi] + \{\!\{\xi\}\!\};$$

for every integer n,

$$[n+\xi] = n + [\xi], \qquad \{\!\{n+\xi\}\!\} = \{\!\{\xi\}\!\},$$

$$[-\xi] = \begin{cases} -[\xi] & \text{if } \xi \in \mathbb{Z}, \\ -[\xi]-1 & \text{if } \xi \notin \mathbb{Z}; \end{cases} \tag{1.30}$$

for every real x and y,

$$[x]+[y] \leqslant [x+y] \leqslant [x]+[y]+1. \tag{1.31}$$

Euclid's algorithm $n = mq + r$, where n, m, q, r are integers and $0 \leqslant r < q$, can be written as

$$n = q\left[\frac{n}{q}\right] + r, \qquad r = q\left\{\!\!\left\{\frac{n}{q}\right\}\!\!\right\}.$$

We define the sequence $\{\xi_n\}_{n\geqslant 0}$ by

$$\xi_0 = \xi\,, \qquad \xi_{n+1} = \frac{1}{\{\!\{\xi_n\}\!\}}\,, \qquad n = 0, 1, \ldots$$

By induction ξ_n is irrational, which implies that $\xi_n > 1$ for every n. Now the algorithm is as follows:

$$\begin{aligned}\xi = [\xi_0] + \frac{1}{1/\{\!\{\xi_0\}\!\}} = [\xi_0] + \frac{1}{\xi_1} = [\xi_0] + \frac{1}{[\xi_1]} {}_{+} \frac{1}{1/\{\!\{\xi_1\}\!\}} \\ = [\xi_0] + \frac{1}{[\xi_1]} {}_{+} \frac{1}{[\xi_2]} {}_{+} \frac{1}{[\xi_3]} {}_{+\cdots+} \frac{1}{[\xi_n]} {}_{+\cdots}\,.\end{aligned} \tag{1.32}$$

Two other useful functions are closely related to the integer and fractional parts of ξ. The first is the *shift* of the integer part, $[\xi + 1/2]$; it determines the closest integer to ξ. The second is

$$\|\xi\| = \min\{|\xi - n| : n \in \mathbb{Z}\} = |\xi - [\xi + 1/2]| = |\{\{x + 1/2\}\} - 1/2|, \tag{1.33}$$

the distance from ξ to $\mathbb{Z}$.

14 Golden ratio. The ancient problem of the golden ratio is to find a rectangle (called golden) with sides a and $a\phi$ such that cutting a square of size $a \times a$ results in a rectangle similar to the initial rectangle. The number ϕ is called the golden ratio and can be found from the proportion

$$\frac{\phi - 1}{1} = \frac{1}{\phi} \Leftrightarrow \phi^2 - \phi - 1 = 0.$$

The quadratic equation for ϕ yields its regular continued fraction:

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1} {}_{+} \frac{1}{\phi} = \cdots = \frac{1+\sqrt{5}}{2} = 1.618\ldots.$$

By Theorem 1.14 below the convergents to ϕ have the very special structure $P_n = Q_{n+1}$, $Q_{n+1} = Q_n + Q_{n-1}$. Hence $Q_n = u_n$, $n = 0, 1, \ldots$, is the sequence of Fibonacci:

$$\begin{aligned}u_0 = 1, \quad u_1 = 1, \quad u_2 = 2, \quad u_3 = 3, \quad u_4 = 5, \\ u_5 = 8, \quad u_6 = 13, \quad u_7 = 21, \quad \ldots,\end{aligned}$$

and $P_n = Q_{n+1}$ is the shifted Fibonacci sequence. It follows that

$$\lim_n \frac{u_{n+1}}{u_n} = \phi.$$

Historically the golden ratio appeared in the proportions of the regular pentagon $ABCDE$ (see Fig. 1.3) inscribed into the unit circle (Timerding 1918). The diagonals AC, AD, BE, BD, EC make a regular five-point star. Since the arcs $\smile ED$, $\smile DC$, $\smile CB$ are equal, the corresponding inscribed angles $\angle EAD$, $\angle DAC$, $\angle CAB$ with vertices at A are equal to $360°/10 = 36°$. Since this holds for every vertex of $ABCDE$, the triangles $\triangle AFB$ and $\triangle BGC$ are equal and therefore $AF = FB = BG = GC = y$. Since the sum of the angles of $\triangle BGC$ is $180°$, we obtain $\angle BGC = 180° - 2 \times 36°$. It follows that $\angle BGA = 72°$. Furthermore, $\angle DBA = 72°$ also,

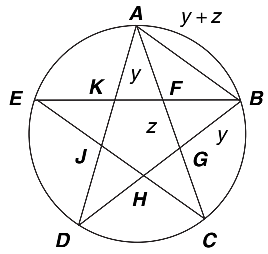

Fig. 1.3. The golden ratio.

which implies that $|AG| = |AB| = y + z$, where $y = |AF|$, $z = |FG|$. Considering the angles of $\triangle AGB$ and $\triangle BFG$, we obtain

$$\angle BGA = 72^\circ = 2 \times 36^\circ = \angle ABG, \qquad \angle GAB = 36^\circ = \angle FBG,$$

implying that $\triangle AGB \backsim \triangle BFG$. It follows that

$$\frac{1}{1 + z/y} = \frac{y}{y+z} = \frac{z}{y} = x.$$

Hence x is the positive root of $x^2 + x - 1 = 0$ and $x = 1/\phi$.

15 Huygens' theory of real numbers. The following corollary explains why all the fractions in (1.3) are in their lowest terms.

Corollary 1.9 (Huygens 1698) *Any convergent P_n/Q_n, with $n \geqslant 1$, of a formal regular continued fraction is a fraction in its lowest terms.*

Proof Indeed, by (1.16) any common divisor of P_n and Q_n must divide $(-1)^{n-1}$, which implies that P_n and Q_n are relatively prime. □

The basic properties of regular continued fractions, indicated in (1.4) and (1.5), are summarized in Brouncker's theorem 1.7 with $a_k = 1$. Let us mention that inequalities (1.21) determine the Dedekind section corresponding to ξ. Therefore this observation can be used as a guidance to Dedekind's theory of real numbers (for this theory, see Rudin 1964).

Lemma 1.10 *The denominators Q_n of the convergents for a formal infinite regular continued fraction satisfy*

$$Q_n > Q_{n-1} \geqslant u_{n-1}, \qquad n = 2, 3, \ldots \tag{1.34}$$

Proof Let $u_{-1} = 0$. Then $Q_{-1} = u_{-1} = 0$, $Q_0 = u_0 = 1$, $Q_1 = b_1 \geqslant 1 = u_1$. Assuming that $Q_k \geqslant u_k$ for $1 \leqslant k < n$ and $n \geqslant 2$, we obtain from (1.15) that $Q_n = b_n Q_{n-1} + Q_{n-2} \geqslant Q_{n-1} + Q_{n-2} \geqslant u_{n-1} + u_{n-2} = u_n$, which proves the lemma. □

Since $\phi > 1$, $u_n = (\phi^{n+1} - (-1)^{n+1}/\phi^{n+1})/\sqrt{5} \geqslant \phi^n/\sqrt{5} \longrightarrow +\infty$.

Theorem 1.11 *Any infinite regular continued fraction converges.*

Proof By (1.20), (1.21) and (1.34)

$$0 < \frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}} = \frac{1}{Q_{2n}Q_{2n+1}}$$
$$\leqslant \frac{1}{u_{2n}u_{2n+1}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty,$$

which obviously implies the result. □

The following two lemmas are very useful in the study of convergents.

Lemma 1.12 *Let a, b, c, d be integers satisfying $b > 0$, $d > 0$, $bc - ad = 1$ and let $a/b < s/t < c/d$, $t > 0$. Then $t > \max(b, d)$.*

Proof

$$\frac{1}{tb} \leqslant \frac{bs - ta}{tb} = \frac{s}{t} - \frac{a}{b} < \frac{c}{d} - \frac{a}{b} = \frac{1}{bd} \Rightarrow t > d,$$
$$\frac{1}{td} \leqslant \frac{ct - sd}{td} = \frac{c}{d} - \frac{s}{t} < \frac{c}{d} - \frac{a}{b} = \frac{1}{bd} \Rightarrow t > b.$$

□

Lemma 1.13 *Let a, b, c, d be integers satisfying $b > 0$, $d > 0$, $bc - ad = \pm 1$. Then the continuous function*

$$\phi(x) = \frac{ax + c}{bx + d}$$

increases on $(0, +\infty)$ from c/d to a/b if $bc - ad = -1$ and decreases from c/d to a/b if $bc - ad = 1$.

Proof We have $\phi'(x) = (ad - bc)(bx + d)^{-2}$. □

Huygens (1698) proved that the distance of P_{n-1}/Q_{n-1} to the set of all fractions p/q with $1 \leqslant q \leqslant Q_n$ excluding P_{n-1}/Q_{n-1}, is attained at P_n/Q_n.

Theorem 1.14 (Huygens 1698) *Let $b_0 + \mathbf{K}_{k=1}^{\infty}(1/b_k)$ be a formal regular continued fraction and $1 \leqslant q < Q_n$, $n \geqslant 1$. Then, for every integer p with $p/q \neq P_{n-1}/Q_{n-1}$,*

$$\left| \frac{p}{q} - \frac{P_{n-1}}{Q_{n-1}} \right| > \left| \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} \right|. \tag{1.35}$$

In particular, such a p/q cannot be between P_n/Q_n and P_{n-1}/Q_{n-1}.

Proof Since $p/q \neq P_{n-1}/Q_{n-1}$, we see that $|pQ_{n-1} - qP_{n-1}| \geqslant 1$. It follows that

$$\left| \frac{p}{q} - \frac{P_{n-1}}{Q_{n-1}} \right| \geqslant \frac{1}{qQ_{n-1}} > \frac{1}{Q_n Q_{n-1}} = \left| \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} \right|. \tag{1.36}$$

□

Huygens' theorem can be strengthened to include the case $q = Q_n$.

Lemma 1.15 *If* $Q_{n-1} > 2$ *then* (1.35) *holds for any fraction* p/q *with* $p/q \neq P_{n-1}/Q_{n-1}$, $p/q \neq P_n/Q_n$, $1 \leqslant q \leqslant Q_n$. *If* Q_{n-1} *is* 1 *or* 2 *then the equality in* (1.35) *is possible.*

Proof The proof reduces to a revision of (1.36) for the case $q = Q_n$. If $|pQ_{n-1} - qP_{n-1}| > 1$ then (1.35) holds with strict inequality. Otherwise we have the two equations

$$P_n Q_{n-1} - Q_n P_{n-1} = (-1)^{n-1},$$

$$pQ_{n-1} - Q_n P_{n-1} = -(-1)^{n-1}.$$

Now subtraction leads to the identity $P_n - p = 2(-1)^{n-1}/Q_{n-1}$, which can only hold if Q_{n-1} is either 1 or 2. In this case (1.36) applies with the possibility of equality. □

Corollary 1.16 *Any infinite continued fraction converges to an irrational number.*

Proof By Theorem 1.11 the limit $\lim_n P_n/Q_n$ exists. By Theorem 1.14 this limit equals an irrational number. □

By Corollary 1.16 the golden ratio ϕ as well as $\sqrt{2}$ are irrational numbers.

Definition 1.17 *A simple fraction* P/Q *is called a best Huygens approximation to a real number* ξ *if*

$$|\xi - P/Q| < |\xi - p/q|$$

for any fraction $p/q \neq P/Q$ *with* $1 \leqslant q \leqslant Q$.

Theorem 1.18 *Every convergent* P_n/Q_n *to* ξ *with* $n \geqslant 1$ *is a best Huygens approximation. In this case we apply (1.36) under the condition that* p/q *may be equal to* P_n/Q_n.

Proof Let P_n/Q_n be a noninteger convergent to ξ, i.e. $Q_n > 1$, and let p/q be a fraction with $1 \leqslant q \leqslant Q_n$. Suppose that n is odd. If $p/q > P_n/Q_n$ then there is nothing to prove. Otherwise, by Lemma 1.12,

$$\frac{p}{q} \leqslant \frac{P_{n-1}}{Q_{n-1}} < \frac{P_{n+1}}{Q_{n+1}} \leqslant \xi < \frac{P_n}{Q_n}.$$

Notice that $Q_n = Q_{n-1}$ implies $n = 1$, $b_1 = 1$, contradicting the assumption that P_n/Q_n is noninteger. Hence $Q_n > Q_{n-1}$. By (1.22),

$$\left|\frac{p}{q} - \xi\right| \geqslant \left|\frac{P_{n-1}}{Q_{n-1}} - \frac{P_{n+1}}{Q_{n+1}}\right| = \frac{b_{n+1}}{Q_{n+1}Q_{n-1}} > \frac{1}{Q_{n+1}Q_n} \geqslant \left|\frac{P_n}{Q_n} - \xi\right|.$$

The case of even n is considered similarly. If $Q_n = 1$ then $n = 1$ and $b_1 = 1$. Hence

$$\xi = b_0 + \frac{1}{1} \, \frac{1}{+ \, x} = b_0 + \frac{x}{1 + x}, \qquad x > 1,$$

and $P_1/Q_1 = b_0 + 1$ is a best approximation to ξ. □

In the case $n = 0$ we have $\xi = b_0 + 1/x$, where $x > 1$. Then P_0/Q_0 is a best integer approximation to ξ if and only if $x > 2$.

16 Lagrange's theory. By Huygens' theory, convergents to irrational numbers are their best rational approximations. Lagrange observed that the sense in which this statement is true can be strengthened.

Definition 1.19 *A simple fraction P/Q is called a best Lagrange approximation to a real number ξ if*

$$|Q\xi - P| < |q\xi - p| \tag{1.37}$$

for any fraction $p/q \neq P/Q$ with $1 \leqslant q \leqslant Q$.

Lemma 1.20 *Every best Lagrange approximation P/Q is a best Huygens approximation.*

Proof If $1 \leqslant q \leqslant Q$ and $p/q \neq P/Q$ then

$$\left|\xi - \frac{P}{Q}\right| \leqslant \frac{Q}{q}\left|\xi - \frac{P}{Q}\right| = \frac{|\xi Q - P|}{q} < \frac{|\xi q - p|}{q} = \left|\xi - \frac{p}{q}\right|,$$

which proves the lemma. □

Theorem 1.21 *Let $\xi \in \mathbb{R}$. Then every convergent P_n/Q_n with $n \geqslant 1$ satisfies*

$$|Q_n\xi - P_n| < |Q_{n-1}\xi - P_{n-1}|, \tag{1.38}$$

$$|Q_n\xi - P_n| + |Q_{n-1}\xi - P_{n-1}| = |q\xi - p|, \tag{1.39}$$

if $p = P_n - P_{n-1}$, $q = Q_n - Q_{n-1}$, and

$$|Q_n\xi - P_n| + |Q_{n-1}\xi - P_{n-1}| < |q\xi - p| \tag{1.40}$$

for any integers p and q such that $0 < q \leqslant Q_n$, $p/q \neq P_n/Q_n$, $p/q \neq (P_n - P_{n-1})/(Q_n - Q_{n-1})$.

Proof Let us prove (1.38) first. By Euler's formula (1.17)

$$Q_n\xi - P_n = \frac{(-1)^n}{\xi_{n+1}Q_n + Q_{n-1}}, \tag{1.41}$$

which implies (notice that $1 < \xi_{n+1}$)

$$\begin{aligned} |Q_n\xi - P_n| &= \frac{1}{(\xi_{n+1}b_n + 1)Q_{n-1} + \xi_{n+1}Q_{n-2}} \\ &\leqslant \frac{1}{(\xi_{n+1}b_n + 1)Q_{n-1} + Q_{n-2}} \\ &< \frac{1}{\xi_n Q_{n-1} + Q_{n-2}} = |Q_{n-1}\xi - P_{n-1}|. \end{aligned}$$

Formula (1.39) is proved by a direct computation.

By (1.16) the system of linear equations

$$p = xP_n + yP_{n-1},$$
$$q = xQ_n + yQ_{n-1},$$

has a unique solution for integer x and y. It follows that

$$p - q\,\xi = x(P_n - Q_n\xi) + y(P_{n-1} - Q_{n-1}\xi).$$

By (1.21) the differences within the parentheses have opposite signs. If x and y are nonzero and also have opposite signs then (1.39) holds, provided at least one of x and y is not ± 1. The case when $\pm 1 = x = -y$ corresponds to (1.39). If x and y are nonzero and have equal signs then $q = xQ_n + yQ_{n-1}$ cannot satisfy $0 < q < Q_n$. The case $x = 0$ contradicts (1.38). The case $y = 0$ contradicts the assumption $p/q \neq P_n/Q_n$. □

Corollary 1.22 (Lagrange 1789) *Every convergent P_n/Q_n to $\xi \in \mathbb{R}$ with $n \geqslant 1$ is a best Lagrange approximation to ξ.*

Corollary 1.23 (Lagrange [1789]) *If P/Q, $Q > 1$, is a best Lagrange approximation to ξ then P/Q is a convergent to ξ.*

Proof Let P/Q be a best Lagrange approximation that is not a convergent to ξ. Then there exists $n \geqslant 1$ such that $Q_{n-1} < Q \leqslant Q_n$. By (1.39) and (1.40)

$$|Q_{n-1}\xi - P_{n-1}| < |Q\xi - P|,$$

which contradicts the assumption that P/Q is a best Lagrange approximation to ξ. □

17 Nonprincipal convergents. We begin with a citation from Euler (1744, **16**):

If the ratio of circumference to diameter is computed more exactly by continued division, just as before, the following sequence of quotients appears: 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14 etc. from which simple fractions[5] are brought to light in the following way:

[5] $\frac{2}{1} = \frac{3-1}{1-0}$, $\frac{19}{6} = \frac{22-3}{7-1}$, $\frac{311}{99} = \frac{333-22}{106-7}$, $\frac{103\,638}{32\,989} = \frac{103\,993-355}{33\,102-113}$ etc.; $\frac{1}{1} = \frac{2-1}{1-0}$, $\frac{16}{5} = \frac{19-3}{6-1}$ etc.

3	7	15	1	292	1	
$\frac{1}{0}$,	$\frac{3}{1}$,	$\frac{22}{7}$,	$\frac{333}{106}$,	$\frac{355}{113}$,	$\frac{103\,993}{33\,102}$	principal convergents
	$\frac{2}{1}$,	$\frac{19}{6}$,	$\frac{311}{99}$,		$\frac{103\,638}{32\,989}$	nonprincipal convergents
	$\frac{1}{1}$,	$\frac{16}{5}$,	$\frac{289}{92}$,		$\frac{103\,283}{32\,876}$	
		$\frac{13}{4}$,	$\frac{267}{85}$,		$\frac{102\,928}{32\,763}$	
		$\frac{10}{3}$,	$\frac{245}{78}$,		etc.	
		$\frac{7}{2}$,	$\frac{223}{71}$,			
		$\frac{4}{1}$,	$\frac{201}{64}$,			

Therefore, in this way two kinds of fraction are obtained, of which one is too large and the other is too small. Namely, those that are too large are written under the indices[6] 3, 15, 292 etc., and the rest are too small. From this it is easy to establish the whole table of Wallis, which is composed of all fractions approximating the true ratio of circumference to diameter more closely than would be possible with smaller or equal numbers.

An analysis of the columns in Euler's table shows that the column under partial denominator 15 satisfies

$$\frac{22}{7} < \frac{19}{6} < \frac{16}{5} < \frac{13}{4} < \frac{10}{3} < \frac{7}{2} < \frac{4}{1} < \frac{1}{0} \qquad (n=2),$$

whereas its right-hand neighbor under 1 satisfies the opposite inequalities

$$\frac{333}{106} > \frac{311}{99} > \frac{289}{92} > \frac{267}{85} > \frac{245}{78} > \frac{223}{71} > \frac{201}{64} > \cdots > \frac{3}{1} \qquad (n=3).$$

Calculations and the above remark of Euler, implying in particular that Euler considered nonprincipal convergents as *equally satisfactory*, suggest that by the term nonprincipal convergents[7] Euler understood the quotients

$$\frac{P_{n,k}}{Q_{n,k}} = \frac{P_{n+1} - kP_n}{Q_{n+1} - kQ_n} = \frac{(b_{n+1} - k)P_n + P_{n-1}}{(b_{n+1} - k)Q_n + Q_{n-1}}, \tag{1.42}$$

where $0 < k < b_{n+1}$. It is clear from (1.42) that the nonprincipal convergents lie between neighboring pairs of principal convergents. Easy algebra with (1.16) shows that

$$P_{n,k+1}Q_{n,k} - P_{n,k}Q_{n,k+1} = (-1)^n.$$

[6] What Euler calls indices are in fact partial denominators of the regular continued fraction for π; see §§**8**, **10** above.

[7] Nowadays they are called intermediate convergents; see Lang (1966).

It follows that for even n

$$\frac{P_{n+1}}{Q_{n+1}} < \cdots < \frac{P_{n,k}}{Q_{n,k}} < \frac{P_{n,k+1}}{Q_{n,k+1}} < \cdots < \frac{P_{n-1}}{Q_{n-1}}, \tag{1.43}$$

whereas for odd n

$$\frac{P_{n+1}}{Q_{n+1}} > \cdots > \frac{P_{n,k}}{Q_{n,k}} > \frac{P_{n,k+1}}{Q_{n,k+1}} > \cdots > \frac{P_{n-1}}{Q_{n-1}}. \tag{1.44}$$

Theorem 1.24 *Either a nonprincipal convergent $P_{n,k}/Q_{n,k}$ to ξ is a best Huygens approximation or only P_n/Q_n can give a better or the same approximation to ξ.*

Proof Let n be odd. Then

$$\frac{P_{n-1}}{Q_{n-1}} < \frac{P_{n,k}}{Q_{n,k}} < \frac{P_{n+1}}{Q_{n+1}} \leqslant \xi < \frac{P_n}{Q_n}.$$

If $P/Q \neq P_{n,k}/Q_{n,k}$ and

$$\left|\xi - \frac{P}{Q}\right| \leqslant \left|\xi - \frac{P_{n,k}}{Q_{n,k}}\right|, \tag{1.45}$$

then $P_{n,k}/Q_{n,k} < P/Q$. If $P/Q < P_n/Q_n$ then P/Q lies between $P_{n,k}/Q_{n,k}$ and P_n/Q_n, satisfying

$$P_nQ_{n,k} - Q_nP_{n,k} = P_nQ_{n+1} - P_{n+1}Q_n = -(-1)^n = (-1)^{n+1}, \tag{1.46}$$

and hence $Q > Q_{n,k}$ by Lemma 1.12. If $P_n/Q_n < P/Q$ then

$$\frac{1}{QQ_n} \leqslant \frac{P}{Q} - \frac{P_n}{Q_n} < \frac{P}{Q} - \xi \leqslant \xi - \frac{P_{n,k}}{Q_{n,k}} < \frac{P_n}{Q_n} - \frac{P_{n,k}}{Q_{n,k}} = \frac{1}{Q_nQ_{n,k}},$$

implying $Q > Q_{n,k}$ again. Even values of n are considered similarly. □

Theorem 1.24 shows that principal and nonprincipal convergents are the best *one-sided* approximations to ξ. See a slightly different proof in Perron (1954, §16, II).

Corollary 1.25 (Lagrange 1798) *Any simple fraction between ξ and a principal or nonprincipal convergent to ξ has a denominator greater than the denominator of this convergent.*

Best Huygens approximations make a subset in the set of principal and nonprincipal convergents.

Theorem 1.26 (Perron 1954, §16, II) *A fraction P/Q with $Q > 0$ having the property that any fraction p/q between ξ and P/Q satisfies $q > Q$ is either a principal or a nonprincipal convergent to ξ.*

Proof Suppose that P/Q is not principal or nonprincipal convergent. If $P/Q < \xi$, since $Q_0 = 1$, we must have $P_0/Q_0 < P/Q$. Then there are an odd n and a k, $0 \leqslant k < b_{n+1}$, such that

$$\xi > \frac{P_{n,k}}{Q_{n,k}} > \frac{P}{Q} > \frac{P_{n,k+1}}{Q_{n,k+1}}.$$

Then $Q_{n,k} < Q$ by Lemma 1.12, which contradicts our choice of P/Q.

If $\xi < P/Q$ then $P/Q < b_0 + 1 = (P_0 + P_1)/(Q_0 + Q_1)$. Hence there are an even n and a k such that

$$\xi < \frac{P_{n,k}}{Q_{n,k}} < \frac{P}{Q} < \frac{P_{n,k+1}}{Q_{n,k+1}}.$$

Then $Q_{n,k} < Q$ by Lemma 1.12, which again contradicts our choice of P/Q. □

The following theorem specifies those nonprincipal convergents which are best Huygens approximations.

Theorem 1.27 (Perron 1954, §16, III) *A nonprincipal convergent $P_{n,k}/Q_{n,k}$ is a best Huygens approximation to ξ if and only if either $2k < b_{n+1}$ or $2k = b_{n+1}$ and*

$$\frac{1}{b_n} + \cdots + \frac{1}{b_1} > \frac{1}{b_{n+2}} + \frac{1}{b_{n+3}} + \cdots . \tag{1.47}$$

Proof By Theorem 1.24 a nonprincipal convergent $P_{n,k}/Q_{n,k}$ is a best Huygens approximation if and only if

$$\left|\xi - \frac{P_{n,k}}{Q_{n,k}}\right| < \left|\xi - \frac{P_n}{Q_n}\right|.$$

Putting $r = b_{n+1} - k$ for brevity and applying (1.42), we can rewrite this inequality as

$$\frac{|r(Q_n\xi - P_n) + (Q_{n-1}\xi - P_{n-1})|}{rQ_n + Q_{n-1}} < \frac{|Q_n\xi - P_n|}{Q_n}. \tag{1.48}$$

Expressing Euler's formula (1.17) in terms of ξ_{n+1} we obtain the proportional relation

$$Q_{n-1}\xi - P_{n-1} = -\xi_{n+1}(Q_n\xi - P_n),$$

reducing (1.48) to $(\xi_{n+1} - r)Q_n < rQ_n + Q_{n-1}$, since $0 < r < b_{n+1} \leqslant \xi_{n+1}$. It follows that (1.48) is equivalent to

$$\xi_{n+1}Q_n < 2rQ_n + Q_{n-1}. \tag{1.49}$$

Since $b_{n+1} \leqslant \xi_{n+1} < b_{n+1} + 1$, inequality (1.49) holds if $b_{n+1} < 2r = 2b_{n+1} - 2k$ and does not hold if $2r < b_{n+1}$. In the case $2r = b_{n+1}$, (1.49) is equivalent to

$$\xi_{n+1}Q_n < b_{n+1}Q_n + Q_{n-1} = Q_{n+1},$$

which is nothing other than (1.47); see (1.18). □

18 Nonprincipal convergents: applications. Using Theorem 1.27, one can easily calculate the best Huygens approximations for π (Perron 1954, §16, IV):

$$\boxed{\frac{3^{l}}{1}}\quad \frac{13^{g}}{4}\quad \frac{16^{g}}{5}\,\frac{19^{g}}{6}\quad \boxed{\frac{22^{g}}{7}}\quad \frac{179^{l}}{57}\quad \frac{201^{l}}{64}\quad \frac{223^{l}}{71}\quad \frac{245^{l}}{78}\quad \frac{267^{l}}{85}\quad \frac{289^{l}}{92}\quad \frac{311^{l}}{99}$$

$$\boxed{\frac{333^{l}}{106}}\quad \boxed{\frac{355^{g}}{113}}\quad \frac{52163^{l}}{16604}\quad \frac{52518^{l}}{16717}\quad \frac{52873^{l}}{16830}\quad \frac{53228^{l}}{16944}\quad \frac{53583^{l}}{17057}\quad \frac{53938^{l}}{17170}\quad \cdots$$

where the principal convergents are boxed.

Let us consider the regular continued fraction of $\log_2 3 - 1$ used in the construction of musical scales in §**7** above. Euler's table of principal and nonprincipal convergents looks as follows:

$$\begin{array}{ccccccccc|l}
0 & 1 & 1 & 2 & 2 & 3 & 1 & 5 & 2 & \\
\frac{1}{0} & \frac{0}{1} & \frac{1}{1} & \frac{1}{2} & \frac{3}{5} & \frac{7}{12} & \frac{24}{41} & \frac{31}{53} & \frac{179}{306} & \text{principal convergents} \\
\hline
 & & & & \frac{2}{3} & \frac{4}{7} & \frac{17}{29} & & \frac{148}{253} & \text{nonprincipal convergents} \\
 & & & & & & \boxed{\frac{10}{17}} & & \frac{117}{200} & \\
 & & & & & & & & \frac{86}{147} & \\
 & & & & & & & & \frac{55}{94} &
\end{array}$$

Since the number of semitones in the scale must be relatively small, Euler's table clearly shows that there are two good candidates for the musical scale: the principal convergent $7/12$, corresponding to a scale with 12 equal semitones (frequency ratios) and the nonprincipal convergent 10/17, corresponding to a scale of 17 equal semitones. The 12-based scale was studied in detail by Mersenne, who followed Descartes' paper 'Musicae compendium'. Brouncker (1655) published its English translation, supplied with extensive remarks of his own. In these remarks Brouncker analyzed the scale of 17 equal semitones. By Theorem 1.27 the nonprincipal convergent 10/17 is not a Huygens approximation to $\log_2 3 - 1$. Clearly

$$\log_2 \frac{3}{2} - \frac{7}{12} = 0.001\,629\,167\,387\,82\ldots,$$

$$\frac{10}{17} - \log_2 \frac{3}{2} = 0.003\,272\,793\,396\,49\ldots,$$

which implies that the scale of 17 equal semitones approximates a perfect fifth with a double error compared with the scale of 12 equal semitones. As to the approximation of $\log_2 5/4$, the 17-based scale more than doubles the error compared with the 12-based scale:

$$\frac{4}{12} - \log_2 \frac{5}{4} = 0.011\dots\ ,$$

$$\log_2 \frac{5}{4} - \frac{5}{17} = 0.027\dots$$

For the 19-based scale, however the errors is the approximation of both logarithms are positive and almost equally small:

$$\log_2 3/2 - 11/19 = 0.006\,015\dots\ ,$$

$$\log_2 5/4 - 6/19 = 0.006\,138\dots\ ,$$

whereas for 31 semitones the approximation to $\log_2 5/4$ is much better:

$$\log_2 3/2 - 18/31 = 0.004\,317\,339\,430\,833\,600\,808\,6\dots,$$

$$10/31 - \log_2 5/4 = 0.000\,652\,550\,273\,927\,974\,710\,33\dots$$

The choice of these two last scales is explained by Euler's table for the regular continued fraction of $\log_2 5/4$:

$$\begin{array}{ccccccc|l}
0 & 3 & 9 & 2 & 2 & 4 & 6 & \\
\frac{1}{0}, & \frac{0}{1}, & \frac{1}{3}, & \frac{9}{28}, & \frac{19}{59}, & \frac{47}{146}, & \frac{207}{643}, & \text{principal convergents} \\
\hline
 & & \frac{1}{2} & \frac{8}{25} & \boxed{\frac{10}{31}}, & \frac{28}{87}, & \frac{160}{497}, & \text{nonprincipal convergents} \\
 & & \frac{1}{1} & \frac{7}{22} & & & \frac{113}{351}, & \\
 & & & \boxed{\frac{6}{19}} & & & \frac{66}{205} & \\
 & & & \frac{5}{16} & & & \frac{19}{59} & \\
 & & & \text{etc.} & & & &
\end{array}$$

By Theorem 1.27 the simple fraction 10/31 is a best Huygens approximation.

19 Farey's sequences. Rational numbers in (0, 1) in their lowest terms can be arranged in sequences. These sequences were introduced into mathematics by J. Farey in 1816. Farey's sequences provide a good geometrical illustration of the properties of principal and nonprincipal convergents.

Definition 1.28 *Let n be a positive integer. The increasing sequence, including* 0/1 *and* 1/1*, of nonnegative fractions p/q in their lowest terms with $p \leqslant q \leqslant n$ is called Farey's sequence $\mathfrak{F}_n$ of order n.*

The following table lists the first few Farey's sequences:

$$\begin{array}{llllllllllll}
\mathfrak{F}_1: & \frac{0}{1} & \frac{1}{1} \\
\mathfrak{F}_2: & \frac{0}{1} & \frac{1}{2} & \frac{1}{1} \\
\mathfrak{F}_3: & \frac{0}{1} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{1} \\
\mathfrak{F}_4: & \frac{0}{1} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{1}{1} \\
\mathfrak{F}_5: & \frac{0}{1} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} & \frac{2}{3} & \frac{3}{4} & \frac{4}{5} & \frac{1}{1}.
\end{array}$$

Subtraction of 1 from the consonant intervals in (1.8) results in a subset of Farey's series $\mathfrak{F}_5$, which is in a clear agreement with the requirement of consonance. The missing values 2/5, 4/5 correspond to the intervals of a *tritone* (7/5) and a *minor seventh* (9/5) (both are dissonant); 3/4 corresponds to 7/4, which is a little bit more than 7/5.

A simple analysis of the above table shows that any pair $a/b < c/d$ of neighboring fractions in Farey's sequence satisfies $bc - ad = 1$.

Theorem 1.29 *If c/d immediately follows a/b in $\mathfrak{F}_m$, then $cb - ad = 1$.*

Proof We suppose that $d \leqslant b$ and represent the continued fraction of a/b as

$$\frac{a}{b} = \frac{1}{b_1} + \cdots + \frac{1}{b_k} = \frac{1}{b_1} + \cdots + \frac{1}{b_k - 1} + \frac{1}{1}.$$

We use the first continued fraction if k is even and the second if k is odd. Then by (1.21) the last convergent r/s, $r/s \neq a/b$, to a/b must be greater than a/b. Since $s < b \leqslant m$ and since r/s is a fraction in its lowest terms, we have $r/s \in \mathfrak{F}_m$. By Lemma 1.12 this implies that

$$\frac{a}{b} < \frac{c}{d} \leqslant \frac{r}{s}.$$

If $r/s = c/d$ then the result follows from (1.16). If $c/d < r/s$ then by Lemma 1.12 $d > b$, which contradicts our assumption $d \leqslant b$. The case $d > b$ is considered similarly by developing c/d into a continued fraction. □

Corollary 1.30 *If c/d immediately follows a/b in $\mathfrak{F}_m$, $m > 1$, then $b \neq d$.*

Proof Since $bc - ad = 1$, the assumption $b = d$ implies b and d are the divisors of 1. □

Definition 1.31 *The fraction $(a + c)/(b + d)$ is called the mediant of a/b and c/d.*

By Lemma 1.13 the mediant of two positive fractions lies between them (put $x = 1$). Analyzing Farey's table we arrive at the conclusion that any fraction in Farey's sequence $\mathfrak{F}_m$ is the mediant of its neighbors.

Corollary 1.32 *If $a/b < p/q < c/d$ are consecutive fractions in $\mathfrak{F}_m$ then $p = a + c$, $q = b + d$.*

Proof By Theorem 1.29 we have $pb - aq = 1$ and $cq - pd = 1$, which implies by subtraction that $p(b+d) = q(a+c)$ as stated. □

Corollary 1.32 gives a simple algorithm for constructing $\mathfrak{F}_{m+1}$ from $\mathfrak{F}_m$. Namely, consider in $\mathfrak{F}_m$ only those pairs $a/b < c/d$ of neighbors that satisfy $b + d = m + 1$ and insert between them their mediant. The sequence obtained is $\mathfrak{F}_{m+1}$.

Definition 1.33 *A pair of fractions $a/b < c/d$ in $(0, 1)$ is called a normal pair if $bc - bd = 1$.*

It is clear that both fractions in any normal pair are fractions in lowest terms. It can be easily checked that for the mediant p/q of a normal pair $a/b < c/d$ both pairs $a/b < p/q$ and $p/q < c/d$ are normal.

Theorem 1.34 *A pair $a/b < c/d$ is a normal pair if and only if for every $s/t \in (a/b, c/d)$ we have $t > \max(b, d)$.*

Proof If $a/b < c/d$ is a normal pair then $t > \max(b, d)$ by Lemma 1.12. If for every fraction $s/t \in (a/b, c/d)$ the inequality $t > \max(b, d)$ holds then a/b and c/d are neighboring elements of Farey's sequence $\mathfrak{F}_m$ with $m = \max(b, d)$. Theorem 1.29 implies that $bc - ad = 1$. □

Corollary 1.35 *A pair $a/b < c/d$ is a normal pair if and only if a/b and c/d are neighbors in the smallest Farey's sequence containing a/b and c/d.*

20 Convergents: quadratic theory. In the approximation theory of irrational numbers by their convergents an important role is played by Lagrange's formula (1774):

$$\boxed{\left|\xi - \frac{P_n}{Q_n}\right| = \frac{1}{Q_n^2}\,\frac{1}{Q_{n-1}/Q_n + \xi_{n+1}} = \frac{1}{Q_n^2}\,\frac{1}{Q_{n+1}/Q_n + 1/\xi_{n+2}}.} \tag{1.50}$$

The proof of (1.50) is easy. By Euler's formula and (1.15),

$$\left|\xi - \frac{P_n}{Q_n}\right| = \left|\frac{\xi_{n+1}P_n + P_{n-1}}{\xi_{n+1}Q_n + Q_{n-1}} - \frac{P_n}{Q_n}\right| = \frac{1}{Q_n^2}\,\frac{1}{Q_{n+1}/Q_n + 1/\xi_{n+2}}. \tag{1.51}$$

Theorem 1.36 (Lagrange 1774) *For every positive irrational ξ the convergents P_n/Q_n satisfy*

$$\frac{1}{Q_n^2(1 + Q_{n+1}/Q_n)} < \left|\xi - \frac{P_n}{Q_n}\right| < \frac{1}{Q_n^2}, \qquad n = 0, 1, 2, \ldots \tag{1.52}$$

Proof The first inequality follows from $Q_n < \xi_{n+1}Q_n + Q_{n-1}$ and (1.51), since $\xi_{n+1} = b_{n+1} + 1/\xi_{n+2} > 1$. The second inequality follows from $\xi_{n+2} = b_{n+2} + 1/\xi_{n+3} > 1$. □

By (1.50) most convergents approximate ξ better than indicated in the right-hand inequality of (1.52).

Theorem 1.37 (Legendre) *If $q > 0$ and*

$$\left|\frac{p}{q} - \xi\right| < \frac{1}{2q^2} \tag{1.53}$$

then p/q is a convergent to ξ.

Proof If p and q satisfy (1.53) then $|p - q\xi| < 1/2q$. Assuming that x and y are integers such that $|x - y\xi| < |p - q\xi| < 1/2q$, we obtain that

$$\frac{1}{yq} \leqslant \left|\frac{x}{y} - \frac{p}{q}\right| \leqslant \left|\frac{x}{y} - \xi\right| + \left|\xi - \frac{p}{q}\right| < \frac{1}{2yq} + \frac{1}{2q^2}.$$

It follows that $y > q$ and p/q is a convergent to ξ by Theorem 1.21. □

Theorem 1.38 (Vahlen 1895) *From any two consecutive convergents to ξ at least one satisfies* (1.53).

Proof The elementary formula

$$0 < \frac{1}{2}\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \frac{1}{ab} = \frac{1}{2a^2} + \frac{1}{2b^2}$$

shows that

$$\left|\xi - \frac{P_{k-1}}{Q_{k-1}}\right| + \left|\frac{P_k}{Q_k} - \xi\right| = \left|\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}}\right| = \frac{1}{Q_k Q_{k-1}} < \frac{1}{2Q_k^2} + \frac{1}{2Q_{k-1}^2},$$

which implies that at least one convergent must satisfy (1.53). □

Lemma 1.39 *Let ξ be a real number and p, $p \neq 0$ and $q > 0$ be integers satisfying $|\xi - p/q| < 1/q^2$. Then p/q is a principal or nonprincipal convergent to ξ.*

Proof If $q = 1$ then either $p/q = P_0/Q_0$ or $p/q = (P_0 + P_{-1})/(Q_0 + Q_{-1})$. If $q > 1$, we assume that $p/q > \xi$ (the case of the opposite inequality is considered similarly). On the one hand, if p/q is not a principal or nonprincipal convergent then

$$\xi < \frac{P}{Q} < \frac{p}{q} < \frac{P'}{Q'}$$

for two consecutive nonprincipal convergents, implying by Lemma 1.12 that $Q < q$, since $P'Q - PQ' = 1$. On the other hand,

$$\frac{1}{q^2} > \frac{p}{q} - \xi > \frac{p}{q} - \frac{P}{Q} \geqslant \frac{1}{qQ} \quad \Rightarrow \quad Q > q.$$

This contradiction proves the lemma. □

Theorem 1.40 *If p, $p \neq 0$ and $q > 0$ are integers satisfying $|\xi - p/q| < 1/q^2$ then p/q is either a principal convergent to ξ or a nonprincipal convergent $P_{n,k}/Q_{n,k}$ with $k = 1$ or $k = b_{n+1} - 1$.*

Proof By (1.52) we may assume that $P_{n,k}/Q_{n,k}$ is a nonprincipal convergent, i.e. $0 < k < b_{n+1}$. Euler's formula (1.17) and (1.16), (1.22), (1.46) imply, cf. (1.41),

$$Q_{n,k}\xi - P_{n,k} = (-1)^{n+1}\frac{\xi_{n+1} - b_{n+1} + k}{\xi_{n+1}Q_n + Q_{n-1}}. \tag{1.54}$$

It follows that

$$\left|\xi - \frac{P_{n,k}}{Q_{n,k}}\right| < \frac{1}{Q_{n,k}^2} \Leftrightarrow \frac{x}{xQ_n + rQ_n + Q_{n-1}} < \frac{1}{rQ_n + Q_{n-1}},$$

where $0 < r = b_{n+1} - k < b_{n+1}$ and $0 < x = \xi_{n+1} - r$.

For $r = 1$ ($k = b_{n+1} - 1$) the latter inequality holds if and only if

$$b_{n+1} + \frac{1}{b_{n+2}}{}_{+\cdots} < 2 + b_n + \frac{1}{b_{n-1}}{}_{+\cdots+}\frac{1}{b_1}.$$

If $r > 1$ then, putting $R = rQ_n + Q_{n-1}$, we can rewrite the inequality as

$$x = \xi_{n+1} - r = k + \frac{1}{\xi_{n+2}} < \frac{R}{R - Q_n} = 1 + \frac{1}{r - 1 + Q_{n-1}/Q_n} \leqslant \frac{r}{r-1},$$

which obviously leaves no choice for k except $k = 1$. □

The proofs of Lemma 1.39 and of Theorem 1.40 follow Lang (1966, §4) with minor changes. Theorem 1.40 shows that quadratic approximations similar to Lagrange and Huygens approximations lie in the set of convergents. However, (1.52) and (1.53) suggest that the constants 1 and 1/2 multiplying q^{-2} may not be optimal. Theorem 1.41 below shows that a quadratic speed of approximation is characteristic for quadratic irrationals. The proof follows Arnold (1939).

Theorem 1.41 *Let ξ be a real quadratic irrational with discriminant $\mathcal{D}$ and $A > \sqrt{D}$. Then the inequality*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{Aq^2} \tag{1.55}$$

has only a finite number of solutions.

Proof Let ξ' be the number algebraically adjoint to ξ. Then

$$f(X) = aX^2 + bX + c = a(X - \xi)(X - \xi'), \qquad D = a^2(\xi - \xi')^2.$$

Since $f(X)$ has only irrational roots, the integer $q^2 f(p/q)$ cannot be zero and hence, for p/q satisfying (1.55),

$$\frac{1}{q^2} \leqslant \left|f\left(\frac{p}{q}\right)\right| = \left|\xi - \frac{p}{q}\right| \left|a\left(\xi' - \frac{p}{q}\right)\right|$$
$$< \frac{1}{Aq^2}\left|a\left(\xi' - \xi + \xi - \frac{p}{q}\right)\right| \leqslant \frac{\sqrt{D}}{Aq^2} + \frac{|a|}{A^2 q^4},$$

which is impossible for large q if $A > \sqrt{D}$. □

21 Ford circles. Theorem 1.37 has a nice geometrical interpretation. A *Ford circle* for $p/q \in \mathfrak{F}_n$ is the circle $C(p/q)$ in the upper half-plane $\mathbb{C}_+$ of radius $1/2q^2$ tangent to $\mathbb{R}$ at p/q. Notice that the orthogonal projection of $C(p/q)$ onto $\mathbb{R}$ is exactly the interval described by (1.53). The distance d between the centers of $C(p/q)$ and $C(r/s)$ is determined by the formula

$$d^2 = \left(\frac{p}{q} - \frac{r}{s}\right)^2 + \left(\frac{1}{2q^2} - \frac{1}{2s^2}\right)^2$$
$$= \frac{(ps-rq)^2}{s^2q^2} + \frac{(s^2-q^2)^2}{4s^4q^4} = \frac{(ps-rq)^2 - 1}{s^2q^2} + \left(\frac{1}{2q^2} + \frac{1}{2s^2}\right)^2.$$

It follows that for any pair p/q and r/s in $\mathfrak{F}_n$ the Ford circles $C(p/q)$ and $C(r/s)$ either do not intersect or are tangent. The latter happens if and only if p/q and r/s are neighbors in the smallest Farey's sequence to which they both belong (see Corollary 1.35). If p/q and r/s are neighbors in $\mathfrak{F}_n$ then $(p+r)/(q+s) \in \mathfrak{F}_{n+1}$ if and only if $C((p+r)/(q+s))$ is tangent to $C(p/q)$ and $C(r/s)$.

Take any irrational $\xi \in (0, 1)$ and consider the vertical line $L(\xi)$ passing through ξ. This line intersects infinitely many Ford circles $C(p/q)$. By Theorem 1.38 the fact that $L(\xi)$ intersects $C(p/q)$ at two points (which is equivalent to $L(\xi) \cap C(p/q) \neq \varnothing$ since ξ is irrational) implies that p/q is a convergent to ξ.

22 Approximation by $\{\!\{n\theta\}\!\}$. This is an application of Theorem 1.38.

Theorem 1.42 *If θ is irrational then the set of all fractional parts $\{\!\{n\theta\}\!\}$ is dense in* $[0, 1]$.

Proof If p/q is a convergent to θ satisfying (1.53) then

$$\frac{kp}{q} - \frac{1}{2q} < k\theta < \frac{kp}{q} + \frac{1}{2q}, \qquad |k| < q.$$

By Euclid's algorithm, $kp = m_k q + r_k$, $1 \leqslant r_k < q$, with integer m_k and r_k. It follows that

$$m_k + \frac{r_k}{q} - \frac{1}{2q} < k\theta < m_k + \frac{r_k}{q} + \frac{1}{2q}, \qquad |k| < q.$$

Hence

$$m_k = \left[\frac{kp}{q}\right] = [k\theta], \quad \left|\frac{kp}{q} - \{\!\{k\theta\}\!\}\right| < \frac{1}{2q}, \qquad |k| < q. \tag{1.56}$$

Notice that $r_{-k} = q - r_k$ and $m_{-k} = -m_k - 1$.

No pair i, j of distinct integers in $[1, q)$ can satisfy $r_i = r_j$, since then q would divide $(i-j)p$, which is not possible. It follows that $\{r_1, \ldots, r_{q-1}\}$ is a permutation of $\{1, \ldots, q-1\}$. By Theorem 1.38 there are infinitely many convergents p/q satisfying (1.53), which proves the theorem. $\square$

23 Parameterization of $\mathbb{R}$. By §**1**, regular continued fractions parameterize real numbers using integers. Later this important property will be extended to other classes of continued fractions by a similar construction. To provide its rigorous background we consider only those sequences $b = \{b_n\}$ whose domain $\mathcal{D}(b)$ is either the set of all nonnegative integers $\mathbb{Z}_+$ or a finite segment $[0, J] \cap \mathbb{Z}_+$.

Definition 1.43 *A sequence $b = \{b_n\}$ with $b_0 \in \mathbb{Z}$ is included in $\mathcal{Z}$ if one of the following conditions holds:*

(a) $\mathcal{D}(b) = \mathbb{Z}_+$ *and* $b_n \in \mathbb{N}$ *for* $n > 0$;

(b) $\mathcal{D}(b) = [0, J] \cap \mathbb{Z}_+$ *for* $J > 0$, $b_n \in \mathbb{N}$, *and* $2 \leqslant b_J$;

(c) $\mathcal{D}(b) = \{0\}$.

To unify the notation we put $J = J(b) = \infty$ if $\mathcal{D}(b) = \mathbb{Z}_+$.

By Theorem 1.11 the mapping ζ maps $\mathcal{Z}$ to $\mathbb{R}$:

$$b \in \mathcal{Z} \xrightarrow{\zeta} b_0 + \mathop{\mathbf{K}}_{k=1}^{J} \left(\frac{1}{b_k}\right) = \lim_n \frac{P_n}{Q_n} = \xi \in \mathbb{R}. \tag{1.57}$$

It determines the correspondence between *parameters* in $\mathcal{Z}$ and real numbers ξ in $\mathbb{R}$. Parameters with finite domain correspond to rational numbers. The number corresponding to a finite sequence b is often denoted by $[b_0; b_1, \ldots, b_n]$. If $\mathcal{D}(b) = \mathbb{Z}_+$ then ξ in (1.57) is denoted by $[b_0; b_1, b_2 \ldots]$.

We consider on $\mathcal{Z}$ the topology of pointwise convergence. Namely, let $a \in \mathcal{Z}$ and $\{b^{(j)}\}_{j \geqslant 0}$ be a sequence in $\mathcal{Z}$. If $J(a) = \infty$ then $\lim_j b^{(j)} = a$ if for every integer $n \geqslant 0$ there is a J_n such that for every $j > J_n$ we have $b_k^{(j)} = a_k$ for $k = 0, \ldots, n$.

If $J(a) < \infty$ then $\zeta(a) = [a_0; a_1, a_2 \ldots, a_n]$ for some n. We write $\lim_j b^{(j)} = a$ in the two following cases: if $b^{(j)} = a$ for $j > J_0$ or if $\lim_j b_{n+1}^{(j)} = +\infty$ with $b_k^{(j)} = a_k$ for $k = 0, 1, \ldots, n$, $j > J_0$.

Any fraction a/b in its lowest terms in $(0, 1)$ is a convergent for some irrational number $\xi \in (0, 1)$. To find ξ one can develop a/b into a finite continued fraction and then continue it arbitrarily up to infinity.

Lemma 1.44 *Let n be a positive integer, a_0 be an integer and $a_1, \dots, a_n$ be positive integers. Let $P_n/Q_n = [a_0; a_1, a_2, \dots, a_n]$. Then the set of all real numbers $[b_0; b_1, b_2, \dots]$ with $b_k = a_k$ for $0 \leqslant k \leqslant n$ is*

$$\left[\frac{P_n}{Q_n}, \frac{P_n + P_{n-1}}{Q_n + Q_{n-1}}\right] \text{ if } n \text{ is even and } \left[\frac{P_n + P_{n-1}}{Q_n + Q_{n-1}}, \frac{P_n}{Q_n}\right] \text{ if } n \text{ is odd.}$$

Proof Apply Euler's formula (1.17) and Lemma 1.13. □

Theorem 1.45 *The mapping ζ in* (1.57) *is a homeomorphism of the topological space $\mathscr{Z}$ onto $\mathbb{R}$ equipped with the Euclidean topology.*

Proof We first prove that $\zeta : \mathscr{Z} \longrightarrow \mathbb{R}$ is one-to-one. Algorithm (1.32) shows that it is onto. If $a, b \in \mathscr{Z}$ and $\zeta(a) = \zeta(b) = x$ then $a_0 = b_0$. Indeed, on the one hand $a_0 - b_0 \in \mathbb{Z}$, but, on the other hand, by (1.57) we have

$$a_0 - b_0 = \mathop{\mathbf{K}}_{k=1}^{J(b)} \left(\frac{1}{b_k}\right) - \mathop{\mathbf{K}}_{k=1}^{J(a)} \left(\frac{1}{a_k}\right) \in (-1, 1).$$

Hence $a_0 = b_0$. If $a \neq b$ then we have the first integer $n > 0$ with $a_{n+1} \neq b_{n+1}$:

$$x_{n+1} = a_{n+1} + \mathop{\mathbf{K}}_{k=n+2}^{\infty} \left(\frac{1}{a_k}\right) \neq y_{n+1} = b_{n+1} + \mathop{\mathbf{K}}_{k=n+2}^{\infty} \left(\frac{1}{b_k}\right).$$

Let $P_n/Q_n = [a_0; a_1, a_2, \dots, a_n]$. Then by (1.17) and Lemma 1.13

$$x = \zeta(a) = a_0 + \frac{1}{a_1} {}_{+\cdots+} \frac{1}{a_n + 1/x_{n+1}} = \frac{x_{n+1} P_n + P_{n-1}}{x_{n+1} Q_n + Q_{n-1}}$$
$$\neq \frac{y_{n+1} P_n + P_{n-1}}{y_{n+1} Q_n + Q_{n-1}} = b_0 + \frac{1}{b_1} {}_{+\cdots+} \frac{1}{b_n + 1/y_{n+1}} = \zeta(b) = x,$$

resulting in a contradiction.

Now let $b^{(j)} \to a$ in $\mathscr{Z}$. If $n = J(a) < \infty$ then either $b^{(j)}$ stabilizes at some point and there is nothing to prove or $b_k^{(j)} = a_k$ for $k = 0, 1, \dots, n$, $j > J_0$, and $\lim_j b_{n+1}^{(j)} = +\infty$. Then by (1.17)

$$\lim_j \zeta(b^{(j)}) = \lim_j \frac{\zeta(b^{(j)})_{n+1} P_n + P_{n-1}}{\zeta(b^{(j)})_{n+1} Q_n + Q_{n-1}} = \frac{P_n}{Q_n} = \zeta(a),$$

since $\zeta(b^{(j)})_{n+1} > b_{n+1}^{(j)} \to +\infty$.

If $J(a) = \infty$ then for every integer n there is an N_n such that $b_k^j = a_k$ for $0 \leqslant k \leqslant n$ and $j > N_n$. By Lemma 1.44 all numbers $\zeta(b^j)$ with $j > N_n$ are placed between the convergents P_n/Q_n and P_{n-1}/Q_{n-1}. Since by Theorem 1.11 $\lim_n P_n/Q_n = \zeta(a)$, we obtain that $\lim_j \zeta(b^j) = \zeta(a)$.

If $\zeta(a) \notin \mathbb{Q}$ then $J(a) = \infty$. If P_n/Q_n are convergents to $\zeta(a)$ then for every n the number $\zeta(a)$ is inside the open interval with ends at P_n/Q_n and $(P_n + P_{n-1})/(Q_n + Q_{n-1})$ (Lemma 1.44). If $\zeta(b^j) \to \zeta(a)$ then the numbers $\zeta(b_j)$ for all $j > N_n$ are in the same open interval, which implies by Lemma 1.44 that $b^j \to a$ in $\mathscr{Z}$.

If $\zeta(a) \in \mathbb{Q}$ then $\zeta(a) = [a_0; a_1, a_2, \dots, a_n]$ with $a_n > 1$. If n is even then by (1.43) and (1.44)

$$\frac{P_{n-2}}{Q_{n-2}} < \frac{P_{n-1}+P_{n-2}}{Q_{n-1}+Q_{n-2}} < \frac{P_n}{Q_n} < \frac{P_n+P_{n-1}}{Q_n+Q_{n-1}} < \frac{P_{n-1}}{Q_{n-1}}.$$

If $\zeta(b^{(j)}) \to \zeta(a) = P_n/Q_n$ from the right, then $\zeta(b^{(j)})$ must be between P_n/Q_n and $(P_n+P_{n-1})/(Q_n+Q_{n-1})$ for all $j > J_n$. Hence by Lemma 1.44 $b_k = a_k$ for $0 \leqslant k \leqslant n$, $j > J_n$. By Lemma 1.13 $\lim_j \zeta(b^{(j)})_{n+1} = +\infty$, implying $\lim_j b^{(j)}_{n+1} = +\infty$. If $\zeta(b^{(j)}) \to \zeta(a) = P_n/Q_n$ from the left, then $\zeta(b^{(j)})$ must be between $(P_{n-1}+P_n)/(Q_{n-1}+Q_n)$ and P_n/Q_n for all $j > J_n$. By Lemma 1.13 $\lim_j \zeta(b^{(j)})_n = a_n$ and therefore either $b^{(j)} = a$ or $\lim_j b^{(j)}_{n+1} = +\infty$. Odd values of n are considered similarly. □

By Theorem 1.45 regular continued fractions have the important property of *correspondence*. Every real number $x \in \mathbb{R}$ can be uniquely expanded into the finite or infinite sum

$$x = x_0 + \sum_{k \geqslant 1} \frac{x_k}{10^k}, \tag{1.58}$$

where $x_0 \in \mathbb{Z}$, $0 \leqslant x_k < 10$ are integers. The case $x_k = 9$ for every $k > K$ is excluded. If $\zeta(b) = x$ then the continued fraction

$$b_0 + \mathop{\mathbf{K}}_{k \geqslant 1} \left(\frac{1}{b_k}\right)$$

corresponds to the series (1.58) in the sense that P_n/Q_n matches the first decimal places of ξ if $n > N_n$. It is this property of correspondence that allows one to compute partial denominators one by one using more accurate decimal representations of irrational numbers.

1.4 Jean Bernoulli sequences

24 Periodic Jean Bernoulli sequences. In his book on astronomy (1772), Jean Bernoulli considered the first differences

$$r_n = r_n(\theta, \delta) = [(n+1)\theta + \delta] - [n\theta + \delta], \qquad n \in \mathbb{Z},$$

of the integer parts of an arithmetic progression $\{n\theta + \delta\}_{n \in \mathbb{Z}}$, where θ and δ are real numbers.[8] Bernoulli observed that for rational θ these sequences are periodic. Since $[k + \xi] = k + [\xi]$ if $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$,

$$r_n(\theta, \delta) = [\theta] + r_n(\{\!\{\theta\}\!\}, \{\!\{\delta\}\!\}). \tag{1.59}$$

[8] In fact Bernoulli actually considered the function $F(x)$ that gives the closest integer to x. Since $F(x) = [x + 1/2]$ we may consider $[x]$ instead, including the term 1/2 with in δ.

This simple formula reduces the study of $\{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$ to that of $\theta,\delta\in[0,1)$. If $\theta\neq 0$ then $0<(n+1)\theta+\delta-n\theta-\delta=\theta<1$ and $\{r_n\}_{n\in\mathbb{Z}}$ is a sequence of 0 and 1. For $\theta=0$ it is the sequence of zeros only. In general $\{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$ is a sequence of $[\theta]$ and $[\theta]+1$; see (1.59).

Theorem 1.46 (Jean Bernoulli) *A sequence $\{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$ is periodic if and only if $\theta=[\theta]+p/q$, $(p,q)=1$, is rational. Here q is the length of the period of $\{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$ and p is the number of terms $[\theta]+1$ in the period.*

Proof Let $\theta=[\theta]+p/q$, $(p,q)=1$. If $p=0$ then $\{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$ is a constant sequence by (1.59). If $p>0$ then $0<\{\!\{\theta\}\!\}=p/q<1$, implying $q>1$. If $n\in\mathbb{Z}$ then, by the Euclidean algorithm, $n=mq+k$, $0\leqslant k<q$. Since $q\theta\in\mathbb{Z}$,

$$r_n=[(n+1)\theta+\delta]-[n\theta+\delta]=[(k+1)\theta+\delta]-[k\theta+\delta]=r_k,$$

implying the periodicity of $\{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$. Let us prove the converse.

Lemma 1.47 *For every real θ,*

$$\theta=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}r_k(\theta,\delta). \tag{1.60}$$

Proof Since

$$\begin{aligned}\sum_{k=1}^{n}r_k(\theta,\delta)&=[2\theta+\delta]-[\theta+\delta]+\cdots+[(n+1)\theta+\delta]-[n\theta+\delta]\\&=[(n+1)\theta+\delta]-[\theta+\delta]\\&=n\theta+\{\!\{\theta+\delta\}\!\}-\{\!\{(n+1)\theta+\delta\}\!\},\end{aligned}$$

(1.60) follows from a direct computation. □

If $\{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$ is periodic then the sum in (1.60) splits into a sum of equal finite blocks of period length q plus a finite number of the negligible terms at the beginning and end. Hence $\theta=[\theta]+p/q$, where p is the number of times that $[\theta]+1$ occurs in the period. □

In view of (1.60) the number θ is called the *mean value* of r.

To clarify the dependence of $\{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$ on δ in the periodic case let us put $\theta=[\theta]+p/q$, $(p,q)=1$.

Lemma 1.48 *If n, q, $q>1$ are integers and $\delta\in[j/q,(j+1)/q)]$, $j\in\mathbb{Z}$, then*

$$\left[\frac{n}{q}+\delta\right]=\left[\frac{n+j}{q}\right]. \tag{1.61}$$

Proof By the Euclidean algorithm, $n+j = mq+s$. Since $0 \leqslant s < q$ and $\delta = j/q+\delta'$, $\delta' < 1/q$, we have $s/q+\delta' < 1$, implying that both sides of (1.61) are equal to m. □

Theorem 1.49 *For $\theta = [\theta]+p/q$, $(p, q) = 1$, all periodic Jean Bernoulli sequences with mean value θ are listed by $\{r_{n+k}(\theta, 0)\}_{n\in\mathbb{Z}}$, $k = 0, 1, \dots, q-1$. If $\delta \in [j/q, (j+1)/q)$, $j = 0, 1, \dots, q-1$, then*

$$r_n(\theta, \delta) = r_{n+k}(\theta, 0) \quad \textit{where} \quad \left\{\!\!\left\{ \frac{kp}{q} \right\}\!\!\right\} = \frac{j}{q}.$$

Proof By Lemma 1.48 every sequence $\{r_n(\theta, \delta)\}_{n\in\mathbb{Z}}$ is equal to one of the q sequences $\{r_n(\theta, j/q)\}_{n\in\mathbb{Z}}$, $j = 0, \dots, q-1$. However, the q sequences $\{r_{n+k}(\theta, 0)\}_{n\in\mathbb{Z}}$, $k = 0, \dots, q-1$, also correspond to $\theta = p/q$. Since obviously $r_{n+k}(\theta, 0) = r_n(\theta, \{\!\{kp/q\}\!\})$, the formula $kp = mq+j$ establishes equality between them. □

Theorem 1.49 implies a simple algorithm, which is called the *Jean Bernoulli algorithm*, to determine whether a given sequence of a and $a+1$, $a \in \mathbb{Z}$, is a Jean Bernoulli sequence. First we determine the period q of r and then count the number p of times that $[\theta]+1$ occurs in the period, to find the mean value $\theta = p/q$. If one of the q shifts $\{r_{n+k}(\theta, 0)\}_{n\in\mathbb{Z}}$, $q = 0, \dots, q-1$ coincides with a tested sequence then it is a Jean Bernoulli sequence; otherwise it is not.

Another consequence of Theorem 1.49 is that, in theory, when θ is rational one may restrict δ to $\delta = 0$. However, as it is clear from below, it is technically very convenient to consider all, especially irrational, δ in (1, 0).

25 The structure of Jean Bernoulli sequences. If $0 < \theta < 1$ then the obvious identity $j = \theta(j-\delta)/\theta+\delta$, $j \in \mathbb{Z}$, indicates that the integers

$$n_j = \left[\frac{j-\delta}{\theta}\right], \qquad j \in \mathbb{Z}, \tag{1.62}$$

are good candidates for the solutions to the equation $r_n = 1$. Since $1/\theta > 1$, the sequence $\{n_j\}_{j\in\mathbb{Z}}$ increases. The sequence (1.62) for $j \geqslant 1$ is also called a Beatty sequence $B(\theta, \delta)$. See Ex. 1.29 for Beatty's theorem.

Lemma 1.50 *For every $j \in \mathbb{Z}$,*

$$[(n_j+1)\theta+\delta] = j, \qquad [n_j\theta+\delta] = \begin{cases} j-1 & \textit{if } \{\!\{(j-\delta)/\theta\}\!\} > 0, \\ j & \textit{if } \{\!\{(j-\delta)/\theta\}\!\} = 0. \end{cases}$$

Proof Since $(j-\delta)/\theta = [(j-\delta)/\theta]+\{\!\{(j-\delta)/\theta\}\!\}$, we see that

$$j = n_j\theta+\delta+\{\!\{(j-\delta)/\theta\}\!\}\theta. \tag{1.63}$$

It follows that $[n_j\theta+\delta]=j$ if $\{\!\{(j-\delta)/\theta\}\!\}=0$.[9] If $\{\!\{(j-\delta)/\theta\}\!\}>0$ then $0<\{\!\{(j-\delta)/\theta\}\!\}\theta<1$. Hence $j-1<n_j\theta+\delta<j$, implying that $[n_j\theta+\delta]=j-1$. By (1.63),

$$j=(n_j+1)\theta+\delta-(1-\{\!\{(j-\delta)/\theta\}\!\})\,\theta,$$

showing that

$$\{\!\{(n_j+1)\theta+\delta\}\!\}-(1-\{\!\{(j-\delta)/\theta\}\!\})\,\theta=j-[(n_j+1)\theta+\delta]$$

is an integer. Since the left-hand side of the above identity is a number in $(-1,1)$, the integer right-hand side must be zero. □

Corollary 1.51 *For* $0<\theta<1$ *and* $0\leqslant\delta<1$,

$$r_{n_j}(\theta,\delta)=[(n_j+1)\theta+\delta]-[n_j\theta+\delta]=\begin{cases}1 & \textit{if } \{\!\{(j-\delta)/\theta\}\!\}>0,\\ 0 & \textit{if } \{\!\{(j-\delta)/\theta\}\!\}=0.\end{cases} \tag{1.64}$$

A computation of the sum

$$\begin{aligned}\sum_{k=n_j+1}^{n_{j+1}-1} r_k(\theta,\delta)&=[(n_j+2)\theta+\delta]-[(n_j+1)\theta+\delta]+[(n_j+3)\theta+\delta]-\cdots\\ &=[n_{j+1}\theta+\delta]-[(n_j+1)\theta+\delta]=\begin{cases}0 & \text{if } \{\!\{(j+1-\delta)/\theta\}\!\}>0,\\ 1 & \text{if } \{\!\{(j+1-\delta)/\theta\}\!\}=0,\end{cases}\end{aligned} \tag{1.65}$$

shows that r_n vanishes in (n_j,n_{j+1}) if $\{\!\{(j+1-\delta)/\theta\}\!\}>0$.

Notice that for an irrational θ the equality $\{\!\{(j+1-\delta)/\theta\}\!\}=0$ may happen only for one j. If θ is rational and δ irrational then this equality never occurs. By Theorems 1.46 and 1.49, for every periodic Jean Bernoulli sequence $\{r_n\}_{n\in\mathbb{Z}}$ of period q there is a unique rational $\theta=p/q$, $p/q\in(0,1)$, and an irrational δ such that $r_n=r_n(\theta,\delta)$ for every $n\in\mathbb{Z}$. Hence we may always assume without loss of generality that $\theta\in\mathbb{Q}$, $\delta\notin\mathbb{Q}$. By Corollary 1.51 and by (1.65) $r_n=1$ if and only if $n=n_j$ for some j.

This method applied with for very small $\delta>0$ allows one to prove a simple formula for $\{n_j\}_{j\in\mathbb{Z}}$ for $\{r_k(\theta,0)\}_{k\in\mathbb{Z}}$ in the periodic case.

Lemma 1.52 *If* $\theta=p/q\in(0,1)$ *then*

$$n_j=\left[\frac{jq}{p}\right],\qquad j=1,\dots,p-1,\quad n_p=q-1.$$

Proof The lemma follows by (1.62) and Theorem 1.49. □

[9] Notice that then $n_j\theta+\delta$ is an integer.

Jean Bernoulli sequences with irrational mean value θ for large values of the indices contain couples of adjusted arbitrarily long periods of rational Jean Bernoulli sequences.

Theorem 1.53 *Let p/q be a convergent to an irrational number θ and δ be an arbitrary number in* $[0, 1)$. *Then there is a nonnegative integer n such that*

$$r_{n+k}(\theta, \delta) = r_k(p/q, 0), \qquad -q+1 \leqslant k < q-1. \tag{1.66}$$

Proof Let us suppose first that $0 < \delta < 1$. By Theorem 1.42 there are infinitely many positive integers n such that $n\theta = N + \delta_n$, where N is a positive integer, $0 < \delta_n = 1 - \delta + \varepsilon_n < 1$ and ε_n is a very small positive number satisfying in addition

$$\{\!\{(n+k)\theta + \delta\}\!\} = \{\!\{k\theta + \delta_n + \delta\}\!\} = \{\!\{k\theta + \varepsilon_n\}\!\} = \{\!\{k\theta\}\!\} + \varepsilon_n$$

for $|k| \leqslant q$. Then for the integer parts and the same k's we have

$$[(n+k)\theta + \delta] = N + 1 + [k\theta + \varepsilon_n] = N + 1 + [k\theta].$$

It follows that $r_{n+k}(\theta, \delta) = r_k(\theta, 0)$, $|k| < q$. However

$$r_k(p/q, 0) = r_k(\theta, 0), \qquad -q+1 \leqslant k < q-1, \tag{1.67}$$

by (1.56), completing the proof both for $\delta > 0$ and $\delta = 0$. □

Notice that if $\delta = 0$ then we need $n = 0$, whereas if $0 < \delta < 1$ then n can be taken to be arbitrary large in modulus.

Periods of periodic Jean Bernoulli sequences have an important symmetry property.

Theorem 1.54 *If $\theta = [\theta] + p/q$, $p/q < 1$, $(p, q) = 1$ then the part $\{r_1, r_2, \ldots, r_{q-2}\}$ of $\{r_n(\theta, 0)\}_{n\in\mathbb{Z}}$ is symmetric:*

$$r_{-k}(\theta) = r_{q-k}(\theta) = r_{k-1}(\theta), \qquad k = 2, 3, \ldots, q-1. \tag{1.68}$$

Proof If $0 < \theta < 1$ then by (1.30) we have, for $j = 1, 2, \ldots$,

$$[-j\theta] = \begin{cases} -[j\theta] - 1 & \text{if } \{\!\{j\theta\}\!\} > 0, \\ -[j\theta] & \text{if } \{\!\{j\theta\}\!\} = 0. \end{cases} \tag{1.69}$$

Since $\{\!\{j\theta\}\!\} = \{\!\{jp/q\}\!\} > 0$ for $j = 1, 2, \ldots, q-1$, (1.69) implies that

$$r_{q-k} = r_{-k}(\theta) = [-(k-1)\theta] - [-k\theta] = [k\theta] - [(k-1)\theta] = r_{k-1}(\theta)$$

for $k = 2, \ldots, q-1$, by the periodicity of $\{r_n(\theta, 0)\}_{n\in\mathbb{Z}}$. □

26 Markoff's algorithm for periodic sequences. This algorithm was discovered originally by Markoff (1879, 1880) in a slightly different form. Markoff's algorithm is applied to periodic sequences to determine whether they are Jean Bernoulli sequences. It is based on the following theorem.

Definition 1.55 *An infinite sequence $\{r_n\}_{n\in\mathbb{Z}}$ of two integers a and $a+1$ is called a ceiling sequence if all roots of the equation $r_n = a+1$ make an increasing infinite sequence $\{n_k\}_{k\in\mathbb{Z}}$.*

Any periodic sequence of two values is a ceiling sequence.

Theorem 1.56 *Given a ceiling sequence $r = \{r_n\}_{n\in\mathbb{Z}}$ of a and $a+1$ let $\{n_j\}_{j\in\mathbb{Z}}$ be the increasing sequence of the solutions n to the equation $r_n = a+1$. Then $\{r_n\}_{n\in\mathbb{Z}}$ is periodic if and only if $s = \{s_j\}_{j\in\mathbb{Z}}$, $s_j = n_{j+1} - n_j$, is periodic. The period of s is smaller than the period of r.*

Proof Let L be the period of $\{r_n\}_{n\in\mathbb{Z}}$ and $n_1 < n_2 < \cdots < n_d$ be the complete list of the n_j in $[0, L)$. Since $\{r_n\}_{n\in\mathbb{Z}}$ takes two values, we must have $d < L$. Since $\{r_n\}_{n\in\mathbb{Z}}$ is periodic, we have $n_{k+d} = n_k + L$, $k \in \mathbb{Z}$. It follows that

$$\begin{aligned}
\ldots, s_0 &= n_1 - n_0 = n_1 - n_d + L = n_{d+1} - n_d = s_d,\\
s_1 &= n_2 - n_1 = n_{d+2} - n_{d+1} = s_{d+1},\\
s_2 &= s_{d+2}, \ldots, s_d = s_{2d}, \ldots
\end{aligned}$$

is periodic with period $d < L$. Suppose that s is periodic with period d. Let us consider the interval $[n_0, n_d)$ containing the numbers n_k with $k = 0, \cdots, d-1$. Its length $L = s_{jd+0} + \cdots + s_{jd+d-1}$ does not depend on j, by the periodicity of s. Moreover, the distance from n_{jd+k} to n_{jd} equals $n_k - n_0 = s_0 + \cdots + s_{k-1}$ for the same reason. This implies that L is the period of r. Since r is a sequence of two values, $L > d$. □

Markoff's algorithm (periodic case). This algorithm applies to a periodic sequence $\{r_n\}_{n\in\mathbb{Z}}$ of 0 and 1 with period L. It has four entries.

Entry 1. The sequence $\{r_n\}_{n\in\mathbb{Z}}$ determines an infinite increasing sequence $\{n_j\}_{j\in\mathbb{Z}}$ of the solutions n to the equation $r_n = 1$.

Entry 2. If $\{n_{j+1} - n_j\}_{j\in\mathbb{Z}}$ is a nonconstant sequence that is not a sequence of two integers $b > 0$ and $b+1$ then Markoff's algorithm fails.

Entry 3. If $n_{j+1} - n_j = b > 0$ for every $j \in \mathbb{Z}$ then Markoff's algorithm stops.

Entry 4. If $n_{j+1} - n_j = b + s_j$, where $\{s_j\}_{j\in\mathbb{Z}}$ is a sequence of 0 and 1, then $\{s_j\}_{j\in\mathbb{Z}}$ is periodic with period $M < L$ by Theorem 1.56.

Any sequence of two integers a_0 and $a_0 + 1$ has the form $\{a_0 + r_n\}_{n\in\mathbb{Z}}$, where $\{r_n\}_{j\in\mathbb{Z}}$ is a sequence of 0 and 1. Sending $\{r_n\}_{j\in\mathbb{Z}}$ to entry 1, we obtain an increasing sequence of integers $\{n_j\}_{j\in\mathbb{Z}}$. Inspecting the differences $n_{j+1} - n_j$, we send them to the corresponding entry of the algorithm. Thus the algorithm may fail, stop or be continued. In the last case we put $r_n^{(1)} = r_n$, $n_j^{(1)} = n_j$, $a_1 = b$, $r_j^{(2)} = s_j$, $j \in \mathbb{Z}$, and send $\{r_n^{(2)}\}_{n\in\mathbb{Z}}$ to entry 1 by putting $r_n = r_n^{(2)}$. This results in a new sequence $\{n_j\}_{j\in\mathbb{Z}}$, which is inspected in order to be sent to the corresponding entry. Again at this stage the algorithm stops or fails or can be continued. If it is continued, we put $n_j^{(2)} = n_j$, $a_2 = b$, $r_j^{(3)} = s_j$, $j \in \mathbb{Z}$. In the case where the algorithm can be continued the period of the outcome sequence strictly drops. Hence in a finite number of steps the *algorithm must either fail or stop*. If it stops in d steps then we obtain d sequences $\{r_n^{(k)}\}_{n\in\mathbb{Z}}$ of 0 and 1, d increasing integer sequences $\{n_j^{(k)}\}_{j\in\mathbb{Z}}$ and a finite set of positive integers $a_1, \dots, a_d$. These parameters of Markoff's algorithm are related by the formulas

$$n_{j+1}^{(k)} - n_j^{(k)} = a_k + r_j^{(k+1)}, \qquad j \in \mathbb{Z}, \quad k = 1, \dots, d, \tag{1.70}$$

with the understanding that $r_j^{(d+1)} \equiv 0$.

Notice that if Markoff's algorithm stops then $b \geqslant 2$. Suppose on the contrary that $b = 1$. Then $\{n_j\}_{j\in\mathbb{Z}}$ is simply an enumeration of $\mathbb{Z}$, implying that the sequence r, to which Markoff's algorithm was applied at the final step is identically 1. But this is impossible since otherwise the algorithm would have stopped earlier. This reminds us of the situation with finite regular continued fractions whose last partial denominator always exceeds 2. This is not just a coincidence, as the following example shows.

Let us apply Markoff's algorithm to $r_n^{(1)} = r_n(\theta, \delta)$, where $\theta \in \mathbb{Q}$, $0 < \theta < 1$. Then $a_0 = 0$. Any such θ is uniquely represented by

$$\theta = \frac{1}{a_1} \, \frac{1}{+a_2} + \cdots + \frac{1}{a_d}, \qquad a_d \geqslant 2.$$

If $d \geqslant 2$ then $1/\theta = \theta_1 = a_1 + 1/\theta_2$. By (1.62) the solutions $n_j = n_j^{(1)}$ to the equation $r_n^{(1)} = 1$ are given by

$$n_j = \left[\frac{j-\delta}{\theta}\right] = \left[ja_1 - \delta a_1 + \frac{j-\delta}{\theta_2}\right] = ja_1 + \left[\frac{j}{\theta_2} - \delta\theta_1\right].$$

It follows that

$$n_{j+1} - n_j = a_1 + r_j(1/\theta_2, -\delta\theta_1). \tag{1.71}$$

If $d = 1$ then we may formally put $\theta_2 = +\infty$ in (1.71) to obtain $n_{j+1} - n_j = a_1$; then $\theta = 1/a_1$ and Markoff's algorithm stops. If this is not the case then we put $r_j^{(2)} = r_j(1/\theta_2, -\delta\theta_1)$. By (1.71) the sequence $\{r_j^{(2)}\}_{j\in\mathbb{Z}}$ is periodic since $1/\theta_2 \in \mathbb{Q}$. Moreover, since the denominator of θ_2 is smaller than that of $\theta = 1/\theta_1$, its period is less than the period of $\{r_j(\theta, \delta)\}_{j\in\mathbb{Z}}$ by Theorem 1.46, which also follows by Theorem 1.56. Finally $-\delta\theta_1 \notin \mathbb{Q}$ and Markoff's algorithm can be continued.

Hence at step k we obtain positive integers $a_1, a_2, \dots, a_k$ and a periodic sequence $\{r_j^{(k+1)}\}_{j\in\mathbb{Z}}$ of 0's and 1's:

$$\begin{aligned} r_j^{(k+1)} &= r_j\left(\frac{1}{\theta_{k+1}},\ (-1)^k\delta\theta_1\theta_2\cdots\theta_k\right), \\ n_{j+1}^{(k)} - n_j^{(k)} &= r_j\left(a_k + \frac{1}{\theta_{k+1}},\ (-1)^k\delta\theta_1\theta_2\cdots\theta_k\right). \end{aligned} \tag{1.72}$$

In d steps, not exceeding the length of the period of $\{r_j(\theta, \delta)\}_{j\in\mathbb{Z}}$, all partial denominators $a_1, a_2, \dots, a_d$ of the continued fraction for θ are determined. At the last step, $n_{j+1} - n_j$ corresponds to entry 3 of the algorithm and it stops. Formula (1.72) holds even for $k = d$ if one assumes that $\theta_{d+1} = \infty$. Theorem 1.56 extends to Jean Bernoulli sequences.

Theorem 1.57 *Given a ceiling sequence $r = \{r_n\}_{n\in\mathbb{Z}}$ of a and $a+1$, let $\{n_j\}_{j\in\mathbb{Z}}$ be the increasing sequence of the solutions n to the equation $r_n = a+1$. Then $\{r_n\}_{n\in\mathbb{Z}}$ is a Jean Bernoulli sequence if and only if $s = \{s_j\}_{j\in\mathbb{Z}}$, $s_j = n_{j+1} - n_j$, is a Jean Bernoulli sequence.*

Proof If r is a Jean Bernoulli sequence then s is a Jean Bernoulli sequence by (1.71).

If $s_j \equiv 0$ then $\{n_j\}_{j\in\mathbb{Z}}$ is an arithmetic progression with difference a. It follows that $r_k = r_k(1/a, \delta)$. Suppose now that $s_k = a + r_k(\xi, \alpha)$, $a > 0$, where both ξ and α are in $(0, 1)$. We define θ and δ by $\theta = 1/(a+\xi)$, $\delta = -\alpha\theta$. Then $\theta_1 = 1/\theta$, $\theta_2 = 1/\xi$. Since $r_j(1/\theta_2, -\delta\theta_1) = r_j(\xi, \alpha)$, formula (1.71) implies that the indicator of $\{n_j\}_{j\in\mathbb{Z}}$ is a shift of $\{r_k(\theta, \delta)\}_{k\in\mathbb{Z}}$.[10] The formula $r_{j+k}(\theta, \delta) = r_j(\theta, k\theta + \delta)$ completes the proof. □

Theorem 1.58 (Markoff's algorithm) *Let $\{r_n^{(1)}\}_{n\in\mathbb{Z}}$ be a periodic sequence of integers a_0 and $a_0 + 1$ with period L_1 such that Markoff's algorithm never fails. If it stops in d steps, i.e. $n_{j+1}^{(d)} - n_j^{(d)} \equiv a_d$, then $\{r_n^{(1)}\}_{n\in\mathbb{Z}}$ is a Jean Bernoulli sequence corresponding to*

$$\theta = \frac{1}{a_1} \frac{}{+} \frac{1}{a_2} \frac{}{+} \frac{1}{a_3} \frac{}{+\cdots+} \frac{1}{a_d},$$

where $a_1, a_2, \cdots, a_d$ are the parameters of Markoff's algorithm.

Proof Since by the assumption Markoff's algorithm never fails if started with $\{r_n^{(1)}\}_{n\in\mathbb{Z}}$, it must stop in a finite number d of iterations. The parameters of Markoff's algorithm are related by (1.70). Applying Theorem 1.57 iteratively in the reversed order k, $k = d$, $d-1, \dots, 1$, we obtain that $\{r_n^{(1)}\}_{j\in\mathbb{Z}}$ is a Jean Bernoulli sequence corresponding to the rational number $\theta = [0; a_1, \dots, a_d]$. □

[10] By the indicator of $\{n_i\}_{j\in\mathbb{Z}}$ is meant the sequence that equals 1 at the point n_j and 0 otherwise.

Let us illustrate this algorithm for the periodic sequence $\{1, 0, 1, 1, 0\}$ with period 5:

$$\begin{array}{rcccccccccccccc}
r_j^{(1)}: & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
n_{j+1}^{(1)} - n_j^{(1)}: & 2 & & 1 & 2 & & 2 & & 1 & 2 & & 2 & & 1 & a_1 = 1 \\
n_{j+1}^{(2)} - n_j^{(2)}: & 2 & & & 1 & & 2 & & & 1 & & 2 & & & a_2 = 1 \\
n_{j+1}^{(3)} - n_j^{(3)}: & 2 & & & & & 2 & & & & & 2 & & & a_3 = 2 \\
r_j^{(4)}: & 0 & & & & & 0 & & & & & 0 & & & \text{stop}
\end{array}$$

Hence

$$\theta = \frac{1}{1} {}_{+} \frac{1}{1} {}_{+} \frac{1}{2} = \frac{3}{5}.$$

Markoff's algorithm can be also applied to nonperiodic Jean Bernoulli sequences $\{r_n(\theta, \delta)\}_{n\in\mathbb{Z}}$. In this case it will never stop or fail and will finally recover all partial denominators of $\theta = [0; a_1, a_2, \ldots]$. Formulas (1.70) and (1.72) are valid in this case as well and take an especially attractive form for $\delta = 0$. Hence if $\delta = 0$ then all further sequences obtained at the steps of Markoff's algorithm also correspond to $\delta = 0$.

27 Markoff conditions for Jean Bernoulli sequences. It looks as though Theorems 1.46, 1.49 and 1.58 provide an exhaustive description of periodic Jean Bernoulli sequences. However, they are all indirect and require an application of some algorithm, which is possible to run in any concrete case but is difficult to apply universally. In his Master's thesis A. Markoff (1879, 1880) gave an internal description of periodic Jean Bernoulli sequences closely related to the regular continued fraction of θ. A special paper (1882) on this topic was published by him a little later. These results of Markoff turned out to be very important in the theory of binary quadratic forms having a positive determinant, see Section 2.5, which is a deep extension of Lagrange's theory presented in §**20** above.

By Theorem 1.49 every rational θ essentially determines a unique Jean Bernoulli sequence (all other sequences corresponding to θ are obtained by shifts). Therefore a description of these sequences in terms of their differences may exist. Since $[n\theta + \delta] = n\theta + \delta - \epsilon_n$, $0 \leqslant \epsilon_n < 1$, $n \in \mathbb{Z}$,

$$\begin{aligned} &r_{n+p} + r_{n+p-1} + \cdots + r_{n+2} + r_{n+1} \\ &\qquad = [(n+p+1)\theta + \delta] - [(n+1)\theta + \delta] = p\theta + \epsilon_{n+1} - \epsilon_{n+p+1} \end{aligned}$$

and similarly

$$\begin{aligned} &r_{n-p+1} + r_{n-p+2} + \cdots + r_{n-1} + r_n \\ &\qquad = [(n+1)\theta + \delta] - [(n-p+1)\theta + \delta] = p\theta + \epsilon_{n-p+1} - \epsilon_{n+1}. \end{aligned}$$

Subtracting the second identity from the first we obtain that the integer

$$\begin{aligned}(r_{n+p} - r_{n-p+1}) + \cdots + (r_{n+2} - r_{n-1}) + (r_{n+1} - r_n)\\ = 2\epsilon_{n+1} - \epsilon_{n+p+1} - \epsilon_{n-p+1}\end{aligned} \tag{1.73}$$

is in $(-2, 2)$ and hence equals either ± 1 or 0.

Definition 1.59 *An infinite integer sequence $\{r_n\}_{n\in\mathbb{Z}}$ is called a Markoff sequence if it satisfies the following properties:*

(1) *the differences $r_n - r_{n+1}$ may have only the values $+1$, 0, -1;*
(2) *if $r_n - r_{n+1} = +1$ then the first nonzero term $r_{n+1+p} - r_{n-p}$ in*

$$r_{n+2} - r_{n-1}, \quad r_{n+3} - r_{n-2}, \quad \ldots, \quad r_{n+1+p} - r_{n-p},$$

if it exists, is positive;
(3) *if $r_n - r_{n+1} = -1$ then the first nonzero term $r_{n+1+p} - r_{n-p}$ in*

$$r_{n+2} - r_{n-1}, \quad r_{n+3} - r_{n-2}, \quad \ldots, \quad r_{n+1+p} - r_{n-p},$$

if it exists, is negative.

Conditions (2) and (3) mean that whenever $r_n - r_{n+1} \neq 0$ the differences $r_{n+1+p} - r_{n-p}$, $p > 0$, of the terms equally distant from $\{n, n+1\}$ remain zero as p increases until they take the sign opposite to that of $r_{n+1} - r_n$.

Definition 1.60 *If $r_n - r_{n+1} \neq 0$ then $l_n(r)$ is the number p of terms in the corresponding Markoff sequence, $l(r) = \sup_n l_n(r)$.*

It is clear that $l_n(r) - 1$ is the maximal length of the zeros in the Markoff sequences (2) and (3). The characteristics $l(r)$ play a significant role in Markoff's theory of quadratic forms having a positive determinant.

Together with $l_n(r)$ we consider the interval $\omega_n(r) = [n-p, n+1+p]$, $p = l_n(r)$, containing all indices of r involved in the Markoff series at $[n, n+1]$. It is clear that the length of $\omega_n(r) = 2l_n(r) + 1$.

Let us observe that if $u = \{u_n\}_{n\in\mathbb{Z}}$ is a Markoff sequence of a and $a+1$ then $v = \{v_n\}_{n\in\mathbb{Z}}$, $v_n = 2a + 1 - u_n$, is also a Markoff sequence of a and $a+1$. The sequence $v = \tilde{u}$ is called the Markoff sequence *conjugate* to u.

If θ and δ satisfy $m\theta + \delta \notin \mathbb{Z}$ for every $m \in \mathbb{Z}$ then the formula

$$2[\theta] + 1 - r_n(\theta, \delta) = r_n\,(2[\theta] + 1 - \theta,\ 1 - \delta) \tag{1.74}$$

shows that the conjugate to $\{r_n(\theta, \delta)\}_{n\in\mathbb{Z}}$ is a Jean Bernoulli sequence too. Theorem 1.49 and (1.74) show that the conjugate to a periodic Jean Bernoulli sequence is a periodic Jean Bernoulli sequence. In this case $\theta = [\theta] + p/q$ is rational and δ may be taken as irrational. If $\theta \notin \mathbb{Q}$ then the combination $m\theta + \delta$ may be in $\mathbb{Z}$ for at most one m. If this is the case then the sequence $\{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$ is a shift of $\{r_n(\theta)\}_{n\in\mathbb{Z}}$, $r_n = r_n(\theta, 0)$. Suppose that $[\theta] = 0$. Then formula (1.74) shows that

$$r_n(\theta) = 1 - r_n(1-\theta), \qquad n \neq -1, 0. \tag{1.75}$$

Hence the conjugate sequence to $\{r_n(1-\theta)\}_{n\in\mathbb{Z}}$ coincides with $\{r_n(\theta)\}_{n\in\mathbb{Z}}$ for $n \neq -1, 0$. Theorem 1.61 shows therefore that the conjugate to $\{r_n(\theta)\}_{n\in\mathbb{Z}}$ is not a Jean Bernoulli sequence.

Theorem 1.61 *If $r_n(\theta,\delta) = r_n(\theta',\delta')$ for $n \geqslant c$ then this is true for $n < c$ and $\theta = \theta'$.*

Proof The equality $\theta = \theta'$ follows by (1.60). If $\theta \in \mathbb{Q}$, then both sequences are periodic and therefore coincide for $n < c$. If θ is irrational and $0 < \theta < 1$ then applying a shift if necessary we may assume that $c = 0$. We have

$$\sum_{k=0}^{n-1} r_k(\theta,\delta) = [n\theta + \delta],$$

implying that $[n\theta+\delta] = [n\theta+\delta']$. Suppose that $0 \leqslant \delta < \delta' < 1$. By Theorem 1.42 there is an integer n and $0 < \epsilon_n < \delta' - \delta$ such that $n\theta = N + 1 - \delta - \epsilon_n$, $N \in \mathbb{Z}$. It follows that

$$[n\theta+\delta] = [N+1-\epsilon_n] = N,$$
$$[n\theta+\delta'] = [N+1+\delta'-\delta-\epsilon_n] = N+1,$$

which is a contradiction. □

Jean Bernoulli sequences constitute a subclass of *oscillating* Markoff sequences.

Definition 1.62 *A sequence $\{r_n\}_{n\in\mathbb{Z}}$ of two integers a and $a+1$ is called oscillating if the set $\{n : r_n - r_{n+1} \neq 0\}$ is infinite in both directions.*

There are two types of nonoscillating Markoff sequences:

$$r_n = \begin{cases} a+1 & \text{if } n = k, \\ a & \text{if } n \neq k \end{cases} \qquad \text{and} \qquad r_n = \begin{cases} a & \text{if } n = k, \\ a+1 & \text{if } n \neq k. \end{cases} \tag{1.76}$$

In both cases $r_{k-1} - r_k = \mp 1$ and $r_k - r_{k+1} = \pm 1$, other differences $r_l - r_m$ being zero. These sequences are not Jean Bernoulli sequences, see Ex. 1.25, but they are obviously conjugate to each other. In what follows, sequences defined by (1.76) are called *triangle sequences*. More precisely, the left-hand sequences in (1.76) are called triangle sequences of type $(a, a+1)$ and the right-hand sequences in (1.76) are called triangle sequences of type $(a+1, a)$. Notice that the first parameter in a type indicates the value taken infinitely often. Any triangle sequence of type $(a+1, a)$ is a ceiling sequence, whereas that of type $(a, a+1)$ is not.

Theorem 1.63 *Any Jean Bernoulli sequence $\{r_n\}_{n\in\mathbb{Z}}$ is an oscillating Markoff sequence.*

Proof Putting $p = 1$ in (1.73), we obtain that $r_n - r_{n+1}$ can only be 0 or ± 1. If $r_n - r_{n+1} = +1$ and $r_{n+p} - r_{n+1-p} \neq 0$, $p > 1$, is the first nonzero term then $(r_{n+p} - r_{n-p+1}) - 1$ is either 0 or ± 1 by (1.73). It cannot be -1, since $r_{n+p} - r_{n-p+1} \neq 0$, and therefore $r_{n+p} - r_{n-p+1} > 0$. The case $r_n - r_{n+1} = -1$ is considered similarly. It follows that $\{r_n\}_{n\in\mathbb{Z}}$ is a Markoff sequence. Any constant sequence corresponds to the Jean Bernoulli sequence $[(n+1)a + \delta] - [na + \delta] = a$ with integer $\theta = a$ and is oscillating by definition. If $\theta \in \mathbb{Q} \setminus \mathbb{Z}$ then $\{r_n\}_{n\in\mathbb{Z}}$ is periodic and is not constant. Hence $r_n - r_{n+1} \neq 0$ for infinitely many n in both directions. Finally, suppose that θ is irrational but $r_n - r_{n+1} \neq 0$ for a finite number of n's in one of the two possible directions. Replacing θ by $-\theta$ if necessary, we may assume that $r_n = a \in \mathbb{Z}$ for all $n > N$. Then $\theta = a \in \mathbb{Z}$ by Lemma 1.47, contradicting the assumption that θ is irrational. □

For triangle Markoff sequences there is obviously an n such that $l_n(r) = l_{n+1}(r) = +\infty$. This index n is the first preceding the index of the vertex. We find now $l(r)$ for periodic sequences.

Theorem 1.64 *If $\{r_n\}_{n\in\mathbb{Z}}$ is a periodic Jean Bernoulli sequence of period q then $l_n(r) < q$ for every n such that $r_n - r_{n+1} = \pm 1$. There exists n such that $l_n(r) = q - 1$. In particular $l(r) = q - 1$.*

Proof By Theorem 1.49 any periodic Jean Bernoulli sequence r with period q is a shift of $\{r_n(p/q, 0)\}_{n\in\mathbb{Z}}$ for some integer p. Since Markoff's conditions are shift invariant, we may assume without loss of generality that $r = \{r_n(p/q, 0)\}_{n\in\mathbb{Z}}$. If $q = 1$ then $\{r_n\}_{n\in\mathbb{Z}}$ is a constant sequence and therefore $r_n - r_{n+1} = \pm 1$ never occurs. If $q > 1$ then $r_{q-1} = 1$ and $r_q = 0$ by Lemma 1.52, implying that $r_{-1} - r_0 = r_{q-1} - r_q = 1$. Let $r_n - r_{n+1} = 1$. Then $r_{n-q} = r_n$ and $r_{n+1+q} = r_{n+1}$ by periodicity. It follows that $r_{n+1+q} - r_{n-q} = r_{n+1} - r_n = -1$. But $\{r_n\}_{n\in\mathbb{Z}}$ is a Markoff sequence by Theorem 1.63. Therefore $r_{n+1+p} - r_{n-p} > 0$ for some $p < q$. It follows that $l_n(r) \leqslant q - 1$. However, if $n = -1$ then Theorem 1.54 shows that $l_{-1}(r) = q - 1$. The case $r_n - r_{n+1} = -1$ reduces to that obtained on replacing p/q by $(q-p)/p$. □

Theorem 1.65 *If $r = \{r_n(\theta, \delta)\}_{n\in\mathbb{Z}}$ is a nonperiodic Jean Bernoulli sequence then $l(r) = \infty$. If $n \in \mathbb{Z}$ then $l_n(r) = \infty$ if and only if $(n+1)\theta + \delta \in \mathbb{Z}$.*

Proof Equality $l(r) = +\infty$ follows by Theorems 1.53 and 1.64. For every $p \geqslant 0$,

$$r_{n-p} = [-p\theta + (n+1)\theta + \delta] - [-p\theta - \theta + (n+1)\theta + \delta],$$
$$r_{n+1+p} = [p\theta + \theta + (n+1)\theta + \delta] - [p\theta + (n+1)\theta + \delta].$$

If $(n+1)\theta + \delta \in \mathbb{Z}$ then on the one hand $r_{n+1+p} = r_{n-p}$ for every $p \geqslant 1$ by (1.30). On the other hand, putting $p = 0$ we obtain that $r_n = -[-\theta] = 1$ and $r_{n+1} = 0$. Hence

$r_n - r_{n+1} = 1$ and $l_n(r) = \infty$. Let $\delta' = \{\{(n+1)\theta + \delta\}\} \neq 0$. Then

$$r_n = [\delta'] - [-\theta + \delta'] = -[-\theta + \delta'] = 1 \Leftrightarrow 0 < \delta' < \theta,$$

$$r_{n+1} = [\theta + \delta'] - [\delta'] = [\theta + \delta'] = 0 \Leftrightarrow \delta' + \theta < 1.$$

If $r_n - r_{n+1} = +1$ and $r_{n+1+p} = r_{n-p}$ for every $p \geqslant 1$ then

$$[p\theta + \theta + \delta'] - [p\theta + \delta'] = [p\theta + \theta - \delta'] - [p\theta - \delta']. \tag{1.77}$$

By Theorem 1.42 any number $x \in (0, 1)$ can be approximated by fractional parts $\{\{p\theta\}\}$. However, the shifts of $[x]$ which are present in (1.77) may have altogether not more than four points of discontinuity. It follows that

$$[x + \theta + \delta'] - [x + \delta'] = [x + \theta + \delta''] - [x + \delta''], \tag{1.78}$$

where $\delta'' = 1 - \delta' \in (0, 1)$ for every $x \in (0, 1)$ except for possibly four values. For small $x > 0$ the left-hand side is zero, since both $\theta + \delta'$ and δ' are in $(0, 1)$. Similarly $[x + \delta''] = 0$ for small $x > 0$, since $0 < \delta'' < 1$. However, $\delta' < \theta$ and therefore $\theta + \delta'' = 1 + \theta - \delta' > 1$, implying that the right-hand side of (1.78) is 1 for sufficiently small positive x. It follows that $l_m(r) < \infty$.

If $r_m - r_{m+1} = -1$ and $r_{m+1+p} = r_{m-p}$ for every $p \geqslant 1$ then $r_m = 0$, implying $\theta \leqslant \delta'$ and $r_{m+1} = 1$ implying $1 \leqslant \delta' + \theta$. The equalities if they occur are equivalent to

$$\theta = \delta' \Leftrightarrow n\theta + \delta = [(n+1)\theta + \delta] \in \mathbb{Z},$$

$$1 - \theta = \delta' \Leftrightarrow (n+2)\theta + \delta = [(n+1)\theta + \delta] \in \mathbb{Z}.$$

Either of these cases contradicts the fact that $n\theta + \delta$ with an irrational θ may be an integer for at most one value of n. Hence $\theta < \delta'$ and $1 < \theta + \delta'$. As above, the identity (1.78) holds. Since $1 < \theta + \delta' < 2$, its right-hand side is 1 for small positive x. Since $0 < \theta + \delta'' = 1 - (\delta' - \theta) < 1$, its left-hand side is 0. It follows that in this case $l_n(r) < \infty$ too. □

It has already been mentioned that $(n+1)\theta + \delta \in \mathbb{Z}$ for some n with irrational θ if and only if $\{r_n(\theta, \delta)\}_{n\in\mathbb{Z}}$ is a shift of $\{r_n(\theta, 0)\}_{n\in\mathbb{Z}}$.

Definition 1.66 *A Jean Bernoulli sequence $\{r_n(\theta, \delta)\}_{n\in\mathbb{Z}}$ is called regular if either $\theta \in \mathbb{Q}$, i.e. it is periodic, or $(n+1)\theta + \delta \notin \mathbb{Z}$ for every $n \in \mathbb{Z}$. Otherwise it is called singular.*

By Theorem 1.65 a Jean Bernoulli sequence is regular if and only if $l_n(r) < \infty$ for every n. A Jean Bernoulli sequence is singular if and only $l_n(r) = \infty$ for exactly one n.

If $r = \{r_n(\theta)\}_{n\in\mathbb{Z}}$, $r_n(\theta) = r_n(\theta, 0)$, then $l_{-1}(r) = \infty$. The conjugate sequence $\widetilde{r} = \{\widetilde{r}_n(\theta)\}_{n\in\mathbb{Z}}$ (which is not a Jean Bernoulli sequence) also satisfies $l_n(\widetilde{r}) = \infty$ only at $n = -1$.

1.5 Markoff sequences

28 The structure of Markoff sequences. Now we will establish that any Markoff sequence takes at most two values.

Lemma 1.67 (Step-down lemma) *Let $\{r_n\}_{n\in\mathbb{Z}}$ be a Markoff sequence and let $r_n = y+1$, $r_{n+1} = y$. Then r_{n-1} and r_{n+2} are both in $[y, y+1]$ and either $r_{n+2} = r_{n-1}$ or $y = r_{n-1} < r_{n+2} = y+1$.*

Proof On the one hand, by Definition 1.59(1), $r_{n-1} \in [y, y+2]$ and $r_{n+2} \in [y-1, y+1]$. On the other hand by (2) with $p = 1$ we have $r_{n-1} \leqslant r_{n+2}$. Hence both r_{n-1} and r_{n+2} are in $[y, y+1]$. The rest follows from (2). □

Lemma 1.68 (Step-up lemma) *Let $\{r_n\}_{n\in\mathbb{Z}}$ be a Markoff sequence and let $r_n = x$, $r_{n+1} = x+1$. Then r_{n-1} and r_{n+2} are both in $[x, x+1]$ and either $r_{n+2} = r_{n-1}$ or $x+1 = r_{n-1} > r_{n+2} = x$.*

Proof On the one hand, by Definition 1.59(1), $r_{n-1} \in [x-1, x+1]$ and $r_{n+2} \in [x, x+2]$. On the other hand by (3) with $p = 1$ we have $r_{n-1} \geqslant r_{n+2}$. Hence both r_{n-1} and r_{n+2} are in $[x, x+1]$. The rest follows from (3). □

Theorem 1.69 *Any step down $r_m - r_{m+1} = 1$ in a Markoff sequence is followed (preceded) either by a step up $r_n - r_{n+1} = -1$, possibly separated by a finite flat sequence, or by an infinite flat sequence. Any step up $r_m - r_{m+1} = -1$ is followed (preceded) either by a step down $r_n - r_{n+1} = +1$, possibly separated by a finite flat sequence, or by an infinite flat sequence.*

Proof Suppose that $r_m = a+1$, $r_{m+1} = a$ and there is no step up to the right of $m+1$ separated by a flat sequence. Since $m-1$ and $m+2$ are equally distant to $[m, m+1]$, condition (2) of Definition 1.59 implies that on the one hand $r_{m-1} \leqslant r_{m+2}$. On the other hand $r_{m-1}, r_{m+2} \in [a, a+1]$ by Lemma 1.67. Hence there is no option for r_{m+2} except for it to equal $r_{m-1} = a$.

Suppose that $r_j = a$ for $m-k \leqslant j \leqslant m-1$ and for $m+1 \leqslant j \leqslant m+1+k$. Since $m-k-1$ and $m+k$ are equally distant to $[m-1, m]$ and $r_{m-1} - r_m = a-(a+1) = -1$, condition (3) of Definition 1.59 implies that $a = r_{m+k} \leqslant r_{m-k-1}$. Since $m-k-1$ and $m+k+2$ are equally distant to $[m, m+1]$, condition (2) implies that $r_{m-k-1} \leqslant r_{m+k+2}$. Since $r_j = a$ for $j \in [m+1, m+1+k]$, our assumption forces r_{m+k+2} to equal a. Hence $r_{m-k-1} = r_{m+k+2} = a$.

Continuing by induction we obtain a triangle sequence centered at m, i.e. the first sequence in (1.76). The second case, $r_m - r_{m+1} = -1$, is considered similarly. Since the mapping $n \to -n$ maps Markoff sequences into Markoff sequences, and steps up into steps down, the second part of the theorem on the following or preceding jumps follows from the first. □

Corollary 1.70 *Any Markoff sequence is either constant or takes only two values, a or $a+1$.*

Proof If $\{r_n\}_{n\in\mathbb{Z}}$ is not constant then there is n such that $r_n - r_{n+1} = \pm 1$. Of the two values r_n and r_{n+1} we denote the smaller by a. Then the other is $a+1$. By Theorem 1.69 every step down is either followed by a constant sequence a or is followed by a step up. The step up is followed by a step down. Hence r_k oscillates between a and $a+1$. The same is true in the left direction. □

Corollary 1.71 *Any Markoff sequence $\{r_n\}_{n\in\mathbb{Z}}$ such that $r_n - r_{n+1} \neq 0$ for at least three n values is oscillating.*

Proof Take the middle n value of these three and apply Theorem 1.69 in both directions. □

We conclude §**28** with a technical lemma.

Lemma 1.72 *Given any nonconstant Markoff sequence $\{r_n\}_{n\in\mathbb{Z}}$ and any integer m there is at least one $n \in \mathbb{Z}$ such that $m \in \omega_n(r)$.*

Proof If $r_m - r_{m+1} = \pm 1$ then $m \in \omega_m(r)$. If $r_{m-1} - r_m = \pm 1$ then $m \in \omega_{m-1}(r)$. So suppose that r_m is surrounded by equal values. If $r_m = a$, there must be a step up to the left of m or to the right or both. If $r_k = a$ for $k \geqslant m$ then there must be n to the left of m such that $l_n(r) = \infty$. The same is true if $r_k = a$ for $k \leqslant m$. If there are $k < m < l$ with $r_k = r_l = a+1$ then $m \in \omega_k(r)$ by the Markoff property at $[k, k+1]$. □

29 Racing algorithm and Markoff sequences.

By Corollaries 1.70 and 1.71 Markoff sequences oscillate between a and $a+1$, where $a \in \mathbb{Z}$, if they are not constant and not triangle sequences.

A triangle sequence of the type $(a+1, a)$ is a ceiling sequence. Any oscillating sequence of two values a, $a+1$ is a ceiling sequence too. Our next goal is to restate Markoff's properties (2) and (3) in Definition 1.59 for ceiling sequences in more transparent terms.

Racing algorithm at $[n, n+1]$. Two racers A and B run in opposite directions with unit speed. Racer A runs from n to $-\infty$, racer B from $n+1$ to $+\infty$. The end result of the algorithm is a winner, who hits a given infinite increasing sequence $\{n_j\}_{j\in\mathbb{Z}}$ first. If they both hit it simultaneously the race is continued. If the race continues up to infinity then both are declared to be winners.

Lemma 1.73 *Given $\{r_n\}_{n\in\mathbb{Z}}$ a ceiling sequence of two values a and $a+1$, the solutions to the equation $r_n = a+1$, make an infinite increasing sequence $\{n_j\}_{j\in\mathbb{Z}}$. Then* (2) *of* Definition 1.59 *corresponding to $r_n - r_{n+1} = 1$ holds if and only if either the racing algorithm at $[n, n+1]$ claims B as a winner or both A and B win. Condition* (3) *of* Definition 1.59 *for $r_n - r_{n+1} = -1$ holds if and only if the racing algorithm at $[n, n+1]$ claims A as a winner or both A and B win.*

Proof Suppose that $r_n - r_{n+1} = 1$. Then $r_n = a + 1$ and therefore $n = n_k$, $n + 1 = n_k + 1 < n_{k+1}$. Clearly r_{n+1+p} indicates the position of B with respect to $\{n_j\}_{j\in\mathbb{Z}}$ in p seconds, whereas r_{n-p} indicates the position of A at the same moment. It follows that $r_{n+1+p} - r_{n-p} > 0$ for the first time if and only if B hits $\{n_j\}_{j\in\mathbb{Z}}$ earlier than A. The case $r_n - r_{n+1} = -1$ is considered similarly. □

Remark Notice that the slope of the graph of r_x at the interval $[n, n+1]$ always faces the winner.

In the following lemma r is a sequence of a and $a + 1$.

Lemma 1.74

(a) *The Markoff series for r at $[n, n+1]$ with $r_n - r_{n+1} = -1$ satisfies Definition 1.59 part* (3) *if and only if $s_k - s_{k+1} = 1$, where $n = n_{k+1} - 1$, and the Markoff series for s at $[k, k+1]$ satisfies* (2).

(b) *The Markoff series for r at $[n, n+1]$ with $r_n - r_{n+1} = 1$ satisfies* (2) *if and only if $s_k - s_{k+1} = -1$, where $n = n_{k+1}$, and the Markoff series for s at $[k, k+1]$ satisfies* (3).

In both cases $l_k(s) \leqslant l_n(r)$. If $l_n(r) < \infty$ then $l_k(s) < l_n(r)$. The values of r within the domain $\omega_n(r)$ are uniquely determined by the values of s within the domain $\omega_k(s)$.

Proof If $r_n - r_{n+1} = -1$ then $r_n = a$ and $r_{n+1} = a + 1$, implying that $n_k < n = n_{k+1} - 1$ for some k.

If $s_k - s_{k+1} > 1$ then the length s_k of $[n_k, n_{k+1})$ exceeds the length s_{k+1} of $[n_{k+1}, n_{k+2})$ by at least 2. Since A and B compete with equal speeds they cover equal distances in equal times. Therefore the racing algorithm at $[n, n+1]$ assigns the victory to racer B, since B will reach n_{k+2} first. By Lemma 1.73 this contradicts the assumption that $\{r_n\}_{n\in\mathbb{Z}}$ is a Markoff sequence and implies that the case $s_k - s_{k+1} > 1$ is impossible. Similarly the symmetric case $s_k - s_{k+1} < -1$ is also impossible.

If $s_k - s_{k+1} = 1$ then the length s_k of $[n_k, n_{k+1})$ exceeds the length s_{k+1} of $[n_{k+1}, n_{k+2})$ by exactly 1. Let us apply the racing algorithm at $[k, k+1]$. Then, since $s_k = s_{k+1} + 1$ and $n = n_{k+1} - 1 \in [n_k, n_{k+1})$, racers A and B reach n_k and n_{k+2} simultaneously so that the race continues. While racer A passes through intervals of length $s_{k-p} \geqslant 1$, racer B passes through intervals of length $s_{k+1+p} \geqslant 1$. Lemma 1.73 applied to a Markoff sequence $\{r_n\}_{n\in\mathbb{Z}}$ claims that there are two options. Either A hits n_{k-p} for some $p > 1$ first, which implies that B is at this moment somewhere in $[n_{k+1+p}, n_{k+2+p})$ and hence the first nonzero difference $s_{k+1+p} - s_{k-p}$ is positive, or both win the race, so that all differences are zero. Since $s_{k-p} \geqslant 1$ and $s_{k+1+p} \geqslant 1$ for every $p = 1, \ldots, l_k(s)$, we see that $l_k(s) \leqslant l_n(r)$. Hence if $l_n(r) < \infty$ then $l_k(s) < l_n(r)$ if and only if there is at least one s_j, $j \in \omega_k(s)$, $j \neq k$, such that $s_j > 1$. But $s_{k+1+p} - s_{k-p} > 0$ for $p = l_k(s)$. Hence such a j always exists. The formula

$$n_{j+1} - n_j = s_j, \qquad j \in \omega_k(s)$$

recovers the values of r in $\omega_n(r)$ as soon as the relationship $n = n_{k+1} - 1$ between n and k is fixed.

The case $s_k - s_{k+1} = -1$ is considered similarly.

Suppose now that $s_k - s_{k+1} = 1$, $n = n_{k+1} - 1$ and the Markoff series for s at $[k, k+1]$ satisfies the Markoff property (2) of Definition 1.59. The adjusting intervals $[n_k, n_{k+1})$, $[n_{k+1}, n_{k+2})$ have lengths s_k and s_{k+1} respectively. Since $n_k < n = n_{k+1} - 1$, we have $r_n - r_{n+1} = a - (a+1) = -1$. It follows also that $s_k > 1$. Then $r_{n-p} = 1$, $p > 0$, for the first time at $p = s_k - 1 > 0$ and $r_{n+1+p} = 1$, $p > 0$, for the first time at $p = s_{k+1} = s_k - 1$. Therefore racers A and B in the racing algorithm, having started at $[n, n+1]$, reach n_k and n_{k+2} simultaneously. Again by property (2) of Definition 1.59, in the series at $[k, k+1]$ for s the first nonzero difference $s_{k+p+1} - s_{k-p}$, $p > 0$, must be positive. Before that ($j < p$) racers A and B hit n_{k-j} and n_{k+2+j} simultaneously. Since $s_{k+p+1} > s_{k-p}$, at the moment when A hits n_{k-p} racer B will be somewhere in $[n_{k+1+p}, n_{k+2+p})$. Hence A wins the race and Lemma 1.73 implies that the Markoff series at $[n, n+1]$ for r satisfies (3). Part (b) is considered similarly. □

Theorems 1.56 and 1.57 extend to Markoff sequences.

Theorem 1.75 *Let $\{r_n\}_{n\in\mathbb{Z}}$ be a ceiling sequence of two integers a and $a+1$ $\{n_k\}_{k\in\mathbb{Z}}$ the increasing sequence of all solutions to $r_n = a+1$ and $\{s_k\}_{k\in\mathbb{Z}}$ the sequence of the first differences $s_k = n_{k+1} - n_k$. Then $\{r_n\}_{n\in\mathbb{Z}}$ is a Markoff sequence if and only if $\{s_k\}_{k\in\mathbb{Z}}$ is a positive Markoff sequence.*

Proof Suppose first that $\{r_n\}_{n\in\mathbb{Z}}$ is a Markoff sequence and check that $\{s_k\}_{k\in\mathbb{Z}}$ is a Markoff sequence too. We establish first that

$$|s_k - s_{k+1}| \leqslant 1.$$

If $s_k - s_{k+1} > 1$ then the length s_k of $[n_k, n_{k+1})$ exceeds the length s_{k+1} of $[n_{k+1}, n_{k+2})$ by at least 2. Therefore if $n = n_{k+1} - 1$ then $r_n = a$ and $r_{n+1} = a+1$. It follows that $r_n - r_{n+1} = -1$. By part (a) of Lemma 1.74 we must have $s_k - s_{k+1} = 1$, which is a contradiction. The case $s_k - s_{k+1} < 1$ is considered similarly. Now if $s_k - s_{k+1} = 1$ then $n_k < n_{k+1} - 1 = n$. It follows that $r_n - r_{n+1} = -1$. By part (a) of Lemma 1.74 the Markoff sequence at $[k, k+1]$ satisfies (2). The case $s_k - s_{k+1} = -1$ is considered similarly.

Suppose now that $\{s_k\}_{k\in\mathbb{Z}}$ is a positive Markoff sequence and check that $\{r_n\}_{n\in\mathbb{Z}}$ is a Markoff sequence. Since $\{r_n\}_{n\in\mathbb{Z}}$ is a ceiling sequence with two integer values a and $a+1$ such that the set $\{n_k : k \in \mathbb{Z}\}$ lists all solutions to the equation $r_n = a+1$, Markoff property (1) for $\{r_n\}_{n\in\mathbb{Z}}$ is obvious.

Consider the Markoff conditions (2), (3) for $\{r_n\}_{n\in\mathbb{Z}}$. If $r_n - r_{n+1} = 1$ then $n = n_{k+1}$ and $n+1 < n_{k+2}$. The adjusting intervals $[n_k, n_{k+1})$, $[n_{k+1}, n_{k+2})$ have lengths s_k and s_{k+1} respectively. Since $r_{n+1} = 0$, we have $s_{k+1} > 1$. Then $r_{n+1+p} = 1$, $p > 0$, for the first time at $p = s_{k+1} - 1 > 0$ and $r_{n-p} = 1$, $p > 0$, for the first time at $p = s_k$. It follows that B wins the race at the moment $p = s_{k+1} - 1$ if $s_{k+1} - 1 < s_k$, i.e. $-1 < s_k - s_{k+1}$. The case $s_k - s_{k+1} < -1$ is excluded by (1) for the Markoff sequence $\{s_j\}_{j\in\mathbb{Z}}$. If now $s_k - s_{k+1} = -1$, the race runs up since $r_{n+1+p} = r_{n_{k+2}} = 1$, $r_{n-p} = r_{n_k} = 1$ for $p = s_{k+1} - 1$. Markoff property (3) of $\{s_j\}_{j\in\mathbb{Z}}$ says that the first nonzero difference $s_{k+l} - s_{k-1-l}$ must be negative. It follows that B has to cover a smaller distance than A and therefore wins the race. If $s_{k+l} - s_{k-1-l} = 0$ for every $l > 0$ then both A and B win and condition (2) holds also. The case $r_n - r_{n+1} = -1$ is considered similarly. □

The following theorem is proved similarly to Theorem 1.75 and is useful in the evaluation of the number of k values satisfying $l_k(r) = \infty$.

Theorem 1.76 *Let $r = \{r_n\}_{n\in\mathbb{Z}}$ be a ceiling Markoff sequence and $s = \{s_k\}_{k\in\mathbb{Z}}$, $s_k = n_{k+1} - n_k$. Then*

(a) *$r_k - r_{k+1} = -1$ and $l_k(r) = \infty$ if and only if $s_m - s_{m+1} = +1$ and $l_m(s) = \infty$, where $k + 1 = n_{m+1}$;*

(b) *$r_k - r_{k+1} = 1$ and $l_k(r) = \infty$ if and only if $s_m - s_{m+1} = -1$ and $l_m(s) = \infty$, where $k = n_{m+1}$.*

Different indices k with $l_k(r) = \infty$ correspond to the different indices m with $l_m(s) = \infty$.

Proof If $r_k - r_{k+1} = -1$ then $r_{k+1} = a + 1$ and therefore $k + 1 = n_{m+1}$ for some m. If $l_k(r) = +\infty$ then the racing algorithm at $[k, k+1]$ will continue up to infinity. It follows that $s_m - s_{m+1} = 1$ and that $s_{m-p} = s_{m+1+p}$ for every $p \geqslant 1$. Hence $l_m(s) = +\infty$. If $r_k - r_{k+1} = 1$ then $r_k = a + 1$ and therefore $k = n_{m+1}$ for some m. If $l_k(r) = +\infty$ then the racing algorithm at $[k, k+1]$ will continue up to infinity. It follows that $s_m - s_{m+1} = -1$ and that $s_{m-p} = s_{m+1+p}$ for every $p \geqslant 1$. Hence $l_m(s) = +\infty$. It is clear that these arguments are reversible.

To prove the last statement let us observe that if $[k, k+1]$ and $[k', k'+1]$ do not intersect then the $m \neq m'$. If on the contrary $k + 1 = k'$ then by Theorem 1.69 the differences $r_k - r_{k+1}$ and $r_{k'} - r_{k'+1}$ have opposite signs and hence the corresponding n_{m+1} cannot be equal. □

30 The derivatives and integrals of Markoff sequences. The relationship between a ceiling sequence $r = \{r_n\}_{n\in\mathbb{Z}}$ and a positive sequence $s = \{s_k\}_{k\in\mathbb{Z}}$,

$$r_n = a + 1 \quad \Leftrightarrow \quad n = n_k, \quad n_{k+1} - n_k = s_k, \qquad k \in \mathbb{Z}, \tag{1.79}$$

recalls the relationship between a function and its derivative. Adding a constant to r does not change s. Similarly shifts of r do not influence s. Therefore we will call $s = \partial r$ the *derivative* of r. To make this definition consistent with classical calculus we define the derivative of any constant sequence to be a zero sequence.

If r_0 is fixed then there still remains a freedom in enumeration of the solutions to $r_n = a + 1$. Any such an enumeration, as is clear from (1.79), corresponds to a shift in s. Therefore ∂r determines not a particular sequence but the class of sequences which differ by a shift.

If $\{s_k\}_{k\in\mathbb{Z}}$ is any positive sequence then

$$n_j = \begin{cases} s_0 + \cdots + s_{j-1} & \text{if } j \neq 1, \\ 0 & \text{if } j = 0, \\ -s_{-1} - \cdots - s_j & \text{if } j < 0, \end{cases} \tag{1.80}$$

is an increasing sequence infinite in both directions. For any integer a we may define a ceiling sequence r_n assigning the value $a+1$ if $n = n_k$ and the value a otherwise. Then clearly $s = \partial r$. The class of all shifts in these sequences is called the *integral* of s and is denoted by $\int s$. The constant a is called a *constant of integration*.

In this terminology Theorem 1.56 claims that a ceiling sequence is periodic if and only if its derivative is a positive periodic sequence. Theorem 1.57 says that a ceiling sequence is a Jean Bernoulli sequence if and only if its derivative is a positive Jean Bernoulli sequence. Finally, Theorem 1.75 states that a ceiling sequence is a Markoff sequence if and only if its derivative is a positive Markoff sequence. The derivative of a triangle sequence of type $(a+1, a)$ is a triangle sequence of type $(1, 2)$. Triangle sequences of type $(a, a+1)$ are nondifferentiable.

Another good example of differentiation is provide by Jean Bernoulli sequences:

$$r = \{r_n(\theta, 0)\}_{n\in\mathbb{Z}}, \qquad \theta = \frac{1}{a_1} + \frac{1}{\theta_2}.$$

By (1.71) and (1.72) the derivative $s = \partial r$ is

$$a_1 + r_n(1/\theta_2, 0), \qquad n \in \mathbb{Z}.$$

In the calculus of sequences the racing algorithm gives a dynamic interpretation of differentiation and integration.

Corollary 1.77 *There are three mutually exclusive possibilities for the derivative $s = \partial r$ of an oscillating Markoff sequence r: s is a triangle sequence, or it is a constant sequence, or it is an oscillating sequence.*

Proof By Theorem 1.75, $r = \partial s$ is a Markoff sequence. By Corollary 1.70, r is either a constant or takes two values a and $a+1$. If r takes two values then by Theorem 1.69 it either oscillates or equals a triangle sequence. □

Corollary 1.78 *A Markoff sequence $r = \{r_n\}_{n\in\mathbb{Z}}$ oscillates if and only if r is differentiable and its derivative is not equal to a nondifferentiable triangle sequence of the type* $(1, 2)$.

Proof There are two types of positive triangle sequence. The first is a triangle sequence $s = \{s_k\}_{k\in\mathbb{Z}}$ of type $(1, 2)$. Applying a suitable shift to s we may assume that $s_0 = 2$.

Then (1.80) shows that $r = \int s$ contains a sequence $r = \{r_n\}_{n\in\mathbb{Z}}$ such that $r_n = a+1$ for $n \neq 1$ and $r_1 = a$. This sequence as well as its shifts is not oscillating.

The second is a triangle sequence $\{s_k\}_{k\in\mathbb{Z}}$ of type $(a, a\pm 1)$, $a > 1$; then there are infinitely many gaps between a value $r_n = b+1$ and the value $r_n = b$, implying oscillation.

If $s_k = 1$ for every k then $r_n = a+1$ for every n and hence $\{r_n\}_{n\in\mathbb{Z}}$, being a constant sequence, oscillates by definition. If $s_k = a > 1$, $k \in \mathbb{Z}$, then $r = \int s$ has infinitely many gaps between two values in both directions and hence r oscillates.

By Corollary 1.71 the last possibility for s is a positive oscillating Markoff sequence. Hence in both directions it must have infinitely many values exceeding 1. By (1.80) this implies that $r = \int s$ oscillates. □

31 Jean Bernoulli and Markoff periods. If $\theta \in \mathbb{Q}\setminus\mathbb{Z}$ then $\{r_n(\theta)\}_{n\in\mathbb{Z}}$ is a periodic sequence of $a = [\theta]$ and $a+1$ with p numbers $a+1$ in a period of length q, where $\{\!\{\theta\}\!\} = p/q$. The period

$$\mathbf{JB}(r) = \mathbf{JB} : a+1, \quad a, \quad r_1, \quad r_2, \cdots, r_{q-2}, \tag{1.81}$$

corresponding to $n = -1, \ldots, q-2$ is called the *Jean Bernoulli period* of $r = \{r_n(\theta)\}_{n\in\mathbb{Z}}$. Notice that by Theorem 1.54 its part $\{r_1, r_2, \ldots, r_{q-2}\}$ is symmetric. In his thesis (1880) Markoff introduced two other important periods:

$$\begin{aligned} \Pi_1 : &\quad a+1, \quad r_1, \quad r_2, \quad \cdots, \quad r_{q-2}, \quad a, \\ \Pi_2 : &\quad a, \quad r_1, \quad r_2, \quad \cdots, \quad r_{q-2}, \quad a+1. \end{aligned} \tag{1.82}$$

We call period Π_1 the *first Markoff period* and Π_2 the *second Markoff period*. Both Markoff periods are easily obtained from the Jean Bernoulli period (1.81). It is clear that Π_2 is obtained from **JB** by a one-step shift along r to the right. Hence Π_2 is also a period of r. This period has the following extremal property discovered by Markoff.

Lemma 1.79 *If* $0 < \theta \in \mathbb{Q}\setminus\mathbb{Z}$ *then* Π_2 *is the only period of* $r = \{r_n(\theta)\}_{n\in II}$ *such that for any other period* $r_0^*, r_1^*, \ldots, r_{q-1}^*$ *the first nonzero term in*

$$a - r_0^*, \quad r_1 - r_1^*, \quad \ldots, \quad a+1-r_{q-1}^* \tag{1.83}$$

is -1 *and the last nonzero term is* $+1$.

Proof Any other period of r equals $\{r_j, \ldots, r_{j+q-1}\}$ for some $j = 1, \ldots, q-1$. Let k be the first positive number such that $r_{k-1} \neq r_{j+k-1}$. Since

$$\begin{aligned} r_0 + \cdots + r_{k-1} &= [k\theta], \\ r_j + \cdots + r_{j+k-1} &= [(k+j)\theta] - [j\theta], \end{aligned}$$

subtraction of the second equality from the first followed by the use of (1.31) implies

$$0 \neq r_{k-1} - r_{j+k-1} = [k\theta] + [j\theta] - [k\theta + j\theta] = -1.$$

Let us consider the "tail" of (1.83). Since $q\theta = p \in \mathbb{Z}$, we obtain by (1.30) that

$$r_{q-k} + \cdots + r_{q-1} = [q\theta] - [(q-k)\theta] = [k\theta] + 1,$$
$$r_{j+q-k} + \cdots + r_{j+q-1} = [j\theta] - [(j-k)\theta].$$

Hence by (1.31)

$$0 \neq r_{q-k} - r_{j+q-k} = 1 + [k\theta] + [(j-k)\theta] - [j\theta] = +1,$$

where k is the first positive integer with $r_{q-k} - r_{j+q-k} \neq 0$. □

There is exactly one period with the extremal property of Π_2 stated in Lemma 1.79. Indeed, if both r and r^* were such periods the change of their places in (1.83) would lead to a contradiction.

The fact that Π_1 is a period of r is not so obvious. For instance for $\theta = 26/19$ it begins at the marked place of **JB**:

$$\mathbf{JB} = \{2, 1, 1, 2, 1, 1, 2, 1, 1, \underset{\bullet}{2}, 1, 2, 1, 1, 2, 1, 1, 2, 1, \},$$

$$\Pi_2 = \{1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2, \},$$

$$\Pi_1 = \{2, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 1, 2, 1, 1, \}.$$

However, Π_2 is clearly a one-step shift of **JB** to the right, as explained earlier. By Lemma 1.79 Π_2 has an extremal property. So if we can prove that Π_1 satisfies an extremal property symmetric to that of Π_2 then it becomes possible to relate this period to **JB**. This can be done with Markoff's integration techniques presented in §**30** above. For $\theta \in \mathbb{Q}$ we denote by $n(\theta)$ the number of partial denominators in the continued fraction of θ.

Theorem 1.80 (Markoff 1880) *If* $0 < \theta \in \mathbb{Q} \setminus \mathbb{Z}$ *then the period* Π_1 *related to* Π_2 *by* (1.82) *is a period of r. Period* Π_1 *is the only period such that for any other period* $r_0^*, r_1^*, \ldots, r_{q-1}^*$ *the first nonzero term in*

$$(a+1) - r_0^*, \quad r_1 - r_1^*, \quad \ldots, \quad a - r_{q-1}^* \tag{1.84}$$

is $+1$*, whereas the last nonzero term is* -1.

Proof As in case of Π_2 there is at most one period with the extremal property specified in (1.84).

If $n(\theta) = 0$ then $\theta \in \mathbb{Z}$ and hence r is a constant sequence. If $n(\theta) = 1$ then $p = 1$ and $q > 1$. If $q = 2$ then the symmetric part of the Jean Bernoulli period is empty, implying that $\Pi_1 = \{a+1, a\}$ and $\Pi_2 = \{a, a+1\}$. Hence $\Pi_1 = \mathbf{JB}$ and Π_2 is the shift of Π_1. Since $q = 2$, there are only two periods in total so that the series in (1.84) is $\{1, -1\}$ and that in (1.83) is $\{-1, 1\}$. If $q > 2$ then

$$\begin{array}{llllll} \Pi_1: & a+1, & a, & \cdots, & a, & a, \\ \Pi_2: & a, & a, & \cdots & a, & a+1, \end{array}$$

so that the first nonzero term in (1.84) is $+1$ whereas the last is -1. Similarly, the first nonzero term in (1.83) is -1 and the last is $+1$. Again Π_1 is the Jean Bernoulli period and Π_2 is its shift.

Suppose that the theorem is true for every $\theta > 1$ with $n(\theta) < n$. If $\xi = b + 1/\theta$, $b \geqslant 0$, then $n(\xi) = n(\theta) + 1$. By the induction hypothesis Π_1 is the period of r satisfying the extremal property indicated in (1.84). To complete the induction we must establish the same property for $t = \{r_n(\xi)\}_{n\in\mathbb{Z}}$, which is the integral of r made up by b and $b+1$. Let us consider two periods

$$P_1 : b+1\underbrace{\ b\ }_{a-1}b+1\underbrace{\ b\ }_{r_1-1}\cdots\underbrace{\ b\ }_{r_{q-2}-1}b+1\underbrace{\ b\ }_{a};$$

$$P_2 : \underbrace{\ b\ }_{a}b+1\underbrace{\ b\ }_{r_1-1}b+1\cdots\underbrace{\ b\ }_{r_{q-2}-1}b+1\underbrace{\ b\ }_{a-1}b+1,$$

where the second term in P_1 is repeated $a-1$ times, etc.; P_1 is constructed by $\Pi_2 = \Pi_2(r)$ and P_2 by $\Pi_1 = \Pi_1(r)$. Extending P_1 and P_2 periodically in both directions, we obtain two infinite periodic sequences t_1 and t_2 of terms b and $b+1$. It is clear that the derivative of t_2 is a periodic sequence with period $\Pi_1(r)$ and hence by the induction hypothesis it is simply r. Similarly $\partial t_1 = r$. It follows that t_1 and t_2 differ from t by a shift and that P_2 and P_1 are the periods of t. Elementary calculations show that the length of both periods is

$$(a+1) + a + r_1 + \cdots + r_{q-2} = aq + p.$$

Any period P of t begins either with $b+1$ or with b. If it begins with $b+1$ then it must be of the form

$$P : b+1\underbrace{\ b\ }_{r_0^*-1}b+1\underbrace{\ b\ }_{r_1^*-1}\cdots\underbrace{\ b\ }_{r_{q-2}^*-1}b+1\underbrace{\ b\ }_{r_{q-1}^*-1}, \tag{1.85}$$

where $\{r_0^*, r_1^*, \ldots, r_{q-1}^*\}$ is a period of r. By the induction hypothesis the first nonzero term in (1.83) is -1 and the last is 1. Since P_1 is constructed from Π_2 and the first nonzero difference in (1.83) is -1, the first nonzero difference for P_1 and P must occur within the first block of b's in P, which is longer than the corresponding block of b's in P_1. Hence it is $b+1-b=1$. The case of the last nonzero difference is considered similarly. Similarly one many check that the first nonzero difference in (1.83) for P_2 and P is -1 and the last is 1.

If P begins with b then the very first difference between P_1 and P is $b+1-b=1$. To investigate the last difference we may assume that this P begins at the first group of b's in (1.85). Since P_1 and P are the periods of r, the difference between r_0^* and a cannot exceed 1. Therefore if a group $b+1, b, \ldots, b$ at the head of P is moved to the tail of P then the number of b's thus borrowed is strictly less than $r_0^* - 1$. But the tail of P_1 consists of a equal b's, which exceeds the number of borrowed b's by more than 1. It follows that the first nonzero difference is $b-(b+1)=-1$ as stated above. The pair P_2 and P for this case is investigated similarly.

By Lemma 1.79 we obtain that $P_2=\Pi_2(t)$. Comparing the formulas for P_1 and P_2, we see that $P_1=\Pi_1(t)$, which proves the theorem. □

32 Markoff's algorithm for oscillating Markoff sequences. Let $\{r_n\}_{n\in\mathbb{Z}}$ be an oscillating nonconstant Markoff sequence. By Corollaries 1.70 and 1.71 it takes both the values a and $a+1$ infinitely often, implying in particular that it is a ceiling sequence and therefore differentiable. Corollary 1.77 allows one to extend Markoff's algorithm as presented in §**26** in Section 1.4 to oscillating Markoff sequences.
Markoff's algorithm (oscillating case). Markoff's algorithm will now be applied to oscillating Markoff sequences $\{r_j\}_{j\in\mathbb{Z}}$.

Entry 1. Since $r=\{r_n\}_{n\in\mathbb{Z}}$ is differentiable the derivative $s=\partial r$ exists.
Entry 2. If $s=\partial r$ is a triangle sequence then Markoff's algorithm fails.
Entry 3. If $s=\partial r$ is a positive constant sequence then Markoff's algorithm stops.
Entry 4. If $s=\partial r$ is a positive oscillating sequence then Markoff's algorithm can continue.

By Corollary 1.77 the result of each application of this algorithm fits exactly one of the entries 2–4. By Theorem 1.75, entry 2 of the Markoff algorithm for periodic sequences (see the text after the proof of Theorem 1.56) cannot occur if one runs the Markoff algorithm in the oscillating case.

As in the periodic case, the oscillating version of Markoff's algorithm when applied to Jean Bernoulli sequences corresponding to an irrational number recovers its regular continued fraction. In this case Markoff's algorithm never stops or fails.

Definition 1.81 *A Markoff sequence is called singular if either it is a triangle sequence or Markoff's algorithm fails in a finite number of steps. Otherwise it is called regular.*

An important class of regular Markoff sequences is the class of periodic Markoff sequences.

Lemma 1.82 *If r is a periodic Markoff sequence then $s=\partial r$ is a periodic Markoff sequence with a smaller period.*

Proof By Theorem 1.56 s is periodic with a smaller period than r. By Theorem 1.75 s is a Markoff sequence. □

Theorem 1.83 (Markoff 1880) *Any periodic Markoff sequence is a periodic Jean Bernoulli sequence.*

Proof A nonconstant periodic Markoff sequence oscillates. By Lemma 1.82 both versions of Markoff's algorithm give the same results and can be continued until a constant sequence is obtained. The use of Theorem 1.58 completes the proof. □

For regular Markoff sequences, on the one hand, the first option in Corollary 1.77 never occurs. Therefore if we apply Markoff's algorithm to such sequences, $n_{j+1} - n_j$ always has two integer values, a and $a+1$, as Corollary 1.70 shows. For instance this is the case for Jean Bernoulli sequences with irrational mean values. By Ex. 1.28 Markoff's algorithm recovers the mean value θ of a regular Markoff sequence.

On the other hand there are many oscillating singular Markoff sequences. Just consider integrals of positive triangle sequences of the type $(a, a+1)$, $a > 1$. Integrating the sequences obtained we arrive at new ones. The process can be continued up to infinity.

33 The classification of Markoff sequences. If applied to any regular Markoff sequence Markoff's algorithm never fails. By Theorem 1.83 it stops if and only if r is a periodic Jean Bernoulli sequence. We are going to study the case when Markoff's algorithm neither stops nor fails. Any nonperiodic regular Markoff sequence can be represented in the form $r = \{a_0 + r_j^{(1)}\}_{j\in\mathbb{Z}}$, where $a_0 \in \mathbb{Z}$ and $r_j^{(1)}$ is either 0 or 1.

Lemma 1.84 *For any nonperiodic regular Markoff sequence $\{r_j^{(1)}\}_{j\in\mathbb{Z}}$ of 0 and 1 and for any positive integer J there is a periodic Jean Bernoulli sequence $\{r_n(\theta_J, \delta_J)\}_{n\in\mathbb{Z}}$ such that*

$$r_j^{(1)} = r_j(\theta_J, \delta_J) \quad \text{for } |j| \leqslant J. \tag{1.86}$$

The rational number θ_J is either a convergent to the irrational mean value

$$\theta = \frac{1}{a_1} \, {}_{+} \, \frac{1}{a_2} \, {}_{+} \, \frac{1}{a_3} \, {}_{+} \cdots$$

of $\{r_j^{(1)}\}_{j\in\mathbb{Z}}$ or the mediant of two consecutive convergents.

Proof Let $\{n_k^{(1)}\}_{k\in\mathbb{Z}}$ be the increasing sequence of solutions to the equation $r_n^{(1)} = 1$ enumerated so that $n_0^{(1)} \leqslant -1 < 0 \leqslant n_1^{(1)}$. This uniquely determines the enumeration of $\{r_k^{(2)}\}_{k\in\mathbb{Z}}$ by the identity

$$n_{k+1}^{(1)} - n_k^{(1)} = a_1 + r_k^{(2)}, \quad a_1 > 0.$$

Hence the choice of $r_0^{(2)}$ is fixed. To insert the enumeration of solutions to $r_n^{(2)} = 1$ into $\{n_k^{(2)}\}_{k\in\mathbb{Z}}$, we demand that $n_0^{(2)} \leqslant 0 < 1 \leqslant n_1^{(2)}$. Proceeding by induction, we obtain for every $s \geqslant 1$

$$n_{k+1}^{(s)} - n_k^{(s)} = a_s + r_k^{(s+1)} \tag{1.87}$$

and

$$\begin{aligned} &n_0^{(s)} \leqslant -1 < 0 \leqslant n_1^{(s)} && \text{if } s \text{ is odd },\\ &n_0^{(s)} \leqslant 0 < 1 \leqslant n_1^{(s)} && \text{if } s \text{ is even.} \end{aligned} \tag{1.88}$$

If $s=2j+1$ is a large odd number then by (1.88)

$$l_{2j+1} \stackrel{\text{def}}{=} n_0^{(2j+1)} \leqslant -1, \qquad m_{2j+1} \stackrel{\text{def}}{=} n_1^{(2j+1)} \geqslant 0.$$

It follows that $[-1,0] \subset [l_{2j+1}, m_{2j+1}]$.

We apply (1.87) with $s=2j$ to $k \in [l_{2j+1}, m_{2j+1}]$. Clearly $r_k^{(2j+1)}=1$ at the ends of this interval. Hence for $k=l_{2j+1} \leqslant -1$ the index $l_{2j} \stackrel{\text{def}}{=} n_k^{(2j)}$ is less than $n_{k+1}^{(2j)} \leqslant n_0^{(2j)} \leqslant 0$ by at least $a_{2j}+1 \geqslant 2$, implying $l_{2j} \leqslant -2$. However, for $k=m_{2j+1} \geqslant 0$ the index $m_{2j} \stackrel{\text{def}}{=} n_{k+1}^{(2j)}$ cannot be smaller than $n_1^{(2j)} \geqslant 1$ by (1.88). It follows that $[-2,1] \subset [l_{2j}, m_{2j}]$.

Now we apply (1.87) with $s=2j-1$ to $k \in [l_{2j}, m_{2j}]$. Again $r_k^{(2j)}=1$ at the ends of this interval. Hence for $k=l_{2j} \leqslant -2$ the index $l_{2j-1} \stackrel{\text{def}}{=} n_k^{(2j-1)}$ is less than $n_{k+1}^{(2j-1)} \leqslant n_{-1}^{(2j-1)} \leqslant -2$ by at least $a_{2j-1}+1 \geqslant 2$. Therefore $l_{2j-1}=n_k^{(2j-1)} \leqslant -4$. On the other hand for $k=m_{2j} \geqslant 1$ the index $m_{2j-1} \stackrel{\text{def}}{=} n_{k+1}^{(2j-1)}$ must be greater than $n_1^{(2j-1)} \geqslant 0$ by at least 2, see (1.87). It follows that $[-4,2] \subset [l_{2j-1}, m_{2j-1}]$.

Lemma 1.85 *For $u=1,2,\ldots,j$*

$$l_{2j+1-(2u-1)} \leqslant 1-3u < -2+3u \leqslant m_{2j+1-(2u-1)}, \tag{1.89}$$

$$l_{2j+1-2u} \leqslant -1-3u < -1+3u \leqslant m_{2j+1-2u}. \tag{1.90}$$

Proof We will prove the lemma by induction. For $u=1$ (1.89) is equivalent to the already proved inclusion $[-2,1] \subset [l_{2j}, m_{2j}]$, whereas (1.90) is equivalent to $[-4,2] \subset [l_{2j-1}, m_{2j-1}]$.

Suppose that (1.89) holds for some u, $1 \leqslant u \leqslant j$. We apply (1.87) with

$$s=2j+1-(2u-1)-1=2j+1-2u$$

to $k \in [l_{2j+1-(2u-1)}, m_{2j+1-(2u-1)}]$. We have $r_k^{(s+1)}=1$ at the ends of this interval. Hence if $k=l_{2j+1-(2u-1)} \leqslant 1-3u$ then

$$\begin{aligned} l_{2j+1-2u} \stackrel{\text{def}}{=} n_k^{(2j+1-2u)} &\leqslant n_{k+1}^{(2j+1-2u)}-2 \\ &\leqslant n_{2-3u}^{(2j+1-2u)}-2 \\ &\leqslant 2-3u-1-2=-1-3u. \end{aligned}$$

If $k=m_{2j+1-(2u-1)} \geqslant -2+3u$ then

$$\begin{aligned} m_{2j+1-2u} \stackrel{\text{def}}{=} n_{k+1}^{(2j+1-2u)} &\geqslant n_k^{(2j+1-2u)}+2 \\ &\geqslant n_{-2+3u}^{(2j+1-2u)}+2 \\ &\geqslant 2+3u-2-1=-1+3u. \end{aligned}$$

This implies (1.90). Suppose now that (1.90) holds for some $u < j$. We apply (1.87) with

$$s = 2j + 1 - 2u - 1 = 2j - 2u$$

to $k \in [l_{2j+1-2u}, m_{2j+1-2u}]$. We have $r_k^{(s+1)} = 1$ at the ends of this interval. Hence if $k = l_{2j+1-2u} \leqslant -1 - 3u$ then

$$\begin{aligned} l_{2j+1-(2u+1)} \stackrel{\text{def}}{=} n_k^{(2j-2u)} &\leqslant n_{k+1}^{(2j-2u)} - 2 \\ &\leqslant n_{-3u}^{(2j-2u)} - 2 \\ &\leqslant -2 - 3u = 1 - 3(u+1). \end{aligned}$$

If $k = m_{2j+1-2u} \geqslant -1 + 3u$ then

$$\begin{aligned} m_{2j+1-(2u+1)} \stackrel{\text{def}}{=} n_{k+1}^{(2j-2u)} &\geqslant n_k^{(2j-2u)} + 2 \\ &\geqslant n_{-1+3u}^{(2j-2u)} + 2 \\ &\geqslant 2 - 1 + 3u = 1 + 3u = -2 + 3(u+1). \end{aligned}$$

□

By Lemma 1.85 we obtain that

$$l_1 \leqslant -1 - 3j < -1 + 3j \leqslant m_1. \tag{1.91}$$

The sequence $r^{(2j+1)}$ vanishes inside $[l_{2j+1}, m_{2j+1}]$ if there are integer points within the interval and equals 1 at the ends of this interval, whose positive length is $a = a_{2j+1} + r_0^{(2j+2)} > 0$. Repeating the same interval periodically in both directions, we obtain either a constant sequence (in the case $a = 1$) or a Jean Bernoulli sequence of two values with mean value $1/a$. Let us denote this sequence by $e^{(2j+1)}$. The important property of $e^{(2j+1)}$ is

$$e_k^{(2j+1)} = r_k^{(2j+1)}, \quad k \in [l_{2j+1}, m_{2j+1}]. \tag{1.92}$$

By (1.87) $r^{(2j)}$ is the integral of $r^{(2j+1)}$ with constant of integration a_{2j}. Let $e^{(2j)}$ be the integral of $e^{(2j+1)}$ with the same constant of integration. Then $e^{(2j)}$ is defined uniquely up to a shift in some increasing sequence $\{p_k\}_{k\in\mathbb{Z}}$ of the solutions to $e_p^{(2j)} = 1$:

$$p_{k+1}^{(2j)} - p_k^{(2j)} = a_{2j} + e_k^{(2j+1)}.$$

We fix the choice of $\{p_k\}_{k\in\mathbb{Z}}$ by setting

$$p_k^{(2j)} = n_k^{(2j)} \quad \text{for } k = l_{2j+1} = n_0^{(2j+1)}.$$

Applying (1.87) with $s = 2j$ to r and e, we obtain by (1.92) that

$$e_k^{(2j)} = r_k^{(2j)}, \qquad k \in [l_{2j}, m_{2j}].$$

By Theorem 1.56 $e^{(2j)}$ is a periodic sequence with mean value

$$\frac{1}{a_{2j}} + \frac{1}{a_{2j+1} + r_0^{(2j+2)}}.$$

Moving up by induction in $2j$ steps we construct a periodic Jean Bernoulli sequence $e^{(1)}$ with mean value

$$\frac{1}{a_1} + \cdots + \frac{1}{a_{2j+1} + r_0^{(2j+2)}},$$

which satisfies $e_k^{(1)} = r_k^{(1)}$, $k \in [l_1, m_1]$. Now apply (1.91). $\square$

Theorem 1.86 *A nonperiodic regular Markoff sequence r with mean value θ is either a Jean Bernoulli sequence $r = \{r_n(\theta, \delta)\}_{n\in\mathbb{Z}}$ or the conjugate to a Jean Bernoulli sequence $\{r_n(1-\theta, 0)\}_{n\in\mathbb{Z}}$.*

Proof Assuming that $r_0^{(s)} = 0$ for $s \geqslant 1$, we obtain by Lemma 1.84 a sequence of rationals $\theta_J \to \theta$ and of $\delta_J \in [0, 1)$ satisfying (1.86). Passing if necessary to subsequences, we may assume that $\delta_J \to \delta$. Then for every integer n and for sufficiently large J,

$$\begin{aligned} r_n &= [(n+1)\theta_J + \delta_J] - [n\theta_J + \delta_J] \\ &= [(n+1)\theta + \delta + \epsilon_J^{(n+1)}] - [n\theta + \delta + \epsilon_J^{(n)}], \end{aligned}$$

where $\epsilon_J^{(n)} = n(\theta_J - \theta) + \delta_J - \delta \to 0$ if $J \to +\infty$. Hence $r_n = r_n(\theta, \delta)$ if neither $n\theta + \delta$ nor $(n+1)\theta + \delta$ are integers. If $n\theta + \delta \notin \mathbb{Z}$ for every integer n, then the proof is completed. Otherwise, since $\theta \notin \mathbb{Q}$ there is only one $m \in \mathbb{Z}$ with $m\theta + \delta \in \mathbb{Z}$. Then

$$\begin{aligned} r_m(\theta, \delta) &= [(m\theta + \delta) + \theta] - [m\theta + \delta] = 0, \\ r_{m-1}(\theta, \delta) &= [m\theta + \delta] - [m\theta + \delta - \theta] = 1. \end{aligned}$$

Since $m\theta + \delta \in \mathbb{Z}$,

$$\begin{aligned} r_m + r_{m-1} &= \left[m\theta + \delta + \theta + \epsilon_J^{(m+1)}\right] - \left[m\theta + \delta - \theta + \epsilon_J^{(m-1)}\right] \\ &= [\theta + \epsilon_J^{(m+1)}] - \left[-\theta + \epsilon_J^{(m-1)}\right] = 0 - (-1) = 1 \end{aligned}$$

for sufficiently large J. Hence $r_m \neq r_{m-1}$. Now the sequences $\{r_n\}_{n\in\mathbb{Z}}$ and $\{r_n(\theta, \delta)\}_{n\in\mathbb{Z}}$ coincide for $n \neq m, m-1$, both are Markoff sequences and both take different values at $n = m, m-1$. Then they either coincide at indices $m-1$ and m or take opposite values. Hence either $r = \{r_n(\theta, 0)\}_{n\in\mathbb{Z}}$ or r is the conjugate to $\{r_n(1-\theta, 0)\}_{n\in\mathbb{Z}}$; see (1.75). $\square$

Theorem 1.86 shows that Markoff sequences are separated into two nonintersecting classes, the class of Jean Bernoulli sequences and their conjugates, corresponding to regular Markoff sequences, and the class of singular Markoff sequences obtained by finite integration of triangular sequences.

It is useful to note that singular Jean Bernoulli sequences (see Definition 1.66) are regular Markoff sequences.

Definition 1.87 *We say that a sequence $r = \{r_n\}_{n\in\mathbb{Z}}$ is a finite perturbation of a periodic sequence if it is obtained from a periodic sequence by splitting the latter up at some point into two parts, moving them a finite distance in opposite directions and filling the gap by some values.*

Theorem 1.88 *Any singular Markoff sequence is a finite perturbation of a periodic Jean Bernoulli sequence.*

Proof The proof goes by induction in the number of integrations. Any triangle sequence is a one-term perturbation of a constant sequence. The integral of a triangle sequence of type $(1, 2)$ is a triangle sequence of type $(1, 0)$, which cannot be further integrated but which is a finite perturbation of a constant sequence. If r is the integral of a triangle sequence of type $(a, a\pm 1)$, $a > 1$, then the integral is a finite perturbation of $\{r(1/a, 0)\}_{n\in\mathbb{Z}}$. Let $s = \{s_n\}_{n\in\mathbb{Z}}$ be a finite perturbation of $\{r_n(\theta)\}_{n\in\mathbb{Z}}$, so that s coincides with a shift in $\{r_n(\theta)\}_{n\in\mathbb{Z}}$ to the left of some $c < d$ and with possibly another shift to the right of d. By the integration formula (1.79) the integral $r = \int s$ with constant of integration $a > 0$ coincides to the left of $c-1$ with $\{r_n(\mu)\}_{n\in\mathbb{Z}}$, where $\mu = 1/(a+1/\theta)$. A finite number of applications of (1.79) to the indices in $[c, d]$ moves the right-hand part of $\{r_n(\mu)\}_{n\in\mathbb{Z}}$ by a finite distance to the right. Hence r is a finite perturbation of a periodic sequence. □

Theorem 1.89 *For any singular Markoff sequence $r = \{r_n\}_{n\in\mathbb{Z}}$ there are exactly two different $m_1 < m_2$ with $r_{m_i} - r_{m_i+1} = \pm 1$ and $l_{m_i}(r) = \infty$, $i = 1, 2$.*

Proof Any triangle sequence is a one-term perturbation of a constant sequence. If r is a triangle Markoff sequence then the statement of the theorem is obvious. By definition any singular Markoff sequence is obtained from a triangle sequence by a finite number of integrations. It follows that we can complete the proof by induction in a number of integrations. By Theorem 1.76 any integration keeps the number of such exceptional indices invariant. □

Theorem 1.65 shows that sequences $r = \{r_n(\theta, 0)\}_{n\in\mathbb{Z}}$ with irrational θ behave in this respect similarly to singular Markoff sequences. An important difference is that the number of such exceptional indices is 1. Notice that this class of sequences is also invariant under integration.

Theorem 1.90 *A Markoff sequence $r = \{r_n\}_{n\in\mathbb{Z}}$ is periodic if and only if $l(r) < \infty$. If $l(r) < \infty$ then r is a Jean Bernoulli sequence.*

Proof By Theorem 1.83 a periodic Markoff sequence is a periodic Jean Bernoulli sequence. By Theorem 1.64 $l(r) = q - 1$, where q is the period of r. Let $l(r) < \infty$. By definition r is either singular or regular. By Theorem 1.76 there are exactly two indices m with $l_m(r) = \infty$ if r is singular. It follows that r is regular. By Theorem 1.86 the sequence r is either a Jean Bernoulli sequence or is a conjugate to a Jean Bernoulli sequence $\{r_n(\theta)\}_{n\in\mathbb{Z}}$. In the latter case $l_{-1}(r) = \infty$. The same is true if r is the sequence $\{r_n(\theta)\}_{n\in\mathbb{Z}}$; see Theorem 1.65. By Theorem 1.65 the sequence r cannot be a nonperiodic Jean Bernoulli sequence, since otherwise $l(r) = \infty$. Hence r is a periodic Jean Bernoulli sequence. □

Remark There is another proof of Theorem 1.90, which does not use the classification of Markoff sequences. Suppose that $r = \{r_n\}_{n\in\mathbb{Z}}$ satisfies $l(r) < \infty$. By Lemma 1.72 the set of integers is a union of finite intervals $\omega_n(r)$, where $n \in E \subset \mathbb{Z}$. Clearly the length of each $\omega_n(r)$ does not exceed $2l(r)+1$. By Lemma 1.74 we have $l(s) < l(r)$, where $s = \partial r$. Differentiating step by step we arrive at a constant sequence s. Integrating back to r we obtain that r is a periodic Jean Bernoulli sequence.

Scholium 1.91 *For any Markoff sequence $r = \{r_n\}_{n\in\mathbb{Z}}$ the limit*

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} r_k = \theta(r)$$

exists.

Any Markoff sequence is either singular or regular.

It is singular if and only if either of the two following equivalent conditions is satisfied:

(a) $\theta(r) \in \mathbb{Q}$ *and r is not periodic;*

(b) *there are two integers $m_1 < m_2$ such that $l_{m_i}(r) = \infty$, $i = 1, 2$.*

Any regular Markoff sequence is either periodic or nonperiodic.

It is periodic if and only if $l(r) < \infty$. In this case $\theta(r) \in \mathbb{Q}$.

It is nonperiodic if and only if r is one of the following types:

(c) $r = \{r_n(\theta,\delta)\}_{n\in\mathbb{Z}}$, *where $m\theta + \delta \notin \mathbb{Z}$ for $m \in \mathbb{Z}$, $\theta = \theta(r) \notin \mathbb{Q}$;*

(d) *r is a shift in $\{r_n(\theta)\}_{n\in\mathbb{Z}}$, $\theta = \theta(r) \notin \mathbb{Q}$;*

(e) *r is conjugate to a sequence in* (d).

A regular nonperiodic Markoff sequence is of type (c) *if and only if $l_n(r) < \infty$ for every $n \in \mathbb{Z}$ with $r_n - r_{n+1} \neq 0$.*

A regular nonperiodic Markoff sequence is of type (d) *if and only if $l_n(r) < \infty$ for all but one index n.*

Proof By Theorem 1.88 any singular Markoff sequence is a finite perturbation of a periodic Jean Bernoulli sequence.

By Lemma 1.47 a limit exists for every Jean Bernoulli sequence. By Theorems 1.86 and 1.88 every Markoff sequence is a Jean Bernoulli sequence or the conjugate to a Jean Bernoulli sequence or a finite perturbation of a periodic Jean Bernoulli sequence. Hence a limit exists for every Markoff sequence. The rest follows from the classification of Markoff sequences. □

Exercises

1.1 Using the Euclidean algorithm, develop $177/233$ into Schwenter's continued fraction (1636):

$$\frac{177}{233} = \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{4} + \frac{1}{2}$$

and write down all its convergents.

1.2 Prove that

$$\frac{1461}{59} = 24 + \frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{1} + \frac{1}{2}$$

(see Euler 1748, §361).

1.3 For arbitrary positive integers p, q, r find all solutions to the Diophantine equation (M. Bachet 1612)

$$px - qy = r.$$

Hint: Develop p/q into a continued fraction and consider the last convergent which is not equal to p/q (see Perron 1954, §10).

1.4 Prove that a rational number $p/q > 1$ in its lowest terms is represented by a symmetric regular continued fraction

$$\frac{p}{q} = b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_2} + \frac{1}{b_1} + \frac{1}{b_0}$$

if and only if either $q^2 + 1$ or $q^2 - 1$ is divisible by p (Serret's theorem, Perron 1954, §11).

1.5 Prove that every divisor of a sum of two squares is a sum of two squares (Euler and Serret, see Perron [1954, §11]).
Hint: If $p > q$ is a divisor of $q^2 + 1$ then by Ex. 1.4 the continued fraction of p/q is symmetric with $2k + 2$ terms. Show that $p = P_{2k+1} = P_k^2 + P_{k-1}^2$. If p is a divisor of $q^2 + 1$ such that $p \leq q$ then p is a divisor of $(q - sp)^2 + 1$ for any integer s. Finally, if $(a, b) = 1$ then there are integers x, y such that $ax - by = 1$. It follows that any divisor p of $a^2 + b^2$ is a divisor of $(a^2 + b^2)(x^2 + y^2) = (ay + bx)^2 + (ax - by)^2 = (ay + bx)^2 + 1$.

1.6 Justify Staudt's construction (Fig. 1.4) of a regular pentagon inscribed into the unit circle (see Weber 1913, §106, Section 5).

- Draw two perpendicular diameters AB and CD and three lines tangent to the circle at points A, C and D.
- Choose a point c on the tangent through C such that $|Cc| = 2|AB|$ and connect S and c by a line.
- The line cS intersects the circle at N and N_1.
- Draw lines through C, N and through C, N_1.
- The tangent line at D intersects CN at Q and CN_1 at Q_1.

- Finally draw the perpendiculars to AB at n and n_1; the perpendiculars intersect the circle at P, P_1, P_3, P_2.

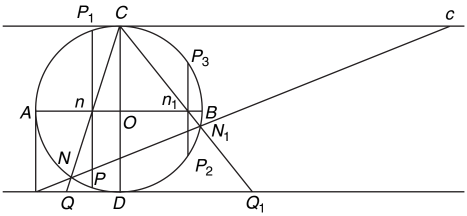

Fig. 1.4. Staudt's construction.

Then the pentagon $AP_1P_3P_2P$ is regular.
Hint: Use $\triangle SNQ \backsim \triangle cNC$[11] to prove that $|SQ|/|Cc| = |NQ|/NC|$. Apply $|QD|^2 = |NQ|\,|QC|$ to show that $|SQ|/|QD| = |Cc|\,|QD|/(|Nc|\,|Qc|)$. Since $|DC|^2 = |NC|\,|QC|$ $(\triangle CND \backsim \triangle CQD)$ and $|Cc| = 2\,|DC| = 4\,|SD|$, this implies that $|SQ|/|QD| = |QD|/SD|$. Assuming that $|SD| = 1$, obtain from here that

$$|On| = \frac{1}{2}|QD| = \frac{-1+\sqrt{5}}{4} = \cos\frac{2\pi}{5}.$$

1.7 Given $a_1 > a_2 > 1$ define a sequence $\{a_k\}_{k\geqslant 1}$ of positive numbers and a sequence $\{n_k\}_{k\geqslant 1}$ of positive integers by $a_{k+1}^{n_k} < a_k < a_{k+1}^{n_k+1}$, $a_{k+2} = a_k/a_{k+1}^{n_k}$. Prove that

$$\log_{a_1} a_2 = \frac{1}{n_1+}\,\frac{1}{n_2+}\,\frac{1}{n_2+\cdots};$$

see Boltyanskii's addendum to the Russian translation of Klein (1932).

1.8 Two graduated rulers have their zero points coincident, and the 100th graduation of one coincides exactly with the 63th of the other. Show that the 27th and 17th coincide more nearly than any other two graduations (Smith 1888, Examples XXXVI, p. 453).

1.9 Ascending continued fractions are defined by

$$\frac{b_1 + \dfrac{b_2 + \dfrac{b_3+\cdots}{a_3}}{a_2}}{a_1} \stackrel{\text{def}}{=} \frac{b_1+}{a_1}\,\frac{b_2+}{a_2}\,\frac{b_3+}{a_3}\cdots.$$

Prove that

$$\frac{b_1+}{a_1}\,\frac{b_2+}{a_2}\,\frac{b_3+}{a_3}\cdots = \frac{b_1}{a_1} + \frac{b_2}{a_1a_2} + \frac{b_3}{a_1a_2a_3} + \cdots$$

(Smith 1888, Examples XXXVI, 9, p. 454).

[11] The symbol $\backsim$ means that the triangles are similar.

1.10 Find the value of

$$x_n = \frac{n}{n\,+}\,\frac{n-1}{n-1\,+}\,\frac{n-2}{n-2\,+\mathit{cdots}+}\,\frac{2}{2\,+}\,\frac{1}{1\,+}\,\frac{1}{2}$$

(Smith 1888, Examples XXXVI, 7, p. 454).

Hint: Observe that $x_0 = 1/2$, $x_1 = 2/3$, $x_2 = 3/4$, $x_3 = 4/5$. Prove by induction that $x_n = (n+1)/(n+2)$.

1.11 Show that

$$\underbrace{\frac{1}{1\,-}\,\frac{1}{4\,-}\,\frac{1}{1\,-}\,\frac{1}{4\,-}\cdots-\,\frac{1}{4}}_{n} = \frac{2n}{n+1}$$

(Smith 1888, Examples XXXVI, 8, p. 454).

Hint: Let x_n be the nth convergent of the continued fraction. Check that $x_1 = 1$, $x_2 = 4/3$ satisfy the formula for $n = 1$ and $n = 2$. Now assuming that $n > 2$, observe that

$$x_n = \frac{1}{1\,-}\,\frac{1}{4-x_n} = \frac{4-x_n}{3-x_{n-2}}.$$

Apply induction.

1.12 Prove that P_n, Q_n in Theorem 1.4 satisfy

$$P_n Q_{n-2} - P_{n-2} Q_n = (-1)^n a_1 \cdots a_{n-1} b_n.$$

1.13 Prove that P_n, Q_n in Theorem 1.4 satisfy

$$P_n Q_{n-3} - P_{n-3} Q_n = (-1)^{n-1} a_1 \cdots a_{n-2} (b_n b_{n-1} + a_n).$$

1.14 Prove that $[\xi] + [\eta] \leqslant [\xi + \eta]$ and $0 \leqslant [\xi] - 2[\xi/2] \leqslant 1$.

1.15 Check that continued fractions for $b/(2b+1)$ and $(b+1)/(2b+1)$, where b is a positive integer, correspond to the exceptional case in Lemma 1.15.

1.16 Check Lagrange's identity

$$\frac{1}{2\,+}\,\frac{1}{b} + \frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{b} = 1, \qquad b \neq -\frac{1}{2}.$$

1.17 Prove that the mediant $(a+b)/(c+d)$ of two positive nonequal fractions a/b and c/d must lie between them.

1.18 If ξ is an irrational number and P_n/Q_n are its convergents then, for every integer k,

$$\lim_n \frac{P_{n+1} - kP_n}{Q_{n+1} - kQ_n} = \xi.$$

Hint:

$$\frac{P_{n+1}-kP_n}{Q_{n+1}-kQ_n}-\xi=\frac{1}{Q_{n+1}/Q_n-k}\left\{\left(\frac{P_{n+1}}{Q_{n+1}}-\xi\right)\frac{Q_{n+1}}{Q_n}-k\left(\frac{P_n}{Q_n}-\xi\right)\right\}.$$

By Theorems 1.5 and 1.7

$$b_{n+1}+\frac{b_{n-1}}{b_nb_{n-1}+1}<\frac{Q_{n+1}}{Q_n}<b_{n+1}+\cfrac{1}{b_n+\cfrac{b_{n-2}}{b_{n-1}b_{n-2}+1}}.$$

Observe that the function $x \to x(bx+1)^{-1}$ increases on $[1,+\infty)$, and deduce from this that $|Q_{n+1}/Q_n-k|>1/3$. For $k=1$ this exercise hints at Stolz's theorem 5.2.

1.19 (Markoff) Prove that, for any regular continued fraction,

$$\left|\frac{1}{b_1}{}_{+\cdots+}\frac{1}{b_n}-\frac{1}{b_1}{}_{+\cdots+}\frac{1}{b_{n-1}}\right|\leqslant\left|\underbrace{\frac{1}{1}{}_{+\cdots+}\frac{1}{1}}_{n}-\underbrace{\frac{1}{1}{}_{+\cdots+}\frac{1}{1}}_{n-1}\right|.$$

Hint: Combine (1.20) with (1.34).

1.20 Check Lagrange's observation (1774) that the fraction 97/400 corresponding to the Gregorian calender is neither principal nor nonprincipal convergent to 4187/17 280. However,

$$\frac{97}{400}=\frac{13P_3-P_2}{13Q_3-Q_2}.$$

1.21 A simple fraction P/Q is a best Lagrange approximation to a real *irrational* number ξ if and only if for every q, $1\leqslant q<Q$,

$$\|Q\xi\|<\|q\xi\|,\qquad P=[Q\xi+1/2]. \tag{E1.1}$$

Hint: Apply (1.37) with $q=Q$ to show that P is the closest integer to $Q\xi$. Hence $P=[Q\xi+1/2]$, $|Q\xi-P|=\|Q\xi\|$ and the inequality of (E1.1) follows by (1.37). If P and Q satisfy (E1.1) then (E1.1) implies (1.37) for $1\leqslant q<Q$. Since ξ is irrational, there is only one best approximation of $Q\xi$ by integers, $P=[Q\xi+1/2]$. Hence (1.37) holds also for $q=Q$, $p/q\neq P/Q$.

1.22 The number of positive integers m such that $1\leqslant m\leqslant n$ and $(m,n)=1$ is denoted by $\varphi(n)$. The function $n\to\varphi(n)$ is called Euler's phi function. Prove that the number of elements in $\mathfrak{F}_n$ is $1+\varphi(1)+\varphi(2)+\cdots+\varphi(n)$.

1.23 Check that for $\theta = 3/2, 4/3, 5/3, 6/5, 7/4, 7/5, 7/6$ the sequences $\{r_n(\theta, 0)\}_{n\in\mathbb{Z}}$ are given by ($n \geqslant 1$):

$$3/2 \to \{2,1,2,1,2,1,2,1,2,1,2,1,\ldots\} = [\overline{2,1}],$$

$$4/3 \to \{1,2,1,1,2,1,1,2,1,1,2,1,\ldots\} = [1,\overline{2,1_2}],$$

$$5/3 \to \{2,2,1,2,2,1,2,2,1,2,2,1,\ldots\} = [\overline{2_2,1}],$$

$$6/5 \to \{1,1,1,2,1,1,1,1,2,1,1,1,1,2,1,\ldots\} = [1,1,1,\overline{2,1_4}],$$

$$7/4 \to \{2,2,2,1,2,2,2,1,2,2,2,1,\ldots\} = [\overline{2_3,1}],$$

$$7/5 \to \{1,2,1,2,1,1,2,1,2,1,1,2,1,2,1,\ldots\} = [1,\overline{2,1,2,1_2}],$$

$$7/6 \to \{1,1,1,1,2,1,1,1,1,1,2,1,1,1,1,1,2,1,\ldots\} = [1_4,\overline{2,1_5}].$$

1.24 Check that the periods for $\theta = 3/5$ and $\theta = 7/12$ ($\delta = 0$) are

$$\{\boxed{1,0,1,}\,1,0\},\ \{\boxed{1,0,1,0,1,1,0,1,0,1,}\,1,0\};$$

the boxed parts are symmetric in agreement with Theorem 1.54.

1.25 Prove that the sequence $\{r_n\}_{n\in\mathbb{Z}}$, where $r_0 = 1$ and $r_n = 0$ for $n \neq 0$, is not a Jean Bernoulli sequence.
Hint: Apply Lemma 1.47.

1.26 Prove (Markoff [1882]) that, for any periodic $\{r_n(p/q, 0)\}_{n\geqslant 0}$,

$$\frac{r_1 + \cdots + r_q}{q} = a_0 + \frac{1}{a_1} \underset{+}{} \frac{1}{a_2} \underset{+\cdots+}{} \frac{1}{a_d} = \frac{p}{q}.$$

1.27 If $\{r_j^{(1)}\}_{j\in\mathbb{Z}}$ is a periodic Markoff sequence of 0's and 1's, then

$$\lim_{k\to+\infty} \frac{r_1^{(1)} + \cdots + r_k^{(1)}}{k} = \frac{1}{a_1 + \lim_{k\to+\infty} \dfrac{r_1^{(2)} + \cdots + r_k^{(2)}}{k}}.$$

Deduce from here the existence of the limit $\lim_{k\to+\infty}(r_1 + \cdots + r_k)/k$ for any Markoff sequence $\{r_k\}_{k\in\mathbb{Z}}$ and find the value of the limit.
Hint: Apply (1.71) and use the identity

$$k = n_1 + (n_2 - n_1) + \cdots + (n_{j_k} - n_{j_k - 1}) + k - n_{j_k},$$

where $n_1, \ldots, n_{j_k}$ is the complete list of n_j's in $[0, k)$.

1.28 If $\{r_j^{(1)}\}_{j\in\mathbb{Z}}$ is a regular Markoff sequence of 0's and 1's then prove that

$$\lim_{k\to+\infty} \frac{r_1^{(1)} + \cdots + r_k^{(1)}}{k} = \frac{1}{a_1} \underset{+}{} \frac{1}{a_2} \underset{+\cdots}{} .$$

1.29 Prove that the sequences $B(\alpha, 0)$ and $B(\beta, 0)$, see (1.62), partition the set of all positive integers $\mathbb{N}$ if and only if $\alpha + \beta = 1$ and $\alpha \in (0, 1)$ is irrational (a theorem of Beatty, 1926).
Hint: Notice that

$$f_x(z) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} z^{[n/x]} = \sum_{k=1}^{\infty} r_k(x, 0) z^k = (1-z) \sum_{k=1}^{\infty} [kx] z^k.$$

Using this and $k\alpha + k\beta = k$, prove that $f_\alpha(z) + f_\beta(z) = z(1-z)^{-1}$; see O'Bryant (2002) for details.

1.30 Recall that a sequence $\{x_n\}_{n \geqslant 0}$ of real numbers is called fundamental if for every $\varepsilon > 0$ there is a positive integer $N(\varepsilon)$ such that $|x_n - x_m| < \varepsilon$ provided that $n, m > N(\varepsilon)$. Prove that partial denominators of any given index of any fundamental sequence stabilize, and deduce from this that every fundamental sequence in $\mathbb{R}$ converges.

2

Continued fractions: algebra

2.1 Euler's algorithm

34 Euler's theorem. In (1685) Wallis proved that common fractions with denominators 2^p5^q expand into finite decimal fractions. He also showed that the length of the period of the decimal expansion of a common fraction m/n cannot exceed $n-1$. Later Lambert proved that a real number has a periodic decimal expansion (starting from some place) if and only if it is rational. Since rational numbers correspond to finite continued fractions a question arises: what are the rational numbers corresponding to periodic continued fractions?

Definition 2.1 *A continued fraction $b_0 + \mathbf{K}_{k=1}^{\infty}(1/b_k)$ is called periodic if there are $h \geqslant 0$ and $d > 0$ such that $b_{j+d} = b_j$, $j = 0, 1, \ldots$ for $j \geqslant h$. If $h = 0$ then the continued fraction is called pure periodic.*

Since $b_k \geqslant 1$ if $k \geqslant 1$, any pure periodic continued fraction satisfies $b_0 \geqslant 1$. Thus the regular continued fraction of $\sqrt{2}$ is periodic ($h = 1$, $d = 1$) and of $\phi = (\sqrt{5}+1)/2$ is pure periodic.

Theorem 2.2 (Euler) *Any periodic regular continued fraction represents a quadratic irrational.*

Proof If $\xi = b_0 + \mathbf{K}_{k=1}^{\infty}(1/b_k)$ is a pure periodic continued fraction with period d then an application of (1.17) with $n = d-1$,

$$\xi = \frac{\xi P_{d-1} + P_{d-2}}{\xi Q_{d-1} + Q_{d-2}},$$

shows that ξ satisfies the quadratic equation

$$Q_{d-1}X^2 + (Q_{d-2} - P_{d-1})X - P_{d-2} = 0\,. \tag{2.1}$$

If ξ is periodic with $h > 0$ then (1.17) for $n = h-1$ shows that ξ is a Mobius transformation with integer coefficients of a pure periodic continued fraction. Hence it is a quadratic irrational too. □

35 Theoretical form of Euler's algorithm. A natural question is whether the converse to Euler's theorem is true. This question seemingly appeared owing to Brouncker's answer to Fermat's question on the Diophantine equation $x^2 - y^2 D = 1$ (see §**43** in Section 2.3). Attacking this problem Euler discovered a convenient algorithm for the development of quadratic surds $\sqrt{D}$ of integers D that are not perfect squares into regular continued fractions. The idea is to apply algorithm (1.32) to $\xi = \sqrt{D}$. If $r_1 = [\sqrt{D}]$ then

$$\sqrt{D} = r_1 + \sqrt{D} - r_1 = r_1 + \cfrac{1}{\cfrac{\sqrt{D}+r_1}{D-r_1^2}} .$$

Let $s_1 = D - r_1^2$ and $b_1 = [(\sqrt{D}+r_1)/s_1]$. Then

$$\begin{aligned}\frac{\sqrt{D}+r_1}{D-r_1^2} = \frac{\sqrt{D}+r_1}{s_1} &= b_1 + \cfrac{1}{\cfrac{s_1(\sqrt{D}+(b_1 s_1 - r_1))}{D-(b_1 s_1 - r_1)^2}} \\ &= b_1 + \cfrac{1}{\cfrac{\sqrt{D}+r_2}{s_2}} ,\end{aligned}$$

where $r_2 = b_1 s_1 - r_1$ and

$$\frac{D-r_2^2}{s_1} = \frac{D - r_1^2 - b_1^2 s_1^2 + 2b_1 s_1}{s_1} = 1 - b_1^2 s_1 + 2b_1 = s_2 \in \mathbb{Z} .$$

Induction yields three sequences $\{r_n\}_{n\geqslant 0}$, $\{b_n\}_{n\geqslant 0}$, $\{s_n\}_{n\geqslant 0}$:

$$\begin{gathered} r_{n+1} = b_n s_n - r_n, \quad s_n s_{n+1} = D - r_{n+1}^2, \\ b_{n+1} = \left[\frac{\sqrt{D}+r_{n+1}}{s_{n+1}}\right], \end{gathered} \tag{2.2}$$

with initial conditions

$$r_0 = 0, \qquad s_0 = 1, \qquad b_0 = \left[\frac{\sqrt{D}+r_0}{s_0}\right] = [\sqrt{D}]. \tag{2.3}$$

Algorithm (2.2), (2.3) is known as Euler's algorithm. It can be run not only for quadratic surds, corresponding to the initial conditions $r_0 = 0$, $s_0 = 1$, but for any integers r_0, s_0, $s_0 \neq 0$, such that s_0 is a divisor of $D - r_0^2$. Notice that any quadratic irrational ξ can be represented as

$$\xi = \frac{\sqrt{D}+r_0}{s_0} \tag{2.4}$$

with integer r_0, s_0 and $(D-r_0^2)/s_0$. Indeed, a quadratic irrational ξ satisfies the irreducible quadratic equation

$$aX^2+bX+c=0\,,\qquad \xi=\frac{-b\pm\sqrt{D}}{2a}\,,\qquad D=b^2-4ac,$$

with integer coefficients a, b, c having greatest common divisor 1. Changing the signs of a, b, c, if necessary, we may assume that

$$\xi=\frac{\sqrt{D}+(-b)}{2a}$$

and may put $r_0=-b$, $s_0=2a$. For instance, for $\phi=(\sqrt{5}+1)/2$ the parameters $r_n=1$, $b_n=1$, $s_n=2$ of Euler's algorithm do not depend on n and are positive integers. The basic properties of the parameters of Euler's algorithm are summarized in the following theorem.

Theorem 2.3 *Let ξ be a quadratic irrational* (2.4) *such that s_0 divides $D-r_0^2$. Then*

(a) *the parameters r_n, b_n and s_n of Euler's algorithm for ξ are integers, $s_n\neq 0$ and $b_n\geqslant 1$ for $n>0$;*
(b) *if $r_0^2<D$, $s_0>0$, then b_n, s_n are positive for $n=1,2,\ldots$, $r_n^2<D$ and $r_n>0$ for $n\geqslant 2$.*

Proof (a) We apply induction. Since $s_{-1}=(D-r_0^2)/s_0$ and b_0 are integers by definition, $r_1=b_0s_0-r_0$ is integer. Assuming that s_k, r_k are integers for $k\leqslant n$, we obtain from (2.2) that r_{n+1} is integer. Next, the identity

$$s_{n+1}=\frac{D-(b_ns_n-r_n)^2}{s_n}=s_{n-1}+b_n(r_n-r_{n+1}) \tag{2.5}$$

shows that s_{n+1} is integer. Since $\sqrt{D}$ is irrational and r_n is integer, $s_n\neq 0$. It follows from

$$\begin{aligned} b_n=\left[\frac{\sqrt{D}+r_n}{s_n}\right]&<b_n+\frac{\sqrt{D}-r_{n+1}}{s_n}\\ &=b_n+\cfrac{1}{\cfrac{\sqrt{D}+r_{n+1}}{s_{n+1}}} \end{aligned} \tag{2.6}$$

that

$$0<\frac{\sqrt{D}-r_{n+1}}{s_n}=\frac{s_{n+1}}{\sqrt{D}+r_{n+1}}<1. \tag{2.7}$$

Therefore $(\sqrt{D}+r_{n+1})/s_{n+1}>1$, implying $b_{n+1}\geqslant 1$.

(b) Suppose now that $r_0^2 < D$ and $s_0 > 0$. Then $b_0 \geqslant 0$ and $b_n \geqslant 1$ for $n \geqslant 1$; see part (a). Suppose that $s_n > 0$ and $r_n^2 < D$. Then on the one hand the first inequality in (2.7) implies that $\sqrt{D} - r_{n+1} > 0$. On the other hand,

$$\sqrt{D} + r_{n+1} = \sqrt{D} + b_n s_n - r_n = s_n \left\{ \frac{\sqrt{D} - r_n}{s_n} + b_n \right\}$$

shows that $\sqrt{D} + r_{n+1} > 0$. Hence $r_{n+1}^2 < D$. Finally

$$s_{n+1} = \frac{D - r_{n+1}^2}{s_n} > 0.$$

Let $n \geqslant 1$. Since $s_n > 0$, we have by the definition of b_n

$$b_n s_n < \sqrt{D} + r_n < b_n s_n + s_n \; .$$

Therefore the inequality $r_n \geqslant b_n s_n$ implies that $s_n > \sqrt{D} > |r_n|$, which contradicts the assumption that $r_n \geqslant b_n s_n \geqslant s_n$, since $b_n \geqslant 1$. Hence $r_n < b_n s_n$ and $r_{n+1} = b_n s_n - r_n > 0$. □

The proof of the above theorem follows Smith (1888). The condition $n \geqslant 2$ in (b) is essential: if $\xi = (\sqrt{5} + 1)/4$ then $r_1 = -1$.

The parameters r_n and s_n of Euler's algorithm are related by the remainders ξ_n of the continued fraction for ξ.

Lemma 2.4 *For every $n \geqslant 0$,*

$$\xi_n = \frac{\sqrt{D} + r_n}{s_n}.$$

Proof Assuming that the formula is true for n, we obtain

$$\frac{1}{\xi_{n+1}} = \xi_n - b_n = \frac{\sqrt{D} + r_n}{s_n} - b_n = \frac{D - r_{n+1}^2}{s_n(\sqrt{D} + r_{n+1})} = \frac{s_{n+1}}{\sqrt{D} + r_{n+1}}$$

and thus that it is true for $n + 1$. □

36 Computational form of Euler's algorithm. Euler's algorithm is very convenient for developing quadratic surds $\sqrt{D}$ into regular continued fractions. This is provided by Euler's observation that

$$b_n = \left[\frac{\sqrt{D} + r_n}{s_n} \right] = \left[\frac{b_0 + r_n}{s_n} \right] \tag{2.8}$$

if $s_n > 0$. Indeed, since $b_0 = [\sqrt{D}]$, we see from

$$b_0 + r_n < \sqrt{D} + r_n < b_0 + r_n + 1$$

that there are no integers between the expressions in these inequalities. Since $s_n \geqslant 1$ is integer, no integer may appear after division by s_n, which implies (2.8). Another useful formula for these computations is (2.5).

We illustrate Euler's computations from (1765, §14) for $D = 31$:

$$\begin{array}{llll}
r_1 = 5 & s_1 = 6 & b_1 = \left[\frac{10}{6}\right] & = 1 \\
r_2 = 6 - 5 = 1 & s_2 = 1 + 1 \times 4 = 5 & b_2 = \left[\frac{6}{5}\right] & = 1 \\
r_3 = 5 - 1 = 4 & s_3 = 6 - 1 \times 3 = 3 & b_3 = \left[\frac{9}{3}\right] & = 3 \\
r_4 = 9 - 4 = 5 & s_4 = 5 - 3 \times 1 = 2 & b_4 = \left[\frac{10}{2}\right] & = 5 \\
r_5 = 10 - 5 = 5 & s_5 = 3 + 5 \times 0 = 3 & b_5 = \left[\frac{10}{3}\right] & = 3 \\
r_6 = 9 - 5 = 4 & s_6 = 2 + 3 \times 1 = 5 & b_6 = \left[\frac{9}{5}\right] & = 1 \\
r_7 = 5 - 4 = 1 & s_7 = 3 + 1 \times 3 = 6 & b_7 = \left[\frac{6}{6}\right] & = 1 \\
r_8 = 6 - 1 = 5 & s_8 = 5 - 1 \times 4 = 1 & b_8 = \left[\frac{10}{1}\right] & = 10
\end{array}$$

One can easily see that Euler's algorithm repeats periodically rows already obtained. Hence

$$\sqrt{31} = 5 + \frac{1}{1} + \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{3} + \frac{1}{1} + \frac{1}{1} + \frac{1}{10} + \frac{1}{1} + \cdots. \tag{2.9}$$

In (1765) Euler found continued fractions for $\sqrt{D}$ for all D up to $D = 120$.

37 Periodicity. We consider first what Euler's algorithm can do.

Theorem 2.5 *Let*

$$\xi = \frac{\sqrt{D} + r}{s},$$

where $D, s \in \mathbb{N}$, D is not a perfect square and $r \in \mathbb{Z}$ satisfies $r^2 < D$. Then the continued fraction of ξ is periodic.

Proof The following identity,

$$\frac{\sqrt{D} + r}{s} = \frac{\sqrt{s^2 D} + sr}{s^2},$$

shows that we may assume that s divides $D - r^2$. By Theorem 2.3 for $n \geqslant 2$ the parameters r_n in Euler's algorithm for ξ satisfy $0 < r_n < \sqrt{D}$. Hence there are at most $[\sqrt{D}]$ possible values for r_n. It follows from $r_{n+1} + r_n = b_n s_n$ that s_n cannot exceed $2[\sqrt{D}]$. These gives not more than $2D$ possible choices for the remainders $\xi_k = (\sqrt{D} + r_k)/s_k$ of the regular continued fraction for ξ. It follows that $\xi_k = \xi_l$ for some pair of positive integers $k < l \leqslant 2D + 1$, implying that ξ is periodic. $\square$

Corollary 2.6 *The continued fraction of any surd $\xi = \sqrt{R}$, $1 < R \in \mathbb{Q} \setminus \mathbb{Q}^2$ is periodic.*

Proof Let $R = p/q$ be the representation of R in lowest terms. Then $\sqrt{R} = \sqrt{D}/q$, where $D = pq$ is a positive integer. Since q divides D, we may apply Theorem 2.5 by putting $s_0 = q$, $r_0 = 0$. $\square$

38 Quadratic surds. A simple analysis of (2.9) shows that the last $b_d = 10$ in Euler's table given above is $2 \times 5 = 2 \times b_0$. Also the part 1, 1, 3, 5, 3, 1, 1, between b_0 and $2b_0$ is symmetric; this is true for other examples of Euler's with $D \leqslant 120$. The proof of this fact follows from a property of real quadratic fields $\mathbb{Q}(\sqrt{R})$, where $R \in \mathbb{Q} \setminus \mathbb{Q}^2$. By definition $\mathbb{Q}(\sqrt{R})$ is the set of all $\xi = (p + q\sqrt{R})/r$, where $p, q, r \in \mathbb{Z}$, $r \neq 0$. There is a Galois automorphism acting on $\mathbb{Q}(\sqrt{R})$ with $\mathbb{Q}$ as a fixed field that sends ξ to the algebraically conjugate element ξ^*:

$$\xi^* = \frac{p - q\sqrt{R}}{r} .$$

Lemma 2.7 *Let the regular continued fraction $b_0 + \mathbf{K}_{k \geqslant 1}(1/b_k)$ of $\sqrt{R}$, $1 < R \in \mathbb{Q}$, have period d. Then $\xi_1 = \xi_{d+1}$.*

Proof By Euler's formula (1.17) $\xi_k \in \mathbb{Q}(\sqrt{R})$, $k = 0, 1, \ldots$ Applying the Galois automorphism to $\xi_n = b_n + \xi_{n+1}^{-1}$, we obtain that

$$\xi_n^* = b_n + \frac{1}{\xi_{n+1}^*} . \tag{2.10}$$

This together with $b_n \geqslant 1$ shows that the inclusion $\xi_n^* \in (-1, 0)$ implies $\xi_{n+1}^* \in (-1, 0)$. However, $\xi_1^* = -(\sqrt{R} + b_0)^{-1} \in (-1, 0)$. It follows that $\xi_n^* \in (-1, 0)$ for $n = 1, 2, \ldots$ In addition (2.10) implies that

$$b_n = -\left[\frac{1}{\xi_{n+1}^*}\right], \qquad n = 1, 2, \ldots \tag{2.11}$$

By the assumption of the lemma, $\xi_k = \xi_{k+d}$ for some $k \geqslant 1$. Let k be the minimal k with this property. If $k > 1$ then (2.11) with $n = k - 1$ shows that $b_{k-1} = b_{l-1}$. Hence $\xi_{k-1} = \xi_{l-1}$, contradicting the choice of k. $\square$

Theorem 2.8 *A periodic regular continued fraction $[b_0; b_1, b_2, \ldots]$ with period $\{b_1, \ldots, b_d\}$ satisfies*

$$b_d = 2b_0, \qquad \{b_1, b_2, \ldots, b_{d-1}\} = \{b_{d-1}, \ldots, b_2, b_1\} \tag{2.12}$$

if and only if it is the regular continued fraction of $\sqrt{R}$, $1 < R \in \mathbb{Q} \setminus \mathbb{Q}^2$.

Proof Comparing the second formula of (1.18) with

$$2b_0 + \frac{1}{b_1} \, {}_{+} \, \frac{1}{b_2} \, {}_{+ \cdots +} \, \frac{1}{b_{d-1}} = \frac{P_{d-1}}{Q_{d-1}} + b_0,$$

we obtain that (2.12) is equivalent to

$$P_{d-1} + b_0 Q_{d-1} = Q_d \,. \tag{2.13}$$

Let $1 < R \in \mathbb{Q}$, $\sqrt{R} \notin \mathbb{Q}$. By Corollary 2.6 the continued fraction of $\sqrt{R}$ has period d. By Lemma 2.7 $\xi_{d+1} = \xi_1 = 1/(\sqrt{R} - b_0)$. Hence by (1.17)

$$\sqrt{R} = \frac{P_d + (\sqrt{R} - b_0)P_{d-1}}{Q_d + (\sqrt{R} - b_0)Q_{d-1}} \,.$$

Comparing the coefficients for $\sqrt{R} \notin \mathbb{Q}$ in this formula, we obtain (2.13).

If the regular continued fraction of $\xi \in \mathbb{R}$ satisfies (2.12) then $\xi_d = b_0 + \xi$. By (1.17)

$$\xi = \frac{P_{d-1}(\xi + b_0) + P_{d-2}}{Q_{d-1}(\xi + b_0) + Q_{d-2}},$$

which is equivalent to the quadratic equation

$$Q_{d-1}\xi^2 + (b_0 Q_{d-1} + Q_{d-2} - P_{d-1})\xi - (b_0 P_{d-1} + P_{d-2}) = 0.$$

By (2.13) and (1.15),

$$b_0 Q_{d-1} + Q_{d-2} - P_{d-1} = b_0 Q_{d-1} + Q_{d-2} - (Q_d - b_0 Q_{d-1}) = 0. \tag{2.14}$$

Hence

$$1 \leqslant b_0 \frac{P_{d-1}}{Q_{d-1}} < \frac{b_0 P_{d-1} + P_{d-2}}{Q_{d-1}} = \xi^2 \in \mathbb{Q}, \tag{2.15}$$

since $P_k > 0$ for every k. □

Theorem 2.8 leaves open the question for which b_0 and symmetric sequences $b_1, b_2, \ldots, b_{d-1}$ the number $R = D$ is integer. In (1765) Euler proposed an interesting method to answer this question for $d \leqslant 8$. We present here Euler's result in a general form, following Perron (1954, Theorem 3.17). Applications of Euler's method to solutions of Pell's equation will be discussed later in §§**43**, **44** in Section 2.3.

To begin with, the b_0 are exactly those integers which make integer the right-hand side of (2.15). To write down formulas showing the dependence of P_{d-1} and P_{d-2} on b_0 we consider the finite continued fraction with convergents

$$b_1 + \frac{1}{b_2} {}_{+\cdots+} \frac{1}{b_k} \equiv \frac{\mathfrak{P}_{k-1}}{\mathfrak{Q}_{k-1}}, \qquad k = 1, \ldots, d-1. \tag{2.16}$$

Since

$$\frac{P_k}{Q_k} = b_0 + \frac{\mathfrak{Q}_{k-1}}{\mathfrak{P}_{k-1}} = \frac{b_0 \mathfrak{P}_{k-1} + \mathfrak{Q}_{k-1}}{\mathfrak{P}_{k-1}},$$

we obtain that

$$\begin{aligned} P_k &= b_0 \mathfrak{P}_{k-1} + \mathfrak{Q}_{k-1}, \\ Q_k &= \mathfrak{P}_{k-1}, \end{aligned} \qquad k = 1, \ldots, d-1. \tag{2.17}$$

Combining (2.17) with (2.15), we get the formula

$$\xi^2 = b_0^2 + \frac{b_0(\mathfrak{Q}_{d-2} + \mathfrak{P}_{d-3}) + \mathfrak{Q}_{d-3}}{\mathfrak{P}_{d-2}}. \tag{2.18}$$

Lemma 2.9 $\mathfrak{P}_{d-3} = \mathfrak{Q}_{d-2}$.

Proof By (2.14), $b_0 Q_{d-1} + Q_{d-2} = P_{d-1}$, which implies by (2.17) that

$$b_0 \mathfrak{P}_{d-2} + \mathfrak{P}_{d-3} = P_{d-1}. \tag{2.19}$$

The proof is completed by comparing this equation with the first equation in (2.17) for $k = d-1$. □

Theorem 2.10 (Euler 1765) *Let $\{b_1, b_2, \ldots, b_{d-1}\}$ be a symmetric sequence of positive integers. Let $\mathfrak{P}_k$ and $\mathfrak{Q}_k$ be the numerators and denominators of the convergents to*

$$b_1 + \frac{1}{b_2} {+ \cdots +} \frac{1}{b_{d-1}}.$$

Then the square of

$$\sqrt{D} = b_0 + \frac{1}{b_1} {+ \cdots +} \frac{1}{b_1} {+} \frac{1}{2b_0} {+} \frac{1}{b_1} {+ \cdots}$$

is a positive integer if and only if there is an integer m such that both the numbers

$$b_0 = \tfrac{1}{2}\left(m\mathfrak{P}_{d-2} - (-1)^d \mathfrak{Q}_{d-3}\mathfrak{P}_{d-3}\right), \qquad D - b_0^2 = m\mathfrak{P}_{d-3} - (-1)^d \mathfrak{Q}_{d-3}^2$$

are positive integers.

Proof Applying Lemma 2.9 to (2.18) and (1.16) to (2.16), we obtain the system

$$\begin{aligned} (D^2 - b_0^2)\mathfrak{P}_{d-2} - 2b_0\mathfrak{Q}_{d-2} &= \mathfrak{Q}_{d-3}, \\ \mathfrak{Q}_{d-3}\mathfrak{P}_{d-2} - \mathfrak{P}_{d-3}\mathfrak{Q}_{d-2} &= (-1)^{d-3}. \end{aligned} \tag{2.20}$$

Multiplying the second equation in (2.20) by $(-1)^{d-3}\mathfrak{Q}_{d-3}$, we see that the pair $x = (-1)^{d-3}\mathfrak{Q}_{d-3}^2$, $y = (-1)^{d-3}\mathfrak{Q}_{d-3}\mathfrak{P}_{d-3}$ is a solution to

$$x\mathfrak{P}_{d-2} - y\mathfrak{Q}_{d-2} = \mathfrak{Q}_{d-3}.$$

It follows that

$$u = (D^2 - b_0^2) - (-1)^{d-3}\mathfrak{Q}_{d-3}^2, \qquad v = 2b_0 - (-1)^{d-3}\mathfrak{Q}_{d-3}\mathfrak{P}_{d-3}$$

is a solution to the homogeneous equation

$$u\mathfrak{P}_{d-2} - v\mathfrak{Q}_{d-2} = 0.$$

Hence $u = m\mathfrak{Q}_{d-2}$ and $v = m\mathfrak{P}_{d-2}$, where m is an integer. Thus

$$\begin{aligned} 2b_0 &= m\mathfrak{P}_{d-2} - (-1)^d \mathfrak{Q}_{d-3}\mathfrak{P}_{d-3}, \\ D - b_0^2 &= m\mathfrak{Q}_{d-2} - (-1)^d \mathfrak{Q}_{d-3}^2 \\ &= m\mathfrak{P}_{d-3} - (-1)^d \mathfrak{Q}_{d-3}^2 \end{aligned}$$

by Lemma 2.9. □

Let us illustrate Theorem 2.10 by the examples of the symmetric sequence 1, 1, 3, 5, 3, 1, 1 corresponding to $\sqrt{31}$ and 1, 1, 1, 1 corresponding to $\sqrt{13}$. By the Euler–Wallis formulas (1.15),

$$\begin{array}{lllllll} \mathfrak{P}_0 = 1, & \mathfrak{P}_1 = 2, & \mathfrak{P}_2 = 7, & \mathfrak{P}_3 = 37, & \mathfrak{P}_4 = 118, & \mathfrak{P}_5 = 155, & \mathfrak{P}_6 = 273, \\ \mathfrak{Q}_0 = 1, & \mathfrak{Q}_1 = 1, & \mathfrak{Q}_2 = 4, & \mathfrak{Q}_3 = 21, & \mathfrak{Q}_4 = 67, & \mathfrak{Q}_5 = 88, & \mathfrak{Q}_6 = 155. \end{array}$$

We have $d = 8$. Notice that $\mathfrak{Q}_6 = 155 = \mathfrak{P}_5$, confirming Lemma 2.9. Now

$$b_0 = 273k + 5, \quad D = 74\,529k^2 + 3040k + 31, \qquad k = 0, 1, 2, \ldots$$

In the case $D = 13$, we have $d = 5$ and

$$\begin{array}{llll} \mathfrak{P}_0 = 1, & \mathfrak{P}_1 = 1, & \mathfrak{P}_2 = 3, & \mathfrak{P}_3 = 5, \\ \mathfrak{Q}_0 = 1, & \mathfrak{Q}_1 = 1, & \mathfrak{Q}_2 = 2, & \mathfrak{Q}_3 = 3. \end{array}$$

By Euler's formulas,

$$b_0 = 5k + 3, \qquad D = 25k^2 + 36k + 13. \qquad k = 0, 1, 2, \ldots$$

Using Euler's method one can easily obtain continued fractions for squared surds of integers starting from any symmetric sequence.

39 Some identities for Euler's parameters. The parameters r_k and s_k from §**35** have a simple algebraic meaning.

Lemma 2.11 *Let $D > 1$ be an integer that is not a perfect square and let ξ be a quadratic irrational* (2.4) *such that s_0 divides $D - r_0^2$. Let $\xi = b_0 + \mathbf{K}_{k\geqslant 1}(1/b_k)$ be the regular continued fraction for ξ. Let $\{r_k\}$, $\{s_k\}$ be the parameters of Euler's algorithm* (2.2)*. Then the remainders ξ_k of the continued fraction for $\sqrt{D}$ satisfy*

$$p_k(X) = s_k X^2 - 2r_k X - s_{k-1} = 0. \tag{2.21}$$

Proof If ξ_k^* is the algebraic conjugate to ξ_k then Viète's formulas show that

$$\begin{aligned} \xi_k + \xi_k^* &= \frac{\sqrt{D} + r_k}{s_k} + \frac{-\sqrt{D} + r_k}{s_k} = \frac{2r_k}{s_k}, \\ \xi_k \xi_k^* &= \frac{\sqrt{D} + r_k}{s_k} \, \frac{-\sqrt{D} + r_k}{s_k} = \frac{-s_{k-1}}{s_k}, \end{aligned}$$

which imply (2.21). □

Corollary 2.12 *The discriminant of* (2.21) *is* $4D$.

Proof By (2.2) the discriminant of (2.21) is $4r_k^2 + 4s_k s_{k-1} = 4D$. □

Euler's algorithm (2.2) can be stated in terms of the quadratic polynomials $p_k(x)$, see (2.21). To see this we substitute the recursion formula for r_{n+1} from (2.2) into (2.5), and rewrite the recursion formula for r_{n+1} from (2.2). This yields

$$\begin{aligned} r_{n+1} &= \tfrac{1}{2} p_n'(b_n) \ , \\ s_{n+1} &= -p_n(b_n), \end{aligned} \tag{2.22}$$

where p_n is defined by (2.21). There are also simple formulas relating r_n, s_n to the numerators P_n and denominators Q_n of the convergents P_n/Q_n to $\xi = \sqrt{D}$. We consider a more general case, $R \in \mathbb{Q} \setminus \mathbb{Q}^2$, $R > 1$. If $R = p/q$ then $\sqrt{R} = \sqrt{D}/q$ where $D = pq$ is an integer. It follows that we may apply Euler's algorithm with $r_0 = 0$ and $s_0 = q$.

Lemma 2.13 (Euler 1765, §23) *For $n = 0, 1, \ldots$ we have*

$$q(Q_n Q_{n-1} R - P_n P_{n-1}) = (-1)^{n+1} r_{n+1},$$
$$q(P_n^2 - Q_n^2 R) = (-1)^{n+1} s_{n+1}.$$

Proof By (1.17),

$$\sqrt{R} = \frac{\xi_{n+1} P_n + P_{n-1}}{\xi_{n+1} Q_n + Q_{n-1}}.$$

Solving this equation in ξ_{n+1}, we obtain

$$\begin{aligned} \xi_{n+1} &= -\frac{P_{n-1} - Q_{n-1}\sqrt{R}}{P_n - Q_n\sqrt{R}} = -\frac{(P_{n-1} - Q_{n-1}\sqrt{R})(P_n + Q_n\sqrt{R})}{P_n^2 - Q_n^2 R} \\ &= \frac{Q_n Q_{n-1} R - P_n P_{n-1} + (-1)^{n-1}\sqrt{R}}{P_n^2 - Q_n^2 R} = \frac{\sqrt{D} + r_{n+1}}{s_{n+1}} \end{aligned} \tag{2.23}$$

by Lemma 2.4. Since $\sqrt{R} = \sqrt{D}/q$ is irrational, this proves the lemma. □

Compare these formulas for $q = 1$ with Brouncker's formulas in the proof of Lemma 2.21 below. Applying the Galois automorphism to (2.23), we obtain

$$\frac{P_n + Q_n\sqrt{R}}{P_{n-1} + Q_{n-1}\sqrt{R}} = \frac{r_{n+1} + \sqrt{D}}{s_n},$$

which results in a beautiful identity,

$$P_n + Q_n\sqrt{R} = \frac{(r_1 + \sqrt{D}) \cdots (r_{n+1} + \sqrt{D})}{s_0 \cdots s_n}. \tag{2.24}$$

Corollary 2.14 *Let $1 < R \in \mathbb{Q} \setminus \mathbb{Q}^2$ and let d be the period of the regular continued fraction for $\sqrt{R}$. Let P_k/Q_k be its convergents. Then*

$$P_{d-1}^2 - Q_{d-1}^2 R = (-1)^d. \tag{2.25}$$

Proof By Lemma 2.7 $\xi_1 = \xi_{d+1}$. By (2.12) $b_d = 2b_0$. Assuming that $R = p/q$, $D = pq$, we obtain

$$\frac{\sqrt{D}}{q} = b_0 + \frac{1}{\xi_1} = b_0 - b_d + b_d + \frac{1}{\xi_{d+1}} = +\xi_d = -b_0 + \frac{\sqrt{D} + r_d}{s_d}.$$

It follows that $s_d = q$, $r_d = b_0 q$. Substituting these values into (2.23), we obtain (2.25). □

For instance, for $p = 3$, $q = 2$,

$$\sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2} = 1 + \frac{1}{4} {}_{+} \frac{1}{2} {}_{+} \frac{1}{4} {}_{+} \cdots .$$

Thus $d = 2$, $P_1 = 5$, $Q_1 = 4$ and $5^2 - 4^2 \times \frac{3}{2} = 25 - 8 \times 3 = 1$ as Corollary 2.14 claims.

2.2 Lagrange's theorem

40 Lagrange's theorem. Euler's theorem, 2.2, raises the question of existing quadratic irrationals with nonperiodic regular continued fractions. This question is answered in the negative by Lagrange's theorem. The crucial role in Lagrange's proof is played by *reduced* quadratic irrationals, which in fact we have already used in the proof of Lemma 2.7.

Definition 2.15 *A quadratic irrational ξ is called reduced* if $\xi > 1$ and the algebraically conjugate irrational ξ^* satisfies $-1 < \xi^* < 0$.

The golden ratio is a reduced quadratic irrational. By the following lemma, Euler's algorithm eventually leads to reduced quadratic irrationals.

Lemma 2.16 *For any real quadratic irrational ξ there is an integer N_ξ such that ξ_n is reduced if $n > N_\xi$.*

Proof Applying Galois' automorphism to Euler' formula (1.17), we obtain

$$\xi^* = \frac{P_{n-1}\xi_n^* + P_{n-2}}{Q_{n-1}\xi_n^* + Q_{n-2}},$$

or equivalently

$$\xi_n^* = -\frac{\xi^* Q_{n-2} - P_{n-2}}{\xi^* Q_{n-1} - P_{n-1}} = -\frac{Q_{n-2}}{Q_{n-1}} \frac{\xi^* - P_{n-2}/Q_{n-2}}{\xi^* - P_{n-1}/Q_{n-1}} .$$

By Theorem 1.11, $\lim_n P_{n-2}/Q_{n-2} = \lim_n P_{n-1}/Q_{n-1} = \xi$. Therefore

$$\xi_n^* = -\frac{Q_{n-2}}{Q_{n-1}}(1+\varepsilon_n),$$

where $\lim_n \varepsilon_n = 0$. It follows that $\xi_n^* < 0$ for $n \geqslant N_\xi$. Hence (2.10) can hold for $n \geqslant N_\xi$ only if $-1 < \xi_{n+1}^* < 0$. □

Lemma 2.17 *Let $0 < D$, $D \in \mathbb{Z}$, $D \neq P^2$, $P \in \mathbb{Z}$. Then there are not more than D reduced quadratic irrationals with discriminant D.*

Proof Any reduced ξ with discriminant D satisfies the quadratic equation

$$aX^2 + bX + c = 0, \qquad D = b^2 - 4ac,$$

where $0 < a \in \mathbb{Z}$, $b, c \in \mathbb{Z}$ and the greatest common divisor of a, b, c is 1. Substituting the formulas

$$\xi = \frac{-b+\epsilon\sqrt{D}}{2a}, \qquad \xi^* = \frac{-b-\epsilon\sqrt{D}}{2a}, \qquad \epsilon = \pm 1$$

into $-1 < \xi^* < 0 < 1 < \xi$, we see that $\epsilon = 1$ and

$$0 < b + \sqrt{D} < 2a < -b + \sqrt{D}. \tag{2.26}$$

It follows that $b < -b$ and therefore $b < 0$. Then the first inequality in (2.26) shows that $0 < |b| < \sqrt{D}$, whereas the second shows that $0 < a < \sqrt{D}$. Any choice of a and b determines c. Hence there are not more than D equations with discriminant D. □

Corollary 2.18 *A quadratic irrational $\xi = (\sqrt{D} + r)/s$, where r, s are integer, is reduced if and only if*

$$0 < \sqrt{D} - r < s < \sqrt{D} + r. \tag{2.27}$$

Theorem 2.19 (Lagrange) *The regular continued fraction of a quadratic irrational is periodic.*

Proof By Lemma 2.16 ξ_n for any $n > N_\xi$ is a reduced quadratic irrational. By Lemma 2.11 ξ_n and ξ_n^* are the roots of (2.21). By Corollary 2.12 the discriminant $4r_n^2 + 4s_n s_{n-1} = 4D$ of ξ_n does not depend on n. By Corollary 2.18 for $n > N_\xi$, $0 < r_n < \sqrt{D}$ and $0 < s_n < 2\sqrt{D}$. Hence in not more than $2D$ steps ξ_n will coincide with some ξ_m, $m > n$. □

41 Ballieu's approach. It turned out that Lagrange's theorem could also be proved in the framework of Euler's algorithm (Perron 1954, Chapter III, §22). The following arguments, which could have been used by Euler, were discovered only by Ballieu (1942). Recall that the parameters of Euler's algorithm are integer coefficients of quadratic equations (2.21). If $s_n s_{n-1} > 0$ for some n then by Viète's formula $\xi_n^* < 0$. Since $\xi_n > b_n \geqslant 1$, we must have $0, b_n \in (\xi_n^*, \xi_n)$. Therefore $p_n(0) = -s_{n-1}$ and $p_n(b_n) = -s_{n+1}$ (see (2.22)) have the same sign. It follows that

$$0 < p_n(0)p_n(b_n) = s_{n-1}s_{n+1} = \frac{s_{n-1}s_n}{s_n^2} s_n s_{n+1} .$$

Hence $s_{k-1}s_k > 0$ for every integer $k \geqslant n$, which implies that $r_k^2 = D - s_{k-1}s_k < D$. This and $\sqrt{D} + r_k/s_k > b_k \geqslant 1$ imply that $s_k > 0$ for $k = n, n+1, \ldots$ Since $r_k^2 < D$ and $s_{k-1}s_k < D$, this gives at most $2[\sqrt{D}]+1$ possibilities for r_k and at most $D-1$ possibilities for s_k. Therefore $\xi_k = \xi_l$ for some pair $n \leqslant k < l \leqslant n + D + 2D\sqrt{D}$. Hence the continued fraction of ξ_0 is periodic.

To complete the proof we need only show that it is not possible that $s_{k-1}s_k < 0$ for every k. Assuming the contrary, we obtain that $\xi_k^* = -s_{k-1}/s_k\xi_k > 0$. It follows that $2r_k/s_k = \xi_k^* + \xi_k > 0$ or that s_k and r_k have the same sign. Hence all terms in $r_k = b_k s_k + (-r_{k+1})$ have the same sign. Iterating we obtain

$$\begin{aligned} |r_1| = b_1|s_1| + |r_2| &= b_1|s_1| + b_2|s_2| + |r_3| = \cdots \\ &= b_1|s_1| + \cdots + b_n|s_n| + |r_n|, \end{aligned}$$

which is obviously impossible for large n.

42 Galois' theorem. This theorem describes continued fractions of reduced quadratic irrationals.

Theorem 2.20 (Galois) *A quadratic irrational ξ can be the value of a pure periodic continued fraction if and only if it is reduced. If the continued fraction of ξ is pure periodic with period d and with the first few partial denominators $b_0, b_1, \ldots, b_{d-1}$, then $-1/\xi^*$ is also pure periodic with the same period d and the first partial denominators $b_{d-1}, b_{d-2}, \ldots, b_0$.*

Proof If the continued fraction of ξ is pure periodic then $\xi = b_0 + 1/\xi_1 > 1$ and ξ is a root of a polynomial $p(X)$ in (2.1). Since $P_{-1} = 1$, $P_0 = b_0 \geqslant 1$, we see that $P_n, Q_n > 0$ for all n and the sequences $\{P_n\}$, $\{Q_n\}$ increase. Therefore

$$p(0) = -P_{d-2} < 0\,, \qquad p(-1) = Q_{d-1} - Q_{d-2} + P_{d-1} - P_{d-2} > 0$$

imply that $p(x)$ must vanish in $(-1, 0)$ by the intermediate value theorem. Hence ξ is a reduced quadratic irrational. Since ξ is reduced, $-1 < \xi^* < 0$ and $-\xi^* = b_0 + 1/\xi_1^*$. Then $\xi^* \in (-1, 0)$ and $b_0 \geqslant 1$ imply $\xi_1^* \in (-1, 0)$. By induction we obtain that ξ_n is a reduced quadratic irrational for every n.

By Lagrange's theorem there exists a minimal n such that $\xi_n = \xi_{n+d}$ for some $d > 0$. If $n = 0$ then ξ is pure periodic. Suppose that $n > 0$. Then

$$\xi_{n-1} - \xi_{n+d-1} = b_{n-1} + \frac{1}{\xi_n} - b_{n+d-1} - \frac{1}{\xi_{n+d}} = b_{n-1} - b_{n+d-1}\,.$$

Passing to the algebraically conjugate irrationals, we obtain that

$$\xi^*_{n-1} - \xi^*_{n+d-1} = b_{n-1} - b_{n+d-1},$$

which is only possible if $b_{n-1} = b_{n+d-1}$, since both ξ^*_{n-1} and ξ^*_{n+d-1} are in $(-1, 0)$. It follows that $\xi_{n-1} = \xi_{n+d-1}$, which contradicts the choice of n.

Let $\xi = \xi_0$ be a pure periodic quadratic irrational. Then

$$\xi_0 = b_0 + \frac{1}{\xi_1}, \quad \ldots, \quad \xi_{d-2} = b_{d-2} + \frac{1}{\xi_{d-1}}, \quad \xi_{d-1} = b_{d-1} + \frac{1}{\xi_d}\,. \tag{2.28}$$

If d is the period of ξ then $\xi_0 = \xi_d$. Applying the Galois automorphism to the last equality of (2.28), we obtain that

$$-\frac{1}{\xi^*_0} = b_{d-1} + \frac{1}{-1/\xi^*_{d-1}}\,.$$

Moving from the right to the left in (2.28) , we obtain

$$-\frac{1}{\xi^*_0} = b_{d-1} + \frac{1}{b_{d-2}} {}_{+\cdots} \frac{1}{b_0} {}_{+} \frac{1}{-1/\xi^*_0},$$

implying the second statement of the theorem since $-1/\xi^*_0 > 1$. □

2.3 Pell's equation

43 Brouncker's solution to Fermat's question. By 1657 John Wallis' *Arithmetica Infinitorum* had reached Pierre de Fermat in Toulouse, Italy. Fermat, interested in number theory, addressed to Wallis the challenge of solving the Diophantine equation

$$x^2 = 1 + y^2 D, \tag{2.29}$$

in positive integers x and y. Here D is a positive integer that is not a square. If D were a perfect square then (2.29) would not have positive-integer solutions. In general squared factors of D can be incorporated in y^2. Here we will pass by the details of the initial misunderstanding of this problem on the part of Brouncker and Wallis. They may be found in an interesting paper by Stedall (2000a) and in Edwards (1977).

Naturally Brouncker examined first the simplest case $D = 2$ and found the following solutions:

$$\begin{array}{rccc} x = & 3 & 17 & 99 \\ y = & 2 & 12 & 70 \end{array}. \tag{2.30}$$

The story of Wallis' product, which is discussed in §**60**, Section 3.2, shows that Brouncker, as the great expert in continued fractions in his time, developed the theory of positive continued fractions, which he successfully applied to the quadrature problem. Therefore he almost certainly must have known the series (1.10), in which the quotients x/y from (2.30) occur at the odd positions. Using his formulas (1.15) Brouncker could easily find the next pair in (1.10):

$$x = 2 \times 239 + 99 = 577,$$
$$y = 2 \times 169 + 70 = 408.$$

Now easy calculations show that

$$1 + 2 \times 408^2 = 332\ 929 = 577^2.$$

The only conclusion which one may derive from this is that the solutions to (2.29) are given by the numerators and denominators of the odd convergents to $\sqrt{D}$, at least for $D = 2$.

There is no direct evidence that Brouncker argued in this way. However, the form in which he sent his solution to Wallis (see Stedall 2000a, pp. 321–2) indicates that it is likely that he found it by continued fractions:

$$2 \times Q: \quad 2 \times 5\frac{1}{1} = 12, \quad 12 \times 5\frac{5}{6} = 70, \quad 70 \times 5\frac{29}{35} = 408 \cdots, \tag{2.31}$$

$$2 \times Q: \quad 2 \times 5\frac{1}{1} \times 5\frac{5}{6} \times 5\frac{29}{35} \times 5\frac{169}{204} \times \cdots \tag{2.32}$$

To break the code of (2.32) let Q_n be the denominator of the nth convergent to $\sqrt{2}$. Then $Q_1 = 2$, $Q_3 = 12$, $Q_5 = 70$, ... Clearly (2.31) relates Q_n and Q_{n+2}. Then (2.32) represents the solutions y as partial products of an infinite product of the Wallis type:

$$Q_1 \frac{Q_3}{Q_1} \frac{Q_5}{Q_3} \frac{Q_7}{Q_5} \frac{Q_9}{Q_7} \cdots$$

The repeated constant 5 in (2.32) is explained by an elementary lemma.

Lemma 2.21 *The recurrence $Q_{n+2} = 6Q_n - Q_{n-2}$ holds for $n \geqslant 1$.*

Proof Consider

$$\begin{aligned} Q_{n+2} &= 2Q_{n+1} + Q_n, \\ 2Q_{n+1} &= 4Q_n + 2Q_{n-1}, \\ -Q_n &= -2Q_{n-1} - Q_{n-2}. \end{aligned} \tag{2.33}$$

The proof follows by adding the three applications of (1.15) in (2.33). □

Adding the first two equations in (2.33) results in $Q_{n+2} = 5Q_n + 2Q_{n-1}$, which together with Lemma 2.21 imply that $5 < Q_{n+2}/Q_n < 6$, as is clearly indicated in (2.32). Now Lemma 2.21 hints that

$$\frac{Q_{n+2}}{Q_n} = 6 - \frac{1}{6} \underset{-}{} \frac{1}{6} \underset{-}{} \frac{1}{6} \underset{-}{} \frac{1}{6} \underset{-}{} \cdots \longrightarrow a,$$

where $a \in (5, 6)$ is the solution to the quadratic equation

$$X = 6 - \frac{1}{X} \Longrightarrow X = 3 + 2\sqrt{2} = 5.828\,427\,124\,746\,19\ldots$$

Notice that $3 = x$ and $2 = y$ is the minimal solution to (2.29) with $D = 2$, whereas the decimal values of the fractions in (2.32) are

$$\frac{5}{6} = 0.83\ldots, \qquad \frac{29}{35} = 0.828\,57\ldots, \qquad \frac{169}{204} = 0.828\,431\ldots$$

One can be fairly sure that these facts did not escape Brouncker's attention.

Using Lemma 2.21 we now can prove that odd convergents to $\sqrt{2}$ give solutions to equation (2.29) with $D = 2$. Let us assume that this is true for all indices $2k - 1$ with $k \leqslant n$. Then by Lemma 2.21

$$\begin{aligned} P_{2n+1}^2 - 2Q_{2n+1}^2 &= (6P_{2n-1} - P_{2n-3})^2 - 2(6Q_{2n-1} - Q_{2n-3})^2 \\ &= 1 + 36 - 12(P_{2n-1}P_{2n-3} - 2Q_{2n-1}Q_{2n-3}). \end{aligned}$$

For the first few values of n the combination within the parentheses is

$$P_{2n-1}P_{2n-3} - 2Q_{2n-1}Q_{2n-3} = 3. \tag{2.34}$$

Compare, by the way, (2.34) with (3.36). So, we may include (2.34) in the induction hypotheses and obtain that

$$\begin{aligned} &P_{2n+1}P_{2n-1} - 2Q_{2n+1}Q_{2n-1} \\ &\quad = (6P_{2n-1} - P_{2n-3})P_{2n-1} - 2(6Q_{2n-1} - Q_{2n-3})Q_{2n-1} \\ &\quad = 6 - (P_{2n-1}P_{2n-3} - 2Q_{2n-1}Q_{2n-3}) = 3, \end{aligned}$$

which completes Brouncker's construction.

For $D = 3$ Brouncker gives the following solution:

$$3 \times Q: \quad 1 \times 3\frac{1}{1} \times 3\frac{3}{4} \times 3\frac{11}{15} \times 3\frac{41}{56} \times \cdots$$

By Euler's algorithm (2.2) we find that

$$\sqrt{3} = 1 + \frac{1}{1} \underset{+}{} \frac{1}{2} \underset{+}{} \frac{1}{1} \underset{+}{} \frac{1}{2} \underset{+}{} \frac{1}{1} \underset{+}{} \frac{1}{2} \underset{+}{} \cdots,$$

and the convergents with odd indices,

$$\begin{array}{rcccc} x = & 2 & 7 & 26 & 97, \\ y = & 1 & 4 & 15 & 56, \end{array} \tag{2.35}$$

again satisfy the equation $x^2 - y^2 D = 1$. In this case

$$\frac{Q_n}{Q_{n-2}} = 4 - \frac{1}{4} {}_{-} \frac{1}{4} {}_{-} \cdots \longrightarrow 2 + 1\sqrt{3} = 3.732\ 050\ 807\ 568\ 877\ldots$$

and

$$3\frac{3}{4} = 3.75, \qquad 3\frac{11}{15} = 3.733\ldots, \qquad 3\frac{41}{56} = 3.732\,142\,857\,14\ldots,$$

since $Q_{n+2} = 4Q_n - Q_{n-2}$.

For $D = 7$ this law must be modified since $x/y = 3/1 = P_1/Q_1$ is not a solution to $x^2 - y^2 D = 1$. However, $P_3/Q_3 = x/y = 8/3$ is a solution; see Ex. 2.1.

If x_1, y_1 is a solution to equation (2.29) then x_n and y_n in

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n \tag{2.36}$$

are also solutions. Indeed, since $\sqrt{D}$ is irrational, (2.36) is still valid with $+$ replaced by $-$. One can also obtain this by an application of the Galois automorphism of $\mathbb{Q}(\sqrt{D})$ to (2.36). Then

$$x_n^2 - y_n^2 D = (x_1 + y_1\sqrt{D})^n (x_1 - y_1\sqrt{D})^n = (x_1^2 - y_1^2 D)^n = 1.$$

We put $x_0 = 1$, $y_0 = 0$, which is also a solution to (2.29).

Theorem 2.22 (Brouncker 1657[1]**)** *The solutions $\{(x_n, y_n)\}_{n\geqslant 1}$ to equation* (2.29) *satisfy*

$$\begin{aligned} x_{n+1} &= (2x_1)x_n - x_{n-1}, x_o = 1, \\ y_{n+1} &= (2x_1)y_n - y_{n-1}, y_o = 0. \end{aligned} \tag{2.37}$$

and the fractions $\{y_n/x_n\}_{n\geqslant 0}$ are the convergents to the continued fraction

$$\frac{1}{\sqrt{D}} = \frac{y_1}{x_1} {}_{-} \frac{1}{2x_1} {}_{-} \frac{1}{2x_1} {}_{-} \frac{1}{2x_1} {}_{-} \cdots. \tag{2.38}$$

Proof By (2.36) for $n = 0, 1, \ldots,$

$$x_{n+1} = x_1 x_n + y_1 D y_n, \qquad y_{n+1} = y_1 x_n + x_1 y_n.$$

[1] Whitford (1912, pp. 51–2).

Iterating these formulas, we obtain

$$\begin{aligned} x_{n+1} &= x_1x_n + y_1^2Dx_{n-1} + x_1y_1y_{n-1}D \\ &= x_1x_n + x_1^2x_{n-1} + x_1y_1y_{n-1}D - x_{n-1} \\ &= x_1x_n + x_1(x_1x_{n-1} + y_1Dy_{n-1}) - x_{n-1} \\ &= (2x_1)x_n - x_{n-1}, \end{aligned}$$

which proves the first identity in (2.37). Similar calculations prove the second. Now (2.37) implies that y_n/x_n are the convergents to the continued fraction (2.38), which converge to $1/\sqrt{D}$ according to

$$\frac{1}{\sqrt{D}} - \frac{y_n}{x_n} = \frac{1}{x_n(x_n + y_n\sqrt{D})} = \frac{1}{x_n(x_1 + y_1\sqrt{D})^n} \underset{n\to\infty}{\longrightarrow} 0.$$

□

Theorem 2.22 shows that Brouncker's method works not only for particular values of D such as $D = 2$, 3, 7 but for any D. Indeed, we may write, following Brouncker,

$$D \times Q: \quad y_1 \times \frac{y_2}{y_1} \times \frac{y_3}{y_2} \times \frac{y_4}{y_3} \times \frac{y_5}{y_4} \times \cdots$$

By (2.37) $y_{n+1} = x_1y_n + y_1x_n$, which implies that

$$\frac{y_{n+1}}{y_n} = x_1 + y_1\frac{x_n}{y_n} \longrightarrow x_1 + y_1\sqrt{D}.$$

The application of (2.37), as in the case $D = 2$, leads to the same conclusion.

Brouncker's formulas (2.37) conveniently list infinitely many solutions provided one solution is known. By (2.38) y_1 divides every y_n. However, the question remains whether this algorithm lists all positive solutions.

Basing his work on Brouncker's hints, Wallis found his own solution to Fermat's problem, which is now called the English method; see Edwards (1977) and Stedall (2000a) for details. In spite of his comments on continued fractions in (1656, §191) Wallis did not follow the lines indicated above. This is further evidence that everything written in Section 191 of Wallis (1656) on continued fractions belongs to Brouncker. By the way, Wallis did not credit this part to himself and moreover, he clearly states this at the beginning of Section 191. See §**60**, Section 3.2.

Euler named equation (2.29) after Pell even in his first papers on this subject, see for instance Euler (1738, §15) or Euler's letter to Goldbach on August 10, 1730.[2] Since Euler (1738) also mentions Fermat and Wallis, it looks probable that Euler could in fact have meant Brouncker and not Pell. However, there is an opinion (Whitford 1912) that Euler mentioned Pell since Pell included the Diophantine equation $x = 12y^2 - z^2$ in the English translation of Rahn's *Algebra* (1668, p. 134). There is some evidence that the first appearance of this problem goes back to Archimedes' cattle problem, which reduces to the Pell equation

$$x^2 - 4729494y^2 = 1,$$

whose minimal solution has thousands of digits. It is not clear how Archimedes could have written the minimal positive solution himself. See Edwards (1977), Koch (2000, p. 3], Vardi (1998) and Whitford (1912) for the history of Pell's equation.

[2] OO723 in the Euler Archive.

44 Euler's method. The fact that Brouncker listed all possible positive solutions to (2.29) may be deduced from the following theorem due to Lagrange (1774, §38, Corollary 4) and from Euler's method, which we describe below. We will follow Arnold (1939) in our proof of Lagrange's theorem, which is a slight modification of original Lagrange's arguments.

Theorem 2.23 (Lagrange 1789) *Let D be a positive integer that is not a perfect square and L be an integer satisfying $|L| < \sqrt{D}$. Then any positive solution x, y to*

$$x^2 - y^2 D = L \tag{2.39}$$

determines an odd convergent x/y to the continued fraction of $\sqrt{D}$ if $L > 0$ and an even convergent if $L < 0$.

Proof For $L = 0$ there are no positive solutions. If $0 < L < \sqrt{D}$ then

$$\frac{x}{y} - \sqrt{D} = \frac{L}{y^2\left(x/y + \sqrt{D}\right)} > 0$$

shows that $x/y > \sqrt{D}$. Combining this with $L < \sqrt{D}$, we obtain that

$$0 < \frac{x}{y} - \sqrt{D} < \frac{\sqrt{D}}{2y^2\sqrt{D}} = \frac{1}{2y^2}\,.$$

By Theorems 1.38 and 1.7 x/y is an odd convergent. If $-\sqrt{D} < L < 0$ then

$$0 < \frac{y}{x} - \frac{1}{\sqrt{D}} = \frac{-L/D}{x^2\left(y/x + \frac{1}{\sqrt{D}}\right)} < \frac{1}{x^2\left(1 + \sqrt{D}y/x\right)} < \frac{1}{2x^2},$$

since $y\sqrt{D}/x > 1$. By Theorems 1.38 and 1.7 y/x is an odd convergent. Now

$$\sqrt{D} = b_0 + \frac{1}{b_1} \mathbin{+} \frac{1}{b_2} \mathbin{+} \frac{1}{b_3} \mathbin{+} \cdots$$

$$\frac{1}{\sqrt{D}} = 0 + \frac{1}{b_0} \mathbin{+} \frac{1}{b_1} \mathbin{+} \frac{1}{b_2} \mathbin{+} \cdots$$

show that the convergents of $\sqrt{D}$ and $1/\sqrt{D}$ are shifted and reciprocal:

$$\begin{array}{ccccc} \frac{b_0}{1} & \frac{P_1}{Q_1} & \frac{P_2}{Q_2} & \frac{P_3}{Q_3} & \cdots \\ \frac{0}{1} & \frac{1}{b_0} & \frac{Q_1}{P_1} & \frac{Q_2}{P_2} & \cdots \end{array}$$

It follows that x/y is an even convergent to $\sqrt{D}$. □

On the basis of numerical experiments Euler (1765) discovered the important role of the parameters s_k in finding solutions to Pell's equations. For instance in the example considered in §**36**, Section 2.1 ($D = 31$) the values of s_k are $s_1 = 6$, $s_2 = 5$, $s_3 = 3$,

$s_4 = 2$, $s_5 = 3$, $s_6 = 5$, $s_7 = 6$, $s_8 = 1$. Euler computed s_k for all $\sqrt{D}$, $D \leqslant 120$, and it turns out that $s_d = 1$ and $s_k > 1$ for $1 \leqslant k \leqslant d-1$.

Lemma 2.24 (Euler 1765) *If d is the period of $\sqrt{D}$ then $s_k = 1$ if and only if $k = md$, $m = 0, 1, 2, \ldots$*

Proof If $s_k = 1$ then $\xi_k = \sqrt{D} + r_k = \xi_0 + r_k$ and therefore $b_k = b_0 + r_k$. Hence

$$r_{k+1} = b_k s_k - r_k = b_0 = r_1, \qquad s_{k+1} = (D - r_{k+1}^2)/s_k = D - b_0^2 = s_1$$

and $\xi_1 = \xi_{k+1}$. Since d is the period of $\sqrt{D}$, $k+1 = 1 + md$, where $m \geqslant 1$. Now if $k = md$ then $s_{md} = s_0$, since the period of $\{s_k\}$ is also d. □

Theorem 2.25 *If the period d of $\sqrt{D}$ is even then the smallest positive solution to Pell's equation* (2.29) *is $x_1 = P_{d-1}$, $y_1 = Q_{d-1}$. If the period d is odd then $x_1 = P_{2d-1}$, $y_1 = Q_{2d-1}$.*

Proof By Lemma 2.13

$$P_n^2 - Q_n^2 D = (-1)^{n+1} s_{n+1}.$$

By Lemma 2.24 $s_{n+1} = 1$ if and only if d divides $n+1$. If d is even then $(-1)^{n+1} = 1$. If d is odd then $(-1)^{n+1} = 1$ for $n+1 = 2d$. □

By Theorem 2.3 both r_{n+1} and s_{n+1} are positive, $r_{n+1} < \sqrt{D}$ and $s_{n+1} = (r_{n+2} + r_{n+1})/b_{n+1} < 2\sqrt{D}$. By Lemma 2.7 the sequence $\{s_n\}_{n\geqslant 1}$ is periodic with period d equal to the period of $\sqrt{D}$. Then the sequence $\{(-1)^n s_n\}_{n\geqslant 1}$ is also periodic. Thus using Euler's algorithm, one can easily obtain a list of all L for which the equation (2.39) has positive solutions.

Let us consider the example $D = 31$ ($d = 8$). Then $P_7 = 1520$ and $Q_7 = 273$; see (2.17). So $1520^2 = 31 \times 273^2 + 1$. If $D = 13$ ($d = 5$) then similarly $P_4 = 18$, $Q_4 = 5$ and $18^2 - 13 \times 5^2 = -1$. Then by Theorem 2.29 below $x_1 = 18^2 + 13 \times 5^2 = 649$, $y_1 = 2 \times 18 \times 5 = 180$.

Corollary 2.26 *Pell's equation* (2.29) *always has positive integer solutions.*

To prove that Brouncker's x_n, y_n, defined by (2.36) starting with the minimal positive solution x_1, y_1, list all solutions to Pell's equation we need some technical lemmas.

Lemma 2.27 *If d is the period of $\sqrt{D}$ then*

$$Q_{(n+1)d-1} = Q_d Q_{nd-1} + Q_{d-1} Q_{nd-2},$$
$$P_{(n+1)d-1} = Q_d P_{nd-1} + Q_{d-1} P_{nd-2}.$$

Proof Since by Theorem 2.8 the period is symmetric, we obtain by the Euler–Wallis formulas

$$Q_{(n+1)d-1} = b_1 Q_{(n+1)d-2} + Q_{(n+1)d-3},$$
$$b_1 Q_{(n+1)d-2} = b_1 b_2 Q_{(n+1)d-3} + b_1 Q_{(n+1)d-4},$$

Observing that $b_1 = Q_1$, $1 = Q_0$ and applying the Euler–Wallis formulas again to the sum of the above equations, we have

$$Q_{(n+1)d-1} = Q_2 Q_{(n+1)d-3} + Q_1 Q_{(n+1)d-4}.$$

Arguing by induction we obtain the first formula and in the same way the second. □

Lemma 2.28 *If d is the period of* $\sqrt{D}$ *then*

$$P_{nd-1} = b_0 Q_{nd-1} + Q_{nd-2},$$
$$DQ_{nd-1} = b_0 P_{nd-1} + P_{nd-2}.$$

Proof Comparing the coefficients of the polynomials of the first degree in $\sqrt{D}$ obtained from the identity

$$\sqrt{D} = \frac{P_{nd} + (\sqrt{D} - b_0) P_{nd-1}}{Q_{nd} + (\sqrt{D} - b_0) Q_{nd-1}},$$

we see that

$$P_{nd-1} + b_0 Q_{nd-1} = Q_{nd} = 2b_0 Q_{nd-1} + Q_{nd-2},$$
$$DQ_{nd} + b_0 P_{nd-1} = P_{nd} = 2b_0 P_{nd-1} + P_{nd-2},$$

which proves the lemma. □

Theorem 2.29 (Euler–Lagrange) *Let d be the period of* $\sqrt{D}$*. Then for any* $n \in \mathbb{N}$

$$(P_{d-1} + Q_{d-1}\sqrt{D})^n = P_{nd-1} + Q_{nd-1}\sqrt{D}.$$

Proof The theorem is obvious for $n = 1$. Assuming that it is valid for n we can complete the proof if

$$P_{(n+1)d-1} = P_{d-1} P_{nd-1} + Q_{d-1} D Q_{nd-1},$$
$$Q_{(n+1)d-1} = P_{d-1} Q_{nd-1} + Q_{d-1} P_{nd-1}.$$

By Lemmas 2.27 and 2.28

$$\begin{aligned} P_{(n+1)d-1} &= Q_d P_{nd-1} + Q_{d-1}(DQ_{nd-1} - b_0 P_{nd-1}) \\ &= (Q_d - b_0 Q_{d-1}) P_{nd-1} + Q_{d-1} D Q_{nd-1} \\ &= P_{d-1} P_{nd-1} + Q_{d-1} D Q_{nd-1}, \end{aligned}$$

and similarly

$$\begin{aligned}Q_{(n+1)d-1} &= Q_d Q_{nd-1} + Q_{d-1}(P_{nd-1} - b_0 Q_{nd-1}) \\ &= (Q_d - b_0 Q_{d-1})Q_{nd-1} + Q_{d-1}P_{nd-1} \\ &= P_{d-1}Q_{nd-1} + Q_{d-1}P_{nd-1},\end{aligned}$$

which proves the theorem. □

Corollary 2.30 *If x_1, y_1 is the minimal positive solution to Pell's equation, then all other solutions are given by the formula* (2.36).

Let us demonstrate the Brouncker–Euler method for the example $D = 61$. Then $d = 11$; see Ex. 2.2 for the continued fraction of $\sqrt{61}$. To find P_{10} and Q_{10} Euler uses the symmetry of the period (see Lemma 2.27):

$$P_{d-1} = Q_k P_{d-k-1} + Q_{k-1} P_{d-k-2}, \qquad Q_{d-1} = Q_k Q_{d-k-1} + Q_{k-1} Q_{d-k-2},$$

where $k < d$. The choice $k = 5$ leads to the simple formulas

$$x = P_{10} = Q_5 P_5 + Q_4 P_4 = 58 \times 453 + 21 \times 164 = 29\,718,$$

$$y = Q_{10} = Q_5 Q_5 + Q_4 Q_4 = 58 \times 58 + 21 \times 21 = 3805.$$

Since $d = 11$ is odd, $x^2 - 61y^2 = -1$. By Theorem 2.29 the minimal solution to $x^2 - 61y^2 = 1$ is given by $x_1 = 2x^2 + 1 = 1766319049$, $y_1 = 2xy = 226153980$, implying that

$$\lim_n \frac{y_{n+1}}{y_n} = 3\,532\,638\,097.999\,999\,999\,7\ldots$$

The importance of Pell's equation in algebra is explained by the fact that it is closely related to the description of the units in the quadratic field $\mathbb{Q}(\sqrt{D})$; for details see Lang (1966).

2.4 Equivalent irrationals

45 Möbius transformations and matrices. Any general continued fraction $b_0 + \mathbf{K}_{k=1}^{\infty}(a_k/b_k)$ generates a sequence of Möbius transformations

$$s_0(w) = b_0 + w, \qquad s_n(w) = \frac{a_n}{b_n + w}, \quad n = 1, 2, \ldots$$

of the Riemann sphere $\hat{\mathbb{C}}$. By (1.12) $a_n \neq 0$ for every n. Hence the transformations s_n are one-to-one mappings of $\hat{\mathbb{C}}$. Let

$$S_n(w) = s_0 \circ s_1 \circ \cdots \circ s_n(w) \tag{2.40}$$

be the composition of the first $n+1$ Möbius transformations. Then

$$\frac{P_n}{Q_n} = S_n(0), \qquad \frac{P_{n-1}}{Q_{n-1}} = S_n(\infty),$$

where P_n/Q_n are the convergents to (1.12). By Euler's formula (1.17),

$$S_n(w) = \frac{P_n + P_{n-1}w}{Q_n + Q_{n-1}w}, \qquad n = 0, 1, 2, \ldots \tag{2.41}$$

Notice that the fractions (2.41) with $w = 1/k$, $k = 1, \ldots, b_{n+1} - 1$ are Euler's non-principal convergents (see §**17** in Section 1.4), which nowadays are called intermediate fractions (see Lang 1966). Lagrange (1798) paid a great deal of attention to these fractions.

By (2.41) $S_{n+1}(w) = S_n(s_{n+1}(w))$ is equivalent to the matrix identity

$$\begin{pmatrix} P_n & P_{n+1} \\ Q_n & Q_{n+1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} \begin{pmatrix} 0 & a_{n+1} \\ 1 & b_{n+1} \end{pmatrix} .$$

This can also be proved by the Euler–Wallis formulas. Iterating, we obtain a matrix version of the Euler–Mindingen formulas for P_n and Q_n:

$$\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a_1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix} . \tag{2.42}$$

Passing to the transpose of this matrix formula results in

$$\begin{pmatrix} P_{n-1} & Q_{n-1} \\ P_n & Q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a_n & b_n \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ a_1 & b_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_0 & 1 \end{pmatrix} . \tag{2.43}$$

An application of the multiplicative functional $C \longrightarrow \det C$ to (2.43) proves (1.16). By (2.40), formula (2.43) can be written as

$$\frac{P_{n-1}w + Q_{n-1}}{P_n w + Q_n} = t_n \circ \cdots \circ t_1 \circ t_0(w), \tag{2.44}$$

where

$$t_k(w) = \frac{1}{a_k w + b_k}, \qquad t_0(w) = \frac{w}{b_0 w + 1} .$$

Hence (2.44) can be written in the form of continued fractions:

$$\frac{P_{n-1}w + Q_{n-1}}{P_n w + Q_n} = \frac{1}{b_n} \, \frac{a_n}{+ \, b_{n-1}} \, \frac{a_{n-1}}{+ \, b_{n-2}} + \cdots + \frac{a_2}{b_1} \, \frac{a_1 w}{+ \, b_0 w + 1} . \tag{2.45}$$

In particular, for $w = 0$ we obtain from (2.45) the first formula of (1.18) and for $w \to \infty$ the second. Both formulas are extensively used in the convergence theory of continued fractions; see Jones and Thron (1980, Chapter 4).

By the definition of continued fractions, $a_n \neq 0$ and therefore by (1.16) S_n in (2.41) is a homeomorphism of the Riemann sphere $\hat{\mathbb{C}}$. In particular the equation $S_n(w) = K$ has a unique solution $w_n = w_n(K)$ for every $K \in \hat{\mathbb{C}}$.

Lemma 2.31 *Let $K \in \hat{\mathbb{C}}$ and $S_k(w_k) = K$ for $k = 1, 2, \ldots, n$. Then*

$$\begin{aligned} P_n + P_{n-1}w_n &= \prod_{k=0}^{n} (b_k + w_k), \\ Q_n + Q_{n-1}w_n &= \prod_{k=1}^{n} (b_k + w_k). \end{aligned} \tag{2.46}$$

Proof Observing that $s_k(w_k) = w_{k-1}$, we obtain from (1.15)

$$\begin{aligned} P_n + P_{n-1}w_n &= (b_n + w_n)P_{n-1} + a_n P_{n-2} \\ &= (b_n + w_n)(P_{n-1} + P_{n-2}w_{n-1}) = \cdots \\ &= (b_n + w_n)\cdots(P_0 + P_1 w_0). \end{aligned}$$

The second identity is proved similarly. □

Theorem 2.32 *In the notation of Lemma* 2.31*,*

$$\begin{aligned} P_n &= \frac{1}{1} \, {}_{+} \, \frac{w_n}{b_n} \, {}_{+} \, \frac{a_n}{b_{n-1}} \, {}_{+} \cdots {}_{+} \, \frac{a_1}{b_0} \prod_{k=0}^{n} (b_k + w_k), \\ Q_n &= \frac{1}{1} \, {}_{+} \, \frac{w_n}{b_n} \, {}_{+} \, \frac{a_n}{b_{n-1}} \, {}_{+} \cdots {}_{+} \, \frac{a_2}{b_1} \prod_{k=1}^{n} (b_k + w_k). \end{aligned} \tag{2.47}$$

Proof Combining the first identity in (2.46) with the first formula in (1.18), we obtain the first identity in (2.47). The second identity is proved similarly. □

46 Generators of $GL_2(\mathbb{Z})$. By Euler's formula (1.17) ξ is the Mobius transformation of ξ_{n+1} corresponding to the matrix (2.42) having determinant $P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}$. The set of all matrices

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \qquad ad - bc = \pm 1$$

with integer entries a, b, c, d is called the general linear group over $\mathbb{Z}$ and is denoted by $GL_2(\mathbb{Z})$.

Theorem 2.33 *Let a, b, c, d be integers such that $ad - bc = \pm 1$, $0 < d < b$, and let α and β, $\beta > 1$, be irrational numbers satisfying*

$$\alpha = \frac{a\beta + c}{b\beta + d} \,.$$

Then there is a positive integer n such that $a = P_n$, $b = Q_n$, $c = P_{n-1}$, $d = Q_{n-1}$, where P_n/Q_n and P_{n-1}/Q_{n-1} are convergents for the continued fraction of α and $\beta = \alpha_{n+1}$.

Proof Since $ad - bc = \pm 1$ the greatest common divisor of a and b is unity. Then

$$\frac{a}{b} = b_0 + \frac{1}{b_1} \, {}_{+} \cdots {}_{+} \, \frac{1}{b_k} = b_0 + \frac{1}{b_1} \, {}_{+} \cdots {}_{+} \, \frac{1}{b_k - 1} \, {}_{+} \, \frac{1}{1} \,.$$

Hence we have two choices of n for $a = P_n$, $b = Q_n$. We will choose n so that $P_n Q_{n-1} - Q_n P_{n-1} = (-1)^{n-1} = ad - bc$. It follows that

$$P_n(Q_{n-1} - d) = Q_n(P_{n-1} - c) \,.$$

Now on the one hand $(P_n, Q_n) = 1$ implies that Q_n divides $Q_{n-1} - d$. On the other hand $0 < d < b = Q_n$ implies that $|Q_{n-1} - d| < Q_n$ and therefore $Q_{n-1} - d$ must be zero. Hence $P_{n-1} = c$. □

Definition 2.34 *Two irrational numbers α and β are called equivalent, $\alpha \sim \beta$, if there is a matrix in $GL_2(\mathbb{Z})$ with entries a, b, c, d, such that*

$$\alpha = \frac{a\beta + c}{b\beta + d}\,.$$

By (1.17) an irrational number α is equivalent to any its remainders α_n.

Theorem 2.35 (Serret) *Two irrational numbers α and β are equivalent if and only if $\alpha_n = \beta_m$ for some $n, m \in \mathbb{Z}_+$.*

Proof If $\alpha \sim \alpha_n$, $\beta \sim \beta_m$ and $\alpha_n = \beta_m$ then $\alpha \sim \beta$, since $GL_2(\mathbb{Z})$ is a group. Suppose now that $\alpha \sim \beta$. Multiplying the entries of

$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

by (-1), if necessary, we may assume that $b\beta + d > 0$. By (1.17)

$$\beta = \frac{\beta_n P_{n-1} + P_{n-2}}{\beta_n Q_{n-1} + Q_{n-2}} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} \beta_n \\ 1 \end{pmatrix} \stackrel{\text{def}}{=} F_n \begin{pmatrix} \beta_n \\ 1 \end{pmatrix}.$$

Since

$$MF_n = \begin{pmatrix} aP_{n-1} + cQ_{n-1} & aP_{n-2} + cQ_{n-2} \\ bP_{n-1} + dQ_{n-1} & bP_{n-2} + dQ_{n-2} \end{pmatrix} = \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix},$$

we may write

$$b' = bP_{n-1} + dQ_{n-1} = Q_{n-1}\left(b\frac{P_{n-1}}{Q_{n-1}} + d\right),$$

$$d' = bP_{n-2} + dQ_{n-2} = Q_{n-2}\left(b\frac{P_{n-2}}{Q_{n-2}} + d\right).$$

Since $\lim_n P_n/Q_n = \beta$, we conclude that $b', d' > 0$ for $n > N$. Since $Q_{n-2} < Q_{n-1}$ and the signs of $P_n/Q_n - \beta$ alternate infinitely often,

$$\begin{aligned} b' - d' =& (Q_{n-1} - Q_{n-2})\,(b\beta + d) \\ &+ b\left\{Q_{n-1}\left(\frac{P_{n-1}}{Q_{n-1}} - \beta\right) - Q_{n-2}\left(\frac{P_{n-2}}{Q_{n-2}} - \beta\right)\right\} > 0. \end{aligned}$$

By Theorem 2.33 β_n is the remainder of α for some n. □

47 Discriminants of equivalent irrationals. First we recall the definitions of quadratic, and more generally algebraic, irrationals.

Definition 2.36 *A number ξ is called an algebraic irrational (over $\mathbb{Q}$) if $\xi \notin \mathbb{Q}$ and ξ is a root of a polynomial equation with integer coefficients*

$$p(X) = c_0 X^n + c_1 X^{n-1} + \cdots + c_n = 0\,. \tag{2.48}$$

An algebraic irrational ξ is called quadratic if in (2.48) $n = 2$.

The set $\mathbb{Z}[X]$ of all polynomials with integer coefficients is a commutative ring. More generally one may consider the commutative ring $\mathfrak{R}[X]$ of all polynomials in X with coefficients in some commutative ring $\mathfrak{R}$. A polynomial $p \in \mathfrak{R}[X]$ is called *irreducible* if $p = ab$, with $a, b \in \mathfrak{R}[X]$, implies that either a or b is invertible in $\mathfrak{R}[X]$. In fact this means that either a or b is an invertible element of $\mathfrak{R}$. This terminology, in particular, allows one to distinguish between irreducible polynomials in $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$. If $p(X)$ is an irreducible polynomial in $\mathbb{Z}[X]$ then the greatest common divisor of its integer coefficients must be 1. It is not so for irreducible polynomials in $\mathbb{Q}[X]$ with integer coefficients.

Since $\mathbb{Z}$ has only two invertible elements, ± 1, for every algebraic irrational ξ there exist exactly two irreducible polynomials in $\mathbb{Z}[X]$ with root ξ. The polynomial with positive leading coefficient is called the *minimal* polynomial of ξ. The degree of the minimal polynomial of ξ is called the degree of the algebraic number ξ.

The main purpose of this section is to prove that equivalent algebraic irrationals have equal discriminants.

Theorem 2.37 *If α and β are two equivalent quadratic irrationals such that*

$$\alpha = \frac{a\beta + c}{b\beta + d},$$

where $a, b, c, d \in \mathbb{Z}$, $ad - bc = \pm 1$ and $p(X) = e'X^2 + f'X + g'$ is an irreducible polynomial in $\mathbb{Z}[X]$ for α, then

$$\begin{aligned} q(X) &= e'(aX + c)^2 + f'(aX + c)(bX + d) + g'(bX + d)^2 \\ &= eX^2 + fX + g \end{aligned}$$

is an irreducible polynomial in $\mathbb{Z}[X]$ for β. The discriminants of $p(X)$ and $q(X)$ are equal:

$$f'^2 - 4e'f' = f^2 - 4ef\ .$$

Proof One can simply say that the results stated in the theorem can be proved by a direct calculation. However, such a proof does not explain why the statement is true. The following matrix identities

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -a & c \\ -b & d \end{pmatrix},$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix},$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c & a \\ d & b \end{pmatrix}$$

show that right multiplication of the initial matrix by the indicated elements of $GL_2(\mathbb{Z})$ reduces the general matrix to a matrix with $0 < d < b$. By Theorem 2.33 this new matrix equals matrix (2.42). However, the last of the above matrix identities and (1.15) show that

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The correspondence

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow -z, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow z,$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \frac{1}{z}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix} \longrightarrow \frac{1}{z+b_n},$$

shows that α is obtained from β by a finite composition of the three following operations: $z \to -z$, $z \to 1/z$, $z \to z+1$. It remains therefore to consider only three possibilities.

If β is a root of $eX^2 + fX + g = 0$ then $-\beta$ is the root of $eX^2 - fX + g = 0$, which is also irreducible and has the same discriminant.

If β is a root of $eX^2 + fX + g = 0$ then $1/\beta$ is the root of $gX^2 + fX + e = 0$, which is also irreducible and has the same discriminant.

If β is a root of $eX^2 + fX + g = 0$ then $\beta + 1$ is the root of

$$e(X-1)^2 + f(X-1) + g = eX^2 + (f-2e)X + (e-f+g) = 0.$$

If $d > 0$ divides e and $f - 2e$ then d divides both e and f. If in addition it divides $e - f + g$, then it divides g, which implies that $d = 1$. Now the discriminant of the equation is

$$(f-2e)^2 - 4e(e-f+g) = f^2 - 4eg\,. \qquad \square$$

The discriminant of (2.21), $4r_k^2 + 4s_k s_{k-1} = 4D$, equals the discriminant of the irreducible polynomial $x^2 - D$. Hence by Theorem 2.37 the quadratic polynomials (2.21) are irreducible in $\mathbb{Z}[X]$ and in particular the greatest common divisor of s_k, r_k, s_{k-1} for $k = 1, 2, \ldots$ is 1.

Theorem 2.37 extends to algebraic numbers. Recall (see Lang 1965) that the discriminant $\mathcal{D}$ of a polynomial

$$q(X) = c_0X^n + c_1X^{n-1} + \cdots + c_{n-1}X + c_n = c_0(X-t_1)\cdots(X-t_n)$$

is defined by

$$\mathcal{D} = c_0^{2n-2} \prod_{i<j} (t_i - t_j)^2, \tag{2.49}$$

which is in agreement with the discriminant formula in the case $n = 2$. By the fundamental theorem on symmetric polynomials (see Lang 1965), $\mathcal{D}$ belongs to the coefficient field of $q(X)$.

Corollary 2.38 *Equivalent algebraic irrationals have equal determinants.*

Proof If $q(X) = c_0X^n + c_1X^{n-1} + \cdots + c_n$ is an irreducible polynomial of β then the irreducible polynomial of $-\beta$ is $c_0(-1)^nX^n + c_1(-1)^{n-1}X^{n-1} + \cdots + c_n$. Since all its roots are symmetric to the roots of $q(X)$, formula (2.49) shows that the discriminant is not changed.

Similarly, the transformation $X \longrightarrow X+1$ does not change the discriminant.

The third transformation $X \longrightarrow 1/X$ sends the roots $t_1, t_2, \ldots, t_n$ of $q(X)$ to $s_j = 1/t_j$, $j = 1, 2, \ldots, n$. Using Viète's formula $t_1t_2\cdots t_n = (-1)^n c_n/c_0$, we obtain that

$$\begin{aligned}
c_0^{2n-2}\prod_{i<j}(t_i - t_j)^2 &= c_n^{2n-2}\left(\frac{c_0}{c_n}\right)^{2n-2}\prod_{i<j}(t_i - t_j)^2 \\
&= c_n^{2n-2}\left(\frac{1}{t_1t_2\cdots t_n}\right)^{2n-2}\prod_{i<j}(t_i - t_j)^2 \\
&= c_n^{2n-2}\prod_{i<j}\left(\frac{1}{t_i} - \frac{1}{t_j}\right)^2 .
\end{aligned}$$

Hence the discriminants of $q(X)$ and $X^nq(1/X)$ are equal. □

2.5 Markoff's theory

48 Motivation. By Legendre's theorem, 1.37, for every irrational ξ any solution in integers p, q, $q > 0$ to the inequality

$$\left|\frac{p}{q} - \xi\right| < \frac{1}{2q^2} \tag{2.50}$$

determines a *convergent* p/q to ξ. By Vahlen's theorem 1.38, of any two consecutive convergents to ξ at least one satisfies (2.50). Therefore (2.50) has infinitely many solutions in integers p, q, $q > 0$ and *all solutions are convergents* to ξ. Using the functional $\xi \to \|\xi\|$ we can eliminate p from (2.50).[3] This p is nothing other than the best approximation of $q\xi$ in $\mathbb{Z}$. Since ξ is irrational p is unique. It follows that (2.50) is equivalent to

$$q\|q\xi\| < \frac{1}{2}, \qquad p = \left[q\xi + \frac{1}{2}\right]. \tag{2.51}$$

Lemma 2.39 *Let ξ be an irrational number. Then the supremum $\mu(\xi)$ of $c > 0$ such that*

$$\left|\frac{p}{q} - \xi\right| < \frac{1}{cq^2} \tag{2.52}$$

has infinitely many solutions in integers p, q, $q > 0$ is given by

$$\mu(\xi) = \left(\liminf_{q\to+\infty} q\|q\xi\|\right)^{-1}. \tag{2.53}$$

[3] Recall that $\|\xi\| = \operatorname{dist}(\xi, \mathbb{Z})$.

Proof By (2.51) inequality (2.52) is equivalent to $q\|q\xi\| < 1/c$. If this is true for an infinite number of values of q then $L = \liminf_{q\to+\infty} q\|q\xi\| \leqslant 1/c$ for every $c < \mu(\xi)$ and hence for $c = \mu(\xi)$. If now $L < \mu(\xi)^{-1}$ then there is $\varepsilon > 0$ such that $L < (\mu(\xi)+\varepsilon)^{-1}$. Then $q\|q\xi\| < 1/(\mu(\xi)+\varepsilon)$ must have infinitely many solutions for q, contradicting the definition of $\mu(\xi)$. □

On the one hand, by Theorem 1.37 every irrational ξ satisfies $\mu(\xi) \geqslant 2$. On the other hand by Theorem 1.41, for any real quadratic irrational ξ with discriminant D and any $\varepsilon > 0$, the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{(\sqrt{D}+\varepsilon)q^2} \tag{2.54}$$

has only a finite number of solutions in integers p, q, $q > 0$. It follows that $\mu(\xi) \leqslant \sqrt{D}$. Since the minimal discriminant D of real quadratic irrationals is 5, see Ex. 2.13, it is natural to ask whether $\sqrt{5}$ is the smallest value of $\mu(\xi)$. The following lemma, which summarizes the observations made above, is useful in the calculation of $\mu(\xi)$.

Lemma 2.40 *For any real ξ*

$$\mu(\xi)^{-1} = \liminf_{n} Q_n|P_n - Q_n\xi| \leqslant \frac{1}{2},$$

where P_n/Q_n are convergents of the continued fraction for ξ.

Proof Since of two consecutive convergents to ξ at least one satisfies (2.50), $\mu(\xi) \geqslant 2$. If $\mu(\xi) \geqslant 2$ then the limit inferior in (2.53) is attained along the denominators of the convergents to ξ. □

Lemma 2.41 *For any irrational ξ*

$$\mu(\xi) = \limsup_{n} \left\{ \left(\frac{1}{b_n} {}_{+\cdots+} \frac{1}{b_1} \right) + b_{n+1} + \left(\frac{1}{b_{n+2}} {}_{+} \frac{1}{b_{n+3}} {}_{+\cdots} \right) \right\}. \tag{2.55}$$

Proof Combine (1.50) with Lemma 2.40 and (1.18). □

The following lemma is helpful for estimating $\mu(\xi)$.

Lemma 2.42 *An irrational ξ considered as a function of its parameters $\{b_n\}$ increases in b_{2n} and decreases in b_{2n+1}.*

Proof Apply Lemma 1.13. □

Lemma 2.43 *If $\mu(\xi) < 3$ and both the values 1 and 2 are taken by b_n infinitely often then $\sqrt{8} < \mu(\xi)$.*

Proof Since $\mu(\xi) < 3$, formula (2.55) shows that for $n \geqslant N$ either $b_n = 1$ or $b_n = 2$. If for $n \geqslant N$ the sequence $\{b_n\}$ is 1 2 1 2 ... then by (2.55)

$$\mu(\xi) \geqslant \left(\frac{1}{1}\,\frac{1}{+\,2}\,{}_{+\cdots}\right) + 2 + \left(\frac{1}{1}\,\frac{1}{+\,2}\,{}_{+\cdots}\right) = 2\sqrt{3} > 3, \tag{2.56}$$

which contradicts the assumption $\mu(\xi) < 3$. Let $\{b_n\}$ contains infinitely many groups 2 2 1. Putting $b_n = b_{n+1} = 2$, $b_{n+2} = 1$, we obtain by (2.55) and Lemma 2.42 that

$$\mu(\xi) \geqslant \left(\frac{1}{2}\,\frac{1}{+\,1}\,{}_{+\cdots}\right) + 2 + \left(\frac{1}{1}\,\frac{1}{+\,1}\,\frac{1}{+\,2}\,{}_{+\cdots}\right) = \frac{9 + 5\sqrt{3}}{6} > \sqrt{8}.$$

If $\{b_n\}$ does not contain the group 2 2 1 infinitely often then for $n > N$ it contains only groups with an isolated 2 separated by 1's. Since $\mu(\xi) < 3$, by (2.56) there must be infinitely many groups 1 1 2 1 in $\{b_n\}$. Then by (2.55)

$$\mu(\xi) \geqslant \left(\frac{1}{1}\,\frac{1}{+\,1}\,\frac{1}{+\,2}\,{}_{+\cdots}\right) + 2 + \left(\frac{1}{1}\,\frac{1}{+\,1}\,\frac{1}{+\,2}\,{}_{+\cdots}\right) = 2\left(1 + \frac{1}{\sqrt{3}}\right) > 3,$$

which cannot be the case happen. This proves the lemma. □

Theorem 2.44 (Markoff 1879, 1880) *For an irrational ξ, either $\mu(\xi) > \sqrt{8}$ or $\mu(\xi) \leqslant \sqrt{8}$. In the latter case, either $\mu(\xi) = \sqrt{8}$ and then $\xi \sim \sqrt{2}$ or $\mu(\xi) = \sqrt{5}$ and then $\xi \sim \phi$.*

Proof For $\xi = \phi$ all the parameters b_n are 1 and by (2.55)

$$\mu(\phi) = \left(\frac{1}{1}\,\frac{1}{+\,1}\,{}_{+\cdots}\right) + 1 + \left(\frac{1}{1}\,\frac{1}{+\,1}\,{}_{+\cdots}\right) = \sqrt{5}. \tag{2.57}$$

So, if $b_n = 1$ for $n \geqslant N$ then $\xi \sim \phi$ and $\mu(\xi) = \sqrt{5}$.

For $\xi = 1 + \sqrt{2}$ all the parameters b_n are 2 and by (2.55)

$$\mu(\xi) = \left(\frac{1}{2}\,\frac{1}{+\,2}\,{}_{+\cdots}\right) + 2 + \left(\frac{1}{2}\,\frac{1}{+\,2}\,{}_{+\cdots}\right) = \sqrt{2} + 1 + \sqrt{2} - 1 = \sqrt{8}. \tag{2.58}$$

So, if $b_n = 2$ for $n \geqslant N$ then $\xi \sim \sqrt{2}$ and $\mu(\xi) = \sqrt{8} > \sqrt{5}$. Thus $\mu(\xi) > \sqrt{5}$ if $\xi \nsim \phi$ and $\mu(\xi) > \sqrt{8}$ if $\xi \nsim \sqrt{2}$. □

Theorems 1.41 and 2.44 may create an impression that quadratic irrationals are exactly those numbers ξ satisfying $\mu(\xi) < +\infty$. This is not the case as the following theorem due to A. Markoff shows.

Theorem 2.45 (Markoff 1879, 1880) *There are uncountably many ξ with $\mu(\xi) = 3$.*

Proof Any increasing sequence $r = \{r_k\}_{k \geqslant 1}$ of positive integers determines a real irrational

$$\xi^{(r)} = [\underbrace{1; 1, \ldots, 1}_{r_1}, 2, 2, \underbrace{1, 1, \ldots, 1}_{r_2}, 2, 2, 1, \ldots] = [b_0; b_1, \ldots, b_n, \ldots].$$

If $b_{n+1}=b_{n+2}=2$ then

$$\begin{aligned}
b_{n+1}+&\left(\frac{1}{b_{n+2}}\,{}_{+}\,\frac{1}{b_{n+3}}\,{}_{+\cdots,}\right)+\frac{1}{b_n}\,{}_{+\cdots+}\,\frac{1}{b_1}\\
&=2+\frac{1}{2}\,{}_{+}\,\frac{1}{1}\,{}_{+\cdots}+\frac{1}{1}\,{}_{+\cdots+}\,\frac{1}{1}\\
&\to 2+\frac{1}{2+(\sqrt{5}-1)/2}+\frac{\sqrt{5}-1}{2}=3, \qquad n\to+\infty, \qquad (2.59)
\end{aligned}$$

since $\lim_k r_k=+\infty$. Similarly, if $b_n=b_{n+1}=2$ then

$$\begin{aligned}
b_{n+1}+&\frac{1}{b_{n+2}}\,{}_{+}\,\frac{1}{b_{n+3}}\,{}_{+\cdots}+\frac{1}{b_n}\,{}_{+\cdots+}\,\frac{1}{b_1}\\
&=2+\frac{1}{1}\,{}_{+}\,\frac{1}{1}\,{}_{+\cdots}+\frac{1}{2}\,{}_{+\cdots+}\,\frac{1}{1}\\
&\to 2+\frac{1}{(1+\sqrt{5})/2}+\frac{1}{2+(\sqrt{5}-1)/2}=3, \quad n\to+\infty.
\end{aligned}$$

Finally, if $b_{n+1}=1$ then $\xi_{n+1}^{(r)}<2$ and

$$b_{n+1}+\frac{1}{b_{n+2}}\,{}_{+}\,\frac{1}{b_{n+3}}\,{}_{+\cdots}+\frac{1}{b_n}\,{}_{+\cdots+}\,\frac{1}{b_1}<3,$$

which proves that $\mu(\xi^{(r)})=3$ for every r. To prove that there are uncountably many $\xi^{(r)}$, we observe that every irrational number x in $(0,1)$ has a unique infinite binary expansion $x=0\,\epsilon_1\epsilon_2\ldots\epsilon_n\ldots$, where the ϵ_n are either 0 or 1. If we put $r_k=\epsilon_1+\cdots+\epsilon_k+k$ then $r=\{r_k\}_{k\geqslant1}$ is an increasing integer sequence. Since obviously $r_{k+1}-r_k=\epsilon_{k+1}+1$, the sequence $r=\{r_k\}_{k\geqslant1}$ recovers x, implying that the number of such sequences is uncountable. □

Corollary 2.46 *There is an uncountable set of transcendental numbers ξ satisfying $\mu(\xi)=3$.*

Proof Since algebraic numbers are the roots of irreducible polynomials with integer coefficients they make a countable set, implying that its complement in the uncountable set considered is also uncountable. □

In his thesis Markoff found the first ten values of $\mu(\xi)$, which are arranged as Table 2.1. This table indicates that $\mu(\xi)$ below 3 may take only discrete values corresponding to quadratic irrationals.

Definition 2.47 *The set of all values of $\mu(\xi)$ defined on irrational ξ is called the Lagrange spectrum.*

Table 2.1. *First points of the Lagrange spectrum*

ξ	$\mu(\xi)$	
$[\overline{1}]$	$\sqrt{5}$	$= 2.236\,067\,977\ldots$
$[\overline{2}]$	$\sqrt{8}$	$= 2.828\,427\,125\ldots$
$[\overline{2_2, 1_2}]$	$\frac{\sqrt{221}}{5}$	$= 2.973\,213\,749\ldots$
$[\overline{2_2, 1_4}]$	$\frac{\sqrt{1517}}{13}$	$= 2.996\,052\,630\ldots$
$[\overline{2_4, 1_2}]$	$\frac{\sqrt{7565}}{29}$	$= 2.999\,207\,188\ldots$
$[\overline{2_2, 1_6}]$	$\frac{\sqrt{2600}}{17}$	$= 2.999\,423\,243\ldots$
$[\overline{2_2, 1_8}]$	$\frac{\sqrt{71\,285}}{89}$	$= 2.999\,915\,834\ldots$
$[\overline{2_6, 1_2}]$	$\frac{\sqrt{257\,045}}{169}$	$= 2.999\,976\,658\ldots$
$[\overline{2_2, 1_2, 2_2, 1_4}]$	$\frac{\sqrt{84\,680}}{97}$	$= 2.999\,982\,286\ldots$
$[\overline{2_2, 1_{10}}]$	$\frac{\sqrt{488\,597}}{233}$	$= 2.999\,987\,720\ldots$

49 The Lagrange spectrum: motivation. By Lagrange's theorem 2.19 every quadratic irrational is periodic. By Serret's theorem 2.35 every periodic irrational is equivalent to a pure periodic irrational, i.e. a reduced quadratic irrational. Then formula (2.55) hints that instead of studying limits at $+\infty$ it is useful to consider two-sided periodic sequences $\xi = \{b_n\}_{n\in\mathbb{Z}}$ of 1's and 2's satisfying for every $n \in \mathbb{Z}$

$$m_n(\xi) = \left(\frac{1}{b_{n-1}} \frac{1}{+\,b_{n-2}} + \cdots\right) + b_n + \left(\frac{1}{b_{n+1}} \frac{1}{+\,b_{n+2}} + \cdots\right) < 3. \tag{2.60}$$

Then the periodicity guarantees $\mu(\xi_+) < 3$ for $\xi_+ = [b_0; b_1, b_2, \ldots]$.

Let $\mathbf{J}$ be the set of all infinite sequences $\xi = \{b_n\}_{n\in\mathbb{Z}}$ of 1's and 2's satisfying

$$m(\xi) \overset{def}{=} \sup_{n\in\mathbb{Z}} m_n(\xi) \leqslant 3, \tag{2.61}$$

and $\mathbf{J}_0 \subset \mathbf{J}$ be the set of ξ such that $m_n(\xi) < 3$ for $n \in \mathbb{Z}$.

The entries of the first column of Table 2.1 contain 1's and 2's with multiplicity two. Reducing the multiplicity to one, we obtain periodic sequences of 1's and 2's. As it is clear from Markoff's reference to Jean Bernoulli's book (1772), he knew that Jean Bernoulli sequences corresponding to rational θ in $[1, 2)$, see Section 1.4, are periodic sequence of 1's and 2's. Therefore it is worthy of our attention to check whether the reduced periodic sequences in the first column of Table 2.1 are Jean Bernoulli sequences.

Table 2.2. *Jean Bernoulli sequences*

$\xi = r^* \to r$	θ	$\{r_n(\theta,0)\}_{n\in\mathbb{Z}}$
$[\overline{1}]$	$\theta = 1$	$[\overline{1}]$
$[\overline{2}]$	$\theta = 2$	$[\overline{2}]$
$[\overline{2,1}]$	$\theta = 1+\frac{1}{2}$	$[\overline{2,1}]$
$[\overline{2,1_2}]$	$\theta = 1+\frac{1}{3}$	$[\overline{2,1_2}]$
$[\overline{2_2,1}]$	$\theta = 1+\frac{2}{3}$	$[\overline{2_2,1}]$
$[\overline{2,1_3}]$	$\theta = 1+\frac{1}{4}$	$[\overline{2,1_3}]$
$[\overline{2,1_4}]$	$\theta = 1+\frac{1}{5}$	$[\overline{2,1_4}]$
$[\overline{2_3,1}]$	$\theta = 1+\frac{3}{4}$	$[\overline{2_3,1}]$
$[\overline{2,1,2,1_2}]$	$\theta = 1+\frac{2}{5}$	$[\overline{2,1,2,1_2}]$
$[\overline{2,1_5}]$	$\theta = 1+\frac{1}{6}$	$[\overline{2,1_5}]$

The parameter $\theta = p/q$ of any periodic Jean Bernoulli sequence can be recovered by (1.60). Recall that p is the total number of 2's and q is the length of the period. By Theorem 1.49 all other periodic Jean Bernoulli sequences corresponding to the same θ are obtained by simple shifts. The calculation of $\{r_n(p/q,0)\}_{n\in\mathbb{Z}}$ following the stated rules completes the test. We arrange its results in Table 2.2 (see Ex. 1.23). It is unlikely that the remarkable control of the discrete part of the Lagrange spectrum by periodic Jean Bernoulli sequences demonstrated in Table (2.2) is an coincidence. And in fact it is not!

There is one more observation based on analysis of the second column in Table 2.1. To simplify the notation we put for any $\xi = \{b_n\}_{n\in\mathbb{Z}}$ and $n \in \mathbb{Z}$:

$$\xi_n = b_n + \frac{1}{b_{n+1}} {}_{+} \frac{1}{b_{n+2}} {}_{+\cdots}, \qquad \eta_n = b_n + \frac{1}{b_{n-1}} {}_{+} \frac{1}{b_{n-2}} {}_{+\cdots}. \tag{2.62}$$

Then

$$m_n(\xi) = \frac{1}{\eta_{n-1}} + \xi_n = \frac{1}{\eta_{n-1}} + b_n + \frac{1}{\xi_{n+1}}. \tag{2.63}$$

If ξ is periodic with period q then ξ_n is a pure periodic quadratic irrational by Euler's theorem 2.2. By Galois' theorem 2.20 the ξ_n^* algebraically conjugate to ξ_n is $-1/\eta_{n-1}$. It follows that

$$m_n(\xi) = \xi_n - \xi_n^* = \frac{\sqrt{D}}{a_n}, \tag{2.64}$$

where D is the discriminant of the irreducible quadratic equation

$$a_n X^2 + b_n X + c_n = 0 \tag{2.65}$$

for ξ_n. By Serret's theorem 2.35 any two ξ_k and ξ_l are equivalent irrationals and therefore their discriminants are equal by Theorem 2.37. Hence D does not depend on n. Thus the problem of finding $m(\xi)$ is equivalent to finding the irreducible equation (2.65) for the minimal $a_n > 0$. If ξ has period q then the number of coefficients to investigate does not exceed q.

For instance, for $\xi = [2_2, 1_2]$ we easily obtain that

$$\xi_2 = 2 + \frac{1}{1} {}_{+} \frac{1}{1} {}_{+} \frac{1}{2} {}_{+} \frac{1}{\xi_2}$$

satisfies the quadratic equation $5X^2 - 11X - 5 = 0$ with discriminant $D = 221$ and $a_1 = 5$. Similarly, for

$$\xi_2 = 2 + \frac{1}{1} {}_{+} \frac{1}{1} {}_{+} \frac{1}{1} {}_{+} \frac{1}{1} {}_{+} \frac{1}{2} {}_{+} \frac{1}{\xi_2}$$

the quadratic equation is $13X^2 - 29X - 13 = 0$ with $D = 1517$, $a_2 = 13$. Theorem 2.37 hints at a way to find the minimal leading coefficient. By this theorem the coefficients a_n and a_m are related by the formula

$$a_m = a_n x^2 + b_n xy + c_n y^2,$$

where x, y are integers such that $xu - yv = 1$ for some integer u and v. On the one hand this Diophantine equation in u and v has a solution if and only if $(x, y) = 1$. On the other hand if a_m is the minimal value, then clearly the latter is the case. Now if we write

$$\min f_n = \min\{|f_n(x, y)| : x, y \in \mathbb{Z},\ |x| + |y| > 0\},$$
$$f_n(x, y) = a_n x^2 + b_n xy + c_n y^2,$$

where $\min f_n$ is the minimal value of the quadratic form f_n on the subset of the lattice $\mathbb{Z} \times \mathbb{Z}$ obtained by dropping the origin $(0, 0)$, then (2.64) implies

$$\frac{1}{m(\xi)} = \frac{\min f_n}{\sqrt{D}}. \tag{2.66}$$

Indeed the minimal integer $|\min f_n|$ is taken on a relatively prime pair x and y. Hence the evaluation of $m(\xi)$ reduces to the problem of finding the minimal value of a quadratic form for a given discriminant D. This approach to the problem was the original point of view taken by Markoff in his thesis.

50 Markoff sequences and J. Since $\mathbf{J}$ is defined by (2.61), it is natural first to specify an exceptional subset $\mathbf{E} = \mathbf{E}(\xi)$ in $\mathbb{Z}$ such that

$$m_n(\xi) \leqslant 3, \qquad n \notin \mathbf{E}(\xi) \Longrightarrow m_n(\xi) \leqslant 3, \quad n \in \mathbb{Z}. \tag{2.67}$$

To indicate such an $\mathbf{E}(\xi)$ we clarify the structure of $\mathbf{J}$.

Lemma 2.48 *If $\xi \in \mathbf{J}$ then it does not have isolated 1's or isolated 2's. Neither of the sequences* $[2_\infty, 1_\infty] = \{\dots, 2, 2, 2, 2, 1, 1, 1, 1, \dots\}$ *and* $[1_\infty, 2_\infty] = \{\dots, 1, 1, 1, 1, 2, 2, 2, 2, \dots\}$ *belongs to* $\mathbf{J}$.

Proof If each 1 were separated by 2's in ξ, then there would be an integer n such that $b_n = 2,\ b_{n+1} = 1,\ b_{n+2} = 2$. Choosing this n in (2.60) and applying Lemma 2.42, we obtain that

$$m_n(\xi) > 2 + \left(\frac{1}{1} \,\frac{1}{+\,2}\right) + \frac{1}{3} = 3.$$

If each 2 were separated by 1's, we put $b_{n-1} = 1,\ b_n = 2,\ b_{n+1} = 1$. Then

$$m_n(\xi) > 2 + \frac{1}{2} + \frac{1}{2} = 3.$$

Hence in both cases $\xi \notin \mathbf{J}$. A simple calculation,

$$\begin{aligned}\left(2 + \frac{1}{2} \,\frac{1}{+\,2} \,+ \cdots\right) + \frac{1}{1} \,\frac{1}{+\,1} \,+ \cdots &= \sqrt{2} + 1 + \frac{\sqrt{5}-1}{2} \\ &= 3.032\ 247\ 551 \ldots > 3\end{aligned}$$

proves the second part of the lemma. □

If $\xi = \{b_n\}_{n\in\mathbb{Z}}$ is an infinite sequence of 1's and 2's, then $n \in \mathbf{E}(\xi)$ if either $b_n = 1$ or $b_{n-1} = 2$, $b_n = 2$, $b_{n+1} = 2$.

Lemma 2.49 *If $\xi = \{b_n\}_{n\in\mathbb{Z}}$ is a sequence of 1's and 2's then*

$$\sup_{n\in\mathbf{E}(\xi)} m_n(\xi) < 2.85.$$

Proof If b_n is surrounded by 2's then by Lemma 2.42

$$\begin{aligned}m_n(\xi) = \frac{1}{2+x} + 2 + \frac{1}{2+y} &\leqslant 2 + \frac{2}{2} \,\frac{1}{+\,2} \,\frac{1}{+\,1} \,\frac{1}{+\,2} \,\frac{1}{+\,1} \,+ \cdots \\ = 2 + \frac{4}{3+\sqrt{3}} &= 4 - \frac{2}{\sqrt{3}} = 2.845\,299\,462 \cdots\end{aligned}$$

If $b_n = 1$ then

$$m_n(\xi) \leqslant 1 + \frac{2}{1} \,\frac{1}{+\,2} \,\frac{1}{+\,1} \,\frac{1}{+\,2} \,+ \cdots = 2\sqrt{3} - 2 = 2.464\ 101\ 615 \ldots.$$

Notice that 2.85 is smaller than the third value of $\mu(\xi)$ in Table 2.1. □

Corollary 2.50 *Both the sequences* $\mathbf{1} = [1_\infty]$ *and* $\mathbf{2} = [2_\infty]$ *are in* $\mathbf{J}$.

Proof For these sequences $\mathbf{E} = \mathbb{Z}$. □

Corollary 2.51 *If $\xi = \{b_n\}_{n\in\mathbb{Z}}$ is a sequence of* 1 *and* 2*, and $n \notin \mathbf{E}(\xi)$, then one of the following two possibilities holds:*

$$b_{n-2} = b_{n-1} = 1, \qquad b_n = b_{n+1} = 2\,; \tag{2.68}$$

$$b_{n-1} = b_n = 2, \qquad b_{n+1} = b_{n+2} = 1. \tag{2.69}$$

Proof By Lemma 2.49, if $n \notin \mathbf{E}(\xi)$ then $b_n = 2$ and either $b_{n-1} = 1$ or $b_{n+1} = 1$. By Lemma 2.48 there are no isolated 1's or 2's. □

If $n \notin \mathbf{E}(\xi)$ then for both cases (2.68) and (2.69),

$$m_n(\xi) = 2 + \frac{1}{2}\,{}_{+}\,\frac{1}{X} + \frac{1}{1}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{Y}. \tag{2.70}$$

Recall that ξ_n and η_n are defined in (2.62). Then in the case (2.68) $X = \xi_{n+2}$, $Y = \eta_{n-3}$ whereas in the case (2.69) $X = \eta_{n-2}$, $Y = \xi_{n+3}$. The following elementary lemma plays a central role in Markoff's arguments.

Lemma 2.52 *For positive X and Y the inequality*

$$\frac{1}{2}\,{}_{+}\,\frac{1}{X} + \frac{1}{1}\,{}_{+}\,\frac{1}{1}\,{}_{+}\,\frac{1}{Y} \leqslant 1 \tag{2.71}$$

holds if and only if $X \leqslant Y$. Equality in (2.71) corresponds to X=Y.

Proof Markoff obtained this lemma by Lagrange's identity (see Ex. 1.16)

$$\frac{1}{2+g} + \frac{1}{1}\,{}_{+}\,\frac{1}{1+g} = 1. \tag{2.72}$$

If $X \leqslant 1/g \leqslant Y$ then (2.72) implies (2.71) by Lemma 2.42. This can also be easily checked by a direct calculation. □

Let $r = \{r_n\}_{n\in\mathbb{Z}}$ be an infinite sequence of 1 and 2. Then, r^* the double of r, is defined by

$$r^* = \{r_n, r_n\}_{n\in\mathbb{Z}} = \{\dots, r_{-n}, r_{-n}, \dots, r_0, r_0, \dots, r_n, r_n, \dots\}.$$

Theorem 2.53 *If r is a regular Jean Bernoulli sequence of 1's and 2's then $r^* \in \mathbf{J}_0$. If r is a singular Markoff or singular Jean Bernoulli sequence of 1's and 2's or its conjugate then $r^* \in \mathbf{J} \setminus \mathbf{J}_0$.*

Proof If r is a constant Markoff sequence then r^* is either **1** or **2**, which are both in **J** by Corollary 2.50.

Let r be a nonconstant regular Jean Bernoulli sequence and $n \notin \mathbf{E}(r^*)$. By Corollary 2.51 we have two cases for n. If n satisfies (2.69), then there is a k such that $r_k = b_{n-1} = b_n = 2$ and $r_{k+1} = b_{n+1} = b_{n+2} = 1$. It follows that $r_k - r_{k+1} = 1$. By Theorems 1.64 and 1.65 the length $l_k(r) = p$ of Markoff's series

$$r_{k+2} - r_{k-1} = \cdots = r_{k+1+p} - r_{k-p} = 0, \qquad r_{k+2+p} - r_{k-p-1} = 1 \tag{2.73}$$

is finite, implying that $r_{k+2+p} = 2$ and $r_{k-p-1} = 1$. By (2.70) $m_n(r^*)$ is a sum of three terms, in which by (2.73)

$$\frac{1}{X} = \frac{1}{r_{k-1}} \, {}_{+} \, \frac{1}{r_{k-1}} \, {}_{+} \cdots {}_{+} \, \frac{1}{r_{k-p}} \, {}_{+} \, \frac{1}{r_{k-p} + x}, \tag{2.74}$$

$$\frac{1}{Y} = \frac{1}{r_{k+2}} \, {}_{+} \, \frac{1}{r_{k+2}} \, {}_{+} \cdots {}_{+} \, \frac{1}{r_{k+1+p}} \, {}_{+} \, \frac{1}{r_{k+1+p} + y}, \tag{2.75}$$

are equal rational functions of x and y. By Lemma 2.42 they are increasing functions of x and y respectively. Since $r_{k-p-1} = 1$ and $r_{k+2+p} = 2$, the parameters x and y have a special form:

$$x = \frac{1}{1} \, {}_{+} \, \frac{1}{1 + x'} \quad \Rightarrow \quad \frac{1}{2} < x < \frac{2}{3}, \tag{2.76}$$

$$y = \frac{1}{2} \, {}_{+} \, \frac{1}{2 + y'} \quad \Rightarrow \quad \frac{2}{5} < y < \frac{3}{7}. \tag{2.77}$$

Since $3/7 < 1/2$ we have $y < x$ and hence $X < Y$. By (2.70) and Lemma 2.52 this shows that $m_n(r^*) < 3$ for every $n \notin \mathbf{E}(r^*)$ and therefore $r^* \in \mathbf{J}_0$. The case of (2.68) is considered similarly.

Let r be a singular Markoff sequence of 1's and 2's. If $n \notin \mathbf{E}(r^*)$ then there are two cases for n. As above we consider the case (2.69). Then $r_k - r_{k+1} = 1$. If $l_k(r) < +\infty$ then $m_n(r^*) < 3$ by arguments already presented. Since r is a singular Markoff sequence, by Theorem 1.89 there must be a k such that $l_k(r) = +\infty$. Then both the continued fractions (2.74) and (2.75) are infinite and equal, implying $m_n(r^*) = 3$ by (2.72). The case of a singular Jean Bernoulli sequence or its conjugate is considered similarly. □

Corollary 2.54 *Let r be a Markoff sequence of 1's and 2's, and let r^* be its double sequence. Then for every $n \notin \mathbf{E}(r^*)$*

$$\frac{1}{560}(1+\sqrt{2})^{-4l_k(r)} < 3 - m_n(r^*) < \frac{5}{4}\phi^{-4l_k(r)}, \tag{2.78}$$

where k and n are related by $r_k = r_n^*$.

Proof By Theorem 2.53 we may assume that $l_k(r) < +\infty$. Suppose that $r_k - r_{k+1} = 1$. Then $r_k = 2$ and $r_{k+1} = 1$. Let $p = l_k(r)$, so that

$$r_{k+2} - r_{k-1} = 0, \qquad \ldots, \qquad r_{k+p+1} - r_{k-p} = 0, \qquad r_{k+p+2} - r_{k-p-1} > 0.$$

We consider the two continued fractions

$$\frac{1}{X} = \frac{1}{r_{k-1}} \, {}_{+} \, \frac{1}{r_{k-1}} \, {}_{+} \cdots {}_{+} \, \frac{1}{r_{k-p}} \, {}_{+} \, \frac{1}{r_{k-p} + x} = \frac{P_{2p} + xP_{2p-1}}{Q_{2p} + xQ_{2p-1}},$$

$$\frac{1}{Y} = \frac{1}{r_{k+2}} \, {}_{+} \, \frac{1}{r_{k+2}} \, {}_{+} \cdots {}_{+} \, \frac{1}{r_{k+p+1}} \, {}_{+} \, \frac{1}{r_{k+p+1} + y} = \frac{P_{2p} + yP_{2p-1}}{Q_{2p} + yQ_{2p-1}},$$

where x satisfies (2.76) and y satisfies (2.77). Then $0 < y < 3/7 < 1/2 < x < 1$, which implies by Lemma 2.42 that $X < Y$. Hence by (1.16)

$$\frac{1}{X} - \frac{1}{Y} = \frac{x-y}{(Q_{2p} + xQ_{2p-1})(Q_{2p} + yQ_{2p-1})}.$$

It follows that ($1 > x - y > 1/2 - 3/7 = 1/14,\ 14 \times 2 \times 2 = 56$)

$$\frac{1}{56Q_{2p}^2} < \frac{1}{X} - \frac{1}{Y} < \frac{1}{Q_{2p}^2}.$$

By (2.70) with $A = 1/X$, $B = 1/Y$

$$\frac{1}{560Q_{2p}^2} < 3 - m_n(r^*) = \frac{A-B}{(2+A)(2+B)} < \frac{1}{4Q_{2p}^2}.$$

The proof is completed by the following elementary inequality, which is valid for the denominators of the convergents of regular continued fractions whose partial denominators are either 1 or 2:

$$\frac{\phi^n}{\sqrt{5}} < Q_n < (1+\sqrt{2})^n. \tag{2.79}$$

The left-hand inequality of (2.79) follows by Lemma 1.10. The right-hand equality is obtained on by induction

$$Q_{n+1} \leqslant 2Q_n + Q_{n-1} < 2(1+\sqrt{2})^n + (1+\sqrt{2})^{n-1} = (1+\sqrt{2})^{n+1}.$$

□

Table 2.2 strongly supports the conjecture that the converse to Theorem 2.53 may be true. It is clear from the proof of Theorem 2.53 that the crucial step in this direction is to prove that every sequence in **J** is a double of some sequence of 1's and 2's.

Theorem 2.55 (Markoff 1879, 1880) *If $\xi \in \mathbf{J}$ then there are five mutually exceptional options for ξ:*

(a) $\xi = \mathbf{1} = [1_\infty]$;
(b) $\xi = \mathbf{2} = [2_\infty]$;
(c) $\xi = [1_\infty, 2, 2, 1_\infty]$;
(d) $\xi = [2_\infty, 1, 1, 2_\infty]$;
(e) $\xi = [\ldots, 2, 2, \underbrace{1}_{2t_{-2}}, 2, 2, \underbrace{1}_{2t_{-1}}, 2, 2, \underbrace{1}_{2t_0}, 2, 2, \underbrace{1}_{2t_1}, 2, 2, \underbrace{1}_{2t_2}, 2, 2, ,\ldots]$, *where* $\{t_n\}_{n\in\mathbb{Z}}$ *is a sequence of nonnegative integers of which at least two are nonzero.*

It is useful to notice that if $\xi = r^*$ for a Markoff sequence r then $t = \partial r - 1$ and therefore t is a Markoff sequence too. Leaving the full proof of this theorem until after the proof of corollary 2.58, we first deduce from it the following theorem.

Theorem 2.56 *If $\xi \in \mathbf{J}$ then $\xi = r^*$ for some Markoff sequence r.*

Proof In cases (a) and (b) ξ is a constant Markoff sequence. It is a triangle sequence of type $(1, 2)$ in case (c) and a triangle sequence of type $(2, 1)$ in case (d).

Obviously any ξ of type (e) is represented as r^* for some sequence r of 1's and 2's. Hence $r_k - r_{k+1}$ may take only the values 0 and ± 1, in agreement with property (1) of Markoff sequences; see Definition (1.59)

If $r_k - r_{k+1} = 1$ for some k then $r_k = 2$ and $r_{k+1} = 1$. Since $\xi = r^*$, there is an n such that $r_k = b_{n-1} = b_n = 2$ and $r_{k+1} = b_{n+1} = b_{n+2} = 1$. Clearly $n \notin \mathbf{E}(\xi)$. Let us consider Markoff's series (2.73) for r at k. Suppose first that there is a p such that $u = r_{k+2+p} \neq r_{k-p-1} = v$. To establish Markoff's property at k $(u - v > 0)$ we must prove that $u = 2$ and $v = 1$. Since $m_n(\xi) = m_n(r^*) \leqslant 3$, (2.70) and Lemma 2.52 show that $X \leqslant Y$, implying $y \leqslant x$. The inequality

$$y = \frac{1}{u}{}_{+}\frac{1}{u + y'} \leqslant \frac{1}{v}{}_{+}\frac{1}{v + y'} = x$$

is equivalent to

$$v + \frac{1}{v + y'} \leqslant u + \frac{1}{u + y'}.$$

Since $u, v = 1, 2$ this may happen only if either $u = v$ or $v < u$. The first option is excluded by the assumption that $u \neq v$. It follows that $u - v > 0$ in agreement with property (2) of Markoff's sequences. If Markoff's series is infinite at k then condition (2) holds automatically. The case $r_k - r_{k+1} = -1$ (property (3) of Markoff sequences) is considered similarly. □

Corollary 2.57 *If $\xi \in \mathbf{J}$ and $m_n(\xi) < 3 - \epsilon$ for some $\epsilon > 0$ for every $n \notin \mathbf{E}(\xi)$ then there is a periodic Jean Bernoulli sequence r of 1's and 2's such that $\xi = r^*$.*

Proof By Theorem 2.56 $\xi = r^*$ for a Markoff sequence r, which is a regular Jean Bernoulli sequence by Theorem 2.53. The right-hand inequality in (2.78) implies that $\sup_k l_k(r) < \infty$. Hence r is a periodic Jean Bernoulli sequence by Theorem 1.65. □

Corollary 2.58 *If r is a non-periodic regular Jean Bernoulli sequence then $r^* \in \mathbf{J}_0$ but*

$$\limsup_{|n| \to +\infty} m_n(r^*) = 3.$$

Proof The inclusion $r^* \in \mathbf{J}_0$ follows by Theorem 2.53. The right-hand inequality in (2.78) and Theorem 1.65 complete the proof. □

Proof of Theorem 2.55 Excluding options (a) and (b), we assume that $\xi = \{b_n\}_{n \in \mathbb{Z}}$ is a nonconstant sequence in $\mathbf{J}$. By Lemma 2.48 there is an n satisfying (2.69). By Lemma 2.52 and (2.70) the condition $m_n(\xi) \leqslant 3$ is equivalent to

$$\eta_{n-2} \leqslant \xi_{n+3}. \tag{2.80}$$

Similarly $m_{n-1}(\xi) \leqslant 3$ is equivalent to

$$m_{n-1}(\xi) - 2 = \frac{1}{2} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\xi_{n+3}} + \frac{1}{\eta_{n-2}} \leqslant 1.$$

Observing that

$$1 - \frac{1}{2+g} = \frac{1}{1} \, {}_{+} \, \frac{1}{1+g},$$

we obtain that $m_{n-1}(\xi) \leqslant 3$ is equivalent to

$$1 + \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\xi_{n+3}} \leqslant \eta_{n-2}. \tag{2.81}$$

By Lemma 2.42 inequalities (2.80) and (2.81) imply that

$$1 + \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\eta_{n-2}} \leqslant \eta_{n-2}, \qquad 1 + \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\xi_{n+3}} \leqslant \xi_{n+3}. \tag{2.82}$$

Now if $\xi \in \mathbf{J}$ and equality holds for one of the relations (2.82) equality holds for the other relation also then implying that $\xi_{n+3} = \eta_{n-2} = \phi$; see §**14**, in Section 1.3. In this case $\xi = r^*$, where r is a triangle Markoff sequence of type $(1,2)$ in agreement with option (c). Excluding option (c) we assume in what follows that both the inequalities in (2.82) are strict:

$$1 + \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\eta_{n-2}} < \eta_{n-2}, \qquad 1 + \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\xi_{n+3}} < \xi_{n+3}. \tag{2.83}$$

If $b_{n+3} = [\xi_{n+3}] = 1$ then the substitution of $\xi_{n+3} = 1 + 1/\xi_{n+4}$ into the second inequality of (2.83) implies after a trivial reduction that $b_{n+4} = [\xi_{n+4}] = 1$. By (2.80) we must have $b_{n-2} = [\eta_{n-2}] = 1$, and similarly the first inequality of (2.83) implies that $b_{n-3} = [\xi_{n-3}] = 1$. Substituting

$$\eta_{n-2} = 1 + \frac{1}{1} \, {}_{+} \, \frac{1}{\eta_{n-4}}, \qquad \xi_{n+3} = 1 + \frac{1}{1} \, {}_{+} \, \frac{1}{\xi_{n+5}},$$

into (2.80) and (2.83), we obtain

$$\eta_{n-4} \leqslant \xi_{n+5},$$

$$1 + \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\eta_{n-4}} < \eta_{n-4}, \qquad 1 + \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\xi_{n+5}} < \xi_{n+5}.$$

Since $\xi_{n+3} \neq \phi$ this process cannot be continued up to infinity and it will stop when $\xi_{n+2s+1} > 2$ for some integer $s = t_0 \geqslant 1$. Then the space between $b_n = 2$ and $b_{n+2s+1} = 2$ is filled with $2t_0$ 1's and to the left of b_{n-1} there will be $2(t_0 - 1)$ 1's. We have

$$\eta_{n-2s} \leqslant \xi_{n+2s+1},$$

$$1 + \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\eta_{n-2s}} < \eta_{n-2s}, \qquad 1 + \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{1} \, {}_{+} \, \frac{1}{\xi_{n+2s+1}} < \xi_{n+2s+1}.$$

Since $\xi_{n+2s+1} > 2$, the last inequality does not provide any new information. In contrast, the first inequality shows that if $b_{n-2s} = [\eta_{n-2s}] = 1$ then $b_{n-2s-1} = [\eta_{n-2s-1}] = 1$ and the process can be continued to the left. Since $\eta_{n-2} \neq \phi$, it must finish in a finite number of steps provided that there is an even number $2t_{-1} \geqslant 2(t_0 - 1)$ of 1's to the left of b_{n-1} until the first 2 appears. If $t_{-1} > 0$, then by Lemma 2.48 this 2 is preceded by 2 and therefore the above arguments can now be run in the left direction starting with the new combination 2 2 1 1. We may then observe that

$$\xi_{n+3} = \underbrace{1 + \frac{1}{1} {}_{+\cdots+} \frac{1}{1}}_{2(t_0-1)} {}_{+} \frac{1}{\xi_{n+2t_0+1(>2)}},$$

$$\eta_{n-2} = \underbrace{1 + \frac{1}{1} {}_{+\cdots+} \frac{1}{1}}_{2t_{-1}} {}_{+} \frac{1}{\eta_{n-2t_{-1}-2(>2)}};$$

substituting these formulas into (2.81) we arrive at

$$\underbrace{1 + \frac{1}{1} {}_{+\cdots+} \frac{1}{1}}_{2(t_0+1)} {}_{+} \frac{1}{\xi_{n+2t_0+1(>2)}} \leqslant \eta_{n-2} = \underbrace{1 + \frac{1}{1} {}_{+\cdots+} \frac{1}{1}}_{2t_{-1}} {}_{+} \frac{1}{\eta_{n-2t_{-1}-2(>2)}},$$

Noting that the expression '(> 2)' in the long subscripts of ξ and η means that the value of the subscript must be greater than 2. Any reduction by an even number of 1's on both sides does not change the inequality. Therefore, if $t_{-1} > t_0 + 1$ then after removing $2(t_0 + 1)$ 1's from each side we arrive at a contradiction:

$$2 < \xi_{n+2t_0+1} \leqslant \underbrace{1 + \frac{1}{1} {}_{+\cdots+} \frac{1}{1}}_{2(t_{-1}-t_0-1)} {}_{+} \frac{1}{\eta_{n-2t_{-1}-2(>2)}}.$$

It follows that the integers ($t_0 \geqslant 1$, $t_{-1} \geqslant 0$) satisfy $|t_0 - t_{-1}| \leqslant 1$ and that inequalities (2.80) and (2.81) hold only in the following cases:

(1) $t_0 = t_{-1} + 1$ and $\xi_{n+2t_0+1} \geqslant \eta_{n-2t_{-1}-1}$;
(2) $t_0 = t_{-1}$;
(3) $t_0 = t_{-1} - 1$ and $\xi_{n+2t_0+1} \leqslant \eta_{n-2t_{-1}-1}$.

If $t_{-1} = 0$ then $b_{n-2} = [\eta_{n-2}] = b_{n-3} = [\eta_{n-3}] = 2$. Since $t_0 \geqslant 1$, there is only one option for t_0, namely $t_0 = 1$. We have

$$\xi_{n+3} = 2 + \frac{1}{2} {}_{+} \frac{1}{\xi_{n+5}}, \qquad \eta_{n+2} = 1 + \frac{1}{1} {}_{+} \frac{1}{2} {}_{+} \frac{1}{2} {}_{+} \frac{1}{\eta_{n-2}}. \tag{2.84}$$

By Lemma 2.52 and (2.70) the condition $m_{n+3}(\xi) \leqslant 3$ is equivalent to

$$\xi_{n+5} \leqslant 2 + \frac{1}{2} {}_{+} \frac{1}{\eta_{n-2}}. \tag{2.85}$$

Since $\eta_{n-2} \leqslant \xi_{n+3}$, substitution of (2.85) into the first equality (2.84) results in

$$\xi_{n+3} \leqslant 2 + \frac{1}{2} \frac{1}{+2} \frac{1}{+2} \frac{1}{+\xi_{n+3}}. \tag{2.86}$$

Substituting the first formula of (2.84) into the inequality $\eta_{n-2} \leqslant \xi_{n+3}$ and combining the result with (2.85), we obtain that

$$\eta_{n-2} \leqslant 2 + \frac{1}{2} \frac{1}{+2} \frac{1}{+2} \frac{1}{+\eta_{n-2}}. \tag{2.87}$$

Inequalities (2.86) and (2.87) and $\eta_{n-2} \leqslant \xi_{n+3}$ show that either

$$\eta_{n-2} = 2 + \frac{1}{2} \frac{1}{+2} \frac{1}{+2} {}_{+\cdots} = 1 + \sqrt{2} = \xi_{n+3},$$

implying that $\xi = r^*$ where r is a triangle Markoff sequence of type (2, 1) in agreement with option (d) of Theorem 2.55, or

$$\xi_{n+3} = \underbrace{2 + \frac{1}{2} {}_{+\cdots+} \frac{1}{2}}_{2\rho} \frac{1}{+\xi_{n+2\rho+3(<2)}},$$

$$\eta_{n-2} = \underbrace{2 + \frac{1}{2} {}_{+\cdots+} \frac{1}{2}}_{2\rho^1} \frac{1}{+\eta_{n-2\rho^1-2(<2)}},$$

where $\rho > 0$ and ρ^1 are integers. This implies that, when option (d) is excluded and $t_{-1} = 0$, a nonzero even number of 2's is added on the left to b_{n-1} and on the right to $b_{n+1} = b_{n+2} = 1$, both followed by 1 1.

If $t_{-1} \geqslant 1$ and $t_0 \geqslant 1$ then to the right of $b_n = 2$ there is the combination 1 1 2 2, which is symmetric to the combination 2 2 1 1 investigated in detail at the beginning. Therefore if in the case of 2 2 1 1 one can essentially move to 1 1 2 2 in the left direction, in the case of 1 1 2 2 one can do the same in the right direction until a new 2 2 1 1 appears. Continuing by induction we prove the theorem. □

It was Theorem 2.55 which led Markoff to the discovery of his characterization of Jean Bernoulli sequences; see Definition 1.59 in §**27**. Namely, using the arguments used in the proof of Theorem 2.55 he established that the sequence $\{t_n\}_{n\in\mathbb{Z}}$ satisfies Definition 1.59.

51 The recovery problem for Markoff sequences. Our analysis of Markoff sequences summarized in Scholium 1.91 together with Theorem 1.61 indicate the possibility of a recovery of regular Markoff sequences from sequences whose positive indices satisfy Markoff conditions from some point. As the example of triangle sequences shows, the assumption of regularity is necessary.

Definition 2.59 *A sequence* $\{r_n\}_{n\geqslant 0}$ *of integers is called a semi-infinite Markoff sequence if there is an integer* $M(r)\geqslant 0$ *such that for every* $n>M(r)$*:*

(a) $r_n - r_{n+1} = \pm 1, 0$*;*

(b) *if* $r_n - r_{n+1} = \pm 1$ *then the Markoff series at* $[n, n+1]$ *ends up by satisfying* $r_{n+1+p} - r_{n-p} = r_n - r_{n+1}$, $p \overset{def}{=} l_n(r)$.

Since r_n is not defined for $n<0$ the length $l_n(r)$ of the Markoff series at $[n, n+1]$ satisfies $n - l_n(r) \geqslant 0$ for every $n > M(r)$. The infimum of $n - l_n(r)$ is denoted by $I(r)$. It is clear that $0 \leqslant I(r) \leqslant M(r)$. For any singular Markoff sequence there are two indices $k<m$ with $l_k = l_m = \infty$; these indices restrict the perturbed part of the singular sequence. If one restricts this sequence to the domain $n \geqslant 0$ then for the sequence obtained r one has $m \leqslant M(r)$.

In contrast with infinite Markoff sequences, Markoff series in the semi-infinite case are always finite. Therefore Theorem 1.69 applied in the forward direction shows that every step down in the domain $\{n > M(r)\}$ is followed by a step up and every step up by a step down. Hence $\{r_n\}_{n\geqslant 0}$ oscillates between a and $a+1$ as soon as there is one n, $n>M(r)$ such that $r_n - r_{n+1} \neq 0$. Moreover, by Markoff's property $r_n = a, a+1$ also for $n \geqslant I(r)$.

We consider now the derivative of r. Let $\{n_k\}_{k\geqslant 0}$ be the increasing sequence of all solutions to the equation $r_n = a+1$ with $n \geqslant 0$. Since $s_k = n_{k+1} - n_k \geqslant 1$, $k \geqslant 0$ and $n_0 \geqslant 0$, we obtain that $n_k \geqslant k$.

Lemma 2.60 *If* r *is a semi-infinite Markoff sequence then* $s = \partial r$ *is a semi-infinite Markoff sequence with* $M(s) \leqslant M(r)$.

Proof If $n > M(r)$ and $r_n - r_{n+1} = -1$ then $r_n = a$ and $r_{n+1} = a+1$. By the definition of semi-infinite Markoff sequence all integers in $\omega_n(r)$ are nonnegative. If j is the left-hand end of $\omega_n(r)$ then $r_j = a+1$. It follows that $j \geqslant n_0$ and therefore there is a $k \geqslant 0$ such that $n = n_{k+1} - 1$.

If $n > M(r)$ and $r_n - r_{n+1} = 1$ then $r_n = a+1$ and $r_{n+1} = a$. If there is a $k \geqslant 0$ such that $n = n_{k+1}$ then we fix it. If such a k does not exist then $r_j \neq a+1$ for $j<n$. If $n > M(r)+1$ then $n-1 > M(r)$ and $r_{n-1} - r_n = -1$. But then at the left-hand end j of $\omega_{n-1}(r)$ we must have $r_j = a+1$ by the Markoff condition at $[n-1, n]$. Since this contradicts our assumption we must have $n \leqslant M(r)+1$. Hence $n_0 = n$. It follows that in any case $n_0 \in [0, M(r)+1]$.

Suppose that $k > M(r)$ and $s_k - s_{k+1} > 1$. Then the length of $[n_k, n_{k+1})$ exceeds the length of $[n_{k+1}, n_{k+2})$ by at least 2. Hence for $n = n_{k+1} - 1 > n_k$ we have $r_n = a$, $r_{n+1} = a+1$, implying that $r_n - r_{n+1} = -1$. Since $n > n_k \geqslant k > M(r)$, the Markoff condition for r at $[n, n+1]$ says that racer A wins (see the racing algorithm, discussed in §**29** of section 1.5. But this is not possible since the distance from n_k to n exceeds that from $n+1$ to n_{k+2} by at least 1. The case $s_k - s_{k+1} < -1$ is considered similarly.

If $s_k - s_{k+1} = 1$ and $k > M(r)$ then $n = n_{k+1} - 1 \geqslant k > M(r)$. The distance from n to n_k is now equal to the distance from $n+1$ to n_{k+2}. Therefore the Markoff conditions imply that $l_j(r) > s_k - 1$. Since $r_n - r_{n+1} = -1$, racer A in the racing algorithm at $[n, n+1]$ for r hits some $n_{k-p} \geqslant I_m(r)$ first, implying that $s_{m+1+p} - s_{m-p} > 0$. The case $s_k - s_{k+1} = -1$ is considered similarly. It follows that $\{s_k\}_{k\geqslant 0}$ is a semi-infinite Markoff sequence and $0 \leqslant I(s) \leqslant M(s) \leqslant M(r)$. □

Theorem 2.61 *Any semi-infinite Markoff sequence $\{r_n\}_{n\geqslant 0}$ of two integers a and $a+1$ extends either to a Jean Bernoulli sequence $\{e_n\}_{n\in\mathbb{Z}}$ or to its conjugate sequence.*

Proof Let $J > M(r)$ be a large number. Then by Lemma 1.72 the segment $[M(r), J]$ can be covered by a finite number of finite intervals $\omega_n(r)$. Differentiating r, we obtain an s such that $l_k(s) < l_n(s)$ for every n. The interval $\omega_k(s)$ covers a smaller segment $[M(r), J_1]$. Proceeding by induction in a finite number of steps we let the derivative be identically 1 on some $[M(r), J_q]$. Let us extend this sequence so that it is identically 1 in both directions. Integrating the latter sequence back with the same constants of integration, we obtain finally a periodic sequence coinciding with r on $[M(r), J]$. Passing to the limit as in Theorem 1.86, we complete the proof. □

On the one hand the set **J** consists of a two-sided infinite sequence. On the other hand any irrational ξ with $\mu(\xi) \leqslant 3$ determines a one-sided sequence $\{b\}_{n\geqslant 1}$ of partial denominators of the regular continued fraction for ξ.

Theorem 2.62 *If $\xi = [b_0; b_1, b_2, \ldots]$ is an irrational such that $m_n(\xi) < 3$ for every $n > N$ then there is a Jean Bernoulli sequence r of 1's and 2's such that $\xi_n = r_n^*$ for all sufficiently large n.*

Proof Applying the algorithm used in the proof of Theorem 2.56, we obtain that, starting from some particular point, ξ is the double of some semi-infinite sequence r. Applying the inequalities (2.78), we derive that r is a semi-infinite Markoff sequence. Applying Theorem 2.61, we obtain that r uniquely extends to a Jean Bernoulli sequence e. □

Theorem 2.63 *Any irrational ξ with $\mu(\xi) < 3$ is a quadratic irrational. The Lagrange spectrum below 3 is discreet, accumulating to 3.*

Proof By Theorem 2.62 there is a double r^* of a Jean Bernoulli sequence r coinciding with the partial denominators of the regular continued fraction for ξ, starting from some point in the latter. By Corollary 2.57 the sequence r is periodic. It follows that ξ is a quadratic irrational. By (2.78) the values of $\mu(\xi)$ approach 3 when the length of the period increases. Since there is only a finite number of sequences of 1's and 2'2s for a given length of period, the theorem is proved. □

52 Calculations of the Lagrange spectrum. Theorem 2.63 obviously implies that one has

$$\mu(\xi) = m(\xi) \tag{2.88}$$

if $\mu(\xi) < 3$. This important formula, discovered by Markoff, allows one to compute the Lagrange spectrum below 3. For $n \in \mathbf{E}(\xi)$ Lemma 2.49 implies $m_n(\xi) < 2.85$. Theorem 1.64 claims that $l(r) = q - 1$ for any Jean Bernoulli sequence r with period q. By Corollary 2.54 we have

$$\begin{aligned}
q &= 1, \qquad 1.75 < m(\xi),\\
q &= 2, \qquad 2.817\,627\,458 < m(\xi),\\
q &= 3, \qquad 2.973\,392\,205 < m(\xi),\\
q &= 4, \qquad 2.996\,117\,975 < m(\xi),\\
q &= 5, \qquad 2.999\,433\,620 < m(\xi).
\end{aligned}$$

The third inequality shows that the first three lines in Table 2.1 do indeed represent the three first values of $\mu(\xi)$. Proceeding by induction, theoretically one can locate with these inequalities the whole Lagrange spectrum below 3. There is however another approach, based on the theory of Markoff's periods of Jean Bernoulli sequences presented in §**31** in Section 1.5. If $\mu(\xi) < 3$ and ξ is not constant, then $\xi = r^*$ for some nonconstant Jean Bernoulli sequence r by Corollary 2.57. Since the number 2.85 is smaller than the third value of $\mu(\xi)$, Lemma 2.49 says that in the evaluation of $m(\xi)$ we need consider only $n \notin \mathbf{E}(\xi)$. For such an n we have $b_n = 2$, $b_{n+1} = 1$ or $b_{n-1} = 1$, $b_n = 2$; see Corollary 2.51. Let us consider the first Markoff period of r:

$$\Pi_1 = \{\alpha_1,\ \alpha_2,\ \dots,\ \alpha_{q-1},\ \alpha_q\}, \qquad \alpha_1 = 2,\ \alpha_q = 1.$$

Recall that $\alpha_2 = \alpha_{q-1}$, $\alpha_3 = \alpha_{q-2}, \dots$, i.e. the middle part of Π_1 is symmetric. Let $\Pi = \{\beta_1,\ \beta_2,\ \dots,\ \beta_{q-1},\ \beta_q\}$ be any other period of r which is just a shift of Π_1. To simplify the notation we introduce the following continued fractions:

$$x = \frac{1}{\alpha_q} + \frac{1}{\alpha_q} + \frac{1}{\alpha_{q-1}} + \frac{1}{\alpha_{q-1}} + \cdots + \frac{1}{\alpha_2} + \frac{1}{\alpha_2} + \frac{1}{\alpha_1} + \frac{1}{\alpha_1 + x}, \tag{2.89}$$

$$u = \frac{1}{\beta_q} + \frac{1}{\beta_q} + \frac{1}{\beta_{q-1}} + \frac{1}{\beta_{q-1}} + \cdots + \frac{1}{\beta_2} + \frac{1}{\beta_2} + \frac{1}{\beta_1} + \frac{1}{\beta_1 + u}, \tag{2.90}$$

$$y = \frac{1}{\alpha_2} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_3} + \cdots + \frac{1}{\alpha_q} + \frac{1}{\alpha_q} + \frac{1}{\alpha_1} + \frac{1}{\alpha_1 + y}, \tag{2.91}$$

$$v = \frac{1}{\beta_2} + \frac{1}{\beta_2} + \frac{1}{\beta_3} + \frac{1}{\beta_3} + \cdots + \frac{1}{\beta_q} + \frac{1}{\beta_q} + \frac{1}{\beta_1} + \frac{1}{\beta_1 + v}. \tag{2.92}$$

Theorem 2.64 (Markoff 1880) *For any nonconstant Jean Bernoulli sequence r,*

$$m(r^*) = x + \alpha_1 + \frac{1}{\alpha_1 + y}. \tag{2.93}$$

Proof There are two types of period of r^*. We consider first the image $\{\beta_1, \beta_1, \dots \beta_q, \beta_q\}$ of Π in r^*. Let n be any index in r^* corresponding to the first β_1 in this double period. Since r is not constant, $m(r^*) > 2.85$. We may assume that $\beta_1 = 2$ and $\beta_q = 1$. Indeed, otherwise either b_n is 1, or b_n is 2 and has a 2 on each side in r^*, implying that $m_n(r^*) < 2.85$ by Lemma 2.49. Since $\alpha_1 = \beta_1$ and $\alpha_q = \beta_q$, by Theorem 1.80 the first nonzero difference $\alpha_i - \beta_i$ is $+1$. Hence $\alpha_i = 2$, $\beta_i = 1$ for $1 < i < q$ and

$$\beta_i + \frac{1}{\beta_i + s_i} = 1 + \frac{1}{1 + s_i} < 2 + \frac{1}{2 + t_i} = \alpha_i + \frac{1}{\alpha_i + t_i}.$$

By Lemma 2.42, (2.91) and (2.92) show that $y < v$ and hence $1/(\beta_1 + v)$ is smaller than $1/(\alpha_1 + y)$. By Theorem 1.80 the last nonzero difference $\alpha_j - \beta_j$ is -1, implying that $\alpha_j = 1$, $\beta_j = 2$ for some $i < j < q$. Hence

$$\alpha_j + \frac{1}{\alpha_j + s_j} = 1 + \frac{1}{1 + s_j} < 2 + \frac{1}{2 + t_j} = \beta_j + \frac{1}{\beta_j + t_j}.$$

By Lemma 2.42, (2.89) and (2.90) show that $u < x$. It follows that

$$u + \beta_1 + \frac{1}{\beta_1 + v} < x + \alpha_1 + \frac{1}{\alpha_1 + y}.$$

To complete the proof we must also test one-step shifts $\{\beta_1, \beta_2, \beta_2 \dots \beta_q, \beta_q, \beta_1\}$ in r^* of the periods considered above. In other words that we must check

$$\frac{1}{\beta_1 + u} + \beta_1 + v < x + \alpha_1 + \frac{1}{\alpha_1 + v}. \tag{2.94}$$

The central part of the period Π_1 is symmetric. Therefore the symmetry of $\mathbb{Z}$ with respect to its center transforms Π_1 onto Π_2. It follows that the sequence obtained is a shift in r^*. However, the transformation of the left-hand side of (2.94) is of the form already considered. This completes the proof. □

Using Euler's formula (2.1) for quadratic irrationals, Markoff simplified the right-hand side of (2.93). Let

$$\frac{P}{Q} = \alpha_1 + \frac{1}{\alpha_q} + \frac{1}{\alpha_q} + \frac{1}{\alpha_{q-1}} + \frac{1}{\alpha_{q-1}} + \cdots + \frac{1}{\alpha_2} + \frac{1}{\alpha_2} + \frac{1}{\alpha_1},$$
$$\frac{P'}{Q'} = \alpha_1 + \frac{1}{\alpha_q} + \frac{1}{\alpha_q} + \frac{1}{\alpha_{q-1}} + \frac{1}{\alpha_{q-1}} + \cdots + \frac{1}{\alpha_2} + \frac{1}{\alpha_2}$$

be odd and even convergents to $\xi = x + \alpha_1$. It follows that

$$PQ' - P'Q = 1. \tag{2.95}$$

By (2.1)

$$Q\xi^2-(P-Q')\xi-P'=0,$$

implying that

$$\xi-\xi^*=\frac{\sqrt{(P-Q')^2+4QP'}}{Q}.$$

If we write $\alpha_1=2=1+1/1$ then

$$\frac{P-2Q}{Q}=\frac{1}{\alpha_q}+\frac{1}{\alpha_q}+\frac{1}{\alpha_{q-1}}+\frac{1}{\alpha_{q-1}}+\cdots+\frac{1}{\alpha_2}+\frac{1}{\alpha_2}+\frac{1}{1}+\frac{1}{1}$$

is symmetric since $\alpha_q=1$. Hence by Corollary 1.6

$$P_k=P-2Q=Q_{k-1}=Q-Q'\Longrightarrow Q'=3Q-P. \tag{2.96}$$

Combining (2.95) and (2.96), we obtain the formula

$$QP'=PQ'-1=3PQ-P^2-1.$$

It follows that

$$(P-Q')^2+4QP'=(2P-3Q)^2+12PQ-4P^2-4=9Q^2-4,$$

and hence

$$m(r^*)=\sqrt{9-\frac{4}{Q^2}}. \tag{2.97}$$

This formula of Markoff formula parameterizes the Lagrange spectrum by Q, which depends on the parameters of the first Markoff period Π_1. By (1.81), (1.82) the parameters α_j of Π_1 are simply related to the Jean Bernoulli period $\mathbf{JB}(r)$ of r. It follows that the Lagrange spectrum is parameterized by the rational numbers p/q in $[1,2]$:

$$\frac{p}{q}\to\frac{P}{Q}=2+\frac{1}{1}+\frac{1}{1}+\frac{1}{r_1}+\frac{1}{r_1}+\cdots+\frac{1}{r_{q-2}}+\frac{1}{r_{q-2}}+\frac{1}{2}, \tag{2.98}$$

where $r_j=r_j(p/q,0)$. By (2.96) the above formula for P/Q can be simplified:

$$\frac{P}{Q}=3-\frac{Q'}{Q}=3-\frac{1}{2}+\frac{1}{r_1}+\frac{1}{r_1}+\cdots+\frac{1}{r_{q-2}}+\frac{1}{r_{q-2}}+\frac{1}{2}. \tag{2.99}$$

If we formally put $Q=1$ in (2.97) then we obtain $m=\sqrt{5}$. If we put $Q=2$ then $m=\sqrt{8}$.

Let us order simple fractions p/q in $[1,2]$ lexicographically. In other words we write $p/q\prec r/s$ if either $p<r$ holds or both $p=r$ and $q<s$ hold. Using this order and (2.97) with (2.98), we can easily find the first points of the Lagrange spectra. Table 2.3 demonstrates that the mapping $p/q\to m(r^*)$ increases up to the simple

Table 2.3. *Calculation of the Lagrange spectrum*

p/q	**JB**	P/Q	$m(r^*)$
1/1	{1}	1/1	2.236 067 977 499 78 ...
2/1	{2}	5/2	2.828 427 124 746 19 ...
3/2	{2, 1}	13/5	2.973 213 749 463 70 ...
4/3	{2, 1, 1}	34/13	2.996 052 629 869 29 ...
5/3	{2, 1, 2}	75/29	2.999 207 188 146 83 ...
5/4	{2, 1, 1, 1}	89/34	2.999 423 243 289 87 ...
6/5	{2, 1, 1, 1, 1}	233/89	2.999 915 834 362 00 ...
7/4	{2, 1, 2, 2}	437/169	2.999 976 658 056 08 ...
7/5	{2, 1, 1, 2, 1}	507/194	2.999 982 286 411 02 ...
7/6	{2, 1, 1, 1, 1, 1}	610/233	2.999 987 720 016 37 ...
8/5	{2, 1, 2, 1, 2}	1120/433	2.999 996 444 233 73 ...
8/7	{2, 1, 1, 1, 1, 1, 1}	1597/610	2.999 998 208 366 39 ...
9/5	{2, 1, 2, 2, 2}	2547/985	2.999 999 312 874 08 ...
9/7	{2, 1, 1, 1, 2, 1, 1}	3468/1325	2.999 999 620 268 16 ...
9/8	{2, 1, 1, 1, 1, 1, 1, 1}	4181/1597	2.999 999 738 604 00 ...
10/7	{2, 1, 1, 2, 1, 2, 1}	7571/2897	2.999 999 920 565 02 ...
10/9	{2, 1, 1, 1, 1, 1, 1, 1, 1}	10 946/4181	2.999 999 961 862 83 ...
11/6	{2, 1, 2, 2, 2, 2}	14 845/5741	2.999 999 979 772 89 ...
11/7	{2, 1, 2, 1, 2, 1, 2}	10 946/6466	2.999 999 984 054 52 ...
11/8	{2, 1, 1, 2, 1, 1, 2, 1}	19 760/7561	2.999 999 988 338 61 ...
11/9	{2, 1, 1, 1, 1, 2, 1, 1, 1}	23 763/9077	2.999 999 991 908 59 ...
11/10	{2, 1, 1, 1, 1, 1, 1, 1, 1, 1}	28 657/10 946	2.999 999 994 435 86 ...
12/7	{2, 1, 2, 2, 1, 2, 2}	38 014/14 701	2.999 999 996 915 28 ...
12/11	{2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}	75 025/28 657	2.999 999 999 188 20 ...
13/7	{2, 1, 2, 2, 2, 2, 2}	86 523/33 461	2.999 999 999 404 56 ...
13/8	{2, 1, 2, 1, 2, 2, 1, 2}	97 427/37 666	2.999 999 999 530 09 ...
13/9	{2, 1, 2, 2, 1, 2, 1, 2, 1}	113 058/43 261	2.999 999 999 643 78 ...
13/10	{2, 1, 1, 1, 2, 1, 1, 2, 1, 1}	135 163/51 641	2.999 999 999 750 01 ...
13/11	{2, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1}	162 867/62 210	2.999 999 999 827 73 ...
13/12	{2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}	196 418/75 025	2.999 999 999 881 56 ...
14/9	{2, 1, 2, 1, 2, 1, 2, 1, 2}	249 755/96 557	2.999 999 999 928 49 ...
14/13	{2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}	514 229/196 418	2.999 999 999 982 71 ...
15/8	{2, 1, 2, 2, 2, 2, 2, 2}	504 293/195 025	2.999 999 999 982 47 ...
15/11	{2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 1}	770 133/294 685	2.999 999 999 992 32 ...
15/13	{2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1}	1 116 300/426 389	2.999 999 999 996 33 ...
15/14	{2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}	1 346 269/514 229	2.999 999 999 997 47 ...
16/9	{2, 1, 2, 2, 2, 1, 2, 2, 2}	1 291 324/499 493	2.999 999 999 997 32 ...
16/11	{2, 1, 1, 2, 1, 2, 1, 2, 1, 2, 1}	1 688 299/646 018	2.999 999 999 998 40 ...
16/13	{2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1}	2 423 593/925 765	2.999 999 999 999 22 ...
16/15	{2, 1, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1}	3 524 578/1 346 269	2.999 999 999 999 63 ...
17/9	{2, 1, 2, 2, 2, 2, 2, 2, 2}	2 939 235/1 136 689	2.999 999 999 999 48 ...
17/10	{2, 1, 2, 2, 1, 2, 2, 1, 2, 2}	3 306 781/1 278 818	2.999 999 999 999 59 ...
17/11	{2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2}	3 729 600/1 441 889	2.999 999 999 999 67 ...
17/12	{2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 2, 1}	4 406 309/1 686 049	2.999 999 999 999 76 ...

fraction 15/8. For further fractions one notices a shift in blocks having the same p value. Nevertheless this method allows one to arrange the calculations very easily.

Theorem 2.45 can be strengthened. Nonperiodic Jean Bernoulli sequences have the form $r = \{r_n(\theta, \delta)\}_{n\in\mathbb{Z}}$, where $\theta \in (1, 2)$ is irrational. By Theorem 1.61 any such sequence determines a unique irrational number $\xi \in (0, 1)$:

$$\xi(\theta, \delta) = \frac{1}{r_1} \, \frac{1}{+\, r_1} \, \frac{1}{+\, r_2} \, \frac{1}{+\, r_2} \, \frac{1}{+\, r_3} \, \frac{1}{+\, r_3} + \cdots + \frac{1}{r_n} \, \frac{1}{+\, r_n} + \cdots$$

These numbers make up an important class of irrational numbers satisfying $\mu(\xi) = 3$. Since the irrational θ values make a continuum, there must be irrational θ such that the numbers $\xi(\theta, 0)$ are transcendental. This example of transcendental numbers is interesting since there is an explicit formula for $r_n = [(n+1)\theta] - [n\theta]$. These transcendental numbers $\xi(\theta, \delta)$ are also interesting since they are approximated by rational numbers the most poorly. They are in a sense the first transcendental numbers one finds in studying the functional $\mu(\xi)$. In contrast, $\mu(\xi) = +\infty$ in a well-known example of Liouville; Liouville's numbers can be approximated by rational numbers very well. So, the classes of algebraic and transcendental numbers have a huge overlap from the point of view of rational approximation. The set of quadratic irrationals with $\mu(\xi) < 3$ lies within the symmetric difference.

Theorem 2.65 (Markoff 1879, 1880) *A number smaller than* 3 *is a point of the Markoff spectrum if and only if it is of the form*

$$\sqrt{9 - \frac{4}{m^2}},$$

where m is a positive integer such that the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz$$

has a solution in positive integers $x = k$, $y = l$, $z = m$ satisfying $k \leqslant m$, $l \leqslant m$.

For instance, the case $m = 1$ (then $k = l = 1$) corresponds to $\mu(\xi) = \sqrt{5}$ and the case $m = 2$ (then $k = l = 1$) corresponds to $\mu(\xi) = \sqrt{8}$; these are considered in Theorem 2.44. The next point of the Lagrange spectrum is $\sqrt{221}/5$, corresponding to $m = 5$ ($k = l$, $l = 2$). We refer the interested reader to Cusick and Flahive (1989) or to the original publications of Markoff (1879, 1880) for the proof of this theorem.

Exercises

2.1 Apply Euler's algorithm to prove that

$$\sqrt{7} = 2 + \frac{1}{1} \, \frac{1}{+\, 1} \, \frac{1}{+\, 1} \, \frac{1}{+\, 4} \, \frac{1}{+\, 1} \, \frac{1}{+\, 1} \, \frac{1}{+\, 1} \, \frac{1}{+\, 4} + \cdots .$$

2.2 Apply Euler's algorithm to prove that

$$\sqrt{61} = 7 + \frac{1}{1} \, {}_{+} \frac{1}{4} \, {}_{+} \frac{1}{3} \, {}_{+} \frac{1}{1} \, {}_{+} \frac{1}{2} \, {}_{+} \frac{1}{2} \, {}_{+} \frac{1}{1} \, {}_{+} \frac{1}{3} \, {}_{+} \frac{1}{4} \, {}_{+} \frac{1}{1} \, {}_{+} \frac{1}{14} \, {}_{+ \cdots}.$$

2.3 Show that

$$7 + \frac{1}{14} \, {}_{+} \frac{1}{14} \, {}_{+ \cdots} = 5\left(1 + \frac{1}{2} \, {}_{+} \frac{1}{2} \, {}_{+ \cdots}\right).$$

2.4 Prove that every mixed regular periodic continued fraction with more than one nonperiodic elements ($h \geqslant 2$) is a root of a quadratic equation with integer coefficients. The other root has the same sign (Smith 1888, §364).

2.5 Prove that $1 < p/q$, $(p, q) = 1$, satisfies

$$\frac{p}{q} = b_0 + \frac{1}{b_1} \, {}_{+} \frac{1}{b_2} \, {}_{+ \cdots +} \frac{1}{b_2} \, {}_{+} \frac{1}{b_1} \, {}_{+} \frac{1}{b_0}$$

if and only if either $q^2 + 1$ or $q^2 - 1$ is divisible by p (Serret's theorem, Perron 1954, Section 1, §11]).

2.6 Prove that every divisor of a sum of two squares is also a sum of two squares (Euler and Serret, see Perron 1954, Section 1, §11]). *Hints:* If $p > q$ is a divisor of $q^2 + 1$ then by Ex. 2.5 the continued fraction of p/q is symmetric in $2k + 2$ terms. Show that $p = P_{2k+1} = P_k^2 + P_{k-1}^2$. If p is a divisor of $q^2 + 1$ such that $p \leq q$ then p is a divisor of $(q - sp)^2 + 1$ for any integer s. Finally, if $(a, b) = 1$ then there are integers x, y such that $ax - by = 1$. It follows that any divisor p of $a^2 + b^2$ is a divisor of $(a^2 + b^2)(x^2 + y^2) = (ay + bx)^2 + (ax - by)^2 = (ay + bx)^2 + 1$.

2.7 If $m \in \mathbb{Z}$ and n is a positive integer then $m \equiv r \pmod{n}$ means that n divides $m - r$. Prove Euler's theorem, which says that

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

for every a with $(a, n) = 1$.

Hints: Let $\{r_1, \ldots, r_s\}$ be the set of all elements in $\{1, \ldots, n\}$ with $(r_j, n) = 1$, $j = 1, 2, \ldots, s$. Notice that $s = \varphi(n)$ and that

$$ar_1 \equiv r_{\sigma 1} \pmod{n}, \qquad \dots \qquad , ar_s \equiv r_{\sigma s} \pmod{n},$$

where σ is a one-to-one mapping of the set $\{1, 2, \dots, s\}$ onto itself. Multiply the above identities.

2.8 Prove that $\varphi(mn) = \varphi(m)\varphi(n)$ for every pair of m and n with $(m, n) = 1$. Applying this formula, show that

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_r}\right) ,$$

where $p_1 < \dots < p_r$ is the complete list of prime divisors of n.

2.9 Following Euler's arguments prove that any prime number of the form $p = 4n + 1$, where n is an integer, can be represented as a sum of two squares (Fermat's statement).

Hints: If $p = 4n + 1$ is prime then $k^{4n} \equiv 1 \pmod{p}$ by Ex. 2.7, $k = 1, \dots, 4n$. Then $2^{4n} - 1, 3^{4n} - 2^{4n}, \dots, (4n)^{4n} - (4n-1)^{4n}$ are all divisible by p. Observing that each of the above differences equals $a^{4n} - b^{4n} = (a^{2n} + b^{2n})(a^{2n} - b^{2n})$, deduce that either p is the sum of two squares by Ex. 2.6, or p is the divisor of each of the differences

$$2^{2n} - 1, \quad 3^{2n} - 2^{2n}, \quad \dots, \quad (4n)^{2n} - (4n-1)^{2n} .$$

Taking consecutive differences of the above series, show that $(2n)!$ is divisible by a prime number of the form $p = 4n + 1$.

2.10 Prove Jirard's theorem: a positive integer is a sum of two integer squares if and only if it is a finite product of multipliers each of which is 2, the square of an integer or a prime number of the form $4n + 1$.

2.11 Prove Euler's formula, Euler (1755, p. 280):

$$\sqrt{2} = \frac{7}{5}\left(1 + \frac{1}{100} + \frac{1 \times 3}{100 \times 200} + \frac{1 \times 3 \times 5}{100 \times 200 \times 300} + \cdots\right) .$$

Hint: Combine the binomial formula

$$(1-x)^{-1/2} = 1 + \left(\frac{x}{2}\right) + \frac{1 \times 3}{2!}\left(\frac{x}{2}\right)^2 + \cdots + \frac{1 \times 3 \times \cdots (2n-1)}{n!}\left(\frac{x}{2}\right)^n + \cdots$$

with the Pell equation $P^2 - Q^2 D = -1$ to get

$$\sqrt{D} = \frac{P}{Q}\left(1 - \frac{1}{P^2 + 1}\right)^{-1/2}$$
$$= \frac{P}{Q}\left(1 + \left(\frac{1}{2Q^2 D}\right) + \frac{1 \times 3}{2!}\left(\frac{1}{2Q^2 D}\right)^2 + \frac{1 \times 3 \times 5}{3!}\left(\frac{1}{2Q^2 D}\right)^3 + \cdots\right),$$

and put $P = 7$, $Q = 5$, $D = 2$ to get $7^2 - 5^2 \times 2 = -1$.

2.12 Prove that for any positive solution to Pell's equation $P^2 - Q^2D = 1$

$$\sqrt{D} = \frac{P}{Q}\left(1 - \left(\frac{1}{2P^2}\right) - \frac{1}{2!}\left(\frac{1}{2P^2}\right)^2 - \frac{1\times 3}{3!}\left(\frac{1}{2P^2}\right)^3 - \cdots\right).$$

2.13 Prove that the minimal value of the discriminants of real quadratic irrationals is 5.
Hint: If $b = 2b_1$ then $D = 4(b_1^2 - ac) \geqslant 5$, since $\sqrt{D} \notin \mathbb{Q}$. If $b = 2b_1 + 1$ then $D = 4(b_1^2 + b_1 - ac) + 1 \geqslant 5$.

2.14 (Markoff). For every irrational ξ show that there exist infinitely many rational numbers p/q such that

$$\xi - \frac{p}{q} < \frac{1}{\sqrt{5}q^2}.$$

Hint: Notice that, for $\xi = \phi$,

$$\phi - \frac{p}{q} = \frac{1}{q^2(2\phi - 1 + \varepsilon_n)} = \frac{1}{q^2(\sqrt{5} + \varepsilon)}.$$

3

Continued fractions: analysis

3.1 Convergence: elementary methods

53 Paradox of Sofronov (1729–60). This was published in Smirnov and Kulyabko (1954). Sofronov was a student of Euler. Applying Bombelli's method, see §**3** in Section 1.1, we obtain

$$\sqrt{-a} = b + \sqrt{-a} - \sqrt{b^2} = b + \frac{-a-b^2}{b+\sqrt{-a}} = b + \frac{-a-b^2}{2b} {}_{+} \frac{-a-b^2}{2b} {}_{+\cdots}.$$

The choice $b=1$, $a=1$ suggests that $i=\sqrt{-1}$ corresponds to a real continued fraction:

$$i = 1 - \frac{2}{2} {}_{-} \frac{2}{2} {}_{-} \frac{2}{2} {}_{-\cdots} \quad !$$

Clearly this identity cannot hold since the left-hand side is purely imaginary. To explain this paradox let us observe that $i+1$ is a continued fraction of the form

$$m + \frac{n}{m} {}_{+} \frac{n}{m} {}_{+\cdots} \approx m + \frac{1}{m/n} {}_{+} \frac{1}{m} {}_{+} \frac{n}{m} {}_{+\cdots} \approx \alpha + \frac{1}{\beta} {}_{+} \frac{1}{\alpha} {}_{+} \frac{1}{\beta} {}_{+\cdots}, \tag{3.1}$$

where $\alpha = m$, $\beta = m/n$ ($n=-2$, $m=2$). The symbol $\approx$ means that the convergents of two continued fractions coincide.

Definition 3.1 *A continued fraction* $\mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ *is called convergent if its convergents* $\{d_n\}_{n\geqslant 0}$ *have finite or infinite limit.*

Let

$$Q(z) = \sum_{n=0}^{\infty} Q_n z^n$$

be the power series whose coefficients are the denominators of (3.1). The Euler–Wallis formulas say

$$Q_{2k+1} = \beta Q_{2k} + Q_{2k-1}, \qquad k = 0, 1, 2, \ldots;$$
$$Q_{2k} = \alpha Q_{2k-1} + Q_{2k-2}, \qquad k = 1, 2, \ldots$$

Now the first recurrence implies that

$$Q_{\text{odd}} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} Q_{2k+1} z^{2k+1} = \beta z \sum_{k=0}^{\infty} Q_{2k} z^{2k} + z^2 \sum_{k=0}^{\infty} Q_{2k+1} z^{2k+1} ,$$

whereas the second that

$$Q_{\text{even}} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} Q_{2k} z^{2k} = 1 + \alpha z \sum_{k=0}^{\infty} Q_{2k+1} z^{2k+1} + z^2 \sum_{k=0}^{\infty} Q_{2k} z^{2k} .$$

It follows that

$$(1-z^2) Q_{\text{odd}} = \beta z Q_{\text{even}} ,$$
$$(1-z^2) Q_{\text{even}} = 1 + \alpha z Q_{\text{odd}} .$$

Solving this linear system, we obtain

$$Q_{\text{even}} = \frac{1-z^2}{(1-z^2)^2 - \alpha\beta z^2} , \qquad Q_{\text{odd}} = \frac{\beta z}{(1-z^2)^2 - \alpha\beta z^2} .$$

It is clear that the roots of the denominators of Q_{even} and Q_{odd} are

$$\pm \frac{\sqrt{\alpha\beta}}{2} \pm \sqrt{\frac{\alpha\beta}{4} + 1}$$

In Sofronov's case $\alpha = m = 2b$, $\beta = m/n = -2b/(a+b^2)$. Therefore $\alpha\beta = 4b^2/(a+b^2)$, which shows that the poles of Q_{even}, Q_{odd} and therefore Q are on $\mathbb{T}$. The function Q, being a linear combination of rational fractions of the form $(1-\zeta z)^{-1}$ where $|\zeta| = 1$, must have uniformly bounded Taylor coefficients. Therefore all numbers Q_n are uniformly bounded. Now the formula

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}}{Q_n Q_{n-1}}$$

implies that Sofronov's fraction diverges. Indeed, its right-hand side does not tend to zero. See other details in Smirnov and Kulyabko (1954).

54 Paradox of quadratic equations. Following Markoff, let us consider the quadratic equation

$$z^2 - 2z - 1 = 0 \quad \Leftrightarrow \quad z = 2 + \frac{1}{z} ,$$

which has two roots, $z_1 = 1 + \sqrt{2}$, $z_2 = 1 - \sqrt{2}$. Iterating, we obtain

$$z_{1,2} = 2 + \frac{1}{2} \, {}_{+} \, \frac{1}{2} \, {}_{+} \, \frac{1}{2} \, {}_{+} \cdots {}_{+} \, \frac{1}{2 + 1/z_{1,2}} . \tag{3.2}$$

By Theorem 1.11 the infinite continued fraction $2 + \mathbf{K}_{n=1}^{\infty}(1/2)$ converges to one of the roots $z_{1,2} = 1 \pm \sqrt{2}$ of the quadratic equation considered. Since all convergents of the continued fraction are positive, it is clear that it converges to $z_1 = 1+\sqrt{2}$, which implies that

$$\sqrt{2} = 1 + \frac{1}{2} \, \frac{1}{+ \, 2} \, \frac{1}{+ \, 2} \, _{+ \cdots} \; .$$

However, for z_2 we have

$$-\sqrt{2} = 1 + \frac{1}{2} \, \frac{1}{+ \, 2} \, \frac{1}{+ \, 2} \, _{+ \cdots +} \, \frac{1}{2+1/(1-\sqrt{2})} \; .$$

The situation is clarified by the following theorem due to Markoff.

Theorem 3.2 (Markoff 1948) *Let $b_0 + \mathbf{K}_{n=1}^{\infty}(a_n/b_n)$ be a positive continued fraction and let z_{n+1}, defined by*

$$z_1 = b_0 + \frac{a_1}{b_1} \, \frac{a_2}{+ \, b_2} \, _{+ \cdots +} \, \frac{a_n}{b_n + z_{n+1}} \; ,$$

be positive infinitely often. Then if the continued fraction converges it converges to z_1.

Proof Let $s_0(w) = b_0 + w$, $s_k(w) = a_k/(b_k + w)$. Then

$$z_1 = S_n(z_{n+1}) = s_0 \circ s_1 \circ \cdots \circ s_n(z_{n+1})$$

with $z_{n+1} \geqslant 0$ infinitely often. All the functions $s_k(z)$ are monotonic and continuous on $[0, +\infty)$. Hence the same is true for their composition S_n. Picking two limit values $w = 0$ and $w = +\infty$, we obtain that z_1 must be in the interval with end-points at $P_n/Q_n = S_n(0)$ and $P_{n-1}/Q_{n-1} = S_n(+\infty)$. Since the continued fraction converges, the proof is complete. □

55 Koch's and Seidel's theorems. These theorems are simple convergent tests for continued fractions with positive terms. Koch's theorem considers more general continued fractions $\mathbf{K}_{n=1}^{\infty}(1/b_n)$ with complex b_n.

Theorem 3.3 (Koch 1895) *If $\sum_{n=1}^{\infty} |b_n| < \infty$ then the limits*

$$\lim_n P_{2n} = P\,, \qquad \lim_n P_{2n+1} = P'\,, \tag{3.3}$$

$$\lim_n Q_{2n} = Q\,, \qquad \lim_n Q_{2n+1} = Q' \tag{3.4}$$

exist, are finite and satisfy $P'Q - PQ' = 1$. In particular, the continued fraction $\mathbf{K}_{n=1}^{\infty}(1/b_n)$ diverges.

Proof Assuming that $P_0 = b_0 = 1$ (notice that $P_1 = 1$) we obtain, by the Euler–Wallis formulas,

$$
\begin{aligned}
|P_1| &\leqslant |b_1||P_0| + |P_{-1}| = (1+|b_1|) \ , \\
|P_2| &\leqslant |b_2||P_1| + |P_0| \leqslant |b_2|(1+|b_1|)+1 \leqslant (1+|b_2|)(1+|b_1|) \ , \\
&\ \vdots \\
|P_n| &\leqslant |b_n||P_{n-1}| + |P_{n-2}| \leqslant (1+|b_1|)\cdots(1+|b_{n-2}|) \\
&\qquad \times (|b_n|(1+|b_{n-1}|)+1) \leqslant (1+|b_1|)\cdots(1+|b_n|) \ ,
\end{aligned}
$$

showing that $\{P_n\}_{n\geqslant 0}$ is bounded. Similarly, $\{Q_n\}_{n\geqslant 0}$ is also bounded. Now the existence of the limits follows from

$$
\begin{aligned}
P_{2n} &= b_{2n}P_{2n-1} + P_{2n-2} = b_{2n}P_{2n-1} + b_{2n-2}P_{2n-3} + \cdots + b_2P_1 + P_0 \ , \\
Q_{2n} &= b_{2n}Q_{2n-1} + Q_{2n-2} = b_{2n}Q_{2n-1} + b_{2n-2}Q_{2n-3} + \cdots + b_2Q_1 + Q_0 \ .
\end{aligned}
$$

Finally

$$
\lim_n \frac{P_{2n}}{Q_{2n}} = \frac{P}{Q} \neq \frac{P'}{Q'} = \lim_n \frac{P_{2n+1}}{Q_{2n+1}} \ ,
$$

since $P'Q - PQ' = 1$ by (1.16). □

Remark Notice that although the continued fraction in Koch's theorem diverges, its even and odd convergents converge – but to different limits.

Theorem 3.4 (Seidel 1846) *If $b_n > 0$ for $n > 1$ then the continued fraction $\mathbf{K}_{n=1}^{\infty}(1/b_n)$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ diverges.*

Proof By (1.18) we have

$$
\begin{aligned}
\frac{Q_n}{Q_{n-2}} = \frac{Q_n}{Q_{n-1}}\,\frac{Q_{n-1}}{Q_{n-2}} &= \left(b_n + \frac{1}{b_{n-1}} {}_{+\cdots+} \frac{1}{b_1}\right) \\
&\quad \times \left(b_{n-1} + \frac{1}{b_{n-2}} {}_{+\cdots+} \frac{1}{b_1}\right) \\
&= 1 + b_n\left(b_{n-1} + \frac{1}{b_{n-2}} {}_{+\cdots+} \frac{1}{b_1}\right) > 1 \ .
\end{aligned}
$$

Consequently

$$
\begin{aligned}
Q_{2n} &> Q_{2n-2} > \cdots > Q_2 > 1 \ ; \\
Q_{2n+1} &> Q_{2n-1} > \cdots > Q_3 > Q_1 = b_1 \ .
\end{aligned}
$$

The Euler–Wallis formulas imply that

$$Q_{2n} = b_{2n}Q_{2n-1} + Q_{2n-2} > b_1 b_{2n} + Q_{2n-2} > b_1(b_2 + \cdots + b_{2n}) ,$$
$$Q_{2n+1} = b_{2n+1}Q_{2n} + Q_{2n-1} > b_1 + b_3 + \cdots + b_{2n+1} .$$

Hence if $\sum b_n = +\infty$ then either $\lim_n Q_{2n} = +\infty$ or $\lim_n Q_{2n+1} = +\infty$. The identity

$$\frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}} = \frac{1}{Q_{2n}Q_{2n+1}}$$

implies the equality of the limits of the even and odd convergents provided that they exist. By Brouncker's theorem (see Theorem 1.7) the even convergents increase and are bounded from above by the odd convergents, which form a decreasing sequence. It follows that the continued fraction converges.

If $\sum b_n < +\infty$ then $\mathbf{K}_{n=1}^{\infty}(1/b_n)$ diverges by Koch's theorem. □

56 The equivalence transform. The transformations applied to the continued fractions in (3.1) changed the partial numerators and denominators but did not affect the convergents.

Definition 3.5 *Two continued fractions*

$$q_0 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{p_n}{q_n}\right) \quad \textit{and} \quad q_0^* + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{p_n^*}{q_n^*}\right)$$

with convergents $\{d_n\}_{n\geqslant 0}$ *and* $\{d_n^*\}_{n\geqslant 0}$ *are called equivalent if* $d_n = d_n^*$ *for* $n = 0, 1, \ldots$ *In this case we write*

$$q_0 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{p_n}{q_n}\right) \approx q_0^* + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{p_n^*}{q_n^*}\right).$$

Notice that since the convergents of equivalent continued fractions coincide they converge or diverge simultaneously. Hence before studying the convergence problem we must clarify the problem of equivalence.

Theorem 3.6 *Two continued fractions are equivalent if and only if there exists a sequence of nonzero constants* $\{r_n\}_{n\geqslant 0}$ *with* $r_0 = 1$ *such that*

$$\begin{aligned} p_n^* &= r_n r_{n-1} p_n , \qquad n = 1, 2, \ldots , \\ q_n^* &= r_n q_n , \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{3.5}$$

Proof The sufficiency follows from the formula

$$\begin{aligned} &q_0 + \frac{p_1}{q_1} \genfrac{}{}{0pt}{}{}{+} \frac{p_2}{q_2} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{p_n}{q_n} \genfrac{}{}{0pt}{}{}{+\cdots} \\ &\approx r_0 q_0 + \frac{r_1 r_0 p_1}{r_1 q_1} \genfrac{}{}{0pt}{}{}{+} \frac{r_2 r_1 p_2}{r_2 q_2} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{r_n r_{n-1} p_n}{r_n q_n} \genfrac{}{}{0pt}{}{}{+\cdots} , \end{aligned}$$

which is obtained by consecutive multiplications. If two continued fractions are equivalent then all Mobius transforms $S_n^{*-1} \circ S_n(w)$ have two fixed points at 0 and ∞, as can easily be seen from

$$\frac{P_n}{Q_n} = S_n(0) = S_n^*(0) = \frac{P_n^*}{Q_n^*} ,$$
$$\frac{P_{n-1}}{Q_{n-1}} = S_n(\infty) = S_n^*(\infty) = \frac{P_{n-1}^*}{Q_{n-1}^*} ,$$

and hence are a multiple of w. It follows that

$$S_n^{*-1}(w) = r_n S_n^{*-1}(w) , \qquad n = 0, 1, \ldots$$

where $r_0 = 1$ and $r_n \neq 0$, $n = 1, 2, \ldots$ Using this formula, we obtain

$$\begin{aligned}\frac{p_n^*}{q_n^*} &= s_n^*(0) = S_{n-1}^{*-1} \circ S_n^*(0) = r_{n-1} S_{n-1}^{-1} \circ S_n^*(0)\\ &= r_{n-1} S_{n-1}^{-1} \circ S_n(0) = r_{n-1} s_n(0) = \frac{r_{n-1} p_n}{q_n} ,\end{aligned}$$

which shows that q_{n+1}^* and q_{n+1} may vanish only simultaneously. Next,

$$\begin{aligned}S_n^{*-1}(w) = s_n^{*-1} \circ S_{n-1}^{*-1}(w) &= -q_n^* + \frac{p_n^*}{r_{n-1} S_{n-1}^{-1}(w)}\\ &= -q_n^* + \frac{p_n^*}{r_{n-1} p_n} (S_n^{-1}(w) + q_n)\\ &= -q_n^* + \frac{p_n^* q_n}{r_{n-1} p_n} + \frac{p_n^*}{r_{n-1} p_n} S_n^{-1}(w) = \frac{p_n^*}{r_{n-1} p_n} S_n^{-1}(w) ,\end{aligned}$$

which proves the first formula in (3.5). Then the second follows from the already proved identity $p_n^* = r_{n-1}(q_n^*/q_n) p_n$ and the observation that q_n and q_n^* may vanish only together. □

The above proof follows Jones and Thron (1980). Theorem 3.6gives two important representatives in any class of equivalent continued fractions.

Corollary 3.7 *If $q_n \neq 0$ for $n = 1, 2, \ldots$ then*

$$q_0 + \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{p_n}{q_n} \right) \approx q_0 + \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{p_n^*}{1} \right) ,$$

where

$$p_1^* = \frac{p_1}{q_1} , \qquad p_n^* = \frac{p_n}{q_n q_{n-1}} , \quad n = 2, 3, \ldots$$

Proof The proof uses the following argument:

$$\begin{aligned}
q_0 + \frac{p_1}{q_1} \, {}_{+} \, \frac{p_2}{q_2} \, {}_{+} \, \frac{p_3}{q_3} \, {}_{+} \, \frac{p_4}{q_4} \, {}_{+\cdots} &\approx q_0 + \frac{p_1/q_1}{1} \, {}_{+} \, \frac{p_2/q_1}{q_2} \, {}_{+} \, \frac{p_3}{q_3} \, {}_{+} \, \frac{p_4}{q_4} \, {}_{+\cdots} \\
&\approx q_0 + \frac{p_1/q_1}{1} \, {}_{+} \, \frac{p_2/(q_1q_2)}{1} \, {}_{+} \, \frac{p_3/q_2}{q_3} \, {}_{+} \, \frac{p_4}{q_4} \, {}_{+\cdots} \approx \cdots \\
&\approx q_0 + \frac{p_1/q_1}{1} \, {}_{+} \, \frac{p_2/(q_1q_2)}{1} \, {}_{+} \, \frac{p_3/(q_2q_3)}{1} \, {}_{+} \, \frac{p_4/(q_3q_4)}{1} \, {}_{+\cdots} ,
\end{aligned}$$

which were used in Sofronov's example; see §**53** at the start of this chapter. □

Corollary 3.8 *For every continued fraction we have*

$$\mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{p_n}{q_n}\right) \approx \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{1}{q_n^*}\right) ,$$

where

$$q_{2k-1}^* = \frac{p_2p_4\cdots p_{2k-2}q_{2k-1}}{p_1p_3\cdots p_{2k-1}} , \qquad q_{2k}^* = \frac{p_1p_3\cdots p_{2k-1}q_{2k}}{p_2p_4\cdots p_{2k}}$$

Proof By Theorem 3.6 there must be r_n such that

$$r_{n-1}r_np_n = 1 , \qquad r_nq_n = q_n^* , \qquad n = 1, 2, \ldots \tag{3.6}$$

By induction we obtain that

$$r_1 = \frac{1}{p_1} , \qquad r_2 = \frac{p_1}{p_2} , \qquad r_3 = \frac{p_2}{p_1p_3} , \qquad r_4 = \frac{p_1p_3}{p_2p_4} , \ \ldots ,$$
$$r_{2n-1} = \frac{p_2p_4\cdots p_{2n-2}}{p_1p_3\cdots p_{2n-1}} , \qquad r_{2n} = \frac{p_1p_3\cdots p_{2n-1}}{p_2p_4\cdots p_{2n}} , \ldots$$

Now the corollary follows from the second formula of (3.6). □

In Corollary 3.7 the partial denominators must be nonzero, whereas in Corollary 3.8 the partial numerators are nonzero by the definition of a continued fraction. Corollary 3.8 gives a representation similar to that for a regular continued fraction. However, in this case the formulas for partial denominators look more complicated than in Corollary 3.7.

Equivalence transforms provide very useful convergence tests for continued fractions with positive terms, which we will often use in what follows.

Corollary 3.9 *A continued fraction $q_0 + \mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ with positive p_n and q_n converges if and only if*

$$\sum_{n=1}^{\infty} \frac{p_1p_3\cdots p_{2n-1}q_{2n}}{p_2p_4\cdots p_{2n}} + \sum_{n=1}^{\infty} \frac{p_2p_4\cdots p_{2n}q_{2n+1}}{p_1p_3\cdots p_{2n+1}} = +\infty .$$

Proof The result follows by Corollary 3.8 and Theorem 3.4. □

Corollary 3.10 *If $p_n > 0$, $q_n > 0$ and $\sum_{n=1}^{\infty}(q_{n-1}q_n/p_n)^{1/2} = +\infty$ then $q_0 + \mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ converges.*

Proof By an elementary inequality, $2\sqrt{u_n v_n} \leqslant u_n + v_n$, the divergence of $\sum_{n=1}^{\infty}\sqrt{u_n v_n} = +\infty$ implies that $\sum_{n=1}^{\infty} u_n + \sum_{n=1}^{\infty} v_n = +\infty$ and the corollary then follows from Corollary 3.9. □

Corollary 3.11 *If $c_n > 0$ and $\sum_{n=1}^{\infty} c_n^{-1/2} = +\infty$ then $\mathbf{K}_{n=1}^{\infty}(c_n/s)$ converges for every $s > 0$.*

Proof Put $q_n = s$ in Corollary 3.10. □

57 Convergence of general continued fractions. Unlike the convergence theory of series, in which the convergence is completely determined by the asymptotic properties of their terms, the convergence of continued fractions depends on the early terms as well. The reason is that if the convergents P_n/Q_n of a continued fraction

$$q_0 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{p_n}{q_n}\right) \tag{3.7}$$

tend to ∞ then the convergents Q_n/P_n of the continued fraction

$$\frac{1}{q_0 + \mathop{\mathbf{K}}\limits_{n=1}^{\infty}(p_n/q_n)}$$

converge to 0. This difficulty may be resolved by introducing the spherical metric on the extended complex plane $\hat{\mathbb{C}}$ via the standard stereographic projection of the sphere onto $\hat{\mathbb{C}}$. Then the convergence of continued fractions may be understood in the spherical metric. However, in practice it is important to distinguish between the convergence of a continued fraction to finite and to infinite values. Therefore Pringsheim introduced the notion of an *unconditionally* convergent continued fraction. A continued fraction (3.7) is said to be unconditionally convergent if all the continued fractions

$$\mathop{\mathbf{K}}_{n=m}^{\infty}\left(\frac{p_n}{q_n}\right), \qquad m = 1, 2, \ldots$$

converge to finite values. In many cases p_n and q_n in (3.7) are functions of some parameters. Therefore our assumption $p_n \neq 0$ turns out to be restrictive. However, if one allows p_n to vanish, there may be cases when both P_n and Q_n are zero and therefore the convergent P_n/Q_n cannot be defined. To eliminate such cases Perron (1957) introduced the term *sinnlos* (meaningless). A finite continued fraction

$$\frac{P_m}{Q_m} = q_0 + \mathop{\mathbf{K}}_{n=1}^{m}\left(\frac{p_n}{q_n}\right)$$

is called *sinnlos* if $Q_m = 0$.

As in series theory one can consider absolutely convergent continued fractions. These are continued fractions satisfying

$$\left|\frac{P_0}{Q_0}\right| + \sum_{n=0}^{\infty}\left|\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n}\right| < +\infty\,. \tag{3.8}$$

As explained above all terms must be included in this series. By (1.16), condition (3.8) is equivalent to

$$\left|\frac{P_0}{Q_0}\right| + \sum_{n=0}^{\infty}\left|\frac{p_1\cdots p_{n+1}}{Q_{n+1}Q_n}\right| < +\infty\,.$$

Any regular continued fraction converges absolutely by (1.34).

3.2 Contribution of Brouncker and Wallis

58 Wallis' product (1656). The first formula for the infinite product

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots \tag{3.9}$$

was found by Viète (1593). The modern proof, which is due to Euler (1763), is easy. Iterating the formula $\sin 2\varphi = 2\sin\varphi\cos\varphi$, we have

$$\cos\frac{\pi}{4}\,\cos\frac{\pi}{8}\cdots\cos\frac{\pi}{2^n} = \frac{2}{\pi}\,\frac{\pi\,/\,2^n}{\sin\pi\,/\,2^n}\,.$$

Therefore $\lim_{x\to 0}\sin x/x = 1$ implies that

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\,\cos\frac{\pi}{8}\,\cos\frac{\pi}{16}\,\cos\frac{\pi}{32}\cdots$$

Since $\cos(\pi/4) = \sqrt{2}/2$, $\cos\varphi/2 = \sqrt{(1+\cos\varphi)/2}$, we get (3.9).

The original proof is very similar to that of Euler on the one hand and to that of Archimedes's theorem 1.2 on the other. Let S_n be the area of a right n-polygon inscribed into the unit circle and let r_n be the radius of the circle inscribed in this polygon. Then by elementary geometry $S_n : S_{2n} = r_n = \cos(\pi/n)$. It follows that

$$\frac{S_4}{S_8} = \cos\frac{\pi}{4}\,,\qquad \frac{S_8}{S_{16}} = \cos\frac{\pi}{8}\,,\qquad \frac{S_{16}}{S_{32}} = \cos\frac{\pi}{16}\,,\qquad \ldots$$

Since $S_4 = 2$ and $S_n \to \pi$, multiplication of the equalities results in (3.9).

A disadvantage of Viète's formula is that it represents π as an infinite product of algebraic irrationals. In his classical treatise Wallis (1656) obtained another formula,

$$\boxed{\frac{2}{\pi} = \frac{1\times 3}{2\times 2}\,\frac{3\times 5}{4\times 4}\,\frac{5\times 7}{6\times 6}\cdots\frac{(2n-1)(2n+1)}{2n\times 2n}\cdots\,,} \tag{3.10}$$

in which all multipliers are rational. It was a really great achievement; furthermore Wallis' formula was helpful for the quadrature problem. In fact, owing to Euler's efforts the ideas generated by Wallis' formula finally resulted in the Lambert–Legendre proof of the irrationality of π. Euler's analysis of Wallis' proof led him to formulas for the gamma and beta functions, see Andrews, Askey and Roy (1999, p. 4), as well as to other important discoveries especially in the theory of continued fractions. We will discuss these topics later.

Nowadays the proof of Wallis' formula can be shortened to a few lines. Integration by parts shows that

$$\begin{aligned} \int_0^{\pi/2} \sin^{2n}\theta\, d\theta &= \frac{\pi}{2}\,\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2\times 4\times 6\times\cdots\times 2n} = \frac{\pi}{2}u_n\,, \\ \int_0^{\pi/2} \sin^{2n+1}\theta\, d\theta &= \frac{2\times 4\times 6\times\cdots\times 2n}{3\times 5\times 7\times\cdots\times(2n+1)} = v_n\,. \end{aligned} \tag{3.11}$$

Combining (3.11) with the trivial inequalities

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta > \int_0^{\pi/2} \sin^{2n+1}\theta\, dx > \int_0^{\pi/2} \sin^{2n+2}\theta\, d\theta \tag{3.12}$$

(observe that $\sin^{2n}\theta > \sin^{2n+1}\theta > \sin^{2n+2}\theta$ on $(0, \pi/2)$), we immediately obtain

$$\frac{u_n}{v_n} > \frac{2}{\pi} > \frac{u_n}{v_n}\left(1-\frac{1}{2n+2}\right),$$

which shows that

$$0 < \frac{1\times 3}{2\times 2}\,\frac{3\times 5}{4\times 4}\,\frac{5\times 7}{6\times 6}\cdots\frac{(2n-1)(2n+1)}{2n\times 2n} - \frac{2}{\pi} < \frac{3}{\pi(2n+2)}\,, \tag{3.13}$$

implying (3.10).

This now standard proof is in fact an improvement on Wallis' original arguments; the usage of the inequalities (3.12) was Euler's idea (1768, Chapter IX, §356). Notice that in 1655–6, when Wallis was working on his book, neither integration by parts nor the change of variable formula were known. Instead Wallis made his discoveries using a simple relation of integrals to areas as well as a method of interpolation. Since $y = \sqrt{1-x^2}$ is the equation of the circular arc in the upper half-plane,

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}\, dx$$

is the formula for the area of one quarter of the unit disc. Motivated by Viète's formula, Wallis introduced a family $I(p, q)$ of reciprocals of these of related integrals, which he was able to compute:

$$I(p, q) = \frac{1}{\displaystyle\int_0^1 (1-x^{1/p})^q\, dx} = \frac{(p+1)(p+2)\cdots(p+q)}{1\times 2\times\cdots\times q} = \binom{p+q}{p}\,. \tag{3.14}$$

Here p and q are positive integers. For $p = 1/2$, $q = n$ we have

$$I(1/2, n) = \frac{1 \times 3 \times \cdots \times (2n+1)}{2 \times 4 \times \cdots \times 2n} = \frac{1}{v_n} . \tag{3.15}$$

Clearly v_n decreases with growth in n. Since the sequence (3.15) is obtained by a very simple law, one may hope that it can be naturally interpolated to all positive real numbers and in particular to $n = 1/2$. The value of v_0 is 1 and that of v_1 is $2/3 = 0.66\ldots$ The value of interest, $v_{1/2} = \pi/4 = 0.78\ldots$, is *regularly* placed between v_0 and v_1. In other words the results of this numerical experiment can only be explained by some simple formula for $v_{1/2}$. Using these arguments and following essentially the lines of the proof given above, Wallis found a representation of $(1/2)I(1/2, 1/2) = 2/\pi$ as the infinite product (3.10) and proved its convergence.

A detailed account of Wallis' logic in obtaining his formula is presented in Kramer (1961) (see a brief version in Kolmogorov and Yushkevich 1970). A recent coverage of the history of (3.10) is given in the English translation of *Arithmetica Infinitorum* by Stedall (2004, pp. xvii–xx).

It is interesting that later Euler (1748, §158) discovered another proof of Wallis' formula, which is based on the development of $(\sin x)/x$ (which played a crucial role in the proof of Viète's formula) into an infinite product:

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2 \pi^2}\right) . \tag{3.16}$$

Setting $x = \pi/2$, Euler (1748, §185) obtains Wallis' product

$$\frac{2}{\pi} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{2k}\right)\left(1 + \frac{1}{2k}\right) = \prod_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{2k \times 2k} .$$

With this method Euler obtained a Wallis-type formula for the reciprocal of the length of the side of a square inscribed into the unit circle

$$\frac{1}{\sqrt{2}} = \frac{1 \times 3}{2^2} \, \frac{5 \times 7}{6^2} \, \frac{9 \times 11}{10^2} \, \frac{13 \times 15}{14^2} \cdots \tag{3.17}$$

Notice that the infinite product (3.17) is obtained from Wallis' product (3.10) by dropping the even multipliers. For the quarter-perimeter of a $2n$-polygon inscribed in the unit circle Euler obtained

$$n \sin\left(\frac{\pi}{2n}\right) = \frac{(2n-1)(2n+1)}{n \times 3n} \, \frac{(4n-1)(4n+1)}{3n \times 5n} \cdots$$

Also, Euler (1746, §158) proved that, cf. (3.16),

$$\cos x = \prod_{k=0}^{\infty} \left(1 - \frac{4x^2}{(2k+1)^2 \pi^2}\right) . \tag{3.18}$$

Setting $x = \pi/2^{j+1}$, $j = 1, 2, \ldots$, we obtain the formula

$$\cos\left(\frac{\pi}{2^{j+1}}\right) = \prod_{k=0}^{\infty}\left(1 - \frac{1}{(2^j(2k+1))^2}\right),$$

which explains the relationship between Viète's and Wallis' formulas.

59 Brouncker's continued fraction. When Wallis found a proof of (3.10) he showed it to Brouncker; see Kramer (1961, p. 86). Brouncker responded with a remarkable continued fraction,

$$\frac{4}{\pi} = 1 + \frac{1^2}{2} {}_+ \frac{3^2}{2} {}_+ \frac{5^2}{2} {}_+ \cdots = 1 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2}\right). \tag{3.19}$$

On the one hand, according to Stedall (2004, pp. xviii–xix) formula (3.10) appeared after 28 February 1655. On the other hand, in April 1655 Wallis responded to Hobbes' threats to reveal a quadrature of the unit circle by publishing some excerpts from Wallis (1656). Therefore it looks as though both formulas were proved in March 1655.

Brouncker didn't publish his result. But some of Brouncker's ideas were included as comments at the end of Wallis (1656). Though not at all clear, these comments had at least one positive outcome. They attracted Euler's attention and two very important papers (Euler 1744, 1750b) on the analytic properties of continued fractions appeared. One can therefore say that the analytic theory of continued fractions began from Brouncker's formula.

If it is known that formula (3.19) holds, then easy arguments known to Brouncker lead to a proof. Indeed, by the Euler–Wallis formulas,

$$P_n = 2P_{n-1} + (2n-1)^2 P_{n-2},$$
$$Q_n = 2Q_{n-1} + (2n-1)^2 Q_{n-2}.$$

These formulas can be rewritten as follows:

$$P_n - (2n+1)P_{n-1} = -(2n-1)\{P_{n-1} - (2n-1)P_{n-2}\},$$
$$Q_n - (2n+1)Q_{n-1} = -(2n-1)\{Q_{n-1} - (2n-1)Q_{n-2}\}.$$

Since $P_1 = 3$ and $P_2 = 2 \times 3 + 9 = 15$, we obtain from $P_2 - 5P_1 = 0$ by induction that $P_n - (2n+1)P_{n-1} = 0$. Hence

$$P_n = 1 \times 3 \times 5 \times \cdots \times (2n+1) = (2n+1)!!, \tag{3.20}$$

since $P_0 = 1$. Now $Q_2 = 13$, $Q_1 = 2$ implying $Q_2 - 5Q_1 = 3$. Therefore

$$Q_n - (2n+1)Q_{n-1} = (-1)^n (2n-1)!!,$$

or equivalently

$$\frac{Q_n}{(2n+1)!!} - \frac{Q_{n-1}}{(2n-1)!!} = \frac{(-1)^n}{(2n+1)}.$$

Remembering that $P_n = (2n+1)!!$, we arrive at the formula

$$\frac{Q_n}{P_n} = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}, \tag{3.21}$$

which states that the convergents of Brouncker's continued fraction coincide with the reciprocals of partial sums of an alternating series:

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \int_0^1 \frac{dx}{1+x^2} \,. \tag{3.22}$$

It is not clear whether Brouncker knew the above formula. However, at least by 1657, i.e. approximately by the time when Wallis completed his treatise, Brouncker definitely knew a similar one (see Kolmogorov and Yushkevich 1970, p. 158):

$$\ln 2 = \sum_{n=1}^{\infty} \frac{1}{(2n-1)\times 2n} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \int_1^2 \frac{dx}{x} \,,$$

which he published with a rigorous proof in Brouncker (1668). Moreover, according to a report of O'Connor and Robertson (2002) there is evidence that Brouncker knew this formula already in 1654, a year before he learned the problem from Wallis.

The parallel between these two cases becomes more clear if we transform the above computations into the following theorem.

Theorem 3.12 *Let $a > 0$ and $\{y_n\}_{n\geqslant 0}$ be a sequence such that $y_0 > 0$, $y_n > a$ for $n = 1, 2, \dots$ and $\sum_{n=1}^{\infty} 1/y_n = +\infty$. Let $p_n = (y_n - a)y_{n-1}$, $n = 1, 2, \dots$ Then*

$$y_0 + \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{p_n}{a}\right) = \frac{y_0}{1 + \sum_{n=1}^{\infty} (-1)^n (y_1 - a)\cdots(y_n - a)/y_1 \cdots y_n} \,. \tag{3.23}$$

Proof By the Euler–Wallis formulas, $P_n - y_n P_{n-1} = (a - y_n)(P_{n-1} - y_{n-1}P_{n-2})$. Since $P_0 - y_0 P_1 = 0$, this implies that $P_n = y_n P_{n-1}$. Hence $P_n = y_n y_{n-1} \cdots y_1 y_0$. Similarly

$$Q_n - y_n Q_{n-1} = (a - y_n)\cdots(a - y_1)(Q_0 - y_0 Q_1) = (a - y_n)\cdots(a - y_1) \,.$$

It follows that

$$\frac{Q_n}{P_n} - \frac{Q_{n-1}}{P_{n-1}} = (-1)^n \frac{(y_n - a)\cdots(y_1 - a)}{y_n y_{n-1} \cdots y_1 y_0} \,,$$

implying the identity

$$\frac{Q_n}{P_n} = \frac{1}{y_0} + \frac{1}{y_0} \sum_{k=1}^{n} (-1)^k \left(1 - \frac{a}{y_k}\right) \cdots \left(1 - \frac{a}{y_1}\right) \,. \tag{3.24}$$

The telescopic series on the right-hand side of (3.24) converges if and only if

$$\lim_n \left(1 - \frac{a}{y_n}\right) \cdots \left(1 - \frac{a}{y_1}\right) = 0 \,,$$

which is the case since $\sum_n 1/y_n = +\infty$. □

If $a=2$ and $y_0=1$, $y_n=2n+1$ then $p_n=(y_n-a)y_{n-1}=(2n-1)^2$ and we obtain Brouncker's formula:

$$1+\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2}\right)=\frac{1}{1+\sum_{n=1}^{\infty}(-1)^n\frac{1\times 3\times\cdots\times(2n-1)}{3\times\cdots\times(2n+1)}}=\frac{4}{\pi}\,.$$

If $a=1$, $y_0=1$, $y_n=n+1$ then $p_n=(y_n-a)y_{n-1}=n^2$ and

$$1+\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{1}\right)=\frac{1}{1+\sum_{n=1}^{\infty}(-1)^n\frac{1\times 2\times\cdots\times n}{2\times\cdots\times(n+1)}}=\frac{1}{\ln 2}\,. \tag{3.25}$$

If $a=2$, $y_0=2$, $y_n=n+2$ then $p_n=(y_n-a)y_{n-1}=n(n+1)$ for $n=1,2,\ldots$ and

$$2+\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n(n+1)}{2}\right)=\frac{1}{2\sum_{n=0}^{\infty}(-1)^n\dfrac{1}{(n+1)(n+2)}}=\frac{1}{\ln 4-1}\,. \tag{3.26}$$

Let us return to (3.22). The left-hand side is $\pi/4$, one quarter of the area of a unit disc or equivalently the area of a disc of diameter 1 (see Fig. 3.1). The right-hand side equals the area under the curve $y=1/(1+x^2)$ bounded by the coordinate axis OX and the two vertical lines $x=0$ and $x=1$ (Fig. 3.1). The curve $y=1/(1+x^2)$ has a name, the *witch of Agnesi*. The Italian mathematician Maria Agnesi included properties of this curve (see Fig. 3.1) in her famous book (1748) on analytic geometry. Known historical materials witness that the first study of this curve was in fact made by Fermat (1601–65). One can easily find many interesting facts on the witch of Agnesi on the Internet.

To prove Brouncker's formula with the tools available in 1655, we need the equality of the area of the disc with radius $1/2$ to the area under the witch of Agnesi for $0\leqslant x\leqslant 1$. A direction to take in proving this is given by Archimedes' theorem 1.2 and the so-called Pythagorean triples, which were well known both to Brouncker and Fermat; see Edwards (1977).

Definition 3.13 *A triple $\{x,y,z\}$ of nonnegative integer numbers is called Pythagorean if it is a solution to the Diophantine equation*

$$x^2+y^2=z^2\,.$$

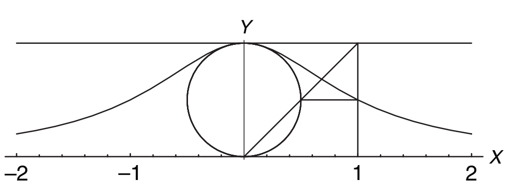

Fig. 3.1. The witch of Agnesi curve, $y=1/(1+x^2)$.

A well-known example of a Pythagorean triple is $\{3,4,5\}$. Pythagorean triples give a complete list of points with rational coordinates on the part of the unit circle in the first quadrant; other rational points are obtained by symmetry. For instance, $(3/5,4/5)$ is a point on the unit circle. The rational points on a circular arc can be listed by the rational parametrization of $\mathbb{T}$

$$x(t) = \frac{2t}{1+t^2}\,, \qquad y(t) = \frac{1-t^2}{1+t^2}\,.$$

The arc of interest corresponds to $0 \leqslant t \leqslant 1$. The formulas

$$\begin{aligned} y(t)-y(s) &= \frac{1-t^2}{1+t^2} - \frac{1-s^2}{1+s^2} = 2\frac{s^2-t^2}{(1+t^2)(1+s^2)}\,; \\ x(t)-x(s) &= \frac{2t}{1+t^2} - \frac{2s}{1+s^2} = 2\frac{(1-ts)(t-s)}{(1+t^2)(1+s^2)}\,, \end{aligned}$$

and the elementary identity $(s+t)^2+(1-st)^2 = (1+s^2)(1+t^2)$ show that the distance between two points $P = P(s)$ and $Q = Q(t)$ on $\mathbb{T}$ corresponding to $0 \leqslant s < t \leqslant 1$ is given by

$$\operatorname{dist}(P,Q) = 2\frac{t-s}{\sqrt{(1+t^2)(1+s^2)}}\,.$$

It follows that the area of the triangle with vertexes at $P(k/n)$, $Q((k+1)/n)$ and the origin equals approximately

$$\begin{aligned} \operatorname{Area}(\Delta OPQ) &= \frac{1}{n}\frac{1+o(1)}{\sqrt{(1+k^2/n^2)(1+(k+1)^2/n^2)}} \\ &= \frac{1+o(1)}{1+k^2/n^2}\,\frac{1}{n}\,, \end{aligned}$$

the corresponding area of the rectangle under the witch of Agnesi. Passing to the limit, we obtain the equality of the required areas. The middle part of (3.22) can be treated using Brouncker's method (1668).

Only 70 years later Euler (1744) found a proof to Brouncker's formula seemingly following these lines. In his paper (1750b) Euler indicates several times (see §§17, 19, 20 in this paper) the importance of finding Brouncker's original proof (see especially the end of §17). It can however, be recovered from the notes made by Wallis (1656, Proposition 191, comment).

60 A recovery of Brouncker's proof (March 1655). Two observations play a key role in the recovery of Brouncker's proof. The first is the formula

$$\begin{aligned}&\frac{1\times 3}{2\times 2}\,\frac{3\times 5}{4\times 4}\,\frac{5\times 7}{6\times 6}\cdots\frac{(2n-1)(2n+1)}{2n\times 2n}\\&\qquad=\frac{1\times 3}{0}\,{}_{+}\,\frac{2\times 2}{0}\,{}_{+}\,\frac{3\times 5}{0}\,{}_{+\cdots+}\,\frac{(2n-1)(2n+1)}{0}\,{}_{+}\,\frac{2n\times 2n}{1}\,,\end{aligned}\tag{3.27}$$

demonstrating the close relationship of these products to continued fractions. Since any formal infinite continued fraction with identically zero partial denominators diverges, something should be done to make them positive. This was known to Brouncker; see Wallis (1656, p. 169, footnote 79).

The second observation is the following comment by Brouncker; see Wallis (1656, p. 168). One look at (3.10) is enough to notice that the numerators are the products of the form $s(s+2)=s^2+2s=(s+1)^2-1$, s being odd, whereas the denominators are whole squares of even numbers. This suggest the idea of increasing s to $b(s)$ and $s+2$ to $b(s+2)$ so that

$$b(s)b(s+2)=(s+1)^2\,. \tag{3.28}$$

Then, to keep (3.27) valid, the odd zero partial denominators on the right-hand side of (3.27) will automatically become positive. That is exactly what we need to complete the proof. The fact that $s+1$ is even is also helpful since it may provide necessary cancellations. Now using (3.28) repeatedly, we may write

$$\begin{aligned}b(1)&=\frac{2^2}{b(3)}=\frac{2^2}{4^2}b(5)=\frac{2^2}{4^2}\frac{6^2}{b(7)}=\frac{2^2}{4^2}\frac{6^2}{8^2}b(9)=\cdots\\&=\frac{2^2}{4^2}\frac{6^2}{8^2}\frac{10^2}{12^2}\cdots\frac{(4n-2)^2}{(4n)^2}\,b(4n+1)\\&=\frac{1^2}{2^2}\frac{3^2}{4^2}\cdots\frac{(2n-1)^2}{(2n)^2}\,b(4n+1)\\&=\frac{1\times 3}{2^2}\,\frac{3\times 5}{4^2}\,\frac{5\times 7}{6^2}\cdots\frac{(2n-1)(2n+1)}{(2n)^2}\,\frac{b(4n+1)}{(2n+1)}\,.\end{aligned}\tag{3.29}$$

Combined with Wallis' formula this implies

$$b(1)=\left(\frac{2}{\pi}+o(1)\right)\frac{b(4n+1)}{(2n+1)}\,. \tag{3.30}$$

Since $s+2<b(s+2)$ and $b(s)b(s+2)=(s+1)^2$, we have

$$s<b(s)<\frac{s^2+2s+1}{s+2}=s+\frac{1}{2+s}\,, \tag{3.31}$$

which together with (3.30) imply

$$b(1) = \lim_{n} \left(\frac{2}{\pi} + o(1) \right) \frac{b(4n+1)}{(2n+1)} = \frac{4}{\pi} \, . \tag{3.32}$$

It remains only to find a formula for $b(s)$.

We thus arrive at the crucial point of Brouncker's arguments, which by the way shows that contrary to other mathematicians, see Wallis (1656, p. xxvii], Brouncker understood very clearly Wallis' interpolation. Wallis' main observation was that the values of functions $f(s)$ represented by analytic formulas can be uniquely recovered (interpolated) by their values $f(n)$ at integer points. Nowadays the uniqueness theorems of complex analysis reduce this to more or less routine applications of the uniqueness principle for analytic functions. But in 1652–5 it was a revolutionary discovery. One cannot exclude, by the way, that Euler obtained his great formula (3.16) using Wallis' interpolation, since *Arithmetica Infinitorum* was always on Euler's desk.

Motivated by (3.31) one can develop $b(s)$ into a series in positive powers of $1/s$,

$$b(s) = s + c_0 + \frac{c_1}{2s} + \frac{c_2}{s^2} + \frac{c_3}{s^3} + \cdots \tag{3.33}$$

and find the coefficients c_0, $c_1, \cdots$ inductively using (3.28). By (3.31) coefficient c_0 vanishes. To find c_1 we assume that

$$b(s) = s + \frac{c_1}{s} + O\left(\frac{1}{s^2}\right), \qquad s \longrightarrow +\infty$$

and then determine c_1 from the equation

$$s^2 + 2s + 1 = b(s)b(s+2) = s^2 + 2s + 2c_1 + O\left(\frac{1}{s}\right), \qquad s \to +\infty \, ,$$

implying that $c_1 = 1/2$. It follows that

$$b(s) = s + \frac{1}{2s} + O\left(\frac{1}{s^2}\right), \qquad s \to +\infty \, .$$

Similarly, elementary calculations show that $c_2 = 0$, $c_3 = -9/8$, $c_4 = 0$, $c_5 = 153/16$, $c_6 = 0$ and therefore

$$b(s) = s + \frac{8s^4 - 18s^2 + 153}{16s^5} + O\left(\frac{1}{s^7}\right) \, .$$

Applying the Euclidean algorithm to the quotient of polynomials, we have

$$\frac{8s^4-18s^2+153}{16s^5}=\cfrac{1}{2s+\cfrac{9(4s^3-34s)}{8s^4-18s^2+153}}=\cfrac{1}{2s+\cfrac{9}{\cfrac{8s^4-18s^2+153}{4s^3-34s}}}$$

$$=\cfrac{1}{2s+\cfrac{9}{2s+\cfrac{25(2s^2+153/25)}{4s^3-34s}}}=\cfrac{1}{2s+\cfrac{9}{2s+\cfrac{25}{2s+\cdots}}}.$$

A remarkable property of the above calculations is that $1^2=1$, $3^2=9$, $5^2=25$, etc appear automatically as common divisors of the coefficients of the polynomials in Euclid's algorithm. The fraction $153/25$ appears only because in c_7 we did not include the term in c_7. Increasing the number of terms in (3.33) we arrive naturally at the conclusion that

$$b(s)=s+\frac{1^2}{2s}{}_{+}\frac{3^2}{2s}{}_{+}\frac{5^2}{2s}{}_{+}\frac{7^2}{2s}{}_{+\cdots+}\frac{(2n-1)^2}{2s}{}_{+\cdots}\;. \tag{3.34}$$

Having obtained (3.34), we may reverse the order of the argument and compute the differences

$$\frac{P_0(s)}{Q_0(s)}\frac{P_0(s+2)}{Q_0(s+2)}-(s+1)^2=s(s+2)-(s+1)^2=\boxed{(-1)=O(1)}\;;$$

$$\frac{P_1(s)}{Q_1(s)}\frac{P_1(s+2)}{Q_1(s+2)}-(s+1)^2=\frac{4s^4+16s^3+20s^2+8s+9}{4s^2+8s}$$

$$-\frac{4s^4+16s^3+20s^2+8s}{4s^2+8s}=\boxed{\frac{9}{4s^2+8s}=O\left(\frac{1}{s^2}\right)}\;;$$

$$\frac{P_2(s)}{Q_2(s)}\frac{P_2(s+2)}{Q_2(s+2)}-(s+1)^2=\frac{16s^6+96s^5+280s^4+480s^3+649s^2+594s}{16s^4+64s^3+136s^2+144s+225}-(s+1)^2$$

$$=\boxed{\frac{-225}{16s^4+64s^3+136s^2+144s+225}=O\left(\frac{1}{s^4}\right)}.$$

One can find these very formulas in Wallis (1656, pp. 169–70), where Wallis writes after the last formula: "$\cdots$ *which is less than the square* F^2+2F+1.[1] *And thus it may be done as far as one likes; it will form a product which will be (in turn) now greater than, now less than, the given square (the difference, however, continually decreasing, as is clear), which was to be proved.*"

[1] In Wallis' notation $s=F$.

To make Wallis' comments clearer, we observe that the denominators of the boxed fractions are polynomials in s with positive coefficients. This can easily be explained, since they are the products $Q_n(s)Q_n(s+2)$ of polynomials which have positive coefficients by (1.15). Next, if

$$P_n(s)P_n(s+2)-(s+1)^2Q_n(s)Q_n(s+2)=b_n\,, \tag{3.35}$$

where b_n is a constant, then $b_n=-(-1)^n[(2n+1)!!]^2$. We have already seen this from the formulas presented by Wallis for $n=0,1,2$. If (3.35) holds then evaluating it at $s=-1$ we obtain $P_n(-1)P_n(1)=b_n$. By (3.20) we have $P_n(1)=(2n+1)!!$ and it remains to observe that $P_n(s)$ is odd for even n and is even for odd n. Assuming that (3.35) holds for every n with $b_n=-(-1)^n[(2n+1)!!]^2$, we obtain for $s>0$

$$\frac{P_{2k}(s)}{Q_{2k}(s)}\,\frac{P_{2k}(s+2)}{Q_{2k}(s+2)}<(s+1)^2<\frac{P_{2k+1}(s)}{Q_{2k+1}(s)}\,\frac{P_{2k+1}(s+2)}{Q_{2k+1}(s+2)}\,.$$

By Corollary 3.10 the continued fraction (3.34) converges. See Ex. 3.8 ($s\geqslant 1$) and Ex. 3.21 ($s>0$) for elementary proofs. Since all terms in continued fraction (3.34) are positive, Theorem 1.7 implies that even convergents increase to $b(s)$ and odd convergents also decrease to $b(s)$.

Passing to the limit in the above inequalities, we obtain that continued fraction $b(s)$ satisfies functional equation (3.28). The inequality $s<b(s)$ is clear from (3.34). Thus the proof is completed by the following lemma.

Lemma 3.14 *Let $P_n(s)/Q_n(s)$ be the nth convergent to Brouncker's continued fraction* (3.34). *Then*

$$P_n(s)P_n(s+2)-(s+1)^2Q_n(s)Q_n(s+2)=(-1)^{n+1}[(2n+1)!!]^2\,. \tag{3.36}$$

Proof The Euler–Wallis formulas for the convergents P_n/Q_n look as follows:

$$\begin{aligned} P_n(s)&=2sP_{n-1}(s)+(2n-1)^2P_{n-2}(s)\,, &\quad P_0(s)&=s\,,\quad P_{-1}(s)=1\,,\\ Q_n(s)&=2sQ_{n-1}(s)+(2n-1)^2Q_{n-2}(s)\,, &\quad Q_0(s)&=1\,,\quad Q_{-1}(s)=0\,, \end{aligned}$$

and identity (1.16) can be written as

$$P_{n+1}(s)Q_n(s)-P_n(s)Q_{n+1}(s)=(-1)^n[(2n+1)!!]^2\,. \tag{3.37}$$

Assuming that (3.36) holds for n and taking into account (3.37), we obtain

$$\begin{aligned}\frac{P_{n+1}(s)}{Q_{n+1}(s)}\,\frac{P_{n+1}(s+2)}{Q_{n+1}(s+2)}-(s+1)^2 &= \frac{P_n(s)}{Q_n(s)}\,\frac{P_n(s+2)}{Q_n(s+2)}-(s+1)^2\\ &\quad+\left\{\frac{P_{n+1}(s)}{Q_{n+1}(s)}-\frac{P_n(s)}{Q_n(s)}\right\}\frac{P_{n+1}(s+2)}{Q_{n+1}(s+2)}\\ &\quad+\left\{\frac{P_{n+1}(s+2)}{Q_{n+1}(s+2)}-\frac{P_n(s+2)}{Q_n(s+2)}\right\}\frac{P_n(s)}{Q_n(s)}\\ &= -\frac{(-1)^n[(2n+1)!!]^2}{Q_n(s)Q_n(s+2)}+\frac{(-1)^n[(2n+1)!!]^2}{Q_{n+1}(s)Q_n(s)}\,\frac{P_{n+1}(s+2)}{Q_{n+1}(s+2)}\\ &\quad+\frac{(-1)^n[(2n+1)!!]^2}{Q_{n+1}(s+2)Q_n(s+2)}\,\frac{P_n(s)}{Q_n(s)}\,. \end{aligned} \tag{3.38}$$

The Euler–Wallis formula for Q_n implies that

$$\begin{aligned}\frac{Q_n(s+2)}{(2s)^n} &= \left(1+\frac{2}{s}\right)\frac{Q_{n-1}(s+2)}{(2s)^{n-1}}+O\left(\frac{1}{s^2}\right)=\cdots\\ &= \left(1+\frac{2}{s}\right)^n+O\left(\frac{1}{s^2}\right)=1+\frac{2n}{s}+O\left(\frac{1}{s^2}\right),\\ \frac{Q_n(s)}{(2s)^n} &= 1+O\left(\frac{1}{s^2}\right), \qquad s\longrightarrow\infty\,.\end{aligned}$$

By (3.37)

$$\frac{P_{n+1}(s)}{Q_{n+1}(s)}-\frac{P_n(s)}{Q_n(s)}=O\left(\frac{1}{s^{2n+1}}\right),$$

implying that

$$\begin{aligned}\frac{P_n(s)}{Q_n(s)} &= \frac{P_1(s)}{Q_1(s)}=s+\frac{1}{2s}+O\left(\frac{1}{s^2}\right),\\ \frac{P_{n+1}(s+2)}{Q_{n+1}(s+2)} &= \frac{P_1(s+2)}{Q_1(s+2)}=s+2+\frac{1}{2s}+O\left(\frac{1}{s^2}\right).\end{aligned}$$

Combining these formulas with (3.38), we obtain

$$\begin{aligned}
&\frac{P_{n+1}(s)}{Q_{n+1}(s)}\,\frac{P_{n+1}(s+2)}{Q_{n+1}(s+2)}-(s+1)^2\\
&\quad=\frac{(-1)^n[(2n+1)!!]^2}{(2s)^{2n}}\times\left\{-\left(1-\frac{2n}{s}\right)+\frac{1}{2}\left(1+\frac{2}{s}\right)\right.\\
&\qquad\qquad\left.+\frac{1}{2}\left(1-\frac{2n+2}{s}-\frac{2n}{s}\right)+O\left(\frac{1}{s^2}\right)\right\}\\
&\quad=O\left(\frac{1}{s^{2n+2}}\right),
\end{aligned}$$

which proves the lemma. For another proof, see Ex. 4.20. □

This proof demonstrates a deep understanding of continued fractions by Brouncker. Since Brouncker did not have much time for his discovery, it is quite likely that by March 1655 he already had the theory of positive continued fractions to hand. Maybe it appeared as an outcome of Brouncker's possible revision of the geometrical proof of the irrationality of $\sqrt{2}$, see (1.3), in the spirit of Viète's *Algebra Nova* (1600). Taking into account Brouncker's mathematical interests, see for instance § **43** at the start of Section 2.3, this looks probable. Or it could have appeared from Brouncker's research on musical scales; see §**18** in Section 1.3. Unfortunately, there are only very few of Brounckers papers by which to judge this. But if it were the case, then many points in the above proof become less mysterious. To begin with, by that time he could have already known that regular continued fractions can be obtained from infinite decimal fractions by consecutive division and from quotients of integers by the Euclidean algorithm. He could also have known that quotients of polynomials can be developed into continued fractions by long division. Also he could have known an important analogy between the quotients of polynomials and rational numbers and have understood that in this correspondence the base $1/10$ corresponds to $1/s$. This is exactly the circle of ideas involved in the proof of Brouncker's formula. A very new idea here is to consider a continuum of *asymptotic series*.

Definition 3.15 [2] *A formal power series*

$$c_0+\frac{c_1}{s}+\frac{c_2}{s^2}+\frac{c_3}{s^3}+\cdots$$

is called an asymptotic expansion of a function $y(s)$ at ∞ if there exist sequences of positive numbers $\{r_n\}_{n\geqslant 0}$ and $\{A_n\}_{n\geqslant 0}$ such that

$$\left|y(s)-\sum_{k=0}^{n}\frac{c_k}{s^k}\right|\leqslant\frac{A_n}{s^{n+1}},\qquad s>r_n\,.$$

[2] This definition is credited to Poincaré (1886).

In other words for every n

$$y(s)=\sum_{k=0}^{n}\frac{c_k}{s^k}+O\left(\frac{1}{s^{n+1}}\right), \qquad s\to\infty .$$

It is clear from the definition that every function may have at most one asymptotic expansion:

$$c_0=\lim_{s\to\infty}y(s), \qquad c_n=\lim_{s\to\infty}s^n\left\{y(s)-\sum_{k=0}^{n-1}\frac{c_k}{s^k}\right\}, \quad n=1,2,\ldots$$

In contrast with decimal fractions it is not true that an asymptotic expansion determines a unique function. For instance the zero asymptotic expansion (all c_k are 0's) in $1/s$ corresponds to the zero function as well as to e^{-s}.

As soon as the basic lines of the proof are established one can reverse the arguments and obtain a rigorous proof of Brouncker's formula. To conclude, let us notice that Wallis' formula can be derived from the functional equation (3.28) satisfied by the continued fraction (3.34). To see this one should just combine (3.29) with the proof of Brouncker's formula given in §**59** above. Then $b(1)=4/\pi$ and by (3.31) we obtain

$$\frac{1\times 3}{2^2}\,\frac{3\times 5}{4^2}\,\frac{5\times 7}{6^2}\cdots\frac{(2n-1)(2n+1)}{(2n)^2}\left(2-\frac{1}{2n+1}\right)$$
$$<b(1)=\frac{4}{\pi}<\frac{1\times 3}{2^2}\,\frac{3\times 5}{4^2}\,\frac{5\times 7}{6^2}\cdots\frac{(2n-1)(2n+1)}{(2n)^2}\,\frac{4n+2}{4n+3}\times 2,$$

which obviously implies (3.10).

61 Brouncker's functional equation. In fact Brouncker proved more than (3.19). His initial idea of constructing a $b(s)$ satisfying $s<b(s)$ and (3.28) eventually turned into the remarkable and important identity

$$s^2=\left((s-1)+\frac{1^2}{2(s-1)}\,{}_{+}\,\frac{3^2}{2(s-1)}\,{}_{+}\,\frac{5^2}{2(s-1)}\,{}_{+}\cdots\right)$$
$$\times\left((s+1)+\frac{1^2}{2(s+1)}\,{}_{+}\,\frac{3^2}{2(s+1)}\,{}_{+}\,\frac{5^2}{2(s+1)}\,{}_{+}\cdots\right), \tag{3.39}$$

valid for $s > 1$. The algebraic identity (3.28), which finally resulted in (3.34), can easily be used to represent $b(s)$ as an infinite product:

$$\begin{aligned} b(s) &= \frac{(s+1)^2}{(s+3)^2} B(s+4) = \frac{(s+1)^2}{(s+3)^2}\,\frac{(s+5)^2}{(s+7)^2} b(s+8) \\ &= \frac{(s+1)^2}{(s+3)^2}\,\frac{(s+5)^2}{(s+7)^2} \cdots \frac{(s+4n-3)^2}{(s+4n-1)^2}\, b(s+4n) \\ &= (s+1)\frac{(s+1)(s+5)}{(s+3)^2} \cdots \frac{(s+4n-3)(s+4n+1)}{(s+4n-1)^2}\,\frac{b(s+4n)}{(s+4n+1)}\,. \end{aligned}$$

Multipliers are grouped in accordance with Wallis' formula,

$$\frac{(s+4n-3)(s+4n+1)}{(s+4n-1)^2} = 1 - \frac{4}{(s+4n-1)^2}\,,$$

which provides the convergence of the product at least for $s > -3$.

Theorem 3.16 (Brouncker) *Let $y(s)$ be a function on $(0, +\infty)$ satisfying (3.28) and the inequality $s < y(s)$ for $s > C$, where C is some constant. Then*

$$y(s) = (s+1)\prod_{n=1}^{\infty} \frac{(s+4n-3)(s+4n+1)}{(s+4n-1)^2} = s + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2s}\right) \tag{3.40}$$

for every positive s.

This theorem, which Brouncker actually proved, completely corresponds to the ideology of *Arithmetica Infinitorum* and interpolates Wallis' result ($s = 1$) to the whole scale of positive s.

62 Brouncker's program. Brouncker's method relates together four different objects:

(1) the functional equation $b(s)b(s+2) = (s+1)^2$;

(2) the asymptotic series (3.33);

(3) the continued fraction (3.34) corresponding to this series;

(4) the infinite product locating singular points of $b(s)$.

63 Brouncker's method for decimal places of π. When Huygens learned of Wallis' and Brouncker's formulas he did not believe them and asked for numerical confirmation. The rate of convergence of Wallis' product is very slow. The same is true for Brouncker's continued fraction, which is not a great surprise since its convergents are partial sums of the telescopic series (3.21). To perform this calculation Brouncker derived from (3.28) the important formulas

$$b(4n+1) = \frac{2^2}{1\times 3}\,\frac{4^2}{3\times 5}\cdots\frac{(2n)^2}{(2n-1)(2n+1)}\,\frac{4}{\pi}\,(2n+1)\,,$$
$$b(4n+3) = \frac{1\times 3}{2^2}\,\frac{3\times 5}{4^2}\cdots\frac{(2n-1)(2n+1)}{(2n)^2}\,(2n+1)\pi\,. \tag{3.41}$$

If $n=6$ then the continued fraction $y(4\times 6+1) = y(25)$ has partial denominators equal to $2\times 25 = 50$, which considerably improves its convergence. Thus we obtain the following boundaries for π:

$$\beta_0\frac{Q_{2k+1}}{P_{2k+1}} < \pi < \beta_0\frac{Q_{2k}}{P_{2k}}\,,$$

where

$$\beta_0 = 4\,\frac{2^2\times 4^2\times 6^2\times 8^2\times 10^2\times 12^2}{3^2\times 5^2\times 7^2\times 9^2\times 11^2} = 78.602\,424\,992\,035\,381\,646\ldots\,,$$

and P_j/Q_j are convergents to $b(25)$. Putting $k=0,1,2$ in the above formula we find that

$$k=0,\qquad 3.141\,583\,732\,695\,26 < \pi < 3.144\,096\,999\,681\,42\,,$$
$$k=1,\qquad 3.141\,592\,651\,947\,82 < \pi < 3.141\,592\,740\,820\,66\,,$$
$$k=2,\qquad 3.141\,592\,653\,587\,59 < \pi < 3.141\,592\,653\,639\,71\,.$$

Notice that even the first convergent to $b(25)$ gives four true places of π. The fifth convergent gives without tedious calculations 11 true places. This was the *first algebraic calculation of* π. Viète in 1593 could not use his formula and instead applied the traditional method of Archimedes to obtain nine decimal places. In 1596 Ludolph van Ceulen obtained 20 decimal places by using a polygon with 60×2^{29} sides. The amount of calculations made by Ludolph is incomparably large in relation to the short and beautiful calculations of Brouncker. A detailed historical report on Brouncker's calculations can be found in Stedall (2000a). It looks as though this achievement of Brouncker remained unnoticed, and even his formulas (3.41) were only later rediscovered by Euler.

3.3 Brouncker's method and the gamma function

64 Euler's gamma function. Daniel Bernoulli and Goldbach posed the problem of finding a formula extending factorial n, $n! \stackrel{\text{def}}{=} 1\times 2\times\cdots\times n$, to real values of n. In his letter of 13 October 1729 to Goldbach Euler solved this problem. There is no doubt that Euler's solution was motivated by Wallis' interpolation method (1656, p. 133).

It is stated in Wallis (1656, p. 169) that Brouncker found the continued fraction (3.34) using the functional equation (3.28). Arguing by analogy one can search for an extension $\Gamma(x)$ to $\Gamma(n+1)=n!$ as a solution to

$$\Gamma(x+1)=x\Gamma(x)\,. \tag{3.42}$$

Let $0\leqslant x<1$. Iterating (3.42), we obtain

$$\Gamma(x)=\frac{\Gamma(x+n+1)}{x(x+1)\cdots(x+n)}\,, \qquad n\in\mathbb{Z}_+\,. \tag{3.43}$$

Now if one can find an asymptotic formula for $\Gamma(t+1)$ as $t\to+\infty$ then it can be used to define $\Gamma(x)$ as a limit of elementary functions. The function $\log\Gamma(t+1)$ interpolates the sequence $z_k=\log k!$ at $t=k$.

Definition 3.17 *A sequence* $\{c_n\}_{n\geqslant 0}$ *is called convex if* $2c_k\leqslant c_{k+1}+c_{k-1}$ *for* $k\geqslant 1$.

The sequence $\{z_k\}_{k\geqslant 0}$ is convex. Since $\log\Gamma(t+1)$ interpolates it, one may assume that the graph of $\log\Gamma(t+1)$ is convex too. We compare the slopes of the three chords on the coordinate plane determined by the following points: $(n-1,z_{n-1})$, (n,z_n); (n,z_n), $(x+n,\log\Gamma(x+n+1))$; (n,z_n), $(n+1,z_{n+1})$. Then we obtain that

$$\log n\leqslant\frac{\log\Gamma(x+n+1)-\log n!}{x}\leqslant\log(n+1)\,,$$

or equivalently

$$n^x n!\leqslant\Gamma(x+n+1)\leqslant(n+1)^x n!\,. \tag{3.44}$$

Substitution of these inequalities into (3.43) shows that

$$\frac{n^x n!}{x(x+1)\cdots(x+n)}\leqslant\Gamma(x)\leqslant\frac{n^x n!}{x(x+1)\cdots(x+n)}\left(1+\frac{1}{n}\right)^x\,,$$

which leads to Euler's definition of the gamma function:

$$\Gamma(x)=\lim_{n\to\infty}\frac{n^x n!}{x(x+1)\cdots(x+n)}\,. \tag{3.45}$$

The above arguments show that if (3.42) has a logarithmic convex solution then it is defined by (3.45). Next,

$$\frac{n^x n!}{x(x+1)\cdots(x+n)}=\frac{1}{x}\left(\frac{n}{n+1}\right)^x\prod_{j=1}^{n}\left(1+\frac{x}{j}\right)^{-1}\left(1+\frac{1}{j}\right)^x \tag{3.46}$$

implies the existence of the limit in (3.45), since

$$\left(1+\frac{x}{j}\right)^{-1}\left(1+\frac{1}{j}\right)^x=1+\frac{x(x-1)}{2j^2}+O\left(\frac{1}{j^3}\right)\,.$$

Finally

$$\frac{d^2}{dx^2}\log\left(\frac{n^x n!}{x(x+1)\cdots(x+n)}\right)=\sum_{j=0}^{n}\frac{1}{(x+j)^2}>0$$

shows that the limit function $\Gamma(x)$ is logarithmic convex and

$$\frac{d^2}{dx^2}\log\Gamma(x)=\sum_{j=0}^{\infty}\frac{1}{(x+j)^2}\,. \tag{3.47}$$

Corollary 3.18 (The Bohr–Mollerup theorem 1922) *If $f(x)$ is a positive logarithmic convex function on $x>0$ satisfying $f(1)=1$, $f(x+1)=xf(x)$ then $f(x)=\Gamma(x)$.*

This completes the first step (the functional equation) of Brouncker's program, given in §**62**.

65 Stirling's formula. This is the asymptotic formula

$$\Gamma(x)=\sqrt{2\pi}e^{-x}x^{x-1/2}\left\{1+\frac{1}{12x}+\frac{1}{288x^2}+\cdots+O\left(\frac{1}{x^n}\right)\right\}\,, \tag{3.48}$$

where $x\to+\infty$. The inequalities (3.44) can be used to conjecture that

$$\Gamma(x)=ax^{x-1/2}e^{-x}e^{\mu(x)} \tag{3.49}$$

where $\mu(x)$ is some function and a is some constant. We first prove a weak version of (3.48) established to a great extent by De Moivre and Stirling in 1730; see Andrews, Askey and Roy (1999, 1.4).

Theorem 3.19 *For $x>0$,*

$$\Gamma(x)=\sqrt{2\pi}x^{x-1/2}e^{-x+\theta(x)/12x}\,,\qquad 0<\theta(x)<1\,.$$

Proof It is a matter of routine calculation to show that the function $x^{x-1/2}e^{-x}e^{\mu(x)}$ satisfies the functional equation (3.42) if and only if

$$\mu(x)-\mu(x+1)=g(x)\,, \tag{3.50}$$

where $g(x)=(x+1/2)\log(1+1/x)-1$.

Lemma 3.20 $0<g(x)<1/(12x(x+1))$ *for* $x>0$.

Proof For $|t|<1$,

$$\frac{1}{2}\ln\frac{1+t}{1-t}=\frac{\ln(1+t)-\ln(1-t)}{2}=\sum_{k=0}^{\infty}\frac{t^{2k+1}}{2k+1}\,.$$

If $t=1/(2x+1)$ then

$$0<g(x)=\frac{1}{2t}\ln\frac{1+t}{1-t}-1=\sum_{k=1}^{\infty}\frac{t^{2k}}{2k+1}<\frac{1}{3}\sum_{k=1}^{\infty}t^{2k}=\frac{1}{12x(x+1)}\,. \qquad \square$$

Notice that $g(x)$ decays at the rate indicated because in the exponent x^{x-c} in Stirling's formula c is taken as 1/2. The sum of shifted equations (3.50) shows that the only solution to (3.50) with $\lim_{x\to+\infty}\mu(x)=0$ satisfies

$$0<\mu(x)=\sum_{n=0}^{\infty}g(x+n)<\frac{1}{12}\sum_{n=0}^{\infty}\left(\frac{1}{x+n}-\frac{1}{x+n+1}\right)=\frac{1}{12x}, \tag{3.51}$$

see Lemma 3.20. Since, differentiating twice,

$$\left(\log(x^{x-1/2}e^{-x})\right)''=\frac{1}{x}+\frac{1}{2x^2}\,, \qquad g''(x)=\frac{1}{2x^2(x+1)^2}\,,$$

the function on the right-hand side of (3.49) is logarithmic convex. Since it also satisfies (3.42), Corollary 3.18 implies (3.49). To find a in (3.49) we apply Wallis' formula

$$\begin{aligned}\frac{2}{\pi}&=\lim_n\frac{1\times3}{2^2}\,\frac{3\times5}{4^2}\cdots\frac{(2n-1)(2n+1)}{(2n)^2}=\lim_n\frac{(2n)!^2}{2^{4n}n!^4}(2n+1)\\&=\lim_n\frac{a^2(2n+1)^{4n+1}e^{-4n-4+O(1/n)}}{a^4 2^{4n}(n+1)^{4n+2}e^{-4n-4+O(1/n)}}(2n+1)=\frac{4}{a^2}\,,\end{aligned}$$

showing that $a=\sqrt{2\pi}$. □

The nth term of the sum in (3.47) equals

$$\frac{1}{(x+n)^2}=\int_0^{+\infty}te^{-t(x+n)}\,dt\,. \tag{3.52}$$

Therefore

$$\frac{d^2}{dx^2}\log\Gamma(x)=\int_0^{+\infty}te^{-tx}\left(\sum_{j=0}^{\infty}e^{-tn}\right)dt=\int_0^{+\infty}\frac{te^{-tx}}{1-e^{-t}}\,dt\,. \tag{3.53}$$

The solutions to $e^{-z}=1$ are given by $z=2\pi ni$, $n\in\mathbb{Z}$. It follows that $z/(1-e^{-z})$ is analytic in $|z|<2\pi$ and is represented in $|z|<2\pi$ by the convergent power series

$$\frac{z}{1-e^{-z}}=\sum_{n=0}^{\infty}\frac{B_n(-1)^n}{n!}z^n\,. \tag{3.54}$$

The coefficients B_n are called *Bernoulli numbers*. It is easy to see that $B_0=1$ and $B_1=-1/2$. Since the function

$$\frac{z}{1-e^{-z}}-1-\frac{z}{2}$$

is even, $B_{2k+1}=0$ for $k=1,2,\ldots$ However, $B_2=1/6$, $B_4=-1/30$, $B_6=1/42$, $B_8=-1/30$, $B_{10}=5/66$, ... See Kudryavcev (1936) for an account of the properties of B_n.

If we formally substitute the power series (3.54) into the integral in (3.53), we obtain an asymptotic series for $\log\Gamma(x)''$:

$$\frac{d^2}{dx^2}\log\Gamma(x) \sim \sum_{n=0}^{\infty}\frac{B_n(-1)^n}{n!}\int_0^{+\infty} t^n e^{-tx}\,dt$$
$$= \sum_{n=0}^{\infty}\frac{B_n(-1)^n}{x^{n+1}} = \frac{1}{x}+\frac{1}{2x^2}+\sum_{n=2}^{\infty}\frac{B_n(-1)^n}{x^{n+1}}\,. \tag{3.55}$$

This substitution is justified by the following lemma.

Lemma 3.21 (Watson 1918) [3] *Let f be a function on $(0,+\infty)$ such that $|f(t)|<M$ for $t>\varepsilon$ and $f(t)=\sum_{k=0}^{\infty}c_k t^k$, $0<t<2\varepsilon$. Then*

$$\int_0^{+\infty} f(t)e^{-st}\,dt \sim \sum_{k=0}^{\infty}\frac{k!c_k}{s^{k+1}}\,,\qquad s\to+\infty \tag{3.56}$$

is the asymptotic expansion for the Laplace transform of f.

Proof It is clear that

$$\left|\int_{\varepsilon}^{+\infty} f(t)e^{-st}\,dt\right| < \frac{M}{\varepsilon}e^{-\varepsilon t}\,.$$

Next, for a given nonnegative integer k,

$$\int_0^{\varepsilon} t^k e^{-st}\,dt = \frac{1}{s^{k+1}}\int_0^{\varepsilon s} t^k e^{-t}\,dt$$
$$= \frac{1}{s^{k+1}}\int_0^{\infty} t^k e^{-t}\,dt - \frac{1}{s^{k+1}}\int_{s\varepsilon}^{\infty} t^k e^{-t}\,dt$$
$$= \frac{\Gamma(k+1)}{s^{k+1}} + O\left(e^{-\varepsilon s/2}\right)\,,$$

which proves (3.56) by Definition 3.15. □

Since

$$\frac{d^2}{dx^2}\left(\log(x^{x-1/2}e^{-x})\right) = \frac{1}{x}+\frac{1}{2x^2}\,,$$

formula (3.49) with $a=\sqrt{2\pi}$ and formula (3.55) imply that

$$\frac{d^2\mu}{dx^2}(x) \sim \sum_{n=2}^{\infty}\frac{B_n(-1)^n}{x^{n+1}}\,.$$

Integrating this asymptotic expansion twice from x to $+\infty$, we obtain

$$\mu(x) \sim \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)x^{2k-1}}\,. \tag{3.57}$$

[3] See also Andrews, Askey and Roy (1999, p. 614).

This implies the important formula

$$\boxed{\Gamma(x) \sim \sqrt{2\pi}x^{x-1/2}e^{-x}\exp\left\{\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)x^{2k-1}}\right\}.} \tag{3.58}$$

Now any number of terms in (3.48) can be found by (3.58). This formula gives the asymptotic expansion for $\Gamma(x)$, completing the second step in Brouncker's program (§**62**).

66 The Newman–Schlömilch formula. We have

$$\frac{n^x n!}{x(x+1)\cdots(x+n)} = e^{-\gamma_n x}\frac{1}{x}\prod_{j=1}^{n}\frac{e^{x/j}}{1+x/j},$$

where (see Ex. 3.13)

$$\gamma_n = \sum_{j=1}^{n}\frac{1}{j} - \log n \to \gamma = 0.577\,215\ldots$$

is the Euler–Mascheroni constant. It follows that

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{j=1}^{\infty}\left\{\left(1+\frac{x}{j}\right)e^{-x/j}\right\}. \tag{3.59}$$

Formula (3.59) locates the singularities of the gamma function and corresponds to the fourth step in Brouncker's program (§**62**). Euler found an integral representation for $\Gamma(x)$. Integration by parts shows that

$$G(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt\,, \qquad x > 0$$

satisfies (3.42). Clearly $G(1) = 1$.

Theorem 3.22 [4] *The sum of two logarithmic convex functions is logarithmic convex.*

By Theorem 19 the Riemann sums of $G(x)$ as well as their limit $G(x)$ are logarithmic convex . Then by Corollary 3.18

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt \overset{t=\log 1/s}{=} \int_0^1 \left(\log\frac{1}{s}\right)^{x-1} ds\,, \qquad x > 0\,, \tag{3.60}$$

where the substitution $t = \log 1/s$ is made in the first integral.

[4] See Ex. 3.16 for a proof.

67 Ramanujan's formula for Brouncker's function. The analogy between the gamma function and Brouncker's function can be demonstrated by an explicit formula.

Lemma 3.23 *Let $g(s)$ be a monotonic function on $(0,+\infty)$ vanishing at infinity and $a>0$ a positive number. Then the functional equation*

$$f(s)+f(s+a)=g(s) \tag{3.61}$$

has a unique solution vanishing at infinity given by the formula

$$f(s)=\sum_{n=0}^{\infty}(-1)^n g(s+na)\,. \tag{3.62}$$

Proof Writing (3.61) for $s:=s$, $s:=s+a$, $s:=s+2a$, ... and taking the alternating sum of the equations thus obtained we get (3.62). The function f defined by (3.62) satisfies $\lim_{x\to+\infty} f(s)=0$ since g is monotonic and series (3.62) is telescopic. □

Taking the logarithm of (3.28) shows that $f(s)=\log b(s)$ satisfies (3.61) with $g(s)=2\log(s+1)$ and $a=2$. This g, however, does not vanish at $+\infty$. Differentiating twice, we find that $f(s)=(\log b)''(s)$ satisfies (3.61) with $g(s)=-2(s+1)^{-2}$.

Theorem 3.24 *The functional equation $f(s)+f(s+2)=-2(s+1)^{-2}$ has a unique solution satisfying $\lim_{s\to+\infty} f(s)=0$:*

$$f(s)=(\log b)''(s)=\sum_{n=0}^{\infty}\frac{2(-1)^{n+1}}{(s+2n+1)^2}=\frac{d^2}{ds^2}\log\left[\frac{\Gamma(s+3/4)}{\Gamma(s+1/4)}\right]^2. \tag{3.63}$$

Proof By Theorem 3.16

$$\frac{b'}{b}(s)=\frac{1}{s+1}+\sum_{n=1}^{\infty}\frac{8}{(s+4n-3)(s+4n-1)(s+4n+1)}\,. \tag{3.64}$$

It follows that $\lim_{s\to+\infty}(\log b)''(s)=0$. By Lemma 3.23 $(\log b)''(s)$ equals

$$\begin{aligned}-2\sum_{n=0}^{\infty}\frac{(-1)^n}{(s+2n+1)^2}&=\sum_{n=0}^{\infty}\frac{2}{(s+4n+3)^2}-\sum_{n=0}^{\infty}\frac{2}{(s+4n+1)^2}\\&=\frac{1}{8}\left\{(\log\Gamma)''\left(\frac{s+3}{4}\right)-(\log\Gamma)''\left(\frac{s+1}{4}\right)\right\}\\&=\frac{d^2}{ds^2}\log\left(\frac{\Gamma(s+3/4)}{\Gamma(s+1/4)}\right)^2\,;\end{aligned}$$

see (3.47). □

It follows from (3.63) that

$$b(s)=e^{as+b}R(s)\,,\qquad R(s)\overset{\text{def}}{=}4\left(\frac{\Gamma(s+3/4)}{\Gamma(s+1/4)}\right)^2\,.$$

Hence

$$R(s)R(s+2) = 4^2 \left(\frac{\Gamma(3+s/4)\Gamma(1+1+s/4)}{\Gamma(1+s/4)\Gamma(3+s/4)} \right)^2 = (s+1)^2 ,$$

so R satisfies the same functional equation as y. It follows that $a = b = 0$. Thus we obtain the following theorem.

Theorem 3.25 (Ramanujan) *For every $s > 0$*

$$b(s) = s + \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{(2n-1)^2}{2s} \right) = 4 \left[\frac{\Gamma(3+s/4)}{\Gamma(1+s/4)} \right]^2 = R(s) .$$

Having found Ramanujan's formula, we can give a short proof of it. By Theorem 3.16 it is sufficient to check that $R(s) > s$ for large s. This is equivalent to

$$\frac{\Gamma^2(s+1/2)}{\Gamma^2(s)} > s - \frac{1}{4} .$$

Furthermore, Stirling's formula implies

$$\frac{\Gamma(s+1/2)}{\Gamma(s)} = \sqrt{s} \left(1 - \frac{1}{8s} + \frac{1}{128s^2} + \cdots \right) , \tag{3.65}$$

which proves the required inequality if $s \to +\infty$. See Ex. 3.18 for the third proof of Theorem 3.25.

Applying Theorem 3.25 to the case $s = 1$ and observing that, by Brouncker's formula, $y(1) = 4/\pi$ we obtain

$$\Gamma(1/2) = \sqrt{\pi} . \tag{3.66}$$

Combining Brouncker's formula with Euler's definition (3.45) of the gamma function, we immediately see that the definition of $b(s)$ as an infinite product (3.40) is completely analogous to Euler's definition of the gamma function. A difference between $\Gamma(x)$ and $b(s)$ is that whereas $\Gamma(x)$ is logarithmic convex, (3.47), $b(s)$ is logarithmic concave, (3.63).

A good introduction to the theory of the gamma function can be found in Artin (1931). See Havil (2003) for a nice elementary presentation and Whittaker and Watson (1902) as well as Andrews, Askey and Roy (1999) for an advanced theory.

Exercises

3.1 Show that the numerator of the nth convergent to the continued fraction

$$a + \frac{b}{c+} \frac{b}{c+} \frac{b}{c+\cdots}$$

is the coefficient of x^n in the expansion of

$$\frac{a+bx}{1-cx-bx^2} = a + P_1 x + P_2 x^2 + \cdots .$$

Show that the denominators Q_n of the nth convergents to this fraction satisfy

$$\frac{1}{1-cx-bx^2} = 1 + Q_1 x + Q_2 x^2 + \cdots$$

(Smith 1888, §360, p. 458).

3.2 Assuming that all $b_k \neq 0$ prove Stern's identity (Perron 1957):

$$\frac{b_1}{b_1} + \frac{b_2}{b_2} + \frac{b_3}{b_3} + \cdots \approx \frac{b_0}{b_0} + \frac{b_0}{b_1} + \frac{b_1}{b_2} + \frac{b_2}{b_3} + \cdots .$$

Hint: Apply the equivalence transform with $r_n = b_{n-1}/b_n$, $n \geqslant 1$, r_0; see Theorem 3.6.

3.3 If $\{a_n\}_{n\geqslant 1}$ is a sequence such that $a_n \neq 0$, r, -1 then the continued fraction

$$1 + \frac{a_1}{1} + \frac{(a_1-r)(a_1+1)}{1} + \frac{a_1 a_2}{1} + \frac{(a_2-r)(a_2+1)}{1} + \frac{a_2 a_3}{1} + \cdots$$

may converge only either to 0 or to $1+r$ (Wall 1948; see Perron [1957, Theorem 1.5]).

3.4 Prove that an infinite continued fraction with positive $\{a_n\}_{n\geqslant 1}$ and $\{b_{2n}\}_{n\geqslant 1}$

$$b_0 + \frac{a_1}{0} + \frac{a_2}{b_2} + \frac{a_3}{0} + \frac{a_4}{b_4} + \frac{a_5}{0} + \frac{a_6}{b_6} + \frac{a_7}{0} + \cdots$$

always diverges (Broman 1877; see Perron 1957, Theorem 2.4)

3.5 Prove that the continued fraction ($s > 0$)

$$\frac{1^p}{s} + \frac{2^p}{s} + \frac{3^p}{s} + \frac{4^p}{s} + \frac{5^p}{s} + \cdots$$

converges if and only if $p \leqslant 2$.

3.6 Show that

$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1\times 2}{3} + \frac{3\times 4}{3} + \frac{5\times 6}{3} + \cdots$$

(Smith 1888, Examples XXXVII, 31, p. 457).

Hint: Put $a = 3$, $y_n = 2n+2$, $n = 0, 1, \ldots$ in Theorem 3.12 and combine it with the binomial formula

$$\sqrt{1+x} = 1 + \frac{1}{2}\left(x - \frac{x^2}{4} + \cdots + (-1)^{n-1}\frac{1\times 3\cdots\times(2n-3)}{4\times 6\times\cdots\times(2n)}x^n + \cdots\right)$$

evaluated at $x = 1$.

3.7 Show that

$$1-\frac{1}{\sqrt{2}}=\frac{1}{2}{}_{+}\ \frac{2\times 3}{1}{}_{+}\ \frac{4\times 5}{1}{}_{+}\ \frac{6\times 7}{1}{}_{+\cdots}$$

(Smith 1888, Examples XXXVII, 32, p. 458). Deduce from here that

$$\sqrt{2}=1+\frac{1}{1}{}_{+}\ \frac{2\times 3}{1}{}_{+}\ \frac{4\times 5}{1}{}_{+}\ \frac{6\times 7}{1}{}_{+\cdots} \tag{E3.1}$$

Hint: Put $a=1$, $y_n=2n+2$, $n=0,1,\ldots$ in Theorem 3.12 and combine it with the binomial formula

$$\begin{aligned}\frac{1}{\sqrt{1+x}}=&1-\frac{1}{2}x+\frac{1\times 3}{2\times 4}x^2+\cdots\\&+(-1)^{n-1}\frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\ldots\times(2n)}x^n+\cdots\end{aligned}$$

evaluated at $x=1$.

3.8 Apply Theorems 1.4 and 1.7 to prove the convergence of continued fraction (3.34) for $s\geqslant 1$.

Since $P_n(s)\geqslant P_n(1)=(2n+1)!!$ for $s\geqslant 1$, for such an s we have

$$\left|\frac{Q_{n-1}}{P_{n-1}}-\frac{Q_n}{P_n}\right|\leqslant\frac{(2n-1)!!^2}{(2n+1)!!(2n-1)!!}=\frac{1}{2n+1},$$

implying the existence of $\lim_n Q_n/P_n$ (and hence of $\lim_n P_n/Q_n$).

3.9 Prove that

$$b(4n)=\frac{3^2}{1\times 5}\frac{7^2}{5\times 9}\cdots\frac{(4n-1)^2}{(4n-3)(4n+1)}(4n+1)b(0),$$

$$b(4n+2)=\frac{1\times 5}{3^2}\frac{5\times 9}{7^2}\cdots\frac{(4n-3)(4n+1)}{(4n-1)^2}\frac{(4n+1)}{b(0)},$$

where $b(0)=1/b(2)$.

3.10 Prove that

$$\frac{\pi}{4}=\frac{2\times 4}{3^2}\frac{4\times 6}{5^2}\frac{6\times 8}{7^2}\frac{8\times 10}{9^2}\cdots$$

Hint: Rearrange the multipliers in Wallis' formula.

3.11 Prove that

$$\pi=3+\frac{1^2}{6}{}_{+}\ \frac{3^2}{6}{}_{+}\ \frac{5^2}{6}{}_{+}\ \frac{7^2}{6}{}_{+}\ \frac{9^2}{6}{}_{+\cdots}.$$

Hint: Apply Brouncker's formula $b(s)b(s+2)=(s+1)^2$ with $s=1$.

3.12 Prove that for any function $f(s)$ defined on $(0,2]$ there is a unique solution $z(s)$ to the equation $z(s)z(s+2)=(s+1)^2$ satisfying $z(s)=f(s)$ on $(0,2]$. If $f(s)\neq y(s)$ for at least at one point in $(0,2]$, then $z(s)\leqslant s$ infinitely often on $s>0$.

Hint: Apply Theorem 3.16.

3.13 Prove that the sequence

$$\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

has a positive limit.
Hint: Consider $\delta_n = \gamma_n - 1/n$ and observe that

$$\gamma_{n+1} - \gamma_n = \frac{1}{n+1} - \log\left(1 + \frac{1}{n}\right) ,$$

$$\delta_{n+1} - \delta_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right) .$$

Apply the elementary inequality

$$\frac{1}{n+1} < \log\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

to deduce that γ_n monotonically decreases and δ_n monotonically increases. Notice that $\delta_n < \gamma_n$.

3.14 Prove that a twice-differentiable function $f(x)$ with continuous second derivative $f''(x)$ on $(0, +\infty)$ is logarithmic convex if and only if $f(x) > 0$ and $f(x)f''(x) - (f'(x))^2 \geqslant 0$.
Hint: Compute the second derivative of $\log f(x)$.

3.15 Prove that if a_i, b_i, c_i $(i = 1, 2)$ are real numbers satisfying

$$a_i > 0 , \qquad a_i c_i - b_i^2 \geqslant 0 ,$$

then

$$a_1 + a_2 > 0 , \qquad (a_1 + a_2)(c_1 + c_2) - (b_1 + b_2)^2 \geqslant 0 .$$

Hint: Consider quadratic polynomials $p_i(x) = a_i x^2 + 2b_i x + c_i$ and observe that $p_i(x) \geqslant 0$. Hence $p_1(x) + p_2(x) \geqslant 0$.

3.16 Prove Theorem 19.
Hint: Assuming that both functions are smooth, apply Ex. 3.14 and Ex. 3.15; see Artin (1931) for details. Apply an approximation to cover the general case.

3.17 Let $a + b = c + d$. Assuming that no multiplier in the infinite product below vanishes, prove that

$$\prod_{j=0}^{\infty} \frac{(a+nj)(b+nj)}{(c+nj)(d+nj)} = \frac{\Gamma(c/n)\Gamma(d/n)}{\Gamma(a/n)\Gamma(b/n)} .$$

Hint: Apply (3.59). See Whittaker and Watson (1902, §12.13) for details.

3.18 Deduce Theorem 3.25 from Ex. 3.17 and Theorem 3.16.

3.19 Prove Euler's functional equation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} .$$

Hint: Combine (3.16) with (3.59). Apply the identity

$$1 = \frac{1}{\Gamma(1)} = e^{\gamma} \prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right) e^{-1/k} .$$

3.20 Check that the first few terms in Stieltjes' continued fraction for $\mu(x)$ are given by

$$\mu(x) = \frac{1}{12x} \, {}_{+} \, \frac{2}{5x} \, {}_{+} \, \frac{53}{42x} \, {}_{+} \, \frac{1170}{53x} \, {}_{+} \, \frac{22\,999}{429x} \, {}_{+} \cdots .$$

See Khovanskii (1958, Chapter III, §11).

3.21 Prove that Brouncker's continued fraction converges for every $s > 0$.
Hint: Put $Q_n = (2n+1)!!D_n$. Convergence occurs if and only if

$$\begin{aligned} \frac{a_1 a_2 \cdots a_n}{Q_n Q_{n-1}} &= \frac{(2n-1)!!^2}{(2n+1)!!(2n-1)!!D_n D_{n-1}} \\ &= \frac{1}{(2n+1)D_n D_{n-1}} \to 0. \end{aligned}$$

The Euler–Wallis formulas

$$D_n = \frac{2s}{2n+1} D_{n-1} + \frac{2n-1}{2n+1} D_{n-2}$$

imply that $(2n+1)D_n D_{n-1} = 2sD_{n-1}^2 + (2n-1)D_{n-1}D_{n-2}$. Hence

$$(2n+1)D_n D_{n-1} = 2s\left(D_0^2 + D_1^2 + \cdots + D_{n-1}^2\right).$$

Next, $D_n^2 > (n+1)^{-1}$ for even n. Indeed if $D_{n-2}^2 > (n-1)^{-1}$ then

$$D_n^2 > \left(\frac{2n-1}{2n+1}\right)^2 D_{n-2}^2 > \left(\frac{2n-1}{2n+1}\right)^2 \frac{1}{n-1} > \frac{1}{n+1},$$

as elementary algebra shows. It follows that

$$(2n+1)D_n D_{n-1} > 2s\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2m+1}\right),$$

where m is the greatest number satisfying $2m \leqslant n-1$.

3.22 Simplify the proof of Ex. 3.21 for $s \geqslant 1$.
Hint: If $s = 1$ then D_n lies between D_{n-1} and D_{n-2}, implying the convergence. If $s > 1$ then $D_n(s) > D_n(1)$, again implying the convergence.

4

Continued fractions: Euler

4.1 Partial sums

68 Euler's first approach. Theorem 3.12 says that convergents of some continued fractions coincide with partial sums of series. This phenomenon was first studied in detail by Euler (1744). Motivated by Wallis' product we will slightly modify Euler's original arguments and relate them to D. Bernoulli's *inverse* problem of reconstructing continued fractions from their convergents. In what follows $\hat{\mathbb{C}}$ denotes the extended complex plane $\mathbb{C} \cup \infty$.

Theorem 4.1 (Bernoulli 1775) *A sequence $\{d_n\}_{n \geqslant 0}$ in $\hat{\mathbb{C}}$ is the sequence of convergents to a continued fraction $q_0 + \mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ if and only if $d_0 \neq \infty$, $d_n \neq d_{n-1}$, $n = 1, 2, 3, \ldots$*

Proof If $d_n = P_n/Q_n$, $n = 0, 1, \ldots$, is a sequence of convergents to a continued fraction then $d_0 = q_0 \neq \infty$ and by (1.16)

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} p_1 \cdots p_n \neq 0 \,.$$

Therefore Q_{n-1} and Q_n cannot both vanish. Similarly, if $Q_n = 0$ then $P_n \neq 0$. This shows that $d_n \neq d_{n-1}$.

To prove the converse we assume that the numerators of the convergents are d_n and the denominators all equal 1. Let all the d_n be finite. Then the Euler–Wallis recurrence (1.15) takes the form

$$d_n = q_n d_{n-1} + p_n d_{n-2},$$
$$1 = q_n + p_n.$$

The determinant of this linear system in two unknowns p_n and q_n is $d_{n-1} - d_{n-2} \neq 0$. It follows that

$$p_n = \frac{d_{n-1} - d_n}{d_{n-1} - d_{n-2}}\,, \qquad q_n = \frac{d_n - d_{n-2}}{d_{n-1} - d_{n-2}}\,, \qquad n = 2, 3, \ldots \tag{4.1}$$

The initial values are $q_0 = d_0$, $p_1 = d_1 - d_0$, $q_1 = 1$. If, say, $d_n = \infty$ then by the assumption both d_{n-1} and d_{n+1} are finite. We put $P_n = 1$, $Q_n = 0$ and by (1.15) obtain the system

$$1 = q_n d_{n-1} + p_n d_{n-2},$$
$$0 = q_n + p_n\,.$$

The second equation shows that $q_n = -p_n$, and

$$q_n = \frac{1}{d_{n-1} - d_{n-2}}\,, \qquad p_n = -\frac{1}{d_{n-1} - d_{n-2}}$$

follows from the first. $\square$

Without stating Bernoulli's theorem explicitly, Euler found a continued fraction with convergents equal to the partial sums of a given series.

Theorem 4.2 (Euler 1744) *Let $\{c_n\}_{n\geqslant 0}$ be a sequence of nonzero complex numbers. Then*

$$\sum_{k=0}^{n} c_k = \frac{c_0}{1} \mathbin{\genfrac{}{}{0pt}{}{}{-}} \frac{c_1/c_0}{1 + c_1/c_0} \mathbin{\genfrac{}{}{0pt}{}{}{-}} \frac{c_2/c_1}{1 + c_2/c_1} \mathbin{\genfrac{}{}{0pt}{}{}{-}} \cdots \mathbin{\genfrac{}{}{0pt}{}{}{-}} \frac{c_n/c_{n-1}}{1 + c_n/c_{n-1}}\,. \tag{4.2}$$

Proof We apply Theorem 4.1 to $d_n = \sum_{k=0}^{n} c_k$, $n \geqslant 0$. Since $c_n \neq 0$, we have $d_n \neq d_{n-1}$ for $n = 1, 2, \ldots$ Next, $d_0 = c_0 \neq \infty$. Since $d_n \neq \infty$, formula (4.1) shows that

$$p_n = \frac{d_n - d_{n-1}}{d_{n-2} - d_{n-1}} = -\frac{c_n}{c_{n-1}}\,,$$
$$q_n = \frac{d_n - d_{n-2}}{d_{n-1} - d_{n-2}} = \frac{c_n + c_{n-1}}{c_{n-1}}\,, \qquad n = 2, 3, \ldots$$

Since $q_0 = d_0 = c_0$, $p_1 = d_1 - d_0 = c_1$, $q_1 = 1$, Theorem 4.1 shows that

$$\sum_{k=0}^{n} c_k = c_0 + \frac{c_1}{1} \mathbin{\genfrac{}{}{0pt}{}{}{-}} \frac{c_2/c_1}{1 + c_2/c_1} \mathbin{\genfrac{}{}{0pt}{}{}{-}} \frac{c_3/c_2}{1 + c_3/c_2} \mathbin{\genfrac{}{}{0pt}{}{}{-}} \cdots \mathbin{\genfrac{}{}{0pt}{}{}{-}} \frac{c_n/c_{n-1}}{1 + c_n/c_{n-1}}\,. \tag{4.3}$$

The application to (4.3) of the elementary identity

$$c_0 + \frac{c_1}{1 + w} = \frac{c_0}{1} \mathbin{\genfrac{}{}{0pt}{}{}{-}} \frac{c_1/c_0}{1 + c_1/c_0 + w}$$

proves (4.2). $\square$

If we put

$$\rho_0 = c_0, \qquad c_k = \rho_0 \rho_1 \cdots \rho_k, \qquad k = 1, 2, \ldots\,, \tag{4.4}$$

then (4.3) and (4.2) turn into the beautiful formula

$$\sum_{k=0}^{n} \rho_0 \rho_1 \cdots \rho_k = \frac{\rho_0}{1} \mathbin{\genfrac{}{}{0pt}{}{}{-}} \frac{\rho_1}{1 + \rho_1} \mathbin{\genfrac{}{}{0pt}{}{}{-}} \cdots \mathbin{\genfrac{}{}{0pt}{}{}{-}} \frac{\rho_n}{1 + \rho_n}\,. \tag{4.5}$$

Definition 4.3 *The continued fraction* (4.5) *for which* ρ_k *satisfies* (4.4) *is called the Euler continued fraction for the series* $\sum c_k$.

An interesting Euler continued fraction is obtained from Ex. 4.41. For this convergent Euler's series we have $\rho_n = (p+ns)/(q+ns+s)$, $n = 0, 1, \dots$ Applying obvious equivalent transforms to (4.5), we obtain

$$\frac{p}{q+s} - \frac{(p+s)(q+s)}{p+q+3s} - \frac{(p+2s)(q+2s)}{p+q+5s} - \cdots = \frac{p}{q-p}\,, \tag{4.6}$$

or

$$p+s = \frac{(p+s)(q+s)}{p+q+3s} - \frac{(p+2s)(q+2s)}{p+q+5s} - \frac{(p+3s)(q+3s)}{p+q+7s} - \cdots.$$

Putting $s = 0$, we see that the periodic continued fraction

$$\frac{pq}{p+q} - \frac{pq}{p+q} - \frac{pq}{p+q} - \frac{pq}{p+q} - \cdots$$

converges to the smallest root, $X = p$, of the quadratic equation

$$X^2 - (p+q)X + pq = 0\,.$$

69 Applications. We need a weak version of Abel's theorem. For Abel's theorem see for instance Hairer and Wanner (1996, p. 248) or Rudin (1964, Theorem 8.2).

Lemma 4.4 *If* $0 < u_{n+1} \leqslant u_n$, $n \geqslant 0$ *and* $\lim_n u_n = 0$ *then*

$$\lim_{r\to 1-0} \sum_{k=0}^{\infty} (-1)^k u_k r^k = \sum_{k=0}^{\infty} (-1)^k u_k\,.$$

Proof For every even n and $0 < r < 1$,

$$\begin{aligned} 0 < U_n(r) &\overset{\text{def}}{=} \sum_{k=n}^{\infty} (-1)^k u_n r^n \\ &= u_n r^n - (u_{n+1} r^{n+1} - u_{n+2} r^{n+2}) - \cdots \leqslant u_n\,. \end{aligned}$$

It follow that for every even n

$$\sum_{k=0}^{n-1} (-1)^k u_k \leqslant \liminf_{r\to 1-0} U_0(r) \leqslant \limsup_{r\to 1-0} U_0(r) \leqslant \sum_{k=0}^{n-1} (-1)^k u_k + u_n\,.$$

Passing to the limit in n completes the proof. □

By Theorem 4.2 every series with nonzero terms corresponds to an Euler continued fraction. Let us compute the Euler continued fraction for Leibniz's series

$$\frac{\pi}{4} = \arctan x \Big|_0^1 = \int_0^1 \frac{dx}{1+x^2} = \lim_{r\to 1-0} \int_0^r \sum_{k=0}^{\infty} (-1)^k x^{2k}\, dx$$
$$= \lim_{r\to 1-0} \sum_{k=0}^{\infty} (-1)^k \int_0^r x^{2k}\, dx = \lim_{r\to 1-0} \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k+1}}{2k+1}$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

(see Lemma 4.4). Let $c_0 = 0$ and $c_n = (-1)^{n-1}/(2n-1)$ for $n \geqslant 1$. Resolving (4.4) in ρ_k, we obtain

$$\rho_1 = c_1 = 1, \qquad \rho_2 = \frac{c_2}{c_1} = -\frac{1}{3}, \qquad \rho_3 = -\frac{3}{5}, \ \ldots,$$
$$\rho_k = -\frac{2k-3}{2k-1}, \qquad k = 2, 3, \ldots$$

It follows from (4.5) that

$$\frac{\pi}{4} = \frac{1}{1} {+} \frac{1/3}{1-1/3} {+} \frac{3/5}{1-3/5} {+} \frac{5/7}{1-5/7} {+} \cdots$$
$$= \frac{1}{1} {+} \frac{1}{3-1} {+} \frac{3^2}{5-3} {+} \frac{5^2}{7-5} {+} \cdots = \frac{1}{1} {+} \frac{1}{2} {+} \frac{3^2}{2} {+} \frac{5^2}{2} {+} \cdots,$$

which is (3.19). Similarly one can obtain (3.25):

$$\ln 2 = \int_0^1 \frac{dx}{1+x} = \int_0^1 \sum_{k=0}^{\infty} (-1)^k x^k\, dx = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1}$$
$$= \frac{1}{1} {+} \frac{1/2}{1-1/2} {+} \frac{2/3}{1-1/3} {+} \frac{3/4}{1-3/4} {+} \cdots = \frac{1}{1 + \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(n^2/1\right)}.$$

Another of Euler continued fractions is obtained from the series

$$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

By Theorem 4.2

$$\frac{1}{e} = 1 - \frac{1}{1} {+} \frac{1/2}{1-1/2} {+} \frac{1/3}{1-1/3} {+} \frac{1/4}{1-1/4} {+} \cdots$$
$$= 1 - \frac{1}{1} {+} \frac{1}{2-1} {+} \frac{2}{3-1} {+} \frac{3}{4-1} {+} \cdots$$
$$= 1 - \frac{1}{1} {+} \frac{1}{1} {+} \frac{2}{2} {+} \frac{3}{3} {+} \cdots {+} \frac{n}{n} {+} \cdots,$$

which implies that

$$\boxed{e = 2 + \frac{2}{2} \, {}_{+} \frac{3}{3} \, {}_{+} \cdots {}_{+} \frac{n}{n} \, {}_{+} \cdots} \,. \tag{4.7}$$

70 Continued fractions and sums: general case. Any convergent P_n/Q_n of a continued fraction $\mathbf{K}_{k=1}^{\infty}(p_k/q_k)$ can be represented as

$$\begin{aligned}\frac{P_n}{Q_n} &= \frac{P_1}{Q_1} - \left(\frac{P_1}{Q_1} - \frac{P_2}{Q_2}\right) + \left(\frac{P_3}{Q_3} - \frac{P_2}{Q_2}\right) - \left(\frac{P_3}{Q_3} - \frac{P_4}{Q_4}\right) + \cdots \\ &= \frac{p_1}{Q_1} - \frac{p_1 p_2}{Q_1 Q_2} + \frac{p_1 p_2 p_3}{Q_2 Q_3} - \frac{p_1 p_2 p_3 p_4}{Q_3 Q_4} + \cdots + (-1)^{n-1} \frac{p_1 \cdots p_n}{Q_{n-1} Q_n} \,,\end{aligned} \tag{4.8}$$

see (1.16). By (1.15) $q_k = (Q_k - p_k Q_{k-2}) {Q_{k-1}}^{-1}$, which, followed by equivalence transforms, gives the continued fraction

$$\begin{aligned}\frac{P_n}{Q_n} &= \mathop{\mathrm{K}}_{k=1}^{n} \left(\frac{p_k}{Q_k - p_k Q_{k-2}/Q_{k-1}}\right) \\ &= \frac{p_1}{Q_1} \, {}_{+} \frac{p_2 Q_1}{Q_2 - p_2} \, {}_{+} \frac{p_3 Q_1 Q_2}{Q_2 - p_3 Q_1} \, {}_{+} \cdots {}_{+} \frac{p_n Q_{n-2} Q_{n-1}}{Q_n - p_n Q_{n-2}} \,.\end{aligned} \tag{4.9}$$

In (1750b) Euler applied these formulas to transform the sum

$$\frac{x_1}{y_1} - \frac{x_2}{y_2} + \frac{x_3}{y_3} - \frac{x_4}{y_4} + \cdots + (-1)^{n-1} \frac{x_n}{y_n} \tag{4.10}$$

into a continued fraction. Suppose that $x_k \neq 0$. Then $p_1, \ldots, p_n$ can be found by comparing the numerators in (4.8) and (4.10):

$$p_1 = x_1, \quad p_2 = \frac{x_2}{x_1}, \quad p_3 = \frac{x_3}{x_2}, \quad \ldots, \quad p_n = \frac{x_n}{y_n} \,.$$

Similarly, comparing the denominators in (4.8) and (4.10) we get

$$Q_1 = y_1, \quad Q_2 = \frac{y_2}{y_1}, \quad Q_3 = \frac{y_1 y_3}{y_2}, \quad Q_4 = \frac{y_2 y_4}{y_1 y_3}, \quad Q_5 = \frac{y_1 y_3 y_5}{y_2 y_4}, \ldots$$

Substituting these expressions into (4.9) and applying equivalence transforms, we obtain for $n = 5$

$$\begin{aligned}\frac{P_5}{Q_5} &= \frac{x_1}{y_1} \, {}_{+} \frac{y_1 x_2/x_1}{y_2/y_1 - x_2/x_1} \, {}_{+} \frac{y_2 x_3/x_2}{y_1 y_3/y_2 - x_3 y_1/x_2} \\ &\quad {}_{+} \frac{y_3 x_4/x_3}{y_2 y_4/y_1 y_3 - x_4 y_2/x_3 y_1} \, {}_{+} \frac{y_4 x_5/x_4}{y_1 y_3 y_5/y_2 y_4 - x_5 y_1 y_3/x_4 y_2} \\ &= \frac{x_1}{y_1} \, {}_{+} \frac{y_1^2 x_2}{y_2 x_1 - y_1 x_2} \, {}_{+} \frac{x_1 y_2^2 x_3}{y_3 x_2 - y_2 x_3} \, {}_{+} \frac{x_2 y_3^2 x_4}{y_4 x_3 - y_3 x_4} \, {}_{+} \frac{x_3 y_4^2 x_5}{y_5 x_4 - y_4 x_5} \,.\end{aligned}$$

Theorem 4.5 (Euler 1750b, §2) *For any nonzero sequences $\{x_n\}_{n\geqslant 0}$, $\{y_n\}_{n\geqslant 1}$, satisfying $x_0 = 1$, and any integer $n \geqslant 1$*

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{x_k}{y_k} = \cfrac{x_1}{y_1 + \underset{k=2}{\overset{n}{\mathbf{K}}}\left(\cfrac{x_{k-2}y_{k-1}^2 x_k}{y_k x_{k-1} - y_{k-1}x_k}\right)}.$$

Proof This follows from Theorem 4.2. Just put $c_{k-1} = (-1)^{k-1}x_k/y_k$ for $k = 1, 2, \ldots$ and apply equivalence transforms. $\square$

Integrating by using the Maclaurin series for the denominator shows that

$$\int_0^1 \frac{x^{n-1}\,dx}{1+x^m} = \frac{1}{n} - \frac{1}{m+n} + \frac{1}{2m+n} - \frac{1}{3m+n} + \cdots.$$

Putting $x_k = 1$, $y_k = (k-1)m + n$ in Theorem 4.5 we obtain

$$\int_0^1 \frac{x^{n-1}\,dx}{1+x^m} = \frac{1}{n\,+}\,\frac{n^2}{m\,+}\,\frac{(m+n)^2}{m\qquad +}\,\frac{(2m+n)^2}{m\qquad +\cdots},$$

which gives Brouncker's formula if $n = 1$, $m = 2$ and (3.25) if $n = 1$, $m = 1$. For $m = n = 2$ we again obtain (3.25). Other applications of Theorem 4.5 can be found in Exs. 4.8–4.10.

4.2 Euler's version of Brouncker's method

71 Euler's quadrature formulas. According to elementary geometry, the half-length $S(x)$ of an arc completing a chord of length $2x$ on $\mathbb{T}$ is

$$S(x) = \arcsin x$$

(apply $\sin\angle AOB = |AP|/|OA|$, see Fig. 1.2 with the x- and y- axes interchanged. Since $S'(x) = (1-x^2)^{-1/2}$, the binomial theorem followed by integration implies

$$\arcsin x = x + \frac{1}{2}\,\frac{x^3}{3} + \frac{1\times 3}{2\times 4}\,\frac{x^5}{5} + \frac{1\times 3\times 5}{2\times 4\times 6}\,\frac{x^7}{7} + \cdots \tag{4.11}$$

Hence

$$\frac{\pi}{2} = \arcsin(1) = 1 + \frac{1}{2}\,\frac{1}{3} + \frac{1\times 3}{2\times 4}\,\frac{1}{5} + \frac{1\times 3\times 5}{2\times 4\times 6}\,\frac{1}{7} + \cdots$$

This shows that the coefficients in (4.11) are related in some way to π. To find this relationship Euler applies the Brouncker–Wallis interpolation method to the Taylor series of $(\arcsin x)' = 1/\sqrt{1-x^2}$, which is given by

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\times 3}{2\times 4}x^4 + \frac{1\times 3\times 5}{2\times 4\times 6}x^6 + \cdots \tag{4.12}$$

Namely, he associates with (4.12) a positive sequence $\{x_n\}_{n\geqslant 0}$ satisfying

$$x_0x_1 = \frac{1}{2}, \quad x_2x_3 = \frac{3}{4}, \quad x_4x_5 = \frac{5}{6}, \quad x_6x_7 = \frac{7}{8}, \quad x_8x_9 = \frac{9}{10}, \quad \ldots$$

The choice of such a sequence is not unique. So an extra requirement is that the intermediate products x_1x_2, x_3x_4, x_5x_6, $\ldots$ interpolate the mediants, namely

$$x_1x_2 = \frac{2}{3} = \frac{1+3}{2+4}, \quad x_3x_4 = \frac{4}{5} = \frac{3+5}{4+6}, \quad x_5x_6 = \frac{6}{7} = \frac{5+7}{6+8}, \quad \ldots,$$

implying that

$$x_nx_{n+1} = \frac{n+1}{n+2}, \qquad n = 0, 1, 2, \ldots \tag{4.13}$$

Then

$$x_0 = \frac{1}{2x_1} = \frac{1\times 3}{2^2}x_2 = \frac{1\times 3^2}{2^2\times 4x_3} = \frac{1\times 3^2\times 5}{2^2\times 4^2}x_4 = \cdots,$$

which can be rewritten as follows

$$\begin{aligned} x_0 &= \frac{1\times 3}{2\times 2}\,\frac{3\times 5}{4\times 4}\cdots\frac{(2n-1)(2n+1)}{2n\times 2n}x_{2n}, \\ x_0 &= \frac{1\times 3}{2\times 2}\,\frac{3\times 5}{4\times 4}\cdots\frac{(2n-1)(2n+1)}{2n\times 2n}\,\frac{2n+1}{2n+2}\,\frac{1}{x_{2n+1}}. \end{aligned} \tag{4.14}$$

By Wallis' formula (3.10)

$$x_0 = \frac{2}{\pi}\lim_n x_{2n}, \qquad x_0 = \frac{2}{\pi}\frac{1}{\lim_n x_{2n+1}}.$$

Hence if $x_0 = 2/\pi$, then $\lim_n x_{2n}$ and $\lim_n x_{2n+1}$ exist and equal 1. With this method Euler in (1750b) developed infinite products of the Wallis type into infinite continued fractions. To pass from the rational numbers in (4.13) to integers Euler considers an auxiliary sequence $y_n = (n+1)x_n$, $n \geqslant 0$, satisfying

$$y_0 = \frac{2}{\pi}, \qquad y_ny_{n+1} = (n+1)^2, \qquad n = 0, 1, 2, \ldots \tag{4.15}$$

Formula (4.15) calls to mind Brouncker's formula (3.28). Although Euler's original calculations were motivated by algebraic identities (1750b, §§38–40), as in Brouncker's case they can be obtained by an asymptotic formula for $\{y_n\}_{n\geqslant 0}$. This formula follows easily from elementary identities known to Euler:

$$\begin{aligned}
\sum_{j>k}\frac{1}{j^2} &= \frac{1}{k}-\sum_{j>k}\left\{\frac{1}{j(j-1)}-\frac{1}{j^2}\right\} = \frac{1}{k}-\sum_{j>k}\frac{1}{j^2(j-1)} \\
&= \frac{1}{k}-\frac{1}{2k(k+1)}+\frac{1}{2}\sum_{j>k}\left\{\frac{1}{j-1}+\frac{1}{j+1}-\frac{2}{j}\right\}-\sum_{j>k}\frac{1}{j^2(j-1)} \\
&= \frac{1}{k}-\frac{1}{2k(k+1)}+\sum_{j>k}\left\{\frac{1}{j(j^2-1)}-\frac{1}{j^2(j-1)}\right\} \\
&= \frac{1}{k}-\frac{1}{2k(k+1)}-\sum_{j>k}\frac{1}{j^2(j^2-1)} = \frac{1}{k}-\frac{1}{2k^2}+O\left(\frac{1}{k^3}\right), \qquad (4.16)
\end{aligned}$$

and

$$\log(1-x) = -\int_0^x \frac{dt}{1-t} = -x+O\left(x^2\right), \qquad x\to 0\,,$$

Lemma 4.6 *The sequence y_n satisfies*

$$y_n = n+\frac{1}{2}+\frac{1}{8n}+O\left(\frac{1}{n^2}\right).$$

Proof Formulas (3.13), (4.14) and (4.16) imply

$$\log x_{2k} = \sum_{j>k}\log\left(1-\frac{1}{4j^2}\right) = -\frac{1}{4k}+\frac{1}{8k^2}+O\left(\frac{1}{k^3}\right).$$

It follows that

$$x_{2k} = \exp\log x_{2k} = 1-\frac{1}{2(2k)}+\frac{5}{8(2k)^2}+O\left(\frac{1}{k^3}\right)$$

and therefore

$$x_{2k+1} = \frac{2k+1}{2k+2}\,\frac{1}{x_{2k}} = 1-\frac{1}{2(2k+1)}+\frac{5}{8(2k+1)^2}+O\left(\frac{1}{k^3}\right).$$

Hence

$$y_n = (n+1)\left\{1-\frac{1}{2n}+\frac{5}{8n^2}+O\left(\frac{1}{n^3}\right)\right\} = n+\frac{1}{2}+\frac{1}{8n}+O\left(\frac{1}{n^2}\right)$$

as stated. □

Now let $\{z_n\}_{n\geqslant 0}$ be defined by

$$y_0 = \frac{1}{z_0}\,, \qquad y_n = n+\frac{1}{z_n}\,, \quad n=1,2,\ldots \qquad (4.17)$$

Lemma 4.7 (Euler 1750b) *The sequence* $\{z_n\}_{n\geqslant 0}$ *in* (4.17) *satisfies*

$$z_n = 1 + \frac{1}{1/(n+1) + z_{n+1} - 1}\,, \qquad n = 0, 1, 2, \dots$$

Proof By (4.15)

$$\begin{aligned} z_n = \frac{1}{y_n - n} &= \frac{y_{n+1}}{(n+1)^2 - n y_{n+1}} = \frac{(n+1) + 1/z_{n+1}}{(n+1) - n/z_{n+1}} \\ &= \frac{(n+1)z_{n+1} + 1}{(n+1)z_{n+1} - n} = 1 + \frac{1}{-n/(n+1) + z_{n+1}}\,, \end{aligned}$$

which proves the lemma. □

Iterating Lemma 4.7, we obtain

$$\begin{aligned} \frac{2}{\pi} = y_0 = \frac{1}{z_0} &= \frac{1}{1} \mathbin{+} \frac{1}{1 + z_1 - 1} = \frac{1}{1} \mathbin{+} \frac{1}{1} \mathbin{+} \frac{1}{1/2 + z_2 - 1} \\ &= \frac{1}{1} \mathbin{+} \frac{1}{1} \mathbin{+} \frac{1}{1/2} \mathbin{+} \frac{1}{1/3 + z_3 - 1} = \frac{1}{1} \mathbin{+} \frac{1}{1} \mathbin{+} \frac{1}{1/2} \mathbin{+} \cdots \mathbin{+} \frac{1}{1/n + z_n - 1} \\ &= \frac{1}{1} \mathbin{+} \frac{1}{1} \mathbin{+} \frac{1\times 2}{1} \mathbin{+} \frac{2\times 3}{1} \mathbin{+} \frac{3\times 4}{1} \mathbin{+} \cdots \mathbin{+} \frac{(n-1)n}{1 + n(z_n - 1)}\,. \end{aligned}$$

By Corollary 3.10 the continued fraction in the above formula converges. Since $z_n - 1 > 0$ for large n ($\lim_n z_n = 2$ by Lemma 4.6) it converges to $2/\pi$ by Markoff's test (Theorem 3.2). In fact, one can show that $z_n - 1 > 0$ for all n. It follows that

$$\frac{\pi}{2} = 1 + \frac{1}{1} \mathbin{+} \frac{1\times 2}{1} \mathbin{+} \frac{2\times 3}{1} \mathbin{+} \frac{3\times 4}{1} \mathbin{+} \cdots \mathbin{+} \frac{(n-1)n}{1} \mathbin{+} \cdots\,. \tag{4.18}$$

Let us consider another sequence $\{z_n\}_{n\geqslant 0}$:

$$y_0 = \frac{1}{z_0}\,, \qquad y_n = n + \frac{1}{2} + \frac{1}{z_n}\,, \qquad n = 1, 2, \dots \tag{4.19}$$

Then $z_n = 8n + O(1)$ by Lemma 4.6.

Lemma 4.8 (Euler 1750b) *The sequence* $\{z_n\}_{n\geqslant 0}$ *in* (4.19) *satisfies*

$$z_n = 4n + 6 + \frac{16(n+1)^2}{z_{n+1} - (4n+2)}\,, \qquad n = 0, 1, 2, \dots$$

Proof By (4.15),

$$
\begin{aligned}
z_n &= \frac{1}{y_n-(n+1/2)} = \frac{y_{n+1}}{(n+1)^2-(n+1/2)y_{n+1}} \\
&= \frac{(n+3/2)+1/z_{n+1}}{(n+1)^2-(n+1-1/2)(n+1+1/2)-(n+1/2)/z_{n+1}} \\
&= \frac{(n+3/2)z_{n+1}+1}{z_{n+1}/4-(n+1/2)} \\
&= \frac{(4n+6)z_{n+1}+4}{z_{n+1}-(4n+2)} = 4n+6+\frac{4+(4n+2)(4n+6)}{z_{n+1}-(4n+2)}\,.
\end{aligned}
$$

Observing that $4+(4n+2)(4n+6)=4+(4n+4)^2-4=16(n+1)^2$, we obtain the required formula. □

Iterating Lemma 4.8, we obtain

$$
\begin{aligned}
\frac{2}{\pi} = y_0 &= \frac{1}{2}+\frac{1}{z_0} = \frac{1}{2}+\frac{1}{6\,+}\,\frac{16}{z_1-2} = \frac{1}{2}+\frac{1}{6\,+}\,\frac{16\times 1^2}{8\quad+}\,\frac{16\times 2^2}{z_2-6} \\
&= \frac{1}{2}+\frac{1}{6\,+}\,\frac{16\times 1^2}{8\quad+}\,\frac{16\times 2^2}{8\quad+}\,\frac{16\times 3^2}{z_3-10} \\
&= \frac{1}{2}+\frac{1}{6\,+}\,\frac{16\times 1^2}{8\quad+}\,\frac{16\times 2^2}{8\quad+\cdots+}\,\frac{16\times n^2}{z_n-(4n-2)} \\
&= \frac{1}{2}+\frac{1}{6\,+}\,\frac{4\times 1^2}{2\quad+}\,\frac{2^2}{2\,+}\,\frac{3^2}{2\,+}\,\frac{4^2}{2\,+\cdots+}\,\frac{n^2}{(z_n-(4n-2))/4}\,.
\end{aligned}
$$

Since

$$
\frac{z_n-(4n-2)}{4}-2 = \frac{z_n-4n-6}{4} > 0
$$

for sufficiently large n (notice that $z_n = 8n+O(1)$), Markoff's test implies the interesting formula

$$
\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{2}\right) = 2\,\frac{\pi-3}{4-\pi} \tag{4.20}
$$

relating the proportion determined by π in $[3,4]$ to a simple continued fraction. Compare this formula with (3.25). We return to this topic later; see Theorem 4.27. An extension of this method can be found in Exercise 4.12. See also Exercise 4.16.

72 The equation $y(s)y(s+1)=(s+1)^2$. We now combine Brouncker's and Euler's methods (see §**61** in Section 3.2 and §**71** above) to obtain analytic formulas for a solution to this equation. If $s \leqslant y(s)$ then $y(s) = (s+1)^2/y(s+1) \leqslant s+1$. Hence

$$
\begin{aligned}
y(s) &= \frac{(s+1)^2}{(s+2)^2} y(s+2) = \frac{(s+1)^2(s+3)^2\cdots(s+2n-1)^2}{(s+2)^2(s+4)^2\cdots(s+2n)^2}\, y(s+2n) \\
&= (s+1)\frac{(s+1)(s+3)}{(s+2)^2}\cdots\frac{(s+2n-1)(s+2n+1)}{(s+2n)^2}\,\frac{y(s+2n)}{s+2n+1} \\
&\to (s+1)\prod_{n=1}^{\infty}\frac{(s+2n-1)(s+2n+1)}{(s+2n)^2} = y(s)\,.
\end{aligned}
$$

Applying formula (3.59) and Ex. 3.13, we obtain that

$$
\begin{aligned}
2\frac{\Gamma^2(s/2+1)}{\Gamma^2((s+1)/2)} &= \frac{(s+1)^2}{2}\, e^{\gamma}\prod_{j=1}^{\infty}\left\{\frac{(s+2j+1)^2}{(s+2j)^2}\, e^{-1/j}\right\} \\
&= \frac{s+1}{2}\lim_{n\to\infty}(2n+1+s)e^{\gamma-\sum_{j=1}^{n}1/j}\prod_{j=1}^{n}\frac{(s+2j-1)(s+2j+1)}{(s+2j)^2} \\
&= (s+1)\prod_{j=1}^{\infty}\frac{(s+2j-1)(s+2j+1)}{(s+2j)^2} = y(s)\,. \qquad (4.21)
\end{aligned}
$$

Elementary calculations using (3.65) show that

$$
2\,\frac{\Gamma^2(s+1/2+1/2)}{\Gamma^2((s+1)/2)} = s+\frac{1}{2}+\frac{1}{8s}+O\left(\frac{1}{s^2}\right)
$$

(compare this with Lemma 4.6). Therefore the only solution $y(s)$ satisfying $s \leqslant y(s)$ for all sufficiently large s to the functional equation is given by the infinite product obtained above. If $z(s) = 1/(y(s)-s)$ then

$$
z(s) = 1+\frac{1}{1/(s+1)+z(s+1)-1}\,.
$$

Iterations result in the continued fraction

$$
y(s) = s+\frac{1}{1}{\,+\,}\frac{(s+1)}{1}{\,+\,}\frac{(s+1)(s+2)}{1}{\,+\,}\frac{(s+2)(s+3)}{1}{\,+\cdots}\,,
$$

which coincides with (4.18) when $s=1$. Similarly,

$$
y(s) = s+\frac{1}{2}+\frac{1}{4s+6}{\,+\,}\frac{4(s+1)^2}{2}{\,+\,}\frac{(s+2)^2}{2}{\,+\,}\frac{(s+3)^2}{2}{\,+\cdots}\,.
$$

The function $y(s)$ is closely related to Brouncker's function $b(s)$, as the following formula shows (see Theorem 3.25):

$$
y(s) = \frac{2s^2}{b(2s-1)}\,, \qquad s > 1/2\,. \qquad (4.22)
$$

We discuss another continued fraction for $y(s)$ in §**95** at the start of Section 4.9.

4.3 An extension of Wallis' formula

73 Euler's form of Wallis' product. *Arithmetica Infinitorum* by Wallis was well known to Euler. It was on his reading list when he studied mathematics under the supervision of Johann Bernoulli in Basel; see Calinger (1966). Moreover, A. P. Yushkevich (see Kramer 1961, p. 84), found in the Archive of the Academy of Sciences of the USSR Euler's copy of this book, which contained numerous annotations by Euler. Therefore the incomplete proof of Brouncker's formula given in §191 of Wallis (1656) could not have escaped Euler's attention. Euler undertook tremendous efforts to recover Brouncker's original proof, see Euler (1750b, §§17–20). He failed to do this but instead made many important discoveries. The first, very natural, idea was to extend Wallis' formula to quotients of integrals of the $I(p,q)$ type. This could give him some freedom in the manipulations with parameters, in the style of §**71** above, to obtain finally Brouncker's formula (3.39). This program was realized in Euler's dissertation (1750a), which was immediately followed by (1750b), where in §20 Euler refers to (1750a). The basic ideas had appeared already, in the third letter of Euler to Goldbach (8 January 1730), **OO0717**,[1] in which Euler obtained his formula (3.45) for $\Gamma(x)$.

Since he was interested in extensions of (3.14) Euler used the integrals

$$\int_0^1 \frac{x^m\,dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} \sin^m\theta\,d\theta$$

in the algebraic, not the trigonometric, form. Euler's approach leads to interesting conclusions. To get some guidance on which quotients to consider Euler first observed that, for every nonnegative integer α, the integrals in (3.11) satisfy

$$I(\alpha) \stackrel{\text{def}}{=} \int_0^1 \frac{x^{2\alpha}}{\sqrt{1-x^2}}\,dx \int_0^1 \frac{x^{2\alpha+1}}{\sqrt{1-x^2}}\,dx = \frac{1}{2\alpha+1}\,\frac{\pi}{2}. \tag{4.23}$$

We prove now a lemma from Euler (1750b, 1768, Chapter VIII, §332), which Euler left without a proof.

Lemma 4.9 *Formula* (4.23) *holds for all α satisfying $\alpha > -1/2$.*

Proof If $m > 0$ then integration by parts,

$$\int_0^1 \frac{x^{m+1}}{\sqrt{1-x^2}}\,dx = -\int_0^1 x^m\,d\sqrt{1-x^2} = m\int_0^1 (1-x^2)\frac{x^{m-1}\,dx}{\sqrt{1-x^2}}$$

results in the formula

$$\int_0^1 \frac{x^{m+1}}{\sqrt{1-x^2}}\,dx = \frac{m}{m+1}\int_0^1 \frac{x^{m-1}}{\sqrt{1-x^2}}\,dx\,.$$

[1] This number gives the index in the Euler Archive www.eulerarchive.com.

Applying this formula for $m = 2\alpha+1$ and $m = 2\alpha+2$, we obtain

$$\begin{aligned}(2\alpha+3)I(\alpha+1) &= (2\alpha+3)\int_0^1 \frac{x^{2\alpha+2}}{\sqrt{1-x^2}}\,dx \int_0^1 \frac{x^{2\alpha+3}}{\sqrt{1-x^2}}\,dx \\ &= (2\alpha+3)\frac{2\alpha+1}{2\alpha+2}\int_0^1 \frac{x^{2\alpha}}{\sqrt{1-x^2}}\,dx\ \frac{2\alpha+2}{2\alpha+3}\int_0^1 \frac{x^{2\alpha+1}}{\sqrt{1-x^2}}\,dx \\ &= (2\alpha+1)I(\alpha)\,.\end{aligned}$$

Hence $\alpha \to (2\alpha+1)I(\alpha)$ is periodic on $(-1/2, +\infty)$ with period 1. Let $n = [\alpha]$ be the greatest integer not exceeding α. Since $x^{2n} \geqslant x^{2\alpha} > x^{2n+2}$ and $x^{2n+1} \geqslant x^{2\alpha+1} > x^{2n+3}$ if $0 < x < 1$,

$$\frac{2n+3}{2n+1}\,\frac{\pi}{2} \geqslant (2\alpha+1)I(\alpha) > \frac{2n+1}{2n+3}\,\frac{\pi}{2},$$

by (4.23). It follows that $\lim_{\alpha\to+\infty}(2\alpha+1)I(\alpha) = \pi/2$. Hence the periodic function $\alpha \to (2\alpha+1)I(\alpha)$ must be constant. □

In these terms Wallis' result looks as follows

$$\begin{gathered}\frac{\int_0^1 x^{2n}(1-x^2)^{-1/2}dx}{\int_0^1 x^{2n+1}(1-x^2)^{-1/2}dx} = \left(\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2\times 4\times 6\times\cdots\times 2n}\right)^2 \frac{\pi}{2}(2n+1)\,, \\ \int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}}\int_0^1 \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{1}{2n+1}\,\frac{\pi}{2}\,.\end{gathered} \tag{4.24}$$

Euler's form of Wallis' formula has an interesting application, indicated by Euler in his letter to Stirling (July 27, 1738).

Corollary 4.10

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\cdots = \frac{\pi^2}{8}\,. \tag{4.25}$$

Proof To get (4.25) Euler multiplied (4.11) by $(1-x^2)^{-1/2}$ and integrated:

$$\int_0^x \frac{\arcsin t\,dt}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty}\frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)(2n+1)}\int_0^x \frac{t^{2n+1}\,dt}{\sqrt{1-t^2}}\,. \tag{4.26}$$

By (4.24),

$$\int_0^1 \frac{t^{2n+1}\,dt}{\sqrt{1-t^2}} = \frac{2\times 4\times 6\times\cdots\times(2n)}{3\times 5\times 7\times\cdots\times(2n+1)}\,,$$

which shows that the series in (4.26) converges uniformly on $[0, 1]$. Passing to the limit $(x \to 1-0)$ in (4.26) and calculating the integral, we obtain (4.25). □

The change of variables $x := r^{n/2}$ in (4.23) followed by substitution of α with $1/n - 1/2 > -1/2$ transforms (4.23) into

$$\int_0^1 \frac{dr}{\sqrt{1-r^n}} \int_0^1 \frac{r^{n/2}dr}{\sqrt{1-r^n}} = \frac{\pi}{n} . \tag{4.27}$$

74 Euler's products and the lemniscate of Bernoulli (see Fig. 4.1). The following lemma segregates a class of integrals suitable for Wallis' interpolation. On the way it generalizes another of Wallis' formulas (see Zeuthen 1903)[2]

$$\int_0^1 \sqrt{x^3}(1-x)^{n-1}\,dx = \frac{2n+5}{2n}\int_0^1 \sqrt{x^3}(1-x)^n\,dx .$$

Lemma 4.11 (Euler 1768, Chapter IX, §360) *For positive m, n, k*

$$\int_0^1 x^{m-1}(1-x^n)^{k/n-1}\,dx = \frac{m+k}{m}\int_0^1 x^{m+n-1}(1-x^n)^{k/n-1}\,dx .$$

Proof The formula of the lemma follows from the identity

$$\int_0^1 x^{m+n-1}(1-x^n)^{k/n-1}\,dx = -\frac{1}{k}\int_0^1 x^m\,d(1-x^n)^{k/n}$$
$$= \frac{m}{k}\int_0^1 x^{m-1}(1-x^n)^{k/n-1}\,dx - \frac{m}{k}\int_0^1 x^{m+n-1}(1-x^n)^{k/n-1}\,dx .$$

□

Theorem 4.12 (Euler 1750a) *For positive m, μ, n and k*

$$\frac{\int_0^1 x^{m-1}(1-x^n)^{(k-n)/n}\,dx}{\int_0^1 x^{\mu-1}(1-x^n)^{(k-n)/n}\,dx} = \prod_{j=0}^{\infty}\frac{(\mu+jn)(m+k+jn)}{(\mu+k+jn)(m+jn)} . \tag{4.28}$$

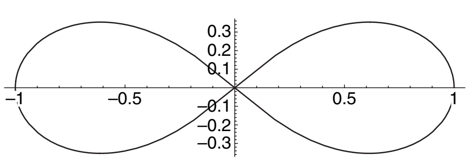

Fig. 4.1. Lemniscate of Bernoulli.

[2] Wallis' and Euler's formulas are related by the change of variables $x := 1-x$

Proof The proof of this theorem follows the proof of Wallis' formula. Iterating Lemma 4.11, we obtain in s steps that

$$\int_0^1 \frac{x^{m-1}\,dx}{(1-x^n)^{1-k/n}}\,dx = u_s \prod_{j=0}^{s-1} \frac{m+k+jn}{m+jn},$$

$$\int_0^1 \frac{x^{\mu-1}\,dx}{(1-x^n)^{1-k/n}} = v_s \prod_{j=0}^{s-1} \frac{\mu+k+jn}{\mu+jn},$$

where

$$u_s = \int_0^1 \frac{x^{m+sn-1}\,dx}{(1-x^n)^{1-k/n}} \quad \text{and} \quad v_s = \int_0^1 \frac{x^{\mu+sn-1}\,dx}{(1-x^n)^{1-k/n}}.$$

If $\mu = m$ then there is nothing to prove. Since μ and m enter the formulas symmetrically, we may assume that $\mu < m$. Euler's trick used in the proof of Wallis' formula shows that $v_s > u_s > v_{s+[(m-\mu)/n]+1}$. It follows that

$$\frac{v_s}{u_s} > 1 > \frac{v_s}{u_s}\,\frac{v_{s+[(m-\mu)/n]+1}}{v_s} = \frac{v_s}{u_s} w_s,$$

where

$$w_s = \prod_{j=s}^{s+[(m-\mu)/n]+1} \frac{\mu+jn}{\mu+k+jn} \to 1$$

as $s \to +\infty$. Hence $0 < v_s/u_s - 1 < 1/w_s - 1 = (1-w_s)/w_s \to 0$. □

Notice that Euler's theorem gives a formula for any convergent product

$$\prod_{j=0}^{\infty} \frac{(a+jn)(b+jn)}{(c+jn)(d+jn)}$$

with positive a, b, c, d. Indeed, the identity

$$\frac{(a+jn)(b+jn)}{(c+jn)(d+jn)} = 1 + \frac{jn(a+b-c-d)+ab-cd}{(c+jn)(d+jn)}$$

shows that the product converges if and only if $a+b = c+d$, i.e. $c-a = b-d = k$. If $k = 0$ then each multiplier is 1. The case $k < 0$ reduces to $k > 0$ by taking reciprocals. Hence $c = a+k$, $b = d+k$. To find the value of the product it remains to put $\mu = a$, $m = d$ in (4.28).

Remark We are obtaining these formulas in the present analysis with the help of the gamma function, see Ex. 3.17. In fact they played a crucial role in Euler's discovery of the gamma function. The change of variables $t = x^n$ on the left-hand side of (4.28) shows that it is the quotient of two beta functions:

$$\frac{\int_0^1 x^{m-1}(1-x^n)^{(k-n)/n}\,dx}{\int_0^1 x^{\mu-1}(1-x^n)^{(k-n)/n}\,dx} = \frac{B(m/n, k/n)}{B(\mu/n, k/n)}. \tag{4.29}$$

Here the beta function $B(p,q)$ is defined by

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}; \tag{4.30}$$

see Ex. 4.18 or Andrews, Askey and Roy (1999, Theorem 1.1.4) for another proof.

The correspondence between the pairs of integrals and the products established in (4.28) is not one-to-one. For instance, putting in (4.28) first $\mu=p$, $m=p+r$, $k=2q$, $n=2r$ and then $\mu=p$, $m=p+2q$, $k=r$, $n=2r$, we obtain

$$\frac{\int_0^1 x^{p+r-1}(1-x^{2r})^{q/r-1}\,dx}{\int_0^1 x^{p-1}(1-x^{2r})^{q/r-1}\,dx}=\prod_{j=0}^{\infty}\frac{(p+2jr)(p+2q+r+2jr)}{(p+2q+2jr)(p+r+2jr)}\,, \tag{4.31}$$

$$\frac{\int_0^1 x^{p+2q-1}(1-x^{2r})^{-1/2}\,dx}{\int_0^1 x^{p-1}(1-x^{2r})^{-1/2}\,dx}=\prod_{j=0}^{\infty}\frac{(p+2jr)(p+2q+r+2jr)}{(p+2q+2jr)(p+r+2jr)}\,. \tag{4.32}$$

This immediately implies an important corollary due to Euler.

Corollary 4.13 *For every positive p, q, r*

$$\frac{\int_0^1 x^{p+r-1}(1-x^{2r})^{q/r-1}\,dx}{\int_0^1 x^{p-1}(1-x^{2r})^{q/r-1}\,dx}=\frac{\int_0^1 x^{p+2q-1}(1-x^{2r})^{-1/2}\,dx}{\int_0^1 x^{p-1}(1-x^{2r})^{-1/2}\,dx}\,.$$

Formula (4.32) with $p=1$, $q=1/2$, $r=1$ is Wallis' formula

$$\frac{2}{\pi}=\frac{\int_0^1 x(1-x^2)^{-1/2}dx}{\int_0^1(1-x^2)^{-1/2}dx}=\frac{1\times 3}{2\times 2}\,\frac{3\times 5}{4\times 4}\,\frac{5\times 7}{6\times 6}\,\frac{7\times 9}{8\times 8}\cdots \tag{4.33}$$

If $p=1$, $q=1$, $r=2$ in (4.32), then

$$\frac{\int_0^1 x^2(1-x^4)^{-1/2}dx}{\int_0^1(1-x^4)^{-1/2}dx}=\frac{1\times 5}{3\times 3}\,\frac{5\times 9}{7\times 7}\,\frac{9\times 13}{11\times 11}\,\frac{13\times 17}{15\times 15}\cdots \tag{4.34}$$

is Euler's infinite product for the *moment of inertia* of the unit mass uniformly distributed along the lemniscate of Bernoulli. With $n=4$, formula (4.27) turns into another of Euler's formulas, see Euler (1750b, 1768, p. 185),

$$\int_0^1\frac{dx}{\sqrt{1-x^4}}\int_0^1\frac{x^2dx}{\sqrt{1-x^4}}=\frac{\pi}{4}, \tag{4.35}$$

called *the lemniscate identity* (McKean and Moll 1999, p. 69). More generally, for $p=1$, $q=n/4$, $r=n/2$ in (4.32) we have

$$\frac{\int_0^1 x^{n/2}(1-x^n)^{-1/2}dx}{\int_0^1(1-x^n)^{-1/2}dx}=\prod_{j=0}^{\infty}\frac{(1+jn)(1+(j+1)n)}{(1+n/2+jn)^2}\,. \tag{4.36}$$

4.4 Wallis' formula for sinusoidal spirals

75 Motivation. Formulas (4.33), (4.34) and (4.36) show that the integrals in their denominators are the lengths of certain curves. Given $n > 0$ let us find a polar curve $r = r(\theta)$ whose element of length ds satisfies

$$ds = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\, dr = \frac{dr}{\sqrt{1-r^n}}\,.$$

This differential equation in θ can easily be integrated,

$$\theta = \pm\int \frac{r^{n/2-1}\,dr}{\sqrt{1-r^n}} = \pm\frac{2}{n}\int \frac{dr^{n/2}}{\sqrt{1-r^n}} = \pm\frac{2}{n}\arcsin\left(r^{n/2}\right) + C\,,$$

to obtain the equation of the corresponding polar curve

$$r^{n/2} = \cos\left(\pm\frac{n\theta}{2} + A\right)\,, \tag{4.37}$$

where the constant A is responsible for rotations and $\pm$ indicates the symmetry with respect to the real axis. With $A = 0$, n rational and the sign is $+$, then the curve (4.37) is known as the *sinusoidal spiral*. For rational values of n it was first studied by Colin Maclaurin in 1718.

76 Classical examples. If $n = 1$, $A = 0$ then the curve is the cardioid, see Fig. 4.2, $\sqrt{r} = \cos(\theta/2)$ or equivalently $2r = 1 + \cos\theta$.

If $n = 2$, $A = \pi/2$ and the sign is $-$, then this curve is the circle $r = \sin\theta$ inscribed in the witch of Agnesi; see Fig. 3.1.

If $n = 4$, $A = 0$ and the sign is $+$, then we obtain the equation $r^2 = \cos 2\theta$ of the *lemniscate of Bernoulli* presented in Fig. 4.1. It was studied in 1694 by Jakob Bernoulli and can also be defined by the algebraic equation $(x^2+y^2)^2 = x^2 - y^2$.

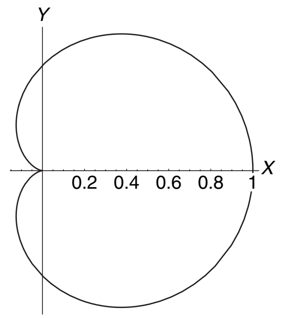

Fig. 4.2. The cardioid, found by setting $n = 1$ and $A = 0$ in (4.37), which becomes $2r = 1 + \cos\theta$.

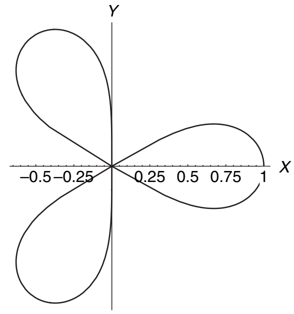

Fig. 4.3. The 3-lemniscate (three-pole lemniscate).

If $n = 6$, $A = 0$ and the sign is $+$, then $r^3 = \cos 3\theta$ and we obtain the 3-lemniscate, see Fig 4.3. If $n = 6$, $A = \pi/2$ and the sign is $-$, it is called *Kiepert's curve* or the *three-pole lemniscate*. It looks very similar to the trifolium $r = \cos 3\theta$.

Since all these curves belong to one family of sinusoidal spirals (4.37), one may expect that their lengths can be expressed as infinite products similar to (3.10). Since n will definitely enter these formulas, the choice of n must be rational. In particular, for a cardioid ($n = 1$ in (4.37)),

$$\frac{\int_0^1 x^{1/2}(1-x)^{-1/2}dx}{\int_0^1 (1-x)^{-1/2}dx} = \frac{2\times 4}{3^2}\,\frac{4\times 6}{5^2}\,\frac{6\times 8}{7^2}\cdots$$

$$= \frac{1}{2}\,\frac{2^2}{1\times 3}\,\frac{4^2}{3\times 5}\cdots = \frac{\pi}{4}\,, \tag{4.38}$$

see (3.10). For the 1/4-lemniscate, see Fig 4.4, ($n = 1/2$ in (4.37)),

$$\frac{9\pi}{32} = \frac{\int_0^1 x^{1/4}(1-x^{-1/2})^{-1/2}dx}{\int_0^1 (1-x^{-1/2})^{-1/2}dx} = \frac{4\times 6}{5^2}\,\frac{6\times 8}{7^2}\,\frac{8\times 10}{9^2}\,\frac{10\times 12}{11^2}\cdots,$$

which again resembles (3.10). This has a simple explanation.

Lemma 4.14 *Let $n = 1/m$ for a positive integer m. Then*

$$\frac{\int_0^1 x^{n/2}(1-x^n)^{-1/2}dx}{\int_0^1 (1-x^n)^{-1/2}dx} = \frac{1\times 3}{2^2}\,\frac{3\times 5}{4^2}\cdots\frac{(2m-1)(2m+1)}{(2m)^2}\,\frac{2m}{2m+1}\,\frac{\pi}{2}$$

$$= \frac{(2m)(2m+2)}{(2m+1)^2}\,\frac{(2m+2)(2m+4)}{(2m+3)^2}\,\frac{(2m+4)(2m+6)}{(2m+5)^2}\cdots$$

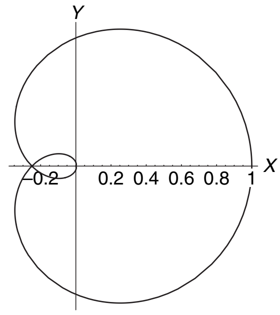

Fig. 4.4. The 1/4-lemniscate.

Proof The change of variables $x := x^{2m}$ and Lemma 4.11 show that

$$\int_0^1 \frac{x^{n/2}dx}{\sqrt{1-x^n}} = 2m\int_0^1 \frac{x^{2m}dx}{\sqrt{1-x^2}}\,,$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^n}} = 2m\int_0^1 \frac{x^{2m-1}dx}{\sqrt{1-x^2}} = \frac{2m+1}{2m}\int_0^1 \frac{x^{2m+1}dx}{\sqrt{1-x^2}}\,.$$

The proof is completed by the use of (4.24). □

77 Wallis-type formulas. Essentially new formulas arise from the sinusoidal spirals corresponding to integer values of n, $n > 2$. Identity (4.27) presents the lengths of sinusoidal spirals as infinite products.

Theorem 4.15 *For every r-lemniscate,*

$$\frac{1}{\int_0^1 (1-x^{2r})^{-1/2}dx} = \frac{1(2r+1)}{2(r+1)}\,\frac{3(4r+1)}{4(3r+1)}\,\frac{5(6r+1)}{6(5r+1)}\cdots \tag{4.39}$$

Proof If $n = 2r$ then by (4.27) and (4.32)

$$\int_0^1 \frac{dx}{\sqrt{1-x^{2r}}} = \frac{\pi/2r}{\int_0^1 x^r(1-x^{2r})^{-1/2}dx} = \frac{\int_0^1 x^{r-1}(1-x^{2r})^{-1/2}dx}{\int_0^1 x^r(1-x^{2r})^{-1/2}dx}$$

$$= \prod_{j=0}^{\infty} \frac{(r+1+2jr)(2r+2rj)}{(r+2jr)(2r+1+2jr)}\,. \tag{4.40}$$

□

For example, the lemniscate of Bernoulli ($r = 2$) consists of four equal arcs (Fig. 4.1) each of length

$$L_1 = \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \prod_{j=0}^{\infty} \frac{(3+4j)(4+4j)}{(2+4j)(5+4j)} = 1.311\,028\,777\ldots \tag{4.41}$$

To obtain Wallis' formula we put $r = 1$ in (4.39). For the cardioid ($r = 1/2$) it turns into the obvious identity

$$\frac{1}{2} = \frac{1\times 2}{1\times 3}\,\frac{3\times 3}{2\times 5}\,\frac{5\times 4}{3\times 7}\,\frac{7\times 5}{4\times 9}\,\frac{9\times 6}{5\times 11}\cdots,$$

and for the 1/4-lemniscate, Fig. 4.4, it gives the more interesting formula

$$\frac{3}{8} = \frac{1\times 3}{1\times 5}\,\frac{3\times 4}{2\times 7}\,\frac{5\times 5}{3\times 9}\,\frac{7\times 6}{4\times 11}\,\frac{9\times 7}{5\times 13}\,\frac{11\times 8}{6\times 15}\cdots$$

In general for $r = 1/2k$ with integer k

$$\frac{1(1+k)}{1(1+2k)}\,\frac{3(2+k)}{2(3+2k)}\,\frac{5(3+k)}{3(5+2k)}\,\frac{7(4+k)}{4(7+2k)}\cdots = \frac{(2k-1)!!}{(2k)!!}\,.$$

This follows from (4.30) with $p = 1/2r$, $q = 1/2$ and the functional equation for the gamma function.

4.5 An extension of Brouncker's formula

78 Brouncker's continued fractions for sinusoidal spirals. Assuming now that r is a positive integer, let us consider the length of the r-lemniscate as an infinite product and also as a quotient of two integrals, (4.40). These formulas resemble those for the unit circle and indicate a possibility of extensions to sinusoidal spirals. Applying (4.32) to

$$\begin{aligned} y(s) &= (q+s)\frac{\int_0^1 x^{r+s+q-1}(1-x^{2r})^{-1/2}\,dx}{\int_0^1 x^{r+s-q-1}(1-x^{2r})^{-1/2}\,dx} \\ &= (q+s)\prod_{j=0}^{\infty}\frac{(r+s-q+2jr)(2r+s+q+2jr)}{(r+s+q+2jr)(2r+s-q+2jr)}\,, \end{aligned} \tag{4.42}$$

we obtain

$$\begin{aligned} y(s) &= (q+s)\,\frac{(r+s-q)(2r+s+q)}{(r+s+q)(2r+s-q)}\,\frac{(3r+s-q)(4r+s+q)}{(3r+s+q)(4r+s-q)}\cdots \\ &= (q+s)\,\frac{r+s-q}{r+s+q}\left\{\frac{(r+(r+s)-q)(2r+(s+r)+q)}{(r+(r+s)+q)(2r+(r+s)-q)}\right\}^{-1}\cdots \\ &= (q+s)(s+r-q)\frac{1}{y(s+r)}\,. \end{aligned}$$

It follows that

$$y(s)y(s+r) = (s+q)(s+r-q)\,, \tag{4.43}$$

which coincides with (3.28) for $q=1$, $r=2$. Moreover,

$$s(s+r)=s^2+sr<s^2+sr+q(r-q)=(s+q)(s+r-q)\ ,$$

if $r>q$. Hence the basic principles of Brouncker's method are valid.

Having arrived at (4.43), for the time being we will forget the formula for y and assume that $y(s)$ is just a positive function on $(0,+\infty)$ satisfying (4.43) and $s<y(s)$ for $s>0$. Then

$$s<y(s)=\frac{(s+q)(s+r-q)}{y(s+r)}<\frac{(s+q)(s+r-q)}{s+r}=s+\frac{q(r-q)}{s+r}\ .$$

Keeping in mind Brouncker's experience for the circle (see §**60** in section 3.2), we may directly develop $y(s)$ into a continued fraction, skipping the intermediate step of asymptotic expansion. Let $y(s)=s+a_0/y_1(s)$. To find a_1 we substitute this expression in to (4.43) and obtain

$$\frac{a_0(s+r)}{y_1(s)}+\frac{a_0 s}{y_1(s+r)}+\frac{a_0^2}{y_1(s)y_1(s+r)}=q(r-q)\ .$$

If $y_1(s)\sim 2s$ as $s\to+\infty$ then the equation shows that $a_0=q(r-q)$. We also obtain the functional equation for y_1

$$y_1(s)y_1(s+r)=sy_1(s)+(s+r)y_1(s+r)+q(r-q)\ .$$

Repeating the above arguments with $y_1(s)=2s+a_1/y_2(s)$, we find that

$$a_1=2r^2+q(r-q)=(r+q)(2r-q)\ ,$$

$$y_2(s)y_2(s+r)=(s-r)y_2(s)+(s+2r)y_2(s+r)+(r+q)(2r-q)\ .$$

Similarly

$$y_k(s)=2s+\frac{a_k}{y_{k+1}(s)},\qquad a_k=(kr+q)(kr+r-q)\ ,$$

$$\begin{aligned}y_k(s)y_k(s+r)=(s-kr+r)y_k(s)+(s+kr)y_k(s+r)\\ +(kr-r+q)(kr-q)\ .\end{aligned}$$

It follows that for every k we have

$$y(s)=s+\frac{q(r-q)}{2s}\underset{+}{}\frac{(r+q)(2r-q)}{2s}\underset{+\cdots+}{}\frac{(kr-r+q)(kr-q)}{y_k(s)}\ . \tag{4.44}$$

Leaving the proof to Ex. 4.20 (compare the above calculations with §§38–40 of Euler 1750b), we deduce from (4.44) Euler's theorem.

Theorem 4.16 (Euler 1750, §47) *For $s,q>0$ and $r>q$*

$$s+\mathop{\mathbf{K}}_{k=0}^{\infty}\left(\frac{(kr+q)(kr+r-q)}{2s}\right)=(q+s)\frac{\int_0^1 x^{r+s+q-1}(1-x^{2r})^{-1/2}\,dx}{\int_0^1 x^{r+s-q-1}(1-x^{2r})^{-1/2}\,dx}\ . \tag{4.45}$$

If $r = 2$, $q = 1$ in Theorem 4.16 then

$$\boxed{s + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2s}\right) = (s+1)\frac{\int_0^1 x^{s+2}(1-x^4)^{-1/2}\,dx}{\int_0^1 x^s(1-x^4)^{-1/2}\,dx}\,.} \tag{4.46}$$

Notice that (4.46) follows from Theorem 3.25 by (4.30). To see this just make the substitution $x := x^{1/4}$ in (4.46).

Euler's theorem for $s = q = 1/2$ combined with (4.40) implies an extension of Brouncker's formula for the r-lemniscate,

$$\frac{2}{\int_0^1 (1-x^{2r})^{-1/2}dx} = 1 + \mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{(2(k-1)r+1)(2kr-1)}{2}\right)\,. \tag{4.47}$$

For $r = 1$ we obtain Brouncker's formula, for $r = 2$ the formula for the lemniscate of Bernoulli,

$$\begin{aligned}\frac{2}{\int_0^1 (1-x^4)^{-1/2}dx} &= 1 + \frac{1\times 3}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{5\times 7}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{9\times 11}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{13\times 15}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \cdots \\ &= 1 + \frac{2^2-1}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{6^2-1}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{10^2-1}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{14^2-1}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \cdots\end{aligned}$$

and for Kiepert's curve (Fig. 4.3, $r = 3$) we have

$$\begin{aligned}\frac{2}{\int_0^1 (1-x^6)^{-1/2}dx} &= 1 + \frac{1\times 5}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{7\times 11}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{13\times 17}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{19\times 23}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \cdots \\ &= 1 + \frac{3^2-2^2}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{9^2-2^2}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{15^2-2^2}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \frac{21^2-2^2}{2} \mathbin{\genfrac{}{}{0pt}{}{}{+}} \cdots\,.\end{aligned}$$

If $y(s, r, q)$ is the continued fraction in (4.45) and the continued fraction $y(s, r)$ is defined as $2y(s/2, r, 1/2)$ then equivalence transforms show that

$$y(s, r) = s + \mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{(2(k-1)r+1)(2kr-1)}{2s}\right)\,. \tag{4.48}$$

It follows from (4.43) that

$$y(s, r)y(s+2r, r) = (s+1)(s+2r-1)\,. \tag{4.49}$$

We can obtain a generalization of Brouncker's theorem as follows.

Theorem 4.17 *If $y(s)$ is a positive continuous function on $(0,+\infty)$ such that $s < y(s)$, $y(s)y(s+2r) = (s+1)(s+2r-1)$, where $1/2 < r$, then*

$$y(s) = (s+1)\prod_{k=0}^{\infty}\frac{(s+2r-1+4kr)(s+4r+1+4kr)}{(s+2r+1+4kr)(s+4r-1+4kr)}$$
$$= s + \mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{(2k-1)^2r^2-(r-1)^2}{2s}\right)$$
$$= (s+1)\frac{\int_0^1 x^{r+s/2-1/2}(1-x^{2r})^{-1/2}\,dx}{\int_0^1 x^{r+s/2-3/2}(1-x^{2r})^{-1/2}\,dx}\,.$$

This reduces to Brouncker's theorem for $r=1$. See Ex. 4.22 for a generalization of Ramanujan's formula.

4.6 On the formation of continued fractions

79 Euler's form of Euclid's method. In (1782) Euler considered a generalization of (1.11):

$$\begin{aligned} c_0x_0 &= b_0x_1 + a_1x_2\,,\\ c_1x_1 &= b_1x_2 + a_2x_3\,,\\ c_2x_2 &= b_2x_3 + a_3x_4\,,\\ &\vdots\\ c_nx_n &= b_nx_{n+1} + a_{n+1}x_{n+2}\,,\\ &\vdots \end{aligned} \tag{4.50}$$

All sequences $\{x_n\}$ satisfying (4.50) are assumed to be nonzero. Then

$$\frac{c_0x_0}{x_1} = b_0 + \frac{a_1c_1}{b_1} + \frac{a_2c_2}{b_2} + \cdots + \frac{a_nc_n}{b_n + a_{n+1}x_{n+2}/x_{n+1}}\,. \tag{4.51}$$

80 The evolution equation $s = x^n(\alpha-\beta x-\gamma x^2)$. For this equation, $s(x)=0$ at $x=0$ and $x=\xi=(\sqrt{\beta^2+4\alpha\gamma}-\beta)/2\gamma > 0$. We assume that α, β, γ are positive. Differentiating the evolution equation, we obtain

$$ds = n\alpha x^{n-1}dx - (n+1)\beta x^n dx - (n+2)\gamma x^{n+1}dx\,,$$

which implies after integration that

$$n\alpha\int_0^{\xi} x^{n-1}dx = (n+1)\beta\int_0^{\xi} x^n dx + (n+2)\gamma\int_0^{\xi} x^n dx\,.$$

Let

$$x_n = \int_0^{\xi} x^n\,dx = \frac{\xi^{n+1}}{n+1}\,,$$

$$c_n = (n+1)\alpha\,, \qquad b_n = (n+2)\beta\,, \qquad a_n = (n+2)\gamma\,,$$

in (4.50). Then by Markoff's test (Theorem 3.2) and (4.51)

$$\frac{2\alpha}{\xi} = 2\beta + \frac{2\times 3\alpha\gamma}{3\beta}\,{}_{+}\,\frac{3\times 4\alpha\gamma}{4\beta}\,{}_{+}\,\frac{4\times 5\alpha\gamma}{5\beta}\,{}_{+\cdots}\;. \tag{4.52}$$

Observing that

$$\frac{4\alpha\gamma}{\sqrt{\beta^2+4\alpha\gamma}-\beta} = \beta + \sqrt{\beta^2+4\alpha\gamma}\,,$$

putting $x = \alpha\gamma/\beta^2$, and applying an equivalence transform to (4.52) we obtain

$$\sqrt{1+4x} = 1 + \frac{2x}{1}\,{}_{+}\,\frac{x}{1}\,{}_{+}\,\frac{x}{1}\,{}_{+}\,\frac{x}{1}\,{}_{+\cdots}\,, \qquad x > 0\,. \tag{4.53}$$

81 The evolution equation $s = x^n(a-x)$. To create three terms that depend on n, $n+1$ and $n+2$, we multiply and divide ds by $\alpha+\beta x$ with positive α and β:

$$ds = \frac{na\alpha x^{n-1}dx + (n\beta a - (n+1)\alpha)x^n dx - \beta(n+1)x^{n+1}dx}{\alpha+\beta x}\,.$$

This implies after integration

$$na\alpha\int_0^a \frac{x^{n-1}dx}{\alpha+\beta x} = ((n+1)\alpha - n\beta a)\int_0^a \frac{x^n dx}{\alpha+\beta x} + (n+1)\beta\int_0^a \frac{x^{n+1}dx}{\alpha+\beta x}\,.$$

Let us put in (4.50)

$$x_n = \int_0^a \frac{x^n dx}{\alpha+\beta x}\,,$$

$$c_n = (n+1)a\alpha\,, \qquad b_n = (n+2)\alpha - (n+1)\beta a\,, \qquad a_n = (n+1)\beta\,,$$

Assuming that $0 < x = (\beta a)/\alpha \leqslant 1$, we see that all terms in (4.51) are positive and therefore the continued fraction converges to $a\alpha x_0/x_1$ by Markoff's test. Integration shows that

$$x_0 = \frac{1}{\beta}\ln\left(1+\frac{\beta a}{\alpha}\right)\,, \qquad x_1 = \frac{a}{\beta} - \frac{\alpha}{\beta^2}\ln\left(1+\frac{\beta a}{\alpha}\right)\,.$$

This leads to the continued fraction

$$\ln(1+x) = \frac{x}{1}\,{}_{+}\,\frac{1^2x}{2-x}\,{}_{+}\,\frac{2^2x}{3-2x}\,{}_{+}\,\frac{3^2x}{4-3x}\,{}_{+}\,\frac{4^2x}{5-4x}\,{}_{+\cdots}\,, \qquad 0 < x \leqslant 1\,, \tag{4.54}$$

interpolating (3.25) at $x = 1$.

82 The evolution equation $s = x^{2n+1}(1-x^2)$. We have

$$ds = \frac{(2n+1)\alpha^2 x^{2n} dx}{\alpha^2+\beta^2 x^2} + \frac{((2n+1)\beta^2-(2n+3)\alpha^2)x^{2n+2}dx-(2n+3)\beta^2 x^{2n+4}dx}{\alpha^2+\beta^2 x^2} .$$

It follows that

$$(2n+1)\alpha^2 \int_0^1 \frac{x^{2n}dx}{\alpha^2+\beta^2x^2} = ((2n+3)\alpha^2-(2n+1)\beta^2)\int_0^1 \frac{x^{2n+2}dx}{\alpha^2+\beta^2x^2} + (2n+3)\beta^2\int_0^1 \frac{x^{2n+4}dx}{\alpha^2+\beta^2x^2} .$$

Hence we obtain system (4.50) with

$$x_n = \int_0^1 \frac{x^{2n}dx}{\alpha^2+\beta^2x^2} , \qquad c_n = (2n+1)\alpha^2 ,$$

$$b_n = (2n+3)\alpha^2-(2n+1)\beta^2 , \qquad a_{n+1} = (2n+3)\beta^2 .$$

Therefore

$$\frac{\alpha^2 x_0}{x_1} = 3\alpha^2-\beta^2 + \frac{3^2\alpha^2\beta^2}{5\alpha^2-3\beta^2} {+} \frac{5^2\alpha^2\beta^2}{7\alpha^2-5\beta^2} {+} \frac{7^2\alpha^2\beta^2}{9\alpha^2-7\beta^2} {+} \cdots ,$$

provided that $\beta^2/\alpha^2 \leqslant 1$. Integrating, we obtain

$$x_1 = \frac{1}{\beta^2}\left(1-\alpha^2x_0\right) , \qquad x_0 = \frac{1}{\beta\alpha}\arctan\frac{\beta}{\alpha}.$$

Putting $x = \beta/\alpha$ and applying equivalence transforms we arrive at

$$\frac{\arctan x}{x} = \frac{1}{1} + \frac{1^2x^2}{3-1x^2} {+} \frac{3^2x^2}{5-3x^2} {+} \frac{5^2x^2}{7-5x^2} {+} \cdots \tag{4.55}$$

which is valid for $|x| \leqslant 1$. From (4.55) with $x = 1$ we obtain (3.19).

83 The evolution equation $s = x^n e^{tx}(1-x)$. Differentiating the evolution equation and then integrating from 0 to 1, we obtain

$$n\int_0^1 x^{n-1}e^{tx}dx = (n+1-t)\int_0^1 x^n e^{tx}dx + t\int_0^1 x^{n+1}e^{tx}dx .$$

It is clear that this is (4.50) with

$$x_n = \int_0^1 x^n e^{tx}dx ,$$

$$c_n = n+1 , \qquad b_n = n+2-t , \qquad a_n = t .$$

After elementary transformations we derive the following continued fraction:

$$\frac{t}{e^t-1} = 1-t+\frac{1t}{2-t}\,{}_{+}\,\frac{2t}{3-t}\,{}_{+}\,\frac{3t}{4-t}\,{}_{+}\,\frac{4t}{5-t}\,{}_{+\cdots}\,. \tag{4.56}$$

This continued fraction converges for every real t.

84 The evolution equation $s = x^n e^{zx}(1-x)^\lambda$. Euler's method results in the recursion

$$n\int_0^1 x^{n-1}(1-x)^{\lambda-1}e^{zx}\,dx = (n+\lambda-z)\int_0^1 x^n(1-x)^{\lambda-1}e^{zx}\,dx + z\int_0^1 x^{n+1}(1-x)^{\lambda-1}e^{zx}\,dx\,.$$

Let $\delta > 0$ and

$$x_0 = \int_0^1 x^{\delta-1}(1-x)^{\lambda-1}e^{zx}\,dx\,, \qquad x_1 = \int_0^1 x^{\delta}(1-x)^{\lambda-1}e^{zx}\,dx\,.$$

Then

$$\frac{\delta x_0}{x_1} = \delta+\lambda-1+\frac{(\delta+1)z}{\delta+1+\lambda-z}\,{}_{+}\,\frac{(\delta+2)z}{\delta+2+\lambda-z}\,{}_{+}\,\frac{(\delta+3)z}{\delta+3+\lambda-z}\,{}_{+\cdots}\,, \tag{4.57}$$

which for $\lambda = \delta = 1/2$ takes the form

$$\frac{1}{2}\frac{\int_0^1 e^{zx}(x-x^2)^{-1/2}\,dx}{\int_0^1 xe^{zx}(x-x^2)^{-1/2}\,dx} = 1-z+\frac{3z}{4-2z}\,{}_{+}\,\frac{5z}{3-z}\,{}_{+}\,\frac{7z}{8-2z}\,{}_{+}\,\frac{9z}{5-z}\,{}_{+\cdots}\,.$$

Let

$$\varphi(z) = \int_0^1 \frac{e^{zx}\,dx}{\sqrt{x-x^2}}\,.$$

Then applying iteratively Euler's formula (see Ex. 4.45), we obtain a beautiful relation:

$$\frac{1}{2}\,\frac{\varphi}{\varphi'}(z) = 1-z+\frac{3z}{4}\,{}_{-}\,\frac{2z}{1}\,{}_{+}\,\frac{5}{1}\,{}_{-}\,\frac{2z}{1}\,{}_{+}\,\frac{7}{1}\,{}_{-}\,\frac{2z}{1}\,{}_{+}\,\frac{9}{1}\,{}_{-}\,\frac{2z}{1}\,{}_{+\cdots}\,.$$

4.7 Euler's differential method

85 The theory. Euler's differential method reduces to a simple lemma followed by the solution of an elementary differential equation.

Lemma 4.18 (Euler 1750b) *Let R and P be two positive functions on $(0,1)$ such that, for $n = 0, 1, 2, \ldots$ and some positive α, β, γ,*

$$(a+n\alpha)\int_0^1 PR^n dx = (b+n\beta)\int_0^1 PR^{n+1}dx + (c+n\gamma)\int_0^1 PR^{n+2}dx\,;$$

then

$$\frac{\int_0^1 PRdx}{\int_0^1 Pdx} = \frac{a}{b} {}_{+} \frac{(a+\alpha)c}{b+\beta} {}_{+} \frac{(a+2\alpha)(c+\gamma)}{b+2\beta} {}_{+} \frac{(a+3\alpha)(c+2\gamma)}{b+3\beta} {}_{+\cdots}\,.$$

Proof The condition of the lemma can be obviously written as follows:

$$\frac{\int_0^1 PR^n dx}{\int_0^1 PR^{n+1}dx} = \frac{b+n\beta}{a+n\alpha} + \left(\frac{a+n\alpha}{c+n\gamma}\frac{\int_0^1 PR^{n+1}dx}{\int_0^1 PR^{n+2}dx}\right)^{-1}\,.$$

Iterating this formula and applying elementary transformations, we get the lemma by Markoff's test (Theorem 3.2). □

The next brilliant idea of Euler was to search P and R as functions satisfying the following identity with indefinite integrals:

$$\begin{aligned}&(a+n\alpha)\int PR^n dx + R^{n+1}S \\ &\quad = (b+n\beta)\int PR^{n+1}dx + (c+n\gamma)\int PR^{n+2}dx\,.\end{aligned}$$

If $R^{n+1}S$ vanishes at 0 and 1 then P and R must satisfy the conditions of Lemma 4.18. Euler's formula in differentials looks as follows:

$$(a+n\alpha)Pdx + RdS + (n+1)SdR = (b+n\beta)PRdx + (c+n\gamma)PR^2dx\,.$$

Considering it as a polynomial in n one can replace it with the system

$$\begin{aligned} aPdx + RdS + SdR &= bPRdx + cPR^2dx\,, \\ \alpha Pdx + SdR &= \beta PRdx + \gamma PR^2dx\,. \end{aligned}$$

Solving these equations for Pdx, we find

$$Pdx = \frac{RdS+SdR}{bR+cR^2-a} = \frac{SdR}{\beta R+\gamma R^2-\alpha}\,. \tag{4.58}$$

It follows from the second equation in (4.58) that

$$\begin{aligned}\frac{dS}{S} &= \frac{(b-\beta)RdR+(c-\gamma)R^2dR-(a-\alpha)dR}{\beta R^2+\gamma R^3-\alpha R} \\ &= \frac{(a-\alpha)dR}{\alpha R} + \frac{(\alpha b-\beta a)dR+(\alpha c-\gamma a)RdR}{\alpha(\beta R+\gamma R^2-\alpha)}\,. \end{aligned} \tag{4.59}$$

86 First example. Formula (4.60) below was obtained by Stieltjes in (1890, **3**). In Khovanskii (1958), for instance, it is derived from Roger's formulas for the Laplace transforms of elliptic functions.

$$\frac{1}{s+\underset{n=1}{\overset{\infty}{\mathbf{K}}}\,(n(n+1)/s)}=\int_0^\infty \frac{e^{-sx}}{\cosh^2 x}\,dx = s\int_0^\infty e^{-sx}\tanh x\,dx\,. \tag{4.60}$$

Euler's differential method provides a very simple proof of Stieltjes's formula. In Lemma 4.18 and (4.59) we put

$$\begin{array}{llll} a=1\,, & b=s\,, & c=1\,, & \alpha b-\beta a=s\,,\\ \alpha=1\,, & \beta=0\,, & \gamma=1\,, & \alpha c-\gamma a=0\,, \end{array}$$

$$\frac{dS}{S}=s\frac{dR}{R^2-1}\qquad\Longrightarrow\qquad S=C\left|\frac{1-R}{1+R}\right|^{s/2}.$$

If $R(x)=x$ then $R^{n+1}S$ vanishes at $x=0$ and $x=1$ for $n\geqslant 0$. If $C=-1$ then by (4.58) $P>0$ and

$$\int_0^1 P\,dx=\int_0^1\left(\frac{1-x}{1+x}\right)^{s/2}\frac{dx}{1-x^2}\overset{x=\frac{1-t}{1+t}}{=}\frac{1}{2}\int_0^1 t^{s/2-1}\,dt=\frac{1}{s}\,,$$

substituting $x=\dfrac{1-t}{1+t}$. Similarly,

$$\begin{aligned}\int_0^1 RP\,dx&=\frac{1}{2}\int_0^1 t^{s/2-1}\left(\frac{1-t}{1+t}\right)dt=\int_0^{+\infty}e^{-sx}\tanh x\,dx\\ &=-\frac{1}{s}\int_0^{+\infty}\tanh x\,de^{-sx}=\frac{1}{s}\int_0^\infty\frac{e^{-sx}}{\cosh^2 x}\,dx\,,\end{aligned}$$

substituting $t=e^{-x}$. This proves (4.60). The second integral is obtained from the first using integration by parts.

Remark It is easy to see that (3.26) and (4.18) follow from (4.60) by putting $s=2$ and $s=1$ respectively.

Corollary 4.19 *For $s>0$*

$$y(s)=\frac{1}{s+\underset{n=1}{\overset{\infty}{\mathbf{K}}}\,(n(n+1)/s)}=2s\sum_{k=0}^{\infty}\frac{(-1)^k}{(s+2k)(s+2k+2)}.$$

Proof We have

$$\int_0^1 t^{s/2-1}\left(\frac{1-t}{1+t}\right)dt = \int_0^1 \frac{t^{s/2-1}-t^{s/2}}{1+t}\,dt$$
$$= \sum_{k=0}^{\infty}(-1)^k\int_0^1 (t^{s/2-1+k}-t^{s/2+k})\,dt$$
$$= 4\sum_{k=0}^{\infty}\frac{(-1)^k}{(s+2k)(s+2k+2)}\,.$$

□

Corollary 4.20 *The continued fraction $y(s)$ of* corollary 4,19 *satisfies*

$$\frac{y(s)}{s}+\frac{y(s+2)}{s+2}=\frac{2}{s(s+2)}\,.$$

Proof Apply Corollary 4.19. □

Notice that (E4.9) in Ex. 4.29 gives the asymptotic expansion at infinity for $y(s)$. This completes Brouncker's program **62** (section 3.2) in this case.

87 Second example. For every $s \geqslant 0$, we have

$$e\int_0^1 x^s e^{-x}\,dx = \frac{1}{s}\,{}_{+}\,\frac{1}{s+1}\,{}_{+}\,\frac{2}{s+2}\,{}_{+}\,\frac{3}{s+3}\,{}_{+\cdots+}\,\frac{n}{s+n}\,{}_{+\cdots}\,. \tag{4.61}$$

Now in Lemma 4.18 and in (4.59) we put

$$\begin{array}{llll} a=1, & b=s+1, & c=1, & \alpha b-\beta a=s, \\ \alpha=1, & \beta=1, & \gamma=0, & \alpha c-\gamma a=1, \end{array}$$

$$\frac{dS}{S}=(s+1)\frac{dR}{R-1}+dR \qquad \Longrightarrow \qquad S=(1-R)^{s+1}e^R\,.$$

If $R(x)=1-x$ then $R^{n+1}S=0$ at $x=0$ and $x=1$. Thus

$$\int_0^1 P dx = e\int_0^1 x^s e^{-x}\,dx\,, \qquad \int_0^1 PR dx = -se\int_0^1 x^s e^{-x}\,dx+1\,,$$

which implies (4.61). Passing to the limit $s\to 0+$, we obtain (4.7).

Corollary 4.21 *For $s \geqslant 0$*

$$\frac{1}{s+\underset{n=1}{\overset{\infty}{\mathbf{K}}}\,(n/(s+n))}=\sum_{n=1}^{\infty}\frac{1}{(s+1)(s+2)\cdots(s+n)}\,.$$

Proof Integrating by parts, we obtain

$$e\int_0^1 x^s e^{-x}\,dx = \frac{1}{s+1} + \frac{e}{(s+1)}\int_0^1 x^{s+1}e^{-x}\,dx$$
$$= \frac{1}{s+1} + \frac{1}{(s+1)(s+2)} + \frac{e}{(s+1)(s+2)}\int_0^1 x^{s+2}e^{-x}\,dx = \cdots$$

□

Finally,

$$e\int_0^1 x^s e^{-x}\,dx = e\int_0^\infty e^{-st}e^{-e^{-t}}e^{-t}\,dt$$

allows one to obtain an arbitrary number of coefficients in the asymptotic expansion using Watson's lemma 3.21.

88 Third example: the arctangent. For every $s > 0$,

$$\arctan\frac{1}{s} = \frac{1}{s}\,\underset{+}{}\,\frac{1^2}{3s}\,\underset{+}{}\,\frac{2^2}{5s}\,\underset{+}{}\,\frac{3^2}{7s}\,\underset{+}{}\,\frac{4^2}{9s}\,\underset{+\cdots+}{}\,\frac{n^2}{(2n+1)s}\,\underset{+\cdots}{}\ . \tag{4.62}$$

Remark Since all convergents to the continued fraction (4.62) are odd, the equality (4.62) holds for every real s, $s \neq 0$.

This time, in Lemma 4.18 and in (4.59) we put

$$\begin{array}{llll} a = 1, & b = 3s, & c = 2, & \alpha b - \beta a = s, \\ \alpha = 1, & \beta = 2s, & \gamma = 1, & \alpha c - \gamma a = 1, \end{array}$$

$$\frac{dS}{S} = \frac{(R+s)dR}{2sR + R^2 - 1} \qquad \Longrightarrow \qquad S = -\sqrt{1+s^2-(R+s)^2}\,.$$

If $R(x) = (\sqrt{1+s^2} - s)x$ then $R^{n+1}S$ vanishes at $x = 0$ and $x = 1$ for every nonnegative n. Notice that the maximum value of R on $[0, 1]$ is $\sqrt{1+s^2} - s$, which implies that $1 + s^2 - (R+s)^2 > 0$ for $0 < x < 1$. Next, by (4.58),

$$\int_0^1 P dx = \arcsin\frac{R+s}{\sqrt{1+s^2}}\bigg|_0^1 = \frac{\pi}{2} - \arcsin\frac{s}{\sqrt{1+s^2}}\ ;$$

$$\int_0^1 PR dx = -\sqrt{1+s^2-(R+s)^2} - s\arcsin\frac{R+s}{\sqrt{1+s^2}}\bigg|_0^1$$
$$= 1 - s\left(\frac{\pi}{2} - \arcsin\frac{s}{\sqrt{1+s^2}}\right).$$

The elementary identity

$$\frac{\pi}{2} - \arcsin\frac{s}{\sqrt{1+s^2}} = \arctan\frac{1}{s}\ ,$$

which can be proved by differentiation, completes the proof of (4.62). Putting $s = 1$ in (4.62), we obtain

$$\frac{\pi}{4} = \frac{1}{1} {}_{+} \frac{1^2}{3} {}_{+} \frac{2^2}{5} {}_{+} \frac{3^2}{7} {}_{+} \frac{4^2}{9} {}_{+\cdots+} \frac{n^2}{2n+1} {}_{+\cdots} \; . \tag{4.63}$$

Differentiation shows that

$$\arctan\frac{1}{s} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)s^{2n+1}} , \tag{4.64}$$

where the series converges uniformly in $|s| > 1+\varepsilon$ for every $\varepsilon > 0$. Hence (4.64) is the asymptotic series for arctangent.

89 Fourth example. Here we establish the following formula:

$$\begin{aligned} \frac{1}{s + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} (n^2/(s+n))} &= \phi\sqrt{5}\int_0^1 \frac{x^{s-1+1/\phi}\,dx}{\phi^2 + x^{\sqrt{5}}} \\ &= \phi\sqrt{5}\int_0^\infty e^{-st}\frac{e^{-t/\phi}}{\phi^2 + e^{-t\sqrt{5}}}\,dt , \end{aligned} \tag{4.65}$$

where $\phi = \phi^2 - 1$ is the golden ratio. Evaluating the first integral in (4.65) at $s = \phi$, we obtain the beautiful relation

$$\frac{1}{\phi + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} (n^2/(\phi+n))} = \phi \ln\left(1 + \frac{1}{\phi^2}\right) . \tag{4.66}$$

Now we put in Lemma 4.18 and in (4.59)

$$\begin{array}{llll} a = 1, & b = s+1, & c = 2, & \alpha b - \beta a = s, \\ \alpha = 1, & \beta = 1, & \gamma = 1, & \alpha c - \gamma a = 1, \end{array}$$

$$\frac{dS}{S} = \frac{s\,dR + R\,dR}{R^2 + R - 1} = \frac{(R+1/2)dR + (s-1/2)dR}{(R+1/2)^2 - 5/4} . \tag{4.67}$$

To simplify the notation we set

$$\delta = \left(s - \frac{1}{2}\right)\frac{1}{\sqrt{5}} - \frac{1}{2} = \frac{s-\phi}{\sqrt{5}} , \qquad \Delta = \phi + 1 .$$

Integrating (4.67) and using the factorization

$$R^2 + R - 1 = (R+\phi)(R - 1/\phi) ,$$

we obtain $S = |R - 1/\phi|^{\delta+1}|R+\phi|^{-\delta}$. Hence, if $R(x) = x/\phi$ then $R^{n+1}S = 0$ at $x = 0$ and $x = 1$. Then

$$\int_0^1 P\,dx = \int_0^1 \left(\frac{1-x}{x+\Delta}\right)^{\delta} \frac{dx}{x+\Delta} \overset{x = \frac{1-t\Delta}{1+t}}{=} \int_0^{1/\Delta} \frac{t^\delta}{1+t}\,dt ,$$

substituting $x = \dfrac{1-t\Delta}{1+t}$. Then

$$\int_0^1 RP\,dx = \frac{1}{\phi}\int_0^1 x\left(\frac{1-x}{x+\Delta}\right)^{\delta}\frac{dx}{x+\Delta} = \frac{1}{\phi}\int_0^{1/\Delta}\frac{t^{\delta}(1-t\Delta)}{(1+t)^2}\,dt\,.$$

Integration by parts shows that

$$\begin{aligned}\int_0^{1/\Delta}\frac{t^{\delta}(1-t\Delta)}{(1+t)^2}\,dt &= \int_0^{1/\Delta}\frac{t^{\delta}\,dt}{1+t} - (1+\Delta)\int_0^{1/\Delta}\frac{t^{\delta+1}\,dt}{(1+t)^2}\\ &= \frac{1}{\Delta^{\delta}} - ((1+\delta)(1+\Delta)-1)\int_0^{1/\Delta}\frac{t^{\delta}\,dt}{(1+t)}\,.\end{aligned}$$

Observing that $(1+\delta)(1+\Delta)-1 = s\phi$, we obtain

$$\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{s+n}\right) = \frac{\int_0^1 RP\,dx}{\int_0^1 P\,dx} = \frac{1}{\phi}\left\{\frac{1}{\Delta^{\delta}\int_0^{1/\Delta}\frac{t^{\delta}\,dt}{1+t}} - s\phi\right\},$$

which completes the proof by the substitution $t = x^{\sqrt{5}}$.

Corollary 4.22 *For $s > 0$*

$$\frac{1}{s + \mathop{\mathbf{K}}_{n=1}^{\infty}(n^2/\,(s+n))} = \frac{\sqrt{5}}{\phi}\sum_{k=0}^{\infty}\frac{1}{s+1/\phi+k\sqrt{5}}\,\frac{(-1)^k}{\phi^{2k}}\,.$$

90 A special case of Euler's formula. Let us compare the continued fraction

$$s + \frac{fh}{s}\,{}_{+}\,\frac{(f+r)(h+r)}{s}\,{}_{+}\,\frac{(f+2r)(h+2r)}{s}\,{}_{+\cdots}$$

with

$$a\frac{\int_0^1 Pdx}{\int_0^1 PRdx} = b + \frac{(a+\alpha)c}{b+\beta}\,{}_{+}\,\frac{(a+2\alpha)(c+\gamma)}{b+2\beta}\,{}_{+}\,\frac{(a+3\alpha)(c+2\gamma)}{b+3\beta}\,{}_{+\cdots}\,.$$

Then it is clear that

$$a = f-r, \quad b = s, \quad c = h, \quad \alpha = r, \quad \beta = 0, \quad \gamma = r,$$

and the differential equation (4.59) for S takes the form

$$\frac{dS}{S} = \frac{(f-2r)dR}{rR} + \frac{s}{r}\frac{dR}{R^2-1} + \frac{(h-f+r)RdR}{r(R^2-1)}\,.$$

The integral of this differential equation is given by

$$\ln S = \frac{f-2r}{r}\ln R + \frac{s}{2r}\ln\left|\frac{R-1}{R+1}\right| + \frac{h-f+r}{2r}\ln|R^2-1| + C\,.$$

It follows that

$$S = CR^{(f-2r)/r} \left|\frac{R-1}{R+1}\right|^{s/2r} |R^2-1|(h-f+r)/2r\,.$$

If $R = x^r$ then $R^{n+1}S$ vanishes at $x = 0$ and $x = 1$ for all nonnegative integers n provided that

$$0 < f - r < h + s\,. \tag{4.68}$$

By (4.58) we obtain a formula for P:

$$Pdx = Cx^{f-r-1}\left(\frac{1-x^r}{1+x^r}\right)^{s/2r}(1-x^{2r})(h-f-r)/2r\,dx\,.$$

In the above C stands for a constant, the value of which is not important for us since we are interested in the quotients of integrals. By Lemma 4.18, setting $\eta = \{(1-x^r)/(1+x^r)\}^{s/2r}$, we have

$$\begin{aligned} s + \mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{(f+nr)(h+nr)}{s}\right) \\ = (f-r)\frac{\int_0^1 x^{f-r-1}\eta(1-x^{2r})^{-(f+r-h)/2r}dx}{\int_0^1 x^{f-1}\eta(1-x^{2r})^{-(f+r-h)/2r}dx}\,. \end{aligned} \tag{4.69}$$

If $f = h = g$ and $g > r$ then (4.69) takes the form

$$s + \mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{(g+nr)^2}{s}\right) = (g-r)\frac{\int_0^1 x^{g-r-1}\eta(1-x^{2r})^{-1/2}dx}{\int_0^1 x^{g-1}\eta(1-x^{2r})^{-1/2}dx}\,.$$

If $g > 0$ the above identity with $g := g + r$ shows that

$$\mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{(g+nr)^2}{s}\right) = g\frac{\int_0^1 x^{g-r-1}\eta(1-x^{2r})^{-1/2}dx}{\int_0^1 x^{g-1}\eta(1-x^{2r})^{-1/2}dx}\,. \tag{4.70}$$

Corollary 4.23 *For $s > 0$*

$$\frac{1}{s + \mathop{\mathbf{K}}_{n=0}^{\infty}((s+n)^2/1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{s+n}\,.$$

Proof Let us put $s = r = 1$, $g = s$ in (4.70). Then by Lemma 4.4

$$\begin{aligned} \mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{(s+n)^2}{1}\right) &= s\frac{\int_0^1 x^s dx/(1+x)}{\int_0^1 x^{s-1}dx/(1+x)} \\ &= s\frac{1/s - \sum_{k=0}^{\infty}\left((-1)^k/(s+k)\right)}{\sum_{k=0}^{\infty}\left((-1)^k/(s+k)\right)} = \frac{1}{\sum_{k=0}^{\infty}\left((-1)^k/(s+k)\right)} - s\,. \end{aligned}$$

□

4.8 Laplace transform of hyperbolic secant

91 The continued fraction. For $s > 0$,

$$\frac{1}{s + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} (n^2/s)} = \int_0^{+\infty} \frac{e^{-sx}\, dx}{\cosh x} . \tag{4.71}$$

Formula (4.71) is obtained by the substitution $x := e^{-x}$ from the following theorem by Euler (1750b, §69). See also Corollary 4.31 below.

Theorem 4.24 *For $s > 0$,*

$$\frac{1}{s + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} (n^2/s)} = 2\int_0^1 \frac{x^s\, dx}{1+x^2} . \tag{4.72}$$

Proof By (4.70) with $g = 1$ and $r = 1$,

$$\mathop{\mathbf{K}}\limits_{n=1}^{\infty} \left(\frac{n^2}{s}\right) = \frac{\int_0^1 x\left((1-x)(1+x)^{-1}\right)^{s/2} (1-x^2)^{-1/2} dx}{\int_0^1 ((1-x)(1+x)^{-1})^{s/2} (1-x^2)^{-1/2} dx} .$$

Integration by parts followed by the substitution $x := (1-x)/(1+x)$ gives

$$\begin{aligned} &\int_0^1 x\left(\frac{1-x}{1+x}\right)^{s/2} \frac{dx}{\sqrt{1-x^2}} \\ &= 1 - s\int_0^1 \left(\frac{1-x}{1+x}\right)^{s/2} \frac{dx}{\sqrt{1-x^2}} = 1 - s\int_0^1 \frac{x^{(s-1)/2}\, dx}{1+x}. \end{aligned}$$

This results in the formula

$$\frac{1}{s + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} (n^2/s)} = \int_0^1 \frac{x^{(s-1)/2}\, dx}{1+x} = 2\int_0^1 \frac{x^s\, dx}{1+x^2} , \tag{4.73}$$

interpolating (3.25) at $s = 1$ and (4.20) at $s = 2$. □

Since

$$\int_0^1 \frac{x^s\, dx}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k \int_0^1 x^{2k+s} dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{s+2k+1} ,$$

we obtain an analogue of Brouncker's formula (see Theorem 3.16)

$$y(s) = \frac{1}{s + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} (n^2/s)} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{s+2k+1} , \tag{4.74}$$

which gives a functional equation for $y(s)$:

$$y(s) + y(s+2) = \frac{2}{s+1} . \tag{4.75}$$

92 The continued fraction and its asymptotic series. Applying Watson's lemma 3.21 to (4.71), one can easily obtain the asymptotic expansion for y(s). Indeed,

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n \,, \tag{4.76}$$

where E_n are the *Euler numbers*: $E_0 = 1$, $E_1 = 0$, $E_2 = -1$, $E_3 = 0$, $E_4 = 5$, $E_5 = 0$, $E_6 = -61$, $E_7 = 0$, $E_8 = 1385$, ... By Lemma 3.21 and (4.71),

$$\frac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}} (n^2/s)} \sim \sum_{k=0}^{\infty} \frac{E_k}{s^{k+1}} \,, \qquad s \to +\infty \,. \tag{4.77}$$

93 The relationship to Brouncker's continued fraction. It is interesting that Brouncker's continued fraction and $\mathbf{K}(n^2/s)$ are related.

Theorem 4.25 *For $s > 0$,*

$$s + \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{(2n-1)^2}{2s} \right) = \frac{8\pi^2}{\Gamma^4(1/4)} \exp\left\{ \int_0^s \frac{dt}{t + \underset{n=1}{\overset{\infty}{\mathbf{K}}} (n^2/t)} \right\} . \tag{4.78}$$

Proof We observe that both the function $y(s)$ defined in (4.74) and the logarithmic derivative of Brouncker's function, $(\log b)'$, satisfy the functional equation (4.75). By (3.64) $(\log b)'(s) \to 0$ as $s \to +\infty$. The function $y(s)$ vanishes at $+\infty$ by definition. Hence by Lemma 3.23 they are equal:

$$y(s) = \frac{1}{s + \underset{n=1}{\overset{\infty}{\mathbf{K}}} (n^2/s)} = \frac{b'}{b}(s) \,.$$

Integrating this differential equation and observing that

$$y(0) = \frac{1}{y(2)} = \frac{1}{4} \left(\frac{\Gamma(3/4)}{\Gamma(5/4)} \right)^2 = 4 \left(\frac{\Gamma(3/4)\,\Gamma(1/4)}{\Gamma^2(1/4)} \right)^2 = \frac{\Gamma(1/4)^4}{8\pi^2} \,,$$

see Ex. 3.19, we get (4.78). □

Corollary 4.26 *The following asymptotic relation holds as $s \to +\infty$:*

$$s + \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{(2n-1)^2}{2s} \right) \sim s \exp\left\{ -\sum_{k=1}^{\infty} \frac{E_{2k}}{2ks^{2k}} \right\} . \tag{4.79}$$

Proof By Theorem 3.16 the left-hand side of (4.79) is divisible by $s+1$. It follows that Brouncker's continued fraction $y(s)$ can be written as

$$y(s) = y(0)(s+1) \exp\left\{ \int_0^{\infty} \gamma(t)\,dt \right\} \exp\left\{ -\int_s^{\infty} \gamma(t)\,dt \right\} ,$$

where

$$\gamma(t) = \frac{1}{t + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} (n^2/t)} - \frac{1}{1+t} \sim \sum_{k=1}^{\infty} \frac{E_k - (-1)^k}{t^{k+1}} , \qquad t \to +\infty ,$$

by (4.77). Integrating this over $(s, +\infty)$, we obtain

$$\int_s^{\infty} \gamma(t)\, dt \sim \sum_{k=1}^{\infty} \frac{E_k - (-1)^k}{k s^k} , \qquad t \to +\infty .$$

Since $y(s) \sim s$ as $s \to +\infty$, we see that

$$\int_0^{+\infty} \left(\frac{1}{t + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} (n^2/t)} - \frac{1}{1+t} \right) dt = \ln \frac{8\pi^2}{\Gamma(1/4)^4} , \tag{4.80}$$

and

$$s + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} \left(\frac{(2n-1)^2}{2s} \right) \sim (s+1) \exp \left\{ - \sum_{k=1}^{\infty} \frac{E_k - (-1)^k}{k s^k} \right\} . \tag{4.81}$$

The proof is completed by observing that

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k s^k} = - \ln \left(1 + \frac{1}{s} \right) , \qquad s > 1 .$$

$\square$

Let us compare (4.81) with Stirling's formula (3.58) for the gamma function. Easy computations with the first few Euler numbers show that

$$\begin{aligned} s + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} \left(\frac{(2n-1)^2}{2s} \right) &\sim s \exp \left\{ \frac{1}{2s^2} - \frac{5}{4s^4} + \frac{61}{6s^6} + O\left(\frac{1}{s^8} \right) \right\} \\ &= s + \frac{1}{2s} - \frac{9}{8s^3} + \frac{153}{16s^5} + O\left(\frac{1}{s^7} \right) . \end{aligned}$$

Compare this with the asymptotic series for Brouncker's continued fraction $b(s)$, obtained in §**60**, Section 3.2, by a different method. See Ex. 4.32 for the relationship to another continued fraction.

94 Leibnitz' series. We will now apply (4.74) to Leibnitz' series.

Theorem 4.27 *For every positive integers,*

$$\pi = 4 \sum_{k=0}^{s-1} \frac{(-1)^k}{2k+1} + \frac{2(-1)^s}{2s + \mathop{\mathbf{K}}\limits_{n=1}^{\infty} (n^2/2s)} . \tag{4.82}$$

Proof If $0 < s \in \mathbb{Z}$ then $y(2s) = 2(-1)^s \sum_{k \geqslant s}((-1)^k/(2k+1))$ by (4.74). Computations with integrals show that

$$2 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{2}\right) = \frac{2}{4-\pi}\,, \qquad 4 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{4}\right) = \frac{2}{\pi - 8/3}\,;$$

$$6 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{6}\right) = \frac{2}{52/15-\pi}\,, \qquad 8 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{8}\right) = \frac{2}{\pi - 105/304}\,.$$

Assuming that $y(2s)^{-1} = 2(-1)^s\{\pi - 4\sum_{k=0}^{s-1}((-1)^k/(2k+1))\}^{-1}$, we deduce from (4.75) that $y(2s+2) = (-1)^{s+1}\{\pi/2 - 2\sum_{k=0}^{s}((-1)^k/(2k+1))\}$. □

Remark If $s = 50$ then taking the second convergent and adding the first 50 terms of Leibnitz' series, we obtain 10 valid places for π. It should also be noticed that by (4.73)

$$\lim_{s \to 0+}\left(s + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{n^2}{s}\right)\right) = \frac{2}{\pi}\,.$$

4.9 Stieltjes' continued fractions

95 Quotients of Laplace transforms. In (1890) Stieltjes obtained the expansion

$$\frac{\int_0^{+\infty} \sinh^{\beta-1} x \cosh^{-\alpha} x\, e^{-sx}\, dx}{(\beta-1)\int_0^{+\infty} \sinh^{\beta-2} x \cosh^{1-\alpha} x\, e^{-sx}\, dx} = \frac{1}{s} \mathop{}_{+} \frac{\alpha\beta}{s} \mathop{}_{+} \frac{(\alpha+1)(\beta+1)}{s} \mathop{}_{+} \frac{(\alpha+2)(\beta+2)}{s} \mathop{}_{+} \cdots\,. \tag{4.83}$$

One can easily deduce (4.83) from Euler's formula (4.69) by putting $r = 1$, $f = \beta$, $h = \alpha$. Notice that $1 < \beta < 1 + \alpha + s$. Then, as in Section 4.7, §**86**, we make two changes of variable, $x := (1-t)/(1+t)$ and $t := e^{-2x}$, which as can be easily seen results in (4.83). In fact Stieltjes obtained his formula along lines similar to those of Euler. However, Euler's differential method makes these calculations more motivated.

96 The Hermite–Stieltjes formula. In (1891) Stieltjes deduced from (4.83) an important continued fraction,

$$\varphi(s) = s\left(\frac{\Gamma(s)}{\Gamma(s+1/2)}\right)^2 = 1 + \frac{2}{8s-1} \mathop{}_{+} \frac{1\times 3}{8s} \mathop{}_{+} \frac{3\times 5}{8s} \mathop{}_{+} \frac{5\times 7}{8s} \mathop{}_{+} \cdots\,. \tag{4.84}$$

This also follows easily from (4.69); see Ex. 4.36. Competely different arguments are presented in Ex. 5.4. Formula (4.84) conveniently gives as many terms in the asymptotic expansion (3.65) as are needed. Another application of (4.84) is a continued fraction expansion for the solution $y(s)$ to the functional equation of in section 4.2 §**72**. By (4.21),

$$y(s) = s\varphi\left(\frac{s}{2}\right) = s + \frac{2s}{4s-1} + \frac{1\times 3}{4s} + \frac{3\times 5}{4s} + \frac{5\times 7}{4s} + \cdots .$$

Combining this with (4.22), we obtain (see also Ex. 5.3):

$$\frac{s+1}{b(s)} = 1 + \frac{2}{2s+1} + \frac{1\times 3}{2(s+1)} + \frac{3\times 5}{2(s+1)} + \frac{5\times 7}{2(s+1)} + \cdots . \tag{4.85}$$

Putting $s = 1$ in (4.85) we get the formula of Euler discussed in Ex. 4.7.

In (1891) Stieltjes mentioned that formula (4.84) can be used to improve an elegant result of Hermite,

$$\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2\times 4\times 6\times\cdots\times(2n)} = \frac{1}{\sqrt{\pi(n+\epsilon)}}, \qquad 0 < \epsilon < 1/2 .$$

Stieltjes' improvement is based on a formula indicated in Euler (1783):

$$\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2\times 4\times 6\times\cdots\times(2n)} = \frac{1}{\sqrt{\pi n\varphi(n)}},$$

which follows from (3.66) by the functional equation (3.42) for the gamma function:

$$\frac{1}{\sqrt{\pi n\varphi(n)}} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)}{n\Gamma(n)} = \frac{1}{\sqrt{\pi}}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2\times 4\times 6\times\cdots\times(2n)}\Gamma\left(\frac{1}{2}\right) .$$

Now (4.84) implies a formula for Hermite's ϵ:

$$\begin{aligned}\epsilon = \epsilon_n &= \frac{2n}{8n-1} + \frac{1\times 3}{8n} + \frac{3\times 5}{8n} + \frac{5\times 7}{8n} + \cdots \\ &= \frac{1}{4} + \frac{1}{32n} - \frac{1}{128n^2} + O\left(\frac{1}{n^3}\right) .\end{aligned}$$

97 Stieltjes' formulas for quotients of gamma functions. The following continued fraction of Stieltjes interpolates (4.84) at $a = 1/2$.

Theorem 4.28 (Stieltjes 1891) *For $s > 0$ and $a > 0$*

$$\begin{aligned}&1 + \frac{2a}{4s-a} + \frac{1^2-a^2}{4s} + \frac{2^2-a^2}{4s} + \cdots \\ &\quad = \frac{\Gamma(s-a/2+1/4)\Gamma(s+a/2+3/4)}{\Gamma(s+a/2+1/4)\Gamma(s-a/2+3/4)}.\end{aligned} \tag{4.86}$$

Proof We will derive (4.86) from Euler's formula (4.69) using Euler's method. Let $r = 1$, $f = 1+a$, $h = 1-a$, $s := 4s$ in (4.69). Then

$$4s + \mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{n^2-a^2}{4s}\right) = a\frac{\int_0^1 x^{a-1}\left((1-x)(1+x)^{-1}\right)^{2s}(1-x^2)^{-a-1/2}dx}{\int_0^1 x^a\left((1-x)(1+x)^{-1}\right)^{2s}(1-x^2)^{-a-1/2}dx} .$$

The substitutions $x=(1-t)/(1+t)$, $dx=-2dt/(1+t)^2$, $1-x^2=4t/(1+t)^2$ transform this formula into

$$4s+\mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{n^2-a^2}{4s}\right)=a\frac{\int_0^1(1-t)^{a-1}(1+t)^a t^\gamma dt}{\int_0^1(1-t)^a(1+t)^{a-1}t^\gamma dt}\ ,\ \gamma=2s-a-1/2\,.$$

Observing that $(1-t)^{a-1}(1+t)^a t^\gamma=(1-t^2)^{a-1}(t^\gamma+t^{\gamma+1})$, compare this with the trick presented in §**59** of Euler (1750b), we obtain by (4.30)

$$\begin{aligned}\int_0^1(1-t)^{a-1}(1+t)^a t^\gamma\,dt&=\int_0^1(1-t^2)^{a-1}(t^\gamma+t^{\gamma+1})\,dt\\&=\frac{1}{2}\int_0^1(1-t)^{a-1}t^{\gamma/2-1/2}\,dt+\frac{1}{2}\int_0^1(1-t)^{a-1}t^{\gamma/2}\,dt\\&=\frac{1}{2}\left\{\frac{\Gamma(a)\Gamma(\gamma/2+1/2)}{\Gamma(a+\gamma/2+1/2)}+\frac{\Gamma(a)\Gamma(\gamma/2+1)}{\Gamma(a+\gamma/2+1)}\right\}\,.\end{aligned}$$

Similarly

$$\int_0^1(1-t)^{a-1}(1+t)^a t^\gamma\,dt=\frac{1}{2}\left\{\frac{\Gamma(a)\Gamma(\gamma/2+1/2)}{\Gamma(a+\gamma/2+1/2)}-\frac{\Gamma(a)\Gamma(\gamma/2+1)}{\Gamma(a+\gamma/2+1)}\right\}\,.$$

Hence

$$4s+\mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{n^2-a^2}{4s}\right)=a\,\frac{1+\Pi}{1-\Pi}\,,\qquad \Pi=\frac{\Gamma(1+\gamma/2)\Gamma(a+\gamma/2+1/2)}{\Gamma(a+\gamma/2+1)\Gamma(\gamma/2+1/2)}\,.$$

The identity $1+2a(a(1+\Pi)/(1-\Pi)-a)^{-1}=\Pi^{-1}$ completes the proof. □

Theorem 4.29 (Stieltjes 1891) *For $s>0$ and $a>0$,*

$$\begin{aligned}&\frac{4}{4s}\,{}_{+}\,\frac{1^2-4a^2}{8s}\,{}_{+}\,\frac{3^2-4a^2}{8s}\,{}_{+}\,\frac{5^2-4a^2}{8s}\,{}_{+}\cdots\\&\quad=\frac{\Gamma(s-a/2+1/4)\Gamma(s+a/2+1/4)}{\Gamma(s-a/2+3/4)\Gamma(s+a/2+3/4)}\,.\end{aligned}$$

Proof Put $r=1$, $f=1/2+a$, $h=1/2-a$, $s:=4s$ in (4.69). □

98 Laplace transforms of some hyperbolic functions. The following result was established by Stieltjes (1890, §3) with a tricky calculation which is close to Euler's evolution method presented in Section 4.6.

Theorem 4.30 *For $s > 0$, $m \geqslant 0$, $1 \geqslant a \geqslant 0$,*

$$\int_0^\infty \frac{e^{-su}\,du}{(\cosh u + a\sinh u)^m} = \frac{1}{s+ma} \mathop{}_{+} \frac{m(1-a^2)}{s+(m+2)a} \mathop{}_{+} \frac{2(m+1)(1-a^2)}{s+(m+4)a} \mathop{}_{+} \frac{3(m+2)(1-a^2)}{s+(m+6)a} \mathop{}_{+} \frac{4(m+3)(1-a^2)}{s+(m+8)a} \mathop{}_{+\cdots} .$$

Proof Stieltjes' idea, originating in trigonometrical integrals (1890, §1), was to integrate the definite integral by parts, where for the sake of brevity $A(u) = \cosh u + a\sinh u$ and $n > 0$, $m > 0$:

$$\begin{aligned} F(m,n) &= \int_0^\infty \frac{e^{-su}\sinh^n u\,du}{A(u)^m} \\ &= -\frac{e^{-su}}{s}\frac{\sinh^n u}{A(u)^m}\bigg|_0^{+\infty} \\ &\quad + \frac{1}{s}\int_0^\infty e^{-su}\left(n\frac{\sinh^{n-1} u\cosh u}{A(u)^m} - m\frac{\sinh^n u(\sinh u + a\cosh u)}{A(u)^{m+1}}\right) du \\ &= \frac{1}{s}\int_0^\infty e^{-su}\frac{\sinh^{n-1} u}{A(u)^{m+1}}\big[nA(u)(A(u)-a\sinh u) - m\sinh^2 u \\ &\qquad\qquad -am(A(u)-a\sinh u)\sinh u\big]\,du \\ &= \frac{n}{s}F(m-1,n-1) - \frac{(n+m)a}{s}F(m,n) - \frac{m(1-a^2)}{s}F(m+1,n+1)\,. \end{aligned}$$

It follows that

$$\frac{nF(m-1,n-1)}{F(m,n)} = s + (n+m)a + \frac{m(1-a^2)(n+1)}{(n+1)F(m,n)/F(m+1,n+1)}\,.$$

Let us pass to the limit $n \to 0^+$ in this recursion. We have

$$nF(m-1,n-1) = -\int_0^{+\infty} \sinh^n u\,d\frac{e^{-su}}{A(u)^{m-1}\cosh u}\,.$$

The derivative of the function under the differential sign decreases at infinity exponentially fast. Therefore we may pass to the limit under the integral sign to obtain that

$$\lim_{n\to 0^+} nF(m-1,n-1) = -\frac{e^{-su}}{A(u)^{m-1}\cosh u}\bigg|_0^{+\infty} = 1\,,$$

which completes the proof by Theorem 3.2 and Corollary 3.10. □

Corollary 4.31 *For any real m and $s > 0$,*

$$\int_0^{+\infty} \frac{e^{-su}\,du}{\cosh^m u} = \frac{1}{s+}\,\frac{m}{s+}\,\frac{2(m+1)}{s+}\,\frac{3(m+2)}{s+\cdots}\,. \tag{4.87}$$

Compare this corollary with (4.71) and Ex. 4.30.

99 The error function. For every $s > 0$,

$$e^{s^2/2}\int_s^{+\infty} e^{-x^2/2}\,dx = \frac{1}{s+}\,\frac{1}{s+}\,\frac{2}{s+}\,\frac{3}{s+}\,\frac{4}{s+\cdots+}\,\frac{n}{s+\cdots}\,; \tag{4.88}$$

see Hardy (1940, p. 8, (1.8)). In fact this formula was first explicitly indicated by Euler in 1754 (1760, §29), and appeared in a more general form earlier in (1750b, §§71–3).

Remark The function defined by

$$\operatorname{erfc} s = \frac{2}{\sqrt{\pi}}\int_s^{+\infty} e^{-x^2}\,dx = \sqrt{\frac{2}{\pi}}\int_{s\sqrt{2}}^{+\infty} e^{-x^2/2}\,dx$$

is called the *complementary error function* (see Andrews, Askey and Roy 1999, p. 196). The identity (4.88) can be used to get good approximations to $\operatorname{erfc}(s)$ for large s.

To prove (4.88) we put $a = 1$, $b = s$, $c = 1$, $\alpha = 1$, $\beta = 0$, $\gamma = 0$ in Lemma 4.18 and use the limit $+\infty$ instead of $x = 1$:

$$\frac{\int_0^{+\infty} PR\,dx}{\int_0^{+\infty} P\,dx} = \frac{1}{s+}\,\frac{2}{s+}\,\frac{3}{s+}\,\frac{4}{s+\cdots+}\,\frac{n}{s+\cdots}\,.$$

The differential equation (4.59) for S becomes $dS/S = -sdR - RdR$, which, with $R(x) = x$, shows that $xS(x) = xe^{-sx-x^2/2}$ vanishes at $x = 0$ and $x = +\infty$. It follows that we may put $Pdx = e^{-sx-x^2/2}dx$ and $RPdx = xe^{-sx-x^2/2}dx$. Let us consider

$$\varphi(s) = \int_0^{+\infty} e^{-sx-x^2/2}\,dx = e^{s^2/2}\int_s^{+\infty} e^{-x^2/2}\,dx\,.$$

Then φ satisfies

$$\varphi'(s) = s\varphi(s) - 1 = -\int_0^{+\infty} xe^{-sx-x^2/2}dx\,. \tag{4.89}$$

Now (4.88) follows from the elementary identity

$$\frac{\int_0^{+\infty} PR\,dx}{\int_0^{+\infty} P\,dx} = \frac{1 - s\varphi(s)}{\varphi(s)} = \frac{1}{\varphi(s)} - s\,.$$

If we set $s := \sqrt{2}s$ in (4.88) and change the variable of integration to $x := x/\sqrt{2}$ then after equivalence transforms we obtain the formula

$$2e^{s^2}\int_s^{\infty} e^{-x^2}dx = \frac{2}{2s+}\,\frac{2}{2s+}\,\frac{4}{2s+}\,\frac{6}{2s+}\,\frac{8}{2s+\cdots+}\,\frac{2n}{2s+\cdots}\,. \tag{4.90}$$

4.10 Continued fraction of hyperbolic cotangent

100 Euler's numerical experiments. In (1744) Euler computed the first partial denominators of the regular continued fraction for

$$e = 2.718\,281\,828\,459\,04\ldots$$

and discovered the remarkable law:

$$e = 2 + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{4} + \frac{1}{1} + \frac{1}{1} + \frac{1}{6} + \frac{1}{1} + \cdots \ . \tag{4.91}$$

It looks no less beautiful than the standard formulas

$$e = \lim_n \left(1 + \frac{1}{n}\right)^n = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\ ,$$

and in addition immediately proves the irrational of e; see Corollary 1.16. After that Euler computed the continued fraction for

$$\begin{aligned}\sqrt{e} &= 1.648\,721\,270\,7\ldots\\ &= 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{5} + \frac{1}{1} + \frac{1}{1} + \frac{1}{9} + \frac{1}{1} + \frac{1}{1} + \frac{1}{13} + \cdots\ ,\end{aligned}$$

obeying a similar progression law of partial denominators. Next,

$$\sqrt[3]{e} = 1 + \frac{1}{2} + \frac{1}{1} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{1} + \frac{1}{14} + \frac{1}{1} + \frac{1}{1} + \frac{1}{20} + \cdots$$

confirms this law again. To concentrate on the arithmetic progressions Euler removed the repeating 1's by an elementary formula:

$$\begin{aligned}a + \frac{1}{m} + \frac{1}{n} + \frac{1}{w} &= a + \frac{nw+1}{mnw+m+w}\\ &= a + \frac{n}{mn+1} + \frac{nw+1}{mnw+m+w} - \frac{n}{mn+1}\\ &= a + \frac{n}{mn+1} + \frac{1}{(mn+1)w+m}\\ &= \frac{1}{mn+1}\left\{(mn+1)a + n + \frac{1}{(m+n)w+m}\right\}\ ,\end{aligned}$$

which shows that

$$\begin{aligned}&a + \frac{1}{m} + \frac{1}{n} + \frac{1}{b} + \frac{1}{m} + \frac{1}{n} + \frac{1}{c} + \cdots\\ &\quad = \frac{1}{mn+1}\left\{(mn+1)a + n + \frac{1}{(mn+1)b+n+m} + \cdots\right\}\ .\end{aligned} \tag{4.92}$$

This interesting identity, by the way, implies the beautiful formula

$$a+\frac{1}{m}\,\frac{1}{+n}\,\frac{1}{+b}\,\frac{1}{+m}\,\frac{1}{+n}\,\frac{1}{+c}+\cdots$$
$$-a-\frac{1}{n}\,\frac{1}{+m}\,\frac{1}{+b}\,\frac{1}{+n}\,\frac{1}{+m}\,\frac{1}{+c}+\cdots=\frac{n-m}{nm+1},$$

calling to mind the addition formula for cotangents. Let us put $m=n=1$, $a=2$, $b=4$, $c=6$, $d=8$, etc. in (4.92). Then

$$2+\frac{1}{1}\,\frac{1}{+1}\,\frac{1}{+4}\,\frac{1}{+1}\,\frac{1}{+1}\,\frac{1}{+6}+\cdots$$
$$=\frac{1}{2}\left\{-1+6+\frac{1}{10}\,\frac{1}{+14}\,\frac{1}{+18}+\cdots\right\}=\frac{\xi-1}{2},$$

where ξ denotes the continued fraction related to the progression $6, 10, 14, 18, \ldots$ Consequently by (4.91)

$$e=2+\frac{1}{1+2/(\xi-1)}=2+\frac{\xi-1}{\xi+1}=1+\frac{2}{1+1/\xi},$$

which leads us to

$$1+\frac{1}{\xi}=\frac{2}{e-1}=-1+\frac{e+1}{e-1},$$

and finally to

$$\coth\frac{1}{2}=\frac{e+1}{e-1}=2+\frac{1}{6}\,\frac{1}{+10}\,\frac{1}{+14}\,\frac{1}{+18}\,\frac{1}{+22}\,\frac{1}{+26}+\cdots.$$

Similarly, Euler obtains

$$\coth\frac{1}{4}=\frac{\sqrt{e}+1}{\sqrt{e}-1}=4+\frac{1}{12}\,\frac{1}{+20}\,\frac{1}{+28}+\cdots,$$
$$\coth\frac{1}{6}=\frac{\sqrt[3]{e}+1}{\sqrt[3]{e}-1}=6+\frac{1}{18}\,\frac{1}{+30}\,\frac{1}{+42}+\cdots,$$
$$\coth(1)=\frac{e^2+1}{e^2-1}=1+\frac{1}{3}\,\frac{1}{+5}\,\frac{1}{+7}+\cdots.$$

A simple analysis of these empirical formulas shows that they can be naturally explained if the following is true:

$$q=\coth p=\frac{1}{p}+\frac{1}{3/p}\,\frac{1}{+5/p}\,\frac{1}{+7/p}+\cdots. \tag{4.93}$$

101 Continued fraction for hyperbolic cotangent: proof. Euler's differential method (see Section 4.7) when applied to (4.93) with $s = 1/p$ corresponds to the choice

$$\begin{aligned} a &= 1\,, \quad b = s\,, \quad c = 1\,, \\ \alpha &= 0\,, \quad \beta = 2s\,, \quad \gamma = 0\,, \end{aligned}$$

$$\frac{dS}{S} = -\frac{1}{2}\frac{dR}{R} + \frac{1}{2s}\left(R + \frac{1}{R}\right) \quad \Rightarrow \quad S = \frac{1}{\sqrt{R}} e^{(1/2s)(R+1/R)}.$$

It is clear from the formula for S that under no choice of a positive function R can $R^{n+1}S$ vanish at any point. So, here this method does not work. Historically it appeared later. However, the first method, which Euler applied in (1744) to develop coth p into a continued fraction, was also differential. Taking into account that $q = \coth p$ satisfies the differential equation

$$\frac{dq}{dp} = 1 - q^2 \quad \Leftrightarrow \quad dq + q^2 dp = dp\,,$$

Euler found by induction the differential equation for the remainders q_n.

Theorem 4.32 (Euler 1744, §28) *For $n = 1, 2, \ldots$ we have*

$$q = \coth p = \frac{1}{p} + \frac{1}{3/p} + \frac{1}{5/p} + \frac{1}{7/p} + \cdots + \frac{1}{(2n-1)/p} + \frac{1}{1/\,x^{2n/(2n+1)}y}\,, \tag{4.94}$$

where $p = (2n+1)x^{1/(2n+1)}$ and y satisfies the differential equation

$$\frac{dy}{dx} + y^2 = x^{-4n/(2n+1)} \tag{4.95}$$

Proof Let $q_0 = \coth p$ and $\{q_n\}_{n\geqslant 0}$ be defined by

$$q_n = \frac{2n+1}{p} + \frac{1}{q_{n+1}}\,. \tag{4.96}$$

Then

$$\frac{dq_n}{dp} = \frac{2n}{p} q_n + 1 - q_n^2\,, \qquad n = 0, 1, \ldots \tag{4.97}$$

For $n = 0$ this equation coincides with the differential equation for coth p. The transition $n \to n+1$ is made using

$$\begin{aligned} &\frac{d}{dp}\left(\frac{2n+1}{p} + \frac{1}{q_{n+1}}\right) \\ &\quad = \frac{2n}{p}\left(\frac{2n+1}{p} + \frac{1}{q_{n+1}}\right) + 1 - \left(\frac{2n+1}{p} + \frac{1}{q_{n+1}}\right)^2\,. \end{aligned}$$

Now the identities $2n+1+2n(2n+1)=(2n+1)^2$ and $2n-2(2n+1)=-2(n+1)$ turn it into (4.97) for $n+1$. Let

$$y(x)=x^{-2n/(2n+1)}q_n\left((2n+1)x^{1/(2n+1)}\right)\,. \tag{4.98}$$

Then

$$\begin{aligned}\frac{dy}{dx}&=-\frac{2n}{2n+1}x^{-(4n+1)/(2n+1)}q_n+x^{-4n/(2n+1)}\frac{dq_n}{dp}\\&=-\frac{2n}{2n+1}x^{-4n+1/(2n+1)}q_n\\&\quad+\frac{2n}{2n+1}x^{-4n+1/(2n+1)}q_n\\&\quad+x^{-4n/(2n+1)}-x^{-4n/(2n+1)}q_n^2=x^{-4n/(2n+1)}-y^2.\end{aligned}$$

□

It looks as though the idea of recruiting Riccati's equation for this purpose goes back to the first paper by D. Bernoulli (1724). In the same year, Jacoppo Riccati (1676–1754), in a paper (Riccati 1724) on the equation

$$dy+y^2dx=ax^m\,dx\,,$$

had posed the problem of finding those m values for which the equation can be integrated in quadratures. In the same volume of *Acta Eruditorum* D. Bernoulli (1726) announced that he had a solution (see the footnote on p. 245 of Euler 1768). Both Euler and Bernoulli worked in St Petersburg and obviously Bernoulli's contribution was known to Euler. In Theorem 4.32 Euler presented it in quite an original way and as a result discovered very interesting relations, which finally resulted in a complete solution of Riccati's problem by Liouville (1839, 1841). We consider this question later.

The continued fraction in (4.93) converges since the partial denominators are positive and make an arithmetic progression. Since

$$\frac{1}{p}+\frac{1}{3/p}\,{}_{+}\,\frac{1}{5/p}\,{}_{+}\,\frac{1}{7/p}\,{}_{+\cdots}\approx\frac{1}{p}+\frac{p}{3}\,{}_{+}\,\frac{p^2}{5}\,{}_{+}\,\frac{p^2}{7}\,{}_{+\cdots}\,,$$

we obtain the convergence of (4.93) by Corollary 3.10 with $p_1=p$, $p_n=p^2$, $n=2,3,\dots$, and $q_n=2n+1$, $n=1,2,\dots$ It is therefore tempting to obtain (4.93) from Theorem 4.32 by passage to the limit in (4.94). However, the paradox of quadratic equations (see §**54** in Section 3.1) urges one to be cautious: in this similar case of quadratic equations the limit may not be equal to the expanded value. Theorem 3.2 guarantees that (4.93) holds provided that $q_n>0$ on $(0,+\infty)$ for every n.

Theorem 4.33 *Every q_n is positive on* $(0,+\infty)$.

Proof The idea of the proof is obvious from Fig. 4.5. By (4.97) the derivatives of q_n at the zeros of q_n equal 1, which forces the graph to cross the p-axis upwards. Therefore it is not possible for q_n to behave like a hyperbola $q=c/p$, $c>0$ in the vicinity of 0 and at the same time to have positive zeros.

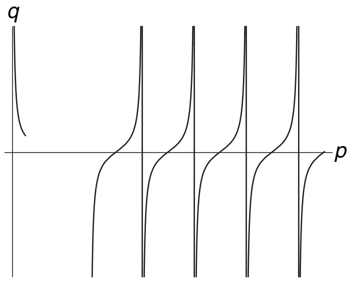

Fig. 4.5. Illustration of the proof of Theorem 4.33.

By Euler's formula (1.17) $q_n(p)$ is a Möbius transformation of $q_0 = \coth p$ with polynomial coefficients in $1/p$ and therefore is meromorphic in $\mathbb{C}$. Observe now that if $u = q_n$ satisfies (4.97) then $v = 1/u$ satisfies

$$\frac{dv}{dp} = -\frac{2n}{p}v + 1 - v^2 \,. \tag{4.99}$$

Therefore poles of u become zeros of v and vice versa. Thus if either u or v vanishes then its derivative must be 1. It follows that all nonzero zeros and poles of q_n are simple and that

$$q_n(p) = \begin{cases} \dfrac{1}{p-a} + r_n(a,p) & \text{if a is a pole ,} \\ p - a + (p-a)^2 r_n(a,p) & \text{if a is a zero ,} \end{cases} \tag{4.100}$$

where $r_n(a,p)$ is analytic at $p = a$. Similar calculations with (4.97) show that if q_n has a pole at $p = 0$ then

$$q_n = \frac{2n+1}{p} + s_n(p) \,, \tag{4.101}$$

where $s_n(p)$ is holomorphic at 0. In fact $s_n(0) = 0$, which leads to the asymptotic formula

$$q_n(p) = \frac{2n+1}{p} + o(1) \,, \qquad p \to 0 \,. \tag{4.102}$$

If $n = 0$ then (4.102) follows from the McLaurin series for e^{2p}. Suppose now that $s_{n-1}(0) = 0$; then by (4.96) for $n-1$

$$\lim_{p \to 0} q_n(p) = \infty \,,$$

showing that q_n has a pole at $p=0$ and therefore implying (4.101). To prove that $s_n(0)=0$ we substitute (4.101) in (4.97):

$$\begin{aligned}&-\frac{2n+1}{p^2}+s_n'\\&\quad=\frac{2n(2n+1)}{p^2}+\frac{2ns_n}{p}+1-\frac{(2n+1)^2}{p^2}-\frac{2(2n+1)s_n}{p}-s_n^2\,.\end{aligned}$$

A calculation of the coefficients at $1/p$,

$$0=(2n-2(2n+1))s_n(0)=-2(n+1)s_n(0)\,,$$

competes the proof of (4.102).

If a poles or zero exists on $(0,+\infty)$ then it must be the zero or pole of the minimal value. Let it be $p=a$. Then q_n is positive near the left end of $(0,a)$ by (4.102) and q_n is negative near the right end of $(0,a)$ by (4.100). It follows that a continuous function q_n on $(0,a)$ must have a zero, which contradicts our choice of a. If there are no zeros or poles for q_n on $(0,+\infty)$ then q_n is continuous on $(0,+\infty)$ and is positive near 0 by (4.102). If $q_n(a)<0$ for some $a, a>0$, then q_n must vanish on $(0,a)$, which contradicts our assumption. Hence $q_n>0$ on $(0,+\infty)$. □

Having proved (4.93) we obtain that

$$\coth\frac{1}{2s}=\frac{e^{1/s}+1}{e^{1/s}-1}=2s+\frac{1}{6s}\,\frac{1}{+\,10s}\,\frac{1}{+\,14s}\,\frac{1}{+\,18s}\,+\cdots \tag{4.103}$$

for $s=1,2,\ldots$

Reverting to arguments used in the proof of (4.92), we get the following formula:

$$\begin{aligned}&a+\frac{1}{b}\,\frac{1}{+\,c}\,\frac{1}{+\,d}\,+\cdots\\&=a-n+\frac{mn+1}{m}\,\frac{1}{+\,n}\,\frac{1}{+\,b-m-n/\,mn+1}\,\frac{1}{+\,m}\,\frac{1}{+\,n}\,\frac{1}{+\,c-m-n/\,mn+1}\,\frac{1}{+\,m}\,+\cdots\,.\end{aligned} \tag{4.104}$$

In particular, for $m=n=1$,

$$\begin{aligned}&a+\frac{1}{b}\,\frac{1}{+\,c}\,\frac{1}{+\,d}\,+\cdots=\\&\qquad=a-1+\frac{2}{1}\,\frac{1}{+\,1}\,\frac{1}{+\,b/2-1}\,\frac{1}{+\,1}\,\frac{1}{+\,1}\,\frac{1}{+\,c/2-1}\,\frac{1}{+\,1}\,\frac{1}{+\,1}\,\frac{1}{+\,d/2-1}\,+\cdots\,,\end{aligned}$$

which implies the remarkable formula of Euler

$$e^{1/s}=1+\frac{1}{s-1}\,\frac{1}{+\,1}\,\frac{1}{+\,1}\,\frac{1}{+\,3s-1}\,\frac{1}{+\,1}\,\frac{1}{+\,1}\,\frac{1}{+\,5s-1}\,\frac{1}{+\,1}\,+\cdots\,. \tag{4.105}$$

Passing to the limit $s \to 1$ in (4.105), we obtain the regular continued fraction (4.91) for e. Formulas (4.91) and (4.105) coupled with Lagrange's theorem (see Theorem 2.19) show that neither e nor any of its integer roots satisfies a quadratic equation with rational coefficients.

102 Comments. In (1744) Euler stated only Theorem 4.32 and just mentioned that the convergence of the continued fraction implies (4.93). It is not clear whether Euler realized that the paradox of quadratic equations could occur in this case as well. Let us formally consider this point. In Theorem 4.32 Euler obtains a formula for q_n, which in terms of the independent variable p looks as follows:

$$q_n(p) = x^{2n/(2n+1)} y_n(x) = \left(\frac{p}{2n+1}\right)^{2n} y_n\left(\left(\frac{p}{2n+1}\right)^{2n+1}\right) .$$

Now y_n itself satisfies a differential equation depending on n. Passage to the limit shows that $y(x) = \lim_n y_n(x)$ satisfies

$$\frac{dy}{dx} + y^2 = \frac{1}{x^2} .$$

This differential equation has at least two solutions $y = a/x$, where a denotes either root of the quadratic equation $a^2 - a - 1 = 0$. Notice that one root is the golden ratio $a = \phi$ and the other is negative, $-1/\phi$. Therefore we are exactly in the situation of the paradox of quadratic equations. The fact that Euler paid attention to the transformation of q_n to a very special form indicates that seemingly he was aware of this difficulty. However, he could have done this in relation to Bernoulli's theorem on Riccati's equation, which we discuss later. In any case the asymptotic formula for $q_n(p)$ at $p = 0$ supports the conjecture of the positiveness of y_n at the points

$$\left(\frac{p}{2n+1}\right)^{2n+1} \longrightarrow 0 ,$$

which are essential for the convergence.

The elementary proof of Theorem 4.33 given above looks very much as though it is logically related to Euler's ideas. Notice that such computations with series were well known to Euler.

As follows from the correspondence between Euler and Goldbach, Euler's famous formulas relating the exponential and trigonometric functions were discovered later (in 1741). The formula $\cot p = i \coth ip$ formally leads to the expression

$$\begin{aligned} \cot p = i \coth ip &= \frac{1}{p} + \frac{i}{3/ip\, +} \, \frac{1}{5/ip\, + \cdots} \\ &= \frac{1}{p} - \frac{1}{3/p\, -} \, \frac{1}{5/p\, -} \, \frac{1}{7/p\, - \cdots} . \end{aligned} \tag{4.106}$$

It is not necessary to use Euler's trigonometric formulas to find this continued fraction, since it can be obtained similarly to the continued fraction of the hyperbolic cotangent. One can easily prove that the remainders u_n for the continued fraction of $\cot p$ satisfy the differential equation

$$\frac{du_n}{dp} = \frac{2n}{p} u_n - 1 - u_n^2 , \qquad n = 0, 1, 2, \ldots \tag{4.107}$$

In 1731–7 Euler did not have any criteria to control the convergence of the continued fraction for $\cot p$ and moreover u_n will not keep the same sign on $(0, +\infty)$, since every u_n has infinitely many poles and zeros on $(0, +\infty)$, as $\cot p$ itself has. Possibly this is the reason why Euler (1744) did not include the development of $\cot p$ into a continued fraction. Later Euler returned to this problem in his paper (1785b) presented to the Russian Academy in 1775 and proved the convergence of the continued fraction for cotangent. We consider this interesting proof later; see Corollary 4.40. Here we mention only that the required tools for this proof were available already in Euler's paper (1744, §§31–2).

Theorem 4.33 was proved in (1857) by Schlömilch; see Ex. 4.54. The fact that the q_n are positive also follows from Legendre's proof; see Rudio (1892) or Lang (1966). Both these proofs completely revised that of Euler, which resulted in a loss of Euler's clear logic.

Euler had these results already in November 1731, almost 45 years before Lagrange (1776) announced (18 July 1776) a method of solution of differential equations with continued fractions. Moreover, in (1744) Euler in fact solved in continued fractions Riccati's equation

$$\frac{dq}{dr}+q^2=nr^{n-2}\,. \tag{4.108}$$

4.11 Riccati's equation

103 Introduction to Riccati's equation. In 1763 D'Alembert named the differential equation

$$\frac{dy}{dx}=P(x)+Q(x)y+R(x)y^2 \tag{4.109}$$

"Riccati's (generalized) equation". It was an attempt to approximate the right-hand side $f(x,y)$ of a general differential equation $y'=f(x,y)$ by the sum of the first three terms of its Taylor series in y. Clearly the differential equations (4.97) are Riccati equations with $P(x)\equiv 1$, $Q(x)=2n/x$ and $R(x)\equiv -1$. The induction step $n\to n+1$ is based on formula (4.96) in which the first term approaches asymptotically the solution of Riccati's equation at the origin. In fact this is a key observation, which provides necessary cancellations. Since this method was successful in the special case investigated by Euler, see §**101**, one can apply it to find an exact (not asymptotic) solution of (4.109). Thus let y_1 satisfy (4.109). Following (4.96) we represent y as

$$y=y_1+\frac{1}{y_2}$$

and substitute this expression into (4.109). Then y_2 satisfies the linear differential equation

$$\frac{dy_2}{dx}=-\left(Q(x)+2y_1(x)R(x)\right)y_2-R(x)\,.$$

Since any linear differential equation $z'=Lz+M$ can be solved in quadratures,

$$z=e^{\int Ldx}\int M(x)e^{-\int Ldx}\,dx\,,$$

the same is true for Riccati's equation provided that there is a formula for one of its solutions. Notice that $\int Ldx$ enters the above formula twice with the same constant of integration. Euler discovered this substitution in 1760.

Let us apply this method to a Riccati equation that we have already considered,

$$\frac{dy}{dx}+y^2=\frac{1}{x^2}\,. \tag{4.110}$$

It has two explicit solutions,

$$y_1(x)=\frac{a_1}{x},\qquad y_0(x)=\frac{a_2}{x}\,,$$

where $a_1 = \phi$ and $a_2 = -1/\phi$ are the solutions to the quadratic equation $a^2 - a - 1 = 0$. Euler's substitution corresponding to $y_1 = a_1/x$ results in the linear differential equation

$$\frac{dy_2}{dx} = \frac{2a_1}{x} y_2 + 1 \, .$$

Hence

$$\begin{aligned} y_2 &= e^{\int (2a_1/x)dx} \int e^{-\int (2a_1/x)dx} \, dx = x^{2a_1} e^C \int x^{-2a_1} e^{-C} dx \\ &= \frac{x}{1-2a_1} + E x^{2a_1} = -\frac{x}{\sqrt{5}} + D x^{1+\sqrt{5}} \, , \end{aligned}$$

where $D \in \mathbb{R}$. This shows that the general solution to (4.110) is given by

$$y(x) = \frac{1+\sqrt{5}}{2x} - \frac{\sqrt{5}}{x(1+Dx^{\sqrt{5}})} \, , \qquad D \in \mathbb{R} \, . \tag{4.111}$$

Putting $D = 0$ we obtain $y = a_2/x$. To make this formula universal we may allow $D = \infty$. Notice that there is only one solution (corresponding to $D = \infty$) which is positive about $x = 0+$. This is exactly the reason why the continued fraction for hyperbolic cotangent converges.

Definition 4.34 *A differential equation is said to be integrable by quadratures if the unknown function can be expressed in terms of algebraic functions, exponentials, logarithms and the operation of integration* (see Ritt 1948, p. 69).

Since (4.110) can be integrated in finite terms (and therefore by quadratures), the natural question arises of how describe those Riccati equations which can be integrated by quadratures. In Theorem 4.32 explicit solutions to some Riccati equations of the type

$$\frac{dy}{dx} = ax^n + by^2 \tag{4.112}$$

were given. From this point of view it is now clear why Euler rewrote the differential equations (4.97) in the form (4.95). This resulted in the following beautiful theorem.

Theorem 4.35 (D. Bernoulli–Euler) *If* $ab = 0$ *or*

$$n = -2 \text{ or } \quad n = -\frac{4m}{2m \pm 1}, \quad m = 0, 1, 2, \dots \, ,$$

then Riccati's equation (4.112) *is integrable by quadratures.*

Proof We start with the differential equation for hyperbolic cotangent

$$\frac{dy}{dx} = 1 - y^2 \, .$$

The continued fraction for $q = \coth p$ generates a sequence of Riccati equations for its remainders. Euler's substitution (4.98) transforms them into differential equations (4.95). The change of variables $z = ky(cx)$ in (4.95) transforms it to the Riccati equation

$$\frac{dz}{dx} = kc^{-(2n-1)/(2n+1)} x^{-4n/(2n+1)} - \frac{c}{k} z^2 \, .$$

By taking the sign of k to be opposite to that of b we make $c = -bk > 0$. Then there is a unique value of k such that the coefficient $kc^{-(2n-1)/(2n+1)}$ equals a. This covers the case $ab < 0$.

If $ab > 0$ then we consider the differential equations (4.107) for the continued fraction of $q = \cot p$. Then substitution (4.98) reduces (4.107) to

$$\frac{dy}{dx} = -x^{-4n/(2n+1)} - y^2$$

and we can easily handle the case $ab > 0$ in the way we did for $ab < 0$.

To cover the remaining case, $n = -4m/(2m-1)$, we simply continue the equations (4.97) by the same formulas, allowing n to be negative. This results in the construction of a so-called *ascendant* continued fraction and at the same time provides a solution by quadratures to (4.112) with negative n and $ab < 0$. The ascendant continued fraction for $q = \cot p$ does the same job for the case $ab > 0$. □

This theorem was stated and proved in the first mathematical paper by D. Bernoulli (1724) but it was Euler who discovered the important relationship of this question to continued fractions. Liouville (1839, 1841) proved that these are the only cases of integrability by quadratures of Riccati's equations (4.112). See Ritt (1948).

104 Evaluation of continued fractions by Riccati's equations. We consider here the continued fraction

$$b_0 + \frac{c}{b_1} {}_+ \frac{c}{b_2} {}_+ \cdots {}_+ \frac{c}{b_n} {}_+ \cdots \tag{4.113}$$

where $b_n = b + nd$, $b_n \neq 0$ and $n = 0, 1, \ldots, d \neq 0$. By the definition of a continued fraction $c \neq 0$. By Corollary 3.10 this continued fraction converges if $d > 0$ and $c > 0$. The parameters $\{b_n\}_{n \geqslant 0}$ and c define two power series in x,

$$\begin{aligned} p(x) = p(x; b, c, d) &= 1 + \sum_{k=1}^{\infty} \frac{x^{kd}}{k!\, b_0 \cdots b_{k-1}} \frac{c^k}{d^k} \, , \\ q(x) = q(x; b, c, d) &= \frac{1}{b_0} + \sum_{k=1}^{\infty} \frac{x^{kd}}{k!\, b_0 \cdots b_k} \frac{c^k}{d^k} \, , \end{aligned} \tag{4.114}$$

which obviously converge by the ratio test. For instance, denoting by a_k the term of the series for p, we have

$$\frac{a_k}{a_{k-1}} = \frac{x^d\, c}{kd(b + (k-1)d)} \quad \to \quad 0 \, , \quad k \to +\infty \, . \tag{4.115}$$

Theorem 4.36 (Euler 1785b) *For any nonzero b, c, d with nonzero $b+nd$, $n = 0, 1, 2, \ldots$ the continued fraction* (4.113) *converges to*

$$b_0 + \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{c}{b_n} \right) = \frac{p(1;\, b,\, c,\, d)}{q(1;\, b,\, c,\, d)}\,.$$

Proof Let P_n/Q_n be convergents of the continued fraction $b_0 + \mathbf{K}_n(1/b_n)$. Putting $\xi_n = P_n/(b_0 b_1 \cdots b_n)$ and using the Euler–Wallis formulas, we will prove by induction that

$$\xi_n = 1 + \sum_{k=0}^{n/2} \frac{(n-k)\cdots(n-2k)\, c^{k+1}}{(1+k)!\, b_0 \cdots b_k b_{n-k} \cdots b_n}\,. \tag{4.116}$$

Since $\xi_0 = 1$ and $\xi_1 = 1 + c/b_0 b_1$, formula (4.116) holds for $n = 0$ and $n = 1$. For $n = 2$ it turns into

$$\frac{2c}{b_0 b_2} = \frac{c}{b_0 b_1} + \frac{c}{b_1 b_2}\,,$$

which is equivalent to the identity $2b_1 = b_0 + b_2$ satisfied by any arithmetic progression.

The Euler–Wallis formula in terms of the ξ_n takes the form

$$\xi_{n+1} = \xi_n + \frac{c}{b_n b_{n+1}} \xi_{n-1}\,.$$

Assuming that (4.116) holds for indices not exceeding n, we may write, grouping terms,

$$\begin{aligned}\xi_{n+1} = 1 + \frac{c}{b_n} \left(\frac{n}{b_0} + \frac{1}{b_{n+1}} \right) + \frac{(n-1)\, c^2}{1!\, b_0 b_{n-1} b_n} \left(\frac{n-2}{2b_1} + \frac{1}{b_{n+1}} \right) \\ + \frac{(n-2)(n-3)\, c^3}{2!\, b_0 b_1 b_{n-2} b_{n-1} b_n} \left(\frac{n-4}{3b_2} + \frac{1}{b_{n+1}} \right) + \cdots\end{aligned}$$

For every j we have

$$\frac{n-2j}{(j+1)b_j} + \frac{1}{b_{n+1}} = \frac{(n-2j)b_{n+1} + (j+1)b_j}{(j+1)b_j b_{n+1}} = \frac{(n-j+1)b_{n-j}}{(j+1)b_j b_{n+1}}\,,$$

since $\{b_k\}$ is an arithmetic progression. Substituting this into the formula for ξ_{n+1}, we obtain (4.116) for $n+1$. Passing now to the limit in (4.116), we see that $\lim_n \xi_n = p(1;\, b,\, c,\, d)$.

Similarly, if $\eta_n = Q_n/(b_0 \cdots b_n)$, then

$$\eta_n = \frac{1}{b_0} + \sum_{k=0}^{(n-1)/2} \frac{(n-k-1)\cdots(n-2k-1)\, c^{k+1}}{(1+k)!\, b_0 \cdots b_{k+1} b_{n-k} \cdots b_n}\,. \tag{4.117}$$

Passing to the limit in (4.117), we obtain $\lim_n \eta_n = q(1;\, b,\, c,\, d)$. $\square$

The functions p and q in (4.114) satisfy the system of differential equations

$$q(x) = \frac{x^{1-d}}{c}\frac{dp}{dx}\,, \tag{4.118}$$

$$\frac{d(x^b q)}{dx} = x^{b-1} p\,. \tag{4.119}$$

To prove (4.118) we differentiate the first formula of (4.114) with respect to x:

$$\frac{x^{1-d}}{c}\frac{dp}{dx} = \sum_{k=1}^{\infty} \frac{k d x^{kd-1} x^{1-d}}{k!\, b_0 \cdots b_{k-1}} \frac{c^{k-1}}{d^k} = \sum_{k=1}^{\infty} \frac{x^{(k-1)d}}{(k-1)!\, b_0 \cdots b_{k-1}} \frac{c^{k-1}}{d^{k-1}} = q\,,$$

whereas (4.119) follows from the second:

$$\frac{d(x^b q)}{dx} = x^{b-1} + \sum_{k=1}^{\infty} \frac{(b+kd) x^{kd+b-1}}{k!\, b_0 \cdots b_k} \frac{c^k}{d^k} = x^{b-1} p\,,$$

since $b + kd = b_k$. The system (4.118), (4.119) implies that p satisfies

$$\frac{d^2 p}{dx^2} + \frac{b-d+1}{x}\frac{dp}{dx} - c\,x^{d-2} p(x) = 0\,,$$

with the obvious boundary conditions $p(0) = 1$ and

$$\frac{dp}{dx}(0) = \begin{cases} 0 & \text{if } d > 1\,, \\ 1/b & \text{if } d = 1\,, \\ \infty & \text{if } d < 1\,. \end{cases}$$

Lemma 4.37 *The function $z(x) = p(x)/q(x)$ is a solution to Riccati's equation*

$$\frac{dz}{dx} = -\frac{1}{x} z^2 + \frac{b}{x} z + c\,x^{d-1} \tag{4.120}$$

satisfying the boundary condition $z(0) = b$.

Proof Since $p = zq$ and $dp = c\,x^{d-1} q dx$ by (4.118),

$$c\,x^{d-1} dx = dz + z\frac{dq}{q} \qquad \Rightarrow \qquad \frac{dq}{q} = \frac{c\,x^{d-1} dx - dz}{z}\,.$$

Combining this with an equivalent form of (4.119),

$$p x^{b-1} dx = b x^{b-1} q dx + x^b dq$$

$$\Leftrightarrow \quad p dx = b q dx + x dq \quad \Leftrightarrow \quad z = b + \frac{x}{q}\frac{dq}{dx}\,,$$

and excluding dq/q, we complete the proof of (4.120). The equality $z(0) = b$ follows from (4.114) with $x = 0$. $\square$

Notice that the substitutions $y = zx^{-b}$, $t = x^b$ transform (4.120) into the Riccati equation

$$b\frac{dy}{dt} + y^2 = c\,t^{d/b-2}\,. \tag{4.121}$$

Theorem 4.38 *For $b > 0$, $d > 0$ and $c \neq 0$,*

$$b + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{c}{b+nd}\right) = y(1)\,, \tag{4.122}$$

where y is the solution of the Riccati equation (4.121) *which has the asymptotic value*

$$y(t) \sim \frac{b}{t}\,, \qquad t \longrightarrow 0^+\,. \tag{4.123}$$

Proof By Theorem 4.36 and Lemma 4.37, formula (4.122) holds for a solution y_1 of (4.121). Since $q(0) = 1/b \neq 0$ in (4.114), elementary computations with power series show that, for some $\varepsilon > 0$ and $r = d/b$,

$$y_1(t) = \frac{z(t^{1/b})}{t} = \frac{b}{t} + \sum_{k=1}^{\infty} c_k t^{kr-1}\,, \qquad 0 \leqslant t \leqslant \varepsilon\,, \tag{4.124}$$

which proves (4.123). We apply Euler's method (see §**103** at the start of Section 4.1) to equation(4.121) and to its solution y_1, which can be represented by Euler's formulas (4.114). We have $L(t) = 2y_1(t)/b$, $M(t) = 1/b$. By (4.124),

$$\int_{\varepsilon}^{x} L(t)dt = \ln x^2 + C + \sum_{k=1}^{\infty}\frac{2c_k}{kd}\,x^{kr} = \ln x^2 + C + \xi(x)\,.$$

This implies that for $0 < x < \varepsilon$

$$\begin{aligned}
\int_{\varepsilon}^{x}\frac{1}{t^2}e^{\xi(x)-\xi(t)}\,dt &= \int_{1}^{x}\frac{1}{t^2}\,dt + O(1)\int_{x}^{\varepsilon}\frac{|\xi(x)-\xi(t)|}{t^2}\,dt \\
&= -\frac{1}{x} + C + O(1)\int_{x}^{\varepsilon}\frac{t^r - x^r}{t^2}\,dt = -\frac{1}{x} + C + O(x^{r-1})\int_{1}^{\varepsilon/x}\frac{t^r-1}{t^2}\,dt \\
&= -\frac{1}{x} + C + O(x^{r-1}) + O\left(\ln\frac{\varepsilon}{x}\right) \\
&= -\frac{1}{x}\left\{1 + O\left(x\ln\frac{\varepsilon}{x} + x^r\right)\right\}\,, \quad x \to 0^+\,.
\end{aligned}$$

Since in our case $M(x) = 1/b$,

$$\begin{aligned}
y_2(x) &= \frac{x^2}{b}\left\{\int_{1}^{x}\frac{e^{\xi(x)-\xi(t)}}{t^2}\,dt + C\right\} \\
&= -\frac{x}{b}\left\{1 + O\left(x\ln\frac{\varepsilon}{x} + x^r\right)\right\}\,, \quad x \to 0^+\,.
\end{aligned}$$

It follows that

$$y(x)=\frac{b}{x}-\frac{b}{x}\frac{1}{1+O\left(x\ln\varepsilon/x+x^{r}\right)}=O\left(\ln\frac{\varepsilon}{x}+x^{r-1}\right)\,,\qquad x\to 0^{+}\,,$$

which does not allow other solutions to have the same asymptotic expansion at $x=0$ as $y_1(x)$. $\square$

The condition that implies $d\neq 0$ in Theorem 4.38 is essential. We have already found all solutions to (4.110); see formula (4.111). Equation (4.110) coincides with (4.121) if $b=c=1$ and $d=0$. In this case (4.122) with $y(t)=\phi/t$ turns into the continued fraction for the golden ratio ϕ.

Corollary 4.39 *For every $s>0$,*

$$\coth\frac{1}{s}=s+\frac{1}{3s}\,\frac{}{+}\,\frac{1}{5s}\,\frac{}{+}\,\frac{1}{7s}\,\frac{}{+\cdots}\,. \tag{4.125}$$

Proof Let $b=s$, $c=1$, $d=2s$. Then equation (4.121) takes the form $s\,dy/dt+y^2=1$. This equation can be easily solved in quadratures by Euler's method, see Ex. 4.48, to find $y(t)=\coth(t/s+\phi)$, where ϕ is a real constant. Among these solutions there is only one, $y(t)=\coth(t/s)$, satisfying (4.123). Thus the proof is completed by Theorem 4.38. $\square$

Equation (4.125) leads to an alternative proof for the continued fraction for hyperbolic cotangent, see (4.93).

Corollary 4.40 *For every $s>0$,*

$$\cot\frac{1}{s}=s-\frac{1}{3s}\,\frac{}{-}\,\frac{1}{5s}\,\frac{}{-}\,\frac{1}{7s}\,\frac{}{-\cdots}\,.$$

Proof Let $b=s$, $c=-1$, $d=2s$. Then equation (4.121) takes the form $s\,dy/dt+y^2=-1$. All solutions to this differential equation are listed by the formula $y(t)=\cot(t/s+\phi)$, see Ex. 4.49. Among them there is only one, $y(t)=\cot(t/s)$, satisfying (4.123). $\square$

By (4.115) the functions $p(x;b,c,d)$ and $q(x;b,c,d)$ in (4.114) are both meromorphic functions of their parameters. In particular, in the case of the cotangent,

$$p\,(x;s,-1,2s)=1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,(2k-1)!!}\left(\frac{x^{2s}}{2s}\right)^k\,,$$

$$q\,(x;s,-1,2s)=\frac{1}{s}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,(2k+1)!!}\left(\frac{x^{2s}}{2s}\right)^k\,,$$

have at most two (essential) singular points at $s=0$ and $s=\infty$. Hence $p\,(1;s,-1,2s)$ and $q\,(1;s,-1,2s)$ may have only isolated zeros of finite order, tending to an accumulation point at $s=0$. By Theorem 4.36 this implies the convergence of the continued fraction for cotangent to the cotangent function *everywhere* in the complex plane except for $s = 0$. In fact Theorem 4.36 claims that the continued fraction of cotangent converges to infinity at its poles.

105 Riccati's equations and continued fractions. The method of Euler lists all solutions of Riccati equations if one is known. Here we present Euler's method of obtaining formulas for such solutions in continued fractions. Following Euler we consider the Riccati equation

$$dy + ay^2 dx = acx^{2m} dx\,. \tag{4.126}$$

We may assume that $m \neq -1$, since otherwise for any γ satisfying $aX^2 - X - ac = 0$ there is a rational solution γ/x. If $1 + 4a^2c < 0$ then this solution is complex, but Euler's method works in this case as well. We may also exclude the trivial case $a = 0$ and consider only the case $a > 0$. The case $a < 0$ reduces to the latter if y is replaced by $-y$. Dividing both sides of (4.126) by a, we arrive at equation (4.121) with $b = 1/a$, $c = c$, $2m = d/b - 1$. Since $m \neq -1$, $d \neq 0$. Moreover, $d > 0$ if $m > -1/2$. Hence if $a > 0$, $m > -1/2$ then by Lemma 4.37 equation (4.126) has a meromorphic solution on $(0, +\infty)$ given by an explicit formula. This solution satisfies at 0^+ the asymptotic formula

$$y(x) \sim (ax)^{-1}\,, \quad x \to 0^+\,.$$

The change of variables $y = t^{m/(m+1)}z(t)$, $x = t^{1/(m+1)}$ transforms (4.126) into another Riccati equation,

$$\frac{dz}{dt} - \alpha\frac{z}{t} + \beta z^2 = \beta c\,, \tag{4.127}$$

where $\alpha = -m/(m+1)$, $\beta = a/(m+1)$. It also transforms the solution y to (4.126), meromorphic on $(0, +\infty)$, to a meromorphic solution $z = z_0$ of (4.127) satisfying the same asymptotic condition at 0. Observing that $(\alpha + 1)/\beta = 1/a$, we represent z_0 as

$$z_0 = \frac{\alpha + 1}{\beta t} + \frac{c}{z_1}\,.$$

Substituting this expression for z_0 in (4.127), we obtain that z_1 satisfies

$$\frac{dz_1}{dt} - (\alpha + 2)\frac{z_1}{t} + \beta z_1^2 = \beta c\,.$$

In general, if

$$\frac{dz_n}{dt} - (\alpha + 2n)\frac{z_n}{t} + \beta z_n^2 = \beta c \tag{4.128}$$

then z_{n+1} defined by

$$z_n = \frac{\alpha + 2n + 1}{\beta t} + \frac{c}{z_{n+1}}$$

satisfies

$$\frac{dz_{n+1}}{dt} - (\alpha + 2n + 2)\frac{z_{n+1}}{t} + \beta z_{n+1}^2 = \beta c\,. \tag{4.129}$$

As in the case of the hyperbolic cotangent this gives the following finite continued fraction

$$z(t) = \frac{\alpha+1}{\beta t} + \frac{c}{(\alpha+3)/\beta t} \genfrac{}{}{0pt}{}{}{+} \frac{c}{(\alpha+5)/\beta t} \genfrac{}{}{0pt}{}{}{+\cdots+} \frac{c}{(\alpha+2n-1)/\beta t + c/z_n}\,.$$

Assuming now that $c > 0$ and taking into account that $\beta = a/(m+1) > 0$, we see that this continued fraction converges. Equation (4.128) shows that the graph of z_n intersects the real axis in the upward direction, which as in the case of the hyperbolic cotangent leads to the conclusion that the z_n are all positive on $(0, +\infty)$. Hence

$$z(t) = \frac{\alpha+1}{\beta t} + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{c}{(\alpha+2n+1)/\beta t}\right)$$

and for $\alpha = 0$, $\beta = c = 1$ we again obtain the continued fraction for coth t. Returning to equation (4.126), we conclude that

$$y(x) = \frac{1}{ax} + \frac{ac\,x^{2m+1}}{2m+3} \genfrac{}{}{0pt}{}{}{+} \frac{a^2c\,x^{2m+2}}{4m+5} \genfrac{}{}{0pt}{}{}{+} \frac{a^2c\,x^{2m+2}}{6m+7} \genfrac{}{}{0pt}{}{}{+\cdots} \tag{4.130}$$

is a solution to (4.126) on $(0, +\infty)$. If here $m = 0$, $a = 1$, $c = 1$ then (4.130) turns into the continued fraction (4.93) for hyperbolic cotangent. Putting $a = 1$, $c = n$, $2m = n-2$ we obtain a solution to equation (4.108):

$$q(r) = \frac{1}{r} + \frac{n\,r^{n-1}}{n+1} \genfrac{}{}{0pt}{}{}{+} \frac{n\,r^n}{2n+1} \genfrac{}{}{0pt}{}{}{+} \frac{n\,r^n}{3n+1} \genfrac{}{}{0pt}{}{}{+\cdots}\,.$$

Exercises

4.1 If $\{b_n\}_{n\geqslant 1}$ is an increasing positive sequence $\lim b_n = +\infty$, show that

$$\frac{1}{b_1} - \frac{1}{b_2} + \frac{1}{b_3} - \frac{1}{b_4} + \cdots = \frac{1}{b_1} \genfrac{}{}{0pt}{}{}{+} \frac{b_1^2}{b_2 - b_1} \genfrac{}{}{0pt}{}{}{+} \frac{b_2^2}{b_3 - b_3} \genfrac{}{}{0pt}{}{}{+} \frac{b_3^2}{b_4 - b_3} \genfrac{}{}{0pt}{}{}{+\cdots}$$

4.2 Prove that

$$\log(1+x) = \frac{1}{1} \genfrac{}{}{0pt}{}{}{+} \frac{1^2}{2-1x} \genfrac{}{}{0pt}{}{}{+} \frac{2^2}{3-2x} \genfrac{}{}{0pt}{}{}{+} \frac{3^2}{4-3x} \genfrac{}{}{0pt}{}{}{+\cdots}\,.$$

4.3 Deduce Theorem 3.12 from Theorem 4.5.

4.4 Using Euler's transformation of series to continued fractions prove the following formula for Catalan's constant:

$$C=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}=\frac{1}{1}{}_{+}\frac{1^4}{8}{}_{+}\frac{3^4}{16}{}_{+}\frac{5^4}{24}{}_{+}\cdots$$

(Bowman and Mc Laughlin 2002).

4.5 Prove that

$$\sin x=\frac{x}{1}{}_{+}\frac{x^2}{2\times 3-x^2}{}_{+}\frac{2\times 3x^2}{4\times 5-x^2}{}_{+}\frac{4\times 5x^2}{6\times 7-x^2}{}_{+}\cdots .$$

4.6 Investigate the convergence of the formal continued fraction (Euler 1750b, §6)

$$\int_0^1\frac{x^{n-1}\,dx}{(1+x^m)^{\mu/\nu}}=\frac{1}{n}{}_{+}\frac{\mu n^2}{\nu m}{}_{+}\frac{\nu(\mu+\nu)(m+n)^2}{(3\nu-\mu)m+(\nu-\mu)n}$$
$${}_{+}\frac{2\nu(\mu+2\nu)(2m+n)^2}{(5\nu-2\mu)m+(\nu-\mu)n}{}_{+}\frac{3\nu(\mu+3\nu)(3m+n)^2}{(7\nu-3\mu)m+(\nu-\mu)n}{}_{+}\cdots .$$

Hint: Observe that by the binomial theorem $(0<r<1)$

$$\int_0^r\frac{x^{n-1}dx}{(1+x^m)^{\mu/\nu}}=\sum_{k=0}^{\infty}(-1)^k\frac{\mu\cdots(\mu+\nu(k-1))}{k!\,\nu^k(n+km)}r^{n+km} .$$

Let u_k be the modulus of the coefficient at r^{n+mk}. Apply Lemma 4.4 to show that

$$\int_0^1\frac{x^{n-1}\,dx}{(1+x^m)^{\mu/\nu}}=\sum_{k\geqslant 0}(-1)^k u_k ,$$

if this series converges. Deduce from the identity

$$\frac{u_{k+1}}{u_k}=1+\frac{km(\mu-2\nu)+n\mu-\nu(n+m)}{(\nu+k\nu)(n+m+km)}$$

that u_k increases if $\mu/\nu>2$ and $\lim u_k\neq 0$ if $\mu/\nu=2$. Apply Theorem 4.5 to prove that the continued fraction diverges. In the case $\mu/\nu=2$, separately indicated in Euler (1750b, **9**) prove that for $m>n$ the even convergents and odd convergents converge to different limits. Prove that $\lim u_k=0$ if $\mu<2\nu$ and deduce from this that the continued fraction converges.

4.7 Prove Euler's formula (see Euler 1785a)

$$\frac{4}{\pi-2}=3+\frac{1\times 3}{4}{}_{+}\frac{3\times 5}{4}{}_{+}\frac{5\times 7}{4}{}_{+}\frac{7\times 9}{4}{}_{+}\cdots .$$

Hint: Apply Theorem 3.12 with $a=4$, $y_0=3$, $y_1=5$, $y_2=7$, ... Use formula (4.64) for $s=1$.

4.8 Justify Euler's claim (see Euler 1785a, 1750b, §9) that

$$1+\frac{1\times 3\times 1^2}{1}\,\underset{+}{}\,\frac{2\times 4\times 2^2}{1}\,\underset{+}{}\,\frac{3\times 5\times 3^2}{1}\,\underset{+}{}\,\frac{4\times 6\times 4^2}{1}\,\underset{+\cdots}{}=\frac{2}{1/2+\log 2}\,.$$

Hint: Apply Corollary 3.9 to check that the continued fraction diverges. Notice that

$$\frac{1}{1+s}+\frac{\log(1+s)}{s}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n}s^{n-1}$$

for $0<s<1$. Apply Theorem 4.5 with $x_n=(n+1)s^{n-1}$, $y_n=n$ to prove that the above expression equals

$$2\left(1+\mathop{\mathbf{K}}_{n=2}^{\infty}\left(\frac{(n-1)^3(n+1)s}{(1-s)n^2+s}\right)\right)^{-1}$$

for $0<s<1$. Check the convergence of this continued fraction using Corollary 3.10. This example shows that even for positive continued fractions there are cases where summation of a series is possible by an analytic continuation and may be not possible by the method of continued fractions. Another such example is given by Brouncker's continued fraction $b(s)$ at $s=0$.

4.9 Prove Euler's formula (Euler 1785a) for alternating triangular numbers $\binom{2n}{2}$:

$$1+\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\binom{2n}{2}\Big/(n+1)\right)=\frac{\sqrt{3}}{2\log\left(1+\sqrt{3}\right)\Big/\sqrt{2}}\,.$$

Hint: Observe that

$$v=\frac{1}{\sqrt{1+z^2}}\int\frac{dz}{\sqrt{1+z^2}}=\frac{\log(z+\sqrt{1+z^2})}{\sqrt{1+z^2}}$$

satisfies the differential equation $(1+z^2)dv/dz+vz=1$ and hence

$$\frac{z\log(z+\sqrt{1+z^2})}{\sqrt{1+z^2}}=z^2-\frac{2}{3}z^4+\frac{2\times 4}{3\times 5}z^6-\frac{2\times 4\times 6}{3\times 5\times 7}z^8+\cdots$$

Putting $z^2=x/y$, obtain the continued fraction

$$\frac{\sqrt{x}(x+y)}{\log(\sqrt{x}+\sqrt{x+y})\Big/\sqrt{y}}=y+\frac{1\times 2xy}{3y-2x}\,\underset{+}{}\,\frac{3\times 4xy}{5y-4x}\,\underset{+}{}\,\frac{5\times 6xy}{7y-6x}\,\underset{+\cdots}{}\,.$$

Now put $x=1$, $y=2$.

4.10 Prove that

$$\frac{4}{\log 3}=3+\frac{1\times 2\times 3}{7}\,\underset{+}{}\,\frac{3\times 4\times 3}{11}\,\underset{+}{}\,\frac{5\times 6\times 3}{15}\,\underset{+}{}\,\frac{7\times 8\times 3}{19}\,\underset{+\cdots}{}\,.$$

Hint: Put $x=1$, $y=3$ in Ex. 4.9.

4.11 Every odd prime number p can be uniquely represented as $4n+1$ or $4n-1$. Hence every odd prime is a neighbor of a multiple of four. Prove Euler's version of Wallis' formula for primes:

$$\frac{\pi}{4} = \frac{3}{4} \times \frac{5}{4} \times \frac{7}{8} \times \frac{11}{12} \times \frac{13}{12} \times \frac{17}{16} \times \frac{19}{20} \times \frac{23}{24} \times \frac{29}{28} \times \cdots$$

Hint: Use Euler's formula, for $s > 1$,

$$\prod_p \left(1 \pm \frac{1}{p^s}\right)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} ,$$

where the positive sign is taken if $p = 4k-1$ and the negative sign if $p = 4k+1$; Landau (1974, vol. 1, §109).

4.12 Prove that for positive p, q, r

$$\begin{aligned} &\frac{p(p+2q-r)}{p+r} \frac{\int_0^1 x^{p+2q-1}(1-x^{2r})^{-1/2}\,dx}{\int_0^1 x^{p+2r-1}(1-x^{2r})^{-1/2}\,dx} \\ &\qquad = p+q-r+\cfrac{q(r-q)}{p+q+\mathop{\mathbf{K}}\limits_{n=0}^{\infty}\{(p+nr)(p+2q+(n-1)r)/2r\}} . \end{aligned} \tag{E4.1}$$

Hints: Apply the Brouncker–Euler interpolation method to the hypergeometric series considered in Euler (1750b, §21). Construct $x_n > 0$ satisfying

$$x_n x_{n+1} = \frac{p+nr}{p+2q+nr} , \qquad n = 0, 1, 2, \ldots .$$

Apply Theorem 4.12 to find x_0 and prove that

$$x_n = 1 - \frac{q}{r}\frac{1}{n} + \frac{q(2p+2q-r)}{2r^2}\frac{1}{n^2} + O\left(\frac{1}{n^3}\right) .$$

Consider $y_n = (p+2q+(n-1)r)x_n$ and check that

$$\begin{aligned} y_n y_{n+1} &= (p+nr)(p+2q+(n-1)r) , \\ y_n &= nr + (p+q-r) + \frac{q(r-q)}{2r}\frac{1}{n} + O\left(\frac{1}{n^2}\right) . \end{aligned} \tag{E4.2}$$

Define z_n by $y_n = p+q+(n-1)r+q(r-q)/z_n$ and check that,

$$z_n - (p+q+nr) = \frac{(p+nr)(p+2q+(n-1)r)}{2r+z_{n+1}-(p+q+(n+1)r)} .$$

Use (E4.2) to show that $z_n = 2nr + O(1/n)$. Apply Markoff's test, Corollary 4.13 and Lemma 4.11.

4.13 Prove that

$$\frac{\int_0^1 (1-x^4)^{-1/2}dx}{\int_0^1 x^2(1-x^4)^{-1/2}dx} = 2 + \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{(2n-1)^2}{4}\,. \tag{E4.3}$$

Hint: Put $p=q=1$, $r=2$ in (E4.1) and apply Lemma 4.11. Notice that this is Brouncker's continued fraction for $s=2$, which is an analogue of Brouncker's continued fraction (3.19) for the lemniscate of Bernoulli.

4.14 Prove that

$$\frac{1\times 5}{3\times 3}\,\frac{5\times 9}{7\times 7}\,\frac{9\times 13}{11\times 11}\,\frac{13\times 17}{15\times 15}\cdots = \frac{1}{2+\mathop{\mathbf{K}}_{n=1}^{\infty}((2n-1)^2/4)}\,. \tag{E4.4}$$

Hint: Apply Euler's product (4.34) to (E4.3).

4.15 Consider an alternative proof of (E4.4). Take Brouncker's sequence y_n satisfying (3.28) and apply Brouncker's method to prove that y_0 equals the infinite product in (E4.4). Apply Euler's method presented in Lemma 4.7 to prove that it equals the value of the continued fraction in (E4.4).

4.16 Prove that

$$\frac{\pi-3}{4-\pi} = \frac{1^2}{4}\,{}_{+}\,\frac{2^2}{1}\,{}_{+}\,\frac{3^2}{4}\,{}_{+}\,\frac{4^2}{1}\,{}_{+}\,\frac{5^2}{4}\,{}_{+}\,\frac{6^2}{1}\,{}_{+}\,\frac{7^2}{4}\,{}_{+}\cdots$$

Hint: Using Lemma 4.11 evaluate the left-hand side of (E4.1) for $p=2$, $q=1$, $r=2$. Apply an equivalence transform to the right-hand side.

4.17 Prove that, for $r>0$ and $m>0$,

$$\int_0^1 \frac{x^{m-1}\,dx}{\sqrt{1-x^{2r}}} = \frac{1}{r}\,\frac{2(m+r)}{3m}\,\frac{4(m+3r)}{5(m+2r)}\,\frac{6(m+5r)}{7(m+4r)}\cdots$$

Hint: Apply Theorem 4.12 with $\mu=n$.

4.18 Prove Euler's formula for the beta function (4.30).

Hint: Apply Theorem 4.12 with $\mu=1$, $n=1$, $m=p$, $k=q$:

$$\begin{aligned} B(p,q) &= \frac{p+q}{(1+q)pq}\prod_{j=1}^{\infty}\frac{(1+1/j)\,e^{-1/j}\,(1+(p+q)/j)\,e^{-(p+q)/j}}{(1+(1+q)/j)\,e^{-(1+q)/j}\,(1+p/j)\,e^{-p/j}} \\ &= \frac{p+q}{(1+q)pq}\,\frac{(1+q)\Gamma(1+q)p\Gamma(p)e^{\gamma(p+1+q)}}{\Gamma(1)(p+q)\Gamma(p+q)e^{\gamma(p+1+q)}} = \frac{\Gamma(q)\Gamma(p)}{\Gamma(p+q)}\,; \end{aligned}$$

see (3.59) and (3.42).

4.19 Prove that for positive p, q, r

$$\frac{p+2q-r}{p}\frac{\int_0^1 x^{p+r-1}(1-x^{2r})^{q/r-1}\,dx}{\int_0^1 x^{p-1}(1-x^{2r})^{q/r-1}\,dx}$$
$$= 1+\frac{2(q-r)}{p+r+\underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\dfrac{(p+nr)(p+2q+(n-2)r)}{r}\right)}\ ,$$

see Euler (1750b, §§21–5).

Hints: Using (4.17) and (4.19) as motivation write down

$$y_n = m+(n-1)r+\frac{1}{z_n}\ , \qquad n=0,1,\dots\ ,$$

with a free parameter m. Apply (E4.2) to prove that

$$z_n = \frac{(m+nr)z_{n+1}+1}{(P+nQ)z_{n+1}-(m-r+nr)}\ ,$$

with P and Q as defined in §22 of the Appendix. Use the asymptotic formula

$$(2q-2r)z_n = 2+\frac{q}{rn}+O\left(\frac{1}{n^2}\right)\ ,$$

and Markoff's test to complete the proof of convergence.

4.20 Give a rigorous proof of Theorem 4.16.

Hints: Extend Brouncker's approach presented in Chapter 3. Namely, let $a_k = (kr+q)(kr+r-q)$ and P_n/Q_n be the convergents to (4.45). Using the Euler–Wallis formulas check that $P_n(x)$ is even for odd n and odd for even n, whereas Q_n is even for even n and odd for odd n. Prove that

$$P_n(q) = q(q+r)\cdots(q+nr)\ , \tag{E4.5}$$

$$P_n(r-q) = (r-q)(2r-q)\cdots(nr+r-q)\ . \tag{E4.6}$$

Lemma 4.41 *For* $n = 0, 1, \dots$

$$P_n(s)P_n(s+r)-(s+q)(s+r-q)Q_n(s)Q_n(s+r) = (-1)^{n+1}\prod_{j=0}^{n} a_j\ .$$

Hints: Since $Q_{n+1}P_n - P_{n+1}Q_n = (-1)^{n+1}a_0\cdots a_n$, the polynomial on the left-hand side of the identity to be proved equals $Q_{n+1}P_n - P_{n+1}Q_n$, which shows that this identity is equivalent to the formula

$$P_n(s)(P_n(s+r)-Q_{n+1}(s))$$
$$= Q_n(s)((s+q)(s+r-q)Q_n(s+r)-P_{n+1}(s))\ .$$

Since P_n and Q_n do not have common factors, we conclude that the above formula is equivalent to the following two:

$$\begin{aligned} P_n(s+r) &= Q_{n+1}(s) + l + n(s)Q_n(s) \,, \\ (s+q)(s+r-q)Q_n(s+r) &= P_{n+1}(s) + l_n(s)P_n(s) \,, \end{aligned} \tag{E4.7}$$

where $l_n(s)$ is a linear polynomial in s. By the Euler–Wallis formulas the polynomial part of Q_{n+1}/Q_n is $2s$. However, the polynomial part of $P_n(s+r)/Q_n$ is $s+(n+1)r$ (observe that $P_n(s) = 2^n s^{n+1} + as^{n-1} + \dots$, $Q_n(s) = 2^n s^n + bs^{n-2} + \cdots$). Hence $l_n(s) = (n+1)r - s$. Now formulas E4.7 are obtained by induction.

The proof can be now completed as in the case of Brounker's formula by Lemma 4.41.

4.21 Deduce Ramanujan's formula, Theorem 3.25, from (4.46).

Hint: Combine (4.29) with (4.30).

4.22 Let $y(s) > s$ for $s > 0$ be a solution to the functional equation

$$y(s)y(s+2r) = (s+1)(s+2r-1) \,, \quad 1/2 < r \,.$$

Prove a generalization of Ramanujan's formula

$$y(s) = 4r\frac{\Gamma\left((s+2r+1)/4r\right)\Gamma\left((s+4r-1)/4r\right)}{\Gamma\left((s+1)/4r\right)\Gamma\left((s+2r-1)/4r\right)} \,. \tag{E4.8}$$

Hint: Apply Theorem 4.17 to locate the poles and zeros of $y(s)$. Use (3.59) to combine four gamma functions to have the same poles and zeros in their totality as $y(s)$. Check that (E4.8) satisfies this functional equation. Apply Stirling's formula to prove that the solution exceeds s.

4.23 Prove that

$$\pi = 2 + \frac{16}{1\times3\times5} + \frac{16}{5\times7\times9} + \frac{16}{9\times11\times13} + \frac{16}{13\times15\times17} + \cdots$$

Hint: Apply (4.18) and Corollary 4.19.

4.24 Euler (1782, §27). Prove that

$$\frac{2}{\sqrt[3]{e^2}-1} = 1 + \frac{6}{4} \genfrac{}{}{0pt}{}{}{+} \frac{12}{7} \genfrac{}{}{0pt}{}{}{+} \frac{18}{10} \genfrac{}{}{0pt}{}{}{+} \frac{24}{13} \genfrac{}{}{0pt}{}{}{+\cdots} \,.$$

Hint: Put $t = 2/3$ in (4.56).

4.25 Using the evolution equation $s = x^n(a - bx^\theta - cx^{2\theta})^\lambda$, find the value of the continued fraction

$$(a+\lambda\theta)b + \frac{(a+\theta)(a+\lambda\theta)ac}{(a+\theta+\lambda\theta)b} \genfrac{}{}{0pt}{}{}{+} \frac{(a+2\theta)(a+\theta+2\lambda\theta)ac}{(a+2\theta+\lambda\theta)b} \genfrac{}{}{0pt}{}{}{+\cdots}$$

(Euler 1782, §29).

4.26 Prove that

$$e^2 = 5 + \frac{8}{2+}\,\frac{6}{3+}\,\frac{8}{4+}\,\frac{10}{5+}\,\frac{12}{6+}\,\frac{14}{7+}\cdots\ .$$

Hint: In Lemma 4.18 and in (4.59) put

$$\begin{aligned} &a = 2\,, && b = s+1\,, && c = 1\,, && \alpha b - \beta a = 2s\,,\\ &\alpha = 2\,, && \beta = 1\,, && \gamma = 0, && \alpha c - \gamma a = 2\,, \end{aligned}$$

$$\frac{dS}{S} = (s+2)\frac{dR}{R-2} + dR \quad \Longrightarrow \quad S = (2-R)^{s+2}e^R\ .$$

Choosing $R(x) = 2x$, prove that

$$s + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{2n}{s+n}\right) = \frac{1}{e^2\int_0^1 x^{s+1}e^{-2x}\,dx}\ .$$

4.27 Prove that for $p > 0$ and $s > 0$

$$\frac{1}{s + \mathop{\mathbf{K}}_{n=1}^{\infty}(pn/(s+n))} = \sum_{n=0}^{\infty}(-1)^n\frac{p^n}{n!}\frac{e^p}{s+p+n}\ .$$

Hint: Prove that

$$s + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{pn}{s+n}\right) = \frac{1}{e^p\int_0^1 x^{s+p-1}e^{-px}\,dx}\ .$$

4.28 If a, b, α, β are positive numbers and $b\alpha - a\beta > 0$ then there are a differentiable function v and a constant $c > 0$ such that $v(0) = 0$, $v' > 0$ on $(0, c)$ and

$$\int_0^c x^n\,dv = \prod_{k=1}^{n}\frac{k\alpha + a}{k\beta + b}\ .$$

Hint: Apply Euler's differential method in the following form:

$$(\alpha n + a)\int_0^x x^{n-1}\,dv = (\beta n + b)\int_0^x x^n\,dv + x^n Q\ .$$

See Euler (1785b) or Kushnir (1957) for details.

4.29 Prove that

$$\frac{1}{s + \mathop{\mathbf{K}}_{n=1}^{\infty}(n(n+1)/s)} \sim \sum_{k=1}^{\infty}\frac{4^k(4^k-1)B_{2k}}{2k}\frac{1}{s^{2k-1}}\ . \qquad \text{(E4.9)}$$

Hint: Apply (4.60), Lemma 3.21 and

$$\tanh x = \sum_{k=1}^{\infty}\frac{4^k(4^k-1)B_{2k}}{(2k)!}x^{2k-1}\,, \qquad |x| < \frac{\pi}{2}\ .$$

4.30 Prove that

$$\int_0^{+\infty} \frac{e^{-sx}\,dx}{\cosh^{3/2}(x)} = \frac{1\times 2}{2s} \, {}_{+} \, \frac{2\times 3}{2s} \, {}_{+} \, \frac{4\times 5}{2s} \, {}_{+} \, \frac{6\times 7}{2s} \, {}_{+} \cdots . \tag{E4.10}$$

Hint: Put $f=4$, $h=5$, $r=2$ in (4.69) and apply the change of variables $x=(1-t)^{1/2}(1+t)^{-1/2}$. Using (E3.1) check (E4.10) by putting $s=1$.

4.31 Show that

$$\int_0^{+\infty} \frac{e^{-sx}\,dx}{\cosh^{3/2} x} = \sum_{n=0}^{+\infty}(-1)^n \frac{(2n+1)!!}{2^n n!} \frac{4\sqrt{2}}{2s+4n+3} . \tag{E4.11}$$

4.32 Prove that

$$\frac{2}{s+\mathop{\mathbf{K}}\limits_{n=1}^{\infty}(n(n+2)/s)} + \frac{s^2-1}{s+\mathop{\mathbf{K}}\limits_{n=1}^{\infty}(n^2/s)} = s .$$

Hint: Put $r=1$, $f=3$, $h=1$ in (4.69) and apply Theorem 4.24.

4.33 Prove that $\lim_{s\to 0^+} \mathbf{K}_{n=1}^{\infty}(n(n+2)/s) = 4/\pi$.

4.34 Prove that $\mathbf{K}_{n=1}^{\infty}(n(n+2)/1) = 1$.

4.35 Prove that

$$\frac{1}{s+\mathop{\mathbf{K}}\limits_{n=1}^{\infty}(n(n+2)/s)} \sim \frac{1}{2}\sum_{k=0}^{\infty} \frac{E_{2k}-E_{2k+2}}{s^{2k+1}} . \tag{E4.12}$$

Hint: Apply (4.77) and Ex. 4.32.

4.36 Prove that

$$1+\frac{2}{2s-1+\mathop{\mathbf{K}}\limits_{n=0}^{\infty}((2n+1)(2n+3)/2s)} = \frac{s}{4}\left(\frac{\Gamma(s/4)}{\Gamma(s+2)/4}\right)^2 .$$

Hints: Apply Euler's formula (4.69) with $f=3$, $h=1$, $r=2$, $s:=2s$. Apply the substitution from Ex. 4.30 to convert the quotient of Euler integrals into the quotient of two beta integrals. Apply (4.30). See Perron (1957, p. 34) for an alternative proof.

4.37 An analytic function may have two different representations by continued fractions, see also (4.6). For $s>0$

$$\mathop{\mathbf{K}}\limits_{n=1}^{\infty}\left(\frac{(s+n)^2}{1}\right) = s+\mathop{\mathbf{K}}\limits_{n=1}^{\infty}\left(\frac{n^2}{2s+1}\right) . \tag{E4.13}$$

Hint: By (4.74) and Corollary 4.23,

$$\frac{1}{2s+1+\overset{\infty}{\underset{n=1}{\mathbf{K}}}\,(n^2/(2s+1))} = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{2s+2k+2} = \frac{1}{s} - \sum_{k=0}^{\infty}\frac{(-1)^k}{s+k}$$

$$= \frac{1}{s} - \frac{1}{s} + \frac{s^2}{1+\overset{\infty}{\underset{n=1}{\mathbf{K}}}\,((s+n)^2/1)}$$

$$= \frac{1}{1+s+\overset{\infty}{\underset{n=1}{\mathbf{K}}}\,((s+n)^2/1)} \,.$$

Notice that the right-hand side of (E4.13) interpolates the corresponding asymptotic series at infinity, whereas the left-hand part does not.

4.38 Prove that

$$\overset{\infty}{\underset{n=k+1}{\mathbf{K}}}\left(\frac{n^2}{1}\right) = k + \overset{\infty}{\underset{n=1}{\mathbf{K}}}\left(\frac{n^2}{2k+1}\right) .$$

Hint: Put $s = k \in \mathbb{N}$ in (E4.13).

4.39 (Ramanujan) For every positive t,

$$\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\frac{\sin ts\,ds}{s+\overset{\infty}{\underset{n=1}{\mathbf{K}}}\,(n/s)} = \frac{1}{t+\overset{\infty}{\underset{n=1}{\mathbf{K}}}\,(n/t)} \,, \qquad \text{(E4.14)}$$

see Levin (1960, p. 369, formula (2)).

Hint: By (4.88) and Fubini's theorem,

$$\int_0^{+\infty}\sin ts\,\varphi(s)\,ds = \int_0^{+\infty}e^{-x^2/2}\,dx\int_0^{+\infty}\sin ts\,e^{-sx}\,ds.$$

Integrating by parts, show that

$$\int_0^{+\infty}\sin ts\,e^{-sx}\,ds = \frac{t}{t^2+x^2} \,.$$

Hence

$$\psi(t) \overset{\text{def}}{=} \int_0^{+\infty}\sin ts\,\varphi(s)\,ds = \int_0^{+\infty}\frac{1}{1+x^2}e^{-x^2t^2/2}\,dx \,.$$

Differentiate with respect to t to obtain that $\psi(t)$ satisfies the differential equation

$$\psi'(t) = t\psi(t) - \int_0^{+\infty}e^{-x^2}\,dx \,.$$

Hence ψ is proportional to φ and satisfies (4.89) with coefficient

$$\int_0^{+\infty}e^{-x^2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty}e^{-x^2}\,dx = \frac{\sqrt{2\pi}}{2} = \sqrt{\frac{\pi}{2}} \,.$$

4.40 Prove another formula of Ramanujan:

$$1+\frac{1}{1\times 3}+\frac{1}{1\times 3\times 5}+\frac{1}{1\times 3\cdot 5\times 7}+\cdots+\frac{1}{1+\mathop{\mathbf{K}}\limits_{n=1}^{\infty}(n/1)}=\sqrt{\frac{\pi e}{2}}\,, \tag{E4.15}$$

see Levin (1960, p. 370, formula (3)).
Hint: Let $y(x)=\sum_{n=0}^{\infty}x^{2n+1}/(2n+1)!!$. Then $y(0)=0$ and $y'(x)=xy(x)+1$, which implies that

$$y(x)=e^{x^2/2}\int_0^x e^{-t^2/2}\,dt\,.$$

Formula (E4.15) follows from (4.88) by putting $x=1$ in

$$\sum_{n=1}^{+\infty}\frac{x^{2n-1}}{(2n-1)!!}+\frac{1}{x+\mathop{\mathbf{K}}\limits_{n=1}^{\infty}(n/x)}=e^{x^2/2}\int_0^{+\infty}e^{-t^2/2}\,dt=e^{x^2/2}\sqrt{\frac{\pi}{2}}\,.$$

4.41 Prove that for every $s\geqslant 0$ and $q>p>0$ the series on the left-hand side of the following equation converges and is equal to the right-hand side:

$$\frac{p}{q+s}+\frac{p(p+s)}{(q+s)(q+2s)}+\frac{p(p+s)(p+2s)}{(q+s)(q+2s)(q+3s)}+\cdots=\frac{p}{q-p}$$

(Euler 1750b, §58).
Hints: (a) Consider the power series in x, where $q>p>0$,

$$y(x)=x^q+\frac{px^{q+s}}{q+s}+\frac{p(p+s)x^{q+2s}}{(q+s)(q+2s)}+\frac{p(p+s)(p+2s)x^{q+3s}}{(q+s)(q+2s)(q+3s)}+\cdots$$

Apply Raabe's test (http://mathworld.wolfram.com) to prove that this series converges uniformly on $[0,1]$ in x to a continuous function $y(x)$ on $[0,1]$ that is infinitely differentiable on $[0,1)$. Check that $y(x)$ satisfies

$$\int x^{p-q-s}\,dy=\frac{qx^{p-s}}{p-s}+x^p+\frac{px^{p+s}}{q+s}+\cdots=\frac{qx^{p-s}}{p-s}+x^{p-q}y(x)+C\,.$$

(b) Deduce from (a) that $y(x)$ satisfies the Riccati equation

$$\frac{dy}{dx}+\frac{(q-p)x^{s-1}}{1-x^s}y=\frac{qx^{q-1}}{1-x^s}\,. \tag{E4.16}$$

(c) Prove that

$$z(x)=q(1-x^s)^{(q-p)/s}\int_0^x\frac{t^{q-1}\,dt}{(1-t^s)^{(q-p+s)/s}}$$

satisfies (E4.16), $z(0)=0$ and therefore $z\equiv y$.

(d) Prove that

$$\frac{y(x)}{(1-x^s)^{(q-p)/s}} = q\int_0^x \frac{t^{q-1}\,dt}{(1-t^s)^{(q-p+s)/s}}$$
$$= \frac{qx^q}{(q-p)(1-x^s)^{(q-p)/s}} - \frac{pq}{q-p}\int_0^x \frac{t^{q-1}\,dt}{(1-t^s)^{(q-p)/s}}\,.$$

Deduce that $y(1) = q/(q-p)$.

4.42 Prove the following generalization of Ex. 4.41. For any positive sequence $\{a_n\}_{n\geqslant 1}$ satisfying $\sum a_n^{-1} = \infty$ and $x > 0$,

$$\frac{1}{x+a_1} + \frac{a_1}{(x+a_1)(x+a_2)} + \frac{a_1a_2}{(x+a_1)(x+a_2)(x+a_3)} \cdots = \frac{1}{x}\,.$$

Hint: To obtain Ex. 4.41 from Ex. 4.42 put $a_n = p+ns$, $x = q-p$. To prove the formula of Ex. 4.42 observe that ($P_0 = 1$)

$$\frac{a_1\cdots a_{k-1}x}{(x+a_1)\cdots(x+a_k)} = P_{k-1} - P_k, \qquad P_k = \frac{a_1\cdots a_k}{(x+a_1)\cdots(x+a_k)}\,.$$

4.43 Prove that for $r > 0$ and $q > p > 0$ the following identity holds:

$$(q-p)\int_0^1 x^{p-1}(1-x^{2r})^{(q-p)/2r}\frac{dx}{1+x^r}$$
$$= q\int_0^1 x^{p-1}(1-x^{2r})^{(q-p)/2r}dx - (q+r)\int_0^1 x^{p+r-1}(1-x^{2r})^{(q-p)/2r}dx\,.$$

Hints: Collect separately the positive and negative terms in

$$(1+x^r)^{-1} = 1 - x^r + x^{2r} - x^{3r} + x^{4r} - \cdots\,,$$

then integrate and apply Lemma 4.11 and Ex. 4.41 to each collection.

4.44 If a, b, c, p, q, r are positive, $a+b-r > c > b-r$ and $g = a+b-c-r$ then

$$\mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{pq(c+nr)(g+nr)}{ap-bq+nr\,(p-q)}\right)$$
$$= \frac{\int_0^1 x^{g+r-1}(1-x^r)^{(c-b)/r}(p+qx^r)^{(c-a)/r}\,dx}{\int_0^1 x^{g-1}(1-x^r)^{(c-b)/r}(p+qx^r)^{(c-a)/r}\,dx}\,.$$

Hint: See Euler (1750b, §§63, 64).

4.45 Show that in order to insert the intermediate fraction

$$\frac{P_k - \rho P_{k-1}}{Q_k - \rho Q_{k-1}}$$

between the convergents P_{k-1}/Q_{k-1} and P_k/Q_k of a continued fraction $q_0 + \mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ it is sufficient to make the replacement

$$\frac{p_k}{q_k} + \frac{p_{k+1}}{q_{k+1}} + \quad \rightarrow \quad \frac{p_k}{q_k - \rho} + \frac{\rho}{1} - \frac{p_{k+1}/\rho}{q_{k+1} + p_{k+1}/\rho} + \cdots$$

in the initial continued fraction.

4.46 Apply Ex. 4.45 with $\rho = p_{k+1}$ to prove that the even convergents to

$$b_0 + \frac{a_1}{b_1 - a_2} + \frac{a_2}{1} - \frac{1}{b_2 - a_3 + 1} + \frac{a_3}{1} - \frac{1}{b_3 - a_4 + 1} + \frac{a_4}{1} - \cdots$$

coincide with the convergents to $b_0 + \mathbf{K}_{n=1}^{\infty}(a_n/b_n)$.

4.47 Using Euler's method (see §**103**, in section 4.11) prove that

$$y(x) = \coth x + \frac{\coth^2 x - 1}{c - \coth x} = \coth(x+a)\,, \qquad c = -\coth a \in \hat{\mathbb{R}}\,,$$

lists all solutions to Riccati's equation $dy/dx = 1 - y^2$. Notice that $y = \coth x$ ($c = \infty$) is the only unbounded solution in the vicinity of $x = 0$.

4.48 Prove that all solutions to Riccati's equation ($s > 0$)

$$s\frac{dy}{dt} + y^2 = 1$$

are listed by $y(t) = \pm 1$ and $y(t) = \coth(t/s + \phi)$, where ϕ is a real number. Notice that $\lim_{t\to\pm\infty} \coth(t/s + \phi) = \pm 1$.

4.49 Using Euler's method prove that all solutions to the Riccati equation ($s > 0$)

$$s\frac{dy}{dt} + y^2 = -1$$

are listed by $y(t) = \cot(t/s + \phi)$, where ϕ is a real number.

4.50 Prove that continued fraction (4.130) is a solution to the Riccati equation (4.126) if $a > 0$, $m > -1/2$, $c < 0$.

4.51 Prove that the continued fraction

$$v(t) = \frac{\beta c t}{\nu + 1} + \frac{\beta^2 t^2 c}{\nu + 3} + \frac{\beta^3 t^3 c}{\nu + 5} + \cdots$$

is a unique solution to the Riccati equation

$$\frac{dv}{dt} - \frac{\nu + 2}{t} + \beta v^2 = bc\,, \qquad v(0) = 0\,.$$

4.52 Prove that the continued fraction

$$z(x) = c + \frac{x^n}{c + an} + \frac{x^n}{c + 2an} + \frac{x^n}{c + 3an} + \cdots$$

is a solution to the Riccati equation

$$ax\frac{dz}{dx} - cz + z^2 = x^n\ .$$

4.53 Prove that the continued fraction

$$y(x) = \frac{a}{x} + \frac{x^{n-1}}{c+an\,+}\ \frac{x^n}{c+2an\,+}\ \frac{x^n}{c+3an\,+}\cdots$$

is a solution to the Riccati equation

$$a\frac{dy}{dx} + \frac{a-c}{x}y + y^2 = x^{n-2}\ .$$

4.54 Recover Schlömilch's (1857) proof of the convergence of (4.103).
Hint: Check that $y(s) = \cosh\sqrt{s}$ satisfies

$$y^{(n+2)} + (4n+2)y^{(n+1)} - y^{(n)} = 0\ . \tag{E4.17}$$

Prove that $u_{n+1} = y^{(n+1)}/y^{(n)}$ satisfies

$$u_{n+1} = \frac{1/2}{2n+1+2su_{n+2}}$$

and

$$\tanh s = \frac{s}{1\,+}\ \frac{s^2}{3\,+}\ \frac{s^2}{5\,+}\cdots+\frac{s^2}{2n+1+s^2u_{n+2}}\ .$$

Use (E4.17) to prove that $s^n y^{(n+1)} = O(1)$ as $s \to 0^+$ and that $[s^{n+1/2}y^{(n+1)}]' = (s^{n-1/2}\big/4)\ y^{(n)}$. Prove by induction that $y^{(n)} > 0$ for $s > 0$.

5

Continued fractions: Euler's influence

106. Here we give a simple extension of Euler's method presented in §**68** at the start of Chapter 4. This, however, results in some interesting formulas.

Theorem 5.1 (Glaisher–Stern) *Let* Π *be an infinite product*

$$\Pi = \prod_{k=0}^{\infty}(1+\gamma_k) \qquad \textit{with} \qquad \Pi_n = \prod_{k=0}^{n}(1+\gamma_k)$$

for nonzero γ_k, $k > 0$*. Then* $\{\Pi_n\}_{n\geqslant 0}$ *is the sequence of convergents to the continued fraction*

$$1+\gamma_0+\frac{(1+\gamma_0)\gamma_1}{1}\,{}_{-}\,\frac{s_2}{1+s_2}\,{}_{-}\,\frac{s_3}{1+s_3}\,{}_{-}\,\frac{s_4}{1+s_4}\,{}_{-}\cdots,$$

where

$$s_n = (1+\gamma_{n-1})\frac{\gamma_n}{\gamma_{n-1}}\ .$$

Proof Put $d_n = \Pi_n$ in (4.1). $\square$

Applying Theorem 5.1 to Euler's infinite product (3.16), we obtain

$$\frac{\sin \pi x}{\pi x} = 1-\frac{x}{1}\,{}_{+}\,\frac{1(1-x)}{x}\,{}_{+}\,\frac{1(1+x)}{1-x}\,{}_{+}\,\frac{2(2-x)}{x}\,{}_{+}\,\frac{2(2+x)}{1-x}$$
$${}_{+}\,\frac{3(3-x)}{x}\,{}_{+}\,\frac{3(3+x)}{1-x}\,{}_{+}\cdots. \tag{5.1}$$

Putting $x = 1/2$, we immediately obtain Euler's formula (4.18). More examples can be found in Perron (1957, pp. 22–5).

5.1 Bauer–Muir–Perron theory

107 A special case of Bernoulli's theorem. By (1.15) the convergents of any continued fraction $b_0 + \mathbf{K}_{k\geqslant 1}(a_k/b_k)$ satisfy

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{P_n + x_n P_{n-1}}{Q_n + x_n Q_{n-1}}\,, \tag{5.2}$$

where $x_n = a_{n+1}/b_{n+1}$. Hence the fractions on the right-hand side of (5.2) converge to the value of the continued fraction if it exists. This resembles a discrete analogue of l'Hôpital's rule, see Ex. 5.1 and Ex. 1.18.

Theorem 5.2 (Stolz) *Let the sequence $\{Q_n\}_{n\geqslant 1}$ monotonically increase to $+\infty$ and $\{P_n\}_{n\geqslant 1}$ be any sequence. Then*

$$\lim_n \frac{P_n}{Q_n} = \lim_n \frac{P_n - P_{n-1}}{Q_n - Q_{n-1}}$$

provided that the limit on the right-hand side exists.

It follows that the choice of x_n indicated after (5.2) is not the only one implying the convergence of such fractions. To study this phenomenon let us apply Bernoulli's theorem 4.1 to the quotients on the right-hand side of (5.2) now assuming the x_n to be arbitrary numbers. To simplify the notation let $C_n = P_n + x_n P_{n-1}$, $D_n = Q_n + x_n Q_{n-1}$. By Theorem 4.1 this application is possible if for every n, $n = 1, 2, \ldots$

$$C_n D_{n-1} - C_{n-1} D_n = (-1)^{n-1} a_1 \cdots a_{n-1} \phi_n \neq 0\,,$$

where

$$\phi_n = a_n - x_{n-1}(b_n + x_n)\,, \qquad n = 1, 2, \ldots \tag{5.3}$$

Then by (4.1) the parameters of the continued fraction $q_0 + \mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ with convergents $\{C_n/D_n\}_{n\geqslant 0}$ are given by

$$\begin{aligned} p_n &= \frac{C_{n-1}D_n - C_n D_{n-1}}{C_{n-1}D_{n-2} - C_{n-2}D_{n-1}}\,\frac{D_{n-2}}{D_n}\,,\\ q_n &= \frac{C_n D_{n-2} - C_{n-2}D_n}{C_{n-1}D_{n-2} - C_{n-2}D_{n-1}}\,\frac{D_{n-1}}{D_n}\,. \end{aligned} \tag{5.4}$$

By Exs. 1.12, 1.13

$$\begin{aligned} &C_n D_{n-2} - C_{n-2} D_n \\ &\quad = (-1)^n a_1 \cdots a_{n-2}\left(a_{n-1}(b_n + x_n) - x_{n-2}(a_n + b_n b_{n-1}) - x_n x_{n-2} b_{n-1}\right) \\ &\quad = (-1)^n a_1 \cdots a_{n-2}\left(\phi_{n-1}(b_n + x_n) - x_{n-2}\phi_n\right)\,. \end{aligned}$$

By (3.5) (put $r_n = D_{n-1}/D_n$ and observe that $r_n r_{n-1} = D_{n-2}/D_n$) the sequence $\{C_n/D_n\}_{n\geqslant 0}$ is the sequence of the convergents to the continued fraction

$$b_0 + x_0 + \frac{\phi_1}{b_1+x_1} + \frac{a_1\phi_2/\phi_1}{b_2+x_2-x_0\phi_2/\phi_1} + \cdots + \frac{a_{n-1}\phi_n/\phi_{n-1}}{b_n+x_n-x_{n-2}\phi_n/\phi_{n-1}} + \cdots . \tag{5.5}$$

Now, applying induction and (1.15), one can check that the numerators of convergents to this continued fraction are C_n and the denominators are D_n. Continued fraction (5.5) is called the *Bauer–Muir transform* of $b_0 + \mathbf{K}_{k\geqslant 1}(a_k/b_k)$, Bauer (1872), Muir (1877). It is clear that the Bauer–Muir transform of a continued fraction is uniquely determined by this continued fraction and the choice of $\{x_n\}_{n\geqslant 0}$.

108 The value of the Bauer–Muir transform. In view of the motivation given in §**107** it is not surprising that the values of a continued fraction and its Bauer–Muir transform are related.

Theorem 5.3 *Suppose that $b_0 + \mathbf{K}_{k\geqslant 1}(a_k/b_k)$ and its Bauer–Muir transform* (5.5) *are convergent continued fractions with positive parameters. Then their values are equal. If the first continued fraction has positive elements and converges and if $x_n \geqslant 0$ starting from some n then the second continued fraction converges to the same value.*

Proof In the first case both the limits

$$\lim_n \frac{P_n}{Q_n} = l_1\,, \qquad \lim_n \frac{P_n + x_n P_{n-1}}{Q_n + x_n Q_{n-1}} = l_2$$

exist by the assumption of the theorem. If $x_n \geqslant 0$ infinitely often and the first continued fraction has positive elements then the fractions in the second limit are placed infinitely often between P_n/Q_n and P_{n-1}/Q_{n-1}, which implies that $l_1 = l_2$. Suppose now that $x_n < 0$ for all $n > N$. By (1.15) and (5.3),

$$P_n + x_n P_{n-1} - \frac{a_n}{x_{n-1}}(P_{n-1} + x_{n-1}P_{n-2}) = -\frac{\phi_n}{x_{n-1}} P_{n-1}\,.$$

A similar formula holds for Q. Hence

$$\frac{P_n + x_n P_{n-1} - (a_n/x_{n-1})(P_{n-1} + x_{n-1}P_{n-2})}{Q_n + x_n Q_{n-1} - (a_n/x_{n-1})(Q_{n-1} + x_{n-1}Q_{n-2})} = \frac{P_{n-1}}{Q_{n-1}}\,.$$

Since $-a_n/x_{n-1} > 0$ we see that P_{n-1}/Q_{n-1} is placed between C_n/D_n and C_{n-1}/D_{n-1}, which again implies $l_1 = l_2$.

If the elements of the first continued fraction are positive and $x_n > 0$ for $n > N$, the convergents C_n/D_n of the second continued fraction (for $n > N$) are placed between P_n/Q_n and P_{n-1}/Q_{n-1}, which implies the convergence of the Bauer–Muir transform to the same value. □

109 Bauer's proof of Brouncker's theorem. For Brouncker's continued fraction $a_n = (2n-1)^2$, $b_0 = s$, $b_n = 2s$, $n = 1, 2, \ldots$ We consider $x_n = An + B$ as a linear polynomial in n and arrange the coefficients A, B so that $\phi_n \equiv$ const. Easy algebra shows that

$$\begin{aligned}\phi_n &= (2n-1)^2 - x_{n-1}(2s+x_n)\\ &= (4-A^2)n^2 + (-4-2sA-2AB+A^2)n + 1 - (2s+B)(B-A)\,.\end{aligned}$$

Hence if $A = 2$ and $B = -s$ then $x_n = 2n - s$, $\phi_n = (s+1)^2$ and $x_{n+1} - x_{n-1} = 4$. By (5.5) and Theorem 5.3,

$$b(s) = \frac{(s+1)^2}{s+2} \, {}_{+} \, \frac{1^2}{2(s+2)} \, {}_{+} \, \frac{3^2}{2(s+2)} \, {}_{+} \, \frac{5^2}{2(s+2)} \, {}_{+\cdots} \; = \frac{(s+1)^2}{b(s+2)}\,,$$

which is equivalent to (3.28).

More sophisticated applications of the Bauer–Muir–Perron theory can be found in Perron (1957, §§7, 8). See also the exercises following this chapter. We include here only Perron's proof of Ramanujan's formula.

110 Perron's proof of Ramanujan's formula. For $s > 1$ Ramanujan stated the following formula:

$$\frac{1}{s^2-1} \, {}_{+} \, \frac{4\times 1^2}{1} \, {}_{+} \, \frac{4\times 1^2}{s^2-1} \, {}_{+} \, \frac{4\times 2^2}{1} \, {}_{+} \, \frac{4\times 2^2}{s^2-1} \, {}_{+\cdots} \; = \int_0^\infty \frac{2te^{-st}}{e^t + e^{-t}}\,dt\,. \tag{5.6}$$

It follows from (4.71) that the continued fraction (5.6) equals minus the derivative of (4.71), which gives an interesting example when the continued fraction of a Laurent series as well as of its derivative can be explicitly found.

To prove (5.6) Perron (1953, 1954 §7) applied a Bauer–Muir transform. We have

$$\begin{aligned} a_{2k-1} &= 4k^2\,, \qquad & b_{2k-1} &= 1\,,\\ a_{2k} &= 4k^2\,, \qquad & b_{2k} &= s^2 - 1\,.\end{aligned}$$

Assuming that x_{2k-1} and x_{2k} are linear polynomials in k, we may choose their coefficients in such a way that ϕ_n does not depent on n:

$$\begin{aligned} x_{2k-1} &= \frac{2k}{s+1} - \frac{1}{2}\,, \qquad x_{2k} = 2k(s+1) + \frac{3+2s-s^2}{2}\,,\\ \phi_{2k-1} &= \phi_{2k} = \frac{(s+1)^2}{4}\,.\end{aligned}$$

The continued fraction (5.6) is positive for $s > 1$, and $x_n > 0$ for large n. Hence by (5.5) and Theorem 5.3,

$$
\begin{aligned}
y(s) &= s^2-1+\frac{4\times 1^2}{1}\,{}_{+}\,\frac{4\times 1^2}{s^2-1}\,{}_{+}\,\frac{4\times 2^2}{1}\,{}_{+}\,\frac{4\times 2^2}{s^2-1}\,{}_{+}\cdots\\
&= \frac{(s+1)^2}{2}+\frac{(s+1)^2/4}{2/(s+1)+1/2}\,{}_{+}\,\frac{4\times 1^2}{(s+1)^2}\\
&\quad {}_{+}\,\frac{4\times 1^2}{1+2/(s+1)}\,{}_{+}\,\frac{4\times 2^2}{(s+1)^2}\,{}_{+}\,\frac{4\times 2^2}{1+2/(s+1)}\,{}_{+}\cdots\\
&= \frac{1}{2}(s+1)^2+\frac{(s+1)^4/4}{2(s+1)+(s+1)^2/2}\,{}_{+}\,\frac{4\times 1^2}{1}\,{}_{+}\,\frac{4\times 1^2}{(s+2)^2-1}\\
&\quad {}_{+}\,\frac{4\times 2^2}{1}\,{}_{+}\,\frac{4\times 2^2}{(s+2)^2-1}\,{}_{+}\cdots \;=\frac{(s+1)^2y(s+2)/2}{y(s+2)-(s+1)^2/2}\,.
\end{aligned}
$$

It follows that $y(s)$, $\lim_{s\to+\infty} y(s)=+\infty$, satisfies

$$\frac{1}{y(s)}+\frac{1}{y(s+2)}=\frac{2}{(s+1)^2}\,.$$

By Theorem 3.24, Theorem 4.25 and (4.71),

$$\frac{1}{y(s)}=-(\log b)''(s)=-\frac{d}{ds}\frac{1}{s+\mathop{\mathbf{K}}\limits_{n=1}^{\infty}(n^2/s)}=\int_0^{+\infty}\frac{xe^{-sx}\,dx}{\cosh\,x}\,. \tag{5.7}$$

Putting $s=\sqrt{5}$ in (5.6), we obtain after obvious equivalence transforms Ramanujan's formula,

$$\frac{1}{1}\,{}_{+}\,\frac{1^2}{1}\,{}_{+}\,\frac{1^2}{1}\,{}_{+}\,\frac{2^2}{1}\,{}_{+}\,\frac{2^2}{1}\,{}_{+}\,\frac{3^2}{1}\,{}_{+}\,\frac{3^2}{1}\,{}_{+}\cdots=\int_0^{+\infty}\frac{8xe^{-\sqrt{5}x}\,dx}{e^x+e^{-x}}\,. \tag{5.8}$$

5.2 From Euler to Scott–Wall

111 Convergence of Euler continued fractions. The convergents of any Euler continued fraction (4.5) are the partial sums of its series $\sum c_k$. Hence an Euler continued fraction and $\sum c_k$ converge or diverge simultaneously.

Lemma 5.4 *For any continued fraction $q_0+\mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ with finite convergents there is a series with nonzero terms whose partial sums coincide with the convergents of this continued fraction.*

Proof For any continued fraction with finite convergents,

$$\frac{P_n}{Q_n}=\frac{P_0}{Q_0}+\sum_{k=1}^{n}\left(\frac{P_k}{Q_k}-\frac{P_{k-1}}{Q_{k-1}}\right)=q_0+\sum_{k=1}^{n}(-1)^{k-1}\frac{p_1\cdots p_k}{Q_kQ_{k-1}}\,, \tag{5.9}$$

where all terms except possibly q_0 are nonzero. If $q_0\neq 0$ then $\rho_0=q_0$,

$$\rho_1=p_1(q_0Q_1)^{-1},\quad \rho_2=-p_2Q_0(Q_2)^{-1},\quad \ldots,\quad \rho_k=-p_kQ_{k-2}(Q_k)^{-1},\ldots$$

and for $q_0 = 0$, $\rho_0 = p_1(Q_1)^{-1}$,

$$\rho_1 = -p_2 Q_0 (Q_2)^{-1}, \quad \ldots, \quad \rho_k = -p_{k+1} Q_{k-1} (Q_{k+1})^{-1}, \quad \ldots$$

Then the ratios P_n/Q_n are the partial sums of the series in (4.5). □

It follows that every continued fraction with finite convergents is equivalent to an Euler continued fraction. Therefore it looks reasonable to study the convergence of Euler continued fractions first. For a positive sequence $\{\rho_n\}_{n\geqslant 0}$ in (4.5) the Euler continued fraction

$$E(\rho) = \frac{\rho_1}{1} \underset{-}{} \frac{\rho_2}{1+\rho_2} \underset{-}{} \cdots \underset{-}{} \frac{\rho_n}{1+\rho_n} \underset{-}{} \cdots = \sum_{k=1}^{\infty} \rho_1 \cdots \rho_k \tag{5.10}$$

always converges either to a finite or an infinite value. To eliminate infinite values let us consider the *modified* Euler continued fraction

$$E^*(\rho) = \frac{1+\rho_1}{1+\rho_1} \underset{-}{} \frac{\rho_2}{1+\rho_2} \underset{-}{} \cdots \underset{-}{} \frac{\rho_n}{1+\rho_n} \underset{-}{} \cdots . \tag{5.11}$$

The convergents E_n^* of (5.11) and E_n of (5.10) are related by a simple formula:

$$E_n^* = (1+\rho_1)\left(1+\rho_1+\frac{\rho_1}{E_n}-1\right)^{-1} = \frac{1+\rho_1}{\rho_1}\left(1-\frac{1}{1+E_n}\right). \tag{5.12}$$

By (4.5) the sequence $\{E_n\}$ increases monotonically. It follows from (5.12) that $\{E_n^*\}$ also increases and is bounded by $1+1/\rho_1$. Hence the continued fraction $E^*(\rho)$ converges to a finite value.

Theorem 5.5 *Any modified Euler continued fraction $E^*(\rho)$ is equivalent to an Euler continued fraction $E(r)$ with positive r_k and $r_0 = 1$.*

Proof By Theorem 3.6 $E^*(\rho)$ is equivalent to

$$\begin{aligned} &\frac{1+\rho_1}{1+\rho_1} \underset{-}{} \frac{\rho_2}{1+\rho_2} \underset{-}{} \cdots \underset{-}{} \frac{\rho_n}{1+\rho_n} \underset{-}{} \cdots \\ &= \frac{1}{1} \underset{-}{} \frac{t_2}{1} \underset{-}{} \cdots \underset{-}{} \frac{t_n}{1} \underset{-}{} \cdots \overset{\text{def}}{=} T(\rho), \end{aligned} \tag{5.13}$$

where

$$t_n = \frac{\rho_n}{(1+\rho_{n-1})(1+\rho_n)} > 0, \qquad n = 2, 3, \ldots \tag{5.14}$$

Hence the convergents E_n^* of $E^*(\rho)$ equal the convergents G_n/H_n of $T(\rho)$. Since the convergents E_n^* increase, we have

$$E_n^* - E_{n-1}^* = \frac{G_n}{H_n} - \frac{G_{n-1}}{H_{n-1}} = (-1)^{n-1}\frac{(-t_2)\cdots(-t_n)}{H_n H_{n-1}} = \frac{t_2\cdots t_n}{H_n H_{n-1}} > 0.$$

Since $H_0 = H_1 = 1$ and $t_n > 0$ for $n \geqslant 2$, this implies by induction that $H_n > 0$ for every n. It follows that if

$$r_k = \frac{t_{k+1}H_{k-1}}{H_{k+1}} > 0\,, \qquad k = 1, 2, \dots\,, \tag{5.15}$$

then

$$r_1 \cdots r_k = \frac{t_2 \cdots t_{k+1}}{H_{k+1}H_k} = E^*_{k+1} - E^*_k = \frac{G_{k+1}}{H_{k+1}} - \frac{G_k}{H_k}\,.$$

Since $G_1/H_1 = 1$,

$$E^*_n = \frac{G_n}{H_n} = 1 + \sum_{k=1}^{n-1} r_1 \cdots r_k\,. \tag{5.16}$$

By (5.16) and (4.5) the continued fraction

$$\frac{1}{1} \underset{-}{} \frac{r_1}{1+r_1} \underset{-}{} \frac{r_2}{1+r_2} \underset{-}{} \cdots \underset{-}{} \frac{r_n}{1+r_n} \underset{-}{} \cdots \,. \tag{5.17}$$

is equivalent to $E^*(\rho)$. □

Corollary 5.6 *Every modified continued fraction $E^*(\rho)$ is equivalent to a continued fraction $T(\rho)$,* (5.13)*, with positive t_n satisfying* (5.14).

Corollary 5.7 *The parameters of a modified Euler continued fraction $E^*(\rho)$ and of the equivalent Euler continued fraction $E(r)$ are related by*

$$1 + \sum_{k=1}^{n} r_1 \cdots r_k = \frac{1+\rho_1}{\rho_1}\left(1 - \frac{1}{1+\sum_{k=1}^{n+1}\rho_1 \cdots \rho_k}\right)\,. \tag{5.18}$$

Proof This follows from (5.12) and (5.16). □

112 Absolute convergence of continued fractions. Since Euler's method reduces the convergence of continued fractions to the convergence of series, it is natural to consider the absolute convergence of continued fractions.

Definition 5.8 *A continued fraction $q_0 + \mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ with convergents $\{f_n\}_{n \geqslant 0}$ is called absolutely convergent if*

$$|f_0| + \sum_{n=1}^{\infty} |f_n - f_{n-1}| < +\infty\,.$$

The proof of Theorem 5.5 involves three equivalent continued fractions $E^*(\rho)$, $E(r)$ and $T(\rho)$, which are all absolutely convergent. Notice that $T(\rho)$ has the form

$$\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{c_n}{1}\right)\,, \tag{5.19}$$

in which in general the c_n are complex numbers. By Corollary 3.7 any continued fraction with nonzero partial denominators is equivalent to a continued fraction of the type (5.19). The formulas for c_n in this equivalence are simple. Therefore any criteria for the convergence of (5.19) is important and the relationships between $T(\rho)$, $E^*(\rho)$ and $E(r)$ can be used to obtain sufficient conditions for the absolute convergence of (5.19) in terms of the parameters $\{r_n\}_{n\geqslant 1}$.

If the continued fraction (5.19) converges absolutely then the convergents f_n to (5.19) are finite and satisfy

$$f_n - f_{n-1} = (-1)^{n-1}\frac{c_1\cdots c_n}{Q_nQ_{n-1}} = (-1)^{n-1}c_1d_1\cdots d_{n-1}\,, \tag{5.20}$$

where

$$d_n = \frac{c_{n+1}Q_{n-1}}{Q_{n+1}}\,, \qquad n = 1, 2, \ldots \tag{5.21}$$

is defined similarly to (5.15). Hence the absolute convergence of (5.19) will follow if one can find a way to deduce the inequalities

$$|d_n| \leqslant r_n\,, \qquad n = 1, 2, \ldots \tag{5.22}$$

from bounds on the partial numerators $\{c_n\}$. Formulas (5.15) hint at a way to do this.

Theorem 5.9 *Suppose that the c_n in* (5.19) *satisfy*

$$|c_n| \leqslant t_n\,, \qquad n = 2,\ 3, \ldots, \tag{5.23}$$

for a sequence $\{t_n\}_{n\geqslant 2}$ defined by (5.14) *with some positive sequence $\{\rho_n\}_{n\geqslant 1}$. Then the continued fraction* (5.19) *converges absolutely,*

$$\begin{aligned}|f_n - f_{n-1}| &\leqslant |c_1|\,r_1\cdots r_{n-1}\\ &= |c_1|\frac{1+\rho_1}{\rho_1}\left(\frac{1}{1+\sum_{k=1}^{n-1}\rho_1\cdots\rho_k} - \frac{1}{1+\sum_{k=1}^{n}\rho_1\cdots\rho_k}\right),\end{aligned} \tag{5.24}$$

and we have

$$\left|\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{c_n}{1}\right)\right| \leqslant |c_1|\frac{1+\rho_1}{\rho_1}\left(1 - \frac{1}{1+\sum_{j=1}^{\infty}\rho_1\cdots\rho_j}\right). \tag{5.25}$$

Proof Excluding H_{n-2} and H_n (see the discussion after (5.14)) from the system

$$\begin{aligned}H_{n-1} &= H_{n-2} - t_{n-1}H_{n-3},\\ H_n &= H_{n-1} - t_nH_{n-2},\\ H_{n+1} &= H_n - t_{n+1}H_{n-1},\end{aligned} \tag{5.26}$$

of Euler–Wallis equations for (5.13), we obtain that

$$H_{n+1} = (1 - t_n - t_{n+1})H_{n-1} - t_nt_{n-1}H_{n-3}\,, \qquad n = 2, 3, \ldots$$

This implies by (5.15) that

$$r_n = \frac{t_{n+1}H_{n-1}}{H_{n+1}} = \left(\frac{1-t_n-t_{n+1}}{t_{n+1}} - \frac{t_n}{t_{n+1}}r_{n-2}\right)^{-1}, \qquad n=2,3,\ldots,$$

where we put $r_0 = 0$ to make the formula valid for $n = 2$. Hence

$$r_n(1-t_n-t_{n+1}) = t_{n+1} + r_n r_{n-2} t_n\,, \qquad n=2,3,\ldots$$

Similarly the denominators Q_n of (5.19) the continued fraction under consideration, satisfy

$$Q_{n+1} = (1+c_n+c_{n+1})Q_{n-1} - c_n c_{n-1} Q_{n-3}\,.$$

It follows that

$$\begin{aligned}|d_n| &= \frac{|c_{n+1}Q_{n-1}|}{|Q_{n+1}|} = \frac{|c_{n+1}||Q_{n-1}|}{|(1+c_n+c_{n+1})Q_{n-1} - c_n c_{n-1}Q_{n-3}|}\\ &\leqslant \frac{|c_{n+1}|}{\big||1+c_n+c_{n+1}| - |c_n||d_{n-2}|\big|}\,, \qquad n=2,3,\ldots\,,\end{aligned}$$

where we must have $d_0 = 0$ to keep the inequality valid for $n = 2$. By (5.23),

$$\begin{aligned}r_n|1+c_n+c_{n+1}| &\geqslant r_n(1-t_n-t_{n+1})\\ &= t_{n+1} + r_n r_{n-2} t_n \geqslant |c_{n+1}| + r_n r_{n-2}|c_n|\,.\end{aligned}$$

Lemma 5.10 *Suppose that a sequence* $\{c_n\}_{n\geqslant 1}$ *satisfies*

$$\begin{aligned}r_1|1+c_2| &\geqslant |c_2|,\\ r_2|1+c_2+c_3| &\geqslant |c_3|,\\ r_n|1+c_n+c_{n+1}| &\geqslant r_n r_{n-2}|c_n| + |c_{n+1}|, \qquad n=3,4,\ldots\end{aligned} \tag{5.27}$$

for positive r_n *and that the* d_n *are defined by* (5.21)*. Then*

$$|d_n| \leqslant r_n, \qquad n=1,2,\ldots$$

Proof For $n = 0$ we formally put $d_0 = r_0$. For $n = 1$ we have

$$|d_1| = \frac{|c_2||Q_0|}{|Q_2|} = \frac{|c_2|}{|1+c_2|} \leqslant r_1\,.$$

For $n = 2$ we have

$$\begin{aligned}|d_2| &= \frac{|c_3||Q_1|}{|Q_3|} = \frac{|c_3|}{|Q_2+c_3Q_1|}\\ &= \frac{|c_3|}{|Q_1+c_2Q_0+c_3Q_1|} = \frac{|c_3|}{|1+c_2+c_3|} \leqslant r_2\,.\end{aligned}$$

Finally, by the expression for $|d_n|$ given before Lemma 5.10, for $n \geqslant 3$ we obtain

$$|d_n| \leqslant \frac{|c_{n+1}|}{\big||1+c_n+c_{n+1}|-|c_n||d_{n-2}|\big|} \leqslant r_n \,,$$

since $|d_{n-2}| \leqslant r_{n-2}$ by the induction hypothesis. □

Now the inequality (5.24) follows by (5.20), (5.22) and (5.18). Inequality (5.25) follows from (5.18). □

Definition 5.11 *The inequalities* (5.27) *are called fundamental inequalities for* $\{c_n\}_{n\geqslant 2}$ *if they hold for some nonnegative* $\{r_n\}$.

See Jones and Thron (1980) or Wall (1948) for more details. Fundamental inequalities play a role in continued fractions similar to that of Kummer's test in the theory of positive series, see Ex. 5.12.

Corollary 5.12 (Scott–Wall; Wall 1948, Jones and Thron 1980) *Let* $\{c_n\}_{n\geqslant 0}$ *satisfy* (5.27) *with* r_n *such that* $\sum r_1 r_2 \cdots r_k < +\infty$. *Then the continued fraction* $\mathbf{K}_{n\geqslant 1}\,(c_n\,/\,1)$ *converges absolutely.*

One choice of the test sequence $\{r_n\}$ is motivated by Theorem 3.12.

Theorem 5.13 (Scott–Wall; Khovanskii 1958, Wall 1948) *Let* $\{g_n\}_{n\geqslant 1}$ *be a sequence of positive reals in* $(0, 1)$ *and* $\{c_n\}$ *a sequence of nonzero complex numbers such that*

$$|c_n| \leqslant (1-g_n)g_{n-1}\,, \qquad n = 2, 3, \ldots \tag{5.28}$$

Then $\mathbf{K}_{n=1}^{\infty}\,(c_n/1)$ *converges absolutely to a value* K *satisfying*

$$|K| \leqslant \frac{1}{1-g_1}\left(1-\left(1+\sum_{n=1}^{\infty}\frac{(1-g_1)\cdots(1-g_n)}{g_1\cdots g_n}\right)^{-1}\right).$$

Proof Let $\rho_n = (1-g_n)/g_n$. Then

$$(1-g_n)g_{n-1} = \frac{\rho_n}{(1+\rho_{n-1})(1+\rho_n)} = t_n\,,$$

implying $|c_n| \leqslant t_n$. The bound for $|K|$ follows from (5.25). □

This theorem modifies Pringsheim's theorem obtained in 1905; see Perron (1957, §14, Theorem 2.21). A particular important result is the following.

Corollary 5.14 (Worpitsky's test) *If* $|c_n| \leqslant 1/4$ *then the continued fraction* $\mathbf{K}_{n\geqslant 1}\,(c_n/1)$ *converges absolutely.*

Proof Put $g_n = 1/2$ in Theorem 5.13. □

Corollary 5.15 (Pringsheim, 1905) *A continued fraction* $\mathbf{K}_{n\geqslant 1}(1\,/\,q_n)$ *converges to a finite value if* $|q_{2n-1}|^{-1} + |q_{2n}|^{-1} \leqslant 1$, $n \geqslant 1$.

Proof By Corollary 3.7 $\mathbf{K}_{n\geqslant 1}(1/q_n) \approx \mathbf{K}_{n\geqslant 1}(c_n/1)$ where $c_n = (q_n q_{n-1})^{-1}$. Let $g_{2n-1} = g_{2n} = |q_{2n}|^{-1}$. Then

$$|c_{2n}| = g_{2n}|q_{2n-1}|^{-1} \leqslant g_{2n}(1-g_{2n}) = g_{2n-1}(1-g_{2n}) \,,$$

$$|c_{2n+1}| = g_{2n}|q_{2n+1}|^{-1} \leqslant g_{2n}(1-g_{2n+2}) = g_{2n}(1-g_{2n+1}) \,.$$

proving the corollary. □

Corollary 5.15 follows from Corollary 5.14 by the equivalence transform used in the proof of Corollary 5.15. Indeed, $ab \leqslant (a+b)^2/4 \leqslant 1/4$.

Let us compare Theorem 5.13 with Theorem 3.12 and especially with formula (3.24). Using equivalence transformations we may set $a = 1$. In Theorem 3.12 $y_n > 1$. However, (3.24) is valid with the restriction that $y_n \neq 1$. Hence, assuming $y_0 = 1$ and $y_n \in (0, 1)$, $n \geqslant 1$, we may write

$$1 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{-(1-y_n)y_{n-1}}{1}\right) = \left(1 + \sum_{n=1}^{\infty}(1-y_1)\cdots(1-y_n)\,/\,y_1\cdots y_n\right)^{-1} . \qquad (5.29)$$

Now let $y_k = k/(k+x)$, $k \geqslant 1$, where $x > 0$. Then obvious equivalence transforms render (5.29) as

$$e^x = \frac{1}{1}\,{}_{-}\,\frac{x}{1+x}\,{}_{-}\,\frac{x}{2+x}\,{}_{-}\,\frac{2x}{3+x}\,{}_{-}\,\frac{3x}{4+x}\,{}_{-}\cdots \,. \qquad (5.30)$$

5.3 The irrationality of π

113 Pringsheim's test. At first glance Pringsheim's test has nothing in common with Euler continued fractions and the fundamental Wall–Scott inequalities. However, this turns out not to be the case, and we will explain the relationships later, but first we prove Pringsheim's theorem following Jones and Thron (1980).

Theorem 5.16 (Pringsheim) *If* $|q_n| \geqslant |p_n| + 1$, $n \geqslant 1$, *then* $\mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ *converges absolutely. Its convergents* f_n *satisfy* $|f_n| < 1$.

Proof To prove that $|f_n| < 1$ we consider $s_n(w) = p_n/(q_n + w)$. Clearly

$$|s_{n+m}(0)| = |p_{n+m}||q_{n+m}|^{-1} \leqslant |p_{n+m}|(|p_{n+m}|+1)^{-1} < 1 \,.$$

Assume that $|s_{n+1}\circ\cdots\circ s_{n+m}(0)| < 1$. Then

$$|s_n \circ s_{n+1}\circ\cdots\circ s_{n+m}(0)| = \frac{|p_n|}{|q_n + s_{n+1}\circ\cdots\circ s_{n+m}(0)|} < \frac{|p_n|}{|q_n|-1} \leqslant 1 \,.$$

It follows by induction that $|f_{n+m}| = |s_1\circ\cdots\circ s_{n+m}(0)| < 1$. Next, by the Euler–Wallis formulas,

$$|Q_n| \geqslant |q_n||Q_{n-1}| - |p_n||Q_{n-2}| \geqslant |q_n||Q_{n-1}| - (|q_n|-1)|Q_{n-2}| \,,$$

which implies that the sequence $|Q_n|$ increases monotonically:

$$|Q_n| - |Q_{n-1}| \geqslant (|q_n| - 1)(|Q_{n-1}| - |Q_{n-2}|) \geqslant \prod_{k=1}^{n} |p_n|\,.$$

Therefore

$$\frac{|p_1 \cdots p_n|}{|Q_n Q_{n-1}|} \leqslant \frac{|Q_n| - |Q_{n-1}|}{|Q_n Q_{n-1}|} = \frac{1}{|Q_{n-1}|} - \frac{1}{|Q_n|}\,,$$

and the continued fraction converges absolutely by (5.9). □

114 Lambert's theorem: $\pi \notin \mathbb{Q}$. The story goes back to Lambert, who applied the method of Euler that we have already discussed (see §**102** in Section 4.10) to obtain the continued fraction for the tangent function:

$$\tan v = \frac{1}{1/v} \, \underset{-}{} \, \frac{1}{3/v} \, \underset{-}{} \, \frac{1}{5/v} \, \underset{-}{} \, \frac{1}{7/v} \, \underset{-}{} \cdots = \frac{v}{1} \, \underset{-}{} \, \frac{v^2}{3} \, \underset{-}{} \, \frac{v^2}{5} \, \underset{-}{} \, \frac{v^2}{7} \, \underset{-}{} \cdots\,. \tag{5.31}$$

In fact, Lambert repeated the argument given by Euler for $\coth p$, which was not completely correct as we saw. A correct proof was given later by Euler, see Corollary 4.40. Assuming that $v = m/n$ is rational let us substitute this value in the above formula. Then after an obvious equivalence transformation we obtain

$$\tan \frac{m}{n} = \frac{m}{n} \, \underset{+}{} \, \frac{-m^2}{3n} \, \underset{+}{} \, \frac{-m^2}{5n} \, \underset{+}{} \, \frac{-m^2}{7n} \, \underset{+}{} \, \frac{-m^2}{9n} \, \underset{+}{} \cdots\,. \tag{5.32}$$

If we consider

$$\xi_k = \frac{-m^2}{(2k+1)n} \, \underset{+}{} \, \frac{-m^2}{(2k+3)n} \, \underset{+}{} \, \frac{-m^2}{(2k+5)n} \, \underset{+}{} \cdots\,,$$

then the continued fraction ξ_k satisfies the conditions of Pringsheim's test for sufficiently large k and so converges absolutely. To prove that ξ_k represents an irrational number one can modify Huygens' arguments (see Theorem 1.14). This clever modification is due to Legendre.

Theorem 5.17 (Legendre; see Rudio 1892) *Let $\xi = \mathbf{K}_{n=1}^{\infty}(p_n/q_n)$ be a continued fraction with integer p_n and q_n. If $|q_n| \geqslant |p_n| + 1$, $n \geqslant N$, for some N and $|q_n| > |p_n| + 1$ infinitely often then ξ is irrational.*

Proof By Pringsheim's test ξ_N converges. By (1.17) $\xi_N \notin \mathbb{Q}$ implies that $\xi \notin \mathbb{Q}$. Therefore we may assume that $N = 1$. An equivalence transformation of ξ followed by $\xi \to \pm\xi$, if necessary, reduces the problem to the case of positive q_n and $p_1 > 0$. By Pringsheim's test $|\xi_n| \leqslant 1$, $n \geqslant 1$. If $\xi_n = \pm 1$ for some n then

$$\xi_{n+1} = p_n/\xi_n - q_n = \xi_n p_n - q_n$$

is an integer. The continued-fractions assumption $p_{n+1} \neq 0$ implies that $\xi_{n+1} \neq 0$ and therefore $\xi_{n+1} = \pm 1$. Then $|q_n| = q_n = \xi_n p_n - \xi_{n+1}$, which together with $|q_n| \geqslant |p_n| + 1$

implies that $|q_n| = |p_n| + 1$. Proceeding by induction, we see that $|q_k| = |p_k| + 1$ for $k = n, n+1, \ldots$, which contradicts the assumption that $|q_k| > |p_k| + 1$ infinitely often. It follows that $|\xi_n| < 1$ for every n.

Suppose that $\xi_1 = B/A$ with integer A and B, $0 < |B| < A$. Then

$$|\xi_2| = \left|\frac{p_1 A - q_1 B}{B}\right| = \left|\frac{C}{B}\right| < 1\,.$$

Proceeding by induction and observing that $\xi_k \neq 0$, $k = 1, 2, \ldots$, we obtain an infinite decreasing sequence $A > |B| > |C| > |D| > \cdots$ of positive integers, which is not possible. Hence ξ is irrational. $\square$

Applying Legendre's theorem to the continued fraction of $\tan(m/n)$ we see that $x \longrightarrow \tan x$ maps $\mathbb{Q}$ to $\mathbb{R} \setminus \mathbb{Q}$. But $\tan(\pi/4) = 1 \in \mathbb{Q}$. Therefore π is irrational. If $v = \pi$ in (5.31) then

$$3 = \frac{\pi^2}{5} \, {}_{-} \, \frac{\pi^2}{7} \, {}_{-} \, \frac{\pi^2}{9} \, {}_{-} \cdots = \frac{m}{5n} \, {}_{-} \, \frac{m}{7} \, {}_{-} \, \frac{m}{9n} \, {}_{-} \, \frac{m}{11} \, {}_{-} \cdots ,$$

provided $\pi^2 = m/n$ with $m, n \in \mathbb{Z}$. Since this contradicts Legendre's theorem, π^2 is irrational.

It is useful to note that (4.93) combined with an obvious equivalence transform shows that

$$\coth \frac{m}{n} = \frac{n}{m} + \frac{n^2}{3m} \, {}_{+} \, \frac{n^2}{5m} \, {}_{+} \cdots$$

is irrational for every pair of integers m and n. It follows that $x \to \exp x$ maps $\mathbb{Q}$ into $\mathbb{R} \setminus \mathbb{Q}$.

The original proof of Lambert was not so brilliantly arranged as that of Legendre. Legendre not only provided rigorous proofs but also simplified the proof of (5.32). Instead of Pringsheim's test in the above arguments Legendre proved its partial case using, in fact, the same proof as that given here (see Rudio 1892 for details). See Wallisser (1998) for an analysis of Lambert's proof.

5.4 The parabola theorem

115 Pringsheim's test and Scott–Wall inequalities. In 1655 Brouncker noticed an important difference in the behavior of the even and odd convergents of regular continued fractions; see Theorem 1.7. This observation combined with Bernoulli's inverse formulas (4.1) leads to two important continued fractions: the *even* and *odd* parts of a continued fraction. If

$$q_0 + \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{p_n}{q_n}\right) \tag{5.33}$$

is a continued fraction with convergents $\{f_n\}_{n\geqslant 0}$ then Bernoulli's formulas (4.1) of Theorem 4.1 show that

$$q_0 + \frac{p_1 q_2}{q_1 q_2 + p_2} - \frac{p_2 p_3 q_4}{(q_2 q_3 + p_3) q_4 + q_2 p_4} - \frac{p_4 p_5 q_2 q_6}{(q_4 q_5 + p_5) q_6 + q_4 p_6} - \cdots$$
$$- \frac{p_{2n-2} p_{2n-1} q_{2n-4} q_{2n}}{(q_{2n-2} q_{2n-1} + p_{2n-1}) q_{2n} + q_{2n-2} p_{2n}} - \cdots \tag{5.34}$$

is the even part of (5.33) with convergents $\{f_{2n}\}_{n\geqslant 0}$. Similarly,

$$\frac{q_0 q_1 + p_1}{q_1} - \frac{p_1 p_2 q_3}{(q_1 q_2 + p_2) q_3 + q_1 p_3}$$
$$- \frac{p_3 p_4 q_1 q_5}{(q_3 q_4 + p_4) q_5 + q_3 p_5} - \frac{p_4 p_5 q_2 q_6}{(q_4 q_5 + p_5) q_6 + q_4 p_6} - \cdots$$
$$- \frac{p_{2n-1} p_{2n} q_{2n-3} q_{2n+1}}{(q_{2n-1} q_{2n} + p_{2n}) q_{2n+1} + q_{2n-1} p_{2n+1}} - \cdots \tag{5.35}$$

is the odd part of (5.33) with convergents $\{f_{2n+1}\}_{n\geqslant 0}$. The even part exists if and only if $q_{2k} \neq 0$ for $k = 1, 2, \ldots$ The odd part exists if and only if $q_{2k+1} \neq 0$ for $k = 0, 1, 2, \ldots$ This follows from the requirement that the partial numerators of continued fractions cannot vanish; see the partial numerators of (115) and (115). Details of these calculations can be found in Jones and Thron (1980, Theorems 2.10 and 2.11). However, the same goal can also be achieved by simply excluding excessive numerators and denominators in the Euler–Wallis formulas; see (5.26) or Lemma 2.21. Let us illustrate this method for $\mathbf{K}_{n=1}^{\infty}(c_n/1)$. Adding the Euler–Wallis formulas,

$$\begin{aligned} P_k &= P_{k-1} + c_k P_{k-2}, && \times 1 \\ P_{k-1} &= P_{k-2} + c_{k-1} P_{k-3}, && \times 1 \\ P_{k-2} &= P_{k-3} + c_{k-2} P_{k-4}, && \times(-c_{k-1}) \end{aligned}$$

multiplied by the weights shown on the right and observing that Q satisfies the same equations, we obtain

$$\begin{aligned} P_k &= (1 + c_{k-1} + c_k) P_{k-2} - c_{k-2} c_{k-1} P_{k-4}\,, \\ Q_k &= (1 + c_{k-1} + c_k) Q_{k-2} - c_{k-2} c_{k-1} Q_{k-4}\,, \end{aligned} \tag{5.36}$$

for $k = 3, 4, \ldots$ We have $P_2 = (1 + c_2) \times 0 + c_1 \times 1$ and $Q_2 = (1 + c_2) \times 1 + c_1 \times 0$. Considering equations (5.36) for $k = 2n$, $n = 2, 3, \ldots$ as the Euler–Wallis equations in n for a new continued fraction, we obtain that

$$\frac{1}{0}, \quad \frac{0}{1}, \quad \frac{P_2}{Q_2}, \quad \frac{P_4}{Q_4}, \quad \ldots, \quad \frac{P_{2n}}{Q_{2n}}, \quad \ldots$$

are the convergents to the even part

$$\mathop{\mathbf{K}_{\mathrm{e}}}_{n=1}^{\infty} (c_n/1) \approx \frac{c_1}{1 + c_2} - \frac{c_2 c_3}{1 + c_3 + c_4} - \cdots - \frac{c_{2n-2} c_{2n-1}}{1 + c_{2n-1} + c_{2n}} - \cdots\,. \tag{5.37}$$

To find the odd part of $\mathbf{K}_{n=1}^{\infty}(c_n/1)$ we apply (5.36) for $k=2n+1$, $n \geqslant 1$.

Hence

$$\frac{1}{0}\,, \quad \frac{P_1}{Q_1}=c_1\,, \quad \frac{P_3}{Q_3}\,, \quad \ldots\,, \quad \frac{P_{2n+1}}{Q_{2n+1}}\,, \quad \ldots$$

are the convergents of the odd part

$$\underset{n=1}{\overset{\infty}{\mathbf{K}_{\mathrm{o}}}}(c_n/1) \approx c_1 - \frac{c_1c_2}{1+c_2+c_3} - \frac{c_3c_4}{1+c_4+c_5} - \cdots - \frac{c_{2n-1}c_{2n}}{1+c_{2n}+c_{2n+1}} - \cdots. \tag{5.38}$$

Theorem 5.18 (Scott–Wall; see Jones and Thron 1980) *For every sequence* $\{c_n\}_{n\geqslant 1}$ *satisfying the fundamental inequalities* (5.27), *both the even and odd parts of* $\mathbf{K}_{n=1}^{\infty}(c_n/1)$ *converge to a finite or infinite value. The value of the even part is finite if* $r_1|1+c_2| > |c_2|$. *The value of the odd part is finite if* $r_2|1+c_2+c_3| > |c_3|$.

Proof If we apply to the even part the equivalence transform $\rho_n = r_{2n-1}c_{2n}^{-1}$, $n = 1, 2, \ldots$, then by Theorem 3.6 the partial numerators a_n and partial denominators b_n of the continued fraction thus obtained satisfy

$$a_1 = \frac{r_1c_1}{c_2}\,, \qquad b_1 = \frac{r_1(1+c_2)}{c_2}\,,$$

$$a_n = \frac{r_{2n-1}}{c_{2n}}r_{2n-3}c_{2n-1}\,, \qquad b_n = \frac{r_{2n-1}}{c_{2n}}(1+c_{2n-1}+c_{2n})\,.$$

By the Scott–Wall fundamental inequalities

$$|b_n| = r_{2n-1}\left|\frac{1+c_{2n-1}+c_{2n}}{c_{2n}}\right| \geqslant \frac{r_{2n-1}r_{2n-3}|c_{2n-1}|}{|c_{2n}|}+1 = |a_n|+1\,,$$

and Pringsheim's test the continued fraction $\mathbf{K}_{n=2}^{\infty}(a_n/b_n)$ converges to a point in the closed unit disc. Consequently the even part of $\mathbf{K}_{n=1}^{\infty}(c_n/1)$ converges to $r_1c_1(r_1(1+c_2)+c_2w)^{-1}$, $|w| \leqslant 1$, which is finite if $r_1|1+c_2| > |c_2|$.

We apply the equivalence transform $\rho_{n+1} = r_{2n}c_{2n+1}{}^{-1}$, $n = 1, 2, \ldots$ to the odd part (5.38). Then for $n = 2, 3, \ldots$,

$$a_{n+1} = \rho_n\rho_{n+1}c_{2n-1}c_{2n} = \frac{r_{2n}r_{2n-2}c_{2n}}{c_{2n+1}}\,,$$

$$b_{n+1} = \frac{r_{2n}}{c_{2n+1}}(1+c_{2n}+c_{2n+1})\,.$$

It follows that

$$|b_{n+1}| = \frac{r_{2n}|1+c_{2n}+c_{2n+1}|}{|c_{2n+1}|} \geqslant \frac{r_{2n}r_{2n-2}|c_{2n}|}{|c_{2n+1}|}+1 = |a_{n+1}|+1\,.$$

By Pringsheim's test the continued fraction $\mathbf{K}_{\mathrm{o}_{n=3}}^{\infty}(a_n/b_n)$ converges to a finite value which does not exceed 1 in modulus. This shows that the odd part of $\mathbf{K}_{n=1}^{\infty}(c_n/1)$ converges to

$$c_1 - r_2c_1c_2(r_2(1+c_2+c_3)-vc_3)^{-1}\,, \qquad |v| \leqslant 1\,,$$

which is finite if $r_2|1+c_2+c_3| > |c_3|$. □

116 The parabola theorem. Let us observe that the domain

$$\mathcal{P}_0 = \{z : 1/2 + \operatorname{Re} z \geqslant |z|\}$$

in the complex plane $\mathbb{C}$ is a parabola directed along the real axis and having its vertex at $z = -1/4$.

Lemma 5.19 *Any sequence $\{c_n\}_{n\geqslant 2}$ in $\mathcal{P}_0$ satisfies the fundamental inequalities* (5.27) *with $r_1 = r_2 = \cdots = 1$ and*

$$|1+c_2| \geqslant 1/2 + |c_2|\,, \quad |1+c_2+c_3| \geqslant |c_2| + |c_3|\,. \tag{5.39}$$

Proof Since the modulus is greater than the real part,

$$|1+c_k+c_{k+1}| \geqslant 1/2 + \operatorname{Re} c_k + 1/2 + \operatorname{Re} c_{k+1} \geqslant |c_k| + |c_{k+1}|$$

for $k = 2, 3, \ldots$ and $|1+c_2| \geqslant 1 + \operatorname{Re} c_2 \geqslant 1/2 + |c_2|$. □

Theorem 5.20 (The parabola theorem) *Suppose that all the complex numbers $\{c_n\}_{n\geqslant 2}$ are nonzero and in the parabolic domain $\mathcal{P}_0$. Then the continued fraction $\mathbf{K}_{n=1}^{\infty}(c_n/1)$ converges if and only if*

$$\sum_{k=1}^{\infty} \left| \frac{c_2 c_4 c_6 \cdots c_{2k}}{c_3 c_5 c_7 \cdots c_{2k+1}} \right| + \sum_{k=1}^{\infty} \left| \frac{c_3 c_5 c_7 \cdots c_{2k-1}}{c_4 c_6 c_8 \cdots c_{2k}} \right| = +\infty\,. \tag{5.40}$$

Proof By Lemma 5.19 the sequence $\{c_n\}_{n\geqslant 2}$ satisfies the fundamental inequalities with $r_k = 1$ for $k = 1, 2, \ldots$ Hence by Theorem 5.18 the even and odd parts of $\mathbf{K}_{n=1}^{\infty}(c_n/1)$ converge. Since $|1+c_2| > |c_2|$, the even part converges to a finite value. Similarly, since $|1+c_2+c_3| \geqslant |c_2| + |c_3| > |c_3|$ the odd part converges to a finite value. Therefore the convergence of the continued fraction will be established if we can prove that

$$\left| \frac{P_{2n}}{Q_{2n}} - \frac{P_{2n+1}}{Q_{2n+1}} \right| = \left| \frac{c_1 \cdots c_{2n+1}}{Q_{2n} Q_{2n+1}} \right| \to 0\,, \qquad n \to +\infty\,. \tag{5.41}$$

Lemma 5.21 *We have $Q_k \neq 0$ for $k \geqslant 0$ and, for $n \geqslant 2$,*

$$\begin{aligned} \left| \frac{Q_{2n}}{c_2 \cdots c_{2n}} \right| - \left| \frac{Q_{2n}}{c_2 \cdots c_{2n-2}} \right| &\geqslant \frac{1}{2} \left| \frac{c_3 \cdots c_{2n-1}}{c_2 \cdots c_{2n}} \right|\,, \\ \left| \frac{Q_{2n+1}}{c_3 \cdots c_{2n+1}} \right| - \left| \frac{Q_{2n-1}}{c_3 \cdots c_{2n-1}} \right| &\geqslant \frac{|c_3|}{2} \left| \frac{c_4 \cdots c_{2n}}{c_3 \cdots c_{2n+1}} \right|\,. \end{aligned} \tag{5.42}$$

Proof By (5.36) and (5.27),

$$\begin{aligned} |Q_k| &\geqslant |1+c_{k-1}+c_k||Q_{k-2}| - |c_{k-2}c_{k-1}Q_{k-4}| \\ &\geqslant (|c_{k-1}| + |c_k|)|Q_{k-2}| - |c_{k-2}c_{k-1}Q_{k-4}|\,, \end{aligned}$$

which implies that

$$|Q_k| - |c_k||Q_{k-2}| \geqslant |c_{k-1}|(|Q_{k-2}| - |c_{k-2}||Q_{k-4}|)\,.$$

Putting $k=2n$, $k=2n+1$ and iterating, we obtain

$$\begin{aligned} |Q_{2n}|-|c_{2n}||Q_{2n-2}| &\geqslant |c_3\cdots c_{2n-1}|(|Q_2|-|c_2||Q_0|)\,, \\ |Q_{2n+1}|-|c_{2n+1}||Q_{2n-1}| &\geqslant |c_4\cdots c_{2n}|(|Q_3|-|c_2||Q_1|)\,. \end{aligned} \tag{5.43}$$

By (5.39) $|Q_2|-|c_2||Q_0|>1/2$ and $|Q_3|-|c_2||Q_1|\geqslant|c_3|$. It follows that

$$|Q_k|-|c_k||Q_{k-2}|\geqslant 1/2|c_3\cdots c_{k-1}|\,, \qquad k=4,5,\ldots \tag{5.44}$$

We conclude by induction that $|Q_k|>0$ for $k\geqslant 0$. Inequalities (5.42) follow from (5.43) by division. □

By (5.42) both the sequences $|Q_{2n}(c_2\cdots c_{2n})^{-1}|$, $|Q_{2n+1}(c_3\cdots c_{2n+1})^{-1}|$ increase. Therefore if one series in (5.40) diverges then their product tends to infinity, implying (5.41). If both series converge then by Corollary 3.8 $\mathbf{K}_{n=1}^{\infty}(c_n/1)\approx\mathbf{K}_{n=1}^{\infty}(1/q_n^*)$ with $\sum_n|q_n^*|<+\infty$, and by Koch's theorem 3.3 the continued fraction diverges. □

The following corollary extends Corollary 3.10 to the parabolic domain $\mathcal{P}_0$. However, since $\mathcal{P}_0$ contains the closed disc $\{z:|z|\leqslant 1/4\}$, it also extends Worpitsky's test, Corollary 5.14.

Corollary 5.22 *Suppose that the complex numbers $\{c_n\}_{n\geqslant 2}$ are all nonzero and are in the parabolic domain $\mathcal{P}_0$. Then the continued fraction $\mathbf{K}_{n=1}^{\infty}(c_n/1)$ converges if $\sum_k|c_{2k+1}|^{-1/2}=+\infty$.*

Proof As in Corollary 3.10 we apply the inequality $2\sqrt{u_k v_k}\leqslant u_k+v_k$ with

$$u_k=\left|\frac{c_2c_4c_6\cdots c_{2k}}{c_3c_5c_7\cdots c_{2k+1}}\right|, \qquad v_k=\left|\frac{c_3c_5c_7\cdots c_{2k-1}}{c_4c_6c_8\cdots c_{2k}\,,}\right|,$$

and observe that $u_kv_k=|c_2/c_{2k+1}|$. □

Exercises

5.1 Prove Theorem 5.2. *Hint:* Sum the inequalities

$$(a-\varepsilon)(Q_k-Q_{k-1})<P_k-P_{k-1}<(a+\varepsilon)(Q_k-Q_{k-1})$$

for $k=N,N+1,\ldots,n$, where a is the limit of the ratio in Theorem 5.2.

5.2 Let $y_n>a>0$, $n=1,2,\ldots$ Prove that for $s>\max_n(y_n-y_{n-1})$

$$y_0+\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{y_{n-1}(y_n-a)}{s}\right)=\frac{y_0(s-a)}{s-y_1}\;{}_{+}\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{y_n(y_n-a)}{s+y_n-y_{n+1}}\right).$$

Hint: Apply the Bauer–Muir transform with parameters $a_n=y_{n-1}(y_n-a)$, $b_n=s$, $x_n=-y_n$ and then apply Theorem 5.3.

5.3 Prove that

$$2s+\underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{4n^2-1}{2s}\right)=1+\frac{2s}{b(s+1)-s},$$

where $b(s)$ is Brouncker's continued fraction.

Hint: Apply Ex. 5.2 with $a=2$, $y_n=2n+1$.

5.4 Prove that for $s>0$

$$1+\frac{2}{2s-1}\,{}_{+}\,\frac{1\times 3}{2s}\,{}_{+}\,\frac{3\times 5}{2s}\,{}_{+}\,\frac{5\times 7}{2s}\,{}_{+\cdots}=\frac{s}{4}\left[\frac{\Gamma(s/4)}{\Gamma((s+2)/4)}\right]^2.$$

Hint: Apply Ex. 5.3 and Theorem 3.25.

5.5 Prove that

$$1+\frac{2}{8k-1}\,{}_{+}\,\frac{1\times 3}{8k}\,{}_{+}\,\frac{3\times 5}{8k}\,{}_{+}\,\frac{5\times 7}{8k}\,{}_{+\cdots}=\left(\frac{(2k)!!}{(2k-1)!!}\right)^2\frac{1}{\pi k},$$

$$1+\frac{2}{8k-5}\,{}_{+}\,\frac{1\times 3}{8k-4}\,{}_{+}\,\frac{3\times 5}{8k-4}\,{}_{+}\,\frac{5\times 7}{8k-4}\,{}_{+\cdots}=\left(\frac{(2k-1)!!}{(2k)!!}\right)^2\frac{2k^2\pi}{2k-1}.$$

Hint: Put $s=4k$ and $s=4k-2$ in Ex. 5.4 and apply (3.66).

5.6 For $s>0$ prove that

$$\underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n^2}{s+1}\right)=\frac{s}{s-1}\,{}_{+}\,\underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{n(n+1)}{s}\right).$$

Hint: Put $a_n=n^2$, $b_n=s+1$, $x_n=-n-1$ and apply the Bauer–Muir transform. Obtain a second proof of the formula using a description of the two continued fractions involved in terms of functional equations; see Corollary 4.20 and (4.75).

5.7 Let $x>1$ and $c_n>0$, $n=1,2,3,\ldots$ Then

$$\underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{x+c_n}{c_n}\right)=1+\frac{x-1}{1+c_1}\,{}_{+}\,\underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{x+c_n}{c_{n+1}}\right)$$

provided that the continued fraction on the left-hand side converges.

Hint: Apply the Bauer–Muir transform with $a_n=x+c_n$, $b_n=c_n$, $x_n=1$. Then apply Theorem 5.3.

5.8 Let $c_n>0$, $n=1,2,3,\ldots$ Prove that

$$\underset{n=1}{\overset{\infty}{\mathbf{K}}}\left(\frac{1+c_n}{c_n}\right)=1 \tag{E5.1}$$

provided that the continued fraction converges.

Hint: Pass to the limit $x\to 1^+$ in Ex. 5.7.

5.9 Prove Ramanujan's formula

$$\frac{x+1}{x} \; {}_{+} \; \frac{x+2}{x+1} \; {}_{+} \; \frac{x+3}{x+2} \; {}_{+\cdots} = 1\,, \qquad x \geqslant 0\,.$$

Hint: Assuming first that $x > 0$, in Ex. 5.8 act $c_n = x+n-1$ and apply Corollary 3.10. Reduce the case $x = 0$ to $x = 1$.

5.10 Prove that (E5.1) holds for any complex sequence $\{c_n\}_{n\geqslant 1}$ with $c_n \neq -1$ such that the continued fraction in (E5.1) converges and the infinite product $\prod_n |1+c_n|$ diverges to $+\infty$ (McLaughlin and Wyshinski 2003).

Hint: Apply Ex. 4.45 to prove that (E5.1) is the even part of

$$\frac{c_1+1}{c_1+1} \; {}_{-} \; \frac{1}{1} \; {}_{+} \; \frac{c_2+1}{0} \; {}_{-} \; \frac{1}{1} \; {}_{+} \; \frac{c_3+1}{0} \; {}_{-} \; \frac{1}{1} \; {}_{+\cdots}\,. \tag{E5.2}$$

Let P_n/Q_n be the convergents to (E5.2). Apply Ex. 1.12 to prove that

$$\left|\frac{P_{2n+2}}{Q_{2n+2}} - \frac{P_{2n}}{Q_{2n}}\right| = \frac{\prod_{k=1}^{n+1}|1+c_k|}{|Q_{2n}Q_{2n+2}|} \to 0\,.$$

The conclusion is that $\lim |Q_{2n}Q_{2n+2}| = +\infty$. Apply the Euler–Wallis recursion to establish that

$$P_{2n+1} = Q_{2n+1} = \prod_{k=1}^{n+1}(1+c_k)$$

and use this identity to prove that

$$\left|1 - \frac{P_{2n}}{Q_{2n}}\right| = \left|\frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}}\right| = \frac{1}{|Q_{2n}|}\,.$$

Since the limit of the left-hand side exists, the limit $\lim_n |Q_{2n}|$ must exist and be equal to $+\infty$.

5.11 Using Bauer's method prove that the continued fraction

$$y(s) = s + \mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{(2k-1)^2r^2-(r-1)^2}{2s}\right)$$

satisfies the functional equation for $s > 0$

$$y(s)y(s+2r) = (s+1)(s+2r-1)\,.$$

Hint: Apply the Bauer–Muir transform with $x_n = 2rn - s$.

5.12 Prove Kummer's test. Let $\{c_n\}_{n\geqslant 0}$ be a positive sequence such that $\sum_{n\geqslant 0} c_n^{-1} = +\infty$. If for a positive sequence $\{u_n\}_{n\geqslant 0}$ we have

$$\liminf_n \left(c_n\frac{u_n}{u_{n+1}} - u_{n+1}\right) > 0\,,$$

then $\sum_n u_n < +\infty$.

Hint: Consider the auxiliary series $\sum_n c_nu_n - c_{n+1}u_{n+1}$.

6

P-fractions

6.1 Laurent series

Brouncker's theorem (see Theorem 3.16) and Euler's formula (4.105) are the most impressive achievements of the early theory of continued fractions. Both Brouncker and Euler obtained their results using Wallis' interpolation method. For instance, in Euler's case formula (4.105) can be considered as a restriction to $\mathbb{N}$ of a polynomial continued fraction

$$e^{1/s} = \mathbf{b}_0(s) + \frac{1}{\mathbf{b}_1(s)} \, {}_{+} \, \frac{1}{\mathbf{b}_2(s)} \, {}_{+ \cdots +} \, \frac{1}{\mathbf{b}_n(s)} \, {}_{+ \cdots} \, ,$$

where the $\mathbf{b}_n(s)$ are linear polynomials in s with integer coefficients. This interpolation method follows an analogy between integers and polynomials: integer partial denominators of regular continued fractions are replaced by polynomials. Then decimal representations of numbers, as Brouncker observed, correspond to asymptotic series; see §**60** in Section 3.2. We are going to consider this correspondence in more detail here.

117 Long division of polynomials. The Euclidean algorithm will run not only in $\mathbb{Z}$ but also in some other rings. The most important examples are the rings $\mathbb{Q}[X]$, $\mathbb{R}[X]$, $\mathbb{C}[X]$ of polynomials

$$p(X) = c_0 X^n + c_1 X^{n-1} + \cdots + c_{n-1} X + c_n$$

with rational, real or complex coefficients $c_0, c_1, \ldots, c_n$. By the long division of polynomials, the basic properties of the Euclidean algorithm extend to some rings of polynomials. Rings in which the Euclidean algorithm works are called *Euclidean*.

A commutative ring $\mathfrak{R}$ with only one element, 0, cannot be an integral domain: a commutative ring is an integral domain if it does not have divisors of zero, i.e. $ab = 0$ implies either $a = 0$ or $b = 0$; Lang (1965).

Definition 6.1 (Koch 2000) *An integral domain $\mathfrak{R}$ is called Euclidean if there exists a function*

$$H : \mathfrak{R} \to \mathbb{N} \cup 0 \stackrel{\text{def}}{=} \mathbb{Z}_+$$

called the height, such that $H(ab) \geqslant \max(H(a), H(b))$ *for any* $a, b \neq 0$ *and for any two elements* a, b *with* $a \neq 0$ *there exists a representation*

$$b = qa + r ,$$

where either $r = 0$ *or* $H(r) < H(a)$.

If K is a field then $K[X]$ denotes the Euclidean domain of polynomials in X with coefficients in K. The *height* H in $K[X]$ is the degree $\deg p$ of a polynomial p, $p \in K[X]$. As usual $K(X)$ is the field of quotients for $K[X]$ and is also called the field of rational functions in X (with coefficients in K). The continued fractions (1.2) with coefficients in a given Euclidean domain determine elements of the quotient field for this domain. So, the situation is very similar to what we have for $\mathbb{Z}$ and its quotient field $\mathbb{Q}$.

A particular important example is obtained if $K = \mathbb{C}$ and $X = z$, z being the identical map of $\mathbb{C}$.

Any pair f_0, f_1 of polynomials in X such that $\deg f_0 > \deg f_1$ generates a sequence of polynomials with decreasing degrees $\deg f_0 > \deg f_1 > \deg f_2 > \cdots$ in the set $\mathbb{C}[X]$ of all polynomials with complex coefficients:

$$\begin{aligned} f_0 &= \mathbf{b}_0 f_1 + f_2 , \\ f_1 &= \mathbf{b}_1 f_2 + f_3 , \\ &\vdots \\ f_{n-2} &= \mathbf{b}_{n-2} f_{n-1} + f_n , \\ f_{n-1} &= \mathbf{b}_{n-1} f_n . \end{aligned} \tag{6.1}$$

Here $\mathbf{b}_j \in \mathbb{C}[X]$, $j = 0, 1, \ldots$ Any sequence in $\mathbb{C}[X]$ with decreasing degrees is finite. Therefore there exists $n \in \mathbb{N}$ such that $f_{n-1} = b_{n-1} f_n$, so the algorithm stops at this step. This algorithm is known as the Euclidean algorithm for the long division of polynomials and leads to the same conclusion as in the number-field case: f_n is the greatest common divisor (f_0, f_1) of f_0 and f_1.

The factorization theory of integers is based on the formula

$$ax + by = d . \tag{6.2}$$

Here a and b are integers with the greatest common divisor d and x, y are some integers. For given a and b their greatest common divisor d can be found by the above Euclidean algorithm, and (1.16) with $P_n = a$ and $Q_n = b$ shows that $x = (-1)^{n-1} Q_{n-1}$ and $y = (-1)^n P_{n-1}$ are solutions to (6.2). The fact that (6.2) always has solutions in the integers results in Euclid's factorization theorem,

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} , \tag{6.3}$$

where $p_1 < p_2 < \cdots < p_r$ is the complete list of prime divisors of n and $k_1, k_2, \ldots, k_r$ are positive integers. Namely, attempts to extend (6.3) to other commutative rings led to the notion of the Euclidean ring, or Euclidean domain.

118 ***P*-fractions.** On the one hand, since $\mathbb{C}[z]$ is an Euclidean domain, any rational function $x_0/x_1 \in \mathbb{C}(z)$ expands into a finite continued fraction of the form (1.2), where $\mathbf{b}_j \in \mathbb{C}[z]$, $\deg \mathbf{b}_j \geqslant 1$, $j = 1, 2, \ldots, n-1$. On the other hand, with any sequence

$\{\mathbf{b}_j\}_{j\geqslant 0}$ of polynomials in z satisfying $\deg \mathbf{b}_j \geqslant 1$, $j = 1, 2, \ldots$, one can associate a formal continued fraction

$$\mathbf{b}_0(z) + \mathop{\mathbf{K}}_{k=1}^{\infty} \left(\frac{1}{\mathbf{b}_k(z)} \right) , \tag{6.4}$$

called a P-fraction. Its nth convergent is defined by

$$\frac{P_n}{Q_n} = \mathbf{b}_0 + \frac{1}{\mathbf{b}_1} {}_{+\cdots+} \frac{1}{\mathbf{b}_n} .$$

By the Euler–Wallis recurrence formulas (1.15),

$$\begin{aligned} &P_n = \mathbf{b}_n P_{n-1} + P_{n-2}, \quad Q_n = \mathbf{b}_n Q_{n-1} + Q_{n-2}, \qquad n = 1, 2, \ldots, \\ &P_1 = 1, \quad P_0 = \mathbf{b}_0, \quad Q_{-1} = 0, \quad Q_0 = 1 . \end{aligned} \tag{6.5}$$

It follows from (6.5) that

$$\deg Q_n = \deg \mathbf{b}_1 + \cdots + \deg \mathbf{b}_n , \qquad n = 1, 2, \ldots .$$

Another corollary of (6.5) is the following formula:

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{n-1}}{Q_n Q_{n-1}}. \tag{6.6}$$

Since the above construction is analogous to the corresponding construction in the field $\mathbb{Q}$ of rational numbers, we can use it to obtain and parameterize the continuum of continued P-fractions, as in §**23**, Section 1.3 for regular continued fractions. So, starting with the Euclidean domain $\mathbb{C}[z]$ and applying the method of continued fractions we obtain another continuum, namely, the field $\mathbb{C}\,([1/z])$ of formal Laurent series at $z = \infty$. Every element f in $\mathbb{C}\,([1/z])$ is a formal series

$$f(z) = \sum_{k\in\mathbb{Z}} \frac{c_k}{z^k} , \tag{6.7}$$

where only a finite number of complex coefficients $c_k \in \mathbb{C}$ with negative indices k differ from 0. We put

$$[\![f]\!] = \sum_{k\leqslant 0} \frac{c_k}{z^k}, \qquad \operatorname{Frac} f = \sum_{k>0} \frac{c_k}{z^k} .$$

As for real numbers, $[\![f]\!]$ is called the *integer* part and $\operatorname{Frac} f$ the *fractional* part of f. The field $\mathbb{C}\,([1/z])$ is equipped with a nonarchimedean norm

$$\|f\| = \exp(\deg f) , \qquad \deg f = -\inf\{k \in \mathbb{Z} : c_k \neq 0\} . \tag{6.8}$$

If f is a polynomial then $\deg f$ in (6.8) is its degree. As usual we put $\deg 0 = -\infty$. It is easy to see that the norm in (6.8) satisfies the following properties:

$$\|r\| = 0 \quad \Leftrightarrow \quad r = 0, \tag{6.9}$$

$$\|rs\| = \|r\| \, \|y\|, \tag{6.10}$$

$$\|r + s\| \leqslant \max(\|r\|, \|s\|). \tag{6.11}$$

Clearly, the open unit ball in this norm consists of elements f satisfying $f = \mathrm{Frac}(f)$. The closed unit ball is the ring $\mathbb{C}[[1/z]]$ of all formal Laurent series f at $z = \infty$ such that $c_k = 0$ in (6.7) for $k < 0$.

Theorem 6.2 *Let f be an element of* $\mathbb{C}\,([1/z])$. *Then*

(a) *the element f can be developed into a continued P-fraction* (6.4);
(b) *the partial denominators Q_n of the continued P-fraction of f satisfy*

$$\lim_n \deg Q_n = +\infty\,;$$

(c) *the continued fraction of f converges in the norm of* $\mathbb{C}\,([1/z])$, *moreover*

$$\left\| f - \frac{P_n}{Q_n} \right\| = \exp\left(-\deg Q_n - \deg Q_{n+1}\right)\,;$$

(d) *the continued fraction of f corresponds to the Laurent series of f in the sense that the Laurent coefficients of f and P_n/Q_n are equal for indices k satisfying $k < s_n + s_{n+1}$*;
(e) *every continued P-fraction converges in* $\mathbb{C}\,([1/z])$;
(f) *only one P-fraction corresponds to a Laurent series*;
(g) *a continued fraction* (6.4) *is finite if and only if it corresponds to a rational function*;
(h) *the field* $\mathbb{C}(z)$ *is dense in* $\mathbb{C}\,([1/z])$.

Proof (a) As in the case of real numbers we put $f_0 = f$ and define $f_n = 1/\mathrm{Frac}\, f_{n-1}$ for $n = 1, 2, \ldots$ Then

$$f = [\![f_0]\!] + \frac{1}{1/\mathrm{Frac}\, f_0} = [\![f_0]\!] + \frac{1}{[\![f_1]\!]} {}_{+} \cdots {}_{+} \frac{1}{[\![f_n]\!] + \mathrm{Frac}\, f_n}\,. \tag{6.12}$$

Clearly $[\![f_k]\!] \in \mathbb{C}[z]$. The algorithm (6.12) stops in a finite number of steps if and only if $\mathrm{Frac}\, f_n = 0$ for some integer n, which happens if and only if f is a rational function. Indeed, if $\mathrm{Frac}\, f_n = 0$ for some n then $f \in \mathbb{C}(z)$, since the $[\![f_k]\!]$ are polynomials. However, if $f = P/Q$, where P and Q are polynomials, then the long division of polynomials results in a finite continued fraction (6.12).

(b) If P_n/Q_n are convergents to the continued P-fraction constructed above, then by the Euler–Wallis formulas

$$\begin{aligned} P_{n+1} &= \mathbf{b}_{n+1} P_n + P_{n-1} \,, \\ Q_{n+1} &= \mathbf{b}_{n+1} Q_n + Q_{n-1} \,, \qquad n = 1, 2, \dots \,, \\ P_0 = 0 \,, &\quad P_1 = 1 \,, \quad Q_0 = 1 \,, \quad Q_1 = \mathbf{b}_1(z) \,. \end{aligned} \tag{6.13}$$

It follows from (6.13) that $s_n \stackrel{\text{def}}{=} \deg Q_n = \deg \mathbf{b}_1 + \cdots + \deg \mathbf{b}_n$. Observing that $\deg b_k \geqslant 1$, we obtain $s_n \geqslant n$.

(c) As in the case of real numbers, the Euler–Wallis formulas imply $P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}$, which shows that P_n/Q_n is a fraction in its lowest terms and, together with

$$f = \frac{P_{n+1} + P_n \operatorname{Frac}(f_{n+1})}{Q_{n+1} + Q_n \operatorname{Frac}(f_{n+1})} \,,$$

implies that

$$f - \frac{P_n}{Q_n} = \frac{(-1)^n}{Q_n Q_{n+1}(1 + \operatorname{Frac}(f_{n+1})\, Q_n/Q_{n+1})} \,. \tag{6.14}$$

Finally, by (6.10) and (6.14) we have

$$\begin{aligned} \deg\left(f - \frac{P_n}{Q_n}\right) &= -s_n - s_{n+1} + \deg\left(\frac{1}{1 + \operatorname{Frac}(f_{n+1})\, Q_n/Q_{n+1}}\right) \\ &= -s_n - s_{n+1} \,. \end{aligned} \tag{6.15}$$

(d) This follows from (c) by (6.8).

(e) Using the elementary identity

$$\frac{P_{n+k}}{Q_{n+k}} - \frac{P_n}{Q_n} = \left(\frac{P_{n+k}}{Q_{n+k}} - \frac{P_{n+k-1}}{Q_{n+k-1}}\right) + \cdots + \frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n} \,,$$

equation (6.6) and the property (6.11) of the norm, we obtain

$$\left\| \frac{P_{n+k}}{Q_{n+k}} - \frac{P_n}{Q_n} \right\| \leqslant \exp\left(-s_n - s_{n+1}\right) ,$$

which implies that the sequence of convergents of any continued P-fraction is a Cauchy sequence in $\mathbb{C}([1/z])$. Since by (d) the Laurent coefficients c_j of P_{n+k}/Q_{n+k} and P_n/Q_n are the same for $j < s_n + s_{n+1}$, we obtain that this sequence converges to some formal Laurent series.

(f) Every element f in $\mathbb{C}([1/z])$ is uniquely decomposed into the sum $f = [\![f]\!] + \operatorname{Frac} f$. By (e) every P-fraction $\mathbf{b}_0(z) + \mathbf{K}_{k=1}^{\infty}(1/\mathbf{b}_k(z))$ converges in $\mathbb{C}([1/z])$ and $[\![P_n/Q_n]\!] = \mathbf{b}_n(z)$, $n \geqslant 0$, by the Euler–Wallis formulas. Since $\mathbb{C}([1/z])$ is a field, the process described in (a) applied to any P-fraction gives this P-fraction itself.

(g) This follows from (a) and (f).

(h) This follows from (c). □

119 Huygens' theorem. The following analogue of Huygens' theorem 1.14 holds in the field $\mathbb{C}([1/z])$.

Theorem 6.3 *Let* $\mathbf{b}_0(z)+\mathbf{K}_{k=1}^{\infty}(1/\mathbf{b}_k(z))$ *be an infinite continued fraction and let* $0<\deg Q<\deg Q_n$. *Then for every polynomial* P *with* $P/Q\neq P_{n-1}/Q_{n-1}$,

$$\left\|\frac{P}{Q}-\frac{P_{n-1}}{Q_{n-1}}\right\|>\left\|\frac{P_{n-1}}{Q_{n-1}}-\frac{P_n}{Q_n}\right\| . \tag{6.16}$$

Proof Since $PQ_{n-1}-QP_{n-1}\neq 0$, we obtain by (6.15) that

$$\begin{aligned}\deg\left(\frac{P}{Q}-\frac{P_{n-1}}{Q_{n-1}}\right)&\geqslant\deg\left(\frac{1}{QQ_{n-1}}\right)=-\deg Q-\deg Q_{n-1}\\&>-\deg Q_n-\deg Q_{n-1}=\deg\left(\frac{P_n}{Q_n}-\frac{P_{n-1}}{Q_{n-1}}\right) .\end{aligned}$$

□

Both Theorems 6.2 and 6.3 imply that infinite P-fractions represent irrational functions. However, the arguments of Theorem 6.3 look more explicit. Let us assume to the contrary that $P/Q=f$, where f is an infinite continued fraction. Choose n to satisfy $\deg Q<\deg Q_n$. Such an n must exist, since f is an infinite continued fraction. Then by (6.15) and (6.16) we obtain a contradiction:

$$-s_{n-1}-s_n=\deg\left(f-\frac{P_{n-1}}{Q_{n-1}}\right)>\deg\left(\frac{P_{n-1}}{Q_{n-1}}-\frac{P_n}{Q_n}\right)=-s_{n-1}-s_n .$$

Remark By Theorem 6.2 the field $\mathbb{C}([1/z])$ can be parameterized similarly to $\mathbb{R}$. In the function-field case the parameters are finite or infinite sequences of polynomials $\{\mathbf{b}_k(z)\}_{k\geqslant 0}$ such that $\deg\mathbf{b}_k(z)\geqslant 1$.

120 Associated fractions and C-fractions. The field $\mathbb{C}([1/z])$ of formal Laurent series at $z=\infty$ is isomorphic to the field $\mathbb{C}([z])$ at $z=0$. The isomorphism is given by the substitution

$$f(z)\sim\sum_{k\in\mathbb{Z}}\frac{c_k}{z^k}\quad\to\quad f(1/z)\sim\sum_{k\in\mathbb{Z}}c_kz^k .$$

This isomorphism keeps the form of a Laurent series but violates the form of P-fractions. However, if $\deg\mathbf{b}_n=1$ for every n, i.e. $\mathbf{b}_n=x_nz+y_n$, then

$$\begin{aligned}\mathbf{b}_0(z)+\mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{1}{\mathbf{b}_k(z)}\right)\quad&\to\quad\mathbf{b}_0(1/z)+\frac{1}{x_1/z+y_1}\,{}_{+\cdots+}\,\frac{1}{x_n/z+y_n}\,{}_{+\cdots+}\\&\approx\mathbf{b}_0(1/z)+\frac{z}{x_1+y_1z}\,{}_{+}\,\frac{z^2}{x_2+y_2z}\,{}_{+}\,\frac{z^2}{x_3+y_3z}\,{}_{+\cdots+}\,\frac{z^2}{x_n+y_nz}\,{}_{+\cdots} .\end{aligned}$$

Since $x_n\neq 0$ the equivalence relation can be continued to

$$\frac{k_1z}{1+l_1z}\,{}_{-}\,\frac{k_2z^2}{1+l_2z}\,{}_{-}\,\frac{k_3z^2}{1+l_3z}\,{}_{-\cdots-}\,\frac{k_nz^2}{1+l_nz}\,{}_{-\cdots} , \tag{6.17}$$

where $k_1 = 1/x_1$, $l_1 = y_1/x_1$, $k_n = -1/x_{n-1}x_n$, $l_n = y_n/x_n$. The continued fraction (6.17) is called an *associated* continued fraction. If $l_n = 0$ for every n then the substitution $z := z^2$ leads to the continued fraction

$$a_0 + \operatorname*{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n z}{1}\right) \tag{6.18}$$

with nonzero complex parameters a_n for $n > 0$. Continued fraction (6.18) is called a *regular C-fraction*. It corresponds to a formal power series centered at $z = 0$ in the sense that its nth convergent matches the first n coefficients of this series, see (1.16). However, if we are given a regular continued fraction (6.18) then obvious equivalence transforms reduce it to a form of a P-fraction:

$$a_0 + \operatorname*{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n/z}{1}\right) \approx c_0 + \frac{a_1}{z} \, \frac{}{+} \, \frac{a_2}{1} \, \frac{}{+} \, \frac{a_3}{z} \, \frac{}{+} \, \frac{a_4}{1} \, \frac{}{+} \, \frac{a_5}{z} \, \frac{}{+} \cdots \, . \tag{6.19}$$

The even part of this latter fraction is given by

$$a_0 + \frac{a_1}{z + a_2} \, \frac{}{-} \, \frac{a_2 a_3}{z + a_3 + a_4} \, \frac{}{-} \cdots \frac{}{-} \, \frac{a_{2n} a_{2n+1}}{z + a_{2n+1} + a_{2n+2}} \, \frac{}{-} \cdots \, . \tag{6.20}$$

This follows easily by an iterative application to (6.19) of the formula

$$\frac{p}{1 + q/(z+w)} = p - \frac{pq}{z + q + w} \, , \tag{6.21}$$

with $p = a_{2n}$, $q = a_{2n+1}$. By Corollary 3.8 the continued fraction (6.20) is equivalent (with constant equivalence parameters) to a P-fraction. These simple observations imply the following corollary.

Corollary 6.4 *No regular C-fraction can correspond to a power series at $z = 0$ of a rational function.*

Proof If $f(z)$ corresponds to (6.18) at $z = 0$ then $f(1/z)$ corresponds to (6.20) at $z = \infty$. The continued fraction (6.20) is infinite. Hence by (7) of Theorem 6.2 $f(1/z)$ cannot be rational and similarly for $f(z)$. □

6.2 Convergents

121 Chebyshev–Markoff theory. Convergents to P-fractions are described by Markoff's theorem. At least in part it was known to Gauss and Chebyshev, but it was Markoff (1948) who clearly stated it in his lectures.

Theorem 6.5 (Chebyshev–Markoff) *If $f \in \mathbb{C}([1/z])$ then a rational fraction P/Q in its lowest terms is a convergent for f if and only if*

$$\left\| f - \frac{P}{Q} \right\| \leqslant \exp(-2 \deg Q - 1) \quad \Leftrightarrow \quad \deg\left(f - \frac{P}{Q}\right) \leqslant -2 \deg Q - 1 \, .$$

Proof Necessity follows from (3) of Theorem 6.2. To prove sufficiency we choose n to satisfy

$$s_n = \deg Q_n \leqslant \deg Q = s < \deg Q_{n+1} = s_{n+1}$$

for two consecutive convergents P_n/Q_n and P_{n+1}/Q_{n+1}. Then

$$\deg\left(\frac{P}{Q} - f\right) \leqslant -2s - 1\,, \qquad \deg\left(\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n}\right) = -s_n - s_{n+1}\,,$$

$$\deg\left(\frac{P_{n+1}}{Q_{n+1}} - f\right) = -s_{n+1} - s_{n+2}\,.$$

Since

$$\frac{P_n}{Q_n} - \frac{P}{Q} = \frac{P_n}{Q_n} - \frac{P_{n+1}}{Q_{n+1}} + \frac{P_{n+1}}{Q_{n+1}} - f + f - \frac{P}{Q}$$

and $s < s_{n+1}$, we have

$$\deg\left(\frac{P_n}{Q_n} - \frac{P}{Q}\right) \leqslant \max(-s_n - s_{n+1}, -2s - 1)\,.$$

Applying (6.10), we obtain

$$\begin{aligned}\deg(P_nQ - Q_nP) &\leqslant s + s_n + \max(-s_n - s_{n+1}, -2s - 1)\\ &= \max(s - s_{n+1}, s_n - s - 1) < 0\,,\end{aligned}$$

since $s_n \leqslant s < s_{n+1}$. This implies that $P_nQ - PQ_n = 0$ and therefore $P_n/Q_n = P/Q$. □

Thus the Chebyshev–Markoff theorem is a functional analogue of the Legendre–Vahlen theorem. In the functional case an analogue of Lagrange's theorem 1.21 follows from Theorem 6.5.

Corollary 6.6 *Let $f \in \mathbb{C}([1/z])$ and P_n/Q_n be a convergent to f. Then any rational P/Q in its lowest terms with $0 < \deg Q \leqslant \deg Q_n$ satisfies*

$$\|P_n - Q_nf\| < \|P - Qf\|$$

unless $P = P_n$ and $Q = Q_n$.

Proof Suppose that $\|P - Qf\| \leqslant \|P_n - Q_nf\|$. Then

$$\begin{aligned}\deg\left(\frac{P}{Q} - f\right) &\leqslant \deg\left(\frac{P_n}{Q_n} - f\right) + \deg Q_n - \deg Q\\ &\quad - \deg Q_n - \deg Q_{n+1} + \deg Q_n - \deg Q\\ &= -\deg Q_{n+1} - \deg Q < -2\deg Q - 1\,.\end{aligned}$$

By Markoff's theorem P/Q is a convergent for f. Since $\deg Q \leqslant \deg Q_n$, we obtain that $Q = Q_k$ for some $0 < k \leqslant n$. But

$$\begin{aligned}\deg(P - Qf) &= \deg(P_k - Q_k f) = \deg Q_k + \deg P_k/Q_k - f \\ &= \deg Q_k - \deg Q_k - \deg Q_{k+1} \\ &= -\deg Q_{k+1} > -\deg Q_{n+1} = \deg(P_n - Q_n f) \ ,\end{aligned}$$

if $k \neq n$. □

122 Padé approximants. By Corollary 6.6 the convergents P_n/Q_n to $f \in \mathbb{C}([1/z])$ have a remarkable extremal property: the norm of the linear form $Qf - P$ in polynomials P and Q attains its minimal value exactly for $Q = Q_n$ and $P = P_n$ provided that $\deg Q \leqslant \deg Q_n$.

Problem 6.7 (Padé problem.) *Given a nonzero $f \in \mathbb{C}([1/z])$ and $n \in \mathbb{Z}_+$ find all polynomials P and $Q \neq 0$ such that $\deg Q \leqslant n$ and*

$$\deg(Qf - P) \leqslant -n - 1 \ .$$

Any pair (P, Q) satisfying the Padé problem is called an *n-Padé pair*. Let $\{P_k/Q_k\}_{k\geqslant 0}$ be the sequence of convergents to f. We put $s_k = \deg Q_k$. Then

$$0 = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} < \cdots \to +\infty \ .$$

Suppose that $s_k \leqslant n < s_{k+1}$. Then $\deg(Q_k f - P_k) = -s_{k+1} \leqslant -(n+1)$, since $n < s_{k+1}$. It follows that (P_k, Q_k) is an n-Padé pair, which implies that the Padé problem always has a solution.

Lemma 6.8 *If (P, Q) and (P', Q') are two n-Padé pairs then*

$$\frac{P}{Q} = \frac{P'}{Q'} = \pi_n(f) \ .$$

Proof Using the properties of the norm, we obtain

$$\begin{aligned}\|QP' - Q'P\| &= \|Q'(Qf - P) + Q(P' - Q'f)\| \\ &\leqslant \max\left(\|Q'(Qf - P)\|,\ \|Q(P' - Q'f)\|\right) \\ &= \max\left(\|Q'\|\,\|Qf - P\|,\ \|Q\|\,\|P' - Q'f\|\right) \\ &\leqslant e^n e^{-(n+1)} = 1/e < 1 \ .\end{aligned}$$

Hence $QP' - Q'P = cz^{-1} + \cdots$, which is only possible if $QP' = Q'P$. □

If (P_k, Q_k) is an n-Padé pair then

$$\pi_n(f) = \frac{P_k}{Q_k} \ , \qquad s_k \leqslant n < s_{k+1} \ . \tag{6.22}$$

Since P_k and Q_k are relatively prime polynomials, (6.22) is the representation of $\pi_n(f)$ in its lowest terms.

Applying Lemma 6.8 to an arbitrary n-Padé pair (P, Q) and to (P_k, Q_k), we conclude that $PQ_k = QP_k$. Since P_k and Q_k are relatively prime,

$$Q = qQ_k\,, \quad P = qP_k\,, \qquad q \in \mathbb{C}[z]\,. \tag{6.23}$$

It follows that

$$\deg(Qf - P) = \deg q + \deg(Q_k f - P_k) = \deg q - s_{k+1}\,. \tag{6.24}$$

Corollary 6.9 *All n-Padé pairs* (P, Q) *for* $f \in \mathbb{C}([1/z])$ *are described by* (6.23), *where q is an arbitrary nonzero polynomial such that*

$$0 \leqslant \deg q \leqslant \min(n - s_k,\ s_{k+1} - (n+1))\,.$$

Proof By the conditions of the Padé problem, $\deg Q = \deg q + s_k \leqslant n$ and $q - s_{k+1} \leqslant -(n+1)$; see (6.24). □

123 Jacobi formulas. Although Markoff's theorem gives an algorithm (the algorithm of P-fractions) for the best rational approximations to a given Laurent series f, it is desirable to find explicit formulas for convergents in terms of the Laurent coefficients $\{c_k\}$ of f. For every n, $n = 1, 2, \ldots$ we consider the Hankel matrix

$$H_n(f) \stackrel{\text{def}}{=} \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_2 & c_3 & \cdots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n-1} \end{pmatrix}. \tag{6.25}$$

and an associated polynomial

$$J_n(z) = \det \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n} \\ 1 & z & z^2 & \cdots & z^n \end{pmatrix}. \tag{6.26}$$

Definition 6.10 *Given* $f \in \mathbb{C}([1/z])$, *an index* $n \in \mathbb{N}$ *is called normal if* $n = \deg Q$ *for a convergent* P/Q *of the P-fraction of* f.

Theorem 6.11 (Jacobi 1846) *For* $f \in \mathbb{C}([1/z])$ *an index n is normal if and only if* $\det H_n(f) \neq 0$. *If n is a normal index then there are a convergent* P/Q *with* $\deg Q = n$ *and a nonzero constant* λ *such that*

$$Q(z) = \lambda J_n(z)\,, \qquad P(z) = \lambda [\![J_n(z) f(z)]\!]\,.$$

Proof By Markoff's theorem 6.5 a nonzero polynomial

$$Q(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is the denominator of a convergent for f if and only if

$$\deg \operatorname{Frac} Qf \leqslant -(n+1) . \tag{6.27}$$

Since

$$f(z)Q(z) = \sum_k \frac{a_0 c_k}{z^k} + \sum_k \frac{a_1 c_k}{z^{k-1}} + \cdots + \sum_k \frac{a_n c_k}{z^{k-n}} = \sum_k \frac{1}{z^k} \sum_{j=0}^{n} a_j c_{k+j} ,$$

(6.27) says that such a polynomial exists if and only if the system

$$\begin{array}{ccccccccc} a_0 c_1 & + & a_1 c_2 & + & \ldots & + & a_{n-1} c_n & = & -a_n c_{n+1} \\ a_0 c_2 & + & a_1 c_3 & + & \ldots & + & a_{n-1} c_{n+1} & = & -a_n c_{n+2} \\ \vdots & & \vdots & & & \vdots & \vdots & \vdots & \vdots \\ a_0 c_n & + & a_1 c_{n+1} & + & \ldots & + & a_{n-1} c_{2n-1} & = & -a_n c_{2n} \end{array} \tag{6.28}$$

has a nonzero solution. If n is not a normal index then $s_k < n < s_{k+1}$ for some k. Since (P_k, Q_k) is an n-Padé approximant with $\deg Q_k < n$, (6.28) has a nonzero solution with $a_n = 0$, implying that $\det H_n(f) = 0$.

If $\det H_n(f) \neq 0$ then by Cramer's theorem there always exists a solution $(a_0, a_1, \ldots, a_n)$ to (6.28) with $a_n \neq 0$. Any other nonzero solution is proportional to this. Therefore the space of n-Padé pairs is one-dimensional and by Corollary 6.9 $n = s_k$, i.e. n is a normal index.

For a normal n there is an explicit formula for Q_k. The cofactor expansion of $J_n(z)$, see (6.26), with respect to the last row is as follows:

$$J_n(z) = (-1)^n \det C_{n+1\,1} + \cdots + z^n (-1)^{2n} \det C_{n+1\,n+1} .$$

Using the cofactor expansion in the reverse direction, we obtain that

$$(-1)^n c_j \det C_{n+1\,1} + \cdots + (-1)^{2n} c_{j+n} \det C_{n+1\,n+1} = 0 ,$$

for $j = 1, 2, \ldots, n$, since the determinant in this case has two equal rows. This shows that the coefficients of the polynomial in (6.26) satisfy the system (6.28). Hence $Q_n(z)$ is proportional to the monic polynomial $J_n(z)(\det H_n(f))^{-1}$, i.e. the polynomial with leading monomial z^n. The polynomial P_n is proportional with the same coefficient to the integer part $[f(z) \det C_n(z)/\det H_n(f)]$. □

124 Kronecker's theorem. It is well known that $\xi \in \mathbb{R}$ belongs to $\mathbb{Q}$ if and only if the decimal expansion for ξ is periodic from some point onwards. In contrast with $\mathbb{R}$ the field of a formal Laurent series does not have this property. It can be easily seen that Laurent series with coefficients that are periodic from some point are the Laurent series of rational functions $z^{-n}P(z) + Q(z)(z^g - 1)^{-1}$, where $g \in \mathbb{N}$, $P, Q \in \mathbb{C}[z]$ and

$\deg Q < g$. The Laurent series of rational functions are described by the following theorem by Kronecker, which shows that the notion of periodicity must be modified in the function-field case.

Theorem 6.12 (Kronecker) *A Laurent series*

$$f(z) \sim b_0(z) + \sum_{k=1}^{\infty} \frac{c_k}{z^k} \tag{6.29}$$

is the Laurent series of a rational function if and only if there exists $N \in \mathbb{N}$ such that the Hankel determinants $\det H_n(f)$ are equal to zero for every $n \geqslant N$.

Proof A formal Laurent series f represents an irrational function if and only if the corresponding P-fraction is infinite. In turn, by Jacobi's theorem this happens if and only if $H_n(f) \neq 0$ infinitely often. □

6.3 Quadratic irrationals

125 Euler's substitutions. In spite of some similarity in the behavior of regular continued fractions and P-fractions, things are more complicated in $\mathbb{C}\,([1/z])$. Let us consider for instance either branch of the square root

$$\sqrt{az^2+bz+c}\,, \qquad a, b, c \in \mathbb{C}\,, \tag{6.30}$$

at $z = \infty$. We assume that the roots of the quadratic polynomial az^2+bz+c are different, i.e. $b^2-4ac \neq 0$. For large $R > 0$ both roots of az^2+bz+c lie in $\{z \in \mathbb{C} : |z| < R\}$. Therefore the argument of this polynomial increases by 4π when z makes one counterclockwise rotation round the origin along $\{z \in \mathbb{C} : |z| = R\}$. Hence both branches of (6.30) are single-valued in $\{z \in \mathbb{C} : |z| > R\}$. It follows that (6.30) can be developed into a convergent Laurent series which by Theorem 6.2 corresponds to the continued fraction (6.4).

Euler observed that an attempt to find $b_0(z)$ and $b_1(z)$ for (6.30) leads to an interesting conclusion. Namely, the identity

$$\begin{aligned}\sqrt{az^2+bz+c}+\sqrt{a}\left(z+\frac{b}{2a}\right)&\\ &= 2\sqrt{a}\left(z+\frac{b}{2a}\right)+\frac{c-b^2/4a}{\sqrt{a}\,(z+b/2a)+\sqrt{az^2+bz+c}}\,,\end{aligned} \tag{6.31}$$

which immediately develops (6.30) into the periodic continued fraction

$$\sqrt{az^2+bz+c} = \sqrt{a}\left(z+\frac{b}{2a}\right)+\mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{c-b^2/4a}{2\sqrt{a}\,(z+b/2a)}\right)\,, \tag{6.32}$$

shows that if we denote by v the denominator in (6.31), i.e. x_1/x_2, then

$$v = 2\sqrt{a}\left(z + \frac{b}{2a}\right) + \frac{c - b^2/4a}{v}\ .$$

It follows that

$$\begin{aligned} z &= \frac{1}{2\sqrt{a}}\left(v - \frac{c - b^2/4a}{v}\right) - \frac{b}{2a}\,, \\ \sqrt{az^2+bz+c} &= \frac{1}{2}\left(v + \frac{c - b^2/4a}{v}\right) \end{aligned} \tag{6.33}$$

are rational functions in v. Hence, we obtain Euler's classical theorem.

Theorem 6.13 (Euler) *For any rational function $R(z,w)$ in two complex variables, the primitive*

$$\int R\left(z, \sqrt{az^2+bz+c}\right)\,dz$$

can be expressed in elementary functions.

These formulas look especially attractive for $az^2+bz+c = z^2-1$, i.e. for $a=1$, $b=0$, $c=-1$. We then obtain the continued fraction

$$\sqrt{z^2-1} = z - \frac{1}{2z}\,\underset{-}{}\,\frac{1}{2z}\,\underset{-}{}\,\frac{1}{2z}\,\underset{-}{}\cdots\ . \tag{6.34}$$

In this case the Euler substitutions are given by

$$z = \frac{1}{2}\left(v - \frac{1}{v}\right)\,, \qquad dz = \frac{1}{2}\left(1 + \frac{1}{v^2}\right)dv$$

$$\sqrt{z^2-1} = \frac{1}{2}\left(v - \frac{1}{v}\right)\ .$$

Notice that the continued fraction in (6.32) is a P-fraction if and only if $4ac - b^2 = 4a$. But in fact it can be easily transformed into a periodic P-fraction. For instance, for the continued fraction (6.34) we have

$$\sqrt{z^2-1} = z - \frac{1}{2z}\,\underset{-}{}\,\frac{1}{2z}\,\underset{-}{}\,\frac{1}{2z}\,\underset{-}{}\cdots = z + \frac{1}{-2z}\,\underset{+}{}\,\frac{1}{2z}\,\underset{+}{}\,\frac{1}{-2z}\,\underset{+}{}\,\frac{1}{2z}\,\underset{+}{}\cdots\ .$$

126 Euler's algorithm for square surds. In §§**35–6** in Section 2.1 we studied Euler's algorithm for quadratic surds of integer and rational numbers. Here we consider it first for quadratic surds of polynomials $\mathbf{D}$ of even degree. In order that $\sqrt{\mathbf{D}} \in \mathbb{C}([1/z])$ it is necessary and sufficient that $\deg \mathbf{D}$ is even. If $\deg \mathbf{D}$ is odd then $\sqrt{\mathbf{D}}$ is an element of an algebraic extension of $\mathbb{C}([1/z])$. We assume that the coefficients of $\mathbf{D}$ belong to a number field K, $\mathbb{Q} \subset K \subset \mathbb{C}$.

Lemma 6.14 (Abel 1826) *Let $\mathbf{D} \in K[z]$, $\deg \mathbf{D} = 2g+2$. Then $\mathbf{S} = \pm[\![\sqrt{\mathbf{D}}\,]\!]$ are the only solutions to the equation*

$$\mathbf{D} = \mathbf{S}^2 + \mathbf{T} \tag{6.35}$$

in $K[z]$, where $\deg \mathbf{S} = g+1$ and $\deg \mathbf{T} \leqslant g$.

Proof If $\mathbf{S}$ and $\mathbf{T}$ satisfy (6.35) then the binomial theorem

$$\sqrt{\mathbf{D}} = \pm\mathbf{S}\left(1+\frac{\mathbf{T}}{\mathbf{S}^2}\right)^{1/2} = \pm\mathbf{S}\pm\frac{\mathbf{T}}{2\mathbf{S}}+\cdots = \pm\mathbf{S}+\xi\,, \quad \deg\xi\leqslant -1\,,$$

implies that $\mathbf{S} = \pm[\![\sqrt{\mathbf{D}}\,]\!] \in K[z]$.

Since $\deg\mathbf{D} = 2g+2$, $\sqrt{\mathbf{D}} \in \mathbb{C}([1/z])$. Hence $\sqrt{\mathbf{D}} = \mathbf{S}+\xi$, where $\mathbf{S} = [\![\sqrt{\mathbf{D}}]\!]$, $\deg\mathbf{S} = g+1$ and $\deg\xi \leqslant -1$. It follows that

$$\mathbf{D} = \mathbf{S}^2+2\mathbf{S}\xi+\xi^2\,.$$

Since $\deg\xi\leqslant -1$, $\deg\mathbf{T} = \deg(2\mathbf{S}\xi+\xi^2)\leqslant g$, proving the lemma. $\square$

Abel's lemma 6.14 shows that Euler's algorithm from §**35** in Section 2.1 can be run for $\mathbf{D}$. This algorithm determines three sequences of polynomials $\{\mathbf{r}_n\}_{n\geqslant 0}$, $\{\mathbf{b}_n\}_{n\geqslant 0}$, $\{\mathbf{s}_n\}_{n\geqslant 0}$:

$$\mathbf{r}_{n+1} = \mathbf{b}_n\mathbf{s}_n-\mathbf{r}_n\,, \qquad \mathbf{s}_n\mathbf{s}_{n+1} = \mathbf{D}-\mathbf{r}_{n+1}^2\,, \tag{6.36}$$

$$\mathbf{b}_n = \left[\!\!\left[\frac{\sqrt{\mathbf{D}}+\mathbf{r}_n}{\mathbf{s}_n}\right]\!\!\right] = \left[\!\!\left[\frac{\mathbf{b}_0+\mathbf{r}_n}{\mathbf{s}_n}\right]\!\!\right]$$

with initial conditions

$$\mathbf{r}_0 = \mathbf{0}\,, \qquad \mathbf{s}_0 = \mathbf{1}\,, \qquad \mathbf{b}_0 = \left[\!\!\left[\frac{\sqrt{\mathbf{D}}+\mathbf{r}_0}{\mathbf{s}_0}\right]\!\!\right] = [\![\sqrt{\mathbf{D}}]\!]\,. \tag{6.37}$$

Euler's algorithm for polynomials runs not only for the initial conditions (6.37) but also for $\mathbf{r}_0$, $\mathbf{s}_0$ such that $\mathbf{s}_0$ divides $\mathbf{D}-\mathbf{r}_0^2$. Indeed, in this case the first formula in (6.36) combined with the second implies that

$$\begin{aligned}\frac{\sqrt{D}+\mathbf{r}_n}{\mathbf{s}_n} &= \mathbf{b}_n+\frac{\sqrt{D}-\mathbf{r}_{n+1}}{\mathbf{s}_n} = \mathbf{b}_n+\frac{\mathbf{s}_n\mathbf{s}_{n+1}}{\mathbf{s}_n(\sqrt{D}+\mathbf{r}_{n+1})}\\ &= \mathbf{b}_n+\frac{1}{(\sqrt{D}+\mathbf{r}_{n+1})/\mathbf{s}_{n+1}}\,.\end{aligned}$$

The third formula in (6.36) implies that we have obtained the polynomial continued fraction for $(\sqrt{\mathbf{D}}+\mathbf{r}_0)/\mathbf{s}_0$.

Let us demonstrate this algorithm for $\mathbf{D} = z^4+4z^3+2z^2+1$. By the binomial theorem,

$$\sqrt{\mathbf{D}} = z^2\left(1+\frac{4}{z}+\frac{2}{z^2}+\frac{1}{z^4}\right)^{1/2} = z^2\left(1+\frac{2}{z}-\frac{1}{z^2}+O\left(\frac{1}{z^3}\right)\right)\,,$$

which implies that $\mathbf{b}_0 = z^2+2z-1$. Applying Euler's algorithm, we obtain

$$\begin{aligned}
&\mathbf{r}_1 = z^2+2z-1\,, && \mathbf{s}_1 = 4z\,, && \mathbf{b}_1 = \tfrac{1}{2}z+1\,;\\
&\mathbf{r}_2 = z^2+2z+1\,, && \mathbf{s}_2 = -(z+1)\,, && \mathbf{b}_2 = -2(z+1)\,;\\
&\mathbf{r}_3 = z^2+2z+1\,, && \mathbf{s}_3 = 4z\,, && \mathbf{b}_3 = \tfrac{1}{2}z+1\,;\\
&\mathbf{r}_4 = z^2+2z-1\,, && \mathbf{s}_4 = 1\,, && \mathbf{b}_4 = 2\mathbf{b}_0\,.
\end{aligned}$$

For $n = 5, 6, \ldots$ the rows are periodically repeated with period 4.

Corollary 6.15 *Let K be a number field. If $\mathbf{D} \in K[z]$, $\deg \mathbf{D} = 2g+2$ and $\mathbf{r}_0$, $\mathbf{s}_0$ are polynomials in $K[z]$ then the parameters $\{\mathbf{r}_n\}_{n\geqslant 0}$, $\{\mathbf{b}_n\}_{n\geqslant 0}$, $\{\mathbf{s}_n\}_{n\geqslant 0}$ of Euler's algorithm are polynomials in $K[z]$.*

Proof Apply Abel's lemma 6.14 and Euler's algorithm (6.36). □

A particularly important case of Euler's algorithm is $\sqrt{\mathbf{R}}$, where $\mathbf{R}$ is a rational function in $K(z)$ of positive even degree $2g+2$. Then $\mathbf{R} = \mathbf{p}/\mathbf{q}$, where $\mathbf{p}, \mathbf{q} \in K[z]$ with no common divisor. If $\mathbf{D} = \mathbf{pq}$, then $\sqrt{\mathbf{R}} = \sqrt{\mathbf{D}}/\mathbf{q}$, so that if $\mathbf{s}_0 = \mathbf{q}$ and $\mathbf{r}_0 = 0$ then $\mathbf{s}_0$ divides $\mathbf{D} - \mathbf{r}_0^2$ and Euler's algorithm may be applied. The degrees of the polynomials $\mathbf{p}$, $\mathbf{q}$ satisfy the system

$$\begin{aligned}
\deg \mathbf{p} - \deg \mathbf{q} &= 2g+2\,,\\
\deg \mathbf{p} + \deg \mathbf{q} &= 2e+2\,,
\end{aligned}$$

which shows that $\deg \mathbf{p} = e+g+2$ and $\deg \mathbf{q} = e-g$. The recurrence relations (6.36) for $\{\mathbf{b}_n\}$, $\{\mathbf{r}_n\}$, $\{\mathbf{s}_n\}$ can be also presented in the form

$$\mathbf{r}_n - \mathbf{r}_{n+1} = \frac{\mathbf{s}_{n+1} - \mathbf{s}_{n-1}}{\mathbf{b}_n}\,, \qquad \mathbf{r}_n + \mathbf{r}_{n+1} = \mathbf{b}_n \mathbf{s}_n\,. \tag{6.38}$$

Lemma 6.16 *For any $\sqrt{\mathbf{R}}$, where $\mathbf{R} \in K(z)$, $\deg \mathbf{R} = 2g+2 > 0$, the parameters of Euler's algorithm are in $K[z]$ and satisfy*

$$\deg \mathbf{r}_n = e+1\,, \qquad \deg \mathbf{s}_n \leqslant e\,, \qquad \deg \mathbf{b}_n \leqslant e+1\,,$$
$$\deg \mathbf{b}_n + \deg \mathbf{s}_n = e+1\,.$$

The two leading coefficients of $\mathbf{r}_n$ for $n > 0$ coincide with those of $\mathbf{S}$. If $\deg \mathbf{s}_n = 0$ then $\mathbf{r}_n = \mathbf{r}_{n+1} = \mathbf{S}$.

Proof Let $\mathbf{S} = \pm[\![\sqrt{\mathbf{D}}\,]\!]$, where $\mathbf{D} = \mathbf{pq}$. By Abel's lemma, $\mathbf{S} \in K[z]$ and $\deg \mathbf{S} = e+1$. Next, $\deg \mathbf{s}_0 = \deg \mathbf{q} = e-g < e$. By (6.36),

$$\left[\!\!\left[\frac{\sqrt{\mathbf{D}}+\mathbf{r}_n}{\mathbf{s}_n}\right]\!\!\right] = \left[\!\!\left[\frac{\mathbf{S}+\mathbf{r}_n}{\mathbf{s}_n}\right]\!\!\right] = \mathbf{b}_n = \frac{\mathbf{S}+\mathbf{r}_n}{\mathbf{s}_n} + \frac{\mathbf{r}_{n+1}-\mathbf{S}}{\mathbf{s}_n}\,,$$

implying that

$$\alpha_n \stackrel{\text{def}}{=} \operatorname{Frac}\left(\frac{\mathbf{S}+\mathbf{r}_n}{\mathbf{s}_n}\right) = \frac{\mathbf{S}-\mathbf{r}_{n+1}}{\mathbf{s}_n} . \tag{6.39}$$

Now since $\deg \alpha_n \leqslant -1$ the formula

$$\mathbf{r}_{n+1} = \mathbf{S} - \alpha_n \mathbf{s}_n \tag{6.40}$$

shows that $\deg \mathbf{r}_{n+1} = e+1$ if $\deg \mathbf{s}_n \leqslant e$. Moreover, it shows that the two leading coefficients in $\mathbf{r}_{n+1}$ are the same as in $\mathbf{S}$, which is clearly seen in the example above. Next, by (6.36) and (6.40),

$$\mathbf{s}_{n+1} = \frac{1}{\mathbf{s}_n}\mathbf{s}_{n+1}\mathbf{s}_n = \frac{\mathbf{D}-\mathbf{S}^2}{\mathbf{s}_n} + 2\alpha_n \mathbf{S} - \alpha_n^2 \mathbf{s}_n .$$

By Abel's lemma 6.14 the degree of the first polynomial in the right-hand expression cannot exceed e. The same holds for the second term and the degree of the third cannot exceed $e-2$. Hence $\deg \mathbf{s}_{n+1} \leqslant e$. Now the second equation in (6.38) shows that

$$\deg \mathbf{b}_n + \deg \mathbf{s}_n = e+1 ,$$

which proves the main part of the lemma. To prove the last statement let us write two equations:

$$\mathbf{s}_n \mathbf{s}_{n-1} = \left(\sqrt{\mathbf{D}} - \mathbf{r}_n\right)\left(\sqrt{\mathbf{D}} + \mathbf{r}_n\right) ,$$

$$\mathbf{s}_{n+1} \mathbf{s}_n = \left(\sqrt{\mathbf{D}} - \mathbf{r}_{n+1}\right)\left(\sqrt{\mathbf{D}} + \mathbf{r}_{n+1}\right) .$$

Since $\mathbf{S}+\mathbf{r}_n = [\![\sqrt{\mathbf{D}}+\mathbf{r}_n]\!]$ and $\mathbf{S}+\mathbf{r}_{n+1} = [\![\sqrt{\mathbf{D}}+\mathbf{r}_{n+1}]\!]$ have degree $e+1$ and $\deg \mathbf{s}_{n\pm 1} \leqslant e$, $\deg \mathbf{s}_n = 0$ implies that $\mathbf{S} = \mathbf{r}_n = \mathbf{r}_{n+1}$. $\square$

127 ***P*-fractions of square surds of rational functions.** Any $\mathbf{R} \in K(z)$, $\deg \mathbf{R} = 2g+2 \geqslant 2$ can be represented in lowest terms as $\mathbf{R} = \mathbf{p}/\mathbf{q}$, so that $\sqrt{\mathbf{R}} = \sqrt{\mathbf{D}}/\mathbf{q}$, where $\mathbf{D} = \mathbf{pq}$. By Theorem 6.2 $\sqrt{\mathbf{R}} \in \mathbb{C}([1/z])$ and $\mathbf{b}_k \in K[z]$, see Corollary 6.15. By Euler's formula (1.17),

$$\mathbf{f}_n = \mathbf{b}_n(z) + \underset{k=n+1}{\overset{\infty}{\mathbf{K}}}\left(\frac{1}{\mathbf{b}_k(z)}\right) = \frac{\sqrt{\mathbf{D}}+\mathbf{r}_n}{\mathbf{s}_n} \tag{6.41}$$

are elements of the quadratic field $K(z, \sqrt{\mathbf{R}})$ and therefore we may consider the sequence $\{\mathbf{f}_n^*\}_{n\geqslant 0}$ of their algebraically conjugate elements.

Clearly $K(z, \sqrt{\mathbf{R}}) = K(z, \sqrt{\mathbf{D}})$. Lemma 2.7 has an analogue in $K(z, \sqrt{\mathbf{R}})$.

Lemma 6.17 *For every $n \geqslant 1$,*

$$\deg \mathbf{f}_n^*(z) < 0 , \qquad \mathbf{b}_n(z) = -[\![(\mathbf{f}_{n+1}^*)^{-1}]\!] . \tag{6.42}$$

If the P-fraction of $\sqrt{\mathbf{R}}$ *is periodic with period d then* $\mathbf{f}_1 = \mathbf{f}_{d+1}$.

Proof We have $\mathbf{f}_0^*(z) = -\sqrt{\mathbf{R}}$, $\mathbf{f}_1^*(z) = -(\sqrt{\mathbf{R}} + \mathbf{b}_0)^{-1}$, which shows that $\deg \mathbf{f}_1^* \leqslant -(2g+2) < 0$, since $\mathbf{b}_0 = [\![\sqrt{\mathbf{R}}]\!]$. Now

$$\mathbf{f}_n^* = \mathbf{b}_n + (\mathbf{f}_{n+1}^*)^{-1} \tag{6.43}$$

shows that if $\deg \mathbf{f}_n^* < 0$ then $\deg \mathbf{f}_{n+1}^* < 0$ as well. Hence $\deg \mathbf{f}_k^* < 0$ for every $k > 0$. Taking the integer parts in (6.43), we obtain (6.42).

Let k be the minimal positive k such that $\mathbf{f}_k = \mathbf{f}_{k+d}$. If $k > 1$ then $\mathbf{b}_{k-1} = \mathbf{b}_{d+k-1}$ by (6.42) and therefore $\mathbf{f}_{k-1} = \mathbf{f}_{k-1+d}$, which contradicts the choice of k. □

The algebraic structure of $\mathbf{f}_n$ is described by the following lemma.

Lemma 6.18 *Let* $\mathbf{R} \in K(z)$, $\sqrt{\mathbf{R}} \notin K(z)$, $\deg \mathbf{R} = 2g+2 \geqslant 0$ *and let* P_k/Q_k *be the convergents for the continued P-fraction of* $\sqrt{\mathbf{R}}$. *Then*

$$\mathbf{f}_{n+1} = \frac{(Q_n Q_{n-1}\mathbf{R} - P_n P_{n-1}) + (-1)^{n-1}\sqrt{\mathbf{R}}}{P_n^2 - Q_n^2 \mathbf{R}} \,. \tag{6.44}$$

Proof See (2.23). □

Corollary 6.19 *The parameters of Euler's algorithm for* $\mathbf{R}$ *and convergents to* $\mathbf{R}$ *are related by the formulas*

$$\mathbf{r}_n = (-1)^n \mathbf{q}(Q_{n-1}Q_{n-2}\mathbf{R} - P_{n-1}P_{n-2}) \,,$$
$$\mathbf{s}_n = (-1)^n \mathbf{q}(P_{n-1}^2 - Q_{n-1}^2\mathbf{R}) \,.$$

Proof Compare (6.41) with (6.44). □

Theorem 2.8 also has an analogue in function fields.

Theorem 6.20 *Let* $\mathbf{R} \in K(z)$ *be a rational function such that* $\sqrt{\mathbf{R}} \notin K(z)$ *and* $\deg \mathbf{R} = 2g+2$, $g \geqslant 0$. *If the P-fraction* $\mathbf{b}_0 + \underset{k\geqslant 1}{\mathbf{K}} (1/\mathbf{b}_k)$ *of* $\sqrt{\mathbf{R}}$ *has period d then*

$$\mathbf{b}_d = 2\mathbf{b}_0 \,, \qquad \{\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_{d-1}\} = \{\mathbf{b}_{d-1}, \mathbf{b}_{d-2}, \ldots \mathbf{b}_1\} \,. \tag{6.45}$$

Conversely, if a P-fraction satisfies (6.45) *with* $\mathbf{b}_k \in K(z)$ *then it is the P-fraction of* $\mathbf{R}$, $\sqrt{\mathbf{R}} \notin K(z)$, $\deg \mathbf{R} = 2g+2$.

Proof The proof follows the proof of Theorem 2.8. The computation in (2.15) should be replaced by

$$\deg\left(\frac{\mathbf{b}_0\mathbf{P}_{d-1} + \mathbf{P}_{d-2}}{\mathbf{Q}_{d-1}}\right) = \max\left(\mathbf{b}_0\,\frac{\mathbf{P}_{d-1}}{\mathbf{Q}_{d-1}},\ \frac{\mathbf{P}_{d-2}}{\mathbf{Q}_{d-1}}\right) = 2\deg \mathbf{b}_0 \,.$$

□

Lemma 6.21 *If* $\mathbf{R} \in K(z)$, $\sqrt{\mathbf{R}} \notin K(z)$, $\deg \mathbf{R} = 2g+2 \geqslant 0$ *and the P-fraction of* $\sqrt{\mathbf{R}}$ *is periodic with period* d *then* $\mathbf{b}_d = 2\mathbf{b}_0$, $\mathbf{s}_d = \mathbf{q}$, $\mathbf{r}_d = \mathbf{q}\mathbf{b}_0$, *and the convergent* P_{d-1}/Q_{d-1} *satisfies the Pell equation*

$$P_{d-1}^2 - Q_{d-1}^2 \mathbf{R} = (-1)^d .$$

Proof By Lemma 6.17 $\mathbf{f}_1 = \mathbf{f}_{d+1}$. By (6.45) $\mathbf{b}_d = 2\mathbf{b}_0$. Hence

$$\sqrt{\mathbf{R}} = \mathbf{b}_0 + \frac{1}{\mathbf{f}_1} = \mathbf{b}_0 + \frac{1}{\mathbf{f}_{d+1}} = \mathbf{b}_0 - \mathbf{b}_d + \mathbf{b}_d + \frac{1}{\mathbf{f}_{d+1}} = -\mathbf{b}_0 + \mathbf{f}_d .$$

It follows that

$$\frac{\sqrt{\mathbf{D}}}{\mathbf{q}} = -\mathbf{b}_0 + \frac{\sqrt{\mathbf{D}} + \mathbf{r}_d}{\mathbf{s}_d} .$$

Hence $\mathbf{s}_d = \mathbf{q}$, $\mathbf{r}_d = \mathbf{q}\mathbf{b}_0$. Applying (6.44) with $n = d-1$, we deduce that

$$\begin{aligned} P_{d-1}^2 - Q_{d-1}^2 \mathbf{R} &= (-1)^d , \\ Q_{d-1} Q_{d-2} \mathbf{R} - P_{d-1} P_{d-2} &= (-1)^d \mathbf{b}_0 . \end{aligned} \tag{6.46}$$

□

Finally, the following analogue of Euler's theorem 2.10 is true.

Theorem 6.22 *Let* $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{d-1}\}$ *be a symmetric sequence of polynomials in* $\mathbb{C}[z]$. *Let* $\mathfrak{P}_k/\mathfrak{Q}_k$ *be the convergents to*

$$\mathbf{b}_1 + \frac{1}{\mathbf{b}_2} {}_{+\cdots+} \frac{1}{\mathbf{b}_{d-1}} .$$

Then the square of

$$\sqrt{\mathbf{R}} = \mathbf{b}_0 + \frac{1}{\mathbf{b}_1} {}_{+\cdots+} \frac{1}{\mathbf{b}_1} {}_{+} \frac{1}{2\mathbf{b}_0} {}_{+} \frac{1}{\mathbf{b}_1} {}_{+\cdots}$$

is a polynomial of an even degree if and only if there is a polynomial $\mathbf{m}$ *such that*

$$2\mathbf{b}_0 = \mathbf{m}\mathfrak{P}_{d-2} - (-1)^d \mathfrak{Q}_{d-3}\mathfrak{P}_{d-3} ,$$

$$\mathbf{R} = \mathbf{b}_0^2 + \mathbf{m}\mathfrak{P}_{d-3} - (-1)^d \mathfrak{Q}_{d-3}^2$$

and

$$\deg \mathbf{b}_0 \geqslant 1 + \deg(\mathbf{m}\mathfrak{P}_{d-3} - (-1)^d \mathfrak{Q}_{d-3}^2) . \tag{6.47}$$

It is easy to see that (6.47) is satisfied if deg $\mathbf{m}$ is sufficiently large.

Let us consider as an example $d = 3$. Then $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b}$, $\mathfrak{P}_1 = \mathbf{b}^2 + 1$, $\mathfrak{P}_0 = \mathbf{b}$, $\mathfrak{Q}_0 = 1$. It follows that $2\mathbf{b}_0 = \mathbf{m}(\mathbf{b}^2 + 1) + \mathbf{b}$,

$$\mathbf{R} = \mathbf{b}_0^2 + \frac{2\mathbf{b}_0\mathfrak{Q}_1 + \mathfrak{Q}_0}{\mathfrak{P}_1} = \mathbf{b}_0^2 + \mathbf{m}\mathbf{b} + 1 .$$

It is clear that (6.47) is valid even if $\mathbf{m} = 2$ provided that $\deg \mathbf{b} > 0$. In this case $\mathbf{b}_0 = \mathbf{b}^2 + \mathbf{b}/2 + 1$ and $\mathbf{R} = (\mathbf{b}^2 + \mathbf{b}/2 + 1)^2 + 2\mathbf{b} + 1$. Hence

$$\sqrt{\mathbf{b}^4 + \mathbf{b}^3 + \frac{9}{4}\mathbf{b}^2 + 3\mathbf{b} + 2} = \mathbf{b}^2 + \frac{1}{2}\mathbf{b} + 1 + \frac{1}{\mathbf{b}} + \frac{1}{\mathbf{b}} + \frac{1}{2\mathbf{b}^2 + \mathbf{b} + 2} + \frac{1}{\mathbf{b}} + \frac{1}{\mathbf{b}} + \cdots .$$

128 Abel's theorem. By Lagrange's theorem 2.19 any quadratic irrational in $\mathbb{R}$ is the value of a periodic continued fraction. Since $\mathbb{R}$ and $\mathbb{C}([1/z])$ are described by similar types of continued fraction, one may expect that Lagrange's theorem is valid in $\mathbb{C}([1/z])$ too. Euler's theorem 6.13 as well as §**127** support this conjecture.

For number fields the validity of Lagrange's theorem is rooted in the fact that any Pell equation has a nontrivial solution in the set of convergents to $\sqrt{D}$. This can be easily seen from formula (2.23).

The following theorem by Abel shows that there is a similar relation between functional Pell equations and P-fractions.

Theorem 6.23 (Abel 1826) *Let $\mathbf{R}$ be a rational function in $\mathbb{C}(z)$ of degree $2g+2 \geqslant 2$, which is not a square in $\mathbb{C}(z)$. Then the Pell equation*

$$P^2 - Q^2\mathbf{R} = c\,,^{1} \tag{6.48}$$

where c is a constant, has a solution in the polynomials P and Q with $Q \neq 0$ if and only if $\sqrt{\mathbf{R}}$ can be developed into a periodic continued P-fraction.

First we relate the solutions to (22) with the convergents to $\sqrt{\mathbf{R}}$. If $\mathbf{R} = \mathbf{p}/\mathbf{q}$ is the representation of $\mathbf{R}$ in the lowest terms then

$$Q^2\mathbf{R} = \frac{Q^2\mathbf{p}}{\mathbf{q}} = P^2 + c \in \mathbb{C}(z)\,,$$

implying that $Q = \mathbf{q}Q_1$. Substituting this into (22), we obtain

$$P^2 - Q_1^2\mathbf{D} = c\,. \tag{6.49}$$

It follows that the mapping $(P, Q) \leftrightarrow (P, Q_1)$ establishes one-to-one correspondence between the solutions to (22) and to (6.49).

Lemma 6.24 *Any solution (P, Q) to (22) with $Q \neq 0$ determines a convergent P/Q to $\sqrt{\mathbf{R}}$.*

Proof First we observe that P/Q is a fraction in its lowest terms. Indeed by (6.49) P and Q_1 are co-prime. Since $\mathbf{D} = \mathbf{pq}$, (6.49) implies that P and $\mathbf{q}$ are co-prime. Since $Q = \mathbf{q}Q_1$, P and Q are co-prime. Next

$$0 \neq \text{constant} = P^2 - Q^2\mathbf{R} = (P - Q\sqrt{\mathbf{R}})(P + Q\sqrt{\mathbf{R}})\,.$$

[1] The constant in (22) cannot be zero since $\sqrt{\mathbf{R}} \notin \mathbb{C}(z)$.

Another corollary of this formula is

$$\deg(P - Q\sqrt{\mathbf{R}}) = -\deg(P + Q\sqrt{\mathbf{R}}) \,. \tag{6.50}$$

By (6.11) the identity $2P = (P - Q\sqrt{\mathbf{R}}) + (P + Q\sqrt{\mathbf{R}})$ implies that

$$\deg P \leqslant \max\Big(\deg(P - Q\sqrt{\mathbf{R}}),\ \deg(P + Q\sqrt{\mathbf{R}})\Big) \,. \tag{6.51}$$

It follows that $\deg(P - Q\sqrt{\mathbf{R}}) \neq 0$ since otherwise (6.50) and (6.51) would imply that $\deg P = 0$ and therefore $Q = 0$.

Let us fix the branch of $\sqrt{\mathbf{R}}$ such that $\deg(P - Q\sqrt{\mathbf{R}}) < 0$. Then

$$P = [\![Q\sqrt{\mathbf{R}}]\!] \,, \qquad P + Q\sqrt{\mathbf{R}} = 2P + \mathrm{Frac}(Q\sqrt{\mathbf{R}}) \,,$$

and

$$\deg(P + Q\sqrt{\mathbf{R}}) = \deg P = \deg Q + g + 1 \,. \tag{6.52}$$

By (6.50),

$$\begin{aligned}
\deg\left(\frac{P}{Q} - \sqrt{\mathbf{R}}\right) &= \deg(P - Q\sqrt{\mathbf{R}}) - \deg Q \\
&= -\deg(P + Q\sqrt{\mathbf{R}}) - \deg Q \\
&= -2\deg Q - g - 1 \leqslant -2\deg Q - 1 \,.
\end{aligned}$$

The proof of the lemma is completed by Markoff's theorem 6.5. □

Proof of Theorem 6.23 Let P and Q be nontrivial solutions to (22). By Lemma 6.24 any nontrivial solution to (22), if exists, determines the convergents P/Q to $\sqrt{\mathbf{R}}$. Hence $P = tP_n$, $Q = tQ_n$, where $n \in \mathbb{Z}_+$ and $0 \neq t \in \mathbb{C}$. To simplify the notation we assume that $t = 1$. Then (6.44) implies that $c\mathbf{f}_{n+1} = (Q_n Q_{n-1}\mathbf{R} - P_n P_{n-1}) + (-1)^{n-1}\sqrt{\mathbf{R}}$ and therefore $c\mathbf{f}^*_{n+1} = (Q_n Q_{n-1}\mathbf{R} - P_n P_{n-1}) - (-1)^{n-1}\sqrt{\mathbf{R}}$. Putting $\mathbf{b} = Q_n Q_{n-1}\mathbf{R} - P_n P_{n-1}$ for brevity, we obtain by (6.42) that $\mathbf{b} = (-1)^{n-1}\mathbf{b}_0$. Let $\gamma = (-1)^{n-1}c$. Then

$$\mathbf{f}_{n+1} = (\mathbf{b}_0 + \sqrt{\mathbf{R}})\gamma^{-1} \,. \tag{6.53}$$

Lemma 6.25 *If* $\mathbf{f}_{n+1}$ *satisfies* (6.53) *for some constant* γ *then*

$$\mathbf{b}_1 = \gamma^{-1}\mathbf{b}_n \,, \qquad \mathbf{b}_2 = \gamma\mathbf{b}_{n-1} \,, \qquad \ldots, \qquad \mathbf{b}_k = \gamma^{(-1)^k}\mathbf{b}_{n-k+1} \,, \ldots \tag{6.54}$$

Proof By (6.53),

$$\sqrt{\mathbf{R}} = \mathbf{b}_0 + \frac{1}{\mathbf{b}_1(z)} \,{}_{+}\, \frac{1}{\mathbf{b}_2(z)} \,{}_{+\cdots+}\, \frac{1}{\mathbf{b}_n(z)} \,{}_{+}\, \frac{\gamma}{\mathbf{b}_0 + \sqrt{\mathbf{R}}} \,. \tag{6.55}$$

By (6.55) and by the Euler–Wallis formulas,

$$\sqrt{\mathbf{R}} = \frac{(\mathbf{b}_0 + \sqrt{\mathbf{R}})P_n + \gamma P_{n-1}}{(\mathbf{b}_0 + \sqrt{\mathbf{R}})Q_n + \gamma Q_{n-1}} \,.$$

Since $\mathbf{R}$ is not a square in $\mathbb{C}(z)$, this implies that $P_n = \mathbf{b}_0 Q_n + \gamma Q_{n-1}$. From this equation and (1.18) we obtain

$$\mathbf{b}_0 + \frac{1}{\mathbf{b}_1} \, {}_{+} \frac{1}{\mathbf{b}_2} \, {}_{+} \cdots {}_{+} \frac{1}{\mathbf{b}_n} = \frac{P_n}{Q_n} = \mathbf{b}_0 + \gamma \frac{Q_{n-1}}{Q_n}$$
$$= \mathbf{b}_0 + \frac{1}{(\mathbf{b}_n/\gamma)} \, {}_{+} \frac{1}{\gamma \mathbf{b}_{n-1}} \, {}_{+} \cdots .$$

Since P_n/Q_n expands into a unique P-fraction, we get (6.54). □

If $n = 2k+1$ is odd then $k = n-k+1$ and $\gamma = 1$ by (6.54). Hence $c = 1$. By (6.55) and (6.54) the period of the P-fraction for $\sqrt{\mathbf{R}}$ is $d = 2k+2$.

If $n = 2k$ is even and $c = -1$ then $\gamma = 1$ and the P-fraction for $\sqrt{\mathbf{R}}$ is periodic with period $d = 2k+1$ by (6.55). If $c \neq -1$ then $\gamma \neq 1$. By Lemma 6.25 equations (6.54) hold. Since $n = 2k$, we cannot conclude that $\gamma = 1$. However, we can continue the expansion of $\sqrt{\mathbf{R}}$:

$$\sqrt{\mathbf{R}} = \mathbf{b}_0 + \frac{1}{\mathbf{b}_1} \, {}_{+} \frac{1}{\mathbf{b}_2} \, {}_{+} \cdots {}_{+} \frac{1}{\mathbf{b}_n} \, {}_{+} \frac{1}{2\mathbf{b}_0/(-c)} \, {}_{+} \frac{1}{(-c)\mathbf{b}_1} \, {}_{+} \cdots$$
$${}_{+} \frac{1}{\mathbf{b}_n/(-c)} \, {}_{+} \frac{1}{(-c)\mathbf{f}_{n+1}}$$
$$= \mathbf{b}_0 + \frac{1}{\mathbf{b}_1} \, {}_{+} \frac{1}{\mathbf{b}_2} \, {}_{+} \cdots {}_{+} \frac{1}{\mathbf{b}_n} \, {}_{+} \frac{1}{2\mathbf{b}_0/(-c)} \, {}_{+} \frac{1}{(-c)\mathbf{b}_1} \, {}_{+} \cdots {}_{+} \frac{1}{\mathbf{b}_0 + \sqrt{\mathbf{R}}}$$
$$= \mathbf{b}_0 + \frac{1}{\mathbf{b}_1} \, {}_{+} \frac{1}{-c\mathbf{b}_{n-1}} \, {}_{+} \cdots {}_{+} \frac{1}{\mathbf{b}_{n-1}} \, {}_{+} \frac{1}{-c\mathbf{b}_1} \, {}_{+} \frac{1}{2\mathbf{b}_0/-c} \, {}_{+} \frac{1}{(-c)\mathbf{b}_1} \, {}_{+} \cdots$$
$${}_{+} \frac{1}{2\mathbf{b}_0} \, {}_{+} \cdots .$$

Hence the period of the P-fraction of $\sqrt{\mathbf{R}}$ is $d = 4k+2$. The proof in the opposite direction is covered by Lemma 6.21. □

Corollary 6.26 *Let* $\mathbf{R} \in K[z]$, $\deg \mathbf{R} = 2g+2$. *If the Pell equation for* $\mathbf{R}$ *has a nontrivial solution for some constant then there is a convergent* P_k/Q_k *to the P-fraction of* $\sqrt{\mathbf{R}}$ *such that* $P_k^2 - Q_k^2 \mathbf{R} = 1$.

Proof The P-fraction of $\sqrt{\mathbf{R}}$ is periodic by Theorem 6.23. Hence $P_{d-1}^2 - Q_{d-1}^2 \mathbf{R} = (-1)^d$, d being the period, by Lemma 6.21. Since $f_1 = f_{d+1}$, we have $f_1 = f_{2d+1}$. Hence $P_{2d-1}^2 - Q_{2d-1}^2 \mathbf{R} = (-1)^{2d} = 1$. □

Corollary 6.27 *The P-fraction of* $\sqrt{\mathbf{R}}$, $\mathbf{R} \in K[z]$, $\mathbf{R} = \mathbf{p}/\mathbf{q}$, $\deg \mathbf{R} = 2g+2$ *is periodic if and only if* $\mathbf{s}_n = \mathbf{q}$ *for some n.*

Proof By Corollary 6.19 $\mathbf{s}_{n+1} = \mathbf{q} \Leftrightarrow Q_n^2 \mathbf{R} - P_n^2 = (-1)^n$. □

129 Chebyshev's example. Exercise 6.9 shows that not every functional quadratic irrational can be expanded into a periodic continued fraction. Moreover, Chebyshev found examples of $\mathbf{R} \in \mathbb{Z}[z]$ such that $\sqrt{\mathbf{R}}$ expands into a nonperiodic P-fraction.

A point $z \in \mathbb{C}$ is called constructible of order 0 if either $z = 0$ or $z = 1$. A point $z \in \mathbb{C}$ which is not a constructible point of order n is called constructible of order $n+1$ if it is an intersection either of two circles or a circle (centered at constructible points of order $< n+1$ with radius equal to the distance between two constructible points of order $< n+1$) and a line or two lines (passing through two constructible points of order $< n+1$). We denote by $\mathcal{C}$ the set of all constructible points of finite order. Then $\mathbb{Q} \subset \mathcal{C}$ is an algebraic field extension. The constructible field $\mathcal{C}$ is crucial for the the solution of ancient problems of Euclid's geometry such as cube duplication and angle trisection.

A polynomial $\mathbf{p} \in K[X]$ is called *separable* if it does not have multiple roots. The multiple roots of $\mathbf{p}$ are located as the roots of the greatest common divisor of $\mathbf{p}$ and of its derivative $\dot{\mathbf{p}}$ in X. This greatest common divisor is the result of the application of the Euclidean algorithm to $\mathbf{p}$ and $\dot{\mathbf{p}}$. Hence it must be in $K[X]$. It follows that any irreducible polynomial (see the definition in §**47**, Section 2.4) is separable.

Theorem 6.28 (Chebyshev 1857) *Let* $\mathbf{D}$ *be an irreducible polynomial in* $\mathcal{C}[z]$, $\deg \mathbf{D} = 2g+2$. *Then* $\sqrt{\mathbf{D}}$ *expands into a nonperiodic P-fraction.*

Proof Since $\mathbf{D}$ is irreducible, it is separable. Therefore $\sqrt{\mathbf{D}} \notin \mathcal{C}[z]$. By the binomial theorem all coefficients c_k in

$$\sqrt{\mathbf{D}} = \sum_k \frac{c_k}{z^k} = \mathbf{b}_0(z) + \mathop{\mathbf{K}}_{k=1}^{\infty} \left(\frac{1}{\mathbf{b}_k(z)} \right)$$

are in $\mathcal{C}$. Then the algorithm for P-fractions shows that $\mathbf{b}_k(z) \in \mathcal{C}[z]$ for $k \geqslant 0$. By the Euler–Wallis formulas, $P, Q \in \mathcal{C}[z]$ for every convergent P/Q to $\sqrt{\mathbf{D}}$. Suppose now that Pell's equation for $\mathbf{D}$ has a nontrivial solution. By Lemma 6.24 there is a convergent P/Q for $\sqrt{\mathbf{D}}$ such that

$$P^2(z) - Q^2(z)\mathbf{D}(z) = c \tag{6.56}$$

and $\deg Q$ is minimal. Putting $z = 0$ in (6.56), we obtain that $c \in \mathcal{C}$. Since $\sqrt{\mathbf{D}} \notin \mathcal{C}[z]$ we have $c \neq 0$. Rewriting (6.56) as

$$(P - \sqrt{c})(P + \sqrt{c}) = Q^2\mathbf{D} \tag{6.57}$$

and taking into account that $\mathbf{D}$ is separable, we obtain the factorizations

$$P + \sqrt{c} = Q_1^2\mathbf{D}_1\,, \qquad P - \sqrt{c} = Q_2^2\mathbf{D}_2\,, \tag{6.58}$$

where $Q = Q_1Q_2$ and $\mathbf{D} = \mathbf{D}_1\mathbf{D}_2$; Q_1 and Q_2 do not have common roots and neither do $\mathbf{D}_1$ and $\mathbf{D}_2$, do not have common roots. The first factorization implies that Q_1 is the greatest common divisor of $P + \sqrt{c}$ and of the derivative $\dot{P}$, whereas the second implies that Q_2 is the greatest common divisor of $P-\sqrt{c}$ and $\dot{P}$. Then the Euclidean algorithm implies that $Q_1, Q_2 \in \mathcal{C}[z]$ and consequently $Q_1^2, Q_2^2 \in \mathcal{C}[z]$. The long division of polynomials implies that $\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{C}[z]$.

Now if $\mathbf{D}_1 \equiv 1$ then $P+\sqrt{c} = Q_1^2$, $P-\sqrt{c} = Q_2^2\mathbf{D}$. It follows that

$$Q_1^2 - Q_2^2\mathbf{D} = P+\sqrt{c}-(P-\sqrt{c}) = 2\sqrt{c}\,.$$

Observing that $\deg Q_2 < \deg Q$, we obtain a contradiction with the extremal choice of P and Q. Thus our assumption that Pell's equation for $\mathbf{D}$ has a nontrivial solution contradicts the irreducibility of $\mathbf{D}$. By Theorem 6.23 the P-fraction of $\sqrt{\mathbf{D}}$ is not periodic. $\square$

Following Chebychev we consider the following example.

Corollary 6.29 (Chebychev 1857) *The P-fraction of*

$$\sqrt{z^4+2z^2-8z+9} = b_0(z) + \mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{1}{b_k(z)}\right)$$

is not periodic.

Proof We first observe that $\mathbf{p}(z) = z^4+2z^2-8z+9 > 0$ on the real line. Indeed, $\dot{\mathbf{p}} = 4s^3+4s-8 = 4(s-1)(s^2+s+2)$, implying that the minimal value of $\mathbf{p}(s)$ on $(-\infty,+\infty)$ is $\mathbf{p}(1) = 4 > 0$. It follows that the roots of $\mathbf{p}$ are two pairs of complex conjugate numbers. So if one of the roots is in $\mathcal{C}$ then the conjugate root is also in $\mathcal{C}$, and $\mathbf{p}$ is factored as a product of two quadratic polynomials with coefficients in $\mathcal{C}$. Hence if $\mathbf{p}$ is reducible over $\mathcal{C}$ then it is a product of two quadratic polynomials:

$$\begin{aligned} z^4+2z^2-8z+9 &= (z^2+pz+q)(z^2+rz+s) \\ &= z^4+(p+r)z^3+(q+s+rp)z^2+(rq+sp)z+qs \end{aligned}$$

with coefficients in $\mathcal{C}$. Comparing the coefficients, we arrive at the system

$$p+r=0\,, \qquad q+s+rp=2\,, \qquad rq+sp=-8\,, \qquad qs=9\,.$$

Elementary calculations show that p^2 satisfies the cubic equation

$$X(2+X)^2-36X=64 \quad\Leftrightarrow\quad X^3+4X^2-32X-64=0\,.$$

Putting $X=4Y$, we reduce this equation to

$$\mathbf{q}(Y) = Y^3+Y^2-2Y-1=0\,.$$

Since $\mathbf{q}(-1)=1$, $\mathbf{q}(0)=\mathbf{q}(1)=-1$, $\mathbf{q}(2)=7$, this equation has two negative and one positive real roots. Since $\mathbf{q}$ is monic and $\mathbf{q}(0)=-1$, q cannot have rational roots. Now if one of the roots is in $\mathcal{C}$ then the degree of the irreducible polynomial over $\mathbb{Q}$ corresponding to this root must be a power of 2. The only possibility is that it is exactly 2, since this polynomial must divide $q(Y)$. But then the long division of polynomials would imply that $q(Y)$ has a rational root, which is not the case. It follows that $p^2 \notin \mathcal{C}$ and consequently $p \notin \mathcal{C}$. $\square$

Chebyshev's original arguments were as follows. First he observed that the algebraic equation $z^4+2z^2-8z+9=0$ has the resolvent of third degree

$$\theta^3+16\theta^2-64\times 8\theta-64^2=0\,,$$

where $\theta=(x_1-x_2+x_3-x_4)^2$ (see Serret 1879, p. 478). Putting $\theta=16y$, Chebyshev obtained the equation $y^3+y^2-2y-1=0$, which also can be obtained from the equation found on dividing the circle $z^6+z^5+z^4+z^3+z^2+z+1=0$ by the substitution $y=z+1/z$.

130 Integration in finite terms

Theorem 6.30 (Abel 1826) *Let* $\mathbf{R}$ *be a separable polynomial of degree* $2g+2$ *such that the Pell equation* $P^2-Q^2\mathbf{R}=c$, $c\in\mathbb{C}$, *has a solution in polynomials* P *and* Q *with* $Q\neq 0$. *Then there exists a polynomial* r *of degree* g *such that the Abelian integral*

$$\int\frac{r}{\sqrt{\mathbf{R}}}\,dz=\frac{1}{2}\log\frac{P+Q\sqrt{\mathbf{R}}}{P-Q\sqrt{\mathbf{R}}}+C$$

can be expressed in elementary functions.

Proof Elementary calculations show that

$$\frac{d}{dz}\left(\log\frac{P+Q\sqrt{\mathbf{R}}}{P-Q\sqrt{\mathbf{R}}}\right)=\frac{2(\dot{Q}P-\dot{P}Q)\mathbf{R}+PQ\dot{\mathbf{R}}}{(P^2-Q^2\mathbf{R})\sqrt{\mathbf{R}}}\,. \tag{6.59}$$

Differentiating the Pell equation $P^2-Q^2\mathbf{R}=\text{constant}$, we obtain

$$2P\dot{P}=2Q\dot{Q}\mathbf{R}+Q^2\dot{\mathbf{R}}=Q(2\dot{Q}\mathbf{R}+Q\dot{\mathbf{R}})\,. \tag{6.60}$$

Since P and Q are relatively prime, we see that

$$\dot{P}=Qr\,. \tag{6.61}$$

In particular (6.61) implies that $\deg r=d$. Substituting these expressions in (6.59),

$$\frac{d}{dz}\left(\log\frac{P+Q\sqrt{\mathbf{R}}}{P-Q\sqrt{\mathbf{R}}}\right)=\frac{p(2\dot{Q}\mathbf{R}+Q\dot{\mathbf{R}})-2\dot{P}Q\mathbf{R}}{(P^2-Q^2\mathbf{R})\sqrt{\mathbf{R}}}$$
$$=\frac{2P^2\dot{P}-2\dot{P}Q^2\mathbf{R}}{Q(P^2-Q^2\mathbf{R})\sqrt{\mathbf{R}}}=\frac{2\dot{P}/Q}{\sqrt{\mathbf{R}}}=\frac{2r}{\sqrt{\mathbf{R}}}\,,$$

we complete the proof. $\square$

The Pell equation

$$P^2-Q^2(az^2+bz+c)=\text{constant}$$

obviously has a nontrivial solution. Just take $Q=1$ and P^2 be the full square in az^2+bz+c, i.e. $P=\sqrt{a}(z+b/2a)$. Then by (6.61) $r=\dot{P}=\sqrt{a}$ and by Abel's theorem we have

$$\int \frac{\sqrt{a}}{\sqrt{az^2+bz+c}}\,dz = \frac{1}{2}\log\left(\frac{\sqrt{a}(z+b/2a)+\sqrt{az^2+bz+c}}{\sqrt{a}(z+b/2a)-\sqrt{az^2+bz+c}}\right)+C$$
$$= \log\left(\sqrt{a}(z+b/2a)+\sqrt{az^2+bz+c}\right)+C\,.$$

Using Ostrogradskii's formula

$$\int \frac{P_n}{\sqrt{az^2+bz+c}}\,dz = Q_{n-1}\sqrt{az^2+bz+c}+\lambda\int \frac{1}{\sqrt{az^2+bz+c}}\,dz\,,$$

where P_n and Q_{n-1} are polynomials, $\deg P_n = n$, $\deg Q_{n-1} = n-1$ and λ is a constant, we obtain Euler's theorem from Theorem 6.30.

For $g \geqslant 1$ Euler's substitution fails. Moreover we face a completely new phenomenon. The indefinite integrals, for $k = 0, 1, \ldots, g-1$,

$$\int \frac{z^k}{\sqrt{\mathbf{R}}}\,dz \tag{6.62}$$

are multiple-valued holomorphic function on the extended complex plane $\hat{\mathbb{C}}$ with the exception of a finite number $2g+2$ of simple zeros of $\mathbf{R}$. Either branch of (6.62) is uniformly bounded about any point of $\hat{\mathbb{C}}$.

Traditionally, elementary functions include rational functions, algebraic functions, i.e. solutions to polynomial equations with polynomial coefficients, exponential functions and logarithms. By Euler's formula, from which we obtain

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}\,, \qquad \cos z = \frac{e^{iz}+e^{-iz}}{2i}\,, \qquad \tan z = \frac{1}{i}\,\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}\,,$$

the trigonometric functions are compositions of rational functions and exponentials. However, the formulas

$$\begin{aligned} \arcsin z &= -i\ln(iz+\sqrt{1-z^2})\,, & \arctan z &= \frac{i}{2}\ln\frac{i+z}{i-z}\,,\\ \arccos z &= -i\ln(z+i\sqrt{1-z^2})\,, & \tanh^{-1} z &= \frac{1}{2}\ln\frac{1+z}{1-z}\,, \end{aligned} \tag{6.63}$$

show that other elementary functions are expressed as compositions of logarithms and algebraic functions.

Compositions of exponential and algebraic functions have at least one essential singular point. Compositions of logarithms and algebraic functions have logarithmic singularities. Since integrals of the type (6.62) do not have these types of singularities, it is likely that they cannot be obtained by compositions of elementary functions.

The integrals (6.62) are called *Abelian integrals of the first kind*.

Definition 6.31 *A multivalued function $F(z_1, z_2, \dots, z_d)$ in complex variables $z_1, z_2, \dots, z_d$ is called algebraic if it satisfies the algebraic equation*

$$a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 = 0, \tag{6.64}$$

where $a_n, a_{n-1}, \dots, a_0$ are polynomials in $z_1, z_2, \dots, z_d$.

In practice the polynomial in (6.64) is of the lowest possible degree, i.e. it is irreducible in the ring of polynomials in X over the field $\mathbb{C}(z_1, z_2, \dots, z_d)$.

If w is an algebraic function in z, then the set $R(z, w)$ of all rational functions in z and w is the field of functions associated with w.

Theorem 6.32 (Liouville) *If for an algebraic $R(z, w)$ we have*

$$\int R(z, w)\, dz = F(z, \log w_1, \log w_2, \dots, \log w_k)\ , \tag{6.65}$$

for some algebraic function F then there are complex constants a_1, $a_2, \dots,$ a_k and algebraic functions w_0, $w_1, \dots$ w_k such that

$$\int u\, dz = w_0 + a_1 \log w_1 + a_2 \log w_2 + \cdots + \log w_k\ . \tag{6.66}$$

A proof of this theorem can be found in Chebotarëv (1948). A more general result can be found in Ritt (1948).

Theorem 6.33 (Abel) *If $\mathbf{R} \in \mathbb{C}[z]$ is separable, with $\deg \mathbf{R} = 2g+2$ and*

$$\int \frac{r}{\sqrt{\mathbf{R}}}\, dz\ ,$$

where $r \in \mathbb{C}[z]$, $\deg r \leqslant g$, is an elementary function then $P^2 - Q^2\mathbf{R} = c$, $c \in \mathbb{C}$, has a solution in polynomials P and Q with $Q \neq 0$.

In particular, for every complex a the indefinite integral

$$\int \frac{a+z}{\sqrt{z^4 + 2z^2 - 8z + 9}}\, dz$$

cannot be expressed in finite terms for any complex a. See Corollary 6.29. Further applications are available in Ptashickii (1888).

6.4 Hypergeometric series

131 Analogues of Markoff's test. A formal Laurent series determines by Theorem 6.2(f) only one corresponding P-fraction. Therefore it is of interest to find these continued fractions at least for special Laurent series. We begin with an analogue of Markoff's test for $\mathbb{C}\,([1/z])$.

Theorem 6.34 *If $f \in \mathbb{C}([1/z])$ for infinitely many n is represented as*

$$f = \mathbf{b}_0 + \frac{1}{\mathbf{b}_1} + \frac{1}{\mathbf{b}_2} + \cdots + \frac{1}{\mathbf{b}_n} + \frac{1}{g_n}, \tag{6.67}$$

where $\mathbf{b}_j \in \mathbb{C}[z]$, $\deg \mathbf{b}_j \geqslant 0$, $\deg \mathbf{b}_n \geqslant 1$ and $\deg g_n \geqslant 0$, then the continued fraction $\mathbf{b}_0 + \mathbf{K}_{k\geqslant 1}(1/b_k(z))$ converges to f in $\mathbb{C}([1/z])$.

Proof If $\deg \mathbf{b}_j \geqslant 1$ infinitely often then $\deg Q_n \to +\infty$. By (1.17),

$$f - \frac{P_n}{Q_n} = \frac{(-1)^n}{Q_n^2\big(g_n + Q_{n-1}/Q_n\big)}.$$

Since $\deg g_n \geqslant 0$, $\deg(Q_{n-1}/Q_n) \leqslant -1$ (see (1.18)), and $\deg Q_n \to +\infty$ this proves the theorem. □

Remark As in the case of $\mathbb{R}$ the condition $\deg g_n \geqslant 0$ is essential. Indeed, let $\{\mathbf{b}_k\}_{k\geqslant 1}$ be any sequence in $\mathbb{C}([z])$ with $\deg \mathbf{b}_k \geqslant 1$ and $\mathbf{b}_1 = z$. Then $\{\mathbf{b}_k\}_{k\geqslant 1}$ determines the P-fraction in $\mathbb{C}[1/z]$. By (1.18),

$$\frac{1}{z} = \frac{1}{2z} + \frac{1}{\mathbf{b}_2} + \cdots + \frac{1}{\mathbf{b}_n} + \frac{1}{g_n}$$

holds with $g_n = -Q_{n-1}/Q_n$, $\deg g_n \leqslant -1$.

P-fractions converge in $\mathbb{C}([1/z])$. But pointwise convergence, if it takes place, is also important. In §**112** in Section 5.2 we obtained conditions of absolute convergence for an important class of C-fractions. By Corollary 6.4 the power series at $z = 0$ of a C-fraction cannot match the power series of any rational function. Hence if a C-fraction converges uniformly about $z = 0$ then the limit function is irrational.

Theorem 6.35 *Let $f(z)$ be a function meromorphic in a connected open set G, $0 \in G$, and let f be a C-fraction* (6.18). *Suppose that one of the following conditions holds:*

(a) *for infinitely many n there are functions h_n analytic in $\{z : |z| < \varepsilon_n\}$ such that $h_{n+1}(0) \neq 0$,*

$$f(z) = c_0 + \frac{c_1 z}{1} + \cdots + \frac{c_n z}{h_n(z)}; \tag{6.68}$$

(b) *a subsequence of convergents to the C-fraction f converges to $f(z)$ on an infinite set E with a limit point in G.*

If $\lim_n c_n = 0$ then f converges to a meromorphic $g(z)$ uniformly on compact subsets of $\mathbb{C}$ not containing poles of g and $g(z) = f(z)$ in G.

Proof For $R > 0$ there is an integer m such that $|c_n z| < 1/5$ for $n > m$ and $|z| < R$. By Corollary 5.14 the C-fraction on $D(R) = \{z : |z| \leqslant R\}$,

$$K_m(z) = \mathop{\mathbf{K}}_{n=m+1}^{\infty}\left(\frac{c_n z}{1}\right),$$

converges absolutely and uniformly to an analytic function $K_m(z)$. Let $P_{m,n}/Q_{m,n}$ be the convergents to $K_m(z)$. By (1.15),

$$(P_{m+n}/Q_{m+n})(z) = \frac{P_m(z) + (P_{m,n}/Q_{m,n})(z)P_{m-1}(z)}{Q_m(z) + (P_{m,n}/Q_{m,n})(z)Q_{m-1}(z)} \,. \tag{6.69}$$

For $z \neq 0$ the equalities below

$$\begin{aligned} Q_{m-1} \times \qquad & P_m(z) + P_{m-1}(z)K_m(z) = 0\,, \\ P_{m-1} \times \qquad & Q_m(z) + Q_{m-1}(z)K_m(z) = 0\,, \end{aligned}$$

cannot both hold, since subtraction of the above formulas, multiplied as shown, contradicts (1.16). It follows that if $\mathfrak{Q}_m = Q_m(z) + Q_{m-1}(z)K_m(z) = 0$ then $P_m(z) + P_{m-1}(z)K_m(z) \neq 0$. Since $K_m(z)$ is an irrational function, $\mathfrak{Q}_m$ cannot be identically zero. A nonzero analytic function on $D(R)$ may have only finite number of zeros. Therefore by (6.69) the continued fraction (6.18) converges to $c_0 + K_0(z)$ uniformly on compact subsets of $\mathbb{C}$ not intersecting the zeros of $\mathfrak{Q}_m$. At the zeros of $\mathfrak{Q}_m$ the continued fraction obviously converges to infinity. By (6.68) $f(z)$ and $g(z)$ have equal Taylor series at $z = 0$. If (b) holds then the result follows from the uniqueness theorem for analytic functions. $\square$

Let us illustrate this method for Euler's continued fraction (4.105).

Theorem 6.36 *The continued fraction*

$$e^z = 1 + \frac{z}{1} \, \frac{z}{-\,2} \, \frac{z}{+\,3} \, \frac{z}{-\,2} \, \frac{z}{+\,5} \, \frac{z}{-\,2} \, {}_{+\cdots} \,, \tag{6.70}$$

converges to e^z *uniformly on compact subsets of* $\mathbb{C}$.

Proof Iterative application of the identity

$$\frac{1}{1} \, \frac{1}{+\,1+w} = 1 - \frac{1}{2+w}$$

to the continued fraction (4.105) shows that its convergents $P_{3n}/Q_{3n}(s)$ equal the convergents $A_{2n}/B_{2n}(s)$ of

$$1 + \frac{1}{s} \, \frac{1}{-\,2} \, \frac{1}{+\,3s} \, \frac{1}{-\,2} \, \frac{1}{+\,5s} \, \frac{1}{-\,2} \, {}_{+\cdots} \approx 1 + \frac{z}{1} \, \frac{z}{-\,2} \, \frac{z}{+\,3} \, \frac{z}{-\,2} \, \frac{z}{+\,5} \, \frac{z}{-\,2} \, {}_{+\cdots} \,, \tag{6.71}$$

where $z = 1/s$. Since

$$\begin{aligned} & 1 + \frac{z}{1} \, \frac{z}{-\,2} \, \frac{z}{+\,3} \, \frac{z}{-\,2} \, \frac{z}{+\,5} \, \frac{z}{-\,2} \, {}_{+\cdots} \\ & \approx 1 + \frac{z}{1} \, \frac{z/2}{-\,1} \, \frac{z/(2\times 3)}{+\,1} \, \frac{z/(2\times 3)}{-\,1} \, \frac{z/(2\times 5)}{+\,1} \, \frac{z/(2\times 5)}{-\,1} \, {}_{+\cdots} \,, \end{aligned}$$

We have that $A_{2n}/B_{2n}(1/z)$ are even convergents of a C-fraction with $\lim_n c_n = 0$. By Euler's theorem, see §**101** in Section 4.10, $P_{3n}/Q_{3n}(s)$ converges to $e^{1/s}$ for $s > 0$, implying the convergence to e^z of the even convergents to (6.70) for $z > 0$. Hence (b) of Theorem 6.35 holds. By Theorem 6.35 the continued fraction (6.70) converges to e^z uniformly on compact subsets of $\mathbb{C}$, since e^z is an entire function. □

By Theorem 6.36 the P-fraction of

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{z^k} \stackrel{\text{def}}{=} {}_0F_0\left(\begin{matrix}\text{-}\\ \text{-}\end{matrix}; \frac{1}{z}\right) = {}_0F_0(\text{-}\,;\text{-}\,;z^{-1})$$

(for an explanation of the notation see the discussion after (6.72)) converges in $\mathbb{C}\,([1/z])$. This function is at the very top of the hierarchy of hypergeometric functions.

132 Wallis' hypergeometric function. The notation for hypergeometric series (see below) is considerably simplified by the use of the so-called Pochhammer symbol

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1)\,.$$

As usual, such a product having an empty set of multipliers has the value 1. The hypergeometric Laurent series

$${}_2F_0\left(\begin{matrix}a,b\\ \text{-}\end{matrix}; \frac{1}{s}\right) = {}_2F_0(a,\ b;\ \text{-}\ ;\ s^{-1}) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!}\frac{1}{s^n} \tag{6.72}$$

is a formal Laurent series in $\mathbb{C}\,([1/s])$. Here on the left-hand side of (6.72) the index 2 on F stands for the number of parameters a and b in the numerator of the series in (6.72) and the index 0 for the number of parameters in the denominator (in the argument of F, a hyphen stands for none). The choice $a = 1$, $b = 1$ gives the divergent Wallis series:

$${}_2F_0\left(\begin{matrix}1,1\\ \text{-}\end{matrix}; -\frac{1}{s}\right) = {}_2F_0(1,1;\text{-}\,;s^{-1}) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{(-1)^k k!}{s^k}\,, \tag{6.73}$$

which was studied by Euler (1760).[2] Euler's idea was to replace (6.73) with a suitable formula, for instance with a related continued fraction, and evaluate the value of this formula at $s = 1$.

Theorem 6.37 *The following identity holds in* $\mathbb{C}\,([1/z])$:

$$\begin{aligned}&\frac{{}_2F_0(a,b;\text{-}\,;z^{-1})}{{}_2F_0(a,b+1;\text{-}\,;z^{-1})}\\ &\quad = 1 + \frac{a}{z\,+}\,\frac{b+1}{1\,+}\,\frac{a+1}{z\,+}\,\frac{b+2}{1\,+}\,\frac{a+2}{z\,+}\cdots\end{aligned} \tag{6.74}$$

2 We shall not use the two-tiered notation for the arguments of the hypergeometric functions F in what follows, to avoid a multiplicity of notation.

Proof Although the continued fraction on the right-hand side of (6.74) is not a P-fraction, since it contains constant partial denominators, it still converges in $\mathbb{C}([1/z])$ because there are infinitely many denominators of the first degree. Elementary algebra for series shows that

$$\begin{aligned}&{}_2F_0\left(a, b+1; - ; -z^{-1}\right)\\&\quad = {}_2F_0\left(a, b; - ; -z^{-1}\right) - az^{-1}{}_2F_0\left(a+1, b+1; - ; -z^{-1}\right) .\end{aligned}$$

It follows that

$$\frac{{}_2F_0\left(a, b; - ; -z^{-1}\right)}{{}_2F_0\left(a, b+1; - ; -z^{-1}\right)} = 1 + \frac{az^{-1}}{{}_2F_0\left(a, b+1; - ; -z^{-1}\right)/{}_2F_0\left(a+1, b+1; - ; -z^{-1}\right)} .$$

Since ${}_2F_0$ is symmetric with respect to a and b, the iterations complete the proof by Theorem 6.34. □

Corollary 6.38 (Euler 1760) *In* $\mathbb{C}([1/z])$, *we have*

$$\frac{1}{z} \underset{+}{} \frac{1}{1} \underset{+}{} \frac{1}{z} \underset{+}{} \frac{2}{1} \underset{+}{} \frac{2}{z} \underset{+}{} \frac{3}{1} \underset{+}{} \frac{3}{z} \underset{+}{} \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{z^{k+1}} . \tag{6.75}$$

Proof This follows from Theorem 6.37 and the obvious identity

$$\begin{aligned}{}_2F_0\left(1, 2; - ; -z^{-1}\right) &= 1 - \frac{2!}{z} + \frac{2!3!}{2!z^2} - \frac{3!4!}{3!z^3} - \cdots\\&= z\left\{1 - {}_2F_0\left(1, 1; - ; -z^{-1}\right)\right\} .\end{aligned}$$

Just substitute this in (6.74) and apply equivalence transforms. □

The continued fraction on the left-hand side of (6.75) converges by Corollary 3.10 for every $z = s > 0$. In (1760) Euler found that its value at $s = 1$ is $0.596\,347\,362\,323\ldots$ Next, Euler formally differentiated the right-hand side $y(s)$ of (6.75) and noticed that it satisfies the differential equation $y' = y - 1/s$. This equation has the solution

$$y(s) = e^s \int_s^{\infty} e^{-t} \frac{dt}{t} = \int_0^{\infty} \frac{e^{-st}\,dt}{1+t} = \int_0^{\infty} \frac{e^{-t}\,dt}{t+s} . \tag{6.76}$$

By Watson's lemma 3.21

$$\int_0^{\infty} \frac{e^{-t}dt}{t+s} \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{s^{n+1}} . \tag{6.77}$$

Euler computed the integral in (6.77) at $s = 1$ and discovered that it coincides with the value $0.596\,347\,362\,323\ldots$ of the continued fraction. Using Wallis' interpolation we arrive at the conclusion that

$$\int_0^{\infty} \frac{e^{-t}\,dt}{t+s} = \frac{1}{s} \underset{+}{} \frac{1}{1} \underset{+}{} \frac{1}{s} \underset{+}{} \frac{2}{1} \underset{+}{} \frac{2}{s} \underset{+}{} \frac{3}{1} \underset{+}{} \frac{3}{s} \underset{+}{} \cdots . \tag{6.78}$$

This of course requires a rigorous justification, which can be easily obtained by analogy with Euler's differential method, Section 4.7. Notice that equation (6.74) involves quotients of hypergeometric series for different values of parameters. Formula (6.77) gives an integral representation for ${}_2F_0(1,1;\text{-}\,;-s^{-1})\,s^{-1}$. Now the binomial theorem and Watson's method imply that

$$E(a,b;s) \stackrel{\text{def}}{=} \frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-x}x^{a-1}}{(1+x/s)^b}\,dx \sim {}_2F_0(a,b;\text{-}\,;-s^{-1})\,. \tag{6.79}$$

In view of (6.79) one may conjecture that

$$E(a,b;s) = E(b,a;s)\,. \tag{6.80}$$

This in fact turns out to be true. Applying the trick (3.52), we can write

$$\frac{1}{(1+x/t)^b} = \frac{1}{\Gamma(b)}\int_0^\infty e^{-(1+x/s)y}y^{b-1}\,dy\,.$$

To handle the apparent asymmetry of $E(a,b;s)$ in a and b we write

$$E(a,b;s) = \frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty e^{-x-y-xy/s}x^{a-1}y^{b-1}\,dxdy = E(b,a;s)\,.$$

Now, integration by parts gives

$$\begin{aligned}
E(a,b;s) = E(b,a;s) &= \frac{1}{\Gamma(b+1)}\int_0^\infty \frac{e^{-t}}{(1+t/s)^a}\,dt^b \\
&= -\frac{1}{\Gamma(b+1)}\int_0^\infty t^b d\{e^{-t}(1+t/s)^{-a}\} \\
&= \frac{1}{\Gamma(b+1)}\int_0^\infty \frac{e^{-t}t^b\,dt}{(1+t/s)^a} + \frac{a}{s\Gamma(b+1)}\int_0^\infty \frac{e^{-t}t^b\,dt}{(1+t/s)^{a+1}} \\
&= E(a,b+1;s) + \frac{a}{s}E(a+1,b+1;s)\,.
\end{aligned}$$

This and the symmetry of $E(a,b;s)$ lead to

$$E(a,b;s) = E(a,b+1;s) + \frac{a}{s}E(a+1,b+1;s)\,, \tag{6.81}$$

$$E(a,b;s) = E(a+1,b;s) + \frac{b}{s}E(a+1,b+1;s)\,. \tag{6.82}$$

Now (6.81) and (6.82), as in Lemma 4.18, imply the following corollary.

Corollary 6.39 *For positive a, b and s,*

$$\frac{E(a,b;s)}{E(a,b+1;s)} = 1 + \frac{a}{s}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{b+1}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{a+1}{s}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{b+2}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{a+2}{s}\,\genfrac{}{}{0pt}{}{}{+\cdots}\,.$$

Putting $b=0$ in (6.81) and $b=1$ in (6.82), we find that

$$X=\frac{E(a+1,1;s)}{E(a+1,2;s)}=\frac{1-E}{(a+s)E-s},$$

where $E=E(a,1;s)$. Resolving this equation in E, we find

$$\frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-x}x^{a-1}}{x+s}\,dx=\frac{E(a,1;s)}{s}=\frac{1}{s}\,{}_{+}\,\frac{a}{1}\,{}_{+}\,\frac{1}{sX}$$
$$=\frac{1}{s}\,{}_{+}\,\frac{a}{1}\,{}_{+}\,\frac{1}{s}\,{}_{+}\,\frac{a+1}{1}\,{}_{+}\,\frac{2}{s}\,{}_{+}\cdots{}_{+}\,\frac{n}{s}\,{}_{+}\,\frac{a+n}{1}\,{}_{+}\cdots, \tag{6.83}$$

which explains Euler's striking calculations. Later Stieltjes proved and used these formulas to develop the theory of moments (1895). However, even in 1780 Euler (1818a) had proved these formulas for $\mathbb{C}([z])$.

Applying (6.21) iteratively, we transform (6.74) to

$$1+\frac{a}{z+b+1}\,{}_{-}\,\frac{(a+1)(b+1)}{z+a+b+3}\,{}_{-}\,\frac{(a+2)(b+2)}{z+a+b+5}\,{}_{-}\cdots \tag{6.84}$$

and (6.75) to

$$\int_0^\infty \frac{e^{-t}\,dt}{t+s}=\frac{1}{s+1}\,{}_{-}\,\frac{1^2}{s+3}\,{}_{-}\,\frac{2^2}{s+5}\,{}_{-}\,\frac{3^2}{s+5}\,{}_{-}\cdots. \tag{6.85}$$

Equations (6.84) and (6.85) are called the *contractions* of the initial continued fractions (6.74) and (6.75), since their convergents obviously make subsequences of the convergents for (6.74) and (6.75). Obvious equivalence transformations represent (6.84) and (6.85) as P-fractions.

133 The hypergeometric function ${}_0F_1$. The hypergeometric function ${}_0F_1$ is defined by the series

$$ {}_0F_1(-;a;z)={}_0F_1(a;z)\stackrel{\text{def}}{=}\sum_{n=0}^{\infty}\frac{z^n}{n!(a)_n} \tag{6.86}$$

converging uniformly on compact subsets of $\mathbb{C}$ to an entire function. In (4.114) we studied it in relation to Euler's solution of Riccati's equations.

Theorem 6.40 *The expansion into a continued fraction*

$$\frac{{}_0F_1(a+1;z)}{{}_0F_1(a;z)}=\frac{a}{a}\,{}_{+}\,\frac{z}{a+1}\,{}_{+}\,\frac{z}{a+2}\,{}_{+}\,\frac{z}{a+3}\,{}_{+}\cdots$$
$$=\frac{a}{a}\,{}_{+}\,\frac{1}{s(a+1)}\,{}_{+}\,\frac{1}{a+2}\,{}_{+}\,\frac{1}{s(a+3)}\,{}_{+}\cdots, \tag{6.87}$$

where $z=1/s$, converges to the expanded value both in $\mathbb{C}([z])$ and uniformly on compact subsets of $\mathbb{C}$ in z not containing zeros of ${}_0F_1(a;z)$.

Proof The identity

$$\frac{{}_0F_1(a+1;z)}{{}_0F_1(a;z)} = \cfrac{1}{1+\cfrac{z}{a(a+1)}\cfrac{{}_0F_1(a+2;z)}{{}_0F_1(a+1;z)}}$$

proves the convergence of the continued fractions in (6.87) to the expanded value in $\mathbb{C}([z])$ by Theorem 6.34 and for $s>0$ by Theorem 3.2. The proof is completed by Theorem 6.35(b). □

This theorem is essentially Euler's theorem 4.36 with $b_0 = a$, $d = 1$, $c = z$. The first continued fraction in (6.87) can be easily transformed into the P-fraction of the hyperbolic cotangent. Indeed putting $z = 1/t^2$ in (6.87) we obtain after obvious equivalence transformations

$$\frac{1}{at}\frac{{}_0F_1(a+1;t^{-2})}{{}_0F_1(a;t^{-2})} = \frac{1}{ta}\,{}_+\,\frac{1}{t(a+1)}\,{}_+\,\frac{1}{t(a+2)}\,{}_+\,\frac{1}{t(a+3)}\,{}_+\cdots .$$

Next, the choice $a=1/2$, $t=2s$ gives

$$\frac{1}{s}\frac{{}_0F_1(\frac{3}{2};\ \frac{1}{4}s^{-2})}{{}_0F_1(\frac{1}{2};\ \frac{1}{4}s^{-2})} = \frac{1}{s}\,{}_+\,\frac{1}{3s}\,{}_+\,\frac{1}{5s}\,{}_+\,\frac{1}{7s}\,{}_+\cdots .$$

Since $n!\left(\frac{3}{2}\right)_n = (2n+1)!\,4^{-n}$, $\quad n!(\frac{1}{2})_n = (2n)!\,4^{-n}$, we have

$${}_0F_1\left(\tfrac{3}{2};\ \tfrac{1}{4}s^{-2}\right)\frac{1}{s} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!s^{2n+1}} = \sinh\frac{1}{s}\,,$$

$${}_0F_1\left(\tfrac{1}{2};\ \tfrac{1}{4}s^{-2}\right) = \sum_{n=0}^{\infty}\frac{1}{(2n)!s^{2n}} = \cosh\frac{1}{s}\,.$$

This proves Corollary 4.39 by Euler's method presented in §**104**, Section 4.11.

134 The hypergeometric function ${}_1F_1$. This is defined by

$${}_1F_1(a;c;z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\frac{(a)_n}{n!(c)_n}z^n\,. \tag{6.88}$$

In (1813) Euler considered the continued fraction

$$K(s) = \frac{s}{1}\,{}_+\,\frac{s+1}{2}\,{}_+\,\frac{s+2}{3}\,{}_+\,\frac{s+3}{4}\,{}_+\cdots . \tag{6.89}$$

We apply his differential method (see Section 4.7) and put in (4.59)

$$\begin{aligned} &a=s\,, \quad b=1\,, \quad c=1\,, \qquad \alpha b-\beta a = 1-s\,,\\ &\alpha=1\,, \quad \beta=1\,, \quad \gamma=0\,, \qquad \alpha c-\gamma a = 1\,, \end{aligned}$$

giving

$$\frac{dS}{S} = (s-1)\frac{dR}{R}+\frac{(1-s)dR+RdR}{R-1} \quad\Rightarrow\quad S = CR^{s-1}(1-R)^{2-s}e^R\,.$$

It follows that $R^{n+1}S = 0$ at $x = 0$ and $x = 1$, if $0 < s < 2$ and $R(x) = x$. By (4.58) and Lemma 4.18

$$K(s) = \frac{\int_0^1 x^s(1-x)^{1-s}e^x\,dx}{\int_0^1 x^{s-1}(1-x)^{1-s}e^x\,dx}\,.$$

Putting $s = 1$, we obtain (4.7). Expanding e^x in a Maclaurin series and applying (4.30), we have

$$\begin{aligned}\int_0^1 x^{a-1}(1-x)^{1-s}e^x\,dx &= \sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1 x^{k+a-1}(1-x)^{1-s}\,dx\\ &= \sum_{k=0}^{\infty}\frac{B(a+k,2-s)}{k!} = \sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(2-s))}{k!\Gamma(2+k+a-s)}\,.\end{aligned}$$

Now putting $a = s+1$ and $a = s$, we obtain

$$\begin{aligned}\int_0^1 x^s(1-x)^{1-s}e^x\,dx &= \frac{s\Gamma(s)\Gamma(2-s)}{2}\,{}_1F_1\,(s+1;3;1)\,,\\ \int_0^1 x^{s-1}(1-x)^{1-s}e^x\,dx &= \Gamma(s)\Gamma(2-s)\,{}_1F_1\,(s;2;1)\,,\end{aligned} \tag{6.90}$$

which implies that

$$\frac{s}{2}\,\frac{{}_1F_1\,(s+1;3;1)}{{}_1F_1\,(s;2;1)} = \mathop{\mathbf{K}}_{k=1}^{\infty}\left(\frac{s+k-1}{k}\right)\,. \tag{6.91}$$

Formulas (6.90) hint that there may be an integral representation for ${}_1F_1$.

Lemma 6.41 *For $a > 0$, $c > a$ and arbitrary complex z,*

$${}_1F_1\,(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{c-a-1}\,dt\,.$$

Proof Expand e^{zt} and integrate using (4.30). □

Theorem 6.42 *In $\mathbb{C}\,([z])$,*

$$\begin{aligned}&\frac{{}_1F_1\,(a+1;c+1;z)}{{}_1F_1\,(a;c;z)}\\ &= \frac{c}{c}\,\underset{-}{}\,\frac{(c-a)z}{c+1}\,\underset{+}{}\,\frac{(a+1)z}{c+2}\,\underset{-}{}\,\frac{(c-a+1)z}{c+3}\,\underset{+}{}\cdots\\ &\quad\underset{+}{}\,\frac{(a+n)z}{c+2n}\,\underset{-}{}\,\frac{(c-a+n)z}{c+2n+1}\,\underset{+}{}\cdots\,.\end{aligned}$$

Proof Use Exs. 6.2 and 6.3 alternately. Apply Theorem 6.34. □

135 Gauss' continued fractions for ${}_2F_1$. The series

$$ {}_2F_1\,(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n \tag{6.92}$$

was introduced by Euler (1769), although he had considered partial cases earlier in relation to Wallis' interpolation method (see for instance Ex. 4.41). The ratio test shows that the series (6.92) converges in the unit disc $|z|<1$. The analytic function in z ${}_2F_1(a,b;c;z)$ is also denoted simply as $F(a,b;c;z)$.

A careful account of what was done by Euler and Gauss to develop quotients of hypergeometric series into continued fractions can be found in Andrews, Askey and Roy (1999, §2.5).

Theorem 6.43 (Gauss 1812) *For $z=1/s$ in $\mathbb{C}([1/s])$,*

$$\begin{aligned}&\frac{{}_2F_1(a,b+1;c+1;z)}{{}_2F_1\,(a,b;c;z)}\\ &\quad=\frac{c}{c\,-}\;\frac{a(c-b)z}{c+1\,-}\;\frac{(b+1)(c-a+1)z}{c+2\,-}\cdots\\ &\qquad\frac{(a+n)(c-b+n)z}{c+2n+1\,-}\;\frac{(b+n+1)(c-a+n+1)z}{c+2n+2\,-}\cdots.\end{aligned}$$

Proof The identity

$$F(a,b;c;z)=F(a,b+1;c+1;z)-\frac{a(c-b)}{c(c+1)}zF(a+1,b+1;c+2;z)$$

follows from (6.92) by comparison of the coefficients at z^n. It can be rewritten in a form generating a continued fraction:

$$\frac{F(a,b;c;z)}{F(a,a+1;c+1;z)}=1-\frac{a(c-b)}{c(c+1)}z\,\frac{1}{\dfrac{F(a,b+1;c+1;z)}{F(a+1,b+1;c+2;z)}}\,.$$

Observing the symmetry of ${}_2F_1$ in a and b, one thus obtains

$$\frac{F(a,b;c;z)}{F(a,b+1;c+1;z)}=1-\frac{u_1}{1\,-}\;\frac{v_1}{1\,-}\;\frac{u_2}{1\,-}\;\frac{v_2}{1\,-}\cdots, \tag{6.93}$$

where

$$u_n=\frac{(a+n-1)(c-b+n-1)}{(c+2n-2)(c+2n-1)}z\,,\qquad v_n=\frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)}z. \tag{6.94}$$

Theorem 6.34 and equivalence transformations complete the proof. □

To obtain convergence in a stronger sense, as in the case of ${}_2F_0$, we need an integral representation for ${}_2F_1$.

Theorem 6.44 (Euler 1769) *For $c > b > 0$*

$$ {}_2F_1\,(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\,dt\,. \tag{6.95} $$

Proof The proof of (6.95) is very similar to the proof of Euler's formula (6.79). Whereas in the case of (6.79) we used Euler's integral formula for the gamma function, in the case of (6.95) we can combine the binomial theorem with Euler's formula (4.30) for the beta function:

$$ \begin{aligned} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\,dt &= \sum_{n=0}^{\infty} z^n \frac{(a)_n}{n!} \int_0^1 t^{n+b-1}(1-t)^{c-b-1}\,dt \\ &= \sum_{n=0}^{\infty} z^n \frac{(a)_n}{n!} B(b+n,c-b) \\ &= \sum_{n=0}^{\infty} z^n \frac{(a)_n (b)_n}{(c)_n n!} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1\,(a,b;c;z)\;. \end{aligned} $$

These calculations are made under the assumption that $|z| < 1$. □

Remark Notice that the right-hand side of (6.95) is analytic about any point z such that $z < 0$ and in particular about $z = -1$. Hence Euler's formula (6.95) sums up the hypergeometric series at $z = -1$.

Since the Maclaurin series for Gauss's hypergeometric function converges in the unit disc, we immediately obtain the symmetry of the expression on the right-hand side of (6.95) in respect of a and b. This can also be proved along the same lines as (6.80). In any case the parameters a and b in the integral (6.95) can exchange without changing the value of the integral.

The appearance of the quotient of hypergeometric functions in Theorem 6.43 is explained by (4.69). The change of variables $x = x^{1/r}$ in Euler's integral (6.95) shows that if

$$ b = (f-r)r^{-1} > 0\,, \qquad c-b = (s-f+r+h)2r^{-1} > 0\,, $$
$$ a = (f+r-h)2s^{-1}\,, $$

then by Theorem 6.44

$$ s + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{(f+nr)(h+nr)}{s}\right) = rc\,\frac{{}_2F_1\,(a,b;c;-1)}{{}_2F_1\,(a,b+1;c+1;-1)}\,. $$

putting $b = 0$ in Theorem 6.43 and observing that $F(a,0;c;-z) = 1$, we arrive at the following important result.

Theorem 6.45 *For $z \geqslant 0$,*

$$
\begin{aligned}
&{}_2F_1(a, 1; c; -z)\\
&= \frac{1}{1} {}_{+} \frac{az}{c} {}_{+} \frac{1(c-a)z}{c+1} {}_{+} \frac{(a+1)cz}{c+2} {}_{+} \frac{2(c-a+1)z}{c+3} {}_{+\cdots}\\
&\quad + \frac{(a+n)(c+n-1)z}{c+2n} {}_{+} \frac{(n+1)(c-a+n)z}{c+2n+1} {}_{+\cdots} .
\end{aligned}
\tag{6.96}
$$

Proof For any continuous function f on [0, 1]

$$\lim_{b\to 0} b\int_0^1 t^{b-1} f(t)dt = \lim_{b\to 0}\int_0^1 f(t^{1/b})dt = f(0) .$$

Combining recursions of Theorem 6.43 with Euler's integral representation for ${}_2F_1$ and Theorem 6.44, and passing to the limit $b \to 0$ in the finite version of (E6.1), we obtain (6.96) by Markoff's test if $c := c+1$. □

Since many known functions can be obtained from ${}_2F_1(a, 1; c; -z)$, Theorem 6.45 gives useful expansions into convergent continued fractions. Let us consider for instance

$$\ln(1+z) = z\sum_{n=0}^{\infty} \frac{1}{n+1}(-z)^n = z\; {}_2F_1(1, 1; 2; -z) . \tag{6.97}$$

By Euler's theorem 6.44 the hypergeometric function on the right-hand side of the above equality in $\mathbb{C}([z])$ extends to an analytic function about the positive half-line $(0, +\infty)$. The same is true for $\ln(1+z)$. Hence they must coincide for $z \geqslant 0$. If $a = 1$, $c = 2$ in Theorem 6.45 then

$$\ln(1+z) = \frac{z}{1} {}_{+} \frac{1^2 z}{2} {}_{+} \frac{1^2 z}{3} {}_{+} \frac{2^2 z}{4} {}_{+} \frac{2^2 z}{5} {}_{+} \frac{3^2 z}{6} {}_{+} \frac{3^2 z}{7} {}_{+\cdots} . \tag{6.98}$$

The substitution $z = 1/s$ shows that

$$\int_0^1 \frac{dt}{t+s} = \ln\left(1+\frac{1}{s}\right) = \frac{1}{s} {}_{+} \frac{1^2}{2} {}_{+} \frac{1^2}{3s} {}_{+} \frac{2^2}{4} {}_{+} \frac{2^2}{5s} {}_{+} \frac{3^2}{6} {}_{+} \frac{3^2}{7s} {}_{+\cdots} , \tag{6.99}$$

which can be to compared with (6.78). The function (6.99) can be used to sum the series in (6.97) for $z = 1/s > 0$.

Applying the same method to (4.64), we see that

$$\arctan\frac{1}{s} = \frac{1}{s}\,{}_2F_1\left(\tfrac{1}{2}, 1; \tfrac{3}{2}; -s^{-2}\right) = \frac{1}{s} {}_{+} \frac{1^2}{3s} {}_{+} \frac{2^2}{5s} {}_{+} \frac{3^2}{7s} {}_{+\cdots} .$$

We obtained this formula in §**88**, Section 4.7, with Euler's differential method. Next

$$\ln\frac{1+z}{1-z} = {}_2F_1\left(\tfrac{1}{2}, 1; \tfrac{3}{2}; z^2\right)2z = \frac{2z}{1} {}_{-} \frac{1^2 z^2}{3} {}_{-} \frac{2^2 z^2}{5} {}_{-} \frac{3^2 z^2}{7} {}_{-\cdots} . \tag{6.100}$$

If $z = 1/s$ in (6.100) then after equivalence transformations we get

$$\ln \frac{s+1}{s-1} = \frac{2}{s} \,\underset{-}{}\, \frac{1^2}{3s} \,\underset{-}{}\, \frac{2^2}{5s} \,\underset{-}{}\, \frac{3^2}{7s} \,\underset{-}{}\, \frac{4^2}{9s} \,\underset{-}{}\, \cdots \,. \tag{6.101}$$

Another application is Jacobi's continued fraction (1859) which Jacobi had actually considered in 1843. Observing that $F(a,0;c;z) \equiv 1$ and replacing c by $c-1$ in (6.94), we obtain

$$F(a,1;c;z) = \frac{1}{1} \,\underset{-}{}\, \frac{u_1 z}{1} \,\underset{-}{}\, \frac{v_1 z}{1} \,\underset{-}{}\, \frac{u_2 z}{1} \,\underset{-}{}\, \frac{v_2 z}{1} \,\underset{-}{}\, \cdots \,, \tag{6.102}$$

where

$$u_n = \frac{(a+n-1)(c+n-2)}{(c+2n-3)(c+2n-2)}\,, \qquad v_n = \frac{n(c-a+n-1)}{(c+2n-2)(c+2n-1)}\,. \tag{6.103}$$

Let us put $a = 1$, $b = \alpha$, $c = \alpha+\beta$ in (6.95) and use the symmetry of ${}_2F_1$ in a and b. Then by Euler's theorem 6.44 and (6.96),

$$\begin{aligned} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{z-t}\,dt &= {}_2F_1\left(\alpha,1;\alpha+\beta;z^{-1}\right) z^{-1} \\ &= \frac{1}{z} \,\underset{-}{}\, \frac{u_1}{1} \,\underset{-}{}\, \frac{v_1}{z} \,\underset{-}{}\, \frac{u_2}{1} \,\underset{-}{}\, \frac{v_2}{z} \,\underset{-}{}\, \cdots \,\underset{-}{}\, \frac{u_n}{1} \,\underset{-}{}\, \frac{v_n}{z} \,\underset{-}{}\, \cdots \,. \end{aligned} \tag{6.104}$$

Applying (6.21) with $p = u_n$, $q = -v_n$, $w := -w$, we obtain the following expansion:

$$\begin{aligned} &\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{z-t}\,dt \\ &= \frac{1}{z-u_1} \,\underset{-}{}\, \frac{u_1 v_1}{z-v_1-u_2} \,\underset{-}{}\, \frac{u_2 v_2}{z-v_2-u_3} \,\underset{-}{}\, \cdots \,\underset{-}{}\, \frac{u_n v_n}{z-v_n-u_{n+1}} \,\underset{-}{}\, \cdots \,. \end{aligned} \tag{6.105}$$

Theorem 6.46 (Laguerre) *For* $-1 < \alpha < 1$ *in* $\mathbb{C}([1/s])$*:*

$$\frac{\sin \pi\alpha}{\pi\alpha} \int_{-1}^1 \left(\frac{1+x}{1-x}\right)^{\alpha} \frac{dx}{s-x} = \frac{2}{s-\alpha+\mathop{\mathbf{K}}\limits_{n=1}^{\infty}\left(\alpha^2-n^2 \,/\, (2n+1)s\right)}\,. \tag{6.106}$$

Proof Let us make the following substitutions in (6.105): $t = (x+1)/2$, $z = (s+1)/2$, $\alpha := \alpha-1$, $\beta := 2-\alpha$. By Ex. 3.19 the integrals on the left-hand sides of (6.105) and (6.106) coincide. Elementary algebra applied to (6.103), for $c = 2$ and $a = \alpha+1$, shows that

$$u_n = \frac{n+\alpha}{4n-2}, \qquad v_n = \frac{n-\alpha}{4n+2}, \qquad u_n + v_{n+1} = \frac{1}{2},$$
$$u_n v_n = \frac{n^2-\alpha^2}{4(2n-1)(2n+1)},$$

and therefore the continued fraction in (6.106) is equivalent to the continued fraction in (6.105). $\square$

6.5 Stieltjes' theory

136 Convergence in the right half-plane. Brouncker's continued fraction (3.34) as well as many continued fractions of Euler have the form

$$b_0 s + c_0 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n}{b_n s + c_n}\right), \tag{6.107}$$

where $a_n > 0$, $b_n \geqslant 0$, $\operatorname{Re} c_n \geqslant 0$. If $s > 0$, such continued fractions converge to the expanded value by Markoff's theorem 3.2. To summarize, the reasons for this are Brouncker's inequalities (1.21) and the observation that $(0, +\infty)$ is mapped by $w \longrightarrow a_n(b_n s + c_n + w)^{-1}$ into itself. Such Möbius transformations map the right half-plane $\mathbb{P}_+ = \{z : \operatorname{Re} z > 0\}$ into itself also. This hints that the convergence of (6.107) may extend to $\mathbb{P}_+$.

Definition 6.47 *A family $\mathfrak{N}$ of analytic functions in an open connected set G is called normal if every sequence $\{f_n\}$ in $\mathfrak{N}$ contains a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of G either to infinity or to a finite function. If the first option, infinity, does not occur then $\mathfrak{N}$ is called a pre-compact family.*

Every family of uniformly bounded analytic functions in an open connected set is pre-compact (Ex. 6.8). The mapping $z \to (1-z)/(1+z)$ maps $\mathbb{P}_+$ conformally onto $\mathbb{D}$, sending the family $\mathfrak{R}(G)$ of all analytic functions in G with positive real part in G into the family $\mathscr{B}(G)$ of all analytic functions in G bounded by 1. Since ∞ is mapped to -1, $\mathfrak{R}(G)$ is normal but is not pre-compact. For instance, $\lim_n nz = \infty$ in $\mathbb{P}_+$. No meromorphic function can have a positive real part about its pole or zero.

Theorem 6.48 *The family of convergents to* (6.107) *with $a_n > 0$, $b_n \geqslant 0$ and* $\operatorname{Re} c_n \geqslant 0$ *is a normal family in $\mathbb{P}_+$.*

Proof Let $s_0(w) = k_0 z + l_0 + w$, $s_k(w) = a_k/(b_k z + c_k + w)$. Every s_k maps the right half-plane into itself. Then (see Theorem 3.2)

$$\frac{P_n(z)}{Q_n(z)} = s_0 \circ s_1 \circ \cdots \circ s_n(0)$$

has a positive real part in $\mathbb{P}_+$. □

Corollary 6.49 (Complex Markoff test) *If* (6.107) *converges to finite values on a subset E of* $(0, +\infty)$ *having a finite nonzero limit point then* (6.107) *converges uniformly on compact subsets of* $\{z : \operatorname{Re} z > 0\}$ *to an analytic function with positive real part.*

Proof The convergents make a normal family in $\mathbb{P}_+$ by Theorem 6.48. Since they converge on an $E \subset (0, +\infty)$, there is no subsequence converging to ∞. All limit points of this sequence of analytic functions have equal restrictions to E. Since E has a limit point in $\mathbb{P}_+$, limit functions of the convergents coincide in $\mathbb{P}_+$ by the uniqueness theorem. $\square$

Corollary 6.50 *Brouncker's continued fraction* (3.34) *converges uniformly on compact subsets of* $\mathbb{P}_+$ *to a function in* $\mathfrak{R}(\mathbb{P}_+)$.

Proof Apply Corollary 3.10. $\square$

Corollary 6.51 *The identity in* (3.40) *holds for* $\operatorname{Re} s > 0$ *and the infinite product in* (3.40) *has a positive real part for* $\operatorname{Re} s > 0$.

By Corollary 6.49 the continued fractions given in (3.34), (4.45), (4.60) – (4.62), (4.65), (4.70), (4.71), (4.83), (4.85)–(4.88), (4.90), (4.103), (4.125), (6.78), (6.83), (6.99) all converge to analytic functions with a positive real part in $\mathbb{P}_+$. We consider the most important examples in §§**138–40**.

137 Van Vleck's theorem. Van Vleck's theorem is obtained by refining the arguments of §**136**. For $0 < \varepsilon < \pi/2$ we denote by $\mathbb{P}_\varepsilon$ the angle $\{z : |\arg z| < \pi/2 - \varepsilon\}$ in $\mathbb{P}_+$.

Theorem 6.52 (Van Vleck 1901) *If there is an* $\varepsilon \in (0, \pi/2)$ *such that* $b_n \in \mathbb{P}_\varepsilon$ *for* $n \geqslant 1$ *then:*

(a) *the nth convergent* f_n *of* $\mathbf{K}_{n=1}^{\infty}(1/b_n)$ *is in* $\mathbb{P}_\varepsilon$*;*
(b) *the limits* $\lim_k f_{2k}$ *and* $\lim_k f_{2k+1}$ *both exist and are finite;*
(c) *the continued fraction converges if* $\sum_n |b_n| = +\infty$*;*
(d) *if* $\mathbf{K}_{n=1}^{\infty}(1/b_n)$ *converges to* f *then* $f \in \mathbb{P}_+$.

Proof[3] Since f_n is a composition of Möbius transforms each of which maps $\mathbb{P}_\varepsilon$ into itself, it must be in $\mathbb{P}_\varepsilon$. This proves (a).

Let z be an auxiliary complex parameter such that the functions

$$b_n(z) = |b_n| e^{i\beta_n z}, \qquad \beta_n = \arg b_n$$

interpolate b_n at $z = 1$. Now $|\arg b_n(z)| = |\beta_n \operatorname{Re} z|$ is smaller than $(\pi - \varepsilon)/2$ in $G = \{z : |\operatorname{Re} z| < 1 + \varepsilon/(\pi - 2\varepsilon), |\operatorname{Im} z| < 1\}$. By (a) the family $\{f_n(z)\}_{n \geqslant 1}$ of the convergents to $\mathop{\mathbf{K}}\limits_{n=1}^{\infty} (1/b_n(z))$ takes values in $\mathbb{P}_+$. Hence it is normal in G. Since $b_n(z) > 0$ on $(-i, i) \subset G$, we obtain (b) and (c) by Theorems 3.3 and 3.4. Now (d) follows from (a). $\square$

[3] Here we follow Jones and Thron (1980).

138 Convergence for $b(s)$. Stirling's formula (3.58),

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z} , \qquad |z| \to +\infty , \tag{6.108}$$

is valid in any angular domain $|\arg z| < \pi - \delta$, $\delta > 0$; see Andrews, Askey and Roy R. (1999, Corollary 1.4.3). Simple calculations using Ramanujan's formula (Theorem 3.25), show that in any such domain

$$b(z) \sim z , \qquad |z| \to +\infty . \tag{6.109}$$

Theorem 6.53 *For* $\operatorname{Re} s > 0$*,*

$$\frac{1}{b(s)} = \int_{\mathbb{R}} \frac{d\mu(t)}{s - it} , \qquad d\mu = \frac{1}{8\pi^3} \left| \Gamma\left(\frac{1+it}{4}\right) \right|^4 dt ,$$

where μ is a probability measure on $\mathbb{R}$.

Proof Euler's functional equation for the gamma function, see Ex. 3.19 or Whittaker and Watson (1902, Section 12.14), with $s = 1/4 - it/4$ shows that

$$\Gamma\left(\frac{3+it}{4}\right) = \Gamma(1-s) = \frac{\pi}{\sin \pi s} \frac{1}{\Gamma(s)}$$

and therefore

$$\frac{1}{4} \left\{ \frac{\Gamma((1+it)/4)}{\Gamma((3+it)/4)} \right\}^2 = \frac{1 - \cos 2\pi s}{8\pi^2} |\Gamma(s)|^4 = \frac{1}{8\pi^2} |\Gamma(s)|^4 \left(1 - i \sinh \frac{\pi t}{2}\right) .$$

Hence by Theorem 3.25

$$\operatorname{Re} \frac{1}{b(it)} = \frac{1}{8\pi^2} \left| \Gamma\left(\frac{1+it}{4}\right) \right|^4 > 0 .$$

By (6.109) we can apply Cauchy's integral formula for the holomorphic function $1/b(s)$ to the right half-plane $\operatorname{Re} s > 0$:

$$\frac{1}{b(s)} = \frac{1}{2\pi} \int_{+\infty}^{-\infty} \frac{1}{b(it)} \frac{1}{it - s} \, d(it) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{b(it)} \frac{1}{s - it} \, dt$$

$$0 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{b(it)} \frac{1}{it + s} \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\bar{b}(it)} \frac{1}{s - it} \, dt .$$

To complete the proof it remains to add the above formulas. Since $1/b(s) = 1/s + o(1/s^2)$ as $s \to +\infty$, we obtain that

$$\frac{1}{8\pi^3} \int_{-\infty}^{+\infty} |\Gamma(1 + it/4)|^4 \, dt = 1 . \tag{6.110}$$

Hence $d\mu$ is a probability measure on $\mathbb{R}$. $\square$

Using (6.108) one can easily obtain an asymptotic formula for the density $d\mu/dt$ (see Andrews, Askey and Roy 1999, Corollary 1.4.4):

$$\frac{1}{8\pi^3}\left|\Gamma\left(1+\frac{it}{4}\right)\right|^4 \sim \frac{2}{|t|\pi}\exp\left(-\frac{\pi}{2}|t|\right), \qquad t \to \pm\infty\,. \tag{6.111}$$

Corollary 6.54 *For* $\operatorname{Im} z > 0$,

$$\frac{1}{8\pi^3}\int_{-\infty}^{+\infty}\left|\Gamma\left(\frac{1+it}{4}\right)\right|^4 \frac{dt}{z-t} = \frac{1}{z}\,\underset{-}{}\,\frac{1^2}{2z}\,\underset{-}{}\,\frac{3^2}{2z}\,\underset{-}{}\,\frac{5^2}{2z}\,\underset{-}{}\cdots\,. \tag{6.112}$$

Proof If $\operatorname{Im} z > 0$ and $s = z/i$ then $\operatorname{Re} s > 0$ and

$$\int_{-\infty}^{+\infty}\frac{d\mu(t)}{s-it} = i\int_{-\infty}^{+\infty}\frac{d\mu(t)}{z+t} = i\int_{-\infty}^{+\infty}\frac{d\mu(t)}{z-t}\,,$$

since μ is symmetric. Brouncker's continued fraction is transformed into (6.112) by obvious equivalence transforms. □

Since $d\mu$ is symmetric, we can write

$$\frac{1}{b(s)} = \int_0^{+\infty}\frac{2s}{s^2+t^2}\,d\mu = \frac{s}{8\pi^3}\int_0^{+\infty}\frac{|\Gamma(1+it)/4|^4 t^{-1/2}dt}{s^2+t}\,,$$

$$\begin{aligned}\frac{1}{8\pi^3}\int_0^{+\infty}\frac{\left|\Gamma\big((1+it)/4\big)\right|^4 t^{-1/2}dt}{s+t} &= \frac{1}{\sqrt{s}b(\sqrt{s})}\\ &= \frac{1}{s}\,\underset{+}{}\,\frac{1^2}{2}\,\underset{+}{}\,\frac{3^2}{2s}\,\underset{+}{}\,\frac{5^2}{2}\,\underset{+}{}\,\frac{7^2}{2s}\,\underset{+}{}\cdots\\ &= \left(s+\frac{1}{2}-\frac{9}{8s}+\frac{153}{16s^2}+\cdots\right)^{-1},\end{aligned} \tag{6.113}$$

which explains why $c_2 = c_4 = \cdots = 0$ in (3.33). The continued fraction in (6.113) converges uniformly on compact subsets of $\mathbb{C}\setminus(-\infty,0]$.

139 Gaussian distribution. To obtain the continued fraction for the Gaussian or the normal distribution we reverse the arguments of §**138**. Since $\Gamma(1/2) = \sqrt{\pi}$, see (3.66), we obtain from (6.83) with $a = 1/2$ that

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{-t^2}dt}{z-it} = \frac{2z}{\sqrt{\pi}}\int_0^{+\infty}\frac{e^{-t^2}dt}{z^2+t^2} = \frac{z}{\Gamma(1/2)}\int_0^{+\infty}\frac{e^{-t}t^{1/2-1}dt}{z^2+t}$$

$$= \frac{z}{z^2}\,\underset{+}{}\,\frac{1/2}{1}\,\underset{+}{}\,\frac{1}{z^2}\,\underset{+}{}\,\frac{3/2}{1}\,\underset{+}{}\,\frac{2}{z^2}\,\underset{+}{}\,\frac{5/2}{1}\,\underset{+}{}\cdots = \frac{1}{z}\,\underset{+}{}\,\frac{1/2z}{1}\,\underset{+}{}\,\frac{1}{z^2}\,\underset{+}{}\,\frac{3/2}{1}\,\underset{+}{}\cdots$$

$$= \frac{1}{z}\,\underset{+}{}\,\frac{1}{2z}\,\underset{+}{}\,\frac{2}{z}\,\underset{+}{}\,\frac{3}{2z}\,\underset{+}{}\,\frac{4}{z}\,\underset{+}{}\,\frac{5}{2z}\,\underset{+}{}\cdots = \frac{2}{2z}\,\underset{+}{}\,\frac{2}{2z}\,\underset{+}{}\,\frac{4}{2z}\,\underset{+}{}\,\frac{6}{2z}\,\underset{+}{}\,\frac{8}{2z}\,\underset{+}{}\,\frac{10}{2z}\,\underset{+}{}\cdots\,.$$

Now by (4.90) we obtain

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{-t^2}\,dt}{z-it}=\frac{2}{2z+\mathop{\mathbf{K}}\limits_{n=1}^{\infty}\big(2n/2z\big)}=2e^{z^2}\int_z^{+\infty}e^{-x^2}\,dx\,. \tag{6.114}$$

Hence the Cauchy integral of the Gaussian distribution can be simply expressed via the error function and extended to an entire function by the use of a Laplace transform. We obtain

$$2\int_0^{+\infty}e^{-2xz-x^2}\,dx\,.$$

Corollary 6.55 *For* $\operatorname{Im} z>0$,

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{-t^2}\,dt}{z-t}=\frac{2}{2z}\,\frac{}{-}\,\frac{2}{2z}\,\frac{}{-}\,\frac{4}{2z}\,\frac{}{-}\,\frac{6}{2z}\,\frac{}{-}\,\frac{8}{2z}\,\frac{}{-}\cdots\,. \tag{6.115}$$

140 Continued fraction for $\arctan(1/s)$**.** By Euler's formulas (6.63),

$$\arctan\frac{1}{s}=\frac{i}{2}\ln\frac{s-i}{s+i}=-\frac{1}{2}\arg\frac{s-i}{s+i}+\frac{i}{2}\ln\left|\frac{s-i}{s+i}\right|.$$

The Möbius transform $s\to(s-i)/(s+i)$ maps $\mathbb{P}_+$ onto $\mathbb{C}_-$. Hence the real part of $\arctan(1/s)$ tends to $-\pi$ if s approaches $(-i,i)$ from $\mathbb{P}_+$ and tends to zero if s approaches the complement of $[-i,i]$ on the imaginary axis. It follows that for $\operatorname{Re} s>0$,

$$\frac{1}{2}\int_{-1}^{1}\frac{dt}{s-it}=\frac{1}{s}\,\frac{}{+}\,\frac{1^2}{3s}\,\frac{}{+}\,\frac{2^2}{5s}\,\frac{}{+}\,\frac{3^2}{7s}\,\frac{}{+}\,\frac{4^2}{9s}\,\frac{}{+}\cdots\frac{}{+}\,\frac{n^2}{(2n+1)s}\,\frac{}{+}\cdots\,,$$

and for $\operatorname{Im} z>0$,

$$\frac{1}{2}\int_{-1}^{1}\frac{dt}{z-t}=\frac{1}{z}\,\frac{}{-}\,\frac{1^2}{3z}\,\frac{}{-}\,\frac{2^2}{5z}\,\frac{}{-}\,\frac{3^2}{7z}\,\frac{}{-}\,\frac{4^2}{9z}\,\frac{}{-}\cdots\frac{}{-}\,\frac{n^2}{(2n+1)z}\,\frac{}{-}\cdots\,.$$

141 Asymptotic expansions. The examples considered in §§**138–40** hint that there may be a general theory including them as partial cases. Let $\mathfrak{P}(\mathbb{R})$ be the set of all Borel measures σ on $\mathbb{R}$ with finite moments

$$-\infty<s_k=\int_{-\infty}^{+\infty}x^k\,d\sigma(x)<+\infty\,,\qquad k\geqslant 0; \tag{6.116}$$

the problem of determining all σ with a given sequence $\{s_n\}_{n\geqslant 0}$ is known as the Hamburger moment problem. If $\sigma\in\mathfrak{P}(\mathbb{T})$ then the Cauchy integral

$$C^{\sigma}(z)=\int_{-\infty}^{+\infty}\frac{d\sigma(t)}{z-it}$$

is obviously in $\mathfrak{R}(\mathbb{P}_+)$. It is natural to consider σ in this formula as a measure placed on the imaginary axis.

Lemma 6.56 *If $\sigma \in \mathfrak{P}(\mathbb{R})$ then*

$$C^\sigma(x) \sim \sum_{k=0}^{\infty} \frac{i^k s_k}{x^{k+1}} \tag{6.117}$$

is the asymptotic expansion of $C^\sigma(x)$ as $x \to +\infty$.

Proof [4] We have

$$C^\sigma(x) - \sum_{k=0}^{2n-1} \frac{i^k s_k}{x^{k+1}} = \frac{(-1)^n}{x^{2n+1}} \int_{\mathbb{R}} \frac{x t^{2n} d\sigma}{x - it} = \frac{(-1)^n}{x^{2n+1}} \left(s_{2n} + \int_{\mathbb{R}} \frac{i t^{2n+1} d\sigma}{x - it} \right) .$$

If $R > 0$ then

$$\left| \int_{\mathbb{R}} \frac{t^{2n+1} d\sigma}{x - it} \right| \leqslant \frac{1}{x} \int_{-R}^{R} |t|^{2n+1} d\sigma + \int_{|t|>R} t^{2n} d\sigma .$$

Given $\varepsilon > 0$ we first find $R > 0$ to make the second integral less than $\varepsilon/2$. Then the first will be smaller than $\varepsilon/2$ if x is large enough. ☐

If σ is an even measure, then

$$\int_{-\infty}^{+\infty} \frac{d\sigma(t)}{x - it} = \int_0^{\infty} \frac{2x d\sigma(\sqrt{t})}{x^2 + t} , \tag{6.118}$$

implying that

$$\int_0^{+\infty} \frac{d\sigma(\sqrt{t})}{x + t} \sim \sum_{j=0}^{\infty} \frac{(-1)^j m_j}{x^{j+1}} , \qquad m_j = \int_0^{+\infty} t^j d\sigma(\sqrt{t}) . \tag{6.119}$$

We are interested in functions in $\mathfrak{R}(\mathbb{P}_+)$ admitting an asymptotic expansion (6.117). A proof of the following theorem can be found in Akhiezer (1961) or Shohat and Tamarkin (1943) or it can be deduced directly from Theorem 8.2 by the standard conformal mapping of $\mathbb{D}$ onto $\mathbb{P}_+$.

Theorem 6.57 *For any function $F \in \mathfrak{R}(\mathbb{P}_+)$ there are a unique finite positive measure σ and real numbers $a > 0$ and b such that*

$$F(z) = az + ib + \int_{-\infty}^{+\infty} \frac{1 - itz}{z - it} d\sigma(t) , \qquad \operatorname{Re} z = x > 0 .$$

Corollary 6.58 *A function $F \in \mathfrak{R}(\mathbb{P}_+)$ equals C^σ for a finite positive σ if and only if $F(x) = O(1/x)$ as $x \to +\infty$.*

Corollary 6.59 *A function $F \in \mathfrak{R}(\mathbb{P}_+)$ has an asymptotic expansion as $x \to +\infty$ if and only if $F = C^\sigma$ with $\sigma \in \mathfrak{P}(\mathbb{R})$.*

The measure σ can be recovered by Stieltjes' inversion formula.

[4] Akhiezer (1961).

Theorem 6.60 (Stieltjes–Perron) *Let $a < b$ and σ be positive and finite. Then*

$$\frac{\sigma(\{a\}) + \sigma(\{b\})}{2} + \sigma((a,b)) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \mathrm{Re} \int_a^b C^\sigma(\varepsilon + it) d\sigma(t) . \tag{6.120}$$

We refer to Akhiezer (1961) for the proof.

Corollary 6.59 extends the analogy first observed by Brouncker; see §**60** in Section 3.2. In this analogy $\mathfrak{P}(\mathbb{R})$ corresponds to the continuum of real numbers $\mathbb{R}$ and the asymptotic expansions (6.117) to their decimal expansions. It follows that simple formulas for the continued fractions corresponding to C^σ may exist. By (6.56),

$$C^\sigma(z) = \frac{s_0}{z}\left(1 + \frac{is_1/s_0}{z} + O\left(\frac{1}{z^2}\right)\right) = \frac{s_0}{z - is_1/s_0 + G(z)} ,$$

where $G(x) = O(1/x)$ if $x \to +\infty$. Expressing the above equation in terms of G, we find

$$\mathrm{Re}\, G(z) = -x + x \int \frac{d\sigma(t)}{|z - it|^2} \frac{s_0}{|C^\sigma(z)|^2} > 0 ,$$

since, by the Cauchy–Schwarz inequality

$$|C^\sigma(z)|^2 \leqslant \int d\sigma \int \frac{d\sigma(t)}{|z - it|^2} .$$

Since $G(x) = O(1/x)$, by Corollary 6.58 $G(z) = C^{\sigma_1}(z)$. similarly to the algorithm presented in §**13** at the start of Section 1.3, this algorithm allows one to find the corresponding P-fraction for every $\sigma \in \mathfrak{P}(\mathbb{T})$:

$$C^\sigma(z) = \frac{a_1}{z + ic_1} \,{}_{+}\, \frac{a_2}{z + ic_2} \,{}_{+ \cdots +}\, \frac{a_n}{z + ic_n} \,{}_{+ \cdots} , \tag{6.121}$$

where $a_n = s_0(\sigma_{n-1})$, $c_n = -s_1(\sigma_{n-1})/a_n$.

Following Stieltjes we assume now that σ is an even measure in $\mathfrak{P}(\mathbb{R})$. By Theorem 6.60 this will be the case if and only if $\overline{C^\sigma(z)} = C^\sigma(\bar{z})$. If σ is even then $s_1 = 0$ and therefore

$$C^\sigma(z) = \frac{s_0}{z + C^{\sigma_1}(z)} , \qquad \text{where } \overline{C^{\sigma_1}(z)} = C^{\sigma_1}(\bar{z}) .$$

Iterating this formula, we obtain that $c_1 = c_2 = \cdots = 0$ and therefore

$$C^\sigma(z) = \frac{a_1}{z} \,{}_{+}\, \frac{a_2}{z} \,{}_{+ \cdots +}\, \frac{a_n}{z + C^{\sigma_{n+1}}(z)} . \tag{6.122}$$

The continued fraction $\mathbf{K}_{n=1}^{\infty}(a_n/z)$ corresponds to the asymptotic expansion (6.56) of σ. By (6.118),

$$\int_0^{+\infty} \frac{2d\sigma(\sqrt{t})}{s + t} \sim \frac{a_1}{s} \,{}_{+}\, \frac{a_2}{1} \,{}_{+}\, \frac{a_3}{s} \,{}_{+}\, \frac{a_4}{1} \,{}_{+ \cdots} . \tag{6.123}$$

Definition 6.61 *The Hamburger moment problem* (6.116) *is called determined if there is only one* $\sigma \in \mathfrak{P}(\mathbb{R})$ *satisfying* (6.116). *Otherwise it is called undetermined. If* σ *is restricted to the class of even measures then the moment problem* (6.116) *is called the Stieltjes moment problem.*

In (1895) Stieltjes considered the following family of functions:

$$\int_0^{+\infty} \frac{(1+\lambda \sin \sqrt[4]{t})\ e^{-\sqrt[4]{t}}}{z+t}\, dt\,, \qquad -1 \leqslant \lambda \leqslant 1\,.$$

Since

$$\varphi(s) = \int_0^{+\infty} \sin t\, e^{-st} dt = \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) = \frac{1}{1+s^2}\,,$$

we have

$$\int_0^{+\infty} t^k \sin \sqrt[4]{t}\, e^{-\sqrt[4]{t}} dt = 4\int_0^{+\infty} t^{4k+3} \sin t\, e^{-t} dt = -\varphi^{(4k+3)}(1)\,.$$

Since $(1-i)(1+i)^{-1} = -i$, the derivative $\varphi^{(n)}(1)$,

$$\frac{(-1)^n 2i\varphi^{(n)}(1)}{(n+1)!} = \frac{1}{(1+i)^{n+1}} - \frac{1}{(1-i)^{n+1}} = \frac{(-i)^{n+1}-1}{(1-i)^{n+1}}$$

vanishes if and only if $n = 4k+3$. It follows that the Stieltjes moments (6.119) for these measures do not depend on λ. See Exs. 6.10 and 6.11 for other examples. Hence there are cases when an even more restricted Stieltjes moment problem is undetermined.

Theorem 6.62 *The Stieltjes moment problem* (6.116) *is determined if and only if the continued fraction* $\mathbf{K}_{n=1}^{\infty}(a_n/x)$ *converges for* $x > 0$.

Proof Let P_n/Q_n be the convergents to $\mathbf{K}_{n=1}^{\infty}(a_n/z)$. Then by (6.122)

$$\frac{P_{2n}(x)}{Q_{2n}(x)} < C^{\sigma}(x) < \frac{P_{2n+1}(x)}{Q_{2n+1}(x)}\,, \qquad x > 0\,.$$

Since by Brouncker's theorem 1.7 the even convergents increase and the odd convergents decrease, the moment problem (6.116) has a unique solution σ if the continued fraction $\mathbf{K}_{n=1}^{\infty}(a_n/z)$ converges.

If on the contrary it diverges then we consider the two functions

$$F_{\text{even}}(z) = \lim_n \frac{P_{2n}(z)}{Q_{2n}(z)}\,, \qquad F_{\text{odd}}(z) = \lim_n \frac{P_{2n+1}(z)}{Q_{2n+1}(z)}\,.$$

Both limits exist for $z = x > 0$. Since the family of convergents is normal in $\mathbb{P}_+$, we obtain that F_{even} and F_{odd} are both in $\mathfrak{R}(\mathbb{P}_+)$ by the complex Markoff test, Corollary 6.49. On the one hand, by (6.122) both F_{even} and F_{odd} correspond to the same continued fractions. Hence they have equal asymptotic series. On the other hand, by Corollary 6.59 they determine two different measures σ_{even} and σ_{odd} with equal moments. □

Notice that the method presented is very close to the one we used in proving Euler's formulas with Markoff's test. Corollary 3.9 states the conditions in terms of the $\{a_n\}_{n\geqslant 0}$ that are necessary and sufficient for the moment problem to be determined. See Ex. 3.5 for examples of nondetermined moment problems. A convenient sufficient condition due to Carleman will be considered in §**154**, Section 7.4. We refer to Shohat and Tamarkin (1943) for the other results.

Exercises

6.1 Prove that (Euler (1813, Chapter V)

$$\frac{2}{1}{}_{+}\frac{3}{2}{}_{+}\frac{4}{3}{}_{+}\frac{5}{4}{}_{+}\frac{6}{5}{}_{+}\frac{7}{6}{}_{+\cdots} = 1\,.$$

Hint: Put $s=2$ in (6.91).

6.2 Prove that

$${}_1F_1\left({a+1 \atop c+1};z\right) = {}_1F_1\left({a \atop c};z\right) + \frac{(c-a)z}{c(c+1)}\,{}_1F_1\left({a+1 \atop c+2};z\right)\,.$$

Hint: Apply the definition (6.88) of ${}_1F_1$ to the left-hand side and use the obvious identities

$$(c+n+1)(c)_{n+1} = c(c+1)(c+2)_n\,, \qquad (a)_{n+1} = a(a+1)_n\,.$$

6.3 Prove that

$${}_1F_1\left({a \atop c+1};z\right) = {}_1F_1\left({a \atop c};z\right) - \frac{az}{c(c+1)}\,{}_1F_1\left({a+1 \atop c+2};z\right)\,.$$

6.4 Theorems 6.43 and 6.44 with Markoff's test imply that for $z\geqslant 0$

$$\begin{aligned}
&\frac{\int_0^1 t^b(1-t)^{c-b-1}\,(1+zt)^{-a}dt}{\int_0^1 t^{b-1}(1-t)^{c-b-1}\,(1+zt)^{-a}dt} \\
&\quad = \frac{b}{c}{}_{+}\frac{a(c-b)z}{c+1}{}_{+}\frac{(b+1)(c-a+1)z}{c+2}{}_{+\cdots} \\
&\quad\quad {}_{+}\frac{(a+n)(c-b+n)z}{c+2n+1}{}_{+}\frac{(b+n+1)(c-a+n+1)z}{c+2n+2}{}_{+\cdots}\,.
\end{aligned} \tag{E6.1}$$

6.5 Prove that

$$b(s) = \frac{(s+1)^2}{s+2\times 1}{}_{+}\frac{1^2}{s+2\times 2}{}_{+}\frac{(s+3)^2}{s+2\times 3}{}_{+}\frac{3^2}{s+2\times 4}{}_{+}\frac{(s+5)^2}{s+2\times 5}{}_{+\cdots}\,. \tag{E6.2}$$

Hint: Apply (E6.1) to Euler's formula (4.46). Put $z=1$, $b=(s+1)/2$, $c-b=1/2$, $a=1/2$ in (E6.1).

6.6 For $s = 1$ (E6.2) turns into a nice formula for π:

$$\pi = 3 + \frac{1^2}{5} {}_{+} \frac{4^2}{7} {}_{+} \frac{3^2}{9} {}_{+} \frac{6^2}{11} {}_{+} \frac{5^2}{13} {}_{+} \frac{8^2}{15} {}_{+\cdots} . \tag{E6.3}$$

Compare (E6.3) with (4.63) and notice that they differ, starting with the second partial numerator, by the permutations $4 \leftrightarrow 3$, $6 \leftrightarrow 5$, $7 \leftrightarrow 8$, ...

6.7 Show that Euler's $K(s)$ (1813), see (6.89), satisfies

$$K(s) = \frac{s}{2} {}_{+} \frac{s-2}{3} {}_{+} \frac{s+1}{4} {}_{+} \frac{s-3}{5} {}_{+} \frac{s+2}{6} {}_{+} \frac{s-4}{7} {}_{+} \frac{s+3}{8} {}_{+\cdots} \tag{E6.4}$$

and in particular that

$$K(1) = (e-1)^{-1}, \quad K(2) = 1, \quad K(3) = 4/3, \quad K(4) = 136/73, \ \ldots$$

as Euler indicated.

Hint: Put $a = s$, $c = 2$, $z = 1$ in Theorem 6.42. This is an interesting example of a function given by a simple formula (6.89) which takes rational values in integers $n \geqslant 2$ and a transcendental value at 1.

6.8 Prove that any family of analytic functions uniformly bounded in an open connected set G is pre-compact.

Hint: Prove this first for $G = \mathbb{D}$. Match each function of the family with its Taylor series centered at $z = 0$. Apply Cauchy's formula to prove that the coefficients of these series are all uniformly bounded. Apply the diagonal process to obtain a subsequence of functions with converging Taylor coefficients at any power of z (for details see Markushevich 1985, Part I, Chapter 17, §86).

6.9 Prove that $\mathbb{C}(z, \sqrt{z^2-1})$ contains nonperiodic quadratic P-fractions.

Hint: If $-1 < \alpha < 1$ then by Cauchy's formula

$$\frac{\sin \pi\alpha}{2\pi\alpha} \int_{-1}^{1} \frac{1}{z-x} \left(\frac{1+x}{1-x}\right)^{\alpha} dx = \frac{1}{2\alpha} \left\{ \left(\frac{z+1}{z-1}\right)^{\alpha} - 1 \right\}, \tag{E6.5}$$

implying by (6.106)

$$\left(\frac{z+1}{z-1}\right)^{\alpha} = 1 + \frac{2\alpha}{z-\alpha} {}_{+} \frac{\alpha^2-1}{3z} {}_{+} \frac{\alpha^2-4}{5z} {}_{+} \frac{\alpha^2-9}{7z} {}_{+\cdots} . \tag{E6.6}$$

If $\alpha = 1/2$ then (E6.6) shows that the P-fraction corresponding to the Jacobi polynomials $\{P_n^{(1/2,-1/2)}\}_{n \geqslant 0}$ is not periodic. Notice that by (E6.5) this Cauchy integral belongs to $\mathbb{C}(z, \sqrt{z^2-1})$.

6.10 (Stieltjes) If f is an odd function such that $f(x+1/2) = \pm f(x)$ then

$$\int_0^{+\infty} x^k x^{-\log x} f(\log x)\, dx = 0, \qquad k \in \mathbb{Z}.$$

Hint: Since f is odd, $\int_{\mathbb{R}} e^{-t^2} f(t) dt = 0$. Put $t = \log x - (k+1)/2$.

6.11 (Stieltjes) Show that the integrals

$$\frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{1 + \lambda \sin(2\pi \log t)}{z + t}\, t^{-\log t}\, dt$$

have the same asymptotic expansion as $z = x \to +\infty$, $\lambda \in [-1, 1]$. Find the asymptotic expansion and the corresponding continued fraction.

7
Orthogonal polynomials

7.1 Euler's problem

142 Orthogonal matrices. A square matrix

$$\begin{Vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{Vmatrix}$$

is called *orthogonal* if its entries a_{ij} satisfy

$$\sum_{j=1}^{n} a_{kj} a_{lj} = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l\,. \end{cases} \tag{7.1}$$

Orthogonal 2×2 matrices are parameterized by independent real parameters a and b as

$$\frac{1}{a^2+b^2} \begin{Vmatrix} a & b \\ b & -a \end{Vmatrix} . \tag{7.2}$$

The entries in this parameterization are rational functions in a and b. Euler (1771) found a rational parameterization for orthogonal 3×3 matrices:

$$\begin{Vmatrix} D^2+A^2-B^2-C^2 & 2(AB-CD) & 2(AC+BD) \\ 2(AB+CD) & D^2-A^2+B^2-C^2 & 2(BC-AD) \\ 2(AC-BD) & 2(BC+AD) & D^2-A^2-B^2+C^2 \end{Vmatrix}$$

The sum of the squares of the columns and rows equals $(D^2+A^2+B^2+C^2)^2$, which shows that to obtain the required parametrization one should divide the entries of the matrix by $D^2+A^2+B^2+C^2$.

Euler also found similar formulas for 4×4 matrices:

$$\begin{Vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{Vmatrix},$$

where

$$\begin{aligned}
A_1 &= ap + bq + cr + ds\ , & A_2 &= ar - bs - cp + dq\ ,\\
B_1 &= -aq + bp + cs - dr\ , & B_2 &= as + br + cq + dp\ ,\\
C_1 &= ar + bs - cp - dq\ , & C_2 &= -ap + bq - cr + ds\ ,\\
D_1 &= -as + br - cq + dp\ , & D_2 &= -aq - bp + cs + dr\ ,\\
A_3 &= -as - br + cq + dp\ , & A_4 &= aq - bp + cs - dr\ ,\\
B_3 &= ar - bs + cp - dq\ , & B_4 &= ap + bq - cr - ds\ ,\\
C_3 &= aq + bp + cs + dr\ , & C_4 &= as - br - cq + dp\ ,\\
D_3 &= -ap + bq + cr - ds\ , & D_4 &= ar + bs + cp + dq\ ,
\end{aligned}$$

As an application Euler presented the square matrix

$$\begin{Vmatrix} +68 & -29 & +41 & -37 \\ -17 & +31 & +79 & +32 \\ +59 & +28 & -23 & +61 \\ -11 & -77 & +8 & +49 \end{Vmatrix}$$

having orthogonal columns and rows. The sum of the squares of the rows and columns equals 8515. The sums of the squares of the corners of the large and central interior squares each have this same value:

$$68^2 + 37^2 + 49^2 + 11^2 = 31^2 + 79^2 + 23^2 + 28^2 = 8515 = 5 \times 1703.\text{[1]}$$

In the conclusion of his paper (1771, §36), Euler writes:

> Solutio haec eo maiorem attentionem meretur, quod ad eam nulla certa methodo, sed potius quasi divinando sum perductus; et quoniam ea adeo octo numeros arbitrarious implicat, qui quidem facta reductione ad unitatem, ad septem rediguntur, vix dubitare licet, quin ista solutio sit universalis et omnes prorsus solutiones possibiles in se complectatur. Si quis ergo viam directam ad hanc solutionem manuducentem investigaverit, insignia certe subsidia Analysi attulisse erit censendus. Utrum autem similes solutiones pro amplioribus quadratis, quae numeris 25, 36, et majoribus constant, expectare liceat, vix affirmare ausim. Non solum autem hinc Algebra communis, sed etiam Methodus Diophantea maxima incrementa adeptura videtur.

[1] 1703 is the year in which St Petersburg was founded.

This translates into English as

> This solution deserves to be paid full attention, as I arrived at it not using a definite method but rather by making guesses; and since in addition it depends on eight arbitrary parameters, which after normalization can be reduced to seven, one can hardly doubt that it is universal and includes all possible cases. If somebody can find a direct way to this solution, then it will be have to admitted that he has made an outstanding contribution to analysis. Whether similar solutions exist for wider squares consisting of 25, 36, etc. numbers, I hardly dare to claim. Here it seems that not only algebra but the Diophantine method would benefit from a contribution of great significance.

D. Grave (1937, 1938), using quaternions, explained Euler's approach for the case $n = 4$. Euler's problem can be stated as follows.

Problem 7.1 *Find rational parameterizations of the manifold of real orthogonal matrices with independent parameters.*

A solution to Euler's problem can be found in Vilenkin (1991, Chapter IX, §1]. Let $g_{jk}(\alpha)$ be a rotation by angle α in a plane (x_j, x_k); a rotation moving the vector $(1, 0)$ to vector $(0, 1)$ is positive. Let $g_{k+1k}(\alpha) = g_k(\alpha)$ for brevity.

Theorem 7.2 *Any rotation g of $\mathbb{R}^n$ can be represented as the product of rotations $g = g^{(n-1)}, \ldots, g^{(1)}$, where $g^{(k)} = g_1(\theta_1^k) \cdots g_k(\theta_k^k)$.*

Rotations of $\mathbb{R}^n$ correspond to orthogonal $n \times n$ matrices. Since any rotation of a plane can be rationally parameterized, see (7.2), their products can be rationally parameterized too.

In 1855 Chebyshev discovered a parameterization which gave rise to the theory of orthogonal polynomials. Chebyshev's solution followed important contributions by Gauss, Jacobi, Legendre and Sturm.

7.2 Quadrature formulas

143 Newton–Cotes formulas. In 1671 Newton discovered the three-eighths rule for definite integrals of continuous functions:

$$\int_{x_1}^{x_4} f(x)\,dx \approx \frac{3h}{8}\left(f(x_1) + 3f(x_2) + 3f(x_3) + f(x_4)\right) . \tag{7.3}$$

Here $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = h > 0$. Formula (7.3) gives an approximate value of the definite integral (that is, a quadrature) and therefore is called a quadrature formula. Since Cotes extended Newton's formula to a greater number of nodes, these formulas are often called the Newton–Cotes formulas.

Newton's quadrature formula has a remarkable property: it turns into an equality for any polynomial f, $\deg f \leqslant 4 - 1 = 3$. To check this evaluate both sides for $f(x) = 1$,

x, x^2, x^3 and then extend the formula by linearity. This property of Newton's formula follows from the formula for Lagrange interpolation polynomials,

$$L(x) = \sum_{k=1}^{n} y_k \frac{Q(x)}{(x-x_k)Q'(x_k)} , \qquad Q(x) = (x-x_1) \cdots (x-x_n) . \tag{7.4}$$

It is clear that $L(x_k) = y_k$, $k = 1, 2, \ldots, n$. Fixing a choice of nodes $\{x_k\}$ and integrating (7.4) we obtain the Newton–Cotes quadrature

$$\int_{x_1}^{x_n} f(x)\, dx \approx \sum_{k=1}^{n} l_k f(x_k) , \tag{7.5}$$

which is a true equality on polynomials of degree $n-1$. The Lagrange coefficients l_k in (7.5) are defined by

$$l_k = \int_{x_1}^{x_n} \frac{Q(x)}{(x-x_k)Q'(x_k)}\, dx . \tag{7.6}$$

144 Gaussian quadrature. Since Newton–Cotes quadratures depend on n nodes, a proper choice of the nodes $\{x_k\}$ may give a formula which is a true equality on any polynomial of degree $n-1+n = 2n-1$. This problem was studied in Gauss (1814). We have

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{k=1}^{n} l_k f(x_k)$$

for nodes $-1 \leqslant x_1 < x_2 < \cdots < x_n \leqslant 1$ (in fact Gauss considered the interval $(0, 1)$, which is obtained from $(-1, 1)$ by a linear transformation). A quadrature formula is an equality for any polynomial of degree $2n-1$ if and only if it is an equality for any monomial $f(x) = x^m$, $m = 0, 1, \ldots, 2n-1$. It follows that for these m values

$$\delta_m \stackrel{\text{def}}{=} \int_{-1}^{1} x^m\, dx - \sum_{k=1}^{n} l_k x_k^m = \frac{1-(-1)^{m+1}}{m+1} - \sum_{k=1}^{n} l_k x_k^m = 0 . \tag{7.7}$$

Since the l_k are defined by (7.6), $\delta_m = 0$ for $m = 0, \ldots, n-1$. Notice also that $\delta_m = O(1)$ as $m \to \infty$. For $|z| > 1$, Gauss considered a convergent Laurent series

$$G(z) = \sum_{m=0}^{\infty} \frac{1-(-1)^{m+1}}{(m+1)z^{m+1}} = \sum_{k=1}^{n} l_k \sum_{m=0}^{\infty} \frac{x_k^m}{z^{m+1}} + \sum_{m=n}^{\infty} \frac{\delta_m}{z^{m+1}}$$

and multiplied it by Q to obtain the formula

$$\begin{aligned} Q(z)G(z) &= \sum_{k=1}^{n} \frac{l_k Q(z)}{z-x_k} + \sum_{m=n}^{\infty} \frac{\delta_m Q(z)}{z^{m+1}} \\ &= P(z) + \frac{(\delta_n z^{n-1} + \cdots + \delta_{2n-1})Q(z)}{z^{2n}} + O\left(\frac{1}{z^{n+1}}\right) , \end{aligned} \tag{7.8}$$

where P is a polynomial. Notice that $\delta_n = \cdots = \delta_{2n-1} = 0$ in (7.8) if and only if $\deg(QG-P) \leqslant -n-1$, where $\deg Q = n$. Then by Markoff's theorem P/Q is the nth convergent to the continued fraction

$$\ln\frac{z+1}{z-1} = \frac{2}{z}\,\underset{-}{}\,\frac{1^2}{3z}\,\underset{-}{}\,\frac{2^2}{5z}\,\underset{-}{}\,\frac{3^2}{7z}\,\underset{-}{}\,\frac{4^2}{9z}\,\underset{-}{}\cdots\ ,$$

see (6.101), of

$$\ln\frac{z+1}{z-1} = \sum_{m=0}^{\infty}\frac{1-(-1)^{m+1}}{(m+1)z^{m+1}} = G(z)\ ,$$

as elementary calculations with power series show. It follows that Q is one of the polynomials defined by the Euler–Wallis formulas

$$Q_{n+1}(x) = (2n+1)xQ_n(x) - n^2Q_{n-1}(x)\,, \qquad Q_0(x) = 1\,, \qquad Q_1(x) = x\,. \tag{7.9}$$

To complete Gauss's construction we must check that all the zeros of Q_n are simple and located in $(-1,1)$. Recall that by the intermediate value theorem, see for instance Hairer and Wanner (1996, Theorem 3.5), every polynomial taking values of opposite sign at $a<b$ has at least one zero in (a,b). Both Newton and Gauss considered this statement for polynomials as obvious.

Theorem 7.3 *The zeros $x_{n,1} < x_{n-1,1} < \cdots < x_{n,n}$ of Q_n are located in* $(-1,1)$ *and interlace the zeros of Q_{n-1}.*

Proof By inspection the statement is true for $Q_0 = 1$, $Q_1 = x$, $Q_2 = 3x^2-1$ and $Q_3 = 3x(5x^2-3)$. Easy induction with (7.9) shows that $Q_n(x)$ is even if n is even and is odd if n is odd, $\deg Q_n(x) = n$ and the leading coefficient of $Q_n(x)$ is $(2n-1)!!$. The identity

$$(2n+1)n! - n^2(n-1)! = (n+1)!$$

and (7.9) imply by induction that $Q_n(1) = n! > 0$. Now the proof is completed by induction and splits into the following two cases.

Case 1: *n is even* We assume that Q_k has k simple zeros for $k \leqslant n$ and that the zeros of Q_{n-1} alternate those of Q_n:

$$-1 \overset{+}{<} x_{n\,1} \overset{-}{<} x_{n-1\,1} \overset{-}{<} x_{n\,2} \overset{+}{<} x_{n-1\,2} \overset{+}{<} \ldots \overset{-}{<} x_{n-1\,n-1} \overset{-}{<} x_{n\,n} \overset{+}{<} 1\,.$$

Here $\pm$ indicate the signs of Q_n on the corresponding intervals. Since the zeros of Q_{n-1} alternate the zeros of Q_n, the sequence $Q_{n-1}(x_{n\,k})$ is alternating, which by (7.9) implies that the sequence $Q_{n+1}(x_{n\,k})$ is alternating too. Since $Q_{n-1}(1) = (n-1)!$, the polynomial Q_{n-1} is positive on $(x_{n-1\,n-1}, x_{n\,n})$ and therefore $Q_{n+1}(x_{n\,n}) < 0$. Since n is even and $Q_{n+1}(x_{n\,k})$ alternates, we obtain that $Q_{n+1}(x_{n\,1}) > 0$.

Since $Q_{n+1}(x_{nn}) < 0$ and $Q_{n+1}(1) = (n+1)! > 0$, the polynomial Q_{n+1} must vanish in $(x_{nn}, 1)$ at least once by Bolzano Theorem.

Since $Q_{n+1}(-1) = -(n+1)! < 0$ and $Q_{n+1}(x_{n1}) > 0$, the polynomial Q_{n+1} must vanish in $(-1, x_{n1})$.

Every open interval $(x_{nk}, x_{n(k+1)})$ contains at least one zero of Q_{n+1}, since the values of $Q_{n+1}(x_{nk})$ alternate.

The total number of intervals is $2+(n-1) = n+1$, the degree of Q_{n+1} is $n+1$. Hence all zeros of Q_{n+1} are simple and alternate those of Q_n.

Case 2: *n is odd* The only difference here is that $Q_{n+1}(x_{n1}) < 0$, which implies that Q_{n+1} has at least one zero in $(-1, x_{n1})$, since $Q_{n+1}(-1) = Q_{n+1}(1) = (n+1)! > 0$. □

Thus we obtain the following theorem.

Theorem 7.4 (Gauss 1814) *For every positive integer n there are n nodes* $-1 < x_1 < \cdots < x_n < 1$ *such that*

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{n} l_k f(x_k) \tag{7.10}$$

for every polynomial f, $\deg f \leqslant 2n-1$. *The nodes x_k are the zeros of the denominator of the nth convergent to the continued fraction* (6.101).

Notice that since $\deg Q_n = n$, the approximation of $G(z)$ by the convergent of nth order cannot give order $2n$ at infinity, which implies that $2n-1$ is the highest possible degree in Gauss's quadrature.

145 Jacobi's contribution. Jacobi (1826) observed that the substitution of $f(x) = x^k Q_n(x)$ in (7.10) with $0 \leqslant k < n$ gives

$$\int_{-1}^{1} Q_n\,dx = \int_{-1}^{1} xQ_n\,dx = \cdots = \int_{-1}^{1} x^{n-1}Q_n\,dx = 0\,, \tag{7.11}$$

since $\deg(x^k Q_n(x)) \leqslant 2n-1$ and $Q_n(x_k) = 0$. However Legendre (1785) introduced orthogonal polynomials $P_n(x)$:

$$\int_{1}^{1} P_n(x)P_m(x)\,dx = \frac{1}{2n+1}\delta_{nm}\,, \qquad \delta_{nm} \overset{def}{=} \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m\,, \end{cases} \tag{7.12}$$

which are called now the Legendre polynomials. A glance at (7.11) and (7.12) reveals that the Q_n in (7.11) are constant multiples of the Legendre polynomials. Integration by parts,

$$\int_{-1}^{1} uv'\,dx = uv\Big|_{-1}^{1} - \int_{-1}^{1} vu'\,dx$$

shows that the polynomial

$$v'(x) = \frac{d^n}{dx^n}(x^2-1)^n$$

of degree n satisfies (7.11) since it has zeros of order n at $x=-1$ and $x=1$. Taking into account the formulas for the leading coefficients of the polynomials, we obtain that

$$Q_n(x) = \frac{(2n-1)!!\; n!}{(2n)!}\frac{d^n}{dx^n}(x^2-1)^n .$$

Applying Rolle's theorem iteratively, we obtain again that all the roots of Q_n are located in $(-1,1)$.

In about 1843 Jacobi returned to these results and extended them to the more general Jacobi's weights. Jacobi's paper (1859) was published by Heine after Jacobi's death. In this paper Jacobi observed that the continued fraction (6.101), which played a crucial role in Gauss's quadratures, is a special case of Gauss's continued fraction (6.96). An important ingredient of Gauss's proof was the fact that the continued fraction (6.101) represents in $\mathbb{C}([1/z])$ a Cauchy-type integral having a constant positive weight on $[-1,1]$. Comparing Euler's formula (6.95) with Gauss's continued fraction (6.96) and keeping in mind the symmetry of ${}_2F_1$ in a and b, see (6.92), we easily obtain a continued fraction for Cauchy integrals with some special weights on $[0,1]$. More precisely, let $\alpha>0$, $\beta>0$. Then by Theorem 6.44

$$\begin{aligned} {}_2F_1(\alpha,1;\alpha+\beta;z) &= {}_2F_1(1,\alpha;\alpha+\beta;z) \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{1-zt}\,dt . \end{aligned}$$

Hence by Theorem 6.45

$$\begin{aligned} &\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{z-t}\,dt = {}_2F_1\left(\alpha,1;\alpha+\beta;Z^{-1}\right)Z^{-1} \\ &= \frac{1}{z}\,\underset{-}{}\,\frac{\alpha}{\alpha+\beta}\,\underset{-}{}\,\frac{1\beta}{(\alpha+\beta+1)z}\,\underset{-}{}\,\frac{(\alpha+1)(\alpha+\beta)}{\alpha+\beta+2}\,\underset{-}{}\,\frac{2(\beta+1)}{(\alpha+\beta+3)z}\,\underset{-}{}\cdots \\ &\quad \underset{-}{}\,\frac{(\alpha+n)(\alpha+\beta+n-1)}{\alpha+\beta+2n}\,\underset{-}{}\,\frac{(n+1)(\beta+n)}{(\alpha+\beta+2n+1)z}\,\underset{-}{}\cdots . \end{aligned}$$

This formula implies quadrature formulas analogous to Gauss's quadrature formula. It also can be used to obtain the recurrence relation for Jacobi orthogonal polynomials; see Szegő (1975, Chapter IV, Section 3.4) for details.

It is interesting that Jacobi knew the paper Euler (1771), which motivated Chebyshev in his discovery of general orthogonal polynomials. Moreover, Jacobi even wrote a manuscript (1884), in which he explained Euler's ideas. This paper, like that of 1859, was published only after Jacobi's death.

7.3 Sturm's method

146 Roots of real polynomials. Two theorems on the zeros of polynomials preceded Sturm's theorem.

Theorem 7.5 (Descartes) *The number of positive roots of a polynomial equals the number of sign variations in its coefficients minus a nonnegative even number.*

By Taylor's formula the coefficient of a polynomial f, $\deg f = n$, at x^k is $f^{(k)}(0)/k!$. Hence in Descartes' theorem, for a polynomial f with $\deg f = n$, one in fact counts the variation in sign in the sequence

$$f(x),\ f^{(1)}(x),\ f^{(2)}(x),\ \ldots,\ f^{(n)}(x)\,, \tag{7.13}$$

for $x = 0$. Since $f^{(k)}(x) \sim n(n-1)\cdots(n-k+1)f^{(n)}(0)x^{n-k}/n!$ as $x \to +\infty$, the number of sign variations $V(x)$ in (7.13) for sufficiently large x is zero. A sequence such as (7.13) is an example of a Budan sequence, since in 1803 Budan proved the following theorem, generalizing Descartes' theorem.

Theorem 7.6 (Budan) *Let f be a polynomial such that $f(a)f(b) \neq 0$ for $a < b$. Then the number of roots of f in (a,b) counting multiplicities is less than $V(a) - V(b)$ by a nonnegative even integer.*

Proof We follow Chebyshev (1856–7, §15) and Grave (1938, §109). Let $c \in (a,b)$ be a zero of f of order m. Then $f(x) = (x-c)^m g(x)$, where $g(c) \neq 0$. Differentiating this formula successively, we see that the sequence

$$f(x),\ f^{(1)}(x),\ \ldots,\ f^{(m)}(x)$$

has m sign changes if $x \to c^-$ and has zero sign changes if $x \to c^+$. Hence when x moves through c to b the sign changes decrease by exactly m. This is true for any zero of f, and it is also true for any zero of any derivative of f. Hence $V(a) - V(b)$ must be greater than or equal to the total number of zeros of f in (a,b) by a nonnegative integer $n(a,b)$.

To prove that $n(a,b)$ is even, we investigate what happens when x goes through c, $f^{(l)}(c) = 0$, $f(c) \neq 0$. An important difference compared with the above case is that in the series of derivatives the first and last terms do not vanish at c. As above the series alternates for $x \to c^-$ and does not alternate for $x \to c^+$ at terms that are zero. The signs are controlled by the sign of the last term, which we may assume positive for definiteness. Two possible cases are as follows. In each of (a) and (b) the top line gives the signs of the derivatives as $x \to c^-$, the second line gives their signs at $x = c$ and the third line gives their signs as $x \to c^+$.

$$\begin{array}{cccccccc} & + & - & + & - & + & - & + \\ \text{(a)} & + & 0 & 0 & 0 & 0 & 0 & + \\ & + & + & + & + & + & + & + \end{array} \qquad \begin{array}{cccccccc} & - & - & + & - & + & - & + \\ \text{(b)} & - & 0 & 0 & 0 & 0 & 0 & + \\ & - & + & + & + & + & + & + \end{array}$$

In (a) the first term has the same sign as the last and counting from the right to the left shows that every minus contributes two sign changes, an even number. In (b) the first term has the opposite sign to the last. The same calculation as for (a) gives an odd number of sign changes in the top line and one sign change in the bottom line. Hence again the number of sign changes is even. □

147 Sturm's theorem. The proof of Theorem 7.3 can easily be extended to cover a more general case.

Theorem 7.7 *Let $\{b_n(x)\}_{n\geqslant 1}$ be any sequence of real linear polynomials with positive coefficients at x and let $\{a_n\}_{n\geqslant 1}$ be any nonzero real sequence. Then the zeros of the denominators Q_n of any convergent to*

$$\frac{a_1^2}{b_1(x)} - \frac{a_2^2}{b_2(x)} - \frac{a_3^2}{b_3(x)} - \cdots - \frac{a_n^2}{b_n(x)} - \cdots \tag{7.14}$$

are all real and interlace the zeros of Q_{n-1}.

Proof The difference from Theorem 7.3 is that in this general case it is not necessary to prove that all zeros lie in $[-1, 1]$. Since $Q_n(x) \sim cx^n$, $c > 0$, if $x \to \infty$, the location of $x_{n,1}$ and $x_{n,n}$ may vary. □

The properties of the polynomials Q_n used in Theorems 7.3 and 7.7 are conveniently summarized in the following theorem.

Theorem 7.8 *Let Q_n be the denominator of the nth convergent to the continued fraction* (7.14). *Then the sequence*

$$f_0(x) = Q_n(x)\,, \qquad f_1(x) = Q_{n-1}(x)\,, \qquad \ldots\,, \qquad f_n(x) = Q_0(x) \tag{7.15}$$

satisfies the following properties:

(a) *$f_0(x)f_1(x)$ changes sign from $-$ to $+$ if x passes any zero of $f_0(x)$ in the positive direction;*
(b) *no polynomials $f_k(x)$, $f_{k+1}(x)$, $k = 0, 1, \ldots, m-1$, may have common zeros;*
(c) *if $f_k(\alpha) = 0$, $1 \leqslant k \leqslant m-1$, then $f_{k-1}(\alpha)f_{k+1}(\alpha) < 0$;*
(d) *$f_n(x)$ has no real zeros.*

Proof Applying the determinant identity (1.16) to the continued fraction (7.7), we obtain that

$$P_k Q_{k-1} - P_{k-1} Q_k = a_1^2 a_2^2 \cdots a_k^2\,,$$

which implies (b). The Euler–Wallis formula $Q_{k+1} = b_{k+1}Q_k - Q_{k-1}$ implies (c). Since $Q_0 = 1$, we have (d). Let us prove (a). Since the zeros $\{x_{n-1,k}\}$ of Q_{n-1} alternate with the zeros $\{x_{n,k}\}$ of Q_n, we have $x_{n-1,k} \in [x_{n\,1}, x_{n\,n}]$. If n is even then $Q_n(x) > 0$, $Q_{n-1}(x) < 0$ for $x < x_{n\,1}$, which proves (a) since the signs alternate. The case of odd n is considered similarly. □

Definition 7.9 *A sequence of nonzero polynomials*

$$f(x) = f_0(x), \quad f_1(x), \quad \ldots, \quad f_m(x) \tag{7.16}$$

in $\mathbb{R}[X]$ *is called a Sturm sequence for a polynomial* $f(x)$ *on* $[a, b]$, $a < b$ *if conditions* (a)–(d) *of Theorem* 7.8 *are valid on* $[a, b]$.

There is a striking similarity between Sturm and Budan sequences, see (7.16) and (7.13) respectively. Gauss's method of continued fractions, see Theorem 7.3, gives a better control of the zeros of polynomials for Sturm sequences than for Budan sequences (7.13), where the zeros of higher derivatives, not counted in the proof of Budan's theorem, increase the number of sign changes. Therefore Budan's theorem can be improved by using Sturm sequences. However, the price to be paid is that the zeros have to be assumed simple. Given $x \in [a, b]$ let $W_S(x)$ be the number of sign changes in (7.16).

Theorem 7.10 (Sturm 1829) *Let* $f \in \mathbb{R}[x]$ *be a separable polynomial with simple roots, let* $f(a)f(b) \neq 0$ *for* $a < b$ *and let (7.16) be the Sturm sequence for* $f(x)$. *Then the number of roots of* $f(x)$ *on* (a, b) *is* $W_S(a) - W_S(b)$.

Proof A sign change in (7.16) may occur only at zeros of the f_j. No sign changes occur in f_m, since by (d) it has no zeros on $[a, b]$. If $0 < j < m$ then by (c) the number of the sign changes in $(f_{j-1}(x), f_j(x), f_{j+1}(x))$ is invariant when x passes through any zero of f_j. If x passes through a zero of f_0 then the product $f_0 f_1$ changes sign from $-$ to $+$, which implies that $W_S(x)$ decreases by unity. □

The case of multiple zeros can be treated using the Euclidean algorithm (6.1) for polynomials. If $f_0 = f(X) = c_0 X^m + \cdots + c_m$ and $f_1 = f'(X) = mc_0 X^{m-1} + (m-1)c_1 X^{m-2} + \cdots + c_1$, then let f_n be the greatest common divisor of f and f' which includes all multiple roots. Therefore f/f_n is a separable polynomial.

The second important ingredient of Sturm's method is the choice of Sturm sequence. Sturm (1829) proposed to take the first two polynomials f and f' from Budan's sequence (7.13) and then apply the method of continued fractions. The idea was to calculate the continued fraction of f'/f in the form of Euler (see (4.106)) and Gauss,

$$\frac{f'}{f} = \frac{1}{b_0} - \frac{1}{b_1} - \frac{1}{b_2} - \cdots \ . \tag{7.17}$$

Namely, let $a_1 = a_2 = \ldots = -1$ in (1.11). Then

$$f_0 = b_0 f_1 - f_2\,, \qquad f_1 = b_1 f_2 - f_3, \qquad \ldots, \qquad f_{k-2} = b_{k-2} f_{k-1} - f_k\,, \tag{7.18}$$

where $f_0 = f$ and $f_1 = f'$. Since f is separable, the greatest common divisor of f and f' must be a nonzero constant polynomial f_k.

Inspection of the system (7.18) shows that no pair of polynomials f_j, f_{j+1} may vanish simultaneously, since otherwise the constant polynomial f_k would have a zero at this point as well.

From the equation $f_{j-1} = b_{j-1}f_j - f_{j+1}$ we see that f_{j+1} and f_{j-1} have opposite signs at any zero of f_j.

Finally, $f(x)f'(x)$ changes sign from $-$ to $+$ when x passes through any zero a of f. Indeed, if $f(a) = 0$ then either $f'(a) > 0$ or $f'(a) < 0$. If $f'(a) > 0$ then the graph of f near a is increasing from a negative value $f(a-\varepsilon)$ to a positive value $f(a+\varepsilon)$. This makes ff' negative to the left from a and positive to the right. The case $f'(a) < 0$ is considered similarly.

It follows that the sequence of polynomials

$$f(x), \quad f'(x), \quad f_2(x), \quad \ldots, \quad f_k(x)$$

is a Sturm sequence. Theorem 7.10 is usually applied to such a sequence.

148 Newton's example. We illustrate Sturm's method on Newton's example (1671), $x^3 - 2x - 5 = 0$. The algorithm for the long division of polynomials yields

$$f_0(x) = x^3 - 2x - 5, \qquad f_1(x) = 3x^2 - 2,$$

$$f_2(x) = \frac{4}{3}x + 5, \qquad f_3(x) = -\frac{643}{16} = -40.1875\,.$$

The evaluations of $(f_0(x), f_1(x), f_2(x), f_3(x))$ at $x = -\infty,\ 0,\ 1,\ 2,\ 3,\ +\infty$ are collected below.

$$\begin{array}{lcll}
-\infty & \longrightarrow & (-\infty, +\infty, -\infty, -40,1875)\,, & W_S(-\infty) = 2\,,\\
0 & \longrightarrow & (-5, -2, 5, -40,1875)\,, & W_S(0) = 2\,,\\
1 & \longrightarrow & (-6, 1, 6\tfrac{1}{3}, -40,1875)\,, & W_S(1) = 2\,,\\
2 & \longrightarrow & (-1, 10, 7\tfrac{2}{3}, -40,1875)\,, & W_S(2) = 2\,,\\
3 & \longrightarrow & (16, 25, 9, -40,1875)\,, & W_S(3) = 1,\\
+\infty & \longrightarrow & (+\infty, +\infty, +\infty, -40,1875)\,, & W_S(+\infty) = 1\,.
\end{array}$$

It can be seen that Newton's equation has only one real root in $(2,3)$:

$$W_S(2) - W_S(3) = 2 - 1 = 1\,.$$

Knowing that $(2,3)$ contains one root of the system $f(x) = 0$, $f(x) = x^3 - 2x - 5$, we may write the independent variable as $x = 2 + 1/y$, where $y > 1$, and consider another polynomial

$$q(y) = y^3 p\left(2 + \frac{1}{y}\right) = -y^3 + 10y^2 + 6y + 1\,.$$

This polynomial has one real root $y > 1$ and to fix it we may apply Sturm's algorithm. The process can be repeated. The result is a finite continued fraction approximating the real root of Newton's equation:

$$\begin{aligned}
\xi &= 2 + \frac{1}{10} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{1} + \frac{1}{12} + \frac{1}{3}\\
&= \frac{P_{10}}{Q_{10}} = \frac{16\,415}{78\,37} = 2.094\,551\,486\,4\ldots
\end{aligned}$$

This method (excepting Sturm's ideas) was proposed by Lagrange.

149 Vincent's theorem. In 1836 Vincent showed that Sturm's theorem is very closely related to Descartes' theorem. Vincent's theorem was hinted at earlier by Fourier. Let us apply Lagrange's method to locate the positive roots of a real separable polynomial f. Any such root x can be represented as a finite continued fraction

$$x = b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_m} + \frac{1}{z}\,,$$

where $b_0, b_1, b_2, \ldots, b_m$ are positive integers and $z > 0$. By Euler's formula (1.17),

$$x = \frac{P_m z + P_{m-1}}{Q_m z + Q_{m-1}} \,. \tag{7.19}$$

Then

$$F_m(z) \stackrel{\text{def}}{=} (Q_m z + Q_{m-1})^n f\left(\frac{P_m z + P_{m-1}}{Q_m z + Q_{m-1}}\right)$$

is a polynomial in z such that $F_m(z) = 0$ if and only if $f(x) = 0$.

Theorem 7.11 (Vincent) *For every real separable polynomial f there is a positive integer $M(f)$ such that for any $m > M(f)$ the number of sign changes in the coefficients of F_m is either zero or one.*

Proof We follow Uspenskii (1948) with minor modifications. Let I_m be the segment of $\mathbb{R}$ spanned by the convergents P_{m-1}/Q_{m-1} and P_m/Q_m. Since positive roots of f are simple and $\lim_m |I_m| = 0$ by (1.20) and (1.34), there is an $M(f)$ such that f has at most one root in I_m for $m > M(f)$.

Case 1. We assume first that there are no roots of f in I_m. By (7.19),

$$z = -\frac{P_{m-1} - Q_{m-1}x}{P_m - Q_m x} \,. \tag{7.20}$$

This shows that any real root x of f can be transformed into a negative root of F_m. Similarly, for a complex root $x = a + bi$, $b \neq 0$, we have for the real part of z

$$\operatorname{Re} z = -\frac{(P_{m-1} - Q_{m-1}a)(P_m - Q_m a) + Q_{m-1}Q_m b^2}{(P_m - Q_m a)^2 + Q_m^2 b^2} \,.$$

It follows that $\operatorname{Re} z < 0$ if $a \notin I_m$. If $a \in I_m$ then by (1.20) and (1.34)

$$|(P_{m-1} - Q_{m-1}a)(P_m - Q_m a)| < \frac{1}{Q_{m-1}Q_m} \leqslant 1$$

for sufficiently large m. Hence $\operatorname{Re} z < 0$ if $a \in I_m$, provided that $Q_{m-1}Q_m b^2 > 1$. Increasing $M(f)$ if necessary, we obtain that for every m, $m > M(f)$, every complex root x of f is mapped to the root z of F_m in $\mathbb{C}_- = \{z : \operatorname{Re} z < 0\}$.

Since I_m does not contain any roots of f, all the complex and real roots of F_m are in $\mathbb{C}_-$. It follows that F_m is proportional to a product of linear and quadratic polynomials with positive coefficients. Hence the number of sign variations in the coefficients of F_m is zero.

Case 2. Let $x \in I_m$ be a root of f in I_m and x' be any other root (real or complex). Then by (7.20) and (1.16)

$$z' = -\frac{Q_{m-1}}{Q_m}\,\frac{P_{m-1}/Q_{m-1} - x'}{P_m/Q_m - x'} = -\frac{Q_{m-1}}{Q_m}(1+\alpha) \,, \tag{7.21}$$

where

$$\alpha = \frac{P_{m-1}Q_m - P_m Q_{m-1}}{Q_m Q_{m-1}\,(P_m/Q_m - x')} = \frac{(-1)^m}{Q_m Q_{m-1}\,(P_m/Q_m - x')} \longrightarrow 0$$

if $m \to +\infty$, since $x' \notin I_m$. By (7.20) $z > 0$ since $x \in I_m$. All other roots of F_m lie in $\mathbb{C}_-$. As in case 1 this implies that all coefficients of $G_m(X) = F_m(X)/(X-z)$ are positive.

Definition 7.12 *A positive sequence $c_0 = 1$, $c_1, \dots, c_{n-1}$ is called logarithmic concave if $c_j^2 > c_{j+1}c_{j-1}$ for $j = 1, 2, \dots, n-2$.*

One can easily check that the sequence of the binomial coefficients $c_j = \binom{n-1}{j}$, $j = 0, 1, \dots, n-1$, is logarithmic concave.

Lemma 7.13 *Let $q(X) = \sum_{j=0}^{n-1} c_j X^{n-1-j}$ be a polynomial with logarithmic convex coefficients. Then for every positive ω the number of sign changes in the coefficients of $p(X) = (X-\omega)q(X)$ is* 1.

Proof The number of sign changes in the coefficients of

$$\begin{aligned} p(X) = X^n + (c_1 - \omega)X^{n-1} + (c_2 - c_1\omega)X^{n-2} + \cdots \\ + (c_{n-1} - c_{n-2}\omega)X - c_{n-1}\omega\,, \end{aligned}$$

equals the number of sign changes in

$$c_0 = 1\,, \qquad \frac{c_1}{c_0} - \omega\,, \qquad \frac{c_2}{c_1} - \omega\,, \qquad \dots\,, \qquad \frac{c_{n-1}}{c_{n-2}} - \omega\,, \qquad -\omega\,. \tag{7.22}$$

By the logarithmic concavity of the c_j the sequence $\{c_{j+1}/c_j\}_{j\geqslant 0}$ decreases. The first term of (7.22) is positive and the last term is negative. Hence there is exactly one sign change. □

We now continue the proof of Theorem 7.11. Let $q_m(X) = G_m(Q_{m-1}/Q_m X)$. Then by (7.21) all the roots of q are close to -1 if $m \to +\infty$. By Viète's formulas this implies that the coefficients of q_m approximate the coefficients of a polynomial $c(X+1)^{n-1}$. The coefficients of $(X+1)^{n-1}$ are the binomial coefficients, which constitute a logarithmic concave sequence. Any sufficiently close sequence is also logarithmic concave. Thus the proof of the theorem is completed by Lemma 7.13. □

150 Chebyshev rational functions. We begin with a definition.

Definition 7.14 *A rational function $r(x)$ is called a Chebyshev rational function if it can be represented as*

$$r(x) = \sum_{j=1}^{k} \frac{c_j}{x - x_j}\,, \tag{7.23}$$

where $c_j > 0$ and $x_1 < x_2 < \cdots < x_k$.

The logarithmic derivative $(\log f)'$ of $f = c(x-x_1)^{n_1} \cdots (x-x_k)^{n_k}$ is a Chebyshev function and, in addition,

$$\frac{1}{N}(\log f)' = \frac{1}{N}\frac{f'}{f} = \sum_{j=1}^{k} \frac{n_j/N}{x-x_j} ,$$

where $N = n_1 + \cdots + n_k$. Since any vector $(c_1, \dots, c_k)$ with positive components c_j, $\sum c_j = 1$, can be approximated by a vector with rational coordinates $(n_1/N, \dots, n_k/N)$, it looks probable that the properties of the continued fractions (7.17) considered in Sturm's method may be extended to Chebyshev rational fractions. Moreover, Theorem 7.7 explains Sturm's method better if the rational function in (7.23) is represented by a continued fraction (7.14). The key to such a representation lies in the *Cauchy index* of a rational function.

Let $r(x)$ be a real rational function on $\mathbb{R}$. For a given interval (a, b) let $n_+(a, b)$ be the total number of singularities x_k of $r(x)$ on (a, b) such that $r(x)$ jumps from $-\infty$ to $+\infty$ when x passes through x_k moving to the right, and let $n_-(a, b)$ be the total number of singularities x_k of a function $r(x)$ on (a, b) such that $r(x)$ jumps from $+\infty$ to $-\infty$ when x passes through x_k in the positive direction. Then the integer

$$\mathrm{I}_a^b\, r(x) = n_+(a, b) - n_-(a, b)$$

is called the *Cauchy index* of $r(x)$ on (a, b). For instance, $I_{-\infty}^{+\infty} f'/f = k$. More generally,

$$\mathrm{I}_a^b \left(\sum_{j=1}^{n} \frac{c_j}{x-x_j} \right) = \sum_{x_j \in (a,b)} \operatorname{sign} c_j ,$$

where sign $c = +1$ if $c > 0$ and -1 if $c < 0$. If all $c_j > 0$ then the Cauchy index equals the number of singularities in (a, b).

Lemma 7.15 (Chebyshev 1855) *If u is a Chebyshev rational function then*

$$u(x) = \frac{1}{b(x) - v(x)} , \tag{7.24}$$

where $b(x)$ is a linear polynomial with a positive coefficient of x and $v(x)$ is either zero or a Chebyshev rational function such that

$$I_{-\infty}^{+\infty}\, v(x) = I_{-\infty}^{+\infty}\, u(x) \ - 1 \, . \tag{7.25}$$

Proof If

$$u(x) = \sum_{j=1}^{n} \frac{c_j}{x-x_j} = \frac{\sum_{j=1}^{n} c_j}{x} + O\left(\frac{1}{x^2}\right) , \qquad x \to \infty ,$$

then the real polynomial $[\![1/u]\!]$ is linear and its coefficient at x is $k = (\sum_{j=1}^{n} c_j)^{-1} > 0$. For any Chebyshev function r and $z \in \mathbb{C}$ the imaginary part of $r(z)$ is given by

$$\operatorname{Im} r(z) = -\operatorname{Im} z \sum_{j=1}^{n} \frac{c_j}{|z - x_j|^2} ,$$

which implies that $r(z)$ can vanish only on $\mathbb{R}$. This and (7.24) imply that the poles of v are located on $\mathbb{R}$ at the zeros of u. The function u decreases on $\mathbb{R}$ since

$$u'(x) = -\sum_{j=1}^{n} \frac{c_j}{(x - x_j)^2} < 0 . \tag{7.26}$$

If $u(c) = 0$ then $\lim_{x\to c}(x - c)v(x) = \lim_{x\to c}(x - c)b(x) - 1/u'(c) > 0$ by (7.26), implying that v is a Chebyshev function. By (7.26) the poles of v interlace the poles of u. This proves (7.25). □

Lemma 7.16 *If v is either zero or a Chebyshev rational function and b is a real linear polynomial in z with positive coefficient at z then u in* (7.24) *is also a Chebyshev rational function.*

Proof Since $\operatorname{Im}(b - v) > 0$ if $\operatorname{Im} z > 0$, the zeros of $b - v$ are located on $\mathbb{R}$. Since $(b - v)' > 0$ on $\mathbb{R}$, we see that $\lim_{x\to c}(x - c)u(x) > 0$ at every pole of u, which proves that u is a Chebyshev function. □

Corollary 7.17 *A rational function r is a Chebyshev function if and only if*

$$r(z) = \frac{1}{b_1(z)} \underset{-}{} \frac{1}{b_2(z)} \underset{-}{} \cdots \underset{-}{} \frac{1}{b_n(z)} , \tag{7.27}$$

where $n = I_{-\infty}^{+\infty} r$ and $b_1, b_2, \ldots, b_n$ are real linear polynomials with positive coefficients at z.

If f is a separable polynomial with real roots then $n_j = 1$ in (150) and therefore f'/f is a Chebyshev function. Hence all real polynomials b_j in (7.17) have positive coefficients at x.

7.4 Chebyshev's approach to orthogonal polynomials

151 Continued fractions and orthogonal polynomials. By (7.12) the Legendre polynomials are orthogonal on $[-1, 1]$ with respect to the unit weight, which is related to Gauss's continued fraction (6.101) by

$$\frac{1}{2}\int_{-1}^{1} \frac{1}{z - t}\, dt = \frac{1}{2} \ln \frac{z + 1}{z - 1} = \frac{1}{z} \underset{-}{} \frac{1^2}{3z} \underset{-}{} \frac{2^2}{5z} \underset{-}{} \frac{3^2}{7z} \underset{-}{} \frac{4^2}{9z} \underset{-}{} \cdots . \tag{7.28}$$

However, as Jacobi observed (see §**145**), the denominators of convergents to the continued fraction in (7.28) are orthogonal with respect to this very weight.

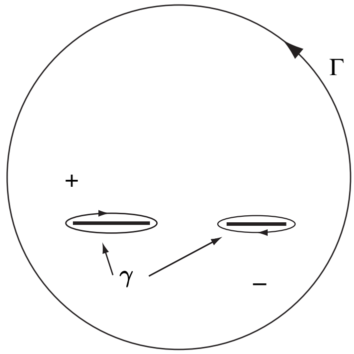

Fig. 7.1. Contours of integration.

Sturm's formula (7.17) and Corollary 7.17 hint that an analogue of Gauss's formula (7.28) holds for any Chebyshev rational fraction. One may expect therefore that the denominators of the convergents to (7.27) are orthogonal with respect to the discrete weight of any Chebyshev rational fraction.

Here we will present the argument for the case of positive Borel measures σ with compact support $\operatorname{supp}\sigma$ in $\mathbb{C}$. Since the Cauchy integral

$$C_\sigma(z) = \int \frac{d\sigma(t)}{t-z}$$

is holomorphic at $z = \infty$ it determines a unique element of $\mathbb{C}([1/z])$.

Theorem 7.18 *Let P/Q be a convergent to C_σ, $n = \deg Q$. Then*

$$\int Q(t)t^k d\sigma = 0\,, \qquad k = 0, 1, \ldots, n-1. \tag{7.29}$$

Proof By Markoff's theorem, $C_\sigma - P/Q = O(z^{-2n-1})$. Following Lagrange (see §**16** in Section 1.3), we consider a linear form in C_σ with polynomial coefficients

$$QC_\sigma - P = \int \frac{Q(z)-Q(t)}{t-z} d\sigma - P + \int \frac{Q(t)}{t-z} d\sigma = O(z^{-n-1})\,. \tag{7.30}$$

It follows that

$$P = -\int \frac{Q(t)-Q(z)}{t-z} d\sigma\,, \qquad \int \frac{Q(t)}{t-z} d\sigma = O(z^{-n-1})\,.$$

Observing that $(t-z)^{-1} = -\sum_{k=1}^{\infty} t^{k-1}z^{-k}$ converges uniformly in t for every z, $|z| > 2|t|$, $t \in \operatorname{supp}\sigma$, we obtain (7.29). □

Remark Compare this proof with Gauss's arguments, see (7.7) and (7.8).

In the case where σ is supported by a smooth curve and is absolutely continuous with respect to the distance along the arc length there is another solution. By Sokhotskii's formulas (see Markushevich 1985, p. 316),

$$F(z) = \int_\gamma \frac{d\sigma(t)}{t-z}, \qquad \sigma'(t) = \frac{F_+(t) - F_-(t)}{2\pi i}.$$

Since $F \in \mathbb{C}([1/z])$, we can develop F into a P-fraction. The moments of Q for $k < n = \deg Q$ are zero by Cauchy's integral theorem (Markushevich 1985, p. 258):

$$\begin{aligned}\int t^k Q(t) d\sigma &= \int_\gamma F(z) z^k Q(z) dz = -\int_\Gamma F(z) z^k Q(z) dz \\ &= -\int_\Gamma \left(F - \frac{P}{Q}\right) z^k Q dz = O\left(\frac{1}{z^{2n+1}}\, z^{k+n+1}\right) = 0.\end{aligned}$$

Chebyshev's theorem 7.18 gives Gauss quadratures for arbitrary measures σ with compact support in $\mathbb{C}$.

Corollary 7.19 *Let P/Q be the convergent of nth order to C_σ such that* $\deg Q = n$ *and all zeros $z_1, z_2, \dots, z_n$ of Q are simple. Then for any polynomial f,* $\deg f \leqslant 2n-1$, *we have*

$$\int f(z) d\sigma(z) = \sum_{k=1}^{n} l_k f(z_k). \tag{7.31}$$

Proof By Lagrange's interpolation formula (see (7.4)), (7.31) is valid for any polynomial f for which $\deg f \leqslant n-1$. If f is any polynomial, $\deg f \leqslant 2n-1$, then $f = Qp + r$ by the long division of polynomials; here $\deg p \leqslant n-1$ and $\deg r \leqslant n-1$. By Theorem 7.18 $\int Qp d\sigma = 0$. Since $f(z_k) = r(z_k)$ if $Q(z_k) = 0$, this implies (7.31). □

Hence Chebyshev's theorem 7.18 is related in the first place to Gauss quadratures. However, for measures supported on lines of the complex plane it gives orthogonal polynomials.

Corollary 7.20 *Let σ be a positive measure with compact support on a line in $\mathbb{C}$. Let P/Q be a convergent to C_σ, $n =$* $\deg Q$. *Then*

$$\int Q(\zeta)\, \overline{\zeta}^k d\sigma(\zeta) = 0, \qquad k = 0, 1, \dots, n-1, \tag{7.32}$$

i.e. the polynomials Q are orthogonal with respect to $d\sigma$.

Proof If ζ is on a line, then $\overline{\zeta}$ will be on the symmetric line with respect to $\mathbb{R}$. Hence there are λ, $|\lambda| = 1$, and a complex b such that $\overline{\zeta} = \lambda\zeta + b$, on the line. The proof is completed by Theorem 7.18 and by the binomial formula $\overline{\zeta}^k = \sum_{j=0}^{k} \lambda^j z^j b^{k-j}$. □

Orthogonal polynomials explain an interesting phenomenon in Gauss quadratures on the location of the nodes of interpolation in $[-1, 1]$.

Lemma 7.21 *Let σ be a probability measure on $\mathbb{C}$. Then the zeros of any orthogonal polynomial p lie in the convex hull* $\operatorname{conv}\sigma$ *of* $\operatorname{supp}\sigma$.

Proof Let $p(\lambda) = 0$. Then $p = (z-\lambda)q$ and $\deg q < \deg p$. Since p is an orthogonal polynomial, it must be orthogonal to q:

$$0 = \int p\bar{q}\, d\sigma = \int (z-\lambda)|q|^2\, d\sigma\, .$$

Let m_q be the total mass of $|q|^2 d\sigma$. We define an auxiliary probability measure $d\mu = m_q^{-1}|q|^2 d\sigma$, which is supported by $\operatorname{supp}\sigma$. It follows that $\lambda = \int_{\operatorname{supp}\sigma} z\, d\mu(z)$. The Riemann sums of the above integral are nothing other than convex combinations of points in $\operatorname{supp}\sigma$. Therefore λ lies in the closure of the convex hull of $\operatorname{supp}\sigma$. □

In Gauss quadratures, σ is the Lebesgue measure on $[-1, 1]$, which is convex. By Lemma 7.21 all zeros of polynomials Q_n lie in $[-1, 1]$.

152 Chebyshev's solution to Euler's problem

Theorem 7.22 (Chebyshev 1855) *Let $\{Q_n\}$ be the denominators of the convergents for the continued fraction*

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{z-t} = \frac{a_1}{b_1(z)} - \frac{a_2}{b_2(z)} - \cdots - \frac{a_n}{b_n(z)} - \cdots\ , \tag{7.33}$$

where $b_n(z) = k_n z + l_n$. Then

$$\int_{\mathbb{R}} Q_n^2(t)d\sigma(t) = \frac{a_1\ \cdots\ a_{n+1}}{k_{n+1}}\, . \tag{7.34}$$

Proof By the Euler–Wallis formulas (1.15), $Q_n(t) = k_n\ \cdots\ k_1 t^n + \cdots + Q(0)$. Observing that $Q_n \perp t^k$ for $k = 0, 1, \ldots, n-1$, we obtain that

$$\int_{\mathbb{R}} Q_n^2 d\sigma = k_n\ \cdots\ k_1 \int_{\mathbb{R}} Q_n t^n\, d\sigma\, .$$

Integration of one of the Euler–Wallis formulas multiplied by t^{n-1},

$$Q_{n+1}t^{n-1} = k_{n+1}t^n Q_n + l_{n+1}Q_n t^{n-1} - a_{n+1}Q_{n-1}t^{n-1},$$

shows by the orthogonality property that

$$k_{n+1}\int_{\mathbb{R}} Q_n t^n\, d\sigma = a_{n+1}\int_{\mathbb{R}} Q_{n-1}t^{n-1}\, d\sigma\ ,$$

implying that

$$\int_{\mathbb{R}} Q_n^2 d\sigma = \frac{k_n\ \cdots\ k_1 a_{n+1}\ \cdots\ a_2}{k_{n+1}\ \cdots\ k_2}\int_{\mathbb{R}} Q_0\, d\sigma = \frac{k_1 a_{n+1}\ \cdots\ a_2}{k_{n+1}}\sigma(\mathbb{R})\, .$$

Comparing the asymptotic formulas for both sides of (7.33) we get $\sigma(\mathbb{R}) = a_1/k_1$. □

By Theorems 7.18 and 7.22 the matrix

$$\begin{pmatrix} \sqrt{k_1 c_1} & \sqrt{k_1 c_2} & \dots & \sqrt{k_1 c_n} \\ \sqrt{k_2 c_1} Q_1(x_1) & \sqrt{k_2 c_2} Q_1(x_2) & \dots & \sqrt{k_2 c_n} Q_1(x_n) \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{k_n c_1} Q_{n-1}(x_1) & \sqrt{k_n c_2} Q_{n-1}(x_2) & \dots & \sqrt{k_n c_n} Q_n(x_n) \end{pmatrix} \tag{7.35}$$

is orthogonal. Applying Theorem 7.22 to (6.115) we obtain (7.40); see §**156**.

153 Asymptotic expansions. Chebyshev's theorem 7.18 extends to arbitrary $\sigma \in \mathfrak{P}(\mathbb{R})$.

Theorem 7.23 *Let P/Q be a convergent to C_σ, $n = \deg Q$, $\sigma \in \mathfrak{P}(\mathbb{R})$. Then $Q \perp z^k$, $0 \leqslant k < n$, in $L^2(d\sigma)$.*

Proof By Markoff's theorem and Lagrange's formula,

$$QC_\sigma - P = \int \frac{Q(z) - Q(t)}{t - z} d\sigma - P + \int \frac{Q(t)}{t - z} d\sigma = O(z^{-n-1})$$

if $z = iy$, $y \to +\infty$. It follows that

$$P = -\int \frac{Q(t) - Q(z)}{t - z} d\sigma \, , \qquad \int \frac{Q(t)}{t - z} d\sigma = O(z^{-n-1}) \, .$$

Since

$$\int \frac{Q(t)}{z - t} d\sigma = \sum_{k=0}^{n} \frac{1}{z^{k+1}} \int_{\mathbb{R}} t^k Q d\sigma + \frac{1}{z^{n+1}} \int_{\mathbb{R}} \frac{t^{n+1} Q d\sigma}{z - t} \, ,$$

the uniqueness of the asymptotic expansion as $y \to +\infty$ proves the theorem. □

154 Carleman's criterion. Here we apply Chebyshev's theorem 7.22 to prove Carleman's criterion.

Theorem 7.24 *If*

$$\sum_{n=1} s_{2n}^{-1/2n} = +\infty \, , \tag{7.36}$$

then Stieltjes' moment problem (6.116) *is determined.*

Proof By Theorem 6.62 the question is reduced to the convergence of $K_{n=1}^{\infty}(a_n/x)$ for $x > 0$, which follows by Corollary 3.11 if we can prove that $\sum_{n=1} a_n^{-1/2} = +\infty$. By Theorem 7.22

$$a_{n+1} \cdots a_1 = \int_{\mathbb{R}} Q_n^2 d\sigma = \int_{\mathbb{R}} t^n Q_n d\sigma \leqslant \left(\int_{\mathbb{R}} Q_n^2 d\sigma \right)^{1/2} s_{2n}^{1/2} \, .$$

Hence $a_1 \cdots a_{n+1} \leqslant s_{2n}$. Now the Carleman inequality

$$\sum_{k=1}^{\infty}(x_1x_2\cdots x_k)^{1/k} < e\sum_{k=1}^{\infty}x_k\,, \tag{7.37}$$

see Akhiezer (1961), completes the proof. □

7.5 Examples of orthogonal polynomials

155 Chebyshev polynomials. Combining (6.34) with the substitution $x = \cos\theta$, we can develop the Cauchy integral

$$\frac{1}{\pi}\int_{-1}^{1}\frac{dx}{(z-x)\sqrt{1-x^2}} = \frac{1}{z}\,\frac{-}{}\,\frac{1}{2z}\,\frac{-}{}\,\frac{1}{2z}\,\frac{-}{}\,\frac{1}{2z}\,\frac{-}{}\cdots$$

into a P-fraction. By the Euler–Wallis formulas (1.15) the denominators $T_n(z)$ of the convergents to (6.34) satisfy

$$\begin{aligned} &T_n(z) = 2zT_{n-1}(z) - T_{n-2}(z)\,, \qquad n = 2,3,\ldots,\\ &T_1(z) = zT_0(z) - T_{-1}(z)\,, \qquad T_0 \equiv 1, \qquad T_{-1} \equiv 0\,. \end{aligned} \tag{7.38}$$

Hence the polynomials

$$T_0(z) \equiv 1, \quad T_1(z) = z, \quad T_2(z) = 2z^2 - 1, \quad T_3(z) = 4z^3 - 3z, \quad \ldots$$

are the Chebyshev classical polynomials, which are orthogonal in the Hilbert space $L^2([-1,1], d\mu)$, $d\mu = \pi^{-1}\,(1-x^2)^{-1/2}\,dx$:

$$\frac{1}{\pi}\int_{-1}^{1} T_j(x)T_k(x)\frac{dx}{\sqrt{1-x^2}} = 0 \text{ if } j \neq k\,.$$

The Cauchy integral associated to with Chebyshev polynomials corresponds to a periodic P-fraction with period 2. Similarly, the formula

$$\frac{2}{\pi}\int_{-1}^{1}\frac{\sqrt{1-x^2}}{z-x}\,dx = \frac{2}{z+\sqrt{z^2-1}} = \frac{1}{z}\,\frac{-}{}\,\frac{1}{4z}\,\frac{-}{}\,\frac{1}{z}\,\frac{-}{}\,\frac{1}{4z}\,\frac{-}{}\,\frac{1}{z}\,\frac{-}{}\cdots \tag{7.39}$$

shows that the Chebyshev polynomials of the second kind also correspond to a periodic P-fraction with period 2.

156 Hermite polynomials. These are defined in Szegő (1975) by

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2}H_n(x)H_m(x)\,dx = 2^n n!\,\delta_{nm}\,. \tag{7.40}$$

To find formulas for the Hermite polynomials we apply Chebyshev's method. By the Euler–Wallis formulas the denominators H_n of the convergents G_n/H_n to (6.115) satisfy the three-term recurrence for Hermite polynomials. We have $H_0(x) = 1$, $H_1(x) = 2x$,

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x)\,, \qquad n \geqslant 2\,. \tag{7.41}$$

Then the $H_n(x)$, $n \geqslant 0$, are orthogonal in $L^2(\pi^{-1/2}e^{-x^2}dx)$ by Theorem 7.23. Formula (7.40) for $n = m$ follows by Theorem 7.22.

157 Brouncker's orthogonal polynomials. Using (1.15) we can easily list the first few of Brouncker's polynomials:

$$P_{-1}(s) = 1\,, \qquad P_0(s) = s\,,$$
$$P_1(s) = 2s^2 + 1\,, \qquad P_2(s) = 4s^3 + 11s\,,$$
$$P_3(s) = 8s^4 + 72s^2 + 25\,, \qquad P_4(s) = 16s^5 + 340s^3 + 589s\,,$$
$$P_5(s) = 32s^6 + 1328s^4 + 3410s^2 + 2025\,.$$

It is clear that $\deg P_n = n+1$ and $P_n(s) = 2^n s^{n+1} + \cdots$ If $s = 0$, then in (1.15)

$$P_n(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (2n-1)^2(2n-5)^2(2n-9)^2 \cdots & \text{if } n \text{ is odd}\,. \end{cases}$$

For instance, $P_5(0) = 9^2 \times 5^2 \times 1^2 = 2025$. For odd n (3.20) implies that

$$\begin{aligned}(2n+1)P_n(0) &= (2n+1)(2n-1)(2n-1)(2n-5)(2n-5)\cdots \\ &> (2n+1)(2n-1)(2n-3)(2n-5)(2n-7)\cdots = (2n+1)!! = P_n(1)\,.\end{aligned}$$

The partial denominators $\mathfrak{B}_n$ of the convergents to the continued fraction in (6.112) satisfy the recurrence relation

$$\mathfrak{B}_n(z) = 2z\mathfrak{B}_{n-1}(z) - (2n-3)^2\mathfrak{B}_{n-2}(z)\,, \qquad n \geqslant 2\,,$$

with $\mathfrak{B}_0(z) = 1$ and $\mathfrak{B}_1(z) = z$. By Theorem 7.23 the $\mathfrak{B}_n$, $n \geqslant 0$, are orthogonal in $L^2(d\mu)$, μ being the measure in Theorem 6.53. By (7.34),

$$\int_{\mathbb{R}} \mathfrak{B}_n^2\, d\mu = \frac{1}{2}(2n-1)!!^2\,, \qquad n = 1, 2, \ldots$$

Observing that $z = is$ and using the recurrence relations for $\mathfrak{B}_n$ and P_n, we see that these polynomials are related by $\mathfrak{B}_n(z) = i^n P_{n-1}(s)$. By Theorem 7.7 all the roots of the polynomials $\mathfrak{B}_n$ are real. Hence all the roots of P_n lie on the imaginary axis. These roots taken in their totality make a barrier between the right half-plane to the left half-plane against the analytic continuation of Brouncker's continued fraction. However, as Theorem 3.16 shows, $b(s)$ can still be extended analytically through the imaginary axis by use of the formula for Wallis' infinite product. The system

$$\mathfrak{V}_n(s) = \frac{\sqrt{2}}{(2n-1)!!}\mathfrak{B}_n(z)\,, \quad n = 1, 2, \ldots, \qquad \mathfrak{V}_0(z) \equiv 1$$

is an orthonormal system in $L^2(d\mu)$. The operation of multiplication by z acts on $\mathfrak{V}_n$ according to the formulas $z\mathfrak{V}_0(z) = 2^{-1/2}\mathfrak{V}_1(z)$,

$$z\mathfrak{V}_n(z) = \left(n + \frac{1}{2}\right)\mathfrak{V}_{n+1} + \left(n - \frac{1}{2}\right)\mathfrak{V}_{n-1}\,, \qquad n \geqslant 1\,.$$

The matrix of this operator in the basis $\{\mathfrak{V}_n\}_{n\geqslant 0}$, i.e. the Jacobi matrix, is given by

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \dots & \dots & 0 & \dots \\ \frac{1}{\sqrt{2}} & 0 & \frac{3}{2} & 0 & 0 & \dots & \dots & 0 & \dots \\ 0 & \frac{3}{2} & 0 & \frac{5}{2} & 0 & \dots & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-\frac{1}{2} & 0 & n+\frac{1}{2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Let us put $z = iy$, $y \to +\infty$ in (6.112). Then

$$\frac{y}{b(y)} \sim \sum_{k=0}^{\infty} (-1)^k \frac{1}{y^{2k}} \int_{\mathbb{R}} t^{2k}\, d\mu\,, \qquad y \to +\infty. \tag{7.42}$$

Comparing this formula with (3.31), we obtain that

$$\int t^2\, d\mu = \frac{1}{2}\,, \qquad \int t^4\, d\mu = \frac{11}{8}\,, \qquad \int t^6\, d\mu = \frac{173}{16}\,.$$

Further moments can be evaluated by the use of (4.79). Since by (6.111)

$$\int_{\mathbb{R}} \frac{\log(d\mu/dx)\, dx}{1+x^2} = -\infty\,,$$

the polynomials are complete in $L^2(d\mu)$; see Akhiezer (1961).

158 Pell's equation and orthogonal polynomials. Let us apply orthogonal polynomials to the study of the solutions of the Pell equation (22) assuming that $\mathbf{R}$ has only real roots.

Lemma 7.25 *If P and Q are solutions to* (22) *then for every $r \in \mathbb{C}[z]$,* $\deg r \leqslant g$*, the rational function Qr/P is a convergent to $r/\sqrt{\mathbf{R}}$.*

Proof It follows from (22) that

$$\deg\left(\frac{1}{\sqrt{\mathbf{R}}} - \frac{Q}{P}\right) + \deg\left(\frac{1}{\sqrt{\mathbf{R}}} + \frac{Q}{P}\right) = -2\deg P - (2g+2)\,. \tag{7.43}$$

Since

$$\frac{2}{\sqrt{\mathbf{R}}} = \left(\frac{1}{\sqrt{\mathbf{R}}} - \frac{Q}{P}\right) + \left(\frac{1}{\sqrt{\mathbf{R}}} + \frac{Q}{P}\right) \tag{7.44}$$

and $\deg(2/\sqrt{\mathbf{R}}) = -g-1$, at least one summand in (7.44), say the second (otherwise change the sign of Q), satisfies the inequality

$$-g-1 \leqslant \deg\left(\frac{1}{\sqrt{\mathbf{R}}} + \frac{Q}{P}\right)\,.$$

By (7.43) we obtain

$$\deg\left(\frac{1}{\sqrt{\mathbf{R}}}-\frac{Q}{P}\right) = -2\deg P - (2g+2) - \deg\left(\frac{1}{\sqrt{\mathbf{R}}}+\frac{Q}{P}\right)$$
$$\leqslant -2\deg P - g - 1 \ .$$

It follows that for every polynomial r, $\deg r \leqslant g$,

$$\deg\left(\frac{r}{\sqrt{\mathbf{R}}}-\frac{Qr}{P}\right) \leqslant -\deg P - 1 \ ,$$

which implies the lemma by Theorem 6.5. □

Since $\mathbf{R}(x) = (x-t_0)(x-t_1)\cdots(x-t_{2g})(x-t_{2g+1})$ is a separable polynomial with real roots, we obtain $g+1$ cuts of the complex plane, (t_0, t_1), (t_2, t_3), ..., (t_{2g}, t_{2g+1}), which together constitute the set $\mathbf{E} = \{x \in \mathbb{R} : R(x) < 0\}$. Let us place in each interval (t_{2k-1}, t_{2k}), $k = 1, 2, \dots, g$, exactly one root of r. Then $\deg r = g$. Counting the arguments along a path moving in $\mathbb{R}$ and passing over the roots of $\mathbf{R}$ along semicircles in the upper half-plane shows that

$$\frac{r(z)}{\sqrt{\mathbf{R}(z)}} = \frac{1}{\pi}\int_{\mathbf{E}} \frac{|r(t)|}{\sqrt{-\mathbf{R}(t)}}\frac{dt}{z-t} \ . \tag{7.45}$$

Here the branch of $\sqrt{\mathbf{R}}$ is chosen to be positive for $z = x > t_{2g+1}$. Since by Lemma 7.25 Qr/P is a convergent for $r/\sqrt{\mathbf{R}}$, we obtain that

$$\int_{\mathbf{E}} \frac{|r(t)|}{\sqrt{-R(t)}} t^j P(t)dt = 0 \qquad \text{for } j = 0, 1, \dots, \deg P - 1. \tag{7.46}$$

Therefore P is an orthogonal polynomial for a family of varying measures supported by $\mathbf{E}$:

$$d\sigma_r = \frac{|r(t)|}{\sqrt{-R(t)}}\mathbf{1}_{\mathbf{E}} \ . \tag{7.47}$$

By Lemma 7.21 all zeros of P are simple and are located on $\mathbb{R}$ in the closed convex hull of $\mathbf{E}$. Then rQ is an orthogonal polynomial of the second kind. It follows that all zeros of rQ are real and simple. Moreover, the zeros of Q must be located in $\mathbf{E}$, since otherwise, picking an r with a zero at the same point as a zero of Q, we would get an orthogonal polynomial with a multiple zero. Now

$$P^2 = 1 + Q^2\mathbf{R} \ , \tag{7.48}$$

which implies that $P^2 > 1$ on the complementary intervals of $\mathbf{E}$ and $0 \leqslant P^2 \leqslant 1$ on $\mathbf{E}$. In particular, all $\deg Q + g + 1$ zeros of P are located in $\mathbf{E}$. Differentiating (7.48), we obtain that $\dot{P}^2 = 2P\dot{P}$, implying that

$$\deg\, P^2 = 2\deg Q + 2g + 1 \ . \tag{7.49}$$

However, by (6.60) Q, which is co-prime with P, must divide $\dot{P}^2$. It follows that the number of zeros in $\mathbf{E}$ of the derivative of P^2 is

$$\deg Q + (g+1) + \deg Q = 2\deg Q + g + 1. \tag{7.50}$$

At the ends of $[t_{2k-1}, t_{2k}]$ the polynomial P^2 equals 1. By Rolle's theorem the derivative of P^2 must vanish at least once in each of g complementary intervals. Combining this with (7.49) and (7.50), all the roots of $\dot{P}^2$ are found. It follows that P must oscillate between -1 and 1 on the intervals of $\mathbf{E}$ and has modulus greater than 1 outside $\mathbf{E}$.

Let us choose r according to (6.61). Then by Theorem 6.30

$$\operatorname{Re} \int \frac{r}{\sqrt{\mathbf{R}}} dz = \ln |P \pm \sqrt{P^2 - 1}| + \text{constant}$$

and we see that the real part of the Abelian integral is constant on $\mathbf{E}$. However,

$$\frac{d}{dz} \int \ln(z-t)\, d\sigma_r = \int \frac{d\sigma_r}{z-t} = \frac{r(z)}{\sqrt{\mathbf{R}}(z)} ,$$

see (7.45). This obviously implies that σ_r with Abel's choice of r is proportional to the equilibrium measure on $\mathbf{E}$. See Akhiezer (1960), Peherstorfer (1991), Peherstorfer and steinbauer (1995) and Tomchuk (1963) for a development of these ideas to the general case when Pell's equation does not have solutions.

Exercises

7.1 Check that the following arguments imply Theorem 7.11. Since from the start of §**149** in Section 7.3 we have

$$\frac{P_m z + P_{m-1}}{Q_m z + Q_{m-1}} = \frac{P_m}{Q_m} + \frac{(-1)^m}{Q_m^2 (z+p)} , \qquad p = \frac{Q_{m-1}}{Q_m} \in (0, 1) ,$$

we can apply to the real separable polynomial f Taylor's formula centered at $\xi_m = P_m/Q_m$ and obtain

$$\begin{aligned} Q_m^{-n} F_m(z) &= (z+p)^n f\left(\frac{P_m z + P_{m-1}}{Q_m z + Q_{m-1}}\right) \\ &= f(\xi_m)(z+p)^n + \frac{(-1)^m}{Q_m^2} f'(\xi_m)(z+p)^{n-1} + \cdots + \frac{(-1)^{nm}}{Q_m^{2n} n!} f^{(n)}(\xi_m) . \end{aligned}$$

The coefficient at z^n is obviously $f(\xi_n)$ and the coefficient at z^0 is

$$p^n \left(f(\xi_m) + \frac{(-1)^m}{Q_m Q_{m-1}} f'(\xi_m) + \cdots + \frac{(-1)^{nm}}{Q_m^n Q_{m-1}^n n!} f^{(n)}(\xi_m) \right) .$$

Similar formulas can be written down for the other coefficients. If ξ_m approaches ξ in such a way that $f(\xi) \neq 0$ then the signs of all coefficients in F_m coincide with the sign of $f(\xi_m)$, since $Q_m \to +\infty$ if $m \to +\infty$.

7.2 Let f be a holomorphic function on a simple closed curve Γ such that it does not vanish on Γ and its integral F has a constant real part on Γ. Then

$$\frac{1}{2\pi i} f(z)dz = \frac{\varepsilon}{2\pi} \mid f \mid ds \tag{E7.1}$$

on Γ, where s is the arc length on Γ and ε is either $+1$ or -1.
Hint: Apply the Cauchy–Riemann equations.

7.3 If f, g are polynomials with real zeros, $\deg g \leqslant \deg f$ and the zeros of g interlace the zeros of f then the polynomial $W(f, g) = f'g - fg'$ does not have real zeros (Vladimir Markoff 1892; see Grave 1938).
Hint: Differentiate the Lagrange interpolation formula, see (7.4),

$$\frac{g}{f}(x) = c + \sum_{f(c_i)=0} \frac{g(c_i)}{f'(c_i)} \frac{1}{x - c_i},$$

and observe counting the zeros that the signs of its coefficients are all the same.

7.4 If the zeros of g and f interlace, $\deg g < \deg f$, then one of g/f and $-g/f$ is a Chebyshev function.
Hint: Observe that any Chebyshev function decreases on $\mathbb{R}$. Count zeros to show that the former is a quotient of two polynomials with interlaced zeros.

7.5 Show that the zeros of g and f interlace then the zeros of g' and of f' also interlace.
Hint: Consider the Wronskian $W(f, g) = f'g - fg'$, which by Ex. 7.3 is either positive or negative on $\mathbb{R}$. Let c_i and c_{i+1} be two consecutive zeros of g'. Then the numbers $f'(c_i)g(c_i) = W(f, g)(c_i)$ have the same sign. But $g(c_i)$ and $g(c_{i+1})$ have opposite signs. Hence the numbers $f'(c_i)$ and $f'(c_{i+1})$ have opposite signs as well.

7.6 For $\operatorname{Re} z > 0$ investigate the orthogonal polynomials associated with

$$\text{(a) } \frac{1}{y(s)} = \int_0^{+\infty} \frac{e^{-sx}\,dx}{\cosh x}, \text{ (b) } \frac{1}{y(s)} = \int_0^{+\infty} \frac{e^{-sx}\,dx}{\cosh^2 x}, \text{ (c) } \coth\frac{1}{s}.$$

7.7 Prove that $|\varphi| \geqslant 1$ in $\mathbb{P}_+$, where φ is defined in (4.84).
Hint: Rewrite (4.84) in the form

$$\frac{\varphi - 1}{\varphi + 1} = \frac{1}{8s+}\,\frac{1 \times 3}{8s} + \frac{3 \times 5}{8s} + \frac{5 \times 7}{8s} + \cdots. \tag{E7.2}$$

7.8 Prove that for the generalized Laguerre polynomials $L_n^{(\alpha)}$,

$$\int_0^{+\infty} e^{-x} x^\alpha L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\,dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{nm}$$

7.9 Prove that Carleman's condition (7.36) implies that the Hamburger moment problem is also determined.
Hint: Make the continued fraction (6.121) fit the conditions of Theorem 6.52 with equivalence transformations. Use the tricks of §**56** in Section 3.1 and Carleman's inequality (7.37) to prove the convergence of the continued fraction (6.121) in $\mathbb{P}_+$.

8

Orthogonal polynomials on the unit circle

8.1 Orthogonal polynomials and continued fractions

159 Herglotz' theorem. In Chebyshev's theory orthogonal polynomials appear as the denominators of convergents to the P-fraction of C_σ, see Corollary 7.20. By §**45** at the start of Section 2.4, continued fractions are compositions of Möbius transforms, which, as is well known in complex analysis, leave invariant the family of circles in $\hat{\mathbb{C}}$. This hints that an analogue of Chebyshev's theory may exist for circles. Since any circle in $\mathbb{C}$ is a linear transformation of the unit circle $\mathbb{T}$, we restrict ourselves to this case.

To determine continued fractions corresponding to $\mathbb{T}$ we consider the spaces $\mathfrak{P}(\mathbb{T})$ of all probability measures supported by $\mathbb{T}$ and $\mathfrak{R}(\mathbb{D})$ of all holomorphic functions $F(z)$ in $\mathbb{D}$, with $\operatorname{Re} F(z) > 0$, $F(0) = 1$. We consider $\mathfrak{P}(\mathbb{T})$ as a convex subset of the Banach space $M(\mathbb{T})$ of all complex Borel measures on $\mathbb{T}$ equipped with the variation norm $\|\mu\| = \operatorname{Var} \mu$. By Riesz' theorem $M(\mathbb{T})$ is the Banach space dual to the Banach space $C(\mathbb{T})$ of all continuous functions f on $\mathbb{T}$ with norm $\|f\|_\infty = \sup_{\zeta \in \mathbb{T}} |f(\zeta)|$. The duality $(C(\mathbb{T}), M(\mathbb{T}))$ given by

$$(f, \mu) \to \int_{\mathbb{T}} f d\mu$$

defines the $*$-weak topology on $\mathfrak{P}(\mathbb{T})$, which makes it compact, Rudin (1973). The continuum $\mathfrak{P}(\mathbb{T})$ corresponds to $[0, 1]$ in the theory of regular continued fractions. We recall the definition of $*$-weak convergence.

Definition 8.1 *A sequence $\{\sigma_n\}_{n\geqslant 0}$ in $M(\mathbb{T})$ is said to converge to $\sigma \in M(\mathbb{T})$ in the $*$-weak topology, $*\text{-}\lim_n \sigma_n = \sigma$, if for every $f \in C(\mathbb{T})$*

$$\lim_n \int_{\mathbb{T}} f \, d\sigma_n = \int_{\mathbb{T}} f \, d\sigma \, . \tag{8.1}$$

Every $\sigma \in \mathfrak{P}(\mathbb{T})$ determines the unique Hilbert space $L^2(d\sigma)$ of functions square integrable on $\mathbb{T}$ with respect to σ. It is equipped with the standard inner product

$$(f,g) = \int_{\mathbb{T}} f\bar{g}\, d\sigma\, .$$

In contrast with the Chebyshev case $\mathfrak{R}(\mathbb{D})$ is not closed under compositions, which is an obstacle in arranging algorithms for continued fractions. However, since the Möbius transform $z \to (1+z)/(1-z)$ maps $\mathbb{D}$ onto $\{z : \operatorname{Re} z > 0\}$, any $F(z) \in \mathfrak{R}(\mathbb{D})$ can be uniquely represented as

$$F(z) = \frac{1+zf(z)}{1-zf(z)} = 1 + \frac{2}{-1+1/zf}\, , \tag{8.2}$$

where f is a holomorphic mapping of $\mathbb{D}$ into itself. The set $\mathcal{B}$ of all holomorphic $f : \mathbb{D} \to \mathbb{D}$ is closed under superpositions. Hence, if f is developed into a continued fraction then the continued fraction for F can be obtained by substitution into (8.2). From another point of view, $\mathcal{B}$ is the unit ball of the Hardy algebra H^∞ consisting of all holomorphic f in $\mathbb{D}$ satisfying

$$\|f\|_\infty \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}} |f(z)| < +\infty\, .$$

This algebra is studied in detail in Garnett (1981). The point of view on $\mathcal{B}$ as the unit ball of H^∞ plays a significant role in what follows. First we establish a relationship between $\mathfrak{P}(\mathbb{T})$, $\mathfrak{R}(\mathbb{D})$ and $\mathcal{B}$.

Theorem 8.2 (Herglotz) *The formula*

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\sigma \stackrel{\text{def}}{=} F^\sigma(z) = \frac{1+zf^\sigma(z)}{1-zf^\sigma(z)} \tag{8.3}$$

defines a one-to-one mapping between $\sigma \in \mathfrak{P}(\mathbb{T})$, $F^\sigma \in \mathfrak{R}(\mathbb{D})$ and $f \in \mathcal{B}$:

$$\sigma \leftrightarrow F^\sigma \leftrightarrow f^\sigma\, . \tag{8.4}$$

Proof That $\sigma \leftrightarrow F^\sigma$ is one-to-one follows from the Taylor expansion of F^σ at $z=0$:

$$F^\sigma(z) = 1 + 2\sum_{k=0}^{\infty} \hat{\sigma}(k) z^k\, , \tag{8.5}$$

where the $\hat{\sigma}(k) = \int_{\mathbb{T}} \bar{\zeta}^k d\sigma$ are the Fourier coefficients of σ. By the Weierstrass approximation theorem, trigonometric polynomials are dense in $C(\mathbb{T})$. Hence the Fourier coefficients $\{\hat{\sigma}(k)\}_{k\in\mathbb{Z}}$ uniquely determine σ.

To prove that for any $F \in \mathfrak{R}(\mathbb{D})$ there is a $\sigma \in \mathfrak{P}(\mathbb{T})$ such that $F = F^\sigma$, we assume first that $G \in \mathfrak{R}(\mathbb{D})$ is analytic on the closed unit disc. Then by Cauchy's formula

$$G(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\zeta}{\zeta-z}\, G(\zeta)d\theta\, , \qquad \zeta = e^{i\theta}\, , \quad |z|<1\, ,$$

$$0 = \overline{\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\zeta}{\zeta-1/\bar{z}}\, G(\zeta)d\theta} = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{z}{z-\zeta}\, \overline{G(\zeta)}d\theta\, .$$

Putting $z = 0$ in the first equation, we see that the mean value of G and $\overline{G}$ against $\mathbb{T}$ is 1. Using an obvious formula, $(\zeta + z)/(\zeta - z) = -1 + 2\zeta/(\zeta - z)$, we obtain with the help of the above equalities that

$$G(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\,\mathrm{Re}\,\{G(\zeta)\}\,d\theta\,, \qquad |z| < 1\,.$$

Since $G(0) = 1$ and $G \in \mathfrak{R}(\mathbb{D})$, the measure $\mathrm{Re}\,\{G(\zeta)\}\,d\theta/2\pi$ is in $\mathfrak{P}(\mathbb{T})$. If $F \in \mathfrak{R}(\mathbb{D})$ then $F_r(z) = F(rz) \in \mathfrak{R}(\mathbb{D})$ for every $0 < r < 1$. Since $G = F_r$ is analytic on $\{z : |z| \leqslant 1\}$, the above formula can be applied. Since $\mathfrak{P}(\mathbb{T})$ is a compact subset of $M(\mathbb{T})$, the family $\mathrm{Re}\,\{F_r(\zeta)\}\,d\theta/2\pi$ has a limit point σ in the weak topology, implying that $F = F^\sigma$. $\square$

Corollary 8.3 *The mappings* (8.4) *are homeomorphisms of* $\mathfrak{P}(\mathbb{T})$ *with the* $*$*-weak topology onto the continuums* $\mathfrak{R}(\mathbb{D})$ *and* $\mathcal{B}$ *equipped with the topology of uniform convergence on compact subsets of* $\mathbb{D}$.

Proof This follows from (8.3) by the compactness principle for analytic functions. $\square$

For $\mu \in M(\mathbb{T})$ and $\zeta \in \mathbb{T}$ we say that μ is differentiable at ζ if the limit

$$\mu'(\zeta) = \lim_{|I|\to 0,\,\zeta\in I} \frac{1}{|I|}\int_I d\mu$$

exists and finite; $\mu'(\zeta)$ is called the Lebesgue derivative of μ at ζ. Here the limit is taken along the family of open arcs contracting to ζ. By the Lebesgue theorem on differentiation (see Garnett 1981) the derivative $\mu'(\zeta)$ exists almost everywhere on $\mathbb{T}$ with respect to Lebesgue measure m; in what follows we write this as m-a.e. on $\mathbb{T}$. Moreover, $d\mu$ can be uniquely represented as $d\mu = \mu' dm + d\mu_{\mathrm{s}}$, where $d\mu_{\mathrm{s}}$ is a measure singular to dm.

Theorem 8.4 (Fatou) *For every* $\sigma \in \mathfrak{P}(\mathbb{T})$ *and every* $\zeta \in \mathbb{T}$ *such that* σ *is differentiable at* ζ,

$$\lim_{z\to\zeta} \mathrm{Re}\,F^\sigma(z) = \sigma'(\zeta)\,,$$

where $z \to \zeta$ *in any angle within* $\mathbb{D}$ *whose vertex is at* ζ.

We refer to Koosis (1998, pp. 11–16) for a proof. Taking the real part of (8.3), we obtain

$$\mathrm{Re}\,F^\sigma(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2}\,d\sigma(\zeta) = \frac{1 - |zf^\sigma(z)|^2}{|1 - zf^\sigma(z)|^2}\,, \qquad |z| < 1\,. \tag{8.6}$$

Any $f \in \mathcal{B}$ has boundary values

$$f(\zeta) = \lim_{r\to 1^-} f(r\zeta) = f(\zeta) \qquad m\text{-a.e. on } \mathbb{T}\,.$$

Moreover, as in the case of F^σ the radial limit can be replaced by the angular limit; see Garnett (1981) or Koosis (1998). Combining this with Fatou's theorem, we obtain an important formula relating σ to f^σ and F^σ:

$$\sigma'(\zeta) = \operatorname{Re} F^\sigma(\zeta) = \frac{1-|f^\sigma(\zeta)|^2}{|1-\zeta f^\sigma(\zeta)|^2} \qquad m\text{-a.e. on } \mathbb{T}\,. \tag{8.7}$$

160 Weak convergence in $\mathfrak{P}(\mathbb{T})$ and Helly's theorems. We begin the systematic study of $\mathfrak{P}(\mathbb{T})$, $\mathfrak{R}(\mathbb{D})$ and $\mathcal{B}$ with $\mathfrak{P}(\mathbb{T})$. The indicators $\mathbf{1}_I$ of open arcs I having no point masses of σ at their ends make a useful set of test functions for (8.1).

Lemma 8.5 *If $*\text{-}\lim_n \mu_n = \mu$ in $\mathfrak{P}(\mathbb{T})$ then*

$$\lim_n \mu_n(I) = \mu(I) \tag{8.8}$$

for any open arc I on $\mathbb{T}$ such that μ vanishes at the end-points of I.

Proof Let $J = \mathbb{T} \setminus \operatorname{clos} I$, where $\operatorname{clos} I$ is the closure of I in $\mathbb{T}$. If $0 \leqslant f \leqslant 1$, $f \in C(\mathbb{T})$ and f is supported by I then

$$\liminf_n \mu_n(I) \geqslant \sup_f \lim_n \int_{\mathbb{T}} f d\mu_n = \sup_f \int_I f d\mu = \mu(I)\,.$$

Similarly, $\liminf_n \mu_n(J) \geqslant \mu(J)$. Since μ vanishes at the end:-points of J,

$$\limsup_n \mu_n(I) = \mu(\mathbb{T}) - \lim_n J \leqslant \mu(\mathbb{T}) - \mu(J) = \mu(I)\,,$$

which obviously implies (8.8). □

Theorem 8.6 (Helly) *Let $\{\mu_n\}_{n\geqslant 0}$ be a sequence in $\mathfrak{P}(\mathbb{T})$ and let $\mu \in \mathfrak{P}(\mathbb{T})$. Then $*\text{-}\lim_n \mu_n = \mu$ if and only if $\lim_n \mu_n(I) = \mu(I)$ for any open arc I on $\mathbb{T}$ such that μ does not have point masses at the end-points of I.*

Proof The necessity follows by Lemma 8.5. Applying (8.8) with $I = \mathbb{T}$ we get $\|\mu_n\| = \mu_n(\mathbb{T}) \leqslant C$ for some $C > 0$. For any $f \in C(\mathbb{T})$ and $\varepsilon > 0$ there is a $\delta = \delta(f, \varepsilon) > 0$ such that

$$\omega_f(\delta) \overset{\text{def}}{=} \sup_{|\zeta - z| \leqslant \delta} |f(\zeta) - f(z)| \leqslant \varepsilon\,.$$

Since $\{\zeta \in \mathbb{T} : \mu(\{\zeta\}) > 0\}$ is either finite or countable, given $\delta > 0$ there is a finite family $\mathfrak{F}$ of disjoint open arcs I on $\mathbb{T}$ such that $|I| < \delta$, $E = \mathbb{T} \setminus \cup_{I \in \mathfrak{F}} I$ is finite and $\mu(E) = 0$. Then $\lim_n \mu_n(E) = 0$ by (8.8). If

$$f_\varepsilon = \sum_{I \in \mathfrak{F}} f_I \mathbf{1}_I + \sum_{\zeta \in E} f(\zeta) \mathbf{1}_{\{\zeta\}}\,, \qquad f_I = \frac{1}{m(I)} \int_I f dm\,,$$

then

$$\lim_n \int_{\mathbb{T}} f_\varepsilon d\mu_n = \sum_{I\in\mathfrak{F}} f_I \lim_n \int_I d\mu_n + \sum_{\zeta\in E} f(\zeta) \lim_n \mu_n(\{\zeta\}) = \int_{\mathbb{T}} f_\varepsilon d\mu ,$$

$$\left|\int_{\mathbb{T}} f d\mu_n - \int_{\mathbb{T}} f_\varepsilon d\mu_n\right| \leqslant C \sup_{\mathbb{T}} |f - f_\varepsilon| \leqslant C\omega_f(\delta) \leqslant C\varepsilon .$$

It follows that

$$\limsup_n \left|\int_{\mathbb{T}} f d\mu_n - \int_{\mathbb{T}} f d\mu\right| \leqslant 2C\varepsilon + \lim_n \left|\int_{\mathbb{T}} f_\varepsilon d\mu_n - \int_{\mathbb{T}} f_\varepsilon d\mu\right| = 2C\varepsilon ,$$

which proves the theorem since ε is an arbitrary positive number. □

Lemma 8.5 and Theorem 8.6 are called Helly's theorems. We turn our attention to the study of $\mathcal{B}$.

161 Hardy spaces. The Hardy class H^p, $p > 0$, is defined as the class of all analytic functions f in $\mathbb{D}$ satisfying

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) = \|f\|_p^p < \infty .$$

It follows from Fatou's theorem 8.4 that for any $f \in H^p$ the radial limits $f(\zeta) = \lim_{r\to 1^-} f(r\zeta)$ exist almost everywhere on $\mathbb{T}$ with respect to Lebesgue measure m. By the uniqueness theorem for analytic functions any $f \in H^p$ can be identified with its boundary values. In (1928) V. I. Smirnov proved the following.

Theorem 8.7 (Smirnov 1928) *For every $F \in \mathfrak{R}(\mathbb{D})$ and $0 < p < 1$,*

$$\|F\|_p^p \leqslant \sec(p\pi/2) .$$

Proof If F is analytic about $\mathbb{D}$ then $F^p = |F|^p(\cos p\Phi + i \sin p\Phi)$ on $\mathbb{T}$. Since $-p\pi/2 \leqslant p\Phi \leqslant p\pi/2$, we obtain by the mean-value theorem applied to $\Re F^p$ that

$$\cos(p\pi/2) \int_{\mathbb{T}} |F|^p dm \leqslant \int_{\mathbb{T}} \operatorname{Re} F^p dm = \operatorname{Re} F(0)^p = 1 .$$

The result follows, since the $F_r = F(rz)$ are all analytic about $\mathbb{D}$. □

By Smirnov's theorem $F^\sigma \in \bigcap_{p<1} H^p$ for every $\sigma \in \mathfrak{P}(\mathbb{T})$.

Theorem 8.8 (Khinchin–Ostrovskii) *Let E be a subset of $\mathbb{T}$ of positive Lebesgue measure and $\{f_n\}_{n\geqslant 0}$ be a bounded sequence in H^p with $p > 0$. If f_n converges in measure to 0 on E, then $f_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$.*

The double arrows signify uniform convergence. A proof of the theorem as well as generalizations to other classes of functions can be found in Khrushchev (1978). See also the original proof in Privalov (1950).

162 Schur's algorithm and Wall continued fractions. Schur introduced and studied basic properties of his algorithm in two fundamental papers (1917, 1918). Starting with $f_0 = f$, Schur's algorithm determines a sequence $\{f_n\}_{n\geqslant 0}$ of functions in $\mathcal{B}$ related by the formulas

$$f_n(z) = \frac{zf_{n+1}(z) + a_n}{1 + \bar{a}_n z f_{n+1}(z)}\,, \quad f_n(0) = a_n\,, \qquad n = 0, 1, \ldots \tag{8.9}$$

Moving f_{n+1} in (8.9) inside the fraction as far as possible, so that

$$f_n(z) = a_n + \frac{(1-|a_n|^2)z}{\bar{a}_n z + 1/f_{n+1}(z)}\,,$$

we obtain by iteration the Wall continued fraction (Wall 1944):

$$f(z) = a_0 + \frac{(1-|a_0|^2)z}{\bar{a}_0 z} \; \frac{1}{+\, a_1} \; \frac{(1-|a_1|^2)z}{+\, \bar{a}_1 z} \; {}_{+\cdots}\,. \tag{8.10}$$

The numbers $a_n = f_n(0)$ in Schur's algorithm are called the *Schur parameters* of f, and the f_n are called its *Schur functions*.

Schur's algorithm can be run until $|a_n| = 1$. If $|a_n| = 1$ then, by the maximum principle, $f_n \equiv a_n$ in $\mathbb{D}$ and the algorithm breaks down. As Schur (1917, §1) noticed, formula (8.9) shows that now we may put $f_k \equiv 0$ for every $k > n$. Then clearly $a_k = 0$ for $k > n$. This agreement, however, contradicts the definition of continued fractions, demanding that the partial numerators of (8.10) must be nonzero. Therefore we stop Schur's algorithm if $|a_n| = 1$. Otherwise it runs to infinity. By the maximum principle, $|a_k| < 1$ for $k < n$ where $|a_n| = 1$ if a_n exists.

Schur's algorithm is to a great extent nothing other than the algorithm of Wall continued fractions. The difference is that whereas the second constructs the convergents, the first constructs the even remainders $\{f_n\}_{n\geqslant 0}$ of the corresponding continued fraction.

The formula (8.9) can be viewed as arising from the substitution of w in the Möbius transformation $\tau_n(w) = (zw + a_n)/(1 + \bar{a}_n w)$ with $f_{n+1}(z)$. Hence

$$f(z) = \tau_0 \circ \tau_1 \circ \cdots \circ \tau_n(f_{n+1})\,. \tag{8.11}$$

Any polynomial p_n of degree n in z determines a conjugate polynomial $p_n^*(z) = z^n\overline{p_n(1/\bar{z})}$. The coefficient vector of p_n^* in $\mathbb{C}^{n+1}$ is the vector of the complex conjugate coefficients of p_n, written in reverse order. Hence $\deg p_n^* \leqslant n$. Since $z = 1/\bar{z}$ on $\mathbb{T}$, this formula is equivalent to

$$p_n^*(z) = z^n\overline{p_n(z)}\,, \qquad z \in \mathbb{T}\,. \tag{8.12}$$

Lemma 8.9 *For every n,*

$$\tau_0 \circ \tau_1 \circ \cdots \circ \tau_n(w) = \frac{A_n + zB_n^* w}{B_n + zA_n^* w}\,, \tag{8.13}$$

where $A_n, B_n \in \mathcal{P}_n$, $B_0 = B_0^* \equiv 1$, $A_0 \equiv a_0$, $A_0^* = \bar{a}_0$.

Proof Suppose that (8.13) holds for n. Then for every $w \in \mathbb{C}$,

$$\tau_0 \circ \tau_1 \circ \cdots \circ \tau_n(\tau_{n+1}(w)) = \frac{A_n + a_{n+1} z B_n^* + z(zB_n^* + \bar{a}_{n+1} A_n) w}{B_n + a_{n+1} z A_n^* + z(zA_n^* + \bar{a}_{n+1} B_n) w} .$$

It follows that

$$\begin{aligned} B_{n+1}^* &= zB_n^* + \bar{a}_{n+1} A_n, & A_{n+1}^* &= zA_n^* + \bar{a}_{n+1} B_n, \\ A_{n+1} &= A_n + a_{n+1} z B_n^*, & B_{n+1} &= B_n + a_{n+1} z A_n^*, \end{aligned} \tag{8.14}$$

completing the proof. □

Putting $w = 0$ in (8.13), we obtain that

$$\tau_0 \circ \tau_1 \circ \cdots \circ \tau_n(0) = \frac{A_n}{B_n} . \tag{8.15}$$

If $w = \infty$ in (8.13) then $\tau_0 \circ \tau_1 \circ \cdots \circ \tau_n(\infty) = zB_n^*/zA_n^*$. Therefore A_n/B_n are the even convergents of order $2n$, and zB_n^*/zA_n^* are the odd convergents of order $2n+1$, to the Wall continued fraction (8.10). It follows that the formulas (8.14) are equivalent to the Euler–Wallis formulas for (8.10). The polynomials $\{A_n\}_{n \geqslant 0}$, $\{B_n\}_{n \geqslant 0}$ associated with the Wall continued fraction (8.10) by (8.14) were first introduced in Schur (1918, §14). However, since the important explicit connection of Schur's algorithm with continued fractions was first considered in Wall (1944), we will call these polynomials the *Wall polynomials* (of (8.10))[1].

Following §**45** at the start of Section 2.4, we represent (8.14) as

$$\begin{pmatrix} zB_n^* & -A_n^* \\ -zA_n & B_n \end{pmatrix} = \prod_{k=0}^{n} \begin{pmatrix} z & -\bar{a}_k \\ -a_k z & 1 \end{pmatrix} . \tag{8.16}$$

The basic properties of A_n, B_n are deduced from the *determinant* identity first obtained in Schur (1918, §14):

$$B_n^* B_n - A_n^* A_n = z^n \prod_{k=0}^{n} (1 - |a_k|^2) = \omega_n z^n . \tag{8.17}$$

To get (8.17) apply the multiplicative functional $C \to \det C$ to both sides of (8.16). Restricting (8.17) to $\mathbb{T}$ and applying (8.12), we obtain

$$|B_n(\zeta)|^2 - |A_n(\zeta)|^2 \equiv \omega_n , \qquad \zeta \in \mathbb{T} , \tag{8.18}$$

which in particular implies that $|A_n/B_n| < 1$ on $\mathbb{T}$ provided that $|a_n| < 1$.

Lemma 8.10 *The Wall polynomials B_n do not vanish in $\{z : |z| \leqslant 1\}$.*

[1] In his papers Schur described his algorithm as "continued-fraction-like".

Proof First we note that $B_0 \equiv 1$. If $B_n \neq 0$ in $|z| \leqslant 1$ then both A_n/B_n and A_n^*/B_n are holomorphic on $\{z : |z| \leqslant 1\}$. By (8.18) they are less than 1 on $\mathbb{T}$. Hence $A_n/B_n, A_n^*/B_n \in \mathcal{B}$ by the maximum principle. If $|z| \leqslant 1$ then by (8.14)

$$|B_{n+1}(z)| = |B_n(z) + a_{n+1} z A_n^*| \geqslant |B_n(z)| \left(1 - |a_{n+1}| \left|\frac{A_n^*}{B_n}\right|\right) > 0\,,$$

implying that $B_{n+1} \neq 0$ in $\{z : |z| \leqslant 1\}$ even if $|a_{n+1}| = 1$. □

Corollary 8.11 *Let $\{a_n\}_{n\geqslant 0}$ be a sequence of Schur parameters that can be finite or infinite. Then for every n such that $|a_n| < 1$,*

$$\left\|\frac{A_n}{B_n}\right\|_\infty < 1\,, \qquad \left\|\frac{A_n^*}{B_n}\right\|_\infty < 1\,.$$

Proof By Lemma 8.10 both functions are analytic on $\{z : |z| \leqslant 1\}$. To complete the proof apply the maximum principle and (8.18). □

By Corollary 8.11 every even convergent A_n/B_n to (8.10) is an interior point of $\mathcal{B}$ if $|a_n| < 1$. If $|a_n| = 1$ then $\omega_n = 0$ and $|A_n/B_n| = 1$ on $\mathbb{T}$ by (8.18). It follows that any function $f \in \mathcal{B}$ with a finite number of Schur parameters is a rational function $f = A_n/B_n$ unimodular on $\mathbb{T}$.

Lemma 8.12 (Schur 1917, §2) *A rational function $f \in \mathcal{B}$ is unimodular, $|f| = 1$, on $\mathbb{T}$ if and only if $f = \lambda p/p^*$, where p is a monic polynomial with roots in $\mathbb{D}$ and $\lambda \in \mathbb{T}$. The factorization $f = \lambda p/p^*$ of f is unique.*

Proof Since $f \in \mathcal{B}$, all poles of f lie in $\{z : |z| > 1\}$. Since f maps $\mathbb{T}$ into $\mathbb{T}$, by Schwarz symmetry the zeros $\lambda_1, \ldots, \lambda_n$ of f are symmetric to the poles $1/\bar{\lambda}_1, \ldots, 1/\bar{\lambda}_n$ with respect to $\mathbb{T}$. Hence $\lambda_k \in \mathbb{D}$, $1 \leqslant k \leqslant n$. If $p(z) = (z - \lambda_1) \cdots (z - \lambda_n)$ then $p^*(z) = z^n \overline{p(1/\bar{z})} = (1 - \bar{\lambda}_1 z) \cdots (1 - \bar{\lambda}_n z)$, implying that

$$b(z) = \frac{p(z)}{p^*(z)} = \prod_{k=1}^{n} \frac{z - \lambda_k}{1 - \bar{\lambda}_k z} \tag{8.19}$$

has the same poles and zeros as f. Then by Liouville's theorem the quotient f/b must be a constant λ. Since $|f| = |b| = 1$ on $\mathbb{T}$, we obtain that $\lambda \in \mathbb{T}$. The inverse statement follows from (8.12). □

In the theory of Hardy spaces, see Garnett (1981), the products (8.19) are called *finite Blaschke products*. They are often normalized by including $-|\lambda_k|/\lambda_k$ in the multipliers to make them positive at $z = 0$. This makes it obvious when the Blaschke products

$$B(z) = \prod_{k=1}^{\infty} -\frac{|\lambda_k|}{\lambda_k} \frac{z - \lambda_k}{1 - \bar{\lambda}_k z} \tag{8.20}$$

converge: $\prod_k |\lambda_k| > 0$. By Lemma 8.12 finite Blaschke products can be described as rational functions unimodular on $\mathbb{T}$ and analytic in $\mathbb{D}$. Notice that the Schur functions and parameters of $(z-\lambda_k)/(1-\bar{\lambda}_k z)$ are $f_1 \equiv 1$, $a_1 = 1$, $a_0 = -\lambda_k$. For a finite Blaschke product b, we denote by $Z(b)$ the total number of zeros of b.

Theorem 8.13 (Schur 1917, §2) *A function $b \in \mathcal{B}$ has a finite number of Schur parameters if and only if it is a finite Blaschke product. The number of parameters of b equals $Z(b)+1$.*

Proof Let $|a_n| = 1$. Then $f = A_n/B_n$ is unimodular on $\mathbb{T}$ and is a finite Blaschke product by Lemma 8.12. If $Z(b) = 1$ then the number of Schur parameters is 2. Suppose the theorem is true for every b with $Z(b) < n$ and consider a b with $Z(b) = n$. Then by Schur's algorithm

$$b = \frac{zb_1(z) + b(0)}{1 + \overline{b(0)}zb_1(z)} \quad \Rightarrow \quad b_1 = \frac{b - b(0)}{z(1 - \overline{b(0)}b)} \,.$$

Since $|b(0)| < 1$ and $b(z) - b(0)$ vanishes at $z = 0$, the rational function b_1 is analytic on $\{z : |z| \leqslant 1\}$. The elementary identity

$$1 - \left| \frac{w+\lambda}{1+\bar{\lambda}w} \right|^2 = \frac{(1-|\lambda|^2)(1-|w|^2)}{|1+\bar{\lambda}w|^2} \,, \qquad |\lambda| \leqslant 1 \,, \quad |w| \leqslant 1 \,, \tag{8.21}$$

shows that $|b_1| = 1$ on $\mathbb{T}$. By Lemma 8.12 b_1 is a finite Blaschke product. To find its Schur representation we observe that $b(0) = \lambda(-\lambda_1)\cdots(-\lambda_n)$. It follows that

$$b_1 = \lambda \frac{p - (-\lambda_1)\cdots(-\lambda_n)p^*}{z(p^* - (-\bar{\lambda}_1)\cdots(-\bar{\lambda}_n)p)} \,.$$

Easy algebra shows that the polynomial $q = (p - (-\lambda_1)\cdots(-\lambda_n)p^*)/z$ is a polynomial of degree $\deg q = \deg p - 1$ and that $b_1(z) = \lambda q/q^*$. Since

$$q^* = p^* - (-\bar{\lambda}_1)\cdots(-\bar{\lambda}_n)p) = p^*(1 - \overline{b(0)}b) \,,$$

the polynomial q^* does not vanish in $\{z : |z| \leqslant 1\}$. Hence we have obtained the Schur factorization of b_1. Since $Z(b_1) = Z(b) - 1 = n - 1$, the result follows by induction. □

Euler's formula (1.17) for Schur's algorithm is

$$f = \frac{A_n + zB_n^* f_{n+1}}{B_n + zA_n^* f_{n+1}} \,, \tag{8.22}$$

from (8.13), putting $w = f_{n+1}$. Taking into account that (8.22) is the Euler formula for Wall continued fractions, we can describe the set $\mathcal{B}_n = \mathcal{B}_n(a_0, \ldots, a_n)$ of all functions in $\mathcal{B}$ whose first $n+1$ Schur parameters $\{a_0, \ldots, a_n\}$ are fixed, $|a_n| < 1$, in terms of the Wall polynomials associated with $\{a_0, \ldots, a_n\}$,

$$\mathcal{B}_n = \left\{ \frac{A_n + zB_n^*\mathcal{E}}{B_n + zA_n^*\mathcal{E}} : \mathcal{E} \in \mathcal{B} \right\} \,. \tag{8.23}$$

Since $\mathcal{E} \equiv 0$ has zero Schur parameters, by putting $\mathcal{E} \equiv 0$ we obtain from (8.23) the Schur parameters of A_n/B_n: $\{a_0, a_1, \ldots, a_n, 0, 0, \ldots\}$. By Corollary 8.11 $A_n^*/B_n \in \mathcal{B}$. Then by

$$\frac{A_n^*}{B_n} = \frac{zA_{n-1}^* + \bar{a}_{n-1}B_{n-1}}{B_{n-1} + a_n z A_{n-1}^*} = \frac{zA_{n-1}^*/B_{n-1} + \bar{a}_n}{1 + a_n z A_{n-1}^*/B_{n-1}}$$

and (8.9), the Schur parameters of A_n^*/B_n are $\{-\bar{a}_n, -\bar{a}_{n-1}, \ldots, -\bar{a}_0, \ldots\}$.

Lemma 8.14 *The elements of $\mathcal{E}_n$ have the same Taylor polynomial of order n at $z = 0$.*

Proof This follows by (8.23) and (8.17) from the formula

$$\frac{A_n + zB_n^*\mathcal{E}}{B_n + zA_n^*\mathcal{E}} - \frac{A_n}{B_n} = z^{n+1}\mathcal{E}\frac{\omega_n}{B_n(B_n + zA_n^*\mathcal{E})}\,. \tag{8.24}$$

□

The lemma below is useful in computations with Wall polynomials.

Lemma 8.15 *For every n,*

$$A_n = a_0 + \cdots + a_n z^n\,, \qquad B_n = 1 + \cdots + a_n \bar{a}_0 z^n\,. \tag{8.25}$$

Proof This follows immediately from (8.14) by induction. □

A number of useful identities related to (8.22) are collected in (E8.1)–(E8.6).

163 The convergence of Schur's algorithm. Schur proved the uniform convergence of A_n/B_n to f on compact subsets of $\mathbb{D}$.

Theorem 8.16 (Schur 1918, §15) *If $f \in \mathcal{B}$ is not a finite Blaschke product and $\{A_n\}_{n\geqslant 0}$, $\{B_n\}_{n\geqslant 0}$ are its Wall polynomials then*

$$\frac{A_n(z)}{B_n(z)} \rightrightarrows f(z)$$

uniformly on compact subsets of $\mathbb{D}$.

Proof By Theorem 8.13 $\{A_n/B_n\}_{n\geqslant 0}$ is infinite. If $\mathcal{E} = f_{n+1}$ in (8.24) then we obtain the Lagrange formula (1.50) for Schur's algorithm:

$$f(z) - \frac{A_n(z)}{B_n(z)} = z^{n+1} f_{n+1}\frac{\omega_n}{B_n^2(1 + z(A_n/B_n)f_{n+1}}\,. \tag{8.26}$$

By Corollary 8.11 $\{A_n/B_n\}_{n\geqslant 0}$ is compact in $\mathbb{D}$. By Lemma 8.14 its limit point is analytic in $\mathbb{D}$ and equals f.

Theorem 8.16 can be also proved constructively. Since $f_{n+1} \in \mathcal{B}$ and $A_n/B_n \in \mathcal{B}$, the right-hand side of (8.26) is analytic in $\mathbb{D}$ and so

$$\left| f(z) - \frac{A_n(z)}{B_n(z)} \right| \leqslant |z|^{n+1} \frac{1}{|1 + zA_n^* B_n^{-1} f_{n+1}|} \leqslant \frac{|z|^{n+1}}{1-|z|} . \tag{8.27}$$

Observe that $\omega_n |B_n(z)|^{-2} \leqslant 1$ in $\mathbb{D}$ by (8.18) and by the maximum principle. □

The odd convergents to any Wall continued fraction are related to the even convergents by a simple formula:

$$\frac{zB_n^*}{zA_n^*} = \frac{\overline{B_n(1/\bar{z})}}{\overline{A_n(1/\bar{z})}} = \overline{\left(\frac{A_n(1/\bar{z})}{B_n(1/\bar{z})} \right)^{-1}} .$$

Hence one may just consider the even convergents A_n/B_n. By (5.34) the even part of a Wall continued fraction exists if and only if $a_k \neq 0$ for $k = 0, 1, \ldots$ If this is the case then the even part of a Wall continued fraction is given by

$$f(z) = \frac{a_0}{1} \underset{-}{} \frac{(1-|a_0|^2)(a_1/a_0)z}{1 + (a_1/a_0)z} \underset{-}{} \cdots \underset{-}{} \frac{(1-|a_n|^2)(a_{n+1}/a_n)z}{1+(a_{n+1}/a_n)z} \underset{-}{} \cdot \tag{8.28}$$

The continued fractions (8.28) were studied by Geronimus (1944) in relation to orthogonal polynomials on $\mathbb{T}$ and are called Geronimus continued fractions. Geronimus continued fractions differ from those of Euler corresponding to the series

$$S(z) = \sum_{n=0}^{\infty} a_n z^n$$

by $(1-|a_n|^2)$ in the partial numerators; see (4.2). Later we show that if $a_n \to 0$ sufficiently fast then $S(z)$ behaves similarly to (8.28).

Notice that the Geronimus continued fraction (8.28), converges uniformly and absolutely on compact subsets of $\mathbb{D}$; see (3.8). This follows from (8.27).

164 The algebra of Schur's algorithm. We turn to the third continuum $\mathfrak{R}(\mathbb{D})$ associated with $\mathbb{T}$. Following their definition, we substitute the convergents

$$\frac{1}{0}, \quad \frac{A_0}{B_0}, \quad \frac{zB_0^*}{zA_0^*}, \quad \frac{A_1}{B_1}, \quad \frac{zB_1^*}{zA_1^*}, \quad \ldots, \quad \frac{A_n}{B_n}, \quad \frac{zB_n^*}{zA_n^*}, \quad \ldots,$$

of the continued fraction (8.10) into (8.2) and obtain the convergents to F^σ,

$$\frac{1}{0}, \quad \frac{1}{1}, \quad \frac{1}{-1}, \quad \frac{\Psi_1^*}{\Phi_1^*}, \quad \frac{z\Psi_1}{-z\Phi_1}, \quad \ldots, \quad \frac{\Psi_n^*}{\Phi_n^*}, \quad \frac{z\Psi_n}{-z\Phi_n}, \quad \ldots, \tag{8.29}$$

Here by (8.3)

$$\begin{aligned} \Phi_{n+1} &= zB_n^* - A_n^*, & \Psi_{n+1} &= zB_n^* + A_n^*, \\ \Phi_{n+1}^* &= B_n - zA_n, & \Psi_{n+1}^* &= B_n + zA_n . \end{aligned} \tag{8.30}$$

These formulas can be conveniently compressed into matrix form:

$$\begin{pmatrix} \Phi_{n+1} & \Psi_{n+1} \\ \Phi^*_{n+1} & -\Psi^*_{n+1} \end{pmatrix} = \begin{pmatrix} zB^*_n & -A^*_n \\ -zA_n & B_n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{8.31}$$

or equivalently, see (8.16),

$$\begin{pmatrix} \Phi_{n+1} & \Psi_{n+1} \\ \Phi^*_{n+1} & -\Psi^*_{n+1} \end{pmatrix} = \prod_{k=0}^{n} \begin{pmatrix} z & -\bar{a}_k \\ -a_k z & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{8.32}$$

Corollary 8.17 *For* $n = 0, 1, \dots,$

$$\Phi_{n+1}\Psi^*_{n+1} + \Phi^*_{n+1}\Psi_{n+1} = 2z^{n+1} \prod_{k=0}^{n} (1 - |a_k|^2). \tag{8.33}$$

Proof Apply the multiplicative functional $C \to \det C$ to (8.32). □

The recurrence form of (8.32) is as follows:

$$\begin{pmatrix} \Phi_{n+1} & \Psi_{n+1} \\ \Phi^*_{n+1} & -\Psi^*_{n+1} \end{pmatrix} = \begin{pmatrix} z & -\bar{a}_n \\ -a_n z & 1 \end{pmatrix} \begin{pmatrix} \Phi_n & \Psi_n \\ \Phi^*_n & -\Psi^*_n \end{pmatrix}.$$

Considering the matrix entries, we obtain

$$\begin{aligned} \Phi_{n+1} &= z\Phi_n - \bar{a}_n \Phi^*_n, & \Psi_{n+1} &= z\Psi_n + \bar{a}_n \Psi^*_n, \\ \Phi^*_{n+1} &= \Phi^*_n - a_n z \Phi_n, & \Psi^*_{n+1} &= \Psi^*_n + a_n z \Psi_n. \end{aligned} \tag{8.34}$$

By Lemma 8.15,

$$\Phi^*_{n+1}(z) = 1 + \cdots - a_n z^{n+1} \quad \Rightarrow \quad \Phi_{n+1} = z^{n+1} + \cdots - \bar{a}_n. \tag{8.35}$$

Hence the Φ_{n+1} are *monic* polynomials. The convergent Ψ^*_n/Φ^*_n is at the $2n$th place in (8.29). Let us assume that all the parameters a_n are nonzero. Then a simple substitution of the Geronimus continued fraction (8.28) for the Schur function f into the second expression in (8.2) shows that

$$\frac{1}{0}, \quad \frac{1}{1}, \quad \frac{\Psi^*_1}{\Phi^*_1}, \quad \dots, \quad \frac{\Psi^*_n}{\Phi^*_n}, \quad \dots \tag{8.36}$$

are the convergents to the continued fraction

$$\begin{aligned} F^\sigma(z) &= \frac{1 + zf^\sigma}{1 - zf^\sigma} = 1 + \frac{2zf^\sigma}{1 - zf^\sigma} \\ &\sim 1 + \frac{2a_0 z}{1 - a_0 z} - \frac{(1 - |a_0|^2)(a_1/a_0)z}{1 + (a_1/a_0)z} - \cdots \\ &\quad - \frac{(1 - |a_{n-1}|^2)(a_n/a_{n-1})z}{1 + (a_n/a_{n-1})z} - \cdots, \end{aligned} \tag{8.37}$$

which is called the *Geronimus continued fraction* for the Herglotz function F^σ. By (8.31) $\Psi_1^*/\Phi_1^* = 1+2a_0z(1-a_0z)^{-1}$ and

$$\frac{\Psi_{n+1}^*}{\Phi_{n+1}^*} = 1 + \frac{2a_0z}{1-a_0z} \, \frac{-}{} \, \frac{(1-|a_0|^2)(a_1/a_0)z}{1+(a_1/a_0)z} \, \frac{-}{} \cdots \frac{-}{} \, \frac{(1-|a_{n-1}|^2)(a_n/a_{n-1})z}{1+(a_n/a_{n-1})z} \tag{8.38}$$

for $n > 0$ by (8.28) and (8.31). Following (8.36) we put $\Psi_0^* = \Phi_0^* = 1$, $\Psi_{-1}^* = 1$, $\Phi_{-1}^* = 0$. The Euler–Wallis formulas (1.15) and equation (8.38) show that $X_n = \Psi_n^*$ and $X_n = \Phi_n^*$ satisfy the recurrence relations

$$X_{n+1} = \big(1+\frac{a_n}{a_{n-1}}z\big)X_n - (1-|a_{n-1}|^2)\frac{a_n}{a_{n-1}}zX_{n-1}\,, \qquad n \geqslant 1\,,$$

$$X_1 = (1-a_0z)X_0 + 2a_0zX_{-1}\,.$$

By (8.31) $\Phi_{n+1}^*(z) = B_n(z)\,(1-zA_n^*(z)/B_n(z)) \neq 0$ for $|z| \leqslant 1$, which agrees with Lemma 7.21. Hence the $\Psi_{n+1}^*/\Phi_{n+1}^*$ are analytic on $|z| \leqslant 1$.

165 Geronimus' theorems. The first is a theorem relating orthogonal polynomials to Schur's parameters. We begin with a lemma.

Lemma 8.18 *Suppose that polynomials P_n and Q_n of degree n satisfy*

$$Q_n(z)+P_n(z)F^\sigma(z) = O\left(z^{-1}\right)\,, \qquad z \longrightarrow \infty\,, \tag{8.39}$$

$$Q_n(z)+P_n(z)F^\sigma(z) = O(z^n)\,, \qquad z \longrightarrow 0\,, \tag{8.40}$$

where $\sigma \in \mathfrak{P}(\mathbb{T})$. Then $P_n \perp \{\mathbf{1}, \zeta, \ldots \zeta^{n-1}\}$ in $L^2(d\sigma)$ and

$$Q_n(z) = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}(P_n(\zeta)-P_n(z))d\sigma(\zeta)\,. \tag{8.41}$$

Proof If $z \to \infty$ in the identity

$$Q_n + P_nF^\sigma = Q_n - \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\{P_n(\zeta)-P_n(z)\}d\sigma + \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}P_n d\sigma\,,$$

then by (8.39) the left-hand side is $O(1/z)$. Since the first summand on the right-hand side is a polynomial and since

$$\lim_{z\to\infty} \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}P_n d\sigma = -\int_{\mathbb{T}} P_n d\sigma\,,$$

we obtain that

$$Q_n = \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\{P_n(\zeta)-P_n(z)\}d\sigma + \int_{\mathbb{T}} P_n d\sigma\,.$$

It follows that

$$Q_n + P_n F^\sigma = \int_{\mathbb{T}} P_n d\sigma + \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} P_n d\sigma = 2\int_{\mathbb{T}} \frac{1}{1-\bar{\zeta} z} P_n d\sigma$$
$$= 2\int P_n d\sigma + 2z\int \bar{\zeta} P_n d\sigma + \cdots + 2z^{n-1}\int \bar{\zeta}^{n-1} P_n d\sigma + \cdots,$$

which completes the proof by (8.40) if $z \to 0$. □

Lemma 8.19 *For any $\sigma \in \mathfrak{P}(\mathbb{T})$,*

$$\Psi_n + \Phi_n F^\sigma = -\frac{2\bar{a}_n \omega_{n-1}}{z} + O(1/z^2), \qquad z \to \infty, \tag{8.42}$$

$$\Psi_n + \Phi_n F^\sigma = 2\omega_{n-1} z^n + O(z^{n+1}), \qquad z \to 0, \tag{8.43}$$

$$-\Psi_n^* + \Phi_n^* F^\sigma = 2a_n \omega_{n-1} z^{n+1} + O(z^{n+2}), \qquad z \to 0, \tag{8.44}$$

$$-\Psi_n^* + \Phi_n^* F^\sigma = -2\omega_{n-1} + O(1/z), \qquad z \to \infty. \tag{8.45}$$

Proof Elementary calculations with (8.22), (8.17) and (8.31) show that

$$\Psi_n + \Phi_n F^\sigma = \frac{2\omega_{n-1} z^n}{(1-zf)(B_{n-1} + zA_{n-1}^* f_n)} = 2\omega_{n-1} z^n + O(z^{n+1}),$$

$$-\Psi_n^* + \Phi_n^* F^\sigma = \frac{2\omega_{n-1} f_n z^{n+1}}{(1-zf)(B_{n-1} + zA_{n-1}^* f_n)} = 2a_n \omega_{n-1} z^{n+1} + O(z^{n+2})$$

as $z \to 0$, implying (8.43), (8.44). Since $\overline{F^\sigma(1/\bar{z})} = -F^\sigma(z)$, two other formulas follow by conjugation. □

Theorem 8.20 (Geronimus 1944) *For every $\sigma \in \mathfrak{P}(\mathbb{T})$ the family of monic polynomials $\{\Phi_n\}_{n\geqslant 0}$ is orthogonal in $L^2(d\sigma)$:*

$$\int_{\mathbb{T}} \Phi_n \bar{\Phi}_k d\sigma = 0 \quad \text{if } n \neq k.$$

Proof $P_n = \Phi_n$ and $\Psi_n = Q_n$ satisfy Lemma 8.19 by (8.42), (8.43). □

By (8.41) we obtain that

$$\Psi_n(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} (\Phi_n(\zeta) - \Phi_n(z)) d\sigma(\zeta). \tag{8.46}$$

Theorem 8.21 (Geronimus 1944) *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ and let $f = f^\sigma$ be the Schur function of σ with parameters $\{a_n\}_{n\geqslant 0}$. Let $\{\Phi_n\}_{n\geqslant 0}$ be a family of monic orthogonal polynomials in $L^2(d\sigma)$ such that* deg $\Phi_n = n$, $\|\Phi_n\| > 0$. *Then*

$$a_n = -\overline{\Phi_{n+1}(0)}.$$

Proof By Theorem 8.20 the polynomials $\{\Phi_n\}_{n\geqslant 0}$ are orthogonal in $L^2(d\sigma)$. By (8.35) $a_n = -\overline{\Phi_{n+1}(0)}$. The proof is completed by Ex. 8.13. □

Notice that $\|\Phi_n\| = 0$ if and only if σ is a finite convex combination of point masses on $\mathbb{T}$ located at the zeros of Φ_n. For $\sigma \in \mathfrak{P}(\mathbb{T})$ we define

$$N(\sigma) = \begin{cases} k & \text{if } \sigma \text{ is a sum of } k \text{ point masses,} \\ \infty & \text{otherwise,} \end{cases}$$

to be the number of growth points for σ. If $n < N(\sigma)$ then $\|\Phi_n\| > 0$, since Φ_n cannot vanish on σ. If $n = N(\sigma) < +\infty$ then there is a unique monic polynomial Φ_n, $\deg \Phi_n = n$, vanishing on σ.

By Theorem 8.20 we can rewrite formulas (8.42)–(8.45). Since Φ_n is orthogonal to $\Phi_0 \equiv 1$, and to ζ^k for $k < n$, we obtain by (8.46) that

$$\Psi_n + \Phi_n F^\sigma = 2\int_{\mathbb{T}} \frac{\Phi_n(\zeta)d\sigma(\zeta)}{1-\bar{\zeta}z} = 2z^n \int_{\mathbb{T}} \frac{\bar{\zeta}^n\Phi_n(\zeta)d\sigma(\zeta)}{1-\bar{\zeta}z} \,. \tag{8.47}$$

Comparing (8.47) with (8.43), we obtain a circular analogue of Chebyshev's formula (7.34):

$$\prod_{k=0}^{n-1}(1-|a_k|^2) = \omega_{n-1} = \int_{\mathbb{T}} \bar{\zeta}^n \Phi_n d\sigma = \int_{\mathbb{T}} |\Phi_n|^2 d\sigma = \int_{\mathbb{T}} \Phi_n^* d\sigma \,. \tag{8.48}$$

Passing to the conjugate polynomials in (8.47), we get by Theorem 8.20

$$F^\sigma - \frac{\Psi_n^*}{\Phi_n^*} = \frac{2z}{\Phi_n^*}\int_{\mathbb{T}} \frac{\Phi_n^*(\zeta)d\sigma(\zeta)}{\zeta - z} = \frac{z^n}{\Phi_n^*}\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, \overline{\Phi_n(\zeta)}\, d\sigma(\zeta) \,. \tag{8.49}$$

Formulas (8.47) and (8.49) are due to Geronimus. We will write (8.49) as

$$F^\sigma - \frac{\Psi_n^*}{\Phi_n^*} = \frac{2z^{n+1}}{\Phi_n^*\Phi_n}\int_{\mathbb{T}} \frac{\overline{\Phi_n(\zeta)}\;\Phi_n(z)}{\zeta-z} d\sigma = \frac{2z^{n+1}}{\Phi_n^*\Phi_n}\int_{\mathbb{T}} \frac{|\Phi_n(\zeta)|^2}{\zeta - z} d\sigma \,. \tag{8.50}$$

8.2 The Gram–Schmidt algorithm

166 Orthogonal vectors in a Hilbert space. By Theorem 8.20 orthogonal polynomials in $L^2(d\sigma)$ satisfy three-term recurrence relations (equivalently, the Euler–Wallis formulas). With this in mind one can develop the theory of orthogonal polynomials on $\mathbb{T}$ starting with the property of orthogonality. This alternative approach to orthogonal polynomials is based on the Gram–Schmidt algorithm, which transforms a family $\{v_n\}_{n\geqslant 0}$ (finite or infinite) of linear independent vectors in a Hilbert space H into a family $\{\varphi_n\}_{n\geqslant 0}$ of orthogonal unit vectors in H. The steps of the Gram–Schmidt algorithm are as follows.

Step 1. For $l_{00} = \|f_0\|^{-1} > 0$ let $\varphi_0 = l_{00} v_0$.

Step 2. Let the vector φ_1 be defined by $\varphi_1 = l_{10} v_0 + l_{11} v_1$, where $l_{11} > 0$ and the values of l_{00} and l_{11} are determined by the following conditions of orthogonality: $(\varphi_1, \varphi_0) = 0$ and $(\varphi_1, \varphi_1) = 1$.

Step $n+1$. If $\varphi_0, \varphi_1, \dots, \varphi_{n-1}$ are already defined then let

$$\varphi_n = l_{n0} v_0 + \cdots + l_{nn} v_n \,, \qquad l_{nn} > 0 \,, \tag{8.51}$$

where the coefficients $\{l_{nj}\}_{0 \leqslant j \leqslant n}$ are uniquely determined by a system of linear equations

$$(\varphi_n, \varphi_k) = \delta_{nk} \,, \qquad k = 0, 1, \dots, n \,. \tag{8.52}$$

Since $\{v_n\}_{n \geqslant 0}$ is a linearly independent family, the number of steps in the Gram–Schmidt algorithm equals the number of elements in this family. The orthogonal family $\{\varphi_n\}_{n \geqslant 0}$ of unit vectors is determined by the property

$$V_n = \operatorname{span}\{v_0, \dots, v_n\} \stackrel{\text{def}}{=} \left\{ \textstyle\sum_{k=0}^{n} \lambda_k v_k : \lambda_k \in \mathbb{C} \right\} = \operatorname{span}\{\varphi_0, \dots, \varphi_n\} \,.$$

By (8.52) φ_n is orthogonal to V_{n-1}, i.e. $\varphi_n \perp V_{n-1}$. By (8.51) we have $l_{nn} v_n - \varphi_n \in V_{n-1}$. It follows that

$$(\varphi_n, v_n) = l_{nn}^{-1} \{ (\varphi_n, l_{nn} v_n - \varphi_n) + (\varphi_n, \varphi_n) \} = l_{nn}^{-1}$$

and that $v_n - l_{nn}^{-1} \varphi_n$ is the orthogonal projection of f_n onto V_{n-1}. Hence

$$\operatorname{dist}(v_n, V_{n-1}) = \| v_n - (v_n - l_{nn}^{-1} \varphi_n) \| = l_{nn}^{-1} \,, \tag{8.53}$$

explaining the meaning of the diagonal coefficients l_{nn} in (8.51).

Again, since $\{v_n\}_{n \geqslant 0}$ is a linearly independent family the quadratic forms in

$$\sum_{i,j=0}^{n} \lambda_i \bar{\lambda}_j v_i v_j = \left\| \sum_{k=0}^{n} \lambda_k v_k \right\|^2 > 0 \tag{8.54}$$

are positive definite, implying that the determinants $D_n = \det\{(v_i, v_j)\}_{i,j=0}^{n}$ are positive. For any matrix $\mathbf{C}$ with two identical rows, $\det \mathbf{C} = 0$. This simple observation results in an explicit formula for φ_n:

$$\varphi_n = (D_n D_{n-1})^{-1/2} \det \begin{vmatrix} (v_0, v_0) & (v_1, v_0) & \dots & (v_n, v_0) \\ (v_0, v_1) & (v_1, v_1) & \dots & (v_n, v_1) \\ \vdots & \vdots & & \vdots \\ (v_0, v_{n-1}) & (v_1, v_{n-1}) & \dots & (v_n, v_{n-1}) \\ v_0 & v_1 & \dots & v_n \end{vmatrix} \,, \tag{8.55}$$

where the determinant is defined by Kramer's rule applied to the last row, $\{v_0 \; v_1 \cdots v_n\}$. Hence (φ_n, v_k) is proportional to the determinant with two identical rows $(v_0, v_k)\;(v_1, v_k) \cdots (v_n, v_k)$ at places k and n for every k, $0 \leqslant k \leqslant n-1$. Hence $\varphi_n \perp v_k$,

$k = 0, 1, \dots, n-1$. By (8.55) the coefficient at v_n equals $(D_{n-1}D_n)^{-1/2}D_{n-1} = (D_{n-1}/D_n)^{1/2}$. Formula (8.55) also implies that

$$(\varphi_n, \varphi_n) = (D_{n-1}/D_n)^{1/2}(\varphi_n, v_n) = \frac{(D_{n-1}/D_n)^{1/2}}{(D_{n-1}D_n)^{1/2}} D_n = 1 .$$

In particular,

$$l_{nn} = (D_{n-1}/D_n)^{1/2} . \tag{8.56}$$

167 Orthogonal polynomials in $L^2(d\sigma)$. The general theory presented in §**166** can be applied to the Hilbert space $L^2(d\sigma)$, $\sigma \in \mathfrak{P}(\mathbb{T})$, and the family of monomials $v_n = z^n$, $n = 0, 1, \dots$ First we determine the vector σ corresponding to linear independent families $\{z^n\}_{n\geqslant 0}$ in $L^2(d\sigma)$. Let $\mathcal{P}_n = \operatorname{span}\{1, z, \dots, z^n\}$.

Lemma 8.22 *The family $\{z^n\}_{n\geqslant 0}$ is linearly dependent in $L^2(d\sigma)$ if and only if $N(\sigma) = k < +\infty$. In this case the vectors $\{1, z, \dots, z^{k-1}\}$ are linearly independent and $z^n \in \mathcal{P}_{k-1}$ for $n \geqslant k$.*

Proof Obviously $\{z^n\}_{n\geqslant 0}$ is linearly dependent in $L^2(d\sigma)$ if and only if $\int_{\mathbb{T}} |p(\zeta)|^2 d\sigma(\zeta) = 0$ for a nonzero polynomial $p(z)$. This identity holds only if σ is supported by the zeros of p. If σ is a sum of k point masses then $\deg p \geqslant k$, implying that $\{1, z, \dots, z^{k-1}\}$ is linearly independent. If $p = z^k + c_{k-1}z^{k-1} + \cdots + c_0$ is the monic polynomial with roots at the point masses of σ then obviously

$$z^k = -c_{k-1}z^{k-1} - \cdots - c_0 \in \mathcal{P}_{k-1}$$

in $L^2(d\sigma)$. Multiplying by z and iterating, we obtain the lemma. $\square$

By Lemma 8.22 the Gram–Schmidt algorithm stops after k steps if $N(\sigma) = k < \infty$ and runs up to infinity otherwise. In any case a sequence $\{\varphi_n\}_{n\geqslant 0}$ of polynomials is obtained such that

$$\begin{gathered} \varphi_n(z) = \alpha_n z^n + \cdots + \varphi_n(0) , \qquad \alpha_n > 0 , \\ \int_{\mathbb{T}} \varphi_n \bar{\varphi}_k \, d\sigma = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases} \end{gathered} \tag{8.57}$$

By Theorem 8.20 and (8.48),

$$\varphi_n(z) = \alpha_n \Phi_n(z) , \qquad \alpha_n = \omega_{n-1}^{-1/2} . \tag{8.58}$$

If $N(\sigma) < \infty$, then $\{\varphi_n\}_{n\geqslant 0}$ consists of $N(\sigma)$ terms: $1 = \varphi_0, \varphi_1, \dots, \varphi_n$, $n = N(\sigma) - 1$. In view of (8.58) and (8.67) below it is natural to put $\varphi_{n+1} = \Phi_{n+1}$, where Φ_{n+1} is the unique monic polynomial vanishing on σ. If $N(\sigma) = \infty$ then $\{\varphi_n\}_{n\geqslant 0}$ is infinite. For instance if $\sigma = m$, m being the normalized ($m(\mathbb{T}) = 1$) Lebesgue measure on $\mathbb{T}$, then the Gram–Schmidt algorithm does not change the family $f_n = z^n$, since the monomials $\{z^n\}_{n\geqslant 0}$ are orthogonal in $L^2(dm)$. The inner products

$$(z^j, z^k) = \int_{\mathbb{T}} z^{j-k}\, d\sigma = c_{k-j} \tag{8.59}$$

where $c_k = \int_{\mathbb{T}} \bar{\zeta}^k\, d\sigma = \hat{\sigma}(k)$ are the Fourier coefficients of σ, depend on $k-j$. Substituting (8.59) into (8.55), we obtain

$$\varphi_n(z) = (D_n D_{n-1})^{-1/2} \det \begin{vmatrix} c_0 & c_{-1} & c_{-2} & \dots & c_{-n} \\ c_1 & c_0 & c_{-1} & \dots & c_{-n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \dots & c_{-1} \\ 1 & z & z^2 & \dots & z^n \end{vmatrix}, \tag{8.60}$$

where D_n is the determinant of the Töplitz matrix $\mathbf{C}_n = \{c_{j-i}\}_{i,j=0}^n$. If $n = N(\sigma)$ then there is a nonzero vector $(\lambda_1, \dots, \lambda_n)$ such that the norm in (8.54) is zero, implying that $D_n = 0$. For $n < N(\sigma)$ we always have $D_n > 0$. By (8.56),

$$l_{nn} = \alpha_n = (D_{n-1}/D_n)^{1/2}\,. \tag{8.61}$$

Since $f \to z^{n-1}\bar{f}$ is an isometry of $\mathcal{P}_{n-1}$ over the field of real numbers, we obtain from (8.53) that

$$\operatorname{dist}(z^n, \mathcal{P}_{n-1}) = \operatorname{dist}(\bar{z}, \mathcal{P}_{n-1}) = \operatorname{dist}(1, z\mathcal{P}_{n-1}) = \frac{1}{\alpha_n}\,. \tag{8.62}$$

Notice that the distance $\operatorname{dist}(z^n, \mathcal{P}_{n-1})$ is attained at $\alpha_n^{-1}\varphi_n - z^n$.

Lemma 8.23 *The sequence $\{\alpha_n\}_{n\geqslant 0}$ is nondecreasing.*

Proof Since $f \to zf$ is an isometry in $L^2(d\sigma)$,

$$\begin{aligned} 1/\alpha_{n+1} &= \operatorname{dist}(z^{n+1}, \mathcal{P}_n) \leqslant \operatorname{dist}(z^{n+1}, z\mathcal{P}_{n-1}) \\ &= \operatorname{dist}(z^n, \mathcal{P}_{n-1}) = 1/\alpha_n\,, \end{aligned}$$

implying that $\alpha_n \leqslant \alpha_{n+1}$ for $n \geqslant 1$. Since $\alpha_0 = 1$ and $\operatorname{dist}(z^n, \mathcal{P}_{n-1})$ cannot exceed 1, we see that $\alpha_0 \leqslant \alpha_1$. □

Corollary 8.24 *The sequence $\{D_n\}_{n\geqslant 0}$ is logarithmic concave:*

$$D_{n-1}D_{n+1} \leqslant D_n^2\,.$$

Proof Apply Lemma 8.23 and (8.61). □

Corollary 8.25 *If $N(\sigma) = +\infty$ then the limits*

$$\lim_{n\to+\infty} \sqrt[n]{D_n} = \lim_{n\to+\infty} \alpha_n^{-2} \tag{8.63}$$

exist, can be finite or infinite and are equal.

Proof Since $N(\sigma) = +\infty$, we have $D_n > 0$ and $\alpha_n < +\infty$ for every $n \geqslant 0$. Since $\log D_0 = 0$, easy geometry with the graph of the concave function $n \to \log D_n$ shows that $\log D_n/n$ decreases with growth in n. However, taking logarithms in (8.61), we see that the sequence

$$\frac{1}{n}\sum_{k=1}^{n}\log\frac{1}{\alpha_k^2} = \frac{\log D_n}{n} \tag{8.64}$$

has a finite or infinite limit. By Lemma 8.23 the sequence $\log(1/\alpha_n^2)$ has the same limit. □

168 Three-term recurrence. The isometry $f \to zf$ in $L^2(d\sigma)$ is responsible for the increasing character of $\{\alpha_n\}_{n\geqslant 0}$. Another important consequence is the Szegő (1939) recurrence relations. We derive them following Landau (1987).

Lemma 8.26 *For $n = 0, 1, \dots, N(\sigma) - 1$, the polynomials φ_{n+1}, $z\varphi_n$, φ_n^* are orthogonal to the set $\{z, z^2, \dots, z^n\}$.*

Proof By the Gram–Schmidt algorithm, $\varphi_{n+1} \perp 1, z, \dots, z^n$. Putting $n := n-1$ in the above formula and applying the isometry $f \to zf$ to both sides of the orthogonality relations obtained, we get that $z\varphi_n$ is orthogonal to $\{z, z^2, \dots, z^n\}$. Finally, the identity

$$(\varphi_n^*, z^k) = \int_{\mathbb{T}} z^n \bar{\varphi}_n \bar{z}^k \, d\sigma = (z^{n-k}, \varphi_n)$$

shows that $\varphi_n^* \perp z, z^2, \dots, z^n$. □

It is clear that $\varphi_{n+1}, z\varphi_n, \varphi_n^* \in \mathcal{P}_{n+1}$. By Lemma 8.26 these polynomials are orthogonal to the subspace $K_n \overset{\text{def}}{=} \operatorname{span}\{z, z^2, \dots, z^n\}$ of $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. Since the dimension $\dim(\mathcal{P}_{n+1} \ominus K_n)$ of the orthogonal complement of K_n in $\mathcal{P}_{n+1}$ cannot exceed 2, the polynomials under consideration are linearly dependent: $\varphi_{n+1} = az\varphi_n + b\varphi_n^*$. Matching the coefficients of z^{n+1} on both sides we find that $a = \alpha_{n+1}/\alpha_n$. Putting $z = 0$, we get $b = \varphi_{n+1}(0)/\alpha_n$.

Corollary 8.27 *For $n = 0, 1, \dots, N(\sigma) - 1$,*

$$\begin{aligned} \alpha_n\varphi_{n+1} &= \alpha_{n+1}z\varphi_n + \varphi_{n+1}(0)\varphi_n^* \,, \\ \alpha_n\varphi_{n+1}^* &= \alpha_{n+1}\varphi_n^* + \overline{\varphi_{n+1}(0)}z\varphi_n \,. \end{aligned} \tag{8.65}$$

Proof The second equality in (8.65) is obtained from the first, already established, by conjugation. □

The formulas (8.65) are called the Szegő recurrence relations. Let us consider the case $n = N(\sigma) - 1 < +\infty$ in more detail. Then

$$\sigma = \sum_{j=0}^{n} p_j\delta_{\zeta_j} \,, \qquad \zeta_j \in \mathbb{T} \,, \quad p_j > 0 \,, \quad p_0 + \cdots + p_n = 1 \,. \tag{8.66}$$

By Lemma 8.26, $z\varphi_n$ and $\varphi_n^* \perp z, \dots, z^n$. It follows that

$$\Phi_{n+1}(z) \stackrel{\text{def}}{=} \prod_{j=0}^{n}(z-\zeta_j) = az\varphi_n + b\varphi_n^* .$$

Observing that $\varphi_n^*(0) = \alpha_n$ and putting $z = 0$ in the above formula, we find that $b = \Phi_{n+1}(0)/\alpha_n$. We obtain $a = 1/\alpha_n$ by comparing the leading coefficients of the two polynomials. Hence

$$\alpha_n\varphi_{n+1}(z) = z\varphi_n + \varphi_{n+1}(0)\varphi_n^* , \tag{8.67}$$

which explains our choice of φ_{n+1} in the discussion following (8.58). Notice that $|\Phi_{n+1}(0)| = |\varphi_{n+1}(0)| = 1$. Lemma 8.26 also implies the orthogonal expansion of φ_n^*.

Corollary 8.28 *For $n = 0, 1, 2, \dots, N(\sigma)-1$,*

$$\alpha_n\varphi_n^*(z) = \sum_{k=0}^{n}\overline{\varphi_k(0)}\varphi_k(z) . \tag{8.68}$$

Proof Since $\varphi_n^* \perp z, z^2, \dots, z^n$,

$$\alpha_n(\varphi_n^*, \varphi_k) = \overline{\varphi_k(0)}\int_{\mathbb{T}}\alpha_n z^n\bar{\varphi}_n\, d\sigma = \overline{\varphi_k(0)}\int_{\mathbb{T}}|\varphi_n|^2\, d\sigma = \overline{\varphi_k(0)} .$$

□

Corollary 8.29 *For $n = 0, 1, 2, \dots, N(\sigma)-1$,*

$$\alpha_{n+1}\varphi_{n+1}^* = \alpha_n\varphi_n^*(z) + \overline{\varphi_{n+1}(0)}\varphi_{n+1}(z) . \tag{8.69}$$

Proof Formula (8.69) is a direct corollary of (8.68). □

If we put $z = 0$ in (8.68), then a useful identity is obtained:

$$\alpha_n^2 = \sum_{k=0}^{n}|\varphi_k(0)|^2 . \tag{8.70}$$

Next, by Theorem 8.21, (8.58) and (8.70),

$$1-|a_k|^2 = 1-|\Phi_{k+1}(0)|^2 = 1-|\varphi_{k+1}(0)|^2\alpha_{k+1}^{-2} = \alpha_k^2/\alpha_{k+1}^2 ,$$

which implies, see (8.17) and (8.58),

$$\omega_n = \prod_{k=0}^{n}(1-|a_k|^2) = \alpha_{n+1}^{-2} , \qquad n \geqslant 0 . \tag{8.71}$$

Using (8.32) and (8.58), we can write down explicit formulas for the orthogonal polynomials in terms of the parameters $\{a_0, \dots, a_n\}$:

$$\begin{pmatrix}\varphi_{n+1} & \psi_{n+1}\\ \varphi_{n+1}^* & -\psi_{n+1}^*\end{pmatrix} = \alpha_{n+1}\prod_{k=0}^{n}\begin{pmatrix} z & -\bar{a}_k\\ -a_k z & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} . \tag{8.72}$$

This and (8.71) show that φ_{n+1} is uniquely determined by these parameters. The relationship with the Wall polynomials is as follows:

$$\begin{aligned} \varphi_{n+1} &= \alpha_{n+1}(zB_n^* - A_n^*), & \psi_{n+1} &= \alpha_{n+1}(zB_n^* + A_n^*), \\ \varphi_{n+1}^* &= \alpha_{n+1}(B_n - zA_n), & \psi_{n+1}^* &= \alpha_{n+1}(B_n + zA_n)\,, \end{aligned} \tag{8.73}$$

see (8.30). Since $A_n/B_n \in \mathcal{B}$, the zeros of φ_n are in $\mathbb{D}$. This also can be derived from the orthogonality relations.

Lemma 8.30 *If $n < N(\sigma)$ then all zeros of φ_n are in $\mathbb{D}$. The polynomial φ_n^* does not vanish in $\{z : |z| \leqslant 1\}$.*

Proof Suppose that $\varphi_n(\lambda) = 0$. Then $\varphi_n(z) = (z-\lambda)p_{n-1}(z)$, where $p_{n-1} \in \mathcal{P}_{n-1}$. Since $\varphi_n \perp \mathcal{P}_{n-1}$, this implies the orthogonal decomposition $zp_{n-1} = \varphi_n \oplus \lambda p_{n-1}$. Then by the Pythagorean theorem

$$1 = \|\varphi_n\|^2 = \|zp_{n-1}\|^2 - |\lambda|^2\|p_{n-1}\|^2 = (1-|\lambda|^2)\|p_{n-1}\|^2\,,$$

which proves the first statement of the lemma. The second is obtained from the first by conjugation. □

Since φ_n^* does not vanish in $\{z : |z| \leqslant 1\}$, we have the factorizations

$$\begin{aligned} \varphi_n(z) &= \alpha_n(z-\lambda_{1n})\cdots(z-\lambda_{nn})\,, \\ \varphi_n^*(z) &= \alpha_n(1-\bar{\lambda}_{1n}z)\cdots(1-\bar{\lambda}_{nn}z)\,, \end{aligned}$$

which determine the finite Blaschke product (see (8.20))

$$b_n(z) = \frac{\varphi_n(z)}{\varphi_n^*(z)} = \prod_{k=1}^{n} \frac{z-\lambda_{kn}}{1-\bar{\lambda}_{kn}z}\,. \tag{8.74}$$

Division of the first equation in (8.65) by the second yields

$$b_{n+1} = \frac{zb_n(z) - \bar{a}_n}{1 - a_n z b_n(z)}\,. \tag{8.75}$$

Hence φ_{n+1} is uniquely recovered by considering (b_{n+1}, α_{n+1}). By (8.67) the points ζ_k in (8.66) coincide with the roots of the equation $zb_n = \bar{a}_n$, $|a_n| = 1$. Since the argument of zb_n increases on $\mathbb{T}$ when z is moving counter-clockwise, this equation has exactly $n+1$ roots for any a_n, $|a_n| = 1$.

169 Orthogonal polynomials and moments. The following theorem is one more illustration of the property of φ_{n+1} to store information on all such polynomials with indices less than $n+1$.

Theorem 8.31 *Let $\{\varphi_n\}_{n\geqslant 0}$ be a sequence of polynomials satisfying* (8.65) *and let m be Lebesgue measure on $\mathbb{T}$. Then $\varphi_0, \varphi_1, \ldots, \varphi_n$ are orthogonal in $L^2(d\sigma_n)$, where $d\sigma_n \stackrel{\text{def}}{=} |\varphi_n|^{-2} dm$:*

$$\int_{\mathbb{T}} \frac{\varphi_k \bar{\varphi}_j}{|\varphi_n|^2}\, dm = \begin{cases} 0 & \text{if } k \neq j, \\ 1 & \text{if } k = j. \end{cases} \tag{8.76}$$

Proof We will show that for every $0 \leqslant k \leqslant n$ and any $p \in \mathcal{P}_{k-1}$

$$\int_{\mathbb{T}} \frac{\bar{p}\varphi_k}{|\varphi_n|^2}\, dm = 0\,, \qquad \int_{\mathbb{T}} \frac{\overline{zp}\ \varphi_k^*}{|\varphi_n|^2}\, dm = 0\,.$$

If $k = n$ then

$$\int_{\mathbb{T}} \frac{\bar{p}\varphi_n}{\varphi_n \bar{\varphi}_n}\, dm = \int_{\mathbb{T}} \frac{z^n \bar{p}}{\varphi_n^*}\, dm = 0$$

by the mean-value theorem, since zp^*/φ_n^* is holomorphic on $\{z : |z| \leqslant 1\}$ by Lemma 8.30. Similarly

$$\int_{\mathbb{T}} \frac{\overline{zp}\,\varphi_n^*}{|\varphi_n|^2}\, dm = \int_{\mathbb{T}} \frac{\overline{zp}\,\varphi_n^*}{\bar{\varphi}_n^* \varphi_n^*}\, dm = \overline{\int_{\mathbb{T}} \frac{zp}{\varphi_n^*}\, dm} = 0\,.$$

It follows that $\varphi_n^* \perp z, z^2, \ldots, z^n$, $\varphi_n \perp 1, z, \ldots, z^{n-1}$ in $L^2(d\sigma_n)$. By (8.69) with $n := n-1$ this implies that $\varphi_{n-1}^* \perp z, z^2, \ldots, z^{n-1}$, which with (8.65) (applied for $n := n-1$) implies that $\varphi_{n-1} \perp 1, z, \ldots, z^{n-2}$. Now the proof can be completed by induction. □

Corollary 8.32 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ and $\varphi_0, \varphi_1, \ldots, \varphi_n$ be polynomials orthogonal in $L^2(d\sigma)$. Then*

$$\int_{\mathbb{T}} \bar{\zeta}^k\, d\sigma(\zeta) = \int_{\mathbb{T}} \bar{\zeta}^k \frac{dm}{|\varphi_n|^2}\,, \qquad |k| \leqslant n\,. \tag{8.77}$$

Proof By Theorem 8.31 the polynomials $\varphi_0, \varphi_1, \ldots, \varphi_n$ are also orthogonal in $L^2(|\varphi_n|^{-2} dm)$. It follows that the inner products in $L^2(d\sigma)$ and $L^2(|\varphi_n|^{-2} dm)$ restricted to $\mathcal{P}_n = \operatorname{span}\{\varphi_0, \varphi_1, \ldots, \varphi_n\}$ are identical. In particular the inner products of 1 and ζ^k in these Hilbert spaces coincide; this equality is expressed by (8.77). □

Equation (8.77) with $k = 0$ shows that $|\varphi_n|^{-2} dm \in \mathfrak{P}(\mathbb{T})$. The following theorem specifies the Herglotz and Schur functions of $|\varphi_n|^{-2} dm$.

Theorem 8.33 *For orthogonal polynomials $\{\varphi_n\}_{n\geqslant 0}$ in $L^2(d\sigma)$,*

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \frac{dm}{|\varphi_n|^2} = \frac{\psi_n^*(z)}{\phi_n^*(z)} = \frac{\Psi_n^*(z)}{\Phi_n^*(z)} = \frac{1 + z\, A_{n-1}/B_{n-1}}{1 - z\, A_{n-1}/B_{n-1}}\,.$$

Proof By (8.33)

$$\Re \frac{\Psi_n^*}{\Phi_n^*} = \frac{\bar{z}^n}{2} \frac{\Phi_n \Psi_n^* + \Phi_n^* \Psi_n}{|\Phi_n|^2} = \frac{\prod_{k=0}^{n-1}(1-|a_k|^2)}{|\Phi_n|^2} = \frac{1}{|\phi_n|^2} \tag{8.78}$$

on $\mathbb{T}$. By Theorem 8.2 Ψ_n^*/Φ_n^* is the Herglotz function of $|\varphi_n|^{-2}dm$. By (8.31) A_{n-1}/B_{n-1} is its Schur function. □

Theorem 8.34 (Favard) *Let $\{a_n\}_{n\geqslant 0}$ be an arbitrary sequence in $\mathbb{D}$. Then there exists a unique $\sigma \in \mathfrak{P}(\mathbb{T})$ such that $\{a_n\}_{n\geqslant 0}$ is the sequence of the parameters of the polynomials orthogonal in $L^2(d\sigma)$.*

Proof Let $\{\varphi_n\}_{n\geqslant 0}$ be defined by (8.65). By Theorem 8.31 the polynomials φ_0, $\varphi_1, \ldots, \varphi_n$ are orthogonal in $L^2(d\sigma_n)$, $d\sigma_n = |\varphi_n|^{-2}dm$. By Corollary 8.32 the Fourier coefficients c_k of σ_n, $\sigma_{n+1}, \ldots,$ are the same if $|k| \leqslant n$. It follows that the limit

$$\lim_n \int_{\mathbb{T}} p \frac{dm}{|\varphi_n|^2} \tag{8.79}$$

exists for any trigonometric polynomial p. By Weierstrass' theorem trigonometric polynomials are dense in $C(\mathbb{T})$. Since $\{\sigma_n\}_{n\geqslant 0}$ is a sequence of probability measures, these measures are in the unit ball of $M(\mathbb{T})$. Hence the limit (8.79) exists for any $p \in C(\mathbb{T})$ and determines a bounded linear functional on $C(\mathbb{T})$. By Riesz' theorem it can be represented by a probability measure σ. □

Theorem 8.34 is usually referred to as Favard's theorem.

170 Verblunsky parameters. By Theorem 8.34,

$$\{a_n\}_{n\geqslant 0} \to \sigma \tag{8.80}$$

is a one-to-one mapping of the infinite product $\prod_{k=0}^{\infty} \mathbb{D}$ to $\sigma \in \mathfrak{P}(\mathbb{T})$ with $N(\sigma) = +\infty$. This mapping extends to finite sums of point masses on $\mathbb{T}$. The measures (8.66) can be uniquely identified with finite sequences of parameters $\{a_0, a_1, \ldots, a_n\}$ satisfying $|a_0| < 1, \ldots, |a_{n-1}| < 1$, $|a_n| = 1$. The parameters $\{a_n\}_{n\geqslant 0}$ viewed as the parameters of measures are called *Verblunsky parameters*; see Simon (2005a).

Theorem 8.31 says that the polynomials $\varphi_0, \ldots, \varphi_n$ are orthogonal in $L^2(|\varphi_n|^{-2}dm)$. Lemma 8.35 lists the remaining orthogonal polynomials.

Lemma 8.35 *Let*

$$\xi_k = \xi_k^{(n)} = \begin{cases} \varphi_k & \text{if } 0 \leqslant k \leqslant n, \\ z^{k-n}\varphi_n & \text{if } k > n. \end{cases} \tag{8.81}$$

Then $\{\xi_k\}_{k\geqslant 0}$ are orthogonal polynomials in $L^2(|\varphi_n|^{-2}dm)$.

Proof Let $k > n$ and $p \in \mathcal{P}_{k-1}$. Then

$$\int_{\mathbb{T}} \frac{\bar{p}\zeta^{k-n}\varphi_n}{\varphi_n\bar{\varphi}_n}\, dm = \int_{\mathbb{T}} \frac{\zeta^k \bar{p}}{\zeta^n \bar{\varphi}_n}\, dm = \int_{\mathbb{T}} \frac{\zeta p^*}{\varphi_n^*}\, dm = 0\,.$$

□

Corollary 8.36 *The Verblunsky parameters of $|\varphi_n|^{-2}dm$ are*

$$a_0, a_1, \dots, a_{n-1}, 0, 0, \dots,\ . \tag{8.82}$$

Given a sequence (8.82) one can choose any $a_n \in \mathbb{T}$ and construct a discrete measure μ_n, see (8.66), corresponding to the Verblunsky parameters $\{a_0, a_1, \dots, a_n\}$. By Corollary 8.32,

$$\sum_{j=0}^{n} p_j \bar{\zeta}_j = \int_{\mathbb{T}} \bar{\zeta}^k \frac{dm}{|\varphi_n|^2}\,, \qquad |k| \leqslant n\,. \tag{8.83}$$

Hence any $\sigma \in \mathfrak{P}(\mathbb{T})$ generates two sequences of probability measures: a sequence $\{|\varphi_n|^{-2}dm\}_{n\geqslant 0}$ of absolutely continuous measures with parameters (8.82) and a sequence $\{\mu_n\}_{n\geqslant 0}$ of discrete measures with parameters $\{a_0, a_1, \dots, a_{n-1}, 1\}$.

Corollary 8.37 *If $\sigma \in \mathfrak{P}(\mathbb{T})$ then $*$-$\lim_n |\varphi_n|^{-2}dm = d\sigma$.*

Proof By (8.77) condition (8.1) holds for any trigonometric polynomial f. Since trigonometric polynomials are dense in $C(\mathbb{T})$ and the variations in $d\sigma_n = |\varphi_n|^{-2}dm$ all equal 1, (8.1) holds for any $f \in C(\mathbb{T})$. □

Corollary 8.38 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ with parameters $\{a_n\}_{n\geqslant 0}$ and μ_n be the measures with parameters $\{a_0, \dots, a_{n-1}, 1\}$. Then $*$-$\lim_n \mu_n = \sigma$.*

Corollary 8.39 *If $\sigma \in \mathfrak{P}(\mathbb{T})$ and $N(\sigma) = +\infty$ then $\psi_n^*/\varphi_n^* \rightrightarrows F^\sigma$ uniformly on compact subsets of $\mathbb{D}$.*

Proof Apply Theorem 8.33 and Corollary 8.37. □

171 Rotations of Verblunsky parameters. If $\{a_n\}_{n\geqslant 0}$ are the Verblunsky parameters of $\sigma \in \mathfrak{P}(\mathbb{T})$, $\lambda \in \mathbb{T}$ and $f = f^\sigma$ then by Favard's theorem 8.34 $\{\lambda a_n\}_{n\geqslant 0}$ are the Verblunsky parameters of some σ_λ. Multiplying (8.9) by λ, we obtain that $f^{\sigma_\lambda} = \lambda f$. A brief analysis of (8.38) for σ_λ indicates a dependence on λ only in the first term of (8.38), which reduces to the substitution $a_0 \to \lambda a_0$.

To obtain a formula for orthogonal polynomials in $L^2(d\sigma_\lambda)$ let $\{s_n\}_{n\geqslant 0}$ be the sequence of the Möbius transform assigned to (8.38) for F^σ and $\{s_n^*\}_{n\geqslant 0}$ a similar sequence associated with F^{σ_λ}; see §**45** at the start of Section 2.4.

Then

$$\begin{gathered} s_0 = s_0^*\,, \qquad s_n = s_n^*\,, \qquad n \geqslant 2\,, \\ s_1(w) = \frac{2a_0 z}{1 - a_0 z + w}\,, \qquad s_1^*(w) = \frac{2\lambda a_0 z}{1 - \lambda a_0 z + w}\,, \end{gathered} \tag{8.84}$$

which implies that

$$S_n^*(0) = s_0^* \circ s_1^* \circ s_1^{-1} \circ s_0^{-1} \circ S_n(0)\,. \tag{8.85}$$

Some easy algebra now shows that

$$s_0^* \circ s_1^{-1} \circ s_0^{-1} \circ (w) = \frac{(w+1)+\lambda(w-1)}{(w+1)-\lambda(w-1)}\,. \tag{8.86}$$

By (8.38) $\Psi_{n+1}^*/\Phi_{n+1}^* = S_{n+1}(0)$. Therefore (8.85) and (8.86) imply the Geronimus formulas (1944, Theorem 7.1, (7.4)) for the orthogonal polynomials in $L^2(d\sigma_\lambda)$:

$$\begin{aligned} 2\varphi_n(z,\lambda) &= (1+\bar{\lambda})\varphi_n(z) + (1-\bar{\lambda})\psi_n(z)\,,\\ 2\varphi_n^*(z,\lambda) &= (1+\lambda)\varphi_n^*(z) + (1-\lambda)\psi_n^*(z)\,,\\ 2\psi_n(z,\lambda) &= (1-\bar{\lambda})\varphi_n(z) + (1+\bar{\lambda})\psi_n(z)\,,\\ 2\psi_n^*(z,\lambda) &= (1-\lambda)\varphi_n^*(z) + (1+\lambda)\psi_n^*(z)\,. \end{aligned} \tag{8.87}$$

In particular, $\varphi_n(z,-1) = \psi_n(z)$. By (8.7)

$$\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\sigma_\lambda = \frac{1+z\lambda f}{1-z\lambda f}\,, \qquad \sigma_\lambda' = \frac{1-|f|^2}{|1-z\lambda f|^2}\,.$$

For $\lambda \in \mathbb{T}$, $(1-\lambda)/(1+\lambda) = i\alpha$, $\alpha = -(2\Im\lambda)/(|1+\lambda|^2)$. It follows that m-a.e. on $\mathbb{T}$,

$$F^{\sigma_\lambda} = \frac{i\alpha + F^\sigma}{1+i\alpha F^\sigma}\,, \quad \sigma_\lambda' = \frac{1+\alpha^2}{|1+i\alpha F^\sigma|^2}\,\sigma'. \tag{8.88}$$

For $\lambda = -1$ we have $\alpha = \infty$ and $\sigma_{-1}' = \sigma' |F^\sigma|^{-2}$ a.e. on $\mathbb{T}$.

8.3 Szegő's alternative

The main problem in the study of orthogonal polynomials is their asymptotic behavior. Szegő measures make a convenient class of measures for which this behavior can be determined relatively easily.

172 Parameters of Szegő measures. By Corollary 8.25 the limit in (8.63) always exists and is either infinite or finite.

Definition 8.40 *A probability measure σ on $\mathbb{T}$ is called a Szegő measure if $N(\sigma) = +\infty$ and $\lim_n \alpha_n < +\infty$.*

Lemma 8.41 *The set $\mathcal{P} = \bigcup_{n\geqslant 0} \mathcal{P}_n$ of all polynomials in z is dense in $L^2(d\sigma)$ if and only if σ is not a Szegő measure.*

Proof If $N(\sigma) < +\infty$ then $\mathfrak{P}_n = L^2(d\sigma)$ for $n \geqslant N(\sigma)$. If $N(\sigma) = +\infty$ then the lemma follows from (8.62). $\square$

Theorem 8.42 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$, $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$ and $\{a_n\}_{n\geqslant 0}$ the parameters of $\{\varphi_n\}_{n\geqslant 0}$. Then σ is a Szegő measure if and only if $N(\sigma) = +\infty$ and $\sum_{k=0}^{\infty} |a_k|^2 < +\infty$.*

Proof By (8.65) the Fourier coefficients of $\bar{z}$ with respect to $\{\varphi_n\}_{n\geqslant 0}$ are

$$\int_{\mathbb{T}} \bar{\zeta}\bar{\varphi}_n d\sigma = \frac{\alpha_n}{\alpha_{n+1}}(\mathbf{1}, \varphi_{n+1}) - \frac{\varphi_{n+1}(0)}{\alpha_n \alpha_{n+1}}(\alpha_n z^n, \varphi_n) = -\frac{\varphi_{n+1}(0)}{\alpha_n \alpha_{n+1}} .$$

By Parseval's identity $\bar{z} \notin \operatorname{span}(\varphi_n : n \geqslant 0)$ if and only if

$$1 = \|\bar{z}\|^2 > \sum_{n=0}^{\infty} \frac{|\varphi_{n+1}(0)|^2}{\alpha_n^2 \alpha_{n+1}^2} = \sum_{n=0}^{\infty} \frac{\alpha_{n+1}^2 - \alpha_n^2}{\alpha_n^2 \alpha_{n+1}^2} = 1 - \lim_n \frac{1}{\alpha_n^2} .$$

By (8.71) $\lim_n \alpha_n^{-2} > 0$ if and only if

$$0 < \prod_{n=0}^{\infty}(1 - |a_n|^2) = \exp\left\{\sum_{n=0}^{\infty} \log(1 - |a_n|^2)\right\} .$$

Since $\log(1-x) = -x + o(x)$ as $x \to 0$, this condition is fulfilled if and only if $\sum_{k=0}^{\infty} |a_k|^2 < +\infty$. □

173 Szegő's entropy theorem. Szegő's theorem states that σ is a Szegő measure if and only if its entropy (see (8.89) below) is finite.

Theorem 8.43 (Szegő–Geronimus) *For any $\sigma \in \mathfrak{P}(\mathbb{T})$,*

$$\log \prod_{k=0}^{\infty}(1 - |a_k|^2) = \log \frac{1}{\alpha^2} = \int_{\mathbb{T}} \log \sigma' dm . \tag{8.89}$$

Hence $\sigma \in \mathfrak{P}(\mathbb{T})$ is a Szegő measure if and only $\int_{\mathbb{T}} \log \sigma' \, dm > -\infty$.

Proof The first equality in (8.89) (due to Geronimus) follows by (8.71). Since by Lemma 8.30 φ_n^* does not vanish in $\{z : |z| \leqslant 1\}$, the function $\log|\varphi_n^*(z)|^{-2}$ being a real part of the analytic function $\log \varphi_n^{*\,-2}$ is harmonic in $\mathbb{D}$. By the mean value theorem for harmonic functions,

$$\int_{\mathbb{T}} \log \frac{1}{|\varphi_n|^2} \, dm = \log \frac{1}{|\varphi_n^*(0)|^2} = \log \frac{1}{\alpha_n^2} . \tag{8.90}$$

If $\int_{\mathbb{T}} \log \sigma' \, dm > -\infty$ then by (8.90) and Jensen's inequality, see Ex. 8.21:

$$\begin{aligned} \int_{\mathbb{T}} \log \sigma' \, dm &= \int_{\mathbb{T}} \log \left(\left| \frac{\varphi_n^*}{\alpha_n} \right|^2 \sigma' \right) dm \leqslant \log \left(\int_{\mathbb{T}} \left| \frac{\varphi_n^*}{\alpha_n} \right|^2 \sigma' \, dm \right) \\ &\leqslant \log \left(\int_{\mathbb{T}} \left| \frac{\varphi_n^*}{\alpha_n} \right|^2 d\sigma \right) = \log \frac{1}{\alpha_n^2} , \end{aligned} \tag{8.91}$$

implying that σ is a Szegő measure by Definition 8.40.

Suppose now that σ is a Szegő measure. Let

$$\log^+ x = \max(\log x, 0)\ , \quad \log^- x = \log^+ x - \log x\ .$$

Observing that $(\log^+ x)^2 \leqslant x$ (see Ex. 8.18) and $|\varphi_n|^{-2} dm \in \mathfrak{P}(\mathbb{T})$, we obtain by Theorem 8.33

$$\int_{\mathbb{T}} \left(\log^+ \frac{1}{|\varphi_n|^2}\right)^2 dm \leqslant \int_{\mathbb{T}} \frac{1}{|\varphi_n|^2} dm = 1\ , \tag{8.92}$$

which by (8.90) implies that

$$\int_{\mathbb{T}} \log^- \frac{1}{|\varphi_n|^2}\ dm = \int_{\mathbb{T}} \log^+ \frac{1}{|\varphi_n|^2}\ dm + \log \alpha_n^2 \leqslant 1 + 2\log\alpha\ . \tag{8.93}$$

For the sequence $d\mu_n = \log|\varphi_n|^{-2} dm$ of real Borel measures on $\mathbb{T}$,

$$d\mu_n^+ = \log^+ \frac{1}{|\varphi_n|^2}\ dm\ , \qquad d\mu_n^- = \log^- \frac{1}{|\varphi_n|^2}\ dm\ .$$

Since $\{\alpha_n\}_{n\geqslant 0}$ is bounded, (8.93) implies that $\{\mu_n^-\}_{n\geqslant 0}$ has a $*$-weak limit point $\nu \in M(\mathbb{T})$:

$$*\text{-}\lim_{n\in\Lambda} d\mu_n^- = \nu' dm + d\nu_{\mathrm{s}}\ , \tag{8.94}$$

where $d\nu_{\mathrm{s}}$ is the singular part of $d\nu$, $\nu' = d\nu/dm$ and $\Lambda \subset \mathbb{Z}_+$.

By (8.92) μ_n^+/dm is in the unit ball of $L^2(\mathbb{T})$, which is compact in the weak topology of $L^2(\mathbb{T})$. It follows that any $*$-limit point of $\{\mu_n^+\}_{n\geqslant 0}$ in $M(\mathbb{T})$ is absolutely continuous with the Lebesgue derivative in this unit ball. Then there exist $\Lambda' \in \Lambda$ and ω' in the unit ball of $L^2(\mathbb{T})$ such that

$$\begin{gathered} *\text{-}\lim_{n\in\Lambda'} d\mu_n^+ = \omega' dm\ , \qquad *\text{-}\lim_{n\in\Lambda'} d\mu_n = d\mu\ , \\ d\mu = (\omega' - \nu')dm - d\nu_{\mathrm{s}}\ . \end{gathered} \tag{8.95}$$

Let I be any open arc on $\mathbb{T}$ such that its end-points do not carry point masses $d\nu_s$ and $d\sigma_s$. By Jensen's inequality, see Ex. 8.21,

$$\exp\left\{\frac{1}{|I|}\int_I \log\frac{1}{|\varphi_n|^2}\ dm\right\} \leqslant \frac{1}{|I|}\int_I \frac{dm}{|\varphi_n|^2}\ . \tag{8.96}$$

Applying Theorem 8.6 separately to $\{\mu_n^+\}_{n\in\Lambda'}$ and $\{\mu_n^-\}_{n\in\Lambda'}$, we obtain

$$\lim_{n\in\Lambda} \frac{1}{|I|}\int_I \log\frac{1}{|\varphi_n|^2}\ dm = \frac{\mu(I)}{|I|}\ . \tag{8.97}$$

Applying Corollary 8.37 and Theorem 8.6, we obtain

$$\lim_n \frac{1}{|I|}\int_I \frac{dm}{|\varphi_n|^2} = \frac{\sigma(I)}{|I|}\ . \tag{8.98}$$

Substitution of (8.97) and (8.98) into (8.96) results in the inequality

$$\frac{\mu(I)}{|I|} \leqslant \log \frac{\sigma(I)}{|I|} .$$

It follows by Lebesgue's theorem on differentiation that

$$\mu' \leqslant \log \sigma' \tag{8.99}$$

almost everywhere on $\mathbb{T}$. Passing to the limit (along Λ') in (8.90), we obtain by (8.99)

$$-\infty < \log \frac{1}{\alpha^2} + \nu_s(\mathbb{T}) = \int_{\mathbb{T}} d\mu + \nu_s(\mathbb{T}) = \int_{\mathbb{T}} \mu' dm \leqslant \int_{\mathbb{T}} \log \sigma' \, dm . \tag{8.100}$$

Combining (8.91) with (8.100) we see that

$$\int_{\mathbb{T}} \log \sigma' \, dm = \log \frac{1}{\alpha^2} , \qquad \nu_s(\mathbb{T}) = 0 , \tag{8.101}$$

which completes the proof of the theorem. □

Corollary 8.44 *Let σ be a Szegő measure and $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then*

$$*\text{-}\lim_n \log \frac{1}{|\varphi_n|^2} \, dm = \log \sigma' \, dm \tag{8.102}$$

in the weak topology of $M(\mathbb{T})$.

Proof Taking (8.99) into account, we obtain

$$\mu' = \log \sigma' \quad \text{a.e. on } \mathbb{T} . \tag{8.103}$$

Substitution of (8.103) into the last formula of (8.95) results in

$$d\mu = \log \sigma' \, dm = (\omega' - \nu')dm .$$

Since ω is an arbitrary $*$-limit point of $\{\mu_n^+\}_{n\in\Lambda}$, this implies that $*\text{-}\lim_{n\in\Lambda} d\mu_n^+ = \omega' dm$. Since ν is an arbitrary $*$-limit point of $\{\mu_n^-\}_{n\in\Lambda}$, we conclude that $*\text{-}\lim_n d\mu_n = \log \sigma' \, dm$. □

Corollary 8.45 *Let σ be a Szegő measure and D_n be the determinant of the matrix $\mathbf{C}_n = \{c_{j-i}\}_{i,j=0}^n$. Then*

$$\lim_n D_n^{1/n} = \exp\left\{ \int_{\mathbb{T}} \log \sigma' \, dm \right\} .$$

Proof By (8.64),

$$\frac{\log D_n}{n} = \frac{1}{n} \left(\log \frac{1}{\alpha_1^2} + \cdots + \log \frac{1}{\alpha_n^2} \right) ,$$

which proves the corollary by (8.89). □

Szegő measures can be described in terms of their Schur's functions.

Theorem 8.46 (Boyd 1979) *A function $f \in \mathcal{B}$ is a Schur function of a Szegő measure σ if and only if*

$$\int_{\mathbb{T}} \log \sigma' \, dm = \int_{\mathbb{T}} \log(1 - |f|^2) \, dm > -\infty \, . \tag{8.104}$$

Proof Considering the real part of (8.3), we obtain

$$\int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} \, d\sigma(\zeta) = \frac{1 - |zf(z)|^2}{|1 - zf(z)|^2} \, , \qquad |z| < 1 \, . \tag{8.105}$$

Taking logarithms in (8.7), we obtain

$$\log \sigma' = \log(1 - |f|^2) - \log |1 - \zeta f(\zeta)|^2 \, . \tag{8.106}$$

Since $\Re\,(1 - zf) > 0$ in $\mathbb{D}$, $1 - zf$ is an outer function. Therefore

$$\int_{\mathbb{T}} \log |1 - \zeta f|^2 \, dm = \log 1 = 0.$$

Integration of (8.106) with respect to dm implies (8.104). □

Corollary 8.44 can be strengthened. The main tool for this is the next lemma, which plays an important role in what follows.

Lemma 8.47 *Let f be the Schur function of $\sigma \in \mathfrak{P}(\mathbb{T})$ and $\{\varphi_n\}_{n \geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Let $b_n = \varphi/\varphi^*$ be the Blaschke product defined in* (8.74) *by the zeros of the orthogonal polynomials $\{\varphi_n\}_{n \geqslant 0}$. Then*

$$|\varphi_n|^2 \sigma' = \frac{1 - |f_n|^2}{|1 - \zeta b_n f_n|^2} \tag{8.107}$$

almost everywhere on $\mathbb{T}$ with respect to Lebesgue measure.

Proof By (8.17) and (8.22) we obtain the following identity on $\mathbb{T}$:

$$1 - |f|^2 = \frac{\omega_{n-1}(1 - |f_n|^2)}{|B_{n-1} + A^*_{n-1} z f_n|^2} \, . \tag{8.108}$$

Similarly

$$|1 - zf|^2 = \left| \frac{(B_{n-1} - zA_{n-1}) - (B^*_{n-1} z - A^*_{n-1}) z f_n}{B_{n-1} + A^*_{n-1} z f_n} \right|^2 \, .$$

By (8.7),

$$\sigma' = \frac{1 - |f|^2}{|1 - zf|^2} = \frac{\omega_{n-1}(1 - |f_n|^2)}{|(B_{n-1} - zA_{n-1}) - (B^*_{n-1} z - A^*_{n-1}) z f_n|^2} \, .$$

From (8.30) we have

$$\varphi_n^* = \frac{B_{n-1} - zA_{n-1}}{\sqrt{\omega_{n-1}}} \, , \qquad \varphi_n = \frac{zB^*_{n-1} - zA^*_{n-1}}{\sqrt{\omega_{n-1}}} \, ; \tag{8.109}$$

thus we obtain

$$\sigma' = \frac{1-|f_n|^2}{|\varphi_n^* - z\varphi_n f_n|^2} \,. \tag{8.110}$$

Since $|\varphi_n^*| = |\varphi_n|$ on $\mathbb{T}$, (8.107) follows from (8.110) on multiplying (8.110) by $|\varphi_n|^2$. $\square$

Theorem 8.48 *A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is a Szegő measure if and only if*

$$\lim_n \int_{\mathbb{T}} \left| \log \frac{1}{|\varphi_n|^2} - \log \sigma' \right| dm = 0 \,. \tag{8.111}$$

Proof Taking logarithms in (8.107) we obtain

$$\log(|\varphi_n|^2\sigma') = \log(1-|f_n|^2) - 2\log|1-\zeta b_n f_n| \,,$$

which implies that

$$\begin{aligned} \left|\log \frac{1}{|\varphi_n|^2} - \log \sigma'\right| &= \log^+ |\varphi_n|^2\sigma' + \log^- |\varphi_n|^2\sigma' \\ \leqslant \log \frac{1}{1-|f_n|^2} &+ 2\log^- |1-\zeta b_n f_n| + 2\log^+ |1-\zeta b_n f_n| \,. \end{aligned} \tag{8.112}$$

The mean value over $\mathbb{T}$ of $\log|1-\zeta b_n f_n|$ is zero. Applying this fact and the elementary inequalities $\log(1+x) \leqslant x \leqslant -\log(1-x)$, $0 \leqslant x \leqslant 1$, we obtain by (8.112)

$$\begin{aligned} &\int_{\mathbb{T}} \left|\log \frac{1}{|\varphi_n|^2} - \log \sigma'\right| dm \\ &\leqslant \int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm + 4\int_{\mathbb{T}} \log^+ |1-\zeta b_n f_n|\, dm \\ &\leqslant \int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm + 4\int_{\mathbb{T}} |f_n|\, dm \\ &\leqslant \int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm + 4\left(\int_{\mathbb{T}} |f_n|^2\, dm\right)^{1/2} \\ &\leqslant \int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm + 4\left(\int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm\right)^{1/2} . \end{aligned}$$

Since $\{a_{n+k}\}_{k\geqslant 0}$ are the Schur parameters of f_n, f_n must be the Schur function of a Szegő measure by (8.89). By (8.89) and (8.104),

$$\int_{\mathbb{T}} \log \frac{1}{1-|f_n|^2}\, dm = \log \prod_{k=n}^{\infty} \frac{1}{1-|a_k|^2} \xrightarrow[n]{} 0 \,, \tag{8.113}$$

which proves the theorem. $\square$

Corollary 8.49 *If σ is a Szegő measure then for every α, $0 < \alpha \leqslant 1$,*

$$\lim_n \int_{\mathbb{T}} (|\varphi_n|^2 \sigma')^\alpha \, dm = 1 \, . \tag{8.114}$$

Proof Jensen's inequality, see Ex. 8.21, implies that

$$\exp\left\{\int_{\mathbb{T}} \alpha \log(|\varphi_n|^2 \sigma') \, dm\right\} \leqslant \int_{\mathbb{T}} (|\varphi_n|^2 \sigma')^\alpha \, dm \leqslant 1 \, .$$

The proof is completed by the use of Corollary 8.44. □

Corollary 8.50 *For every Szegő measure σ we have $\int_{\mathbb{T}} |\varphi_n|^2 \, d\sigma_s = 0$.*

Proof Applying Corollary 8.49 with $\alpha = 1$, we obtain that

$$\lim_n \int_{\mathbb{T}} |\varphi_n|^2 \, d\sigma_s = 1 - \lim_n \int_{\mathbb{T}} (|\varphi_n|^2 \sigma') \, dm = 0 \, .$$

□

Given a Szegő measure σ we define the *Szegő function* of σ by

$$D(z) = D(\sigma, z) = \exp\left\{\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \sqrt{\sigma'} \, dm(\zeta)\right\} , \qquad z \in \mathbb{D} \, . \tag{8.115}$$

In Smirnov's factorization theory for functions in H^p this D is called the outer function and is defined by $|D| = \sqrt{\sigma'}$ a.e. on $\mathbb{T}$ and $D(0) > 0$. In what follows we assume that $D^{-1} \equiv 0$ in $L^2(d\sigma_s)$. The reason for such an agreement is explained by the following theorem.

Theorem 8.51 *If σ is a Szegő measure then*

$$\lim_n \int_{\mathbb{T}} |\varphi_n^* - D^{-1}|^2 \, d\sigma = 0 \, . \tag{8.116}$$

Proof By Corollary 8.50, by the definition of D and by the mean value theorem for harmonic functions,

$$\begin{aligned}
\int_{\mathbb{T}} |\varphi_n^* - D^{-1}|^2 \, d\sigma &= \int_{\mathbb{T}} |\varphi_n^* - D^{-1}|^2 \sigma' \, dm + o(1) \\
&= \int_{\mathbb{T}} |\varphi_n^* D - 1|^2 \, dm + o(1) \\
&= 2 - 2\,\mathrm{Re} \int_{\mathbb{T}} \varphi_n^* D \, dm + o(1) \\
&= 2 - 2\,\mathrm{Re}\ \varphi_n^* D(0) + o(1) = o(1) \, ,
\end{aligned}$$

since $\lim_n (\varphi_n^* D(0)) = 1$ by Corollary 8.44 (observe that the φ_n^* are outer functions satisfying $|\varphi_n^*| = |\varphi_n|$ on $\mathbb{T}$). □

Corollary 8.52 (Geronimus 1958) *If σ is a Szegő measure, then*

$$\lim_n \int_{\mathbb{T}} \left| |\varphi_n|^2 \sigma' - 1 \right| dm = 0 ;$$

in particular $*\text{-}\lim_n |\varphi_n|^2 d\sigma = dm$.

Proof Apply Corollary 8.50 and the triangle inequality to (8.116). □

Theorem 8.53 *If σ is a Szegő measure with $\sigma_s = 0$ then*

$$\lim_n \int_{\mathbb{T}} |\varphi_n^{*-1} - D| \, dm = 0 .$$

Proof By Corollary 8.32 (putting $k = 0$) the function φ_n^{*-1} is a point of the unit sphere in $L^2(\mathbb{T})$. Next, $\varphi_n^{*-1} \rightrightarrows D$ uniformly on compact subsets of $\mathbb{D}$ by Corollary 8.44. Hence $\varphi_n^{*-1} \to D$ in the weak topology of $L^2(\mathbb{T})$. It follows that

$$\int_{\mathbb{T}} \left| \frac{1}{\varphi_n^*} - D \right| dm = 2 - 2\,\mathrm{Re} \int_{\mathbb{T}} \bar{D} \frac{1}{\varphi_n^*} \, dm \longrightarrow 2 - 2\,\mathrm{Re} \int_{\mathbb{T}} \sigma' dm = 0 ,$$

since $\sigma_s = 0$. □

174 Szegő measures and nonextreme points of $\mathcal{B}$. Condition (8.104) can be stated in terms of the convex geometry of $\mathcal{B}$.

Theorem 8.54 *A function f is an extreme point of $\mathcal{B}$ if and only if*

$$\int_{\mathbb{T}} \log(1 - |f|^2) \, dm = -\infty . \tag{8.117}$$

Proof The function $f \in \mathcal{B}$ is not an extreme point of $\mathcal{B}$ if and only if there are two different functions f_1 and f_2 in $\mathcal{B}$ such that $f = (f_1 + f_2)/2$. Putting $g = f_1 - f$, we see that this condition is equivalent to the existence of a nonzero function g such that $f \pm g \in \mathcal{B}$. By the parallelogram identity,

$$2(|f|^2 + |g|^2) = |f + g|^2 + |f - g|^2 \leqslant 2 .$$

In other words, if $f \in \mathcal{B}$ is not an extreme point of $\mathcal{B}$ then there exists a nonzero function $g \in H^\infty$ such that $|g|^2 \leqslant 1 - |f|^2$. Then the integral on the left-hand side of (8.117) converges since $\int_{\mathbb{T}} \log |g|^2 \, dm > -\infty$.

If the integral in (8.117) converges then there is an outer function $g \in H^\infty$ satisfying $|g| = 1 - |f|$ almost everywhere on $\mathbb{T}$. It follows that $z^n g \in \mathcal{B}$ and $f \pm z^n g \in \mathcal{B}$ for every nonnegative integer n. □

Let

$$P(z_1, z_2) = \log \frac{1 + \rho(z_1, z_2)}{1 - \rho(z_1, z_2)} \,, \text{ where } \rho(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right| ,$$

be the *Poincaré metric*; ρ is the *pseudo-hyperbolic* distance in $\mathbb{D}$. There is a beautiful description of Szegő measures in terms of Poincaré's model of Lobachevskii's geometry. We begin with the following simple lemma.

Lemma 8.55 *If $f \in \mathcal{B}$ and A_n/B_n is an even convergent to the Wall continued fraction of f then $\rho(f, A_n/B_n) = |f_{n+1}|$ on $\mathbb{T}$.*

Proof The pseudo-hyperbolic distance on $\mathbb{D}$ is invariant under Möbius transformations. Since for $z \in \mathbb{T}$ the Möbius transform $\tau_k(w) = (zw + a_k)(1 + \bar{a}_k zw)^{-1}$ is a conformal isomorphism of $\mathbb{D}$, we obtain the lemma by induction from (8.11) and (8.15). □

Theorem 8.56 *A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is a Szegő measure if and only if*

$$\lim_n \int_{\mathbb{T}} P(f, A_n/B_n)\, dm = 0\,. \tag{8.118}$$

Proof By Lemma 8.55, on $\mathbb{T}$ we have

$$P(f, A_n/B_n) = \log \frac{1 + |f_{n+1}|}{1 - |f_{n+1}|}\,. \tag{8.119}$$

By Theorem 8.46 $\int_{\mathbb{T}} \log \sigma'\, dm = \int_{\mathbb{T}} \log(1 - |f|^2)\, dm$. By Theorem 8.43

$$\int_{\mathbb{T}} \log(1 - |f|^2)\, dm = \log \omega_n + \int_{\mathbb{T}} \log(1 - |f_{n+1}|^2)\, dm\,.$$

It follows from (8.89) that σ is a Szegő measure if and only if

$$\lim_n \int_{\mathbb{T}} \log \frac{1}{1 - |f_{n+1}|^2}\, dm = 0\,. \tag{8.120}$$

Next, by elementary calculus, for $0 \leqslant x \leqslant 1$

$$x^2 \leqslant \log \frac{1}{1 - x^2} \leqslant \log \frac{1 + x}{1 - x} \leqslant 2x + \log \frac{1}{1 - x^2}\,. \tag{8.121}$$

By (8.119) and the second inequality in (8.121), condition (8.118) implies (8.120) showing that σ is a Szegő measure. If σ is a Szegő measure then (8.120) and the first inequality in (8.121) imply that

$$\begin{aligned} 0 \leqslant \varlimsup_n \int_{\mathbb{T}} |f_{n+1}|\, dm &\leqslant \varlimsup_n \left(\int_{\mathbb{T}} |f_{n+1}|^2\, dm \right)^{1/2} \\ &\leqslant \varlimsup_n \left(\int_{\mathbb{T}} \log \frac{1}{1 - |f_{n+1}|^2}\, dm \right)^{1/2} = 0\,. \end{aligned}$$

Together the last inequality in (8.121) this implies (8.118). □

175 Bary's theorem. In this paragraph we follow Khrushchev (1988). Non-Szegő measures admit an interesting description in terms of Hilbert-space geometry. Combining (8.65) and (8.68) we get

$$z\varphi_n = \frac{\alpha_n}{\alpha_{n+1}}\varphi_{n+1} - \frac{\varphi_{n+1}(0)}{\alpha_n\alpha_{n+1}}\sum_{k=0}^{n}\overline{\varphi_k(0)}\varphi_k\,. \tag{8.122}$$

Since $f \to zf$ is an isometry in $L^2(d\sigma)$, the system $\{z\varphi_n\}_{n\geqslant 0}$ is also orthogonal in $L^2(d\sigma)$. By (8.62) and Lemma 8.41 it is complete in $L^2(d\sigma)$ if and only if σ is not a Szegő measure. Elementary computations with (8.122) show that

$$\begin{aligned}\|z\varphi_n - \varphi_{n+1}\|^2 &= 2\,(1 - \operatorname{Re}(z\varphi_n, \varphi_{n+1}))\\ &= 2\left(1 - \frac{\alpha_n}{\alpha_{n+1}}\right) = 2\left(1 - (1-|a_n|^2)^{1/2}\right)\,.\end{aligned} \tag{8.123}$$

Notice that $\{\varphi_{n+1}\}_{n\geqslant 0}$ is not complete in $L^2(d\sigma)$.

Theorem 8.57 (Bary 1944) *Let $0 \leqslant d_n \leqslant \sqrt{2}$, $\sum_n d_n^2 = +\infty$. Then for every orthogonal system $\{e_n\}_{n\geqslant 0}$ of unit vectors in a Hilbert space H there exists a noncomplete orthogonal system $\{g_n\}_{n\geqslant 0}$ of unit vectors in H such that $\|e_n - g_n\| = d_n$, $n \geqslant 0$.*

Proof We assume first that $d_n < \sqrt{2}$ for every n and put

$$a_n = \left(1 - (1 - d_n^2/2)^2\right)^{1/2}\,, \qquad n = 0, 1, \ldots$$

Since $d_n < \sqrt{2}$, the parameters a_n satisfy $0 \leqslant a_n < 1$ for every n. By Theorem 8.34 there exists a unique probability measure σ on $\mathbb{T}$ such that $\{a_n\}_{n\geqslant 0}$ is the sequence of the parameters of the orthogonal polynomials $\{\varphi_n\}_{n\geqslant 0}$ in $L^2(d\sigma)$. Let $e_n = z\varphi_n$, $g_n = \varphi_{n+1}$, $n \geqslant 0$. It follows from (8.123) that $\|e_n - g_n\|^2 = d_n^2$. Since $g_n \perp \varphi_0$, $n = 0, 1, \ldots$, the system $\{g_n\}_{n\geqslant 0}$ cannot be complete in $L^2(d\sigma)$. However,

$$\lim_n \frac{1}{\alpha_n^2} = \lim_n \prod_{k=0}^{n}(1 - |a_k|^2) = \lim_n \prod_{k=0}^{n}\left(1 - \frac{d_k^2}{2}\right)^2 = 0\,,$$

which shows that σ is not a Szegő measure and therefore the system $\{e_n\}_{n\geqslant 0}$ is complete in $L^2(d\sigma)$.

Suppose now that $\Lambda = \{n : d_n = \sqrt{2}\} \neq \varnothing$ and put $d_n' = \min(d_n, 1)$. Then $\sum_n {d_n'}^2 = +\infty$ and the above construction can be applied to $\{d_n'\}_{n\geqslant 0}$. If $n \in \Lambda$ and $\lambda \in \mathbb{T}$ then $\|e_n - \lambda g_n\|^2 = 2 - 2\operatorname{Re}\bar{\lambda}(e_n, g_n)$. For $\lambda = 1$, then the left-hand side of the last formula is ${d_n'}^2 = 1$, implying that $\operatorname{Re}(e_n, g_n) = 1/2$. It follows that there exists a $\lambda_n \in \mathbb{T}$ such that $\operatorname{Re}(e_n, \lambda_n g_n) = 0$. Then for $n \in \Lambda$ we have $\|e_n - \lambda_n g_n\|^2 = 2$. □

8.4 Erdös measures

176 Schur functions of Erdös measures. A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is called an *Erdös measure* if $\sigma' = d\sigma/dm > 0$ almost everywhere on $\mathbb{T}$. The set $\mathcal{E}(\mathbb{T})$ of all Erdös measures on $\mathbb{T}$ is a convex subset of $\mathfrak{P}(\mathbb{T})$. It follows from (8.7) that $\sigma \in \mathcal{E}(\mathbb{T})$ if and only if the Schur function f of σ satisfies $|f| < 1$ a.e. on $\mathbb{T}$. Erdös measures can be described in terms of their Schur functions $\{f_n\}_{n\geqslant 0}$.

Theorem 8.58 *If $f \in \mathcal{B}$ then $|f| < 1$ a.e. on $\mathbb{T}$ if and only if*

$$\lim_n \int_{\mathbb{T}} |f_n|^2 \, dm = 0 \,, \tag{8.124}$$

where $\{f_n\}_{n\geqslant 0}$ are the Schur functions of f.

Proof We observe first that the condition $|f| < 1$ a.e. is necessary for (8.124). Indeed, if $|f| = 1$ on $E \subset \mathbb{T}$, $m(E) > 0$, then by (8.108) $|f_n| = 1$ on E for every n, which does not allow (8.124) to hold.

Suppose now that $|f| < 1$ a.e. on $\mathbb{T}$. Multiplying (8.107) by

$$|1 - \zeta b_n f_n|^2 = 1 + |f_n|^2 - 2\,\mathrm{Re}\,(\zeta b_n f_n) \,,$$

we obtain after elementary algebra

$$|f_n|^2 = \frac{1 - |\varphi_n|^2\sigma'}{1 + |\varphi_n|^2\sigma'} + \mathrm{Re}\,(\zeta b_n f_n) + \frac{|\varphi_n|^2\sigma' - 1}{1 + |\varphi_n|^2\sigma'}\,\mathrm{Re}\,(\zeta b_n f_n) \,. \tag{8.125}$$

The mean value theorem for analytic functions,

$$\int_{\mathbb{T}} \mathrm{Re}\,(\zeta b_n f_n) = \mathrm{Re} \int_{\mathbb{T}} \zeta b_n f_n \, dm = 0 \,,$$

combined with (8.125) results in the inequality

$$\int_{\mathbb{T}} |f_n|^2 \, dm \leqslant 2 \int_{\mathbb{T}} \left| 1 - \frac{2|\varphi_n|^2\sigma'}{1 + \varphi_n|^2\sigma'} \right| dm \,. \tag{8.126}$$

It is clear that

$$g_n \overset{\text{def}}{=} \frac{2|\varphi_n|^2\sigma'}{1 + |\varphi_n|^2\sigma'} \leqslant |\varphi_n|\sqrt{\sigma'} \,. \tag{8.127}$$

It follows that for any arc $I \subset \mathbb{T}$,

$$\frac{1}{|I|} \int_I g_n^2 \, dm \leqslant \frac{1}{|I|} \int_I |\varphi_n|^2 \, d\sigma \,.$$

For $I = \mathbb{T}$ this implies that

$$\int_{\mathbb{T}} g_n \, dm \leqslant \left(\int_{\mathbb{T}} g_n^2 \, dm \right)^{1/2} \leqslant \left(\int_{\mathbb{T}} |\varphi_n|^2 \, d\sigma \right)^{1/2} = 1 \,. \tag{8.128}$$

By Cauchy's inequality

$$\left(\frac{1}{|I|}\int_I \sqrt{\sigma'}\,dm\right)^2 \leqslant \left(\frac{1}{|I|}\int_I g_n\,dm\right)\left(\frac{1}{2|I|}\int_I \frac{1}{|\varphi_n|^2}+\sigma'\,dm\right). \tag{8.129}$$

Since $0 \leqslant g_n \leqslant 2$ on $\mathbb{T}$, the sequence $\{g_n\}_{n\geqslant 0}$ is bounded in $L^\infty(\mathbb{T})$. Let g be any $*$-weak limit point of $\{g_n\}_{n\geqslant 0}$ in $L^\infty(\mathbb{T})$. Passing to the limit in the above inequality along a subsequence corresponding to g, and applying Corollary 8.37 and Lemma 8.5, we obtain that

$$\left(\frac{1}{|I|}\int_I \sqrt{\sigma'}\,dm\right)^2 \leqslant \left(\frac{1}{|I|}\int_I g\,dm\right)\left(\frac{\sigma(I)}{2|I|}+\frac{1}{2|I|}\int_I \sigma'\,dm\right),$$

if we assume additionally that the ends of I do not carry point masses of σ. By Lebesgue's theorem on differentiation,

$$\sigma' \leqslant g\left(\tfrac{1}{2}\sigma' + \tfrac{1}{2}\sigma'\right) \text{ a.e. on } \mathbb{T}.$$

Since $\sigma' > 0$, it follows that $1 \leqslant g$ a.e. on $\mathbb{T}$ and hence that

$$1 = \int_{\mathbb{T}} g\,dm = \lim_n \int_{\mathbb{T}} g_n\,dm$$

by the $*$-weak convergence. Comparing this inequality with (8.128), we obtain

$$\lim_n \int_{\mathbb{T}} g_n\,dm = \lim_n \int_{\mathbb{T}} g_n^2\,dm = 1.$$

Then the elementary calculation

$$\lim_n \int_{\mathbb{T}} (1-g_n)^2\,dm = 1 - 2\lim_n \int_{\mathbb{T}} g_n\,dm + \lim_n \int_{\mathbb{T}} g_n^2\,dm = 0$$

completes the proof. □

Let us compare Theorems 8.46 and 8.58. Since obviously

$$\log\frac{1}{1-x} = \log(1+x+x^2+\cdots) \geqslant \log\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right) = x,$$

(8.113) implies that

$$\int_{\mathbb{T}} |f_n|^2\,dm \leqslant \int_{\mathbb{T}} \log\frac{1}{1-|f_n|^2}\,dm = \log\prod_{k=n}^{\infty}\frac{1}{1-|a_k|^2} \xrightarrow[n]{} 0,$$

demonstrating a subtle difference between Erdös and Szegő measures.

177 Rakhmanov's theorem. No simple descriptions of Erdös measures in terms of parameters similar to Szegő measures are known. However, there is a beautiful theorem of Rakhmanov (1977, 1983).

Theorem 8.59 (Rakhmanov) *If* $\sigma \in \mathcal{E}(\mathbb{T})$ *with parameters* $\{a_n\}_{n\geqslant 0}$ *then* $\lim_n a_n = 0$.

Proof Since

$$|a_n| = |f_n(0)| = \left|\int_{\mathbb{T}} f_n\,dm\right| \leqslant \left(\int_{\mathbb{T}} |f_n|^2\,dm\right)^{1/2}, \tag{8.130}$$

the proof is completed by Theorem 8.58. □

Theorem 8.60 *Let* $\sigma \in \mathfrak{P}(\mathbb{T})$, $\{\varphi_n\}_{n\geqslant 0}$ *be the orthogonal polynomials in* $L^2(d\sigma)$ *and* f *the Schur function of* σ. *Then*

$$\int_{\mathbb{T}} \left||\varphi_n|^2\sigma' - 1\right|\,dm \leqslant 12\int_{\mathbb{T}} |f_n|^2\,dm, \qquad n = 0, 1, \ldots \tag{8.131}$$

Proof It follows from (8.107) that

$$(|\varphi_n|^2\sigma' - 1)|1 - \zeta b_n f_n|^2 = 2\,\mathrm{Re}\,(\zeta b_n f_n - |f_n|^2) \qquad \text{a.e. on } \mathbb{T}\,. \tag{8.132}$$

Let $\zeta \in \mathbb{T}$ be such that

$$(|\varphi_n|^2|\sigma' - 1) = -(|\varphi_n|^2|\sigma' - 1)_- < 0\,.$$

It follows from (8.132) that $\mathrm{Re}\,(\zeta b_n f_n) < |f_n|^2$ and therefore

$$|1 - \zeta b_n f_n| \geqslant 1 - \mathrm{Re}\,(\zeta b_n f_n) > 1 - |f_n|^2\,.$$

Since $|\mathrm{Re}\,(\zeta b_n f_n)| \leqslant |f_n| \leqslant 1$ a.e. on $\mathbb{T}$, (8.132) implies that

$$(|\varphi_n|^2\sigma' - 1) - (1 - |f_n|^2)^2 \leqslant 2|f_n| + 2|f_n|^2\,. \tag{8.133}$$

Notice that $(|\varphi_n|^2\sigma' - 1)_- \leqslant 1$, since $|\varphi_n|^2\sigma' \geqslant 0$. Keeping this in mind and applying the elementary identity $(1 - |f_n|^2)^2 = 1 - 2|f_n|^2 + |f_n|^4$ to (8.133), we arrive at the inequality

$$(|\varphi_n|^2\sigma' - 1)_- \leqslant 2|f_n| + 2|f_n|^2 + 2|f_n|^2 \leqslant 6|f_n|^2\,, \tag{8.134}$$

implying

$$\int_{\mathbb{T}} (|\varphi_n|^2\sigma' - 1)_-\,dm \leqslant 6\int_{\mathbb{T}} |f_n|\,dm\,, \tag{8.135}$$

since $|f_n| \leqslant 1$. Finally,

$$\begin{aligned}&\int_{\mathbb{T}} (|\varphi_n|^2\sigma' - 1)_+\,dm\\ &\quad\leqslant \int_{\mathbb{T}} (|\varphi_n|^2\sigma' - 1)\,dm + \int_{\mathbb{T}} (|\varphi_n|^2\sigma' - 1)_-\,dm \leqslant 6\int_{\mathbb{T}} |f_n|\,dm\,,\end{aligned}$$

since $\int |\varphi_n|^2\sigma'\,dm \leqslant \int |\varphi_n|^2 d\sigma = 1$. □

If $\sigma \in \mathfrak{P}(\mathbb{T})$ is a Szegő measure then the sequence $\{\varphi_n^*\}_{n\geqslant 0}$ converges in $L^2(d\sigma)$ to D^{-1} by Theorem 8.51. Since any Szegő measure is an Erdös measure, this sequence converges to D^{-1} in measure on $\mathbb{T}$. However, if some subsequence of $\{\varphi_n^*\}_{n\geqslant 0}$ converges in measure on some measurable subset $E \subset \mathbb{T}$ of positive Lebesgue measure, then σ is a Szegő measure. This observation is due to Geronimus (1958, Theorem 5.9) and makes use of the Khinchin–Ostrovskii theorem 8.8. Although it is meaningless to talk of the convergence of $\{\varphi_n^*\}_{n\geqslant 0}$ on $\mathbb{T}$ if σ is not a Szegő measure, no obstacles exist for the convergence of $\{|\varphi_n|\}_{n\geqslant 0}$ on $\mathbb{T}$ if σ is an Erdös measure. And in fact this convergence holds by (8.131).

Theorem 8.61 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$, $f = f^\sigma$ and $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then the following conditions are equivalent:*

(a) *σ is an Erdös measure;*
(b) *the sequence $\{f_n\}_{n\geqslant 0}$ converges to* 0 *in measure on $\mathbb{T}$;*
(c) $\lim_n \int_{\mathbb{T}} \left||\varphi_n|^2\sigma' - 1\right| dm = 0$*;*
(d) *there exists α, $0 < \alpha < 1$, such that*

$$\lim_n \int_{\mathbb{T}} (|\varphi_n|^2\sigma')^\alpha \, dm = 1\,; \tag{8.136}$$

(e) *the equality* (8.136) *holds for every α, $0 < \alpha \leqslant 1$.*

Proof $(a) \Rightarrow (b)$ by Theorem 8.58. $(b) \Rightarrow (c)$ by Theorem 8.60. $(c) \Rightarrow (d)$. Using the elementary inequality

$$|\sqrt{a} - \sqrt{b}| \leqslant \sqrt{|a-b|}\,, \qquad a,\ b > 0\,,$$

we obtain by Jensen's inequality, Ex. 8.21,

$$\int_{\mathbb{T}} \left||\varphi_n|\sqrt{\sigma'} - 1\right| dm \leqslant \int_{\mathbb{T}} \sqrt{\left||\varphi_n|^2\sigma' - 1\right|}\,dm \leqslant \sqrt{\int_{\mathbb{T}} \left||\varphi_n|^2\sigma' - 1\right| dm},$$

implying (8.136) with $\alpha = 1/2$.

$(d) \Rightarrow (e)$. Let $\beta_n(\alpha) = \int_{\mathbb{T}} (|\varphi_n|^2\sigma')^\alpha \, dm$, $0 < \alpha \leqslant 1$. The function β_n is logarithmic convex, while $\alpha \to \beta_n(\alpha)^{1/\alpha}$ increases on (0, 1]. Since obviously $\beta - n(1) \leqslant 1$, we obtain that $\lim_n \beta_n(\alpha) = 1$ if $\lim_n \beta_n(\alpha_0) = 1$ and $\alpha_0 \leqslant \alpha \leqslant 1$. Let now $0 < \alpha < \alpha_0 < 1$, $\lim_n \beta_n(\alpha_0) = 1$. Then $\alpha_0 = \alpha t_0 + t_1$, where $t_0 + t_1 = 1$, $t_i > 0$. The logarithmic convexity of β_n implies $\beta_n(\alpha_0) \leqslant \beta_n(\alpha)^{t_0}\beta_n(1)^{t_1}$. Hence $\lim_n \beta_n(\alpha) = 1$.

$(e) \Rightarrow (a)$. Applying the identity $|a-b| = \left|\sqrt{a} - \sqrt{b}\right| \left|\sqrt{a} + \sqrt{b}\right|$ followed by Cauchy's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{T}} \left||\varphi_n|^2\sigma' - 1\right| dm &\leqslant 2\left(\int_{\mathbb{T}} (|\varphi_n|\sqrt{\sigma'} - 1)^2 dm\right)^{1/2} \\ &= 2(1 + \beta_n(1) - 2\beta_n(1/2))^{1/2} \to 0\,,\end{aligned}$$

which obviously implies that $\sigma' > 0$ a.e. on $\mathbb{T}$. □

178 Erdös measures and nonexposed points of $\mathcal{B}$. A point x in the unit ball, $\operatorname{ball} X$, of a Banach space X is called an *exposed point* of $\operatorname{ball} X$ if there is an x^* in the conjugate space X^* such that $\|x^*\| = x^*(x) = 1$ but $|x^*(y)| < 1$ for every $y \neq x$, $y \in \operatorname{ball} X$.

Exposed points of $\mathcal{B}$ were described by Amar and Lederer (1971), who applied the approach developed by Fisher (1969). By Fatou's theorem, Garnett (1981), every element f, $f \in H^\infty$, can be identified with its radial limits on $\mathbb{T}$. Therefore every $f \in H^\infty$ determines the set $\mathcal{U} = \{t \in \mathbb{T} : |f(t)| = 1\}$ up to a subset of m-measure zero.

Theorem 8.62 (Amar–Lederer) *A function f is an exposed point of $\mathcal{B}$ if and only if $m(\mathcal{U}(f)) > 0$.*

Although historically the theory of the Hardy algebra H^∞ goes back to Nevanlinna's interpolation problem, which in fact Nevanlinna solved with an analogue of Thiele continued fractions, see Garnett (1981, Chapter IV, Section 6), we do not have the space to discuss this interesting topic here. Instead we refer the interested reader to Garnett (1981) for the theory of Hardy algebra and to Havin (1974) for a simple proof of the Amar–Lederer theorem.

Theorem 8.63 *A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is an Erdös measure if and only if the Schur function f of σ satisfies $P(f, A_n/B_n) \Rightarrow 0$ in measure on $\mathbb{T}$.*

Proof Apply Lemma 8.55 and Theorem 8.58. □

8.5 The continuum of Schur parameters

179 The convergence of the parameters. Following the analogy between regular and Wall continued fractions we extend the results of §**23**, Section 1.3, to Schur's parameters. By (8.9) the Schur parameters

$$\mathcal{S}f \stackrel{\text{def}}{=} \{a_0, a_1, a_2, \dots\}$$

of $f \in \mathcal{B}$ make either an infinite sequence $a = \{a_n\}_{n\geqslant 0}$ with domain $\mathcal{D}(a) = \mathbb{Z}_+$ (in this case $|a_n| < 1$ for all n) or a finite sequence $a = \{a_n\}_{n=0}^k$ with domain $\mathcal{D}(a) = [0, k]$ (in this case $|a_n| < 1$ for $n \neq k$ and $|a_k| = 1$). As in the case of the real numbers, $\mathcal{S}f_n = \{a_n, a_{n+1}, a_{n+2}, \dots\}$. We denote by $\mathcal{S}^\infty$ the set of all sequences in $\mathbb{D}$ satisfying

$$|a_n| < 1 \qquad \begin{cases} n \in \mathbb{Z}_+, & \text{if } \mathcal{D}(a) = \mathbb{Z}_+\,; \\ 0 \leqslant n < k\,,\ |a_k| = 1, & \text{if } \mathcal{D}(a) = [0, k]\,. \end{cases}$$

We equip $\mathcal{S}^\infty$ with the topology of point-wise convergence. Similar to $\mathbb{R}$ a sequence $\{a^j\}_{j\geqslant 0}$ in $\mathcal{S}^\infty$ is said to converge to $a \in \mathcal{S}^\infty$ if

$$\lim_j a_n^j = a_n\,, \qquad 0 \leqslant n \leqslant k\,, \qquad \mathcal{D}(a) = [0, k]\,.$$

In this topology $\mathcal{S}^\infty$ is a compact space.

Theorem 8.64 *The mapping $\mathcal{S} : \mathcal{B} \to \mathcal{S}^\infty$ is a homeomorphism.*

The following lemma is crucial for the proof of Theorem 8.64.

Lemma 8.65 *Let $\{f^k\}_{k\geqslant 0}$ be a sequence in $\mathcal{B}$, $a^k = \{a_n^k\}_{n\geqslant 0}$ the Schur parameters of f^k and $\{f_n^k\}_{n\geqslant 0}$ the Schur functions of f^k. Let $\{a_n\}_{n\geqslant 0}$ be the Schur parameters of $f \in \mathcal{B}$ and $\{f_n\}_{n\geqslant 0}$ its Schur functions. Suppose that*

$$\lim_k f^k(z) = f(z) \tag{8.137}$$

for every $z \in \mathbb{D}$. Then for every n

$$\lim_k f_n^k(z) = f_n(z) \tag{8.138}$$

uniformly on compact subsets of $\mathbb{D}$ and in particular

$$\lim_k a_n^k = a_n\,. \tag{8.139}$$

Proof By Cauchy's integral formula (8.137) is equivalent to (8.138) for $n = 0$. Next, we have

$$zf_{n+1}^k = \frac{f_n^k - a_n^k}{1 - \bar{a}_n^k f_n^k}\,. \tag{8.140}$$

If (8.138) holds for some n then $\lim_k a_n^k = a_n$ (put $z = 0$ in (8.138)). If $|a_n| = 1$ then there is nothing to prove since $f_n \equiv a_n$ and f is a finite Blaschke product of order n. In this case f_{n+1} does not exist. If $|a_n| < 1$ then

$$\begin{aligned}&\frac{f_n^k - a_n^k}{1 - \bar{a}_n^k f_n^k} - \frac{f_n - a_n}{1 - \bar{a}_n f_n}\\ &\quad = \frac{(f_n^k - f_n) + (a_n - a_n^k) + (\bar{a}_n^k - \bar{a}_n) f_n f_n^k + a_n^k \bar{a}_n f_n - a_n \bar{a}_n^k f_n^k}{(1 - \bar{a}_n^k f_n^k)(1 - \bar{a}_n f_n)}\,.\end{aligned}$$

It follows that for any compact subset $F \subset \mathbb{D}$, $0 \in \mathbb{F}$, we have

$$\sup_F \left| \frac{f_n^k - a_n^k}{1 - \bar{a}_n^k f_n^k} - \frac{f_n - a_n}{1 - \bar{a}_n f_n} \right| \leqslant \frac{6 \sup_F |f_n^k - f_n|}{(1 - |a_n^k|)(1 - |a_n|)}\,,$$

which by (8.140) implies that $\sup_F |f_{n+1}^k - f_{n+1}| \to 0$ as $k \to \infty$. □

Proof of Theorem 8.64 By Schur's theorem 8.16, the Schur parameters uniquely determine the corresponding function $f \in \mathcal{B}$. This fact, combined with the compactness of $\mathcal{B}$ in the topology of uniform convergence, implies that the converse of Lemma 8.65 is also true. Indeed, suppose that (8.139) holds for every n. Let g be any limit point of $\{f^k\}_{k\geqslant 0}$. Applying Lemma 8.65 to any subsequence of $\{f^k\}_{k\geqslant 0}$ converging to g we conclude that $g = f$, since g has the same Schur parameters as f. □

Since $\mathcal{B}$ is a metric space, the metric can be defined by, for instance,

$$\rho(f, g) = \max_{|z| \leqslant 1/2} |f(z) - g(z)| \, ;$$

$\mathcal{S}^\infty$ is a metric space too.

Corollary 8.66 *Let $f \in \mathcal{B}$. Then $\lim_n a_n = 0$ if and only if $f_n(z) \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$.*

Combining Theorem 8.64 with Corollary 8.3 and statement (8.4) of Theorem 8.2, we obtain a sequence of homeomorphisms

$$\mathfrak{P}(\mathbb{T}) \xrightarrow{\mathcal{H}} \mathfrak{R}(\mathbb{D}) \xrightarrow{\mathcal{F}} \mathcal{B} \xrightarrow{\mathcal{S}} \mathcal{S}^\infty \, .$$

The composite map $\mathcal{S} \circ \mathcal{F} \circ \mathcal{H}$ assigns Verblyunsky parameters $\{a_n\}_{n \geqslant 0}$ to every probability measure σ. By (8.75),

$$\mathcal{S} b_n = \{-\bar{a}_{n-1}, -\bar{a}_{n-2}, \dots , -\bar{a}_0, 1\} \, . \tag{8.141}$$

The Blaschke products $\{b_n\}_{n \geqslant 0}$ together with the Schur functions $\{f_n\}_{n \geqslant 0}$ play an important role in Schur's algorithm. Therefore we call them *inverse* Schur functions.

180 An analogue of Lagrange's formula. By (8.50),

$$F^\sigma(z) - \frac{\psi_n^*(z)}{\phi_n^*(z)} = \frac{z^n}{\varphi_n(z)\varphi_n^*(z)} \left\{ \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} |\varphi_n(\zeta)|^2 \, d\sigma - 1 \right\} . \tag{8.142}$$

It follows from (8.142) that the expression in the braces must vanish at the zeros of φ_n or equivalently at the zeros of the inverse Schur function b_n. Hence if we can express the expression in the braces in terms of f_n and b_n then we will obtain an analogue of Lagrange's formula (1.50) for $\mathfrak{R}(\mathbb{D})$. See also (E8.10).

Theorem 8.67 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ and $\{\varphi_n\}_{n \geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Let $f = \mathcal{F} \circ \mathcal{H}(\sigma)$ be the Schur function of σ. Then*

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} |\varphi_n(\zeta)|^2 \, d\sigma = \frac{1 + z f_n b_n}{1 - z f_n b_n} \, , \qquad z \in \mathbb{D} \, . \tag{8.143}$$

Proof Observe first that for $n = 0$ (8.143) reduces to (8.3). Suppose first that $d\sigma = \sigma' dm$. By Lemma 8.47 and by Fatou's theorem,

$$|\varphi_n|^2 \sigma' = \frac{1 - |f_n|^2}{|1 - \zeta b_n f_n|^2} = \operatorname{Re} \frac{1 + \zeta b_n f_n}{1 - \zeta b_n f_n} \qquad \text{a.e. on } \mathbb{T} \, . \tag{8.144}$$

The analytic function $z \longmapsto (1 + z b_n f_n)(1 - z b_n f_n)^{-1}$ equals 1 at $z = 0$ and its real part is positive in $\mathbb{D}$. By Theorem 8.2 it is the Schwartz integral of $\mu \in \mathfrak{P}(\mathbb{T})$. Now (8.144) implies that $\mu' = |\varphi_n|^2 \sigma'$ a.e. on $\mathbb{T}$:

$$\int_{\mathbb{T}} |\varphi_n|^2 \sigma' dm = \int_{\mathbb{T}} |\varphi_n|^2 d\sigma = 1 \, .$$

It follows that $d\mu = \mu' dm = |\varphi_n|^2 \sigma' dm$.

Let $\sigma \in \mathfrak{P}(\mathbb{T})$ be an arbitrary measure. By Corollary 8.32 the Fourier coefficients of $|\varphi_n|^{-2} dm$ and $d\sigma$ coincide for $|k| \leqslant n$. By Theorem 8.31 $\varphi_0, \dots, \varphi_n$ are orthogonal in $L^2(d\sigma)$ and $L^2(|\varphi_n|^{-2} dm)$. Since the theorem holds for $|\varphi_{n+k}|^{-2} dm$, $k \geqslant 0$, we obtain

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} |\varphi_n|^2 \frac{dm}{|\varphi_{n+k}|^2} = \frac{1 + z b_n g_n^k}{1 - z b_n g_n^k}\,, \qquad |z| < 1\,, \tag{8.145}$$

where g_n^k are the Shur functions of order n of A_{n+k-1}/B_{n+k-1}. By Theorem 8.16, $\lim_k A_{n+k-1} \big/ B_{n+k-1} = f$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 8.65,

$$g_n^k(z) \rightrightarrows f_n(z)\,, \qquad k \to \infty\,, \qquad z \in \mathbb{D}\,. \tag{8.146}$$

Taking into account Corollary 8.37 and (8.146) we complete the proof by taking the limit of (8.145) as $k \to \infty$. □

Corollary 8.68 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$, $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$ and $\{a_n\}_{n\geqslant 0}$ the parameters of σ. Then for $|z| < 1$,*

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \left| \frac{\varphi_n}{\varphi_{n+1}} \right|^2 dm = \frac{1 + z a_n b_n(z)}{1 - z a_n b_n(z)}\,, \tag{8.147}$$

$$\frac{1}{2} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \left(1 + \left| \frac{\varphi_n}{\varphi_{n+1}} \right|^2 \right) dm = \frac{1}{1 - z a_n b_n(z)} = \frac{\Phi_n^*(z)}{\Phi_{n+1}^*(z)}\,. \tag{8.148}$$

Proof Since the Schur function of order n for A_n/B_n is the constant a_n, (8.147) is a direct corollary of Theorem 8.67. Formula (8.148) follows from (8.147). □

181 Expansions of Schur functions

Theorem 8.69 *Let $f \in \mathcal{B}$, $\{f_n\}_{n\geqslant 0}$ be its Schur functions and $\{a_n\}_{n\geqslant 0}$ its Schur parameters and A_n/B_n the even convergents to the Wall continued fraction of f. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \prod_{k=0}^{\infty} (1 - \bar{a}_k f_k)\,, \tag{8.149}$$

$$f(z) = a_0 + \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \frac{\omega_n}{B_n + z A_n^* f_{n+1}}\,, \tag{8.150}$$

where both series converge uniformly on compact subsets of $\mathbb{D}$.

Proof Iterating the identity $f_n(z) = a_n + (1 - \bar{a}_n f_n) z f_{n+1}$, we obtain

$$\begin{aligned} f(z) = a_0 &+ (1 - \bar{a}_0 f_0) a_1 z + (1 - \bar{a}_0 f_0)(1 - \bar{a}_1 f_1) a_2 z^2 + \cdots \\ &+ (1 - \bar{a}_0 f_0) \cdots (1 - \bar{a}_{n-1} f_{n-1}) z^n f_n\,. \end{aligned} \tag{8.151}$$

By (E8.1) and (E8.7) we have

$$\frac{\omega_n}{B_n + zA_n^* f_{n+1}} = \prod_{k=0}^{n} (1 - \bar{a}_k f_k) . \tag{8.152}$$

Finally,

$$\left| \frac{\omega_n}{B_n + zA_n^* f_{n+1}} \right| = \frac{\sqrt{\omega_n}}{|B_n|} \frac{\sqrt{\omega_n}}{|1 + z f_{n+1} A_n^*/B_n|} \leqslant \frac{\sqrt{\omega_n}}{1 - |z|} \tag{8.153}$$

completes the proof. □

Theorem 8.70 *Let $f = f^\sigma$, $\sigma \in \mathfrak{P}(\mathbb{T})$ and $\{a_n\}_{n\geqslant 0}$ be the Schur parameters of f. Then the series $\sum_{n=0}^{\infty} \bar{a}_n f_n(z)$ converges uniformly on compact subsets of $\mathbb{D}$ if and only if σ is a Szegő measure.*

Proof If $z = 0$ then the series turns into $\sum_{n=0}^{\infty} |a_n|^2 < +\infty$, implying that σ is a Szegő measure. Suppose now that $\sum_{n=0}^{\infty} |a_n|^2 < +\infty$ and $|z| < 1 - \varepsilon$, $\varepsilon > 0$. Applying Theorem 8.69 to f_n, we obtain by (8.149), (8.152) and (8.153) that $f_n(z) = \sum_{k=0}^{\infty} a_{n+k} z^k h_{n,k}(z)$, where $|h_{n,k}(z)| \leqslant \varepsilon^{-1}$ in $|z| \leqslant 1 - \varepsilon$. It follows that

$$\sum_{k=n}^{n+p} |\bar{a}_n f_n(z)| \leqslant \sum_{k=n}^{n+p} \sum_{j=0}^{\infty} |a_n a_{n+j}| \varepsilon^{-1} (1 - \varepsilon)^j \leqslant \varepsilon^{-2} \sum_{k=n}^{\infty} |a_n|^2$$

for $|z| \leqslant 1 - \varepsilon$, which proves the theorem. □

8.6 Rakhmanov measures

182 A test for Rakhmanov measures. A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is called a Rakhmanov measure if

$$*\text{-}\lim_n |\varphi_n|^2 d\sigma = dm . \tag{8.154}$$

By Theorem 8.61(c) any Erdös measure is a Rakhmanov measure.

Lemma 8.71 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ and let $\{f_n\}_{n\geqslant 0}$ be the Schur functions and $\{b_n\}_{n\geqslant 0}$ the inverse Schur functions of σ. Then σ is a Rakhmanov measure if and only if $f_n b_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$.*

Proof Apply Theorem 8.67. □

It follows from (8.154) that Rakhmanov measures cannot be supported by compact set smaller than $\mathbb{T}$.

183 An extension of Theorem 8.58. Given any $\sigma \in \mathfrak{P}(\mathbb{T})$ we denote by $E = E(\sigma)$ the Lebesgue support $\{\zeta \in \mathbb{T} : \sigma' > 0\}$ of the absolutely continuous part of σ. The following theorem plays a crucial role in the solution of the convergence problem for Schur's algorithm.

Theorem 8.72 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ be a Rakhmanov measure and $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then*

$$\lim_n \int_{E(\sigma)} \left| 1 - \frac{2|\varphi_n|^2\sigma'}{1+|\varphi_n|^2\sigma'} \right|^2 dm = 0\,. \tag{8.155}$$

Proof By (8.127),

$$\frac{1}{|I|}\int_I g_n^2\, dm \leqslant \frac{1}{|I|}\int_I |\varphi_n|^2 d\sigma\,. \tag{8.156}$$

Since σ is a Rakhmanov measure, Helly's theorem 8.6 implies that

$$\overline{\lim_n} \frac{1}{|I|}\int_I g_n^2\, dm \leqslant 1 \tag{8.157}$$

(where $\overline{\lim}$ means lim sup) and consequently that

$$\overline{\lim_n} \frac{1}{|I|}\int_I g_n\, dm \leqslant 1\,. \tag{8.158}$$

Let g be any limit point of $\{g_n\}_{n\geqslant 0}$ and G any limit point of $\{g_n^2\}_{n\geqslant 0}$ in the $*$-weak topology of $L^\infty(\mathbb{T})$. By (8.129),

$$\frac{1}{|I|}\int_I \sqrt{\sigma'}\, dm \leqslant \left(\frac{1}{|I|}\int_I g_n\, dm\right)^{1/2} \left(\frac{1}{2|I|}\int_I \frac{1}{|\varphi_n|^2} + \sigma'\, dm\right)^{1/2}\,,$$

which obviously implies that

$$\frac{1}{|I|}\int_I \sqrt{\sigma'}\, dm \leqslant \left(\frac{1}{|I|}\int_I g_n^2\, dm\right)^{1/4} \left(\frac{1}{2|I|}\int_I \frac{1}{|\varphi_n|^2} + \sigma'\, dm\right)^{1/2}\,.$$

Passing to the limit in the above inequalities along the corresponding sequences, we obtain by Theorem 8.6 and Corollary 8.37

$$\begin{aligned} \frac{1}{|I|}\int_I \sqrt{\sigma'}\, dm &\leqslant \left(\frac{1}{|I|}\int_I g\, dm\right)^{1/2} \left(\frac{\sigma(I)}{2|I|} + \frac{1}{2|I|}\int_I \sigma'\, dm\right)^{1/2}, \\ \frac{1}{|I|}\int_I \sqrt{\sigma'}\, dm &\leqslant \left(\frac{1}{|I|}\int_I G\, dm\right)^{1/4} \left(\frac{\sigma(I)}{2|I|} + \frac{1}{2|I|}\int_I \sigma'\, dm\right)^{1/2} \end{aligned} \tag{8.159}$$

for every open arc I whose end-points are not point masses of σ. Applying Lebesgue differentiation to (8.159), we obtain

$$\sqrt{\sigma'} \leqslant \sqrt{g}\sqrt{\sigma'}\,, \quad \sqrt{\sigma'} \leqslant \sqrt{G}\sqrt{\sigma'} \qquad \text{a.e. on } \mathbb{T}\,.$$

It follows that

$$1 \leqslant \min(g, G) \qquad \text{a.e. on } E(\sigma)\,. \tag{8.160}$$

However, by (8.157) and (8.158),

$$\frac{1}{|I|}\int_I g dm \leqslant 1\,, \qquad \frac{1}{|I|}\int_I G dm \leqslant 1 \tag{8.161}$$

for any open arc I. Applying Lebesgue's theorem on differentiation, we conclude that

$$\max(g, G) \leqslant 1 \qquad \text{a.e. on } \mathbb{T}\,. \tag{8.162}$$

Obviously $g_n = g_n^2 = 0$ on $\mathbb{T} \setminus E(\sigma)$, which implies that $g \equiv G \equiv 0$ on $\mathbb{T} \setminus E(\sigma)$. It follows from (8.161) and (8.162) that $g = G = \mathbf{1}_E$, where $\mathbf{1}_E$ is the indicator of E. Since g and G were chosen to be arbitrary $*$-weak limits points of $\{g_n\}_{n\geqslant 0}$ and $\{g_n^2\}_{n\geqslant 0}$ respectively, we may conclude that both these sequences converge to $\mathbf{1}_E$ in the $*$-weak topology of $L^\infty(\mathbb{T})$. Hence

$$\int_E (1-g_n)^2\, dm = |E| + \int_{\mathbb{T}} g_n^2\, dm - 2\int_{\mathbb{T}} g_n\, dm \longrightarrow 0$$

as $n \to \infty$. □

184 The Máté–Nevai condition. A sequence $\{a_n\}_{n\geqslant 0}$ in $\mathbb{D}$ is said to satisfy the Máté–Nevai condition if

$$\lim_n a_n a_{n+k} = 0 \qquad \text{for} \;\; k = 1, 2, \ldots \tag{8.163}$$

Theorem 8.73 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ with parameters $\{a_n\}_{n\geqslant 0}$. Then σ is a Rakhmanov measure if and only if $\{a_n\}_{n\geqslant 0}$ satisfies the Máté–Nevai condition.*

Proof Suppose first that $\{a_n\}_{n\geqslant 0}$ satisfies the Máté–Nevai condition. We have

$$\int_{\mathbb{T}} |\zeta\varphi_n - \varphi_{n+1}|^2 d\sigma = 2(1 - \sqrt{1-|a_n|^2}) \leqslant 2|a_n|^2\,. \tag{8.164}$$

Since obviously $\zeta^k\varphi_n \perp \zeta^k\varphi_{n-k}$, $\varphi_{n+k} \perp \zeta^k\varphi_{n-k}$ and $\varphi_{n+k} \perp \varphi_n$, we obtain

$$\int_{\mathbb{T}} \zeta^k |\varphi_n|^2 d\sigma = -\int_{\mathbb{T}} (\zeta^k\varphi_n - \varphi_{n+k})\overline{(\zeta^k\varphi_{n-k} - \varphi_n)}\, d\sigma \tag{8.165}$$

for $k = 1, 2, \ldots$ The following identities are obvious:

$$\begin{aligned}
\zeta^k\varphi_n - \varphi_{n+k} &= (\zeta^k\varphi_n - \zeta^{k-1}\varphi_{n+1}) \\
&\qquad (\zeta^{k-1}\varphi_{n+1} - \zeta^{k-2}\varphi_{n+2}) + \cdots \\
&\qquad + (\zeta\varphi_{n+k-1} - \varphi_{n+k})\,, \\
\zeta^k\varphi_{n-k} - \varphi_n &= (\zeta^k\varphi_{n-k} - \zeta^{k-1}\varphi_{n-k+1}) \\
&\qquad + (\zeta^{k-1}\varphi_{n-k+1} - \zeta^{k-2}\varphi_{n-k+2}) + \cdots \\
&\qquad + (\zeta\varphi_{n-1} - \varphi_n)\,.
\end{aligned} \tag{8.166}$$

Taking into account (8.164) and (8.166), we obtain from (8.165) by Cauchy's inequality (for $k = 1, 2, \dots$) that

$$\begin{aligned}\left|\int_{\mathbb{T}} \zeta^k |\varphi_n|^2 d\sigma\right| &\leqslant \|\zeta^k \varphi_n - \varphi_{n+k}\| \, \|\zeta^k \varphi_{n-k} - \varphi_n\| \\ &\leqslant 2(|a_n| + |a_{n+1}| + \dots + |a_{n+k-1}|)(|a_{n-k}| + \dots + |a_{n-1}|) .\end{aligned} \tag{8.167}$$

By the Máté–Nevai condition the right-hand side of (8.167) tends to zero as $n \to \infty$. Hence σ is a Rakhmanov measure.

Suppose now that σ is a Rakhmanov measure. Then by Lemma 8.71 $f_n b_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$. By (8.9) and (8.75),

$$zb_n f_n = \frac{b_{n+1} + \bar{a}_n}{1 + a_n b_{n+1}} \, \frac{zf_{n+1} + a_n}{1 + \bar{a}_n z f_{n+1}} .$$

It follows that

$$zb_n f_n (1 + a_n b_{n+1})(1 + \bar{a}_n z f_{n+1}) = zb_{n+1} f_{n+1} + a_n b_{n+1} + |a_n|^2 + \bar{a}_n z f_{n+1} ,$$

which obviously implies that

$$a_n b_{n+1} + |a_n|^2 + \bar{a}_n z f_{n+1} \rightrightarrows 0 . \tag{8.168}$$

Multiplying (8.168) by f_{n+1}, we obtain the more symmetric condition

$$a_n f_{n+1}(a_n + z f_{n+1}) \rightrightarrows 0 . \tag{8.169}$$

Lemma 8.74 *If $f \in \mathcal{B}$ satisfies* (8.169) *then its Schur parameters $\{a_n\}_{n \geqslant 0}$ satisfy the Máté–Nevai condition.*

Proof We will prove that

$$a_n f_{n+k}(a_n + z f_{n+1}) \rightrightarrows 0 , \qquad n \to \infty , \tag{8.170}$$

uniformly on compact subsets of $\mathbb{D}$ for $k = 1, 2, \dots$ If $k = 1$ then (8.170) coincides with (8.169). Suppose now that (8.170) holds for some k and prove that it holds for $k + 1$. By (8.9),

$$f_{n+k}(1 + \bar{a}_{n+k} z f_{n+k+1}) = z f_{n+k+1} + a_{n+k} . \tag{8.171}$$

It follows from (8.170), putting $z = 0$, that

$$|a_n a_{n+k}| \leqslant \sqrt{|a_n^2 a_{n+k}|} \to 0 , \qquad n \to \infty .$$

Multiplying (8.171) by $a_n(a_n + z f_{n+1})$, we obtain from (8.170) that

$$a_n f_{n+k+1}(a_n + z f_{n+1}) \rightrightarrows 0$$

as $n \to \infty$, proving the lemma. □

To complete the proof of the theorem apply Lemma 8.74. □

Theorem 8.75 *The following are equivalent:*

(a) *σ is a Rakhmanov measure;*
(b) *the parameters $\{a_n\}_{n\geqslant 0}$ of σ satisfy the Máté–Nevai condition;*
(c) *$a_n f_{n+1} \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$;*
(d) *$a_n b_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$;*
(e) *$b_n f_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$;*
(f) *$*\text{-}\lim_n |\varphi_n/\varphi_{n+1}|^2 dm = dm$;*
(g) *$*\text{-}\lim_n |\varphi_n/\varphi_{n+l}|^2 dm = dm$ for every $l \geqslant 0$;*
(h) *$\Phi^*_{n+1}(z)/\Phi^*_n(z) \rightrightarrows 1$, $n \to \infty$, uniformly on compact subsets of $\mathbb{D}$.*

Proof (a) $\Leftrightarrow$ (b) by Theorem 8.73. (b) $\Rightarrow$ (c): by (8.150) and (8.153) we have

$$f_{n+1} = a_{n+1} + \sum_{k=1}^{\infty} a_{n+1+k} z^k h_{k,n}(z) , \qquad |h_{k,n}(z)| \leqslant (1-|z|)^{-1} . \tag{8.172}$$

Multiplying (8.172) by a_n, we obtain (c).

(c) $\Rightarrow$ (b): if $a_n f_{n+1} \rightrightarrows 0$ then (8.169) is true, which implies (b) by Lemma 8.74. (b) $\Rightarrow$ (d) is similar to (b) $\Rightarrow$ (c), since the Schur parameters of b_n are given in (8.141). (d) $\Rightarrow$ (b) If $a_n b_n(z) \rightrightarrows 0$ then $a_n a_{n-1} \to 0$ (putting $z = 0$). By (8.75) we have

$$b_n(z)(1 - z a_{n-1} b_{n-1}(z)) = z b_{n-1}(z) - \bar{a}_{n-1} . \tag{8.173}$$

Multiplying (8.173) by a_n, we obtain that $a_n b_{n-1}(z) \rightrightarrows 0$ and therefore $a_n a_{n-2} \to 0$. Now the proof is completed by induction.

(a) $\Leftrightarrow$ (e) by Lemma 8.71. (b) $\Leftrightarrow$ (f) and (b) $\Leftrightarrow$ (h) by Corollary 8.68 and by the already proved equivalence (d) $\Leftrightarrow$ (b). (g) $\Rightarrow$ (f) is obvious.

(b) $\Rightarrow$ (g) By Theorem 8.33 the parameters of $|\varphi_{n+l}|^{-2} dm$ are

$$\{a_0, \ldots , a_{n+l-1}, 0, 0 \ldots\} .$$

Putting $d\sigma = |\varphi_{n+l}|^{-2} dm$ in (8.167), we get

$$\left| \int_{\mathbb{T}} \zeta^k \left| \frac{\varphi_n}{\varphi_{n+l}} \right| dm \right| \leqslant 2(|a_n| + |a_{n+1}| + \cdots + |a_{n+l-1}|)(|a_{n-k}| + \cdots + |a_{n-1}|) \to 0 ,$$

where $n \to \infty$, for $k = 1, 2, \ldots$ □

8.7 Convergence of Schur's algorithm on $\mathbb{T}$

185 The convergence theorem. Since $A_n/B_n \rightrightarrows f$ in $\mathbb{D}$ by Schur's theorem 8.16, it is natural to investigate the convergence of A_n/B_n on $\mathbb{T}$. By Theorem 8.56 Szegő measures σ can be characterized by the A_n/B_n to $f = f^\sigma$ in terms of the integrated Poincaré metric. The Schur functions of Erdös measures are described from this point of view by Theorem 8.63. Now we are going to study the convergence of A_n/B_n in $L^2(dm)$.

Lemma 8.76 *If $f \in \mathcal{B}$ is the Schur function of $\sigma \in \mathfrak{P}(\mathbb{T})$ and is such that*

$$\lim_n \int_{E(\sigma)} \left| f - \frac{A_n}{B_n} \right|^2 dm = 0 \tag{8.174}$$

then $A_n/B_n \to f$ in $L^2(dm)$.

Proof By (8.26) f and A_n/B_n have matching Taylor polynomials at $z = 0$ of order n. Hence by Parseval's identity and Cauchy's inequality,

$$\begin{aligned}\int_{\mathbb{T}} \left| f - \frac{A_n}{B_n} \right|^2 dm &= \int_{\mathbb{T}} |f|^2\, dm + \int_{\mathbb{T}} \left| \frac{A_n}{B_n} \right|^2 dm - 2\,\mathrm{Re} \int_{\mathbb{T}} f\, \overline{\frac{A_n}{B_n}}\, dm \\ &= \int_{\mathbb{T}} \left| \frac{A_n}{B_n} \right|^2 dm - \int_{\mathbb{T}} |f|^2\, dm + o(1)\,. \end{aligned} \tag{8.175}$$

By the triangle inequality,

$$\int_E \left| f - \frac{A_n}{B_n} \right|^2 dm \geqslant \left\{ \left(\int_E |f|^2\, dm \right)^{1/2} - \left(\int_{\mathbb{T}} \left| \frac{A_n}{B_n} \right|^2 dm \right)^{1/2} \right\}^2 ,$$

which by (8.174) implies that

$$\int_E \left| \frac{A_n}{B_n} \right|^2 dm - \int_E |f|^2\, dm = o(1)\,, \qquad n \to \infty\,. \tag{8.176}$$

Since $|f| = 1$ on $\mathbb{T} \setminus E$ and $|A_n/B_n| < 1$ on $\mathbb{T}$ by (8.18), we have

$$\int_{\mathbb{T}\setminus E} \left| \frac{A_n}{B_n} \right|^2 dm - \int_{\mathbb{T}\setminus E} |f|^2\, dm < 0\,. \tag{8.177}$$

Taking the sum of (8.176) and (8.177), we obtain from (8.175) that

$$0 \leqslant \int_{\mathbb{T}} \left| f - \frac{A_n}{B_n} \right|^2 dm \leqslant o(1)\,, \qquad n \to +\infty\,,$$

proving the lemma. □

Theorem 8.77 *For the Schur function f of a Rakhmanov measure,*

$$\lim_n \int_{\mathbb{T}} \left| f - \frac{A_n}{B_n} \right|^2 dm = 0\,.$$

Proof By Lemma 8.71 $f_n b_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$. The space $L^\infty(\mathbb{T})$ is the dual space to $L^1(\mathbb{T})$. Since the linear span of Poisson kernels is dense in $L^1(\mathbb{T})$ and $f_n b_n \in \mathcal{B}$, we have $\lim_n \int f_n b_n G dm = 0$ for every $G \in L^1(\mathbb{T})$. Hence $f_n b_n \to 0$ in the $*$-weak topology of $L^\infty(\mathbb{T})$. In particular,

$$\lim_n \int_E \zeta b_n f_n \, dm = 0 \tag{8.178}$$

for every measurable $E \subset \mathbb{T}$. Integration of (8.125), together with (8.178), gives

$$\begin{aligned} \int_E |f_n|^2 \, dm &= \int_E (1 - g_n) \, dm + \operatorname{Re} \int_E \zeta b_n f_n \\ &\quad + \int_E (g_n - 1) \operatorname{Re}(\zeta b_n f_n) \, dm \\ &\leqslant 2 \int_E |1 - g_n| \, dm + o(1) \end{aligned} \tag{8.179}$$

as $n \to \infty$. Resolving (8.22) in f_{n+1}, see also (E8.3) and (E8.5), we obtain

$$|f_{n+1}| \left| 1 - \frac{\overline{A}_n}{\overline{B}_n} \right| = \left| f - \frac{A_n}{B_n} \right| \tag{8.180}$$

on $\mathbb{T}$. Since $A_n/B_n \in \mathcal{B}$ and $f \in \mathcal{B}$, (8.179) and (8.180) imply that

$$\int_E \left| f - \frac{A_n}{B_n} \right|^2 dm \leqslant 4 \int_E |f_{n+1}|^2 \, dm \leqslant 8 \left(\int_E (1 - g_n)^2 \, dm \right)^{1/2} + o(1) \, .$$

To complete the proof apply Theorem 8.72 and Lemma 8.76. □

Theorem 8.78 *Let $f \in \mathcal{B}$, let $\{a_n\}_{n \geqslant 0}$ be its Schur parameters and let A_n/B_n be the even convergents of the Wall continued fraction for f. Then*

$$\lim_n \int_{\mathbb{T}} \left| f - \frac{A_n}{B_n} \right|^2 dm = 0$$

if and only if either f is an inner function or $\lim_n a_n = 0$.

Proof If f is an inner function then $|E(\sigma)| = 0$ and we conclude that $\lim_n A_n/B_n = f$ in $L^2(\mathbb{T})$ by Lemma 8.76. If $\lim_n a_n = 0$ then $f_n \rightrightarrows 0$ uniformly on compact subsets of $\mathbb{D}$ by Corollary 8.66. By Lemma 8.71 σ is a Rakhmanov measure. Hence $\lim_n A_n/B_n = f$ in $L^2(dm)$ by Theorem 8.77. The necessity follows from the lemma.

Lemma 8.79 *Let $f \in \mathcal{B}$ and $|f| < 1$ on a set $E \subset \mathbb{T}$, $|E| > 0$. If*

$$\lim_n \int_E \left| f - \frac{A_n}{B_n} \right| dm = 0 \tag{8.181}$$

then $\lim_n a_n = 0$.

Proof By (8.14),

$$\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = a_{n+1} z^{n+1} \frac{\omega_n}{B_n B_{n+1}} = \frac{a_{n+1} z^{n+1}}{\sqrt{1-|a_n|^2}} \frac{\sqrt{\omega_n}}{B_n} \frac{\sqrt{\omega_{n+1}}}{B_{n+1}} .$$

Using (8.18) we now obtain

$$\int_E \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| dm$$
$$= \frac{|a_{n+1}|}{\sqrt{1-|a_n|^2}} \int_E \left(1 - \left|\frac{A_n}{B_n}\right|^2\right)^{1/2} \left(1 - \left|\frac{A_{n+1}}{B_{n+1}}\right|^2\right)^{1/2} dm .$$

It follows from (8.181) that $A_n/B_n \rightrightarrows f$ on E. Therefore

$$\lim_n \int_E \left(1 - \left|\frac{A_n}{B_n}\right|^2\right)^{1/2} \left(1 - \left|\frac{A_{n+1}}{B_{n+1}}\right|^2\right)^{1/2} dm = \int_E (1-|f|^2)\, dm > 0$$

by Lebesgue's dominated convergence theorem. However,

$$\lim_n \int_E \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| dm = 0$$

by (8.181). It follows that $\lim_n |a_{n+1}|(1-|a_{n+1}|^2)^{-1/2} = 0$. □

The theorem is thus proved. □

8.8 Nevai's class

186 Basic properties. A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is called a Nevai measure if $\lim_n a_n = 0$. The class $\mathfrak{N}(\mathbb{T})$ of all Nevai measures is called Nevai's class. By Rakhmanov's theorem 8.59 any Erdös measure is a Nevai measure. By Theorem 8.73 any Nevai measure is a Rakhmanov measure. Therefore $\operatorname{supp} \sigma = \mathbb{T}$ for any $\sigma \in \mathfrak{N}(\mathbb{T})$, see (8.154). The results of §**185** imply an important corollary.

Corollary 8.80 *A Rakhmanov measure $\sigma \notin \mathfrak{N}(\mathbb{T})$ is singular.*

Proof If σ is a Rakhmanov measure then by Theorem 8.77 the even convergents A_n/B_n of the Wall continued fraction for $f = f^\sigma$ converge to f in $L^2(dm)$. By Theorem 8.78 either f is an inner function or $\lim_n a_n = 0$. The second possibility is excluded by the assumption that σ does not belong to Nevai's class. It follows that f is an inner function and therefore σ is a singular measure. □

There is a nice description of nonsingular Nevai measures.

Theorem 8.81 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$, let $\{f_n\}_{n\geqslant 0}$ be its Schur functions and let $|E(\sigma)| > 0$. Then $\sigma \in \mathfrak{N}(\mathbb{T})$ if and only if*

$$\lim_n \int_{E(\sigma)} |f_n|^2 \, dm = 0 \,. \tag{8.182}$$

Proof On the one hand, since $|E(\sigma)| > 0$, (8.182) implies by the Khinchin–Ostrovskii theorem 8.8 that $f_n \rightrightarrows 0$ in $\mathbb{D}$. It follows that $\lim_n a_n = \lim_n f_n(0) = 0$. On the other hand, if $\lim_n a_n = 0$ then $f_n \rightrightarrows 0$ by Corollary 8.66. By Lemma 8.71 σ is a Rakhmanov measure. Now (8.182) follows from (8.179). □

Corollary 8.82 *Let $\{n_k\}_{k\geqslant 1}$ be any gap sequence of positive integers satisfying $\lim_k(n_{k+1} - n_k) = +\infty$ and let $\{a_n\}_{n\geqslant 0}$ be any sequence of points in $\mathbb{D}$ such that $a_n = 0$ if $n \neq n_k$ and $\overline{\lim}_n |a_n| > 0$. Then $\{a_n\}_{n\geqslant 0}$ is a sequence of Verblunsky parameters of a singular measure.*

By Theorem 8.42 the Verblunsky parameters $\{a_n\}_{n\geqslant 0}$ of any Szegő measure satisfy $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. For every Szegő measure σ and any $1 > \varepsilon > 0$ we define a gap sequence $\{n_k\}_{k\geqslant 1}$ such that $n_1 > 1/\varepsilon$, $\lim_k(n_{k+1} - n_k) = +\infty$. Then we put

$$a_n^* = \begin{cases} a_n & \text{if } n \neq n_k \,, \\ \varepsilon & \text{if } n = n_k \end{cases}$$

and denote by σ^* the measure in $\mathfrak{P}(\mathbb{T})$ with Verblunsky parameters $\{a_n^*\}_{n\geqslant 0}$. Then σ^* is a Rakhmanov measure which is not in Nevai's class. So it is singular. However, if ε is very small it is hard to distinguish the Szegő measure σ from this singular Rakhmanov measure σ^* by their respective Verblunsky parameters. Therefore when solving inverse spectral problems related to wave propagation in stratified media one cannot be sure that the spectral measure σ behaves regularly. What one may at most expect is that the average

$$\frac{1}{|I|} \int_I |\varphi_n|^2 \, dm - 1$$

is small for large n, for any arc I. Therefore the very existence of a Rakhmanov class is closely related to the instability of the solutions of such inverse spectral problems.

Theorem 8.83 *Suppose that $\sigma \in \mathfrak{P}(\mathbb{T})$ with $|E(\sigma)| > 0$ does not belong to Nevai's class. Then $\{A_n/B_n\}_{n\geqslant 0}$ diverges in measure on any subset of positive Lebesgue measure in $E(\sigma)$.*

Proof By Theorem 8.16 $A_n/B_n \rightrightarrows f$ uniformly on compact subsets of $\mathbb{D}$. It follows that $*$-$\lim_n A_n/B_n = f$ in the $*$-weak topology of $L^\infty(\mathbb{T})$. Hence

$$\lim_n \int_{\mathbb{T}} (A_n/B_n) h \, dm = \int_{\mathbb{T}} hf \, dm \tag{8.183}$$

for any $h \in L^1(\mathbb{T})$. Suppose now that $A_n/B_n \Rightarrow g$ on $E \subset E(\sigma)$, $|E| > 0$. Then by Lebesgue's dominant convergence theorem, Vulikh (1973, Theorem VII.3.1) and by (8.183) we obtain

$$\int_{\mathbb{T}} gh\, dm = \int_{\mathbb{T}} fh\, dm$$

for every h supported by E, which implies that $g = f$ a.e. on E. Applying Lebesgue's theorem again we obtain (8.181), which by Lemma 8.79 implies that σ is in Nevai's class. □

187 Nevai's theorems. We summarize some useful inequalities in the following lemma.

Lemma 8.84 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$, $\{a_n\}_{n\geqslant 0}$ be the Verblunsky parameters of σ, $\{f_n\}_{n\geqslant 0}$ the Schur functions of σ and $\{\varphi_n\}_{n\geqslant 0}$ the orthogonal polynomials in $L^2(d\sigma)$. Then*

$$\frac{|a_n|^2}{2} \leqslant \frac{1}{2}\int_{\mathbb{T}} |f_n|^2\, dm \leqslant \int_{\mathbb{T}} \left|1 - |\varphi_n|^2\sigma'\right|\, dm \leqslant 12\int_{\mathbb{T}} |f_n|\, dm\,. \tag{8.184}$$

Proof Since $a_n = f_n(0)$ the first inequality is obtained by the mean value theorem followed by Cauchy's inequality. The second follows by (8.126) and the obvious inequality

$$\int_{\mathbb{T}} \left|1 - \frac{2|\varphi_n|^2\sigma'}{1+|\varphi_n|^2\sigma'}\right|\, dm \leqslant \int_{\mathbb{T}} |1 - |\varphi_n|^2\sigma'|\, dm\,.$$

The third inequality is just (8.131). □

By Theorem 8.33 A_{n+l}/B_{n+l} is the Schur function of $|\varphi_{n+l+1}|^{-2}dm$. We denote by f_n^l the Schur function of order n for A_{n+l}/B_{n+l}. Clearly

$$a_n = f_n^l(0)\,, \quad f_n^0 \equiv a_n\,, \qquad n, l = 0, 1, \ldots \tag{8.185}$$

Theorem 8.85 (Nevai 1991) *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ and $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then $\sigma \in \mathfrak{N}(\mathbb{T})$ if and only if*

$$\lim_n \inf_{l\geqslant 0} \int_{\mathbb{T}} \left|\frac{|\varphi_n|^2}{|\varphi_{n+l+1}|^2} - 1\right|\, dm = 0\,.$$

Proof This is immediate from (8.184) if $\sigma' = |\varphi_{n+1+1}|^{-2}$. Indeed, by (8.185) $a_n = f_n^l(0)$ and $\int_{\mathbb{T}} |f_n^0|\, dm = |a_n|$. □

Theorem 8.86 (Nevai 1991) *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ and $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then σ is an Erdös measure if and only if*

$$\lim_n \sup_{l\geqslant 0} \int_{\mathbb{T}} \left|\frac{|\varphi_n|^2}{|\varphi_{n+l+1}|^2} - 1\right|\, dm = 0\,. \tag{8.186}$$

Proof By (8.9), see also (E8.3) and (E8.5),

$$
\begin{aligned}
zf_n^l &= \frac{B_{n-1}(A_{n+l}/B_{n+l}) - A_{n-1}}{B_{n-1}^* - A_{n-1}^*(A_{n+l}/B_{n+l})} \\
&= \frac{A_{n+l}/B_{n+l} - A_{n-1}/B_{n-1}}{(B_{n-1}^*/B_{n-1} - (A_{n-1}^*/B_{n-1})(A_{n+l}/B_{n+l}))} \,.
\end{aligned} \tag{8.187}
$$

Suppose first that (8.186) holds. Then $L^1 - \lim_n |\varphi_n/\varphi_{n+1}|^2 = 1$ and therefore σ is a Rakhmanov measure by (a) $\Leftrightarrow$ (f) of Theorem 8.75. By Theorem 8.77 $A_n/B_n \rightrightarrows f$ on $\mathbb{T}$. Passing to the limit $l \to \infty$ in (8.187), we obtain that $f_n^l \rightrightarrows f_n$, $l \to \infty$.

Now let $\sigma' = |\varphi_{n+l+1}|^{-2}$ in (8.184). Applying Lebesgue's dominant convergence theorem, we obtain from the second inequality in (8.184) that

$$
\int_{\mathbb{T}} |f_n|^2 \, dm \leqslant 2 \sup_{l \geqslant 0} \int_{\mathbb{T}} \left| \frac{|\varphi_n|^2}{|\varphi_{n+l+1}|^2} - 1 \right| dm \,,
$$

which implies that σ is an Erdös measure by Theorem 8.58.

Now let σ be an Erdös measure. Then $\lim_n a_n = 0$ by Rakhmanov's theorem 8.59. It follows that σ is a Rakhmanov measure. By Theorem 8.77 $A_n/B_n \rightrightarrows f$ on $\mathbb{T}$.

Multiplying both sides of (8.187) by the denominator of the fraction in the second line of (8.187) and using the triangle inequality, we obtain

$$
\int_{\mathbb{T}} |f_n^l|(1 - |f|) \, dm \leqslant \int_{\mathbb{T}} \left| \frac{A_{n+l}}{B_{n+l}} - \frac{A_{n-1}}{B_{n-1}} \right| dm + \int_{\mathbb{T}} \left| \frac{A_{n-1}}{B_{n-1}} - f \right| dm \,.
$$

It follows that $\lim_n \sup_l \int_{\mathbb{T}} |f_n^l|(1 - |f|) \, dm = 0$ and consequently

$$
\lim_n \sup_l \int_E |f_n^l| \, dm = 0 \tag{8.188}
$$

for any measurable $E \subset E(\sigma)$. Since $|f| < 1$ a.e. on $\mathbb{T}$, for every $\varepsilon > 0$ there is an E with $\sup_E |f| < 1$ such that $|\mathbb{T} \setminus E| < \varepsilon$. Observing that $f_n^l \in \mathcal{B}$, we obtain by (8.188) that

$$
\lim_n \sup_l \int_{\mathbb{T}} |f_n^l| \, dm = 0 \,.
$$

The result now follows by the third inequality in (8.184) with $\sigma' = |\varphi_{n+l+1}|^{-2}$. □

The following corollary is immediate from (8.184) and Theorem 8.86.

Corollary 8.87 *The measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is an Erdös measure if and only if*

$$
\lim_n \sup_l \int_E |f_n^l|^2 \, dm = 0 \,.
$$

188 Theorems on Nevai's class. On the Riemann sphere $\hat{\mathbb{C}}$ we consider the metric

$$k(w_1, w_2) = \frac{|w_1 - w_2|}{\sqrt{1+|w_1|^2}\sqrt{1+|w_2|^2}} .$$

It is easy to prove, see Evgrafov (1991, Chapter X, Section 6), that k is invariant under the transforms $w = (1+z\bar{a})(z-a)^{-1}$, $a \in \hat{\mathbb{C}}$, corresponding to rotations of the Riemann sphere.

Theorem 8.88 *If $\sigma \in \mathfrak{P}(\mathbb{T})$ then $\psi_n^*/\varphi_n^* \Rightarrow F^\sigma$ if and only if either σ is singular or $\sigma \in \mathfrak{N}(\mathbb{T})$.*

Proof By (8.73)

$$\frac{\psi_{n+1}^*(z)}{\varphi_{n+1}^*(z)} = \frac{1+zA_n/B_n}{1-zA_n/B_n} .$$

If $z \in \mathbb{T}$ then $\tau_z(w) = (1+zw)/(1-zw) = -\bar{z}(1+zw)/(w-\bar{z})$, being a composition of two rotations, leaves invariant the spherical metric $k(w_1, w_2)$. It follows that

$$k\left(\frac{\psi_{n+1}^*}{\varphi_{n+1}^*}, F^\sigma\right) = k\left(\frac{A_n}{B_n}, f\right) \tag{8.189}$$

a.e. on $\mathbb{T}$. Let η_n be the function on $\mathbb{T}$ defined by the left-hand side of (8.189). Since $f, A_n/B_n \in \mathcal{B}$ we obtain from (8.189) that

$$\frac{1}{2}\left|f - \frac{A_n}{B_n}\right| \leqslant \eta_n \leqslant \left|f - \frac{A_n}{B_n}\right| .$$

The proof is completed by Theorem 8.78. □

Nevai's class is closely related to the Hardy spaces H^p with $0 < p < 1$.

Theorem 8.89 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ be either a singular measure or a measure in Nevai's class. Then for every $0 < p < 1$*

$$\lim_n \int_{\mathbb{T}} \left|\frac{\psi_n^*}{\varphi_n^*} - F^\sigma\right|^p dm = 0 .$$

Proof By Theorem 8.33, $\operatorname{Re}(\psi_n^*/\varphi_n^*) > 0$ in $\mathbb{D}$. Hence

$$\int_{\mathbb{T}} \left|\frac{\psi_n^*}{\varphi_n^*}\right|^s dm \leqslant \frac{1}{\cos(\pi s/2)} , \qquad 0 < s < 1 ,$$

by Smirnov's theorem 8.7. Given p, $0 < p < 1$, we fix any $r > 1$ with $rp = s < 1$. Then for every $e \subset \mathbb{T}$ we have by Hölder's inequality

$$\int_e \left|\frac{\psi_n^*}{\varphi_n^*}\right|^p dm \leqslant \frac{|e|}{|e|^{1/r}}\left(\int_e \left|\frac{\psi_n^*}{\varphi_n^*}\right|^{pr} dm\right)^{1/r} \leqslant \frac{|e|^{1-1/r}}{\{\cos(\pi rp/2)\}^{1/r}} . \tag{8.190}$$

If σ is singular or is in Nevai's class then $\psi_n^*/\varphi_n^* \Rightarrow F^\sigma$ by Theorem 8.88. For every $\varepsilon > 0$ we put $e(\varepsilon, n) = \{\zeta \in \mathbb{T} : |\psi_n^*/\varphi_n^* - F^\sigma| > \varepsilon\}$. It follows that $\lim_n |e(\varepsilon, n)| = 0$. Now

$$\begin{aligned}\overline{\lim_n} \int_{\mathbb{T}} \left| \frac{\psi_n^*}{\varphi_n^*} - F^\sigma \right|^p dm &\leqslant \varepsilon^p + \overline{\lim_n} \int_{e(\varepsilon,n)} \left| \frac{\psi_n^*}{\varphi_n^*} \right|^p dm \\ &\quad + \overline{\lim_n} \int_{e(\varepsilon,n)} |F^\sigma| dm \\ &\leqslant \varepsilon^p + \frac{\overline{\lim}_n |e(\varepsilon, n)|^{1-1/r}}{\{\cos(\pi r p/2)\}^{1/r}} = \varepsilon^p ,\end{aligned}$$

by (8.190) and by Lebesgue's dominated convergence theorem. □

Corollary 8.90 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ be a singular measure or a measure in Nevai's class. Then for every $p > 0$*

$$\lim_n \int_{\mathbb{T}} \left| \log \frac{\psi_n^*}{\varphi_n^*} - \log F^\sigma \right|^p dm = 0 . \tag{8.191}$$

Proof By (8.78) there exists $A_n \in C(\mathbb{T})$ such that $\|A_n\|_\infty < \pi/2$ and $\arg(\psi_n^*/\varphi_n^*) = A_n$ on $\mathbb{T}$. Then

$$\log \frac{\psi_n^*}{\varphi_n^*} = \log \left| \frac{\psi_n^*}{\varphi_n^*} \right| + i A_n .$$

It follows that $\log|\psi_n^*/\varphi_n^*|$ is the harmonic conjugate to $-A_n$. By Theorem 8.88 $\psi_n^*/\varphi_n^* \Rightarrow F^\sigma$ and therefore $A_n \Rightarrow A = \arg F^\sigma$. Since $\{A_n\}_{n\geqslant 0}$ is uniformly bounded, we obtain (8.191) for $p > 1$ by the Lebesgue dominant convergence theorem and by Riesz's theorem (Garnett 1981, Chapter III, Theorem 2.3). For $0 < p \leqslant 1$ the result follows by the Hölder inequality. □

Let us compare Corollary 8.90 with Theorem 8.48. By §**171** in Section 8.2, σ_{-1} corresponds to the Verblunsky parameters $\{-a_n\}_{n\geqslant 0}$, where $\{a_n\}_{n\geqslant 0}$ are the Verblunsky parameters of σ. Then $-f = f^{\sigma_{-1}}$. It follows that $F^{\sigma_{-1}} = 1/F^\sigma$ and therefore $\sigma'_{-1} = \sigma'|F^\sigma|^{-2}$ a.e. on $\mathbb{T}$. Applying Theorem 8.48 separately to σ and σ_{-1}, we get

$$\lim_n \int_{\mathbb{T}} \left| \log \frac{|\psi_n|}{|\varphi_n|} - \log |F^\sigma| \right|^p dm = 0 .$$

Corollary 8.90 says that although for singular measures and Nevai measures we cannot guarantee the convergence of $\log|\varphi_n^*|$ in the L^1 metric, as we can for Szegő measures, more than that holds for $\log|\psi_n^*/\varphi_n^*|$.

Theorem 8.91 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ be either a singular measure or a Nevai measure. Then for $0 < p < 1$,*

$$\lim_n \int_{\mathbb{T}} ||\varphi_n|^{-2} - \sigma'|^p \, dm = 0 . \tag{8.192}$$

Proof Suppose first that $p < 1/4$. Then by Theorem 8.33 and Cauchy's inequality,

$$\int_{\mathbb{T}} \left| \frac{1}{|\varphi_n|^2} - \sigma' \right|^p dm \leqslant \left(\int_{\mathbb{T}} \left| 1 - \left| \frac{A_n}{B_n} \right|^2 - \sigma' \left| 1 - z \frac{A_n}{B_n} \right|^2 \right|^{2p} dm \right)^{1/2} \times \left(\int_{\mathbb{T}} \frac{dm}{|1 - zA_n/B_n|^{4p}} \right)^{1/2} . \tag{8.193}$$

The second multiplier in (8.193) is uniformly bounded by Smirnov's theorem 8.7, since $4p < 1$ and $\operatorname{Re}(1 - zA_n/B_n) > 0$. The first multiplier tends to zero as $n \to \infty$ by Lebesgue dominant convergence, since by Theorem 8.78

$$1 - \left| \frac{A_n}{B_n} \right|^2 - \sigma' \left| 1 - z \frac{A_n}{B_n} \right|^2 \Rightarrow 1 - |f|^2 - \sigma' |1 - zf| = 0 .$$

The proof can now be completed by convexity arguments. Let

$$\delta_n(p) = \int_{\mathbb{T}} \left| |\varphi_n|^{-2} - \sigma' \right|^p dm , \qquad 0 < p \leqslant 1 .$$

Then $\delta_n(1) \leqslant 2$. By Zygmund (1977, Theorem 10.12) the function $p \to \delta_n(p)$ is logarithmic convex. For $p < 1$ we choose any $p_0 < \min(1/4, p)$. Then $p = t_0 p_0 + t_1$, where $t_0 + t_1 = 1$, $t_i > 0$. Then by logarithmic convexity $\delta_n(p) \leqslant \delta_n(p_0)^{t_0} \delta_n(1)^{t_1} \leqslant 2^{t_1} \delta_n(p_0)^{t_0} \to 0$, since $p_0 < 1/4$. □

Corollary 8.92 *If $\sigma \in \mathfrak{N}(\mathbb{T})$ then*

$$|\varphi_n|^{-2} \Rightarrow \sigma' \text{ on } \mathbb{T} , \qquad |\varphi_n|^2 \sigma' \Rightarrow \mathbf{1}_{E(\sigma)} .$$

Theorem 8.93 *If $\sigma \in \mathfrak{N}(\mathbb{T})$ then for every $0 < p < 1$*

$$\lim_n \int_{\mathbb{T}} \left| |\varphi_n|^2 \sigma - \mathbf{1}_{E(\sigma)} \right|^p dm = 0 .$$

Proof This is similar to the proof of Theorem 8.91. If $p < 1/4$ then

$$\int_{\mathbb{T}} \left| |\varphi_n|^2 \sigma - \mathbf{1}_{E(\sigma)} \right|^p dm \leqslant \left(\int_{E(\sigma)} |1 - |f_n|^2 - |1 - \zeta b_n f_n|^2|^{2p} \right)^{1/2} \times \left(\int_{\mathbb{T}} \frac{dm}{|1 - \zeta b_n f_n|^4 p} \right)^{1/2} .$$

The proof is completed by Smirnov's theorem 8.7 and convexity arguments. □

Corollary 8.94 *Let $\sigma \in \mathfrak{N}(\mathbb{T})$; then for $0 < p < 1$*

$$\lim_n \int_{\mathbb{T}} (|\varphi_n|^2 \sigma)^p \, dm = |E(\sigma)| .$$

Proof Apply Theorem 8.93 and $|a^p - b^p| \leqslant |a - b|^p$, $0 < p < 1$. □

189 Totik's theorem. Totik (1992), answering a question of A. Gonchar, gave an elegant construction showing that Nevai measures are ubiquitous in $\mathfrak{P}(\mathbb{T})$. Totik's construction is based on matrices $\{\nu_{jk}\}_{j,k\geqslant 1}$ whose entries ν_{jk} are finite nonnegative Borel measures on $\mathbb{T}$ such that

$$*\text{-}\lim_k \nu_{jk} = j^{-r}\,dm \tag{8.194}$$

for some $r > 1$ and every $j \geqslant 1$. We call $\{\nu_{jk}\}_{j,k\geqslant 1}$ a *T-matrix*. It is clear that for every $r > 1$ there are infinitely many T-matrices.

Theorem 8.95 *For any T-matrix $\{\nu_{jk}\}_{j,k\geqslant 1}$ there is a subsequence $\{k_j\}_{j\geqslant 1}$ such that the normalized version $\|\nu\|^{-1}\nu$ of $\nu = \sum_{j=1}^{\infty} \nu_{jk_j}$ is a Nevai measure.*

Proof By (8.71) $\nu \in \mathfrak{N}(\mathbb{T})$ if and only if

$$\lim_n \frac{\alpha_{n+1}(\nu)}{\alpha_n(\nu)} = \lim_n (1 - |a_n(\nu)|^2)^{-1/2} = 1 .$$

By (8.62)

$$\alpha_n^{-2}(\nu) = \inf_{P_n(z)=z^n+\cdots} \int_{\mathbb{T}} |P_n|^2 d\nu ,$$

implying that $\alpha_n(\nu)$ increases in n and decreases in ν. Passing to subsequences if necessary we may assume that $\|\nu_{jk}\| < 2j^{-r}$ for all k. Suppose now that $k_1, \dots, k_{j-1}$ are constructed. Then

$$\mu_{j0} = \sum_{l=1}^{j-1} \nu_{lk_l} + \left(\sum_{l=j+1}^{\infty} l^{-r}\right) m ,$$
$$\mu_{jk} = \mu_{j0} + \nu_{jk} , \qquad \mu_{j\infty} = \lim_k \mu_{jk} = \mu_{j0} + j^{-r} m . \tag{8.195}$$

Lemma 8.96 *There exists N_j such that for $n \geqslant N_j$ and all $k \geqslant 0$*

$$1 \leqslant \frac{\alpha_{n+1}(\mu_{jk})}{\alpha_n(\mu_{jk})} \leqslant 1 + r2^r j^{-1} .$$

Proof By Theorem 8.43, for every $k \geqslant 1$ the sequence $\{\alpha_n(\mu_{jk})^2\}_{n=1}^{\infty}$ monotonically increases to

$$\alpha^*(\mu_{jk})^2 = \exp\left\{-\int_{\mathbb{T}} \log w_{jk}\, dm\right\} ,$$

where $w_{jk} = d\mu_{jk}/dm = w_{j0} + u_{jk}$, see (8.195), and $u_{jk} \geqslant 0$ satisfies

$$\int_{\mathbb{T}} u_{jk}\, dm \leqslant \|\nu_{jk}\| < 2j^{-r} .$$

Since by (8.195)

$$w_{j0} \geqslant \sum_{l=j+1}^{\infty} l^{-r} \geqslant \int_{j+1}^{\infty} \frac{dx}{x^r} = \frac{1}{(r-1)(j+1)^{r-1}} > \frac{1}{r(2j)^{r-1}} ,$$

we obtain applying Jensen's inequality, see Ex. 8.21, that

$$\frac{\alpha^*(\mu_{j0})^2}{\alpha^*(\mu_{jk})^2} = \exp\left\{\int_{\mathbb{T}} \log \frac{w_{jk}}{w_{j0}}\right\} \leqslant \int_{\mathbb{T}} \frac{w_{jk}}{w_{j0}} dm \leqslant 1 + \int_{\mathbb{T}} \frac{u_{jk}}{w_{j0}} dm$$
$$\leqslant 1 + r2^{r-1} j^{r-1} \int_{\mathbb{T}} u_{jk} dm$$
$$< 1 + r2^r j^{-1} < (1 + r2^r j^{-1})^2 . \tag{8.196}$$

Since $*\text{-}\lim_k \mu_{jk} = \mu_{j\infty}$, the set $S_j = \{\mu_{j0}, \mu_{j1}, \dots \mu_{j\infty}\}$ is compact in the $*$-weak topology. We consider on S_j a sequence of continuous functions

$$f_n(\mu_{jk}) = \min\left(\alpha_n(\mu_{jk}), \frac{\alpha^*(\mu_{j0})}{1 + r2^r j^{-1}}\right) .$$

By (8.196) the sequence $\{f_n\}_{n\geqslant 0}$ converges to $\alpha^*(\mu_{j0})(1+r2^{r+1}j^{-1})^{-1}$. Since $f_n \leqslant f_{n+1}$, it converges to this constant uniformly on S_j. Then

$$\frac{\alpha^*(\mu_{j0})}{1 + r2^{r+1} j^{-1}} \quad \underset{n}{\leftleftarrows} \quad f_n(\mu_{jk}) \leqslant \alpha_n(\mu_{jk}) \leqslant \alpha_n(\mu_{j0}) \leqslant \alpha^*(\mu_{j0}) ,$$

implying that

$$\limsup_n \frac{\alpha_{n+1}(\mu_{jk})}{\alpha_n(\mu_{jk})} \leqslant 1 + r2^r j^{-1}$$

uniformly in k. □

If now $k_j, k_{j+1}, \dots$ all tend to $+\infty$ then for every fixed n we have $\alpha_n(\nu) \to \alpha_n(\mu^{j-1}_{k_{j-1}})$. It follows that we can choose the numbers $K^j_j, K^j_{j+1}, \dots$ so that for $k_j > K^j_j$, $k_{j+1} > K^j_{j+1}, \dots$ we have

$$1 \leqslant \frac{\alpha_{n+1}(\nu)}{\alpha_n(\nu)} < 1 + \frac{r2^{r+1}}{j-1}$$

for every $N_{j-1} < n \leqslant N_j$. Now let $k_j = \max\{K^1_j, \dots, K^j_j\} + 1$, where $K^1_j, \dots, K^j_j$ were determined at the previous steps of the construction. Notice that this choice of k_j does not influence the numbers N_j and therefore the induction can be continued. □

If we choose the entries ν_{jk} of a T-matrix to be discrete measures then ν is a discrete Nevai measure. If the entries are singular continuous measures then we obtain a continuous singular measure in $\mathfrak{N}(\mathbb{T})$. If we choose the ν_{jk} to equal $w_{jk}dm$ with continuous densities w_{jk} satisfying

$$|w_{jk}| \leqslant \frac{1}{j^2} , \quad |\operatorname{supp} w_{jk}| < \frac{\varepsilon}{(1+j)^2} ,$$

where $\varepsilon > 0$, then $\nu \in \mathfrak{N}(\mathbb{T})$ and $d\nu/dm = w \in C(\mathbb{T})$, $m(\{w > 0\}) < \varepsilon$. However, $\operatorname{supp} \nu = \mathbb{T}$, since $\nu \in \mathfrak{N}(\mathbb{T})$.

Let μ be any measure with $\operatorname{supp}\mu = \mathbb{T}$; then by Totik's theorem there is a measure in $\mathfrak{N}(\mathbb{T})$ that is absolutely continuous with respect to μ.

8.9 Inner functions and singular measures

190 Holland's theorem. By (8.7) $\sigma \in \mathfrak{P}(\mathbb{T})$ is singular if and only if $f = f^\sigma$ is unimodular on $\mathbb{T}$, i.e. f is an inner function. By Smirnov's factorization theorem any inner function is the product of a unimodular constant, a Blaschke product (8.20) and a singular function

$$\exp\left\{-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta)\right\},$$

where μ is a positive singular measure. Since no singular measure can be a Szegő measure, $\sum_{n=0}^{\infty} |a_n|^2 = \infty$. The rate of divergence of this series can be estimated by the following beautiful theorem of Holland (1974).

Theorem 8.97 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ be a singular measure and $f = f^\sigma$. Then*

$$\int_{\mathbb{T}} \left|1 - z\sum_{k=0}^{n-1} \hat{f}(k) z^k\right|^2 d\sigma = \sum_{k=n}^{\infty} |\hat{f}(k)|^2 .$$

Proof Let $c_k = 2\hat{\sigma}(k)$, $k \geqslant 1$. If $d_k = \hat{f}(k-1)$, $k \geqslant 1$, then the multiplication of the power series (8.5) by $1 - zf$ gives $1 + zf = (1 - zf)(1 + \sum_{k=1}^{\infty} c_k z^k)$ and $d_k = c_k - \sum_{j=1}^{k-1} d_j c_{k-j}$. Substituting $c_k = 2\hat{\sigma}(k)$, we obtain after obvious calculations

$$d_k = \int_{\mathbb{T}} \left(\bar{\zeta} - \sum_{j=1}^{k-1} d_j \bar{\zeta}^{(k-j)}\right) d\sigma = \int_{\mathbb{T}} \bar{\zeta}^k \left(1 - \sum_{j=1}^{k-1} d_j \zeta^j\right) d\sigma . \tag{8.197}$$

Let $s_n(z) = 1 - \sum_{j=1}^{n} d_j z^j = 1 - z\sum_{j=0}^{n-1} \hat{f}(j) z^j$. Since $s_n = s_{n-1} - d_n z^n$, using (8.197), we obtain

$$\begin{aligned}\int_{\mathbb{T}} |s_n|^2\, d\sigma &= \int_{\mathbb{T}} |s_{n-1}|^2\, d\sigma + d_n^2 - 2\operatorname{Re}\int_{\mathbb{T}} d_n \bar{\zeta}^n s_{n-1}\, d\sigma \\ &= \int_{\mathbb{T}} |s_{n-1}|^2\, d\sigma - d_n^2 = 1 - \sum_{j=1}^{n} |d_j|^2 = \sum_{k=n}^{\infty} |\hat{f}(k)|^2 ,\end{aligned} \tag{8.198}$$

proving the theorem. □

If σ is discrete then by (8.6) it is supported by the level set $zf = 1$. Theorem 8.97 extends this to arbitrary singular σ. Comparing (8.62) with Theorem 8.97, we obtain on the one hand

$$\prod_{k=0}^{n-1} (1 - |a_k|^2) = \frac{1}{\alpha_n^2} = \operatorname{dist}(\mathbf{1}, z\mathcal{P}_{n-1}) \leqslant \sum_{k=n}^{\infty} |\hat{f}(k)|^2 .$$

On the other hand, by (E8.3) we have for any inner f:

$$\int_{\mathbb{T}} \log|B_n f - A_n|\,dm = \sum_{k=0}^{n} \int_{\mathbb{T}} \log|1-\bar{a}_k f_k|\,dm\,.$$

Observing that $B_n(0) = 1$, we obtain

$$\int_{\mathbb{T}} \log\left|f - \frac{A_n}{B_n}\right|\,dm = \sum_{k=0}^{\infty} \log(1-|a_k|^2)\,.$$

By Lemma 8.76 $A_n/B_n \Rightarrow f$, again implying that $\sum_n |a_n|^2 = \infty$ if f is an inner function.

191 Riesz products in the Nevai class. A Riesz product is a measure $\sigma \in \mathfrak{P}(\mathbb{T})$ with the formal Fourier series

$$d\sigma \sim \prod_{k=1}^{\infty}(1+\mathrm{Re}\ \beta_k\zeta^{n_k})\,,$$

where $0 < |\beta_k| \leqslant 1$, $k = 1, 2, \ldots$, and $\{n_k\}_{k\geqslant 1}$ is an increasing sequence of positive integers. Its partial products are nonnegative trigonometric polynomials with unit integrals against $\mathbb{T}$ and corresponding Fourier coefficients. Therefore the $*$-weak limit of this sequence in $M(\mathbb{T})$ exists and determines a Riesz measure σ. See Zygmund (1977) for details. In what follows we assume that

$$n_{k+1} > 2(n_k + n_{k-1} + \cdots + n_1)\,, \tag{8.199}$$

$$\sum_{k=1}^{\infty} |\beta_k|^{2p} < \infty \qquad \text{for every } p,\ p > 1\,, \tag{8.200}$$

$$\sum_{k=1}^{\infty} |\beta_k|^2 = \infty\,. \tag{8.201}$$

Here (8.199) says that every $\hat{\sigma}(k) \neq 0$, $k \neq 0$, is a product of a finite number of multipliers $\beta_j/2$ with different indices j. This together with (8.200) implies that $\{\hat{\sigma}(k)\}_{k\in\mathbb{Z}} \in \bigcap_{p>2} l^p$. Finally, (8.201) implies that σ is a singular measure; see Zygmund (1977).

Theorem 8.98 *There exists a singular Riesz product σ with Verblunsky parameters satisfying $\sum_{n=0}^{\infty} |a_n|^p < +\infty$ for every $p > 2$.*

Proof We construct the required measure in the class of Riesz products σ satisfying (8.199)–(8.201) by specifying the growth in $\{n_k\}_{k\geqslant 1}$. Suppose that $n_1, \ldots, n_k$ have already been chosen. We have that the partial product

$$p_k(\zeta) = \prod_{j=1}^{k}(1+\mathrm{Re}\,\beta_j\zeta_j^{n}) \geqslant 0 \quad \text{ on } \mathbb{T}\,. \tag{8.202}$$

By Feijer's theorem, see Szegő (1975), $p_k = |h_k|^2$, where h_k is a polynomial in z of degree $\mathcal{N} = n_1 + \cdots + n_k$ which does not vanish in $\mathbb{D}$. It follows that the Fourier spectrum $\operatorname{spec}\sigma_k$ of the probability measure $d\sigma_k = |h_k|^2 dm$ is in $[-\mathcal{N}, \mathcal{N}]$ and that σ_k is a Szegő measure.

Without specifying the choice of n_{k+1}, we observe that by (8.199) and (8.202) the measure $d\sigma_{k+1}$ is a linear combination of three measures with disjoint Fourier spectra:

$$d\sigma_{k+1} = (\bar{\beta}_{k+1}/2)\bar{\zeta}^{n_{k+1}} d\sigma_k + d\sigma_k + (\beta_{k+1}/2)\zeta^{n_{k+1}} d\sigma_k \,.$$

Hence the Hilbert spaces $L^2(d\sigma_k)$ and $L^2(d\sigma_{k+1})$, and consequently $L^2(d\sigma)$, induce identical inner products on $\mathcal{P}_n$ if $n < n_{k+1} - \mathcal{N}$. Since the orthogonal polynomials are obtained by the Gram–Schmidt algorithm, the polynomials $\varphi_0, \dots, \varphi_n$ in $\mathcal{P}_n$ are orthogonal in $L^2(d\sigma_k)$, $L^2(d\sigma_{k+1})$ and $L^2(d\sigma)$. It follows that any future choice of n_{k+1} cannot influence the Verblunsky parameters a_n of σ with $n \leqslant \mathcal{N} < n_{k+1} - \mathcal{N}$.

Keeping for a moment the notation $\{a_n\}$ for the parameters of σ_k, we can apply (8.89) and find an integer $\mathcal{N}_k$ satisfying $\mathcal{N}_k > n_1 + \cdots + n_k$ and

$$\begin{aligned} \exp\left\{\int_{\mathbb{T}} \log\, \sigma_k'\, dm\right\} &\leqslant \prod_{j=0}^{\mathcal{N}_k}(1-|a_j|^2) \\ &\leqslant (1+|\beta_{k+1}|^2)\exp\left\{\int_{\mathbb{T}} \log\, \sigma_k'\, dm\right\}. \end{aligned} \tag{8.203}$$

We put $n_{k+1} = 2\mathcal{N}_k$. Since $n_{k+1} - \mathcal{N} > 2\mathcal{N}_k - \mathcal{N}_k = \mathcal{N}_k$, we obtain that $\varphi_0, \dots, \varphi_{\mathcal{N}_k}$ are orthogonal in $L^2(d\sigma_k)$ and in $L^2(d\sigma)$.

Lemma 8.99 *Let* $0 \leqslant \alpha \leqslant 1$ *and* n *be an arbitrary integer. Then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(1+\alpha\cos nx)\, dx = \log\frac{1}{1+a^2}\,,$$

where $a = \alpha(1+\sqrt{1-\alpha^2})^{-1}$.

Proof Apply the mean value theorem to the harmonic function $\log|p(z)|^2$, where $p(z) = (1+az^n)/\sqrt{1+a^2}$. ☐

By (8.203),

$$\begin{aligned} \prod_{j=\mathcal{N}_k+1}^{\mathcal{N}_{k+1}}(1-|a_j|^2) &= \prod_{j=0}^{\mathcal{N}_{k+1}}(1-|a_j|^2)\prod_{j=0}^{\mathcal{N}_k}(1-|a_j|^2)^{-1} \\ &\geqslant \frac{1}{1+|\beta_{k+1}|^2}\exp\left\{\int_{\mathbb{T}} \log\frac{\sigma_{k+1}'}{\sigma_k'}\, dm\right\} \\ &= \frac{1}{1+|\beta_{k+1}|^2}\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log(1+|\beta_{k+1}|\cos(n_{k+1}\theta+\theta_{k+1}))d\theta\right\}, \end{aligned}$$

where $\theta_{k+1} = \arg\beta_{k+1}$. By Lemma 8.99 this implies that

$$\prod_{j=\mathcal{N}_k+1}^{\mathcal{N}_{k+1}}(1-|a_j|^2) \geqslant \frac{1}{(1+|\beta_{k+1}|^2)^2}\,.$$

It follows that

$$\sum_{j=\mathcal{N}_k+1}^{\mathcal{N}_{k+1}} |a_j|^2 \leqslant -\sum_{j=\mathcal{N}_k+1}^{\mathcal{N}_{k+1}} \log(1-|a_j|^2) \leqslant 2\log(1+|\beta_{k+1}|^2) \leqslant 2|\beta_{k+1}|^2$$

and finally, for every $p > 1$,

$$\begin{aligned}\sum_{j=\mathcal{N}_1+1}^{\infty} |a_j|^{2p} &= \sum_{k=1}^{\infty}\left(\sum\nolimits_{j=\mathcal{N}_k+1}^{\mathcal{N}_{k+1}} |a_j|^{2p}\right) \\ &\leqslant \sum_{k=1}^{\infty}\left(\sum\nolimits_{j=\mathcal{N}_k+1}^{\mathcal{N}_{k+1}} |a_j|^{2}\right)^p \leqslant 2^p \sum_{k=1}^{\infty} |\beta_{k+1}|^{2p} ,\end{aligned}$$

which proves the theorem. □

192 Singular measures

Theorem 8.100 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ with Schur function $f = f^\sigma$. For every $p < 1/4$ there is a constant $c_p > 0$ such that*

$$\left(\int_E \sigma'^p \, dm\right)^{1/p} \leqslant c_p \int_E (1-|f_n|^2)\, dm \tag{8.204}$$

for every measurable $E \subset \mathbb{T}$.

Proof Applying (8.107) and Hölder's inequality for powers $1/p$ and $1/(1-p)$, we obtain

$$\int_E \sigma'^p dm \leqslant \left(\int_E (1-|f_n|^2)\, dm\right)^p \left(\int_E \frac{dm}{\{|\varphi_n||1-\zeta b_n f_n|\}^{2p/(1-p)}}\right)^{1-p} .$$

Applying Hölder's inequality to the second integral with $(1-p)/p$ and $(1-p)/(1-2p)$, we obtain

$$\begin{aligned}&\int_E \frac{dm}{\{|\varphi_n||1-\zeta b_n f_n|\}^{2p/(1-p)}} \\ &\qquad \leqslant \left(\int_E \frac{dm}{|\varphi_n|^2}\right)^{p/(1-p)} \left(\int_E \frac{dm}{|1-\zeta b_n f_n|^{2p/(1-2p)}}\right)^{(1-2p)/(1-p)} .\end{aligned}$$

The first integral on the right is bounded by 1 since $|\varphi_n|^{-2} dm \in \mathfrak{P}(\mathbb{T})$. The second integral is bounded by Smirnov's theorem 8.7. □

Corollary 8.101 *A function $f \in \mathcal{B}$ is inner if and only if*

$$\limsup_n \int_{\mathbb{T}} |f_n|^2 \, dm = 1 . \tag{8.205}$$

Proof Let $E = \mathbb{T}$ in (8.204). Then (8.205) implies that $\sigma' = 0$ a.e. on $\mathbb{T}$. If f is an inner function then $|f_n| = 1$ a.e. on $\mathbb{T}$, see (8.108). □

Corollary 8.102 (Rakhmanov 1983, Lemma 4) *Let $f \in \mathcal{B}$ and $\limsup_n |a_n| = 1$. Then f is an inner function.*

Proof Apply (8.130). □

Theorem 8.103 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ satisfy $\lim_n |a_n| = 1$. Then the following are equivalent:*

(a) τ is in the derived set of $\operatorname{supp}\sigma$;
(b) there exists an infinite set $\Lambda \subset \mathbb{N}$ such that $\lim_{n\in\Lambda} \bar{a}_n a_{n-1} = -\tau$.

Proof Let F be the closed set of limit points of $\{-\bar{a}_n a_{n-1}\}_{n\geqslant 1}$. Since $\lim_n |a_n| = 1$, the set F is a subset of $\mathbb{T}$. Since $f_n b_n(0) = -a_n \bar{a}_{n-1}$ the set of limit points of $\{|\varphi_n|^2 d\sigma\}_{n\geqslant 0}$ equals $\{\delta_\tau : \tau \in F\}$, see Theorem 8.67. If $\tau \in F$ then for every $h \in C(\mathbb{T})$,

$$\lim_{n\in\Lambda} \int_{\mathbb{T}} h|\varphi_n|^2 \, d\sigma = h(\tau) \, ,$$

implying that $F \subset \operatorname{supp}\sigma$. If $\tau \in F$ is an isolated point of $\operatorname{supp}\sigma$ then $\sigma(\{\tau\}) > 0$ and by Ex. 8.23 $\lim_n \varphi_n(\tau) = 0$, contradicting the formula above.

To prove that $(\operatorname{supp}\sigma)' \subset F$ we apply Worpitsky's test, Corollary 5.14. By obvious equivalence transforms we reduce the Geronimus continued fraction (8.28) to $\mathbf{K}_{n=1}^{\infty}(c_n(z)/1)$, where $c_1(z) \equiv a_0$,

$$c_2(z) = -\frac{(1-|a_0|^2)(a_1/a_0)z}{1+(a_1/a_0)z} \, ,$$

$$c_n(z) = -\frac{(1-|a_{n-2}|^2)(a_{n-1}/a_{n-2})z}{(1+(a_{n-1}/a_{n-2})z)(1+(a_{n-2}/a_{n-3})z)}$$

for $n \geqslant 3$. If I is any closed arc in $\mathbb{T}\setminus F$ then the denominators of $c_n(z)$ are bounded away from zero in an open neighborhood V of I. Since $\lim_n |a_n| = 1$, this implies that $\lim_n \sup_{z\in V} |a_n(z)| = 0$. Then by Worpitsky's test $\mathbf{K}_{n=N}^{\infty}(c_n(z)/1)$ converges absolutely and uniformly to a holomorphic function in V. Since by Schur's theorem the Geronimus continued fraction converges to f^σ uniformly on compact subsets of $\mathbb{D}$, we see that f^σ extends analytically to V. The allowable points of $\operatorname{supp}\sigma$ in I are located at the zeros of $1 - zf^\sigma$. It follows that $\operatorname{supp}\sigma \cap (\mathbb{T}\setminus F)$ consists only of isolated points. □

Theorem 8.104 *Suppose that the Verblunsky parameters $\{a_n\}_{n\geqslant 0}$ of $\sigma \in \mathfrak{P}(\mathbb{T})$ satisfy*

$$\liminf_n |a_n| > 0 \, , \quad \lim_n \arg(\bar{a}_n a_{n-1}) = \theta \in \mathbb{R} \, .$$

Then there exists an open arc I on $\mathbb{T}$ centered at $e^{i\theta}$ such that $\operatorname{supp}\sigma \cap I$ is finite.

Proof Using rotations if necessary we may assume that $\theta = 0$. We apply Pringsheim's theorem 5.16 to

$$K_N(z) = \mathop{\mathbf{K}}_{n=N}^{\infty} \left(\frac{-(1-|a_{n-1}|^2)(a_n/a_{n-1})z}{1+(a_n/a_{n-1})z} \right) .$$

The condition of Pringsheim's theorem takes the form

$$\left| \frac{a_n}{a_{n-1}} + z \right| \geqslant (1 - |a_{n-1}|^2)|z| + \left| \frac{a_{n-1}}{a_n} \right| . \tag{8.206}$$

Let Δ_n be the open disc centered at $c_n = -a_{n-1}/a_n$ with radius $(1 - |a_{n-1}|^2) + |a_{n-1}/a_n|$. The conditions imposed on $\{a_n\}_{n\geqslant 0}$ imply that the c_n are located in an arbitrary small angle with vertex at $z = 0$ and with the negative real semi-axis as the bisectrix. Since the a_n are separated from $\mathbb{T}$, there is a neighborhood U of $z = 1$ such that $\Delta_n \cap U = \varnothing$ for $n \geqslant N$. Squeezing U if necessary, we obtain that (8.206) holds in U for $n \geqslant N$. By Prigsheim's theorem $K_n(z)$ converges to a bounded analytic function in U. It follows that $f(z) = a_0(1 + K_1(z))^{-1}$ is meromorphic in U. Since f is bounded in $\mathbb{D} \cap U$, we conclude that the Geronimus continued fraction of f converges to f uniformly on I. Since $\liminf_n |a_n| > 0$, we obtain by Lemma 8.79 that $|f| = 1$ on I. Since $\operatorname{supp} \sigma \cap I$ is in the zero set of the analytic function $1 - zf$ on I, the proof is complete. □

193 Cesàro conditions. A sequence $\{a_n\}_{n\geqslant 0}$ is called a Cesàro sequence if

$$\lim_n \frac{|a_0| + \cdots + |a_n|}{n+1} = 0 .$$

We say that $\sigma \in \mathfrak{P}(\mathbb{T})$ is a Cesàro–Nevai measure if its parameters make a Cesàro sequence.

Theorem 8.105 *Any Rakhmanov measure is a Cesàro–Nevai measure.*

Proof Given $\varepsilon > 0$ let $\Lambda(\varepsilon) = \{n : |a_n| \geqslant \varepsilon\}$. By Theorem 8.73 the Verblunsky parameters $\{a_n\}_{n\geqslant 0}$ of σ satisfy the Matè–Nevai condition (8.163). It follows that for every $K > 0$ there is a positive integer $L = L(\varepsilon, K)$ such that

$$|a_{n+k} a_n| < \varepsilon^2 , \qquad k = 1, 2, \ldots , K , \quad n \geqslant L . \tag{8.207}$$

Let $M(\varepsilon) = \Lambda(\varepsilon) \cap [L, +\infty)$. The sets

$$M(\varepsilon) , \quad M(\varepsilon) + 1 , \quad \ldots , \quad M(\varepsilon) + K \tag{8.208}$$

do not intersect. Indeed, if $(M(\varepsilon) + j) \cap ((M(\varepsilon) + i)) \neq \emptyset$ for $i < j \leqslant K$ then there exists $n \in M(\varepsilon)$ such that $n + (j - i) \in M(\varepsilon)$. It follows that $\varepsilon^2 \leqslant |a_n a_{n+(j-i)}|$, which contradicts (8.207) since $1 \leqslant j - i \leqslant K$. We denote by

$$d = d(\varepsilon) = \overline{\lim_n} \frac{1}{n} \operatorname{Card} \{\Lambda(\varepsilon) \cap [0, n]\}$$

the upper density of $\Lambda(\varepsilon)$. Since the sets listed in (8.208) do not intersect, we have

$$\begin{aligned} n &\geqslant L+\sum_{j=0}^{K}\operatorname{Card}\{(M(\varepsilon)+j)\cap[L,n]\} \\ &\geqslant L+(K+1)\operatorname{Card}\{M(\varepsilon)\cap[L,n]\}-K(K+1) \\ &= L+(K+1)\operatorname{Card}\{\Lambda(\varepsilon)\cap[L,n]\}-K(K+1) \\ &= L+(K+1)\operatorname{Card}\{\Lambda(\varepsilon)\cap[0,n]\} \\ &- K(K+1)-(K+1)\operatorname{Card}\{\Lambda(\varepsilon)\cap[0,n]\} \\ &\geqslant (K+1)\operatorname{Card}\{\Lambda(\varepsilon)\cap[0,n]\}-K(K+1)-KL\,. \end{aligned}$$

Dividing both sides by n and passing to the limit $n\to+\infty$, we obtain that $(K+1)d\leqslant 1$. Since K is an arbitrary positive integer, $d=d(\varepsilon)=0$. Since $\{a_n\}_{n\geqslant 0}$ is bounded and ε is an arbitrarily small positive number, we see that $\{a_n\}_{n\geqslant 0}$ is a Cesàro sequence. $\square$

194 Universal measures. Given $\sigma\in\mathfrak{P}(\mathbb{T})$ we denote by $\operatorname{Lim}\sigma$ the derived set of $\{|\varphi_n|^2d\sigma\}_{n\geqslant 0}$, i.e. the set of all limit points of this sequence in $\mathfrak{P}(\mathbb{T})$. A probability measure σ is called *universal* if $\operatorname{Lim}\sigma=\mathfrak{P}(\mathbb{T})$. It is clear that the orthogonal polynomials of a universal measure, if such a measure exists, have very poor asymptotic properties. By Theorem 8.67 σ is universal if and only if the sequence $\{f_nb_n\}_{n\geqslant 0}$ is dense in $\mathcal{B}$.

Theorem 8.106 *There are Cesàro–Nevai universal measures.*

Proof
Step 1. Let $P,Q\in\mathcal{B}$ have an infinite number of Schur parameters:

$$\mathcal{S}P=\{p_0,p_1,\dots\}\,,\qquad \mathcal{S}Q=\{q_0,q_1,\dots\}\,.$$

That is, P and Q are not finite Blaschke products. We put $H=PQ$. Pick a union Γ of disjoint intervals of nonnegative integers:

$$\Gamma=\bigcup_{k=1}^{\infty}[n_k-m_k,n_k+m_k],$$

with $\lim_k m_k=\infty$. Define a function $f\in\mathcal{B}$ by $\mathcal{S}f=\{a_0,a_1,\dots\}$, where

$$a_j\overset{\text{def}}{=}\begin{cases}p_{j-n_k} & \text{if } n_k\leqslant j\leqslant n_k+m_k,\\ -\bar q_{n_k-j-1} & \text{if } n_k-m_k\leqslant j\leqslant n_k-1\end{cases}$$

is arbitrary on the complement $\Gamma^c=\mathbb{Z}_+\setminus\Gamma$. Let f_n and b_n be the Schur functions of f. Then for $n=n_k$

$$\mathcal{S}f_n=\{p_0,p_1,\dots,p_{m_k},\dots\}\,,\qquad \mathcal{S}f_n=\{q_0,q_1,\dots,q_{m_k-1},\dots\}\,.$$

By Theorem 8.64

$$\lim_{n\in\Delta} f_n(z) = P(z)\,, \quad \lim_{n\in\Delta} b_n(z) = Q(z)\,, \qquad \Delta \stackrel{\text{def}}{=} \{n_k : k \geqslant 1\}\,,$$

implying that

$$\lim_{n\in\Delta} f_n(z) b_n(z) = H(z)\,. \tag{8.209}$$

Let $\sigma, \nu \in \mathfrak{P}(\mathbb{T})$ be the measures with Schur functions f and H and $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. It follows from (8.209) that $\nu = *\text{-}\lim_{n\in\Lambda} |\varphi_n|^2 d\sigma$ by Theorems 8.67 and 8.64.

Step 2. Given positive integers r and s we denote pairwise disjoint subsets of $\mathbb{N}$ by

$$\Lambda_{rs} \stackrel{\text{def}}{=} [2^{2^r(2s-1)}, 2^{2^r(2s-1)} + 2s] = [n_{rs} - m_{rs}, n_{rs} + m_{rs}].$$

Let

$$\Lambda \stackrel{\text{def}}{=} \bigcup_{r,s=1}^{\infty} \Lambda_{rs}\,.$$

For a large positive integer N the number of solutions of the inequality $2^k < N$ for positive integer k does not exceed $\log_2 N$. If $k = 2^r(2s-1)$ then $s \leqslant k < \log_2 N$. It follows that

$$\operatorname{Card}\{\Lambda \cap [0, N]\} \leqslant (2\log_2 N + 1)\log_2 N\,,$$

implying that $d(\Lambda) = 0$.

Let $\{H_l\}_{l\geqslant 0}$ be any dense sequence in $\mathcal{B}$. Then the measures ν_l with Schur functions H_l are dense in $\mathfrak{P}(\mathbb{T})$. Each H_l can be factored as $H_l = P_l Q_l$, where neither P_l nor Q_l is a finite Blaschke product.

Now we apply now step 1 to

$$\Lambda_l \stackrel{\text{def}}{=} \bigcup_{s=1}^{\infty} \Lambda_{ls} = \bigcup_{s=1}^{\infty} [n_{ls} - m_{ls}, n_{ls} + m_{ls}]\,, \qquad l = 1, 2, \ldots$$

Thus the Schur parameters $\{a_n\}_{n\geqslant 0}$ of f are defined for $n \in \Lambda$. For $n \notin \Lambda$ we define a_n to satisfy the condition $\lim_{n\notin\Lambda} a_n = 0$. By step 1, for every $l \geqslant 1$ there is a subset $\Delta_l \subset \mathbb{Z}_+$ such that

$$\lim_{n\in\Delta_l} f_n(z) = P_l(z)\,, \qquad \lim_{n\in\Delta_l} b_n(z) = Q_l(z)\,, \qquad \lim_{n\in\Delta_l} f_n(z) b_n(z) = H_l(z)$$

in $\mathcal{B}$. It follows that each measure ν_l is a limit point of the sequence $\{|\varphi_n|^2\, d\sigma\}_{n\in\Lambda}$, implying that σ is a universal measure. $\square$

8.10 Schur functions of smooth measures

195 Pointwise estimates. The method of Schur functions can also be applied to the study of measures σ with smooth σ'.

Theorem 8.107 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$, $\{f_n\}_{n\geqslant 0}$ be the Schur functions of σ and $\{\varphi_n\}_{n\geqslant 0}$ the orthogonal polynomials in $L^2(d\sigma)$. If $\|f_n\|_\infty < 1/2$ then on $\mathbb{T}$*

$$\left| |\varphi_n|^2\sigma' - 1 \right| < 6|f_n|\,(1-2|f_n|)^{-1}\,. \tag{8.210}$$

Proof If $|\varphi_n|^2\sigma' - 1 < 0$ then (8.210) follows by (8.134). It follows from (8.107) that

$$|\varphi_n|^2\sigma' - 1 = 2\,\frac{\mathrm{Re}\,(\zeta b_n f_n) - |f_n|^2}{|1-\zeta b_n f_n|^2}\,, \tag{8.211}$$

which implies that $|\varphi_n|^2\sigma' - 1 \geqslant 0$ if and only if $|f_n|^2 \leqslant \mathrm{Re}\,(\zeta b_n f_n)$. Now let $|\varphi_n|^2\sigma' - 1 \geqslant 0$ for some $\zeta \in \mathbb{T}$. By (8.125)

$$\frac{|\varphi_n|^2\sigma' - 1}{|\varphi_n|^2\sigma' + 1} = \mathrm{Re}\ \zeta b_n f_n - |f_n|^2 + \frac{|\varphi_n|^2\sigma' - 1}{|\varphi_n|^2\sigma' + 1}\,\mathrm{Re}\ \zeta b_n f_n\,. \tag{8.212}$$

Notice that the fraction on the left-hand side of (8.212) is nonnegative and is bounded by 1. Since $|f_n|^2 \leqslant \mathrm{Re}\ \zeta b_n f_n \leqslant |f_n|$, we obtain from (8.212) that $0 \leqslant (|\varphi_n|^2\sigma' - 1)(|\varphi_n|^2\sigma' + 1)^{-1} \leqslant 2|f_n|$, which obviously implies $|\varphi_n|^2\sigma' - 1 \leqslant 4|f_n|(1-2|f_n|)^{-1}$, proving the theorem. □

196 Hölder classes: inverse theorems. We define the Hölder class as follows. For $0 < \alpha < 1$ we put

$$\Lambda_\alpha = \{f \in C(\mathbb{T}) : |f(e^{i(x+t)}) - f(e^{ix})| \leqslant C_f|t|^\alpha,\ x, t \in \mathbb{R}\}\,.$$

The Zygmund class Λ_1 is defined by

$$\Lambda_1 = \{f \in C(\mathbb{T}) : |f(e^{i(x+t)}) + f(e^{i(x-t)}) - 2f(e^{ix})| \leqslant C_f|t|^\alpha,\ x, t \in \mathbb{R}\}\,.$$

Now let $n < \alpha \leqslant n+1$, where n is a positive integer. Then Λ_α denotes the space of all functions on $\mathbb{T}$ with nth derivative $f^{(n)} \in \Lambda_{\alpha-n}$.

Theorem 8.108 *If $\sigma \in \mathfrak{P}(\mathbb{T})$ with Schur functions $\{f_n\}_{n\geqslant 0}$ satisfying $\|f_n\|_\infty = O(n^{-\alpha})$, $\alpha > 0$, then σ is absolutely continuous and $\sigma' \in \Lambda_\alpha$.*

Proof By (8.210) we obtain $\||\varphi_n|^2\sigma' - 1\|_\infty = O(n^{-\alpha})$, which implies that $\inf_{\mathbb{T}} \sigma' > 0$. It follows that $\||\varphi_n|^2 - 1/\sigma'\|_\infty = O(n^{-\alpha})$. Notice that $|\varphi_n|^2 = \varphi_n\bar{\varphi}_n$ is a trigonometric polynomial of order n. Now the result follows by the Bernstein–Zygmund theorem. □

197 Direct theorems. We begin with a technical lemma.

Lemma 8.109 *For every $z \in \mathbb{T}$ the map $\lambda \to \lambda b_n(z,\lambda)$ is a homeomorphism of $\mathbb{T}$.*

Proof By (8.87) and (8.73)

$$\begin{aligned}\lambda b_n(z,\lambda) &= \lambda \frac{\varphi_n(z,\lambda)}{\varphi_n^*(z,\lambda)} = \frac{(1+\lambda)\varphi_n - (1-\lambda)\psi_n}{(1+\lambda)\varphi_n^* - (1-\lambda)\psi_n^*} \\ &= \frac{(\varphi_n - \psi_n) + \lambda(\varphi_n + \psi_n)}{(\varphi_n^* + \psi_n^*) + \lambda(\varphi_n^* - \psi_n^*)} = \frac{-A_{n-1}^* + \lambda z B_{n-1}^*}{B_{n-1} - \lambda z A_{n-1}} \\ &= \frac{B_{n-1}^*}{B_{n-1}} \frac{\lambda z - A_{n-1}^*/B_{n-1}^*}{1 - \lambda z (A_{n-1}/B_{n-1})} \,.\end{aligned}$$

Observing that $A_{n-1}^*/B_{n-1}^* = \bar{A}_{n-1}/\bar{B}_{n-1}$ for $z \in \mathbb{T}$ and that by Corollary 8.11 this complex number lies in $\mathbb{D}$, we see that $\lambda \to \lambda b_n(z,\lambda)$ is a Möbius transform. □

Corollary 8.110 *If $\sigma \in \mathfrak{P}(\mathbb{T})$ then $|f_n| \leqslant \sup_{\lambda\in\mathbb{T}} ||\varphi_n(\zeta,\lambda)|^2\sigma_\lambda' - 1|$ on $\mathbb{T}$.*

Proof By Lemma 8.109, given $\zeta \in \mathbb{T}$ there exists $\lambda \in \mathbb{T}$ such that

$$\zeta b_n(z,\lambda)\lambda f_n(\zeta) = \operatorname{Re}(\zeta b_n(z,\lambda)\lambda f_n) = -|f_n| \,.$$

By (8.211)

$$\sup_{\lambda\in\mathbb{T}} \left| |\varphi_n(z,\lambda)|^2 \sigma_\lambda' - 1 \right| \geqslant 2\frac{|f_n| + |f_n|^2}{(1+|f_n|)^2} = \frac{2|f_n|}{1+|f_n|} \geqslant |f_n| \,,$$

as stated. □

Theorem 8.111 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ be an absolutely continuous measure with $(\sigma')^{-1} \in \Lambda_\alpha$ and Schur functions $\{f_n\}_{n\geqslant 0}$. Then*

$$\|f_n\|_\infty = O(n^{-\alpha}\log n) \,.$$

Proof The harmonic conjugate $-(2\pi)^{-1}\int_{-\pi}^{\pi} \cot\frac{1}{2}(t-x)\,d\sigma(t)$ of σ' is in Λ_α; see Zygmund (1977, Chapter VII, (1.8), Chapter III, Theorem 13.29). It follows that $F^\sigma \in \Lambda_\alpha$. Since

$$zf = \frac{F^\sigma - 1}{F^\sigma + 1} \qquad \text{and} \qquad \frac{1}{F^\sigma + 1} \in \Lambda_\alpha \,,$$

we obtain that $f \in \Lambda_\alpha$. Next, $\|f\|_\infty < 1$ since $\inf \sigma' > 0$. Observing that λf is the Schur function of σ_λ, we obtain that $\lambda \to (\sigma_\lambda)^{-1}$ is a homeomorphism of $\mathbb{T}$ into $\Lambda_\alpha \setminus \{\mathbb{O}\}$. By Szegő's theorem,

$$\varphi_n(z,\lambda) = \frac{z^n}{\overline{D(\sigma_\lambda, z)}} + O\left(\frac{\log n}{n^\alpha}\right), \qquad n \to \infty$$

uniformly in z and λ; $z, \lambda \in \mathbb{T}$ (Grenander and Szegő 1958). Notice that the proof given Section. 3.5 of the latter work extends to $\alpha > 1$ by Bernstein's theorem on best polynomial approximation. It follows that

$$\sup_{\lambda\in\mathbb{T}}\left||\varphi_n(z,\lambda)|^2\sigma'_\lambda - 1\right| = O(n^{-\alpha}\log n)$$

uniformly in $\zeta \in \mathbb{T}$. This proves the theorem by Corollary 8.110. □

Corollary 8.112 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ be an absolutely continuous measure with $(\sigma')^{-1} \in \Lambda_\alpha$ and Verblunsky parameters $\{a_n\}_{n\geqslant 0}$. Then*

$$a_n = O(n^{-\alpha}\log n)\ .$$

Proof This follows by $|a_n| = \left|\int_{\mathbb{T}} f_n\, dm\right| \leqslant \|f_n\|_\infty$. □

The following results are immediate by (8.26) from Theorems 8.108 and 8.111. They demonstrate a remarkable similarity between the behavior of A_n/B_n and the partial Fourier sums of f (Zygmund 1977, Chapter II, Theorem 10.8). Compare this with the discussion in §**163**, Section 8.1.

Corollary 8.113 *If $\sigma \in \mathfrak{P}(\mathbb{T})$ is absolutely continuous with $(\sigma')^{-1} \in \Lambda_\alpha$ and A_n, B_n are its Wall polynomials then*

$$\|f - A_n/B_n\|_\infty = O(n^{-\alpha}\log n)\ .$$

Corollary 8.114 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$ with $f = f^\sigma$ satisfying $\|f\|_\infty < 1$. If*

$$\|f - A_n/B_n\|_\infty = O(n^{-\alpha})\ , \qquad \alpha > 0\ ,$$

then σ is absolutely continuous and $(\sigma')^{-1}, f \in \Lambda_\alpha$.

Proof By (8.18) $\omega_n|B_n|^{-2} = 1 - |A_n/B_n|^2 \rightrightarrows 1 - |f|^2$ uniformly on $\mathbb{T}$. Hence $\sup_{\mathbb{T}} ||f| - |A_n^*/B_n|| \to 0$ as $n \to \infty$. It follows from (8.26) that $\|f_n\|_\infty = O(1/n^\alpha)$, which completes the proof by Theorem 8.108. □

8.11 Periodic measures

198 Ratio-asymptotic measures. A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is called *ratio asymptotic* if the limit

$$G_\sigma(z) \stackrel{\text{def}}{=} \lim_n \frac{\Phi^*_{n+1}(z)}{\Phi^*_n(z)} \tag{8.213}$$

exists for every $z \in \mathbb{D}$. By Theorem 8.75(h) every Rakhmanov measure is ratio asymptotic.

Definition 8.115 *A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is said to satisfy the López condition if there exist $a \in (0, 1]$ and $\lambda \in \mathbb{T}$ such that the Verblunsky parameters of σ satisfy*

$$\lim_n |a_n| = a\,, \qquad \lim_n \frac{a_{n+1}}{a_n} = \lambda\,.$$

Theorem 8.116 *A measure $\sigma \in \mathfrak{P}(\mathbb{T})$ is ratio asymptotic if and only if either σ is a Rakhmanov measure or σ satisfies the López condition.*

The proof of Theorem 8.116 splits into three lemmas.

Lemma 8.117 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$, $\{a_n\}_{n\geqslant 0}$ be the Verblunsky parameters of σ and $\{b_n\}_{n\geqslant 0}$ the inverse Schur functions for σ. If for every $z \in \mathbb{D}$*

$$\lim_n a_n b_n(z) = B(z) \tag{8.214}$$

then for every $k \geqslant 1$ the following limits exist:

$$-\lim_n \bar{a}_n a_{n+k} = c_k\,. \tag{8.215}$$

Proof Since $\{a_n b_n\}_{n\geqslant 0}$ is normal in $\mathbb{D}$, the function $B(z)$ is holomorphic in $\mathbb{D}$. Putting $z = 0$ in (8.214) implies that $c_1 = B(0)$. By (8.75),

$$b_{n+1}(z)(1 - z a_n b_n(z)) = z b_n(z) - \bar{a}_n\,. \tag{8.216}$$

Multiplying both sides of (8.216) by a_{n+1}, we obtain that

$$\lim_n a_{n+1} b_n(z) = (B(z) - B(0))z^{-1} - B^2(z) \stackrel{\text{def}}{=} B_{(1)}(z)$$

uniformly on compact subsets of $\mathbb{D}$. Suppose now that $-\lim_n a_{n+j}\bar{a}_n = c_j$ for $j \leqslant k$ and $\lim_n a_{n+k} b_n(z) = B_{(k)}(z)$ uniformly on compacts of $\mathbb{D}$. For $z = 0$ this implies that $-\lim_n a_{n+k+1}\bar{a}_n = B_{(k)}(0)$. Multiplying both sides of (8.216) by a_{n+k+1}, we obtain

$$\lim_n a_{n+k+1} b_n(z) = (B_{(k)}(z) - B_{(k)}(0))z^{-1} - B_{(k)} B \stackrel{\text{def}}{=} B_{(k+1)}(z)\,,$$

which completes the proof by induction. □

Lemma 8.118 *Let $\{a_n\}_{n\geqslant 0}$ be a sequence in $\mathbb{C}$ such that $\lim_n a_{n+k}\bar{a}_n = c_k$ exists for every $k \geqslant 1$. Then either $c_k = 0$ for every $k \geqslant 1$ or $c_k \neq 0$ for every $k \geqslant 1$. If $c_k \neq 0$ for $k \geqslant 1$ and $|a_n| < 1$ for $n \geqslant 0$ then $\{a_n\}_{n\geqslant 0}$ satisfies the López condition.*

Proof For $k > s$ we have

$$c_k \bar{c}_s = \lim_n a_{n+k}\bar{a}_n \lim_n \bar{a}_{n+s} a_n = \lim_n |a_n|^2 (a_{n+k}\bar{a}_{n+s})\,. \tag{8.217}$$

It is clear that $\lim_n a_{n+k}\bar{a}_{n+s} = \lim_n a_{n+s+(k-s)}\bar{a}_{n+s} = c_{k-s}$. If $c_j \neq 0$ for some $j \geqslant 1$ then for any pair (k, s) with $k - s = j$ we obtain by (8.217) that $c_k \bar{c}_s = c_j \lim_n |a_n|^2$. Since $c_j \neq 0$, this shows that the limit $\lim_n |a_n|^2 = a^2 = c_k \bar{c}_s / c_j$ exists. If $a = 0$ then $c_j = \lim_n a_{n+j}\bar{a}_n = 0$, contradicting $c_j \neq 0$. It follows that $a > 0$. Now $a_n \in \mathbb{D}$ for

$n \geqslant 0$ implies that $a \in (0,1)$. Finally, $\lim_n a_{n+1}/a_n = \lim_n (a_{n+1}a_n)/(\bar{a}_n a_n) = c_1/a^2 \in \mathbb{T}$, which proves the lemma. □

Lemma 8.119 *Let σ satisfy the López condition for some $a \in (0,1)$ and $\lambda \in \mathbb{T}$. Then the ratio-asymptotic limit* (8.213) *holds uniformly on compact subsets of $\mathbb{D}$ and is given by*

$$G_\sigma(z) = \frac{1}{2}\{(1+\lambda z) + \sqrt{(1-\lambda z)^2 + 4\lambda a^2 z}\}\,. \tag{8.218}$$

Proof It follows from the second recurrence formula in (8.65) that

$$\frac{\Phi_{n+1}^*(z)}{\Phi_n^*(z)} = 1 - z a_n b_n(z)\,. \tag{8.219}$$

Although the mapping $\mathcal{S}$ is not linear, we still have by (8.9)

$$\mathcal{S}(cf) = (ca_0, ca_1, \dots)$$

for any unimodular constant c, $|c| = 1$. Taking this observation into account and applying it to (8.141) with $c = a_n/|a_n|$, we obtain that

$$\mathcal{S}\left(\frac{a_n}{|a_n|} b_n(z)\right) = \left(-\frac{a_n \bar{a}_{n-1}}{|a_n|}, -\frac{a_n \bar{a}_{n-2}}{|a_n|}, \dots, \frac{a_n}{|a_n|}\right)\,. \tag{8.220}$$

It follows from the López condition that

$$\lim_n \frac{a_n \bar{a}_{n-k}}{|a_n|} = \lim_n \frac{a_n}{a_{n-k}} \lim_n \frac{|a_{n-k}|^2}{|a_n|} = \lambda^k a\,, \qquad k = 1, 2, \dots\,.$$

Hence the right-hand side of (8.220) converges to

$$x = (-\lambda a, -\lambda^2 a^2, -\lambda^3 a^3, \dots)$$

in $\mathcal{S}^\infty$ if $0 < a < 1$, and to $x = -\lambda$, $\mathcal{D}(x) = \{0\}$, if $a = 1$. In the latter case, by Theorem 8.64 the sequence $\{a_n b_n(z)\}_{n\geqslant 0}$ converges to $-\lambda$ uniformly on compact subsets of $\mathbb{D}$. Therefore $G_\sigma = 1 + \lambda z$, which coincides with (8.218) for $a = 1$. If $0 < a < 1$, then it is easy to see that $\mathcal{S}(-\lambda f_a(\lambda z)) = x$ and by Theorem 8.64 $G_\sigma(z) = 1 + a\lambda f_a(\lambda z)$, whereas the convergence in (8.213) is uniform in $\mathbb{D}$. □

Proof of Theorem 8.116 It follows from (8.219) that $\sigma \in \mathfrak{P}(\mathbb{T})$ is ratio asymptotic if and only if (8.214) holds. If (8.214) holds then the limits (8.215) exist by Lemma 8.117. By Lemma 8.118 either all these limits are zero or the Verblunsky parameters $\{a_n\}_{n\geqslant 0}$ of σ satisfy the López condition. If the first possibility occurs, then by Theorem 8.75 σ is a Rakhmanov measure and is ratio asymptotic with $G_\sigma \equiv 1$. If σ satisfies the López condition then σ is ratio asymptotic by Lemma 8.119. □

Corollary 8.120 *If $\sigma \in \mathfrak{P}(\mathbb{T})$ then*

$$\frac{\Phi^*_{n+1}(z)}{\Phi^*_n(z)} \quad \rightrightarrows \quad \frac{1}{2}\{(1+\lambda z)+\sqrt{(1-\lambda z)^2+4\lambda a^2 z}\} \tag{8.221}$$

in $\mathbb{D}$ for some $a \in (0,1]$, $\lambda \in \mathbb{T}$, if and only if σ satisfies the López condition for exactly these values of a and λ.

Proof If (8.221) holds then σ is ratio asymptotic. Since $a > 0$ in (8.221), we obtain that $G_\sigma \not\equiv 1$. This implies by Theorem 8.116 that σ satisfies the López condition. Next, the values a and λ are uniquely determined by a given G_σ. However, if σ satisfies the López condition then the desired conclusion follows by Lemma 8.119. □

In fact, the convergence in (8.221) is uniform on compact subsets of $\mathbb{C} \setminus \operatorname{supp} \sigma$; see Barrios and López (1999, Theorem 1).

199 Measures with constant parameters. Let $f = f(z;a)$ be the Schur function corresponding to constant parameters $a_k = a = |a|e^{i\theta_a}$, $k \geqslant 0$, where $0 \neq a \in \mathbb{D}$. Then $f_1 = f$ and by (8.9) f satisfies the quadratic equation

$$\bar{a}zX^2 + (1-z)X - a = 0\,. \tag{8.222}$$

Together with f we consider the Schur function $f^\#$ corresponding to the constant parameters $a_k = -\bar{a}$, $k \geqslant 0$. Then $f^\#$ satisfies the equation

$$azX^2 - (1-z)X - \bar{a} = 0\,. \tag{8.223}$$

Hence the solutions to (8.222) are $x_1 = f$ and $x_2 = (zf^\#)^{-1}$. By Viète's theorem,

$$\frac{f}{zf^\#} = x_1 x_2 = -\frac{\bar{a}}{az} \quad \Rightarrow \quad f^\# = -e^{2i\theta_a} f\,. \tag{8.224}$$

The discriminant $\mathcal{D}_a$ of (8.222) and of (8.223),

$$\mathcal{D}_a = (z-1)^2 + 4|a|^2 z = z\left(4|a|^2 - |1-z|^2\right)\,, \qquad z \in \mathbb{T}\,,$$

determines the real trigonometric polynomial

$$\mathcal{T}_a = \mathcal{D}_a/z = 4|a|^2 - |1-z|^2$$

on $\mathbb{T}$. The roots of $\mathcal{D}_a$ are at the two points of intersection of the circle $|1-z| = 2|a| < 2$ with $\mathbb{T}$:

$$z_\pm = 1 - 2|a|^2 \pm 2|a|i\sqrt{1-|a|^2}\,.$$

The arc $\Delta_\alpha = \{\exp(i\theta) : \alpha \leqslant \theta \leqslant 2\pi - \alpha\}$, where $\sin(\alpha/2) = |a|$, $0 < \alpha < \pi$, connects z_+ with z_- counterclockwise along the part of $\mathbb{T}$ outside the disc $\{z : |z-1| < 2|a|\}$. It follows that $\mathcal{T}_a(e^{i\theta}) = 4\left(|a|^2 - \sin^2\theta/2\right) = 2(\cos\theta - \cos\alpha)$ is negative on Δ_α and is positive on the open arc $\gamma_\alpha = \mathbb{T} \setminus \Delta_\alpha$. By (8.21),

$$|1+\bar{a}zf|^2\left(1-|f(z)|^2\right) = \left(1-|a|^2\right)\left(1-|f(z)|^2\right)\,.$$

Hence zf maps $\mathbb{T}$ into itself and into the circle orthogonal to $\mathbb{T}$ centered at $-1/\bar{a}$ with radius $\sqrt{1/|a|^2-1}$. By (8.222),

$$x_1 = f(z;\, a) = \frac{(z-1)+\sqrt{\mathcal{D}_a}}{2\bar{a}z} = \frac{2i\sin(\theta/2)+\sqrt{\mathcal{T}_a}}{2\bar{a}e^{i\theta/2}}\,, \qquad z = e^{i\theta}\,, \tag{8.225}$$

where the branch of the square root is taken to satisfy $\sqrt{1}=1$; this by the Taylor formula guarantees that $f(z;\, a) \sim a$ when $z \to 0$. Hence

$$zf(z;\, a) \;=\begin{cases} \left(\exp\dfrac{i\theta}{2}\right)\dfrac{1}{\bar{a}}\left(\sqrt{|a|^2-\sin^2\dfrac{\theta}{2}}+i\sin\dfrac{\theta}{2}\right) & \text{if } |\theta| \leqslant \alpha\,, \\ \left(i\exp\dfrac{i\theta}{2}\right)\left(\dfrac{1}{\bar{a}}\sin\dfrac{\theta}{2}-\sqrt{\sin^2\dfrac{\theta}{2}-|a|^2}\right) & \text{if } \alpha < |\theta| \leqslant \pi\,. \end{cases} \tag{8.226}$$

Therefore $|f| < 1$ on Δ_α except at the ends and $|f| = 1$ on $\gamma_\alpha = \mathbb{T} \setminus \Delta_\alpha$. Applying complex conjugation to (8.222) for $z \in \Delta_\alpha$, we obtain that $\bar{f} = zf^{\#}$ on Δ_α and hence

$$zff^{\#} = |f|^2 \qquad \text{on } \Delta_\alpha\,. \tag{8.227}$$

Applying (8.7), we easily find σ_a' on Δ_α by Viète's formulas:

$$\begin{aligned} \sigma_a' = \frac{1-|f|^2}{|1-zf|^2} &= \frac{zf^{\#}f-1}{(1-zf)(f^{\#}-1)} = \frac{z(x_1-x_2)}{(1-zx_1)(1-zx_2)} \\ &= \frac{\sqrt{\mathcal{D}_a}}{1+\bar{a}-z(1+a)} = \frac{\sqrt{|\mathcal{D}_a|}}{|2\ \mathrm{Im}((1+a)\sqrt{z})|}\,. \end{aligned} \tag{8.228}$$

Lemma 8.121 *For* $f(z;\, a)$

(a) $zf(z;\, a) : \mathbb{D} \longrightarrow \mathbb{D} \cap \{w : |w+1/\bar{a}| > \sqrt{|a|^{-2}-1}\}$*;*
(b) $f(z;\, a) : \mathbb{D} \longrightarrow \mathbb{D} \cap \{w : |w-1/\bar{a}| < \sqrt{|a|^{-2}-1}\}$*;*
(c) $G_\sigma = 1+\bar{a}zf(z;\, a) : \hat{\mathbb{C}} \setminus \Delta_\alpha \longrightarrow \{w : |w| > \sqrt{1-|a|^2}\}$,

are conformal onto mappings.

Proof (a) This follows from (8.226), which says that the image $\mathbb{T}$ under $zf(z;\, a)$ is the union of two circular arcs.

(b) This follows from (a), since f is zf followed by a Möbius transform (by Shur's algorithm $f = f_1$).

(c) This follows from (a) by the Schwarz reflection principle. □

Since $w = zf(z;\, a)$ is a conformal mapping, there is at most one solution $z_a \in \gamma_\alpha$ to the equation $zf(z;\, a) = 1$. By Lemma 8.121(a) the solution exists if and only if $|1+a|^2 \geqslant 1-|a|^2$. In turn this means that the parameter $0 \neq a \in \mathbb{D}$ must be outside the open disc $|w+1/2| < 1/2$. Next $|f_a| = 1$ on $\mathbb{T} \setminus \Delta_\alpha$, which shows that $\sigma_a' = 0$

everywhere on γ_α except for $z = z_a$. To find the location of z_a we put $w = 1$ in the quadratic equation

$$\bar{a}w^2 + (1-z)w - az = 0 \quad \Rightarrow \quad z_a = \frac{1+\bar{a}}{1+a} .$$

The measure σ_a has a point mass at $z = z_a$ if $z_a \in \gamma_\alpha$. To find $\sigma_a(\{z_a\})$ we observe that by (8.6)

$$F^{\sigma_a}(rz_a) = \frac{1-r^2}{(1-r)^2}\sigma_a(\{z_a\}) + o\left(\frac{1}{1-r}\right) = \frac{2\sigma_a(\{z_a\})}{1-r} + o\left(\frac{1}{1-r}\right),$$

where $r \to 1^-$. Now applying (8.7), we obtain

$$\sigma_a(\{z_a\}) = \operatorname{Re} \lim_{r\to 1^-} \frac{z_a - rz_a}{2z_a} \frac{1+w(rz_a)}{1-w(rz_a)} = \operatorname{Re} \frac{1}{z_a(w)'(z_a)} .$$

Differentiating the quadratic equation for $w = zf$, we easily find that

$$\frac{1}{z_a(w)'(z_a)} = \frac{a+\bar{a}+2|a|^2}{z_a(1+a)^2} = \frac{2(\operatorname{Re} a + |a|^2)}{|1+a|^2} = \sigma_a(\{z_a\}) .$$

Again, if $|1/2+a| > 1/2$ then $\sigma_a(\{z_a\}) > 0$. There is a simple geometric construction due to Geronimus (1941, §4), which allows one to find the arc Δ_α and z_a for any nonzero a easily, see Fig. 8.1. With its center at 0 we plot the circle of radius $|a|$ and continue the tangents to this circle from the point $(-1, 0)$ to the intersection with $\mathbb{T}$. This gives the ends A and B of Δ_α. Then $D = z_a$ between them is the intersection with

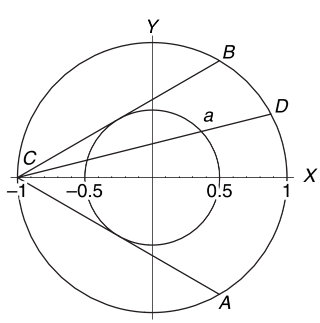

Fig. 8.1. Δ_α and $D = z_a$

$\mathbb{T}$ of the line passing through $(-1,0)$ and (a,a). For an a with $|a+1/2| \leqslant 1/2$ there are no point masses at z_a. Using (8.228), we can derive Geronimus's formula for σ_a':

$$\frac{\sqrt{|\mathcal{D}_a|}}{|1+\bar{a}-z(1+a)|} = \frac{2\sqrt{\sin^2(\theta/2)-\sin^2(\alpha/2)}}{|1+a||z-z_a|}$$
$$= \frac{\sqrt{\sin(\theta-\alpha)/2\,\sin(\theta+\alpha)/2}}{|1+a||\sin(\theta-\beta)/2|}\,.$$

200 Logarithmic potential theory on $\mathbb{T}$[2]. If μ is a nonnegative Borel measure on $\mathbb{T}$ then

$$U^\mu(z) = \int_{\mathbb{T}} \log\frac{e}{|\zeta - z|}\, d\mu(\zeta)$$

is the logarithmic potential of μ. In the above formula $e = 2.71\ldots$. Hence the logarithmic kernel is positive on $\mathbb{T}$. We have

$$\log\frac{e}{|1-\zeta|} = 1 + \sum_{n\neq 0}\frac{\zeta^n}{2|n|}\,.$$

Then the potential energy of μ is nonnegative and equals

$$I(\mu) \overset{\text{def}}{=} \int_{\mathbb{T}} U^\mu\, d\mu = |\hat{\mu}(0)|^2 + \sum_{n\neq 0}\frac{|\hat{\mu}(n)|^2}{2|n|}\,.$$

The logarithmic capacity $\operatorname{cap}\mathcal{K}$ of a compact $\mathcal{K}\subset\mathbb{T}$ is defined as

$$\operatorname{cap}\mathcal{K} \overset{\text{def}}{=} \sup\{\mu(\mathcal{K}) : \operatorname{supp}\mu \subset \mathcal{K}\,,\ U^\mu(\zeta)\leqslant 1 \text{ for } \zeta\in\mathcal{K}\}\,.$$

It is well known that $\operatorname{cap}\mathcal{K} = (\inf I(\mu))^{-1}$, where the infimum is taken over all $\mu \in \mathfrak{P}(\mathbb{T})$ supported by $\mathcal{K}$: $\operatorname{supp}\mu\subset\mathcal{K}$. By duality,

$$\operatorname{cap}\mathcal{K} = \inf\left\{|\hat{\varphi}(0)|^2 + \sum_{n\neq 0} 2|n||\hat{\varphi}(n)|^2\right\}\,,$$

where the infimum is taken over all smooth φ, $\varphi \geqslant 1$, on $\mathcal{K}$. It follows that $\operatorname{cap}\mathbb{T} = 1$. To avoid the deeper implications of potential theory, which are not necessary for our purposes, we will restrict our attention to compacts $\mathcal{K}\subset\mathbb{T}$ which are unions of a finite number of closed arcs. They all are regular in the sense of this theory. Namely, for every regular $\mathcal{K}$ there exists a unique equilibrium measure $\nu_{\mathcal{K}} \in \mathfrak{P}(\mathbb{T})$ such that

$$U^{\nu_{\mathcal{K}}} = (\operatorname{cap}\mathcal{K})^{-1} \text{ on } \mathcal{K}\,, \qquad U^{\nu_{\mathcal{K}}} < (\operatorname{cap}\mathcal{K})^{-1} \text{ on } \mathbb{C}\setminus\mathcal{K}\,. \tag{8.229}$$

If $\mathcal{K}$ is a regular compact subset of $\mathbb{T}$, then

$$g(z) = g_{\mathcal{K}}(z) = (\operatorname{cap}\mathcal{K})^{-1} - U^{\nu_{\mathcal{K}}}(z) \tag{8.230}$$

[2] See Nikishin and Sorokin (1988, Chapter 5, Section 5) for a brief introduction.

is the Green's function for the domain $\mathbb{C}\setminus\mathcal{K}$ (with pole at ∞). Clearly $g>0$ in $\mathbb{C}\setminus\mathcal{K}$ and

$$\begin{aligned}
&\lim_{z\to\zeta} g(z)=0 \qquad \text{for } \zeta\in\mathcal{K}\,,\\
&g(z)=\log|z|+\frac{1}{\operatorname{cap}\mathcal{K}}-1+o(1) \qquad \text{as } z\to\infty\,.
\end{aligned} \tag{8.231}$$

If $\mathcal{K}=\mathbb{T}$ then $\nu_{\mathbb{T}}=m$, $U^m(z)=1$ for $z\in\mathbb{T}$ and $U^m(z)=1-\log|z|$ for $|z|>1$. Hence $\log|z|$ is the Green's function for $\mathbb{T}$.

Another example is a single arc $\mathcal{K}=\Delta_\alpha$. Since $|f_a|<1$ on Δ_α, by (8.226), except for the ends, we have $|1+azf_a(z)|=|G_\sigma(z)|=\sqrt{1-a^2}$ on Δ_α by (8.226) and (8.218). By Lemma 8.121(c),

$$\begin{aligned}
g_\alpha(z)&=\log|G_\sigma(z)|-\log\sqrt{1-a^2}\\
&=\log|z|-\log\sqrt{1-a^2}+\log\left|af_a+\frac{1}{z}\right|\\
&=\log|z|+\log\frac{1}{\sqrt{1-a^2}}+o(1)
\end{aligned}$$

as $z\to\infty$, implying that g_α is the Green's function of $\mathbb{C}\setminus\Delta_\alpha$. Hence

$$\operatorname{cap}\Delta_\alpha=\left(\log\frac{e}{\sqrt{1-a^2}}\right)^{-1}.$$

Especially important is the case when $\mathcal{K}=\cup_{j=1}^r e_j$ is the union of nonintersecting closed arcs e_j on $\mathbb{T}$. Each arc e_j has its interior side e_j^+ facing the origin (the southern pole of $\hat{\mathbb{C}}$) and its exterior side e_j^- facing ∞ (the northern pole of $\hat{\mathbb{C}}$). Since $g>0$ in $\mathbb{C}\setminus\mathcal{K}$ and $g=0$ on $\mathcal{K}$, both normal derivatives $\partial g/\partial n^\pm$ in the directions of ∞ and 0 are nonnegative. Then the density of the equilibrium measure on $\mathcal{K}$ is given by the formula

$$d\nu_{\mathcal{K}}(\zeta)=\frac{1}{2\pi}\left(\frac{\partial g(\zeta)}{\partial n^-}+\frac{\partial g(\zeta)}{\partial n^+}\right)ds(\zeta)\,, \qquad \zeta\in\mathcal{K}\,, \tag{8.232}$$

where ds is the arc length on $\mathcal{K}$. For $|z|\geqslant 1$,

$$g(z)=\log|z|+g\left(\frac{1}{\bar z}\right). \tag{8.233}$$

Indeed $g(1/\bar z)$, being the real part of an analytic function $\log\bar\Phi(1/\bar z)$, is harmonic and bounded in $|z|\geqslant 1$. However, it obviously coincides with another bounded harmonic function $g(z)-\log|z|$ everywhere on $\mathbb{T}$. By the maximum principle they coincide in $|z|>1$. It follows from (8.233) that $\partial g/\partial n^-=1+\partial g/\partial n^+$ and therefore

$$d\nu_{\mathcal{K}}(\zeta)=\frac{1}{2\pi}\left(1+2\frac{\partial g(\zeta)}{\partial n^+}\right)ds(\zeta)\,, \qquad \zeta\in\mathcal{K}\,. \tag{8.234}$$

Logarithmic potentials of $\sigma \in \mathfrak{P}(\mathbb{T})$ can be expressed in terms of their Schur functions $f = f^\sigma$. Indeed,

$$\frac{d}{dz}\int_{\mathbb{T}} \log \frac{1}{\zeta - z}\, d\sigma(\zeta) = \int_{\mathbb{T}} \frac{d\sigma(\zeta)}{\zeta - z} = \frac{f(z)}{1 - zf(z)} \tag{8.235}$$

implies that

$$\int_{\mathbb{T}} \log \frac{1}{|\zeta - z|}\, d\sigma(\zeta) = \operatorname{Re} \int_0^z \frac{f^\sigma(\zeta)}{1 - \zeta f^\sigma(\zeta)}\, d\zeta\,. \tag{8.236}$$

Let us apply the above formula to $\sigma = \nu_{\mathcal{K}}$. We will keep the notation f for the Schur function of $\nu_{\mathcal{K}}$. If $\tilde{g}$ is harmonically conjugate to g then by (8.236)

$$g + i\tilde{g} = \frac{1}{\operatorname{cap} \mathcal{K}} - 1 - \int_0^z \frac{f(\zeta)}{1 - \zeta f(\zeta)}\, d\zeta\,.$$

Following Widom (1969), together with g we consider the multivalued function

$$\Phi(z) = \exp\{g(z) + i\tilde{g}(z)\}\,. \tag{8.237}$$

It is analytic in $\mathbb{C} \setminus \mathcal{K}$ and has a single-valued modulus. We normalize the choice of the multivalued harmonic conjugate function $\tilde{g}(z)$ by the requirement that $\tilde{g}(0) = 0$ for at least one branch of $\tilde{g}(z)$. It is clear that

$$\frac{\Phi'}{\Phi}(z) = (\log \Phi)'(z) = -\frac{f(z)}{1 - zf(z)} \tag{8.238}$$

is single-valued in $\mathbb{C} \setminus \mathcal{K}$ because f extends to $\mathbb{C} \setminus \mathcal{K}$ from $\mathbb{D}$ through the complementary arcs of $\mathcal{K}$ by Schwarz's reflection. Hence if $z \to \infty$ then

$$\frac{\Phi'}{\Phi}(z) = \frac{1}{z}\left(1 - \frac{1}{z}\bar{f}\left(\frac{1}{z}\right)\right)^{-1} = \frac{1}{z}\left\{1 + \frac{\overline{f(0)}}{z} + O\left(\frac{1}{z^2}\right)\right\} \tag{8.239}$$

is analytic at $z = \infty$. It follows that

$$\log \Phi(z) = \log z + c - \frac{\overline{f(0)}}{z} + O\left(\frac{1}{z^2}\right)\,. \tag{8.240}$$

The level curves $C_1, \ldots, C_r$ of $|\Phi| = 1 + \delta$ become concentrated to $e_1, \ldots, e_r$ as $\delta \to 0^+$. By Cauchy's theorem,

$$1 = \frac{1}{2\pi i}\oint_{|z|=R} \frac{\Phi'}{\Phi}\, dz = \sum_{k=1}^{r} \frac{1}{2\pi i}\oint_{\Delta_k} \frac{\Phi'}{\Phi}\, dz = \sum_{k=1}^{r} \frac{1}{2\pi} \underset{C_k}{\Delta} \tilde{g}\,.$$

Now, by the Cauchy–Riemann equations,

$$2\pi\nu_{\mathcal{K}}(e_k) = \int_{e_k} \frac{\partial g}{\partial n_\zeta} |d\zeta| = \int_{e_k} \frac{\partial \tilde{g}}{\partial t_\zeta} |d\zeta| = \underset{e_k}{\Delta} \tilde{g} = \underset{C_k}{\Delta} \arg \Phi\,. \tag{8.241}$$

Here t_ζ is the tangent vector obtained by rotating the normal vector n_ζ counterclockwise through an angle $\pi/2$. The harmonic measures $\nu_{\mathcal{K}}(e_k)$ play a significant role in the theory of periodic measures. For $r = 1$ we have $\nu_{\mathcal{K}}(e_1) = 1$ and

$$\Phi = \frac{G_\sigma}{\sqrt{1-a^2}} . \tag{8.242}$$

201 The equilibrium measure of Δ_α. We first compute the limit measure of $\{|\varphi_n|^2 d\sigma\}_{n\geqslant 0}$ for any ratio-asymptotic σ.

Theorem 8.122 *Let $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$ for $\sigma \in \mathfrak{P}(\mathbb{T})$ with Verblunsky parameters $\{a_n\}_{n\geqslant 0}$. Suppose that $\{a_n\}_{n\geqslant 0}$ satisfies the López condition with parameters (a, λ). Then the limit $*$-$\lim_n |\varphi_n|^2 d\sigma = d\nu_{a,\lambda}$ exists and the Schur function $f^{\nu_{a,\lambda}}$ of $\nu_{a,\lambda} \in \mathfrak{P}(\mathbb{T})$ is $-\lambda f^2(\lambda z; a)$, where $f(z; a)$ is given in* (8.225).

Proof We consider$\{(\bar{a}_n / |a_n|) f_{n+1}(z)\}_{n\geqslant 0}$ and $\{(a_n / |a_n|) b_{n+1}(z)\}_{n\geqslant 0}$ in $\mathcal{B}$. Since $\{a_n\}_{n\geqslant 0}$ satisfies the López condition,

$$\begin{aligned}
&\lim_n \frac{\bar{a}_n}{|a_n|} a_{n+k} = \lim_n |a_n| \lim_n \frac{a_{n+k}}{a_n} = a\lambda^k , \qquad k \geqslant 1 ; \\
&\lim_n -\frac{a_n}{|a_n|} \bar{a}_{n-k} = \lim_n |a_n| \lim_n -\overline{\left(\frac{a_{n-k}}{a_n}\right)} = -a\lambda^k , \qquad k \geqslant 0 .
\end{aligned} \tag{8.243}$$

Notice that

$$\begin{aligned}
\mathcal{S}\left(\frac{\bar{a}_n}{|a_n|} f_{n+1}\right) &= \left(\frac{\bar{a}_n}{|a_n|} a_{n+1} , \ \frac{\bar{a}_n}{|a_n|} a_{n+2} , \dots\right) , \\
\mathcal{S}\left(\frac{a_n}{|a_n|} b_{n+1}\right) &= \left(-\frac{a_n}{|a_n|} \bar{a}_n , \ -\frac{a_n}{|a_n|} \bar{a}_{n-1} , \dots\right) .
\end{aligned} \tag{8.244}$$

By (8.243) the sequences on the right-hand sides of (8.244) converge in $\mathcal{S}^\infty$ to the Schur parameters of $\lambda f(\lambda z; a)$ and of $-\lambda f(\lambda z; a)$ respectively. By Theorem 8.64,

$$\lim_n \frac{\bar{a}_n}{|a_n|} f_{n+1}(z) = \lambda f(\lambda z; a) , \qquad \lim_n \frac{\bar{a}_n}{|a_n|} b_{n+1}(z) = -f(\lambda z; a)$$

uniformly on compact subsets of $\mathbb{D}$. Hence

$$\lim_n f_{n+1}(z) b_{n+1}(z) = -\lambda f^2(\lambda z; a) ,$$

which completes the proof by Theorem 8.67. □

To study $\nu_{a,\lambda}$ in more detail let us put $t = \zeta\lambda$, $w = z\lambda$ in (8.9). Then $\nu_{a,\lambda}(E) = \nu_{a,1}(\lambda E)$ for any Borel subset E of $\mathbb{T}$. Hence we may assume that $\lambda = 1$ and study only $\nu_a \overset{\text{def}}{=} \nu_{a,1}$, $0 < a < 1$. By (8.224) $f^\# = -f$, implying by (8.227) that

$$z f^{\nu_a} = -z f^2 = z f f^\# = |f|^2 , \qquad \nu_a' = \frac{1+|f|^2}{1-|f|^2}$$

on Δ_α. By (8.225), in the neighborhood of $z = 0$ we have

$$f = \frac{1-z}{2az}\left(\frac{\sqrt{\mathcal{D}_a}}{1-z} - 1\right) = \frac{2a}{1-z}\left(1 + \frac{\sqrt{\mathcal{D}_a}}{1-z}\right)^{-1} = -f^{\#} .$$

Then

$$zff^{\#} = \frac{1-z-\sqrt{\mathcal{D}_a}}{1-z+\sqrt{\mathcal{D}_a}} , \qquad F^{\nu_a} = \frac{1+zff^{\#}}{1-zff^{\#}} = \frac{1-z}{\sqrt{\mathcal{D}_a}} . \tag{8.245}$$

It follows from (8.245) that $\lim_{r\to 1^-}(1-r)F^{\nu_a}(r\zeta) = 0$ for every $\zeta \in \mathbb{T}$, implying that ν_a has no discrete masses.

Lemma 8.123 *The function $-zf^2(z;a)$ conformally maps $\mathbb{D}$ onto the sliced disc $\mathbb{D} \setminus (c,1)$, where $c = a^2(1+\sqrt{1-a^2})^{-2}$.*

Proof By the first formula of (8.226) $\theta \to -(\exp i\theta)f^2(\exp i\theta; a)$ moves clockwise along $\mathbb{T}$ (from -1 to 1) as θ runs from 0 to α. By the second formula of (8.226), $\theta \to -(\exp i\theta)f^2(\exp i\theta; a)$ maps (α, π) onto $(c, 1)$, where

$$c = \left(1 - \sqrt{1-a^2}\right)^2 a^{-2} = a^2\left(1+\sqrt{1-a^2}\right)^{-2} .$$

The mapping of $(\pi, 2\pi)$ is symmetric to that of $(0, \pi)$. By the argument principle $-zf^2(z;a)$ is univalent in $\mathbb{D}$. □

Theorem 8.124 *The measure ν_a is the equilibrium measure for Δ_α with respect to the logarithmic kernel.*

Proof By (8.235), (8.238) and (8.242),

$$\int_{\Delta_\alpha} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta) = 1 + 2z \int_{\Delta_\alpha} \frac{d\mu(\zeta)}{\zeta - z} = 1 - 2z\frac{G_\sigma'}{G_\sigma} . \tag{8.246}$$

Let $w(z) = zf(z;a)$, $\mathcal{D} = (z-1)^2 + 4a^2 z$. We have $G_\sigma = 1 + aw$, where w satisfies the algebraic equation

$$aw^2 + (1-z)w - az = 0 . \tag{8.247}$$

Differentiating (8.247), we obtain $w' = (w+a)\mathcal{D}_a^{-1/2}$ and

$$\int_{\Delta_\alpha} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta) = 1 - 2az\frac{w+a}{(1+az)\sqrt{\mathcal{D}_a}} = 1 - \frac{2aw}{\sqrt{\mathcal{D}_a}} = \frac{1-z}{\sqrt{\mathcal{D}_a}} .$$

Now $\mu = \nu_a$ by (8.245). □

202 The parameters of equilibrium measures. In the theory of orthogonal polynomials on the circular arc Δ_α the equilibrum measure ν_a replaces the Lebesgue measure m on $\mathbb{T}$. Therefore it is important to have formulas for the parameters of ν_a. We obtain them by the method of Pell's equation.

Symmetrizing (8.42)–(8.45), we see that if (Q_n, P_n) is a solution to

$$Q_n + P_n F^\sigma = \begin{cases} O(1), & z \to \infty\,, \\ O(z^n), & z \to 0 \end{cases} \tag{8.248}$$

then $(-Q_n^*, P_n^*)$ is also a solution. Since (8.248) is a linear system,

$$\begin{aligned} P_n &= \lambda_n \Phi_n + (1-\lambda_n)\Phi_n^*/\overline{\Phi_n(0)}\,, \\ Q_n &= \lambda_n \Psi_n + (1-\lambda_n)\Psi_n^*/\overline{\Psi_n(0)} \end{aligned} \tag{8.249}$$

for some complex λ_n, by (8.42)–(8.45). Hence if one finds a solution to (8.248) then the formulas for the Verblunsky parameters follow by Geronimus' theorem 8.21. Notice that the pair $p(z) = z+1$, $q(z) = 1$ satisfies the Pell equation $p^2 - q^2\mathcal{D}_a = 4z(1-a^2)$.

Lemma 8.125 *Let P_n and Q_n be polynomials defined by*

$$P_n + Q_n\sqrt{\mathcal{D}_a} = 2^{1-n}(p + q\sqrt{\mathcal{D}_a})^n\,. \tag{8.250}$$

Then, if $P_n = P_n^*$ and $Q_n = Q_n^*$ are self-adjoint monic polynomials of degree $\deg P_n = n$ *and* $\deg Q_n = n-1$ *respectively, we have*

$$P_n^2 - Q_n^2\mathcal{D}_a = 4(1-a^2)^n z^n\,. \tag{8.251}$$

Proof By (8.250)

$$2P_{n+1} = P_n p + Q_n q\mathcal{D}_a\,, \qquad 2Q_{n+1} = P_n q + Q_n p\,.$$

Since p, q, $\mathcal{D}_a$ are self-adjoint, these relations show that P_{n+1} and Q_{n+1} are self-adjoint provided that $P_n = P_n^*$ and $Q_n = Q_n^*$. It is also clear that P_{n+1} and Q_{n+1} are monic if P_n and Q_n are monic. Finally,

$$P_n^2 - Q_n^2\mathcal{D}_a = 2^{2-2n}\big(p - q\sqrt{\mathcal{D}_a}\big)^n\big(p + q\sqrt{\mathcal{D}_a}\big)^n = 4(1-a^2)^n z^n$$

completes the proof. □

By Lemma 8.125,

$$P_n - Q_n\sqrt{\mathcal{D}_a} = \frac{4(1-a^2)^n z^n}{P_n + Q_n\sqrt{\mathcal{D}_a}}\,.$$

Taking into account (8.245), we see that

$$P_n F^{\nu_a} + Q_n(z-1) = \frac{4(1-a^2)^n z^n}{P_n F^{\nu_a} - Q_n(z-1)}(F^{\nu_a})^2\,. \tag{8.252}$$

Since $P_n(0) = Q_n(0) = 1 = F^{\nu_a}(0)$ and as $z \to \infty$ $P_n(z) \sim z^n$, $Q_n(z) \sim z^n$, $F^{\nu_a}(z) \sim -1$, we obtain that

$$P_n F^{\nu_a} + Q_n(z-1) = \begin{cases} 2(1-a^2)^n z^n + o(z^n)\,, & z \to 0\,, \\ -2(1-a^2)^n + o(1)\,, & z \to \infty\,. \end{cases} \tag{8.253}$$

Combining (8.249) with (8.253), we arrive at the system

$$\begin{aligned} (1-a^2)^n &= \lambda_n \omega_{n-1}\,, \\ -(1-a^2)^n &= -\frac{(1-\lambda_n)\omega_{n-1}}{\overline{\Phi_n(0)}}\,. \end{aligned}$$

Eliminating λ_n and applying Theorem 8.21, we obtain the recurrence for the parameters $\{a_n\}_{n\geqslant 0}$ of ν_a:

$$a_n = \frac{(1-a^2)^{n+1}}{\prod_{k=0}^{n-1}(1-a_k^2)} - 1\,, \qquad n = 0, 1, \ldots$$

It follows that

$$1 - a_{n+1} = 2 - \frac{1-a^2}{1-a_n}\,. \tag{8.254}$$

Lemma 8.126 *Suppose that $X^2 - AX + B = 0$, where $A > 0$ and $B > 0$, has two different real roots $r < r'$ and that the sequence $\{X_n\}_{n\geqslant 0}$ is defined by*

$$X_{n+1} = A - \frac{B}{X_n}\,, \qquad n = 0, 1, 2, \ldots\,, \tag{8.255}$$

where X_0 satisfies $X_0 \in (r, r')$. Then $\{X_n\}_{n\geqslant 0}$ is an increasing sequence in (r, r') such that for some $c > 0$

$$X_n = r' - \left(\frac{r}{r'}\right)^n c(1 + o(1))\,, \qquad n \to +\infty\,. \tag{8.256}$$

Proof By Viète's theorem the mapping $\tau(w) = A - B/w$ satisfies $\tau(r) = r$, $\tau(r') = r'$. Since $\tau'(x) = Bx^{-2} > 0$ on (r, r'), we conclude that τ maps the interval onto itself. It follows that $X_n \in (r, r')$, $n = 1, 2, \ldots$ Since by (8.255) the inequality $X_{n+1} > X_n$ is equivalent to the inclusion $X_n \in (r, r')$, we obtain that $\{X_n\}_{n\geqslant 0}$ is an increasing sequence. Passing to the limit in (8.255), we conclude that $\lim_n X_n = r'$.

Let $s_n = r' - X_n$. Then by (8.255) we have

$$(r' - \varepsilon_{n+1})(r' - \varepsilon_n) = A(r' - \varepsilon) - B\,.$$

Since r' is a root of the equation $X^2 - AX + B = 0$, this implies that

$$\frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{r}{r' - \varepsilon_n}\,, \qquad n = 0, 1, \ldots$$

Therefore $\lim_n \varepsilon_{n+1}\varepsilon_n^{-1} = r/r' < 1$. It follows that $\varepsilon_n = o(q^n)$, $n \to +\infty$, for every $q > r/r'$. Hence the infinite product

$$\varepsilon_0 \prod_{k=0}^{\infty} \left(1 - \frac{\varepsilon_k}{r'}\right)^{-1} = c$$

converges to a finite value $c > 0$. Now the identity

$$\varepsilon_n = \frac{\varepsilon_n}{\varepsilon_{n-1}} \frac{\varepsilon_{n-1}}{\varepsilon_{n-2}} \varepsilon_{n-2} \cdots \frac{\varepsilon_1}{\varepsilon_0}\varepsilon_0 = \left(\frac{r}{r'}\right)^n \varepsilon_0 \prod_{k=0}^{n-1} \left(1 - \frac{\varepsilon_k}{r'}\right)^{-1}$$

obviously yields (8.256). □

Theorem 8.127 *The parameters $\{a_n\}_{n\geqslant 0}$ of the equilibrium measure ν_a for Δ_α make a negative decreasing sequence in $(-a, -a^2)$ such that*

$$a_n = -a + c\left(\frac{1-a}{1+a}\right)^n (1 + o(1)), \qquad n \to \infty. \tag{8.257}$$

Proof The quadratic equation $X^2 - 2X + (1-a^2) = 0$ has two different roots, $r = 1 - a < r' = 1 + a$. The sequence $X_n = 1 - a_n$ satisfies (8.255) with $A = 2$ and $B = 1 - a^2$. Since $0 < a < 1$, we have $X_0 = 1 + a^2 \in (r, r')$. The proof is completed by Lemma 8.126. □

203 Wall pairs. A pair (A, B) of relatively prime polynomials is called a *Wall pair* if there are a Wall continued fraction (8.10) and an integer $n \in \mathbb{N}$ such that $A_n = A$, $B_n = B$.

Theorem 8.128 *A pair (A, B) of relatively prime polynomials is a Wall pair if and only if:*

(a) $B(0) = 1$ *and* $\inf\{|B(z)| : z \in \mathbb{D}\} > 0$*;*

(b) *there exists a positive constant ω such that*

$$|B|^2 - |A|^2 \equiv \omega, \qquad z \in \mathbb{T}. \tag{8.258}$$

Proof The necessity is an easy corollary of the properties of the even convergents to a Wall continued fraction. To prove the converse statement we assume that it holds for polynomials A, deg $A < n$. Since $B(0) = 1$, the degree of the trigonometric polynomial $|B|^2 = B\overline{B}$ equals deg B. Now (8.258) implies that deg $B \leqslant$ deg A. Hence

$$A = az^n + \cdots + a_0, \quad a \neq 0, \qquad B = bz^n + \cdots + 1. \tag{8.259}$$

By (8.258),

$$\omega z^n = z^n \bar{B}B - z^n \bar{A}A = B^*B - A^*A = (b - \bar{a}_0 a)z^{2n} + \cdots, \qquad z \in \mathbb{T},$$

which implies that $b - \bar{a}_0 a = 0$. By Corollary 8.11 $\|A/B\|_\infty < 1$. Hence $|a_0| < 1$. Applying Schur's algorithm to $f = A/B$, we consider the Schur function f_1:

$$f_1 = \frac{f - a_0}{z(1-\bar{a}_0 f)} = \frac{(A - a_0 B)/z}{B - \bar{a}_0 A} = \frac{A'}{B'} ,$$

where the polynomials A' and B' are defined by

$$A' = \frac{A - a_0 B}{z(1-|a_0|^2)} , \qquad B' = \frac{B - \bar{a}_0 A}{1 - |a_0|^2} .$$

It follows that $A' = az^{n-1} + \cdots$, $\deg B' \leqslant n-1$. By the definition, $B'(0) = 1$ and $|B'(z)| > 0$ if $|z| \leqslant 1$. Next, on $\mathbb{T}$,

$$|B'|^2 - |A'|^2 = \frac{|B - \bar{a}_0 A|^2 - |A - a_0 B|^2}{(1-|a_0|^2)^2} = \frac{|B|^2 - |A|^2}{1 - |a_0|^2} = \frac{\omega}{1-|a_0|^2} ,$$

which implies that $(B')^* B' - (A')^* A' = z^{n-1}\omega(1-|a_0|^2)^{-1}$. Hence A' and B' are relatively prime and by the induction hypothesis make a Wall pair. It follows from the definition of Schur's algorithm that (A, B) is a Wall pair. □

Observe that $\omega \in (0, 1)$ for every Wall pair. This follows directly from

$$\omega = \prod_{k=0}^{n} (1 - |a_k|^2) ,$$

or it can be obtained by the mean value theorem applied to the harmonic function $\log|B|$:

$$\log\omega = \int_{\mathbb{T}} \log\omega \, dm < \int_{\mathbb{T}} \log|B|^2 \, dm = \log|B(0)|^2 = 0 .$$

If (A, B) is a Wall pair then $(-A^*, B)$ also is a Wall pair, with the same ω.

Theorem 8.129 *In order that a polynomial A be the first component of a Wall pair (A, B) it is necessary and sufficient that*

$$\int_{\mathbb{T}} \log|A| \, dm < 0 . \tag{8.260}$$

The second component B is uniquely determined by A.

Proof We have $A/B \in \mathcal{B}$ and moreover $\|A/B\|_\infty < 1$. By the mean value theorem applied to the harmonic function $\log|B|^2$,

$$\int_{\mathbb{T}} \log|A|^2 \, dm = \int_{\mathbb{T}} \log\left|\frac{A}{B}\right|^2 dm + \log|B(0)|^2 < 0 .$$

Suppose now that (8.260) holds and consider an auxiliary continuous increasing function

$$I(\omega) = \int_{\mathbb{T}} \log(|A|^2 + \omega) \, dm \tag{8.261}$$

on $[0, +\infty)$. Since $I(0) < 0$ and $I(1) > 0$, there exists a unique ω in $(0,1)$ such that $I(\omega) = 0$. By Fejér's theorem, see Szegő (1939, Theorem 1.2.2), there exists a unique algebraic polynomial B such that $\deg B \leqslant \deg A$, $B(0) > 0$, B does not vanish in $\{z : |z| \leqslant 1\}$ and $|B|^2 = |A|^2 + \omega$ on $\mathbb{T}$. By the mean value theorem

$$\log B(0)^2 = \int_{\mathbb{T}} \log(|A|^2 + w)\, dm = I(\omega) = 0,$$

implying $B(0) = 1$. Thus (A, B) must be a Wall pair by Theorem 8.128. □

Corollary 8.130 *A polynomial A is the first component of a Wall pair (A, B) if and only if there exists $p > 0$ such that*

$$\int_{\mathbb{T}} |A|^p\, dm < 1\,. \tag{8.262}$$

Proof Combining a well–known formula Garnett (1981, Chapter IV, Section 6, Ex. 6(c)),

$$\lim_{p\to 0+} \left(\int_{\mathbb{T}} |A|^p\, dm\right)^{1/p} = \exp\left\{\int_{\mathbb{T}} \log|A|\, dm\right\},$$

with Theorem 8.129 we obtain the result. □

Theorem 8.131 *Let B be an arbitrary polynomial such that $B(0) = 1$ and $0 < \omega \leqslant \inf\{\,|B(z)|^2 : |z| \leqslant 1\,\}$. Then there exists a polynomial A, not vanishing in $\mathbb{D}$, such that $|B|^2 - |A|^2 \equiv \omega$ on $\mathbb{T}$, $A(0) > 0$ and (A, B) is a Wall pair.*

Proof By Fejér's theorem the polynomial A is defined by

$$A(z) = \exp\left\{\frac{1}{2}\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log\left(|B(\zeta)|^2 - \omega^2\right)\, dm(\zeta)\right\}\,, \qquad |z| < 1\,, \tag{8.263}$$

which completes the proof by Theorem 8.128. □

Definition 8.132 *The Wall pairs (A, B) and $(\tilde{A}, \tilde{B})$ are said to be equivalent if $A^*A = \tilde{A}^*\tilde{A}$ and $B = \tilde{B}$.*

The degree d of a Wall pair (A, B) is defined as $d = \deg A$. Every pair (A, B) is determined by the rational fraction $A/B \in \mathcal{B}$ with Schur parameters $a_0, a_1, \dots, a_d, 0, 0, \dots$ The parameters $\{a_0, a_1, \dots, a_d\}$ are called the parameters of (A, B). The parameters of $(-A^*, B)$ are $-\bar{a}_d, -\bar{a}_{d-1}, \dots, -\bar{a}_0$.

204 Periodic measures. If (A, B) is a Wall pair with parameters $\{a_0, a_1, \dots, a_d\}$ then there is a unique $f \in \mathcal{B}$ with periodic Schur parameters $\mathcal{S}f = \overline{\{a_0, a_1, \dots, a_d\}}$. Since $f = f_{d+1}$, the function f is a solution to the quadratic equation

$$zA^*X^2 + (B - zB^*)X - A = 0\,. \tag{8.264}$$

Lemma 8.133 *For any Wall pair (A, B) the roots of the polynomials $B - zB^*$ and $B + zB^*$ are simple, lie on $\mathbb{T}$ and are interlaced.*

Proof *We have* $B - zB^* = 0$ exactly at the roots of $zB^*/B = 1$. The argument of the Blaschke product zB^*/B increases in the counterclockwise direction of $\mathbb{T}$, which proves the lemma. □

To simplify the notation we put $b_+ = B + zB^*$ and $b_- = B - zB^*$. The polynomial b_+ is self-adjoint: $b_+^* = b$, $b_+(0) = 1$. The polynomial b_- is anti-self-adjoint: $b_-^* = -b_-$, $b_-(0) = 1$. By (8.17), on the unit circle

$$|b_+|^2 + |b_-|^2 = 4|B|^2 = 4|A|^2 + 4\omega\,. \tag{8.265}$$

Theorem 8.134 *The discriminant* $\mathcal{D}$, $\deg \mathcal{D} = 2d + 2$, *of* (8.264) *is given by*

$$\mathcal{D} = b_+^2 - 4\omega z^{d+1}\,. \tag{8.266}$$

All roots of $\mathcal{D}$ *are lie on* $\mathbb{T}$ *and coincide with the roots of* $|b_+|^2 - 4\omega$. *They are either simple or of second order. The latter is the case if and only if* $|b_+|^2 - 4\omega$ *has a local maximum at this root.*

Proof By (8.17) the discriminant $\mathcal{D}$ of (8.264) is

$$\mathcal{D} = (B - zB^*)^2 + 4zA^*A = (B + zB^*)^2 - 4\omega z^{d+1}\,, \tag{8.267}$$

where ω is a parameter of the Wall pair (A, B). The discriminant $\mathcal{D}$ has degree $2d + 2$ and is self-adjoint:

$$\mathcal{D}^* = z^{2d+2}\left\{\left(\overline{B\,(1/\bar{z})} + z^{-d-1}B(z)\right)^2 - 4\omega z^{-d-1}\right\} = \mathcal{D}\,.$$

Formula (8.266) follows by (8.267). By Lemma 8.133 the polynomial b_+ is separable and its roots lie on $\mathbb{T}$. Let $\mathcal{T}_{\mathcal{D}}(z) = \bar{z}^{(d+1)}\mathcal{D}(z)$ for $z \in \mathbb{T}$. Then by (8.266) and (8.265)

$$\mathcal{T}_{\mathcal{D}}(z) = \begin{cases} |b_+|^2 - 4\omega\,, \\ 4|A|^2 - |b_-|^2\,. \end{cases} \tag{8.268}$$

The first formula in (8.268) shows that $\mathcal{T}_{\mathcal{D}}$ takes the minimal value -4ω at the $d+1$ different zeros $\{t_j^+\}_{j=0}^d$ of b_+. By the second formula of (8.268) $\mathcal{T}_{\mathcal{D}}$ takes nonnegative values at the $d+1$ different zeros $\{t_j^-\}_{j=0}^d$ of b_-, which interlace the zeros of b_+. It follows that on each open arc

$$\left(t_j^+, t_{j+1}^+\right) = \left(t_j^+, t_j^-\right) \cup \left\{t_j^-\right\} \cup \left(t_j^-, t_{j+1}^+\right)$$

there are either two simple zeros τ_j^+, τ_j^- of $\mathcal{T}_{\mathcal{D}}$ in each open interval or one zero of multiplicity 2, $\tau_j^+ = \tau_j^-$. The superscript plus on τ_j^+ indicates that $\mathcal{T}_{\mathcal{D}}$ increases when passing through τ_j^+. The minus on τ_j^- indicates that it correspondingly decreases. Counting zeros, we get $2d + 2$ in total. □

The polynomial $\mathcal{T}_{\mathcal{D}}$ in (8.268) is called the trigonometric polynomial *associated* with (A, B). Since $\deg A = d$ and $\deg b_- = d+1$ by (8.268), there is at least one t_j^- such that $|A(t_j^-)| > 0$. Hence $\mathcal{T}_{\mathcal{D}} > 0$ on an interval containing t_j^-. The graph of the periodic function $\theta \to \mathcal{T}_{\mathcal{D}}(\exp\ i\theta)$ is obtained by subtracting $4\ \omega$ from the graph of $\theta \to |b_+(\exp\ i\theta)|^2$. Clearly $|b_+|^2$ has local maxima. We denote the smallest by m_b. To keep the number of roots of $|b_+|^2 - 4\omega$ consistent with the degree of $\mathcal{D}$, as in Theorem 8.134, we must have $4\omega \leqslant m_b$. It is the case $4\omega = m_b$ which generates the roots of $\mathcal{D}$ of second degree.

Choosing the positive root and observing that $f(0) = a_0$, $|a_0| < 1$, we obtain an explicit formula for f:

$$f(z) = \frac{b_-}{2zA^*}\left(\sqrt{\frac{\mathcal{D}}{b_-^2}} - 1\right) = a_0 + o(1)\,, \qquad z \to 0\,. \tag{8.269}$$

On the unit circle $\mathbb{T}$,

$$\frac{\mathcal{D}}{b_-^2} = 1 + \frac{4z^{1+d}|A|^2}{b_-^2} = 1 - \frac{4|A|^2}{|b_-|^2}\,.$$

Hence on $\mathbb{T}$

$$|f| = \frac{b_-}{2A_-}\left|1 - \sqrt{\frac{4|A|^2}{|b_-|^2}}\right|\,. \tag{8.270}$$

Since $1 - \sqrt{1-x^2} < x$ for $x \in (0, 1)$, (8.270) implies that

$$|f| < 1 \quad \text{if } \mathcal{T}_{\mathcal{D}} < 0\,, \qquad |f| = 1 \quad \text{if } \mathcal{T}_{\mathcal{D}} \geqslant 0\,. \tag{8.271}$$

Hence any periodic f is unimodular on an nonempty interval on $\mathbb{T}$. The sets $\mathcal{E}(f) = \{z \in \mathbb{T} : |f(z)| < 1\}$ and $\mathcal{U}(f) = \{z \in \mathbb{T} : |f(z)| = 1\}$ can be easily described in terms of the roots of $\mathcal{T}_{\mathcal{D}}$:

$$\mathcal{E}(f) = \bigcup_{j=0}^{d} \delta_j, \qquad \mathcal{U}(f) = \bigcup_{j=0}^{d} \gamma_j\,.$$

Here $\delta_j = (\tau_j^-, \tau_j^+)$ and $\gamma_j = [\tau_j^+, \tau_{j+1}^-]$. The periodic measure σ is absolutely continuous on $\mathcal{E}(f)$ with density defined by (8.7). We say that σ is essentially supported on $\mathcal{E}(f)$. The closure of $\mathcal{E}(f)$ in $\mathbb{T}$ is denoted by $\Delta(\sigma)$.

Theorem 8.135 *Given a pair σ, μ of measures having periodic Schur functions f^σ and f^μ, $\mathcal{E}(f^\sigma) = \mathcal{E}(f^\mu)$ if and only if f^σ and f^μ have equal discriminants.*

Proof Since $\mathcal{T}_{\mathcal{D}}(z) = \bar{z}^{(d+1)}\mathcal{D}(z)$, periodic Schur functions with equal discriminants have a common polynomial $\mathcal{T}$, which implies $\mathcal{E}(f) = \mathcal{E}(g)$. Conversely, if $\mathcal{E}(f) = \mathcal{E}(g)$ then $|b_f|^2 - 4\omega_f = |b_g|^2 - 4\omega_g$, since the zero sets of both trigonometrical polynomials coincide. This implies by $\mathcal{T}_{\mathcal{D}}(z) = \bar{z}^{(d+1)}\mathcal{D}(z)$ that the discriminants are equal as well. □

A periodic measure σ may have point masses on arcs γ_j.

Theorem 8.136 *Every closed arc γ_j, $j = 0, 1, \dots d$, contains at most one point mass of the periodic measure σ with Schur function f.*

Proof If $x_1 = f$, x_2 are the roots of (8.264) then by Viète's theorem,

$$A^*(zx_1 - 1)(zx_2 - 1) = B + A^* - z(A + B^*) \, . \tag{8.272}$$

The polynomial $B + A^* = B(1 + A^*/B)$ does not vanish for $|z| \leqslant 1$, since $A/B \in \mathcal{B}$ and $|A| = |A^*|$ on $\mathbb{T}$. Therefore the roots of the adjoint polynomial $(B + A^*)^* = A + B^*$ must be in $\mathbb{D}$. It follows that the set

$$1 = z\frac{(B + A^*)^*}{B + A^*} = z\frac{B^*}{B}\frac{\overline{h}}{h} \tag{8.273}$$

contains the roots of $zf - 1 = 0$. Here $h = 1 + A^*/B$ is a smooth outer function in $\{z : |z| \leqslant 1\}$ with values in the *open* right half-plane. Notice that the middle expression in (8.273) is a finite Blaschke product. Since its argument increases when t moves clockwise, (8.273) has exactly $d + 1$ solutions. Since $\mathcal{T}_{\mathcal{D}} = |B + zB^*|^2 - 4\omega$, the roots of $B + zB^* = 0$ are located inside open intervals δ_j such that each δ_j has exactly one root t_j^+. On $[t_j^+, t_{j+1}^+]$ the argument $\alpha(t)$ of the Blaschke product zB^*/B increases from $-\pi$ to $+\pi$. For h we have

$$h = |h| \exp i\beta t, \qquad -\frac{\pi}{2} < \beta(t) < \frac{\pi}{2} \, , \qquad \frac{\overline{h}}{h} = \exp\{-2i\beta(t)\} \, . \tag{8.274}$$

It follows that $u(t) = \alpha(t) - 2\beta(t)$ is continuous and $u(t_j)u(t_{j+1}) < 0$. Therefore $u(t)$ must vanish inside $[t_j^+, t_{j+1}^+]$ and by (8.273) the right-hand side of (8.272) also vanishes. There is exactly one closed arc of $\mathcal{U}(f)$ in (t_j^+, t_{j+1}^+), namely γ_j. Therefore every arc γ_j contains at most one root of $B + A^* - z(A + B^*) = 0$. Since the number of roots equals the number of arcs the result follows. $\square$

Remark The result on the location of point masses was obtained in a preliminary form by Geronimus (1944). In the form in which it is stated here, the result was obtained by Simon (2005). See Simon (2005) for the inverse problem of the possible locations of point masses.

Discrete masses of σ are located at the roots of $1 - zf = 0$. Since $\operatorname{Re}(1 - zf) \geqslant 0$, the order of these roots does not exceed 1. The end-points of the essential support, where the discriminant has simple zeros, are branching points of order 2 and therefore the order of the zero is 1/2. This implies that $(1 - zf)^{-1}$ is locally integrable at these points. The following theorem gives an explicit formula for the density of the periodic measure associated with a Wall pair.

Theorem 8.137 *Let σ be the periodic measure with Schur function f associated with a Wall pair (A, B) of degree d. Then*

$$\sigma' = \frac{\sqrt{\mathcal{D}}}{B + A^* - z(B^* + A)} = \frac{\sqrt{\mathcal{D}}}{\Phi_{d+1}^* - \Phi_{d+1}} = \frac{\sqrt{|\mathcal{D}|}}{|\Phi_{d+1}^* - \Phi_{d+1}|} \, , \tag{8.275}$$

where Φ_{d+1} is the monic orthogonal polynomial of order $d+1$ and the formula holds on $\mathcal{E}(f)$.

Proof Let $u = 2A^*/(B - zB^*)$. Then by (8.270)

$$zf = -\left(1 - \sqrt{1-|u|^2}\right) u^{-1}$$

on $\mathbb{T}$. It follows that

$$1 - |f|^2 = 2\sqrt{1-|u|^2}\left(1 - \sqrt{1-|u|^2}\right)|u|^{-2}\,.$$

Next,

$$\frac{1-|f|^2}{|1-zf|^2} = \frac{1-|f|^2}{1+|f|^2 - 2\,\mathrm{Re}\,zf} = \left(2\,\frac{1-\mathrm{Re}\,zf}{1-|f|^2} - 1\right)^{-1}.$$

It follows that

$$2\,\frac{1-\mathrm{Re}\,zf}{1-|f|^2} - 1 = \frac{1+\mathrm{Re}\,u}{\sqrt{1-|u|^2}}\,.$$

To find $\mathrm{Re}\,u$ we observe that $\bar{u} = 2\overline{A^*}/\bar{b}_- = -2zA/b_-$, which implies that

$$1 + \mathrm{Re}\,(u) = 1 + \frac{1}{2}(u+\bar{u}) = \frac{B + A^* - z(B^* - A)}{B - zB^*}\,.$$

Combining the above formulas, we prove the first part of (8.275). The formula in terms of orthogonal polynomials follows by (8.30). □

Remark A formula for σ' was first obtained by Geronimus (1944) in a similar form. However, Geronimus' formula for the denominator of the density was not particularly simple. Later it was used by Peherstorfer and Steinbauer (1996) to develop the theory of periodic measures on the unit circle. The formula with $\Phi^*_{d+1} - \Phi_{d+1}$ was obtained only recently in Simon (2005).

Theorem 8.138 *Let σ be the periodic measure associated with a Wall pair (A, B) of degree d. Then*

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\,d\sigma(\zeta) = \frac{A^* + zA + \sqrt{\mathcal{D}}}{B + A^* - z(B^* + A)} = \frac{A^* + zA + \sqrt{\mathcal{D}}}{\Phi^*_{d+1} - \Phi_{d+1}}\,.$$

Proof Let $x_1 = f$ and x_2 be two roots of (8.264). Then by Viète's formulas we have

$$z(x_1 + x_2) = -b_-/A^*\,, \qquad z(x_1 - x_2) = \sqrt{\mathcal{D}}/A^*\,, \qquad z^2 x_1 x_2 = -zA/A^*\,.$$

It follows that

$$\frac{1+zf}{1-zf} = \frac{(1+zx_1)(1-zx_2)}{(1-zx_1)(1-zx_2)} = \frac{A^* + zA + \sqrt{\mathcal{D}}}{(B+A^*) - z(B^* + A)}$$

as stated. □

205 Galois' Theorem for Schur's algorithm. If f is the periodic Schur function associated with a Wall pair (A, B) then the Schur function $f^{\#}$ associated with $(-A^*, B)$ is called the *Galois dual* function for f. The reason for such a terminology is explained by the following theorem of Galois:

Theorem 8.139 (Galois) *Let $f = \{\overline{a_0, \dots, a_d}\}$ be a Schur function in $\mathcal{B}$ with periodic Schur parameters having $\max_j |a_j| > 0$, let w be the algebraic function corresponding to f, and let τ be an automorphism in the Galois' group $\mathrm{Gal}(\mathbb{C}(z,w)/\mathbb{C}(z))$ with $\tau f \neq f$. Then*

$$-\frac{1}{z\tau f} = \{\overline{\bar{a}_d, \bar{a}_{d-1}, \dots, \bar{a}_0}\} = -f^{\#} .$$

Proof By (8.74) and (8.73)

$$b_{n+1} = -\frac{A_n^*}{B_n} + \frac{\omega_n z^{n+1}}{B_n(B_n - zA_n)} , \tag{8.276}$$

implying by (8.75) that $\mathcal{S}(-A_n^*/B_n) = (-\bar{a}_n, -\bar{a}_{n-1}, \dots, -\bar{a}_0, 0, \dots)$. If $n = d$, $A_n = A$, $B_n = B$ then the Wall pair $(-A^*, B)$ corresponds to the quadratic equation, see (8.264),

$$zAX^2 - (B - zB^*)X - A^* = 0 , \tag{8.277}$$

satisfied by $f^{\#} \in \mathcal{B}$. Then $-f^{\#} = \{\overline{\bar{a}_d, \bar{a}_{d-1}, \dots, \bar{a}_0}\}$ is periodic. Since $\tau f \neq f$, we have $\tau f = x_2 = 1/zf^{\#}$. □

206 The measures $\sigma_{ff^{\#}}$. By Galois' theorem 8.139 the roots of (8.264) are given by

$$x_1 = f , \qquad x_2 = (zf^{\#})^{-1} . \tag{8.278}$$

Viète's formulas applied to (8.264) imply that $f^{\#} = -A^*(A)^{-1}f$. Solving (8.264) in the neighborhood of $z = 0$, we obtain

$$1 - zff^{\#} = \frac{x_2 - x_1}{x_2} = -\frac{\sqrt{\mathcal{D}}}{zA^*x_2} , \qquad 1 + zff^{\#} = -\frac{b_-}{zA^*x_2} ,$$

implying that

$$\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\sigma_{ff^{\#}}(\zeta) = \frac{1 + zff^{\#}}{1 - zff^{\#}} = \frac{b_-}{\sqrt{\mathcal{D}}} \tag{8.279}$$

in $\mathbb{D}$. Since $|f| = |f^{\#}|$ on $\mathbb{T}$ and $\Delta(\sigma_{ff^{\#}}) = \Delta(\sigma)$ this formula can be extended to $\mathbb{C} \setminus \Delta_\sigma$ by symmetry. Using Viète's theorem, it is easy to check that $ff^{\#}$ is a quadratic irrational with discriminant $b_-^2\mathcal{D}$. It corresponds to a probability measure $\sigma_{ff^{\#}}$ which can be written down explicitly.

Theorem 8.140 *If $f \in \mathcal{B}$ is periodic then $\sigma_{ff^\#} \in \mathfrak{P}(\mathbb{T})$ is absolutely continuous and is supported by $\mathcal{E}(f)$, and the following holds:*

$$\int_{\mathcal{E}(f)} \frac{\zeta + z}{\zeta - z}\, d\sigma_{ff^\#}(\zeta) = \frac{b_-}{\sqrt{\mathcal{D}}}\,, \qquad (\sigma_{ff^\#})' = \frac{b_-}{\sqrt{\mathcal{D}}} \text{ on } \mathcal{E}(f)\,. \tag{8.280}$$

Proof By (8.267) $\mathcal{D}(b_-)^{-2} = 1 - 4|A|^2|b_-|^{-2}$ on $\mathbb{T}$; it is nonnegative on $\mathcal{E}(f)$ and nonpositive on $\mathcal{U}(f)$ by (8.270), (8.271). Hence $b_-/\sqrt{\mathcal{D}}$ is real on $\mathcal{E}(f)$ and imaginary on $\mathcal{U}(f)$. By Fatou's theorem 8.4 we obtain the second formula of (8.280).

By Theorem 8.134 the roots of $\mathcal{D}$ coincide with those of $|b_+|^2 - 4\omega$ on $\mathbb{T}$. Only roots of the second order may generate point masses. If t_0 is such a root then by Theorem 8.134 it occurs at a local maximum for $|b_+|^2 - 4\omega$ and its value is zero. Hence $t_0 \in (t_j^+, t_{j+1}^+)$, where $b_+(t_j^+) = b_+(t_{j+1}^+) = 0$. Since the zeros of b_- and b_+ interlace, the zero t_j^- of b_- is also in (t_j^+, t_{j+1}^+). Then $t_j^- = t_0$ by (8.268) and $A(t_0) = 0$. It follows from (8.267) that $b_-/\sqrt{\mathcal{D}}$ does not have poles. □

Corollary 8.141 *For any Wall pair (A, B) the quotient $b_-/\sqrt{\mathcal{D}}$ is positive on $\mathcal{E}(f)$ and is pure imaginary on $\mathcal{U}(f)$.*

Corollary 8.142 *The quotient $b_-/\sqrt{\mathcal{D}} \geqslant 1$ on $\mathcal{E}(f)$.*

Proof By (8.280) The quotient $b_-/\sqrt{\mathcal{D}} > 0$ on $\mathcal{E}(f)$. Hence

$$\frac{B - zB^*}{\sqrt{\mathcal{D}}} = \left|\frac{B - zB^*}{\sqrt{\mathcal{D}}}\right| = \frac{|B - zB^*|}{\sqrt{|B - zB^*|^2 - 4|A|^2}} \geqslant 1$$

on $\mathcal{E}(f)$. □

Since $b_+/\sqrt{\mathcal{D}} = 1 + O(z^{d+1})$ as $z \to 0$ and

$$\frac{b_-}{\sqrt{\mathcal{D}}} = \frac{1 - zB^*/B}{1 + zB^*/B}\,\frac{b_+}{\sqrt{\mathcal{D}}}\,, \tag{8.281}$$

the first d parameters of the finite Blaschke product $-B^*B^{-1}$ coincide with those of $ff^\#$. Since $-B^*/B$ is a finite Blaschke product, it is the Schur function of a discrete measure $\mu_B \in \mathfrak{P}(\mathbb{T})$ whose point masses are located at the zeros of $b_+ = B + zB^*$.

Corollary 8.143 *For every pair of polynomials p, q such that the spectrum of the trigonometric polynomial $p\overline{q}$ lies in $[-d, d]$ we have*

$$\int_{\mathbb{T}} p\,\overline{q}\, d\sigma_{ff^\#} = \int_{\mathbb{T}} p\,\overline{q}\, d\mu_B\,. \tag{8.282}$$

In particular, the measures μ_B and $\sigma_{ff^\#}$ have the same orthogonal polynomials up to and including that of order d.

Proof It follows from (8.281) that $F^{\sigma_{ff^\#}}(z) = F_{\mu_B}(z) + O(z^{d+1})$ as $z \to 0$, which says that the Fourier coefficients of the two measures coincide up to the index d. This is equivalent to (8.282). □

Corollary 8.144 *Let f be the periodic Schur function corresponding to a Wall pair (A,B). Then*

$$\int_{\mathbb{T}} (B+\zeta B^*)\bar{\zeta}^k \, d\sigma_{ff^\#}(\zeta) = 0\,, \qquad k = 1,2,\dots,d\,.$$

Proof Apply (8.282) to $p = b_+$, $q = \zeta^k$. Notice that $b_+ = 0$ on μ_B. □

Corollary 8.145 *Let $\sigma \in \mathfrak{P}(\mathbb{T})$, let $f^\sigma = \{\overline{a_0,\dots,a_d}\}$ with $\max|a_j| > 0$ and let $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$. Then $ff^\#$ is the Schur function of the limit measure*

$$d\sigma^{(0)} = *\text{-}\lim_n |\varphi_n|^2 d\sigma\,, \quad n \equiv 0 \pmod{d+1}\,.$$

Proof By (8.276) $b_n = f^\# + O(z^{n+1})$, $z \to 0$, if $n \equiv 0 \pmod{d+1}$. However, $f_n \equiv f$ for such an n in view of the periodicity of the Schur parameters of f. The result follows by Theorem 8.67. □

Theorem 8.146 *Let $P, Q \in \mathbb{C}[z]$,* deg $P =$ deg $Q = d+1$, *and let Q be separable, the roots of P and Q be placed on $\mathbb{T}$ and*

$$P^* = -P\,, \qquad Q^* = Q\,, \qquad P(0) = Q(0) = 1\,. \tag{8.283}$$

Then, for some real δ_j,

$$\frac{P(z)}{Q(z)} = \sum_{j=0}^{d} \frac{z_j + z}{z_j - z}\,\delta_j\,. \tag{8.284}$$

Proof Since (8.284) decomposes P/Q into partial fractions,

$$2\delta_j = -\bar{z}_j \frac{P(z_j)}{Q'(z_j)}\,, \qquad j = 0,1,\dots,d\,.$$

Next, $\dot{Q} = \partial Q/\partial\theta = Q'(z)\, iz$ shows that $2\delta_j = -i\; P(z_j)/\dot{Q}(z_j)$. By (8.283), on $\mathbb{T}$

$$\bar{z}^{(d+1)/2} Q = z^{(d+1)/2}\overline{Q}\,, \qquad \bar{z}^{(d+1)/2} P = -z^{(d+1)/2}\overline{P}\,.$$

It follows that $\varphi(z) = \bar{z}^{(d+1)/2}Q(z)$ and $\psi(z) = i\bar{z}^{(d+1)/2}P(z)$ are real for $z \in \mathbb{T}$. Since $Q(z_j) = 0$, $\dot{\varphi}(z_j) = \bar{z}_j^{(d+1)/2}\dot{Q}(z_j)$ and $\psi(z_j)$ are also real. Consequently,

$$\sum_{j=0}^{d} \delta_j = P(0)/Q(0) = 1 \tag{8.285}$$

and all the δ_j are real. □

Corollary 8.147 *If P and Q are separable polynomials of degree $d+1$ satisfying* (8.283) *and their zeros alternate on $\mathbb{T}$, then in* (8.285) *all $\delta_j > 0$.*

Proof Since Q is separable, φ is separable too. Then the signs of the derivatives $\dot{\varphi}(z_j)$ alternate. Hence in order that the the numbers δ_j be positive it is necessary and sufficient that $\psi(z_j)$ be alternating. $\square$

Definition 8.148 *A pair (P, Q) of polynomials satisfying the conditions of Corollary* 8.147 *is said to be alternating.*

Corollary 8.149 *A pair of polynomials (P, Q) is alternating if and only if there exists a polynomial B, $B(0) = 1$, $B(z) \neq 0$ for $|z| \leqslant 1$, such that*

$$P = B - zB^* , \qquad Q = B + zB^* .$$

Proof By Corollary 8.147 $0 < 1 \leqslant 1 + \operatorname{Re}(P/Q)\,2 = \operatorname{Re}(P+Q)/Q$. Hence $B(z) = \{P(z) + Q(z)\}/2 \neq 0$ in $|z| \leqslant 1$ and $B(0) = 1$. Next, $\deg P = \deg Q = d+1$ and $P^* = -P$, $Q^* = Q$. Since $P(0) = Q(0) = 1$, the coefficient of z^{d+1} in B is zero. Passing to the adjoint polynomials, we obtain $zB^* = \{P(z) - Q(z)\}/2$, which proves the corollary. $\square$

The description of $\sigma_{ff^\#}$ given in Theorem 8.140 is now complete.

Theorem 8.150 *Let $b \in \mathbb{C}[z]$ be separable,* $\deg b = d+1$ *and $b(0) = 1$ and $b^* = b$, with roots on $\mathbb{T}$, $0 < 4\omega \leqslant m_b$. Let $\mathcal{D} = b^2 - 4\omega z^{d+1}$ and $\mathcal{E} = \{t \in \mathbb{T} : |b|^2 < 4\omega\}$. If there is an $r \in \mathbb{C}[z]$,* $\deg r \leqslant d+1$, *such that $r(0) = 1$, $r^* = -r$ and $r/\sqrt{\mathcal{D}} \geqslant 1$ on $\mathcal{E}$, then there is a periodic f with discriminant $\mathcal{D}$ essentially supported by $\mathcal{E}$ and for which $r/\sqrt{\mathcal{D}} = \sigma_{ff^\#}$.*

Proof If $4\omega < m_b$ then $\mathcal{E}$ is a union of $d+1$ open arcs $\delta_j = (\tau_j^-, \tau_j^+)$, $0 \leqslant j \leqslant d$, interlaced with closed arcs $\gamma_j = [\tau_j^+, \tau_{j+1}^-]$ not reducing to points. By the assumption, the boundary values of $r/\sqrt{\mathcal{D}}$ *from inside* $\mathbb{D}$ *are real and* $\geqslant 1$ on δ_j and δ_{j+1}. Let Γ be a smooth path starting in δ_j, and ending in δ_{j+1}, passing along the interior side of $\mathbb{T}$ and a round the zeros of r and $\sqrt{\mathcal{D}}$ in $\mathbb{D}$. Since $\mathcal{D}$ has only two zeros along this path and r has n_j in γ_j, the increment of the argument must be

$$\Delta_\Gamma \left(\frac{r}{\sqrt{\mathcal{D}}} \right) = \pi(1 - n_j) .$$

It follows that n_j is odd and therefore that every γ_j has at least one zero. Since the total number of arcs is $d+1$, each γ_j contains exactly one zero. It follows that (r, b) is an alternating pair. By Corollary 8.149 there is a polynomial B, not vanishing in the closure $\operatorname{clos} \mathbb{D}$, such that $B(0) = 1$ and

$$r = B - zB^* , \qquad b = B + zB^* . \tag{8.286}$$

Since $|r|^2 \geqslant |\mathcal{D}|$ on $\{t \in \mathbb{T} : \mathcal{T}_\mathcal{D} < 0\}$, we have that $\bar{z}^{d+1}(\mathcal{D} - r^2) = |r|^2 + \mathcal{T}_\mathcal{D} \geqslant 0$ everywhere on $\mathbb{T}$. It follows that there exists a polynomial A such that

$$\bar{z}^{d+1}(\mathcal{D} - r^2) = |r|^2 + \mathcal{T}_\mathcal{D} = |r|^2 + |b|^2 - 4\omega = 4|A|^2 \tag{8.287}$$

on $\mathbb{T}$. By (8.286) and (8.287), $4|B|^2 = |r|^2 + |b|^2 = 4|A|^2 + 4\omega$ on $\mathbb{T}$, which implies that (A, B) is a Wall pair. Next $\mathcal{D} = (B - zB^*)^2 + 4zA^*A$ by (8.287), which shows that the Wall pair (A, B) has discriminant $\mathcal{D}$. The proof is completed by Theorem 8.140. □

207 The Green's function of $\mathbb{C} \setminus \Delta(\sigma)$. By (8.268) all periodic Schur functions of a given discriminant, i.e. with fixed $b_+(z) = B + zB^*$ and ω, have the same essential open support $\mathcal{E} = \{t \in \mathbb{T} : \mathcal{T}_{\mathcal{D}} < 0\}$ in $\mathbb{T}$, the closure of which is $\Delta(\sigma)$. If $\mathcal{E}$ essentially supports a periodic $f \in \mathcal{B}$ then there are other periodic Schur functions with the same support. For instance, this is the case for

$$f^\# = \frac{-A^* + zB^* f^\#}{B - zAf^\#} ,$$

corresponding to $(-A^*, B)$, so that $f^\#$ satisfies (8.277). If $\{a_0, \dots, a_d\}$ is a period of f then $\{-\bar{a}_d, \dots, -\bar{a}_0\}$ is the period of $f^\#$. More examples for $d > 0$ can be obtained by a cyclic shift of parameters:

$$\begin{array}{llllll} f_1 = \{a_1, & a_2, & \dots, & a_d, & a_0\} \\ f_2 = \{a_2, & a_3, & \dots, & a_0, & a_1\} \\ \vdots & & & & \\ f_d = \{a_d, & a_0, & \dots, & a_{d-2}, & a_{d-1}\} \end{array} \tag{8.288}$$

The formula

$$1 - |f_n|^2 = \frac{(1 - |a_n|^2)(1 - |f_{n+1}|^2)}{|1 + \bar{a}_n z f_{n+1}|^2} , \qquad z \in \mathbb{T} ,$$

shows that $\mathcal{U}(f) = \mathcal{U}(f_n)$. Hence all the periodic functions in (8.288) have the same essential support. By Theorem 8.135 they have a common discriminant. Since by (E8.1)

$$B + zA^* f = \prod_{k=0}^{d} (1 + z\bar{a}_k f_{k+1}) , \tag{8.289}$$

$B + zA^* f$ is invariant under the above cyclic shift. Moreover, the same holds for any periodic f with discriminant $\mathcal{D}$, since by (8.267) both are functions of b_+ and ω:

$$\rho_1(z) = \frac{B + zA^* f}{\sqrt{\omega}} = \frac{b_+ + \sqrt{\mathcal{D}}}{2\sqrt{\omega}} , \qquad |z| < 1 . \tag{8.290}$$

Hence $\rho_1(z)$ controls the periodic Schur functions with a given discriminant. Since $|f| = 1$ on $\mathbb{T} \setminus \Delta(\sigma) \neq \varnothing$, $f = f^\sigma$ extends to $\mathbb{C} \setminus \Delta(\sigma)$ by the Schwarz reflection principle through $\mathbb{T} \setminus \Delta(\sigma)$. If $z \in \mathbb{T} \setminus \Delta(\sigma)$ then

$$z^{1+d}\{\overline{B(z)} + \overline{zA^*(z)}\,\overline{f(z)}\} = (zB^*(z)f + A(z))f^{-1} = B + zA^* f ,$$

implying that ρ_1 extends analytically through $\mathbb{T} \setminus \Delta(\sigma)$ and that for $|z| > 1$

$$\rho_1(z) = z^{1+d}\overline{\rho_1(\bar{z}^{-1})} \,. \tag{8.291}$$

Compare this formula with (8.234) in §**200**. The following theorem shows that $\rho_1 = \Phi^{d+1}$ in $\mathbb{C} \setminus \Delta(\sigma)$; see §**200**.

Theorem 8.151 *The function set as* $(1+d)^{-1} \log|\rho_1|$ *is the Green's function for* $\mathbb{C} \setminus \Delta(\sigma)$*: for* $z \to \infty$

$$\frac{\log|\rho_1(z)|}{d+1} = \log|z| + \frac{1}{2d+2} \log \frac{1}{\omega} + o(1) \,. \tag{8.292}$$

Proof Since f is periodic and corresponds to (A, B), the identity on $\mathbb{T}$

$$1 - |f|^2 = 1 - \left| \frac{A + zB^* f}{B + zA^* f} \right|^2 = \frac{(1 - |f|^2)\omega}{|B + zA^* f|^2}$$

implies that $|\rho_1| = 1$ on $\mathcal{E}(f)$ and hence on its closure $\Delta(\sigma)$. The algebraic conjugate to ρ_1 is given by

$$\rho_2(z) = \frac{B + zA^* x_2}{\sqrt{\omega}} = \frac{1}{2\sqrt{\omega}} \left(b_+ - \sqrt{\mathcal{D}} \right) , \qquad |z| < 1 \,. \tag{8.293}$$

By Viète's formulas, ρ_1 and ρ_2 are the roots of the quadratic equation

$$\sqrt{\omega} X^2 - b_+ X + \sqrt{\omega} z^{d+1} = 0 \,. \tag{8.294}$$

Since $\rho_1(0) = \omega^{-1/2} \neq 0$ and $\rho_1 \rho_2 = z^{1+d}$, $\rho_1(z) \neq 0$ for every $z \in \mathbb{C}$. Again by Viète's formulas,

$$\frac{1 + \rho_2/\rho_1}{1 - \rho_2/\rho_1} = \frac{\rho_1 + \rho_2}{\rho_1 - \rho_2} = \frac{b_+}{\sqrt{\mathcal{D}}} \,, \tag{8.295}$$

implying that the boundary values of $b_+/\sqrt{\mathcal{D}}$ on $\mathcal{E}(f)$ are pure imaginary. However, on $U(f) = \mathbb{T} \setminus \mathcal{E}(f)$ we have

$$\left(\frac{b_+}{\sqrt{\mathcal{D}}} \right)^2 = \frac{b_+^2}{\mathcal{D}} = \frac{|b_+|^2}{|b_+|^2 - 4\omega} \geqslant 0 \,,$$

implying that $b_+/\sqrt{\mathcal{D}}$ is real on $U(f)$. If $4\omega < m_b$ then $\mathcal{D}$ is separable and any arc δ_j of $\mathcal{E}(f)$ is surrounded by two closed consecutive arcs γ_j, γ_{j+1} of $U(f)$, that do not reduce to a point. Then $\delta_j = (\tau_j^-, \tau_j^+)$ contains only one zero t_j^+ of b_+, $\mathcal{D}(\tau_j^\pm) = 0$ and $\mathcal{D} \neq 0$ on δ_j. Let Γ be a path starting in γ_j and ending in γ_{j+1}. Suppose that it goes counterclockwise, passing t_j^+ and $\tau_j^\pm$ by small semicircles inside $\mathbb{D}$. There are three such semicircles on Γ. The increment of the argument along the first and the last is $\pi/2$ for each whereas along the second it is $-\pi$. It follows that all signs of $b_+/\sqrt{\mathcal{D}}$ on γ_j are equal. By the mean value theorem applied to $\operatorname{Re} b_+(z)/\sqrt{\mathcal{D}(z)}$ this sign is positive since $b_+(0)/\sqrt{\mathcal{D}(0)} = 1$. It follows that $\operatorname{Re} b_+/\sqrt{\mathcal{D}} > 0$ in $\mathbb{D}$. Then

$z^{1+d}/\rho_1^2 = \rho_2/\rho_1 \in \mathcal{B}$ by (8.295), implying that $|\rho_1| > 1$ in $\mathbb{D}$. Returning to the boundary values of $b_+/\sqrt{\mathcal{D}}$ we get the formula

$$\frac{b_+}{\sqrt{\mathcal{D}}}(z) = \begin{cases} \dfrac{|b_+|}{\sqrt{|b_+|^2 - 4\omega}} & \text{if } z \in U(f)\,, \\ i\dfrac{|b_+|}{\sqrt{4\omega - |b_+|^2}} & \text{if } z \in (\tau_j^-, t_j^+)\,, \\ -i\dfrac{|b_+|}{\sqrt{4\omega - |b_+|^2}} & \text{if } z \in (t_j^+, \tau_j^+)\,. \end{cases} \tag{8.296}$$

Hence $|\rho_1(z)| > 1$ in $\mathbb{C}\setminus\Delta(\sigma)$. Passing to the limit in (8.291) and observing that $\rho_1(0) = \omega^{-1/2}$, we obtain (8.292). If $4\omega = m_b$ then we first consider $4\omega < m_b$ and then pass to the limit as $4\omega \to m_b^-$. □

Since (8.294) is not an equation for a periodic Schur function yet its discriminant is $\mathcal{D}$, we conclude that even if $\mathcal{D}$ is the discriminant of a periodic Schur function there is an entire nonperiodic Schur function in $\mathbb{C}(z, \sqrt{\mathcal{D}})$.

Corollary 8.152 *If $\sigma \in \mathfrak{P}(\mathbb{T})$ is a periodic measure, then* $\log|\rho_1(z)|/(d+1)$ *is the Green's function of $\mathbb{C}\setminus\Delta(\sigma)$ with pole at ∞. In particular*

$$\operatorname{cap}\Delta(\sigma) = \left(\log\frac{e}{\omega^{1/(2d+2)}}\right)^{-1}$$

(*see the start of* §**200**).

Corollary 8.153 *If f is periodic then the harmonic measure of every connected component of $\mathcal{E}(f)$ is rational.*

Proof If $4\omega < m_b$ then every such component is a single interval δ_j. Since $\sqrt{\mathcal{D}}/b_+ > 0$ on $U(f)$ it extends to $\mathbb{C}\setminus\Delta(\sigma)$ to an analytic function with positive real part. Now by (8.290),

$$\log\rho_1(z) = \log b_+(z) + \log\left(1 + \frac{\sqrt{\mathcal{D}}}{b_+}\right) - \log(2\sqrt{\omega})\,. \tag{8.297}$$

The increment of the argument of $\log\rho_1(z)$ along a simple closed contour surrounding δ_j is 2π, since b_+ has only one zero on δ_j. Since $\rho_1 = \Phi^{d+1}$, it follows from (8.241) that the harmonic measure of δ_j is $1/(d+1)$. Passing to the limit as $4\omega \to m_b^-$ or counting double zeros, we conclude that the harmonic measure of any component e_k is $k/(d+1)$, where k is the number of δ_j in e_k. □

208 The inverse problem for periodic measures. By Corollary 8.153 the support of every periodic measure $\mathcal{K} = \cup_{j=1}^{r} e_j$ is a union of nonintersecting closed arcs e_j with rational harmonic measures $\nu_{\mathcal{K}}(e_j)$, $1 \leqslant j \leqslant r$. A compact $\mathcal{K}$ with such a property is called *rational*. Rational compacts are invariant under rotations $z \to \lambda z$, $|\lambda| = 1$. However, $\mathcal{S}(f(\lambda z)) = \{\lambda^n a_n\}_{n \geqslant 0}$ for $|\lambda| = 1$ shows that periodic measures are not. The key to understanding which rational compacts may support a periodic measure is given by the formula (8.291).

Suppose that $\mathcal{K}$ is rational with r components $\{e_j\}$ and that $\nu_{\mathcal{K}}(e_j) \in \mathbb{Q}$ for every $j \leqslant r$. Then there is a smallest integer $d \geqslant 0$ such that $(d+1)\nu_{\mathcal{K}}(e_j) \in \mathbb{Z}$ for $1 \leqslant j \leqslant r$. By (8.241) and (8.237) $\rho_1 = \Phi^{d+1}$ is single-valued in $\mathcal{G}_1 = \mathbb{C} \setminus \mathcal{K}$. It is unimodular on $\mathcal{K}$ and therefore can be extended through $\mathcal{K}$ by the Schwarz reflection to another single-valued analytic function,

$$\rho_2(z) = \frac{1}{\overline{\Phi(\bar{z}^{-1})^{d+1}}} ,$$

in $\mathcal{G}_1$. Since $|\Phi| > 1$ everywhere in $\mathbb{C} \setminus \mathcal{K}$, we see that $|\rho_1| > 1$ in $\mathcal{G}_1$ and $|\rho_2| < 1$ in $\mathcal{G}_1$. Since ρ_1 and ρ_2 mutually extend each other through $\mathcal{K}$, we obtain that $\rho_1\rho_2$ is analytic on $\mathbb{C}$ and asymptotically equivalent to cz^{d+1} as $z \to \infty$. This implies that $\rho_1\rho_2$ is a polynomial which is unimodular on $\mathbb{T}$. Hence

$$\rho_1\rho_2 = \lambda_{\mathcal{K}} z^{d+1} , \qquad |\lambda_{\mathcal{K}}| = 1 . \tag{8.298}$$

By (8.291), if $\mathcal{K}$ supports a periodic measure then $\lambda_{\mathcal{K}} = 1$. However, it is clear that any rational compact can be rotated so as to force the corresponding Φ to satisfy (8.298) with $\lambda_{\mathcal{K}} = 1$.

Theorem 8.154 *A compact $\mathcal{K} = \cup_{j=1}^{r} e_j \subset \mathbb{T}$ supports a periodic measure if and only if it is rational and $\lambda_K = 1$.*

Proof The necessity has already been proved. Again, since ρ_1 and ρ_2 extend each other through $\mathcal{K}$ and have polynomial growth at infinity,

$$b = \frac{\rho_1 + \rho_2}{\rho_1(0)} , \qquad \rho_1(0) = \Phi(0) > 0 , \tag{8.299}$$

is a polynomial of degree $d+1$. Since $|\rho_2| < 1 < |\rho_1|$ on $\mathcal{G}_1$, this polynomial has all its roots on $\mathcal{K} \subset \mathbb{T}$. It is clear that $b(0) = 1$. To prove that $b^* = b$, by the uniqueness theorem we can check the identity $z^{d+1}\bar{b} = b$ on $\mathcal{K}$, where both ρ_1 and ρ_2 are unimodular and satisfy (8.298) with $\lambda_{\mathcal{K}} = 1$:

$$z^{d+1}\bar{b} = \frac{1}{\rho_1(0)} \left(\frac{z^{d+1}}{\rho_1} + \frac{z^{d+1}}{\rho_2} \right) = \frac{\rho_1 + \rho_2}{\rho_1(0)} = b .$$

Hence ρ_1 and ρ_2 are the roots of the quadratic equation

$$\sqrt{\omega}X^2 - b(z)X + \sqrt{\omega}z^{d+1} = 0 , \ \sqrt{\omega} = \Phi(0)^{-1} , \tag{8.300}$$

with discriminant $\mathcal{D} = b^2 - 4\omega z^{d+1}$. It remains to prove that there is a Wall pair (A, B) with discriminant $\mathcal{D}$. By Theorem 8.131 such a Wall pair exists if and only if the equation $b = B + zB^*$ has a solution B not equal to zero in $\mathbb{D}$ and such that $|B|^2 \geqslant \omega$ on $\mathbb{T}$. First we describe all solutions B to $b = B + zB^*$.

Lemma 8.155 *Let b be a polynomial of degree $d+1$ with roots on $\mathbb{T}$ such that $b^* = b$, $b(0) = 1$. Then all solutions to $b = B + zB^*$ in polynomials B not vanishing in $|z| \leqslant 1$ are obtained from the formula*

$$\frac{B - zB^*}{B + zB^*} = \int_{b(\zeta)=0} \frac{\zeta + z}{\zeta - z}\, d\tau\,, \tag{8.301}$$

where τ runs over the set of probability measures distributed through the zero set of b in such a way that each root of b carries a positive mass.

Proof The Schur function of any measure τ located at the zeros of b is a Blaschke product $f = \lambda\varphi/\varphi^*$, $|\lambda| = 1$, see (8.74). Therefore,

$$\int_{b(\zeta)=0} \frac{\zeta + z}{\zeta - z}\, d\tau = \frac{1 + zf}{1 - zf} = \frac{\varphi^* + \lambda z\varphi}{\varphi^* - \lambda z\varphi}\,.$$

The requirement that $\tau(\{s\}) > 0$ for every zero of b implies that the polynomial $\varphi^* - \lambda z\varphi$ has the same zero set as b. Observing that $\varphi^*(0) = b(0) = 1$, we obtain that $b = \varphi^* - \lambda z\varphi$. Now, taking into account that $b^* = b$, we conclude that

$$b = (\varphi^* - \lambda z\varphi)^* = z\varphi - \bar{\lambda}\varphi^* = -\bar{\lambda}(\varphi^* - \lambda z\varphi) = -\bar{\lambda} b\,. \tag{8.302}$$

It follows that $-\lambda = 1$, so that $B = \varphi^*$. □

Thus any choice of τ defines the polynomial B, $\deg B = d$, $B(0) = 1$:

$$B = \frac{b}{2}\left(1 + \int_S \frac{\zeta + z}{\zeta - z}\, d\tau\right), \qquad zB^* = \frac{b}{2}\left(1 - \int_S \frac{\zeta + z}{\zeta - z}\, d\tau\right), \tag{8.303}$$

where $S = \{s_0, \dots, s_d\}$ is the zero set of B.

Lemma 8.156 *If $b = B + zB^*$ then $|B|^2 \geqslant \omega$ on $\mathbb{T} \setminus \mathcal{K}$.*

Proof Let $\mathcal{E} = \{t \in \mathbb{T} : |b|^2 - 4\omega \geqslant 0\}$. Then $4|B|^2 \geqslant |b|^2 \geqslant 4\omega$ on $\mathcal{E}$, since the integral in (8.303) is pure imaginary on $\mathbb{T}$. Since ρ_1 and ρ_2 satisfy (8.300), we have

$$\rho_1(z) = \frac{b}{2\sqrt{\omega}}\left(1 + \frac{\sqrt{D}}{b}\right), \qquad \rho_2(z) = \frac{b}{2\sqrt{\omega}}\left(1 - \frac{\sqrt{D}}{b}\right)$$

in $\mathbb{D}$. The boundary values of $\sqrt{D}b^{-1}$ on the unit circle are either pure imaginary or real: $\mathcal{D}b^{-2} = 1 - 4\omega|b|^{-2}$. If they are pure imaginary then $|\rho_1| = |\rho_2| = 1$, otherwise $|\rho_1| \neq 1$ and $|\rho_2| \neq 1$. It follows that $\mathcal{E} = \mathcal{K}$, completing the proof. □

By Lemma 8.156 it is sufficient to determine a measure τ in (8.303) for which the corresponding B satisfies $|B|^2 \geqslant \omega$ on $\mathcal{K}$. The logarithmic derivative indicates that (8.303) simplifies if $\tau(\{s_j\}) = (1+d)^{-1}$ for $s_j \in S$. Indeed,

$$b\sum_{j=0}^{d} \frac{z+s_j}{z-s_j} = b\sum_{j=0}^{d}\left(\frac{2z}{z-s_j} - 1\right) = 2zb' - (d+1)b \tag{8.304}$$

converts (8.303) into

$$B = b - \frac{zb'}{1+d}\,, \qquad B^* = \frac{b'}{1+d}\,. \tag{8.305}$$

Lemma 8.157 *We have* $|b'| \geqslant (1+d)\sqrt{\omega}$ *on* $\mathbb{T}$.

Proof Since $|B| = |B^*|$ on $\mathbb{T}$, it is sufficient to prove this inequality on $\mathcal{K}$. Both ρ_1 and ρ_2 satisfy (8.300). Differentiating it in z, we obtain that

$$\frac{\rho_1'}{\rho_1} = \frac{zb' - \sqrt{\omega}(d+1)\rho_2}{z\sqrt{D}}\,, \qquad \frac{\rho_2'}{\rho_2} = -\frac{zb' - \sqrt{\omega}(d+1)\rho_1}{z\sqrt{D}}\,. \tag{8.306}$$

It follows that

$$\begin{aligned} 2z(\log\rho_1)' = z(\log\rho_1^2)' &= (d+1) + \left(\log\frac{\rho_1}{\rho_2}\right)' \\ &= (d+1) + \frac{2zb' - (d+1)b}{\sqrt{\mathcal{D}}} \\ &= (d+1) + \frac{b}{\sqrt{\mathcal{D}}}\sum_{j=0}^{d}\frac{z+s_j}{z-s_j}\,. \end{aligned} \tag{8.307}$$

Since the boundary values of $\log\rho_1$ from the zero side on $\mathcal{K}$ are $i(d+1)\tilde{g}$ (observe that $g = 0$ on $\mathcal{K}$), we obtain by the Cauchy–Riemann equations

$$\begin{aligned} 0 < 2\frac{\partial g}{\partial n^+} = -2\frac{\partial \tilde{g}}{\partial\theta} &= -\frac{2}{i(d+1)}\frac{d}{d\theta}\log\rho_1 = -\frac{2z}{d+1}(\log\rho_1)' \\ &= -1 - \frac{b}{\sqrt{\mathcal{D}}(d+1)}\sum_{j=0}^{d}\frac{z+s_j}{z-s_j} \\ &= -1 + \frac{b}{\sqrt{\mathcal{D}}(d+1)}\sum_{j=0}^{d}\frac{s_j+z}{s_j-z} \end{aligned}$$

on $\mathbb{T}^-$. It follows that the combination on the right-hand part of the above identity equals $2\partial g/\partial n^+ + 1 \geqslant 1$. Since $b/\sqrt{\mathcal{D}}$ is pure imaginary on $\mathcal{K}$, we obtain by (8.303) that

$$\begin{aligned} 4|B|^2 &= |\mathcal{D}| \left| \frac{b}{\sqrt{D}} + \frac{b}{\sqrt{\mathcal{D}}(d+1)} \sum_{j=0}^{d} \frac{s_j + z}{s_j - z} \right|^2 \\ &= |b|^2 + \left(2\frac{\partial g}{\partial n^+} + 1 \right)^2 (4\omega - |b|^2) \geqslant |b|^2 + 4\omega - |b|^2 = 4\omega\,, \end{aligned} \tag{8.308}$$

completing the proof of the lemma. □

By (8.308) and (8.234) we have

$$d\nu_{\mathcal{K}}(\zeta) = \left(\frac{4|B|^2 - |b|^2}{4\omega - |b|^2} \right)^{1/2} \frac{ds(\zeta)}{2\pi}\,, \qquad \zeta \in \mathcal{K}\,. \tag{8.309}$$

Notice that Lemma 8.157 states the reverse of Bernshtein's inequality for the polynomial b. □

Thus we have found that all supports of periodic measures admit the following algorithmic description. Take any separable polynomial b with roots on $\mathbb{T}$ such that $b^* = b$ and $b(0) = 1$. Let m_b be the minimal local maximum of $|b|^2$ on $\mathbb{T}$. Then for every $\omega \leqslant m_b$ the set $\{t \in \mathbb{T} : |b|^2 \leqslant 4\omega\}$ is the support of a periodic measure.

Remark. The description of the supports of periodic measures in terms of harmonic measures and polynomials was first obtained in Peherstorfer and Steinbauer (2000) with the help of a result of Widom (1969). The elementary approach presented here is based on the theory of Wall pairs developed in Khrushchev (2006a), see also Khrushchev (2006b). More details can be found in Simon (2005).

Exercises

8.1 Show that

$$A_n = a_0 + \left(a_1 + a_0 \sum_{k=1}^{n-1} \bar{a}_k a_{k+1} \right) z + \cdots + a_n z^n\,,$$

$$B_n = 1 + \left(\sum_{k=0}^{n-1} \bar{a}_k a_{k+1} \right) z + \cdots + a_n \bar{a}_0 z^n\,.$$

8.2 Prove the following identity for the Taylor coefficients of f:

$$\hat{f}(n+1) = \frac{\hat{A}_n}{B_n}(n+1) + \omega_n a_{n+1}\,.$$

Hint: Apply (8.26) and observe that $B_n(0) = 1$ by (8.25).

8.3 For every $f \in \mathcal{B}$ show that the Taylor coefficient $\hat{f}(n)$ is uniquely determined by $a_0, \dots, a_n$ and that the Schur parameter a_n is uniquely determined by $\hat{f}(0), \dots, \hat{f}(n)$.

Hint: Apply the equation of Ex. 8.2 and induction.

8.4 If A_n, B_n are Wall polynomials then the nonlinear mapping

$$f \longrightarrow \frac{A_n + zB_n^* f}{B_n + zA_n^* f}$$

maps a convex set $\mathcal{B}$ onto a convex set $\mathcal{B}_n$ and sends the extreme points of $\mathcal{B}$ to the extreme points of $\mathcal{B}_n$.

Hint: Apply (8.24) to check the convexity of $\mathcal{B}_n$. Apply (8.21) to obtain the formula

$$1 - \frac{A_n + B_n^* w}{B_n + A_n^* w}^2 = \frac{\omega_n (1 - |w|^2)}{|B_n + wA_n^*|^2}$$

for $z \in \mathbb{T}$. Apply Theorem 8.54.

8.5 Show that the set $\mathfrak{D}(z) = \{f(z) : f \in \mathcal{B}_n\}$, where $\mathcal{B}_n$ is as in Ex. 8.4, is a circle with center $c_n(z)$ and radius $\rho_n(z)$:

$$c_n(z) = \frac{A_n \overline{B}_n - |z|^2 \overline{A_n^*} B_n^*}{|B_n|^2 - |z|^2 |A_n^*|^2}, \qquad \rho_n(z) = \frac{\omega_n |z|^{n+1}}{|B_n|^2 - |z|^2 |A_n^*|^2}.$$

8.6 Prove that for $d\sigma = (1 + \cos\theta) d\theta / 2\pi$ and $|z| < 1$,

$$1 + z = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} (1 + \cos\theta) d\theta,$$

which implies that $f^\sigma(z) = 1/(2+z)$.

8.7 (Schur 1917) Prove that the parameters a_n and the Schur functions for $f(z) = 1/(2+z)$ are given by

$$f_n(z) = \frac{1}{(n+1)z + (n+2)}, \qquad a_n = \frac{1}{n+2}.$$

8.8 Prove the following identities for Wall polynomials and Schur functions:

$$B_n + A_n^* z f_{n+1} = \prod_{k=0}^{n} (1 + z\bar{a}_k f_{k+1}), \tag{E8.1}$$

$$A_n + B_n^* z f_{n+1} = (a_0 + zf_1) \prod_{k=1}^{n} (1 + z\bar{a}_k f_{k+1}), \tag{E8.2}$$

$$B_n f - A_n = z^{n+1} f_{n+1} \prod_{k=0}^{n} (1 + z\bar{a}_k f_{k+1})^{-1} , \tag{E8.3}$$

$$B_n f - A_n = z^{n+1} f_{n+1} \prod_{k=0}^{n} (1 - \bar{a}_k f_k) , \tag{E8.4}$$

$$B_n^* - A_n^* f = z^n \omega_n \prod_{k=0}^{n} (1 - z\bar{a}_k f_{k+1})^{-1} , \tag{E8.5}$$

$$B_n^* - A_n^* f = z^n \prod_{k=0}^{n} (1 - \bar{a}_k f_k) . \tag{E8.6}$$

Hint: Formula (E8.1) follows from Lemma 2.31 or can be proved by induction with the aid of the useful formula

$$1 - |a_k|^2 = (1 - \bar{a}_k f_k)(1 + \bar{a}_k z f_{k+1}) . \tag{E8.7}$$

Formula (E8.2) follows from (E8.1) by (8.22). To obtain (E8.3), we multiply (E8.2) by B_n and subtract (E8.1) multiplied by A_n from the identity obtained. Then (E8.3) follows by (8.17) and (E8.7). The identity (E8.5) is proved similarly.

8.9 Let $f \in \mathcal{B} \cap C(\mathbb{T})$. Prove that $\lim_n A_n(\zeta)/B_n(\zeta) = f(\zeta)$ for $\zeta \in \mathbb{T}$ with $|f(\zeta)| < 1$ if and only if $\lim_n f_n(\zeta) = 0$.
Hint: apply (8.26).

8.10 Let $f \in \mathcal{B}$ and $F = \{\zeta \in \mathbb{T} : |f(\zeta)| = 1\}$. If $mF > 0$ then show that $A_n/B_n \rightrightarrows f$ on F if and only if $|A_n/B_n| \rightrightarrows 1$ on F.
Hint: Take the modulus in (8.26) and apply the Cauchy inequality to the pth power with $p < 1/2$; for details in Khrushchev (2001a).

8.11 Prove that for every $f \in \mathcal{B}$ the series (8.150) converges to f in $L^p(\mathbb{T})$, $0 < p < 1$.
Hint: Use Theorem 1.14 of Khrushchev (2001a).

8.12 (Njåstad) If σ is a Szegő measure with Schur function $f = f^\sigma$ then $\lim_n (B_n^*/A_n^*)$ exists at $z \in \mathbb{D}$ if and only if $\lim_n \{z^n (A_n^*(z))^{-1}\} = 0$.
Hint: Apply (E8.5), (E8.6).

8.13 Let $\{\Phi_n\}_{n\geqslant 0}$ and $\{\Theta_n\}_{n\geqslant 0}$ be two families of monic orthogonal polynomials in $L^2(d\sigma)$ with positive norms such that $\deg \Phi_n = \deg \Theta_n = n$. Then $\Phi_n = \Theta_n$ for every $n = 0, 1, \ldots$
Hint: Apply induction in n.

8.14 Prove that

$$\frac{\Psi_{n+1}^*}{\Phi_{n+1}^*} - \frac{\Psi_n^*}{\Phi_n^*} = \frac{2z^{n+1} a_n \omega_{n-1}}{\Phi_n^{*2}(1 + a_n z b_n)} . \tag{E8.8}$$

Hint: Apply the determinant identity to (8.38) to show that

$$\Psi_{n+1}^* \Phi_n^* - \Psi_n^* \Phi_{n+1}^* = 2z^{n+1} a_n \omega_{n-1} .$$

8.15 Prove that the Geronimus continued fraction for F^σ converges uniformly and absolutely on any compact subset of $\mathbb{D}$.

Hint: Using (8.30) and (E8.8), prove that

$$\frac{\Psi_{n+1}^*}{\Phi_{n+1}^*} - \frac{\Psi_n^*}{\Phi_n^*} \leqslant \frac{2|z|^{n+1}}{(1-|z|)^3} . \tag{E8.9}$$

8.16 Identify polynomials $\mathbf{A} \in \mathcal{P}_n$ with the Hilbert space $\mathbb{C}^{n+1}$ of their coefficients $\mathbf{a}$. Check that

$$(\mathbf{A}, \mathbf{B}) = \int_{\mathbb{T}} \mathbf{A}\bar{\mathbf{B}}\, d\sigma = \sum_{k,j=0}^{n} a_j \bar{b}_k c_{k-j} = (\mathbf{C}_n \mathbf{a}, \mathbf{b}) .$$

Since $D_n = \det \mathbf{C}_n > 0$, the matrix $\mathbf{C}_n$ is invertible. Hence there exists a unique vector $\mathbf{a}$ in $\mathbb{C}^{n+1}$ such that

$$\mathbf{C}_n \mathbf{a} = (0, 0, \dots, \alpha_n^{-1}) .$$

Show that $\mathbf{A} = \varphi_n$.

8.17 Prove that φ_n is the unique polynomial of degree n for which there is a gap in the Fourier spectrum of the product

$$\varphi_n\, d\sigma \sim \sum_{j>0} d_j \bar{z}^j + \frac{z^n}{\alpha_n} + \sum_{j>n} d_j z^j .$$

8.18 Prove that $(\log^+ x)^2 < x$ for $x > 0$.

Hint: Consider $y(x) = x - \log^2 x$. Prove that $y'(x)$ for $x > 1$ attains its minimum $1 - 2/e > 0$ at $x = e$.

8.19 Prove that

$$b_{n+1} = -\frac{A_n^*}{B_n} + \frac{\omega_n z^{n+1}}{B_n(B_n - zA_n)} .$$

8.20 Prove that

$$F^\sigma(z) + \frac{\psi_n(z)}{\phi_n(z)} = \frac{z^n}{\varphi_n(z)\varphi_n^*(z)} \left\{ \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} |\varphi_n(\zeta)|^2\, d\sigma + 1 \right\} . \tag{E8.10}$$

Hint: Put $z = 1/\bar{z}$ in (8.142) and apply complex conjugation.

8.21 (Jensen's inequality) Let (X, μ) be a probability space, $v \in L^1(\mu)$ a real function and φ a concave function on $\mathbb{R}$. Then

$$\int_X \varphi(v)\, d\mu \leqslant \varphi\left(\int_X v\, d\mu \right) .$$

Hint: The result is trivial for linear φ. Any concave function is the infimum of linear functions. See Garnett (1981, Chapter I, Section 6).

8.22 If σ is a Szegő (Erdős) measure then σ_λ is a Szegő (Erdős) measure for every $\lambda \in \mathbb{T}$.
Hint: Apply (8.88). Observe that, by Smirnoff's theorem, $F^\sigma \in H^p$, $p < 1$. Since $C = 1 + i\alpha F^\sigma$ is in the Nevanlinna class, $\int_{\mathbb{T}} \log|C|^2 dm$ is finite; see Garnett (1981) and Koosis (1989).

8.23 Let $\sigma \in \mathfrak{P}(\mathbb{T})$, let $\{\varphi_n\}_{n\geqslant 0}$ be the orthogonal polynomials in $L^2(d\sigma)$ and let $\tau \in \mathbb{T}$ satisfy $\sigma(\{\tau\}) > 0$. Prove that

$$\sum_{n=0}^{\infty} |\varphi_n(\tau)|^2 \leqslant \sigma(\{\tau\})^{-1},$$

the equality holding if σ is not a Szegő measure.
Hint: Apply Parseval's inequality to the indicator $\mathbf{1}_{\{\tau\}}$ of the set $\{\tau\}$.

8.24 Let $\sigma \in \mathfrak{P}(\mathbb{T})$ with $f^\sigma = f$. Then σ is a Rakhmanov measure if and only if

$$\frac{1}{\varphi_{n+1}^*} = \frac{1 + o(1)}{\sqrt{\omega_n}(1 - zf)} \prod_{k=0}^{n} (1 - \bar{a}_k f_k) \tag{E8.11}$$

uniformly on compact subsets of $\mathbb{D}$.
Hint: Apply (8.73), (8.152) and Theorem 8.67. See Khrushchev (2001a).

8.25 Prove that if σ is a Szegő measure then

$$D(\sigma, z) = \frac{1}{\sqrt{\omega}(1 - zf)} \prod_{n=0}^{\infty} (1 - \bar{a}_n f_n)\,, \qquad z \in \mathbb{D}\,, \tag{E8.12}$$

or equivalently $\prod_{n=0}^{\infty}(1 - z\bar{a}_{n-1} f_n) = \sqrt{\omega}/D(\sigma, z)$, $a_{-1} = -1$.
Hint: Apply Theorem 8.70.

8.26 (Khrushchev 2001b) Prove that there is a Blaschke product with Schur parameters $\{a_n\}_{n\geqslant 0}$ satisfying $\sum_{n\geqslant 0} |a_n|^p < +\infty$ for every $p > 2$.
Hint: Let $f = f^\sigma$, where σ is a Riesz product as in Theorem 8.98. By Frostman's theorem, see Garnett (1981), there is an $\alpha \in \mathbb{D}$ such that the Möbius transform $f_\alpha = \tau_\alpha \circ f$ is a Blaschke product. Let $\lambda_\alpha = (1 - \alpha\bar{a}_0)(1 - \bar{\alpha} a_0)^{-1}$. Since $|\lambda_\alpha| = 1$ and

$$f_\alpha = \frac{\tau_\alpha(a_0) + z\lambda_\alpha f_1(z)}{1 + \overline{\tau_\alpha(a_0)} z\lambda_\alpha f_1(z)}\,,$$

the Schur parameters of f_α are $\{\tau_\alpha(a_0), \lambda_\alpha a_1, \lambda_\alpha a_2, \ldots\}$.

8.27 Let $\{A_n\}_{n\geqslant 0}$ and $\{B_n\}_{n\geqslant 0}$ be the Wall polynomials corresponding to the constant parameters $a_n = a > 0$, $n \geqslant 0$. Show that

$$\frac{A_n}{B_n} = a\frac{zB_n^*/B_n - 1}{z - 1}\,. \tag{E8.13}$$

Hint: From the fact that A_n/B_n are the convergents to (8.28) deduce that A_n and B_n are the solutions to the recurrence

$$U_n = (1 + z)U_{n-1} - (1 - a^2)zU_{n-2}\,, \qquad n \geqslant 2\,. \tag{E8.14}$$

Using (E8.14), $A_0 = a = A_0^*$ and $A_1 = a(1+z) = A_1^*$, show that $A_n^* = A_n$. Since B_n^* satisfies (E8.14), it is a combination of A_n and B_n. Determine the coefficients in order to show that

$$B_n^* = \frac{1}{z} B_n + z - \frac{1}{az} A_n \ ,$$

which is equivalent to (E8.13).

8.28 Prove that all zeros of the polynomials A_n from Ex. 8.27 are located inside the arc Δ_α.
Hint: Use (E8.13) to check that all roots of A_n are located on $\mathbb{T}$. Prove that

$$A_n = \frac{aX_1^{n+1}}{\sqrt{D}} \left\{ 1 - \left(\frac{X_2}{X_1} \right)^{n+1} \right\} \ ,$$

where $X_1 = 1 + azf_a$, $X_2 = z - azf_a$ are two solutions of the recurrence (E8.14). Observe that $|X_1| > |X_2|$ on $\gamma_\alpha = \mathbb{T} \setminus \Delta_\alpha$ and $X_1 = X_2$on $\mathbb{T}$.

8.29 (Perron) Prove that the Wall continued fraction corresponding to constant parameters converges uniformly on compact subsets of $\mathbb{D}$.
Hint: By Ex. 8.27,

$$\frac{zB_n^*}{zA_n^*} = \frac{B_n^*}{A_n} = \frac{1}{z}\frac{B_n}{A_n} + \frac{z-1}{az} \quad \rightrightarrows \quad \frac{1}{zf} + \frac{z-1}{az} = f \ .$$

8.30 Prove that there are Cesàro sequences not satisfying the Máte–Nevai condition.
Hint: Consider $\Lambda = \{2^n + k : n = 0, 1, \ldots, k = 0, 1, \ldots, n\}$ and take any sequence supported by Λ with values separate from 0.

Appendix

Continued fractions, observations
L. Euler (1739)[1]

1. Last year I began an examination of continued fractions, which constitute a relatively new part of analysis. During this time some observations have appeared, which possibly in the future will be not without their usefulness for this theory. With this in mind I will show to everybody who reads this that the basic tools of analysis are important in this science. Let a continued fraction

$$A+\frac{B}{C+}\,\frac{D}{E+}\,\frac{F}{G+}\,\frac{H}{I}\,+\cdots,$$

be given, the true value of which can be found by the extension to infinity of the series

$$A+\frac{B}{1P}-\frac{BD}{PQ}+\frac{BDF}{QR}-\frac{BDFH}{RS}+\cdots,$$

where P, Q, R, S, ... take the values

$$P=C\,,\quad Q=EP+D\,,\quad R=GQ+FP\,,\quad S=IP+HQ\,,\quad \cdots$$

This series always converges independently of the increase or decrease of the quantities B, C, D, E, F, ... only if they are all positive; every term is smaller than the preceding term but greater than the subsequent term, which immediately follows by the above laws giving the values of P, Q, R, S, ...[2][3]

2. If consequently in turn an infinite series

$$\frac{B}{1P}-\frac{BD}{PQ}+\frac{BDF}{QR}-\frac{BDFH}{RS}+\cdots,$$

is given, its sum can easily be expressed by a continued fraction. Then

$$C=P\,,\quad E=\frac{Q-D}{P}\,,\quad G=\frac{R-FP}{Q}\,,\quad I=\frac{S-HQ}{R}\,,\quad \cdots$$

[1] Translated from Latin to Russian by Igor' Gashkov in Moscow (2005) and from Russian to English by the present author. To save space, Rogers' notation has been used, as in the main part of the book; furthermore the multiplication point used by Euler has been replaced by the multiplication sign × used elsewhere in the book.

[2] This statement if understood literally formally contradicts Corollary 3.9. However, see the first example in §9 of this appendix and Exercise 4.6.

[3] See Theorem 1.4 and formula (1.20) for this section.

determine a continued fraction equal to this series, namely,

$$\frac{B}{P+\frac{D}{\frac{Q-D}{P}+\frac{F}{\frac{R-FP}{Q}+\frac{H}{\frac{S-HQ}{R}+\frac{K}{\cdots}}}}}$$

or

$$\frac{B}{P+\frac{DP}{Q-D+\frac{FPQ}{R-FP+\frac{HQR}{S-HQ+\frac{KRS}{\cdots}}}}}\,.$$

Thus if a series is given by

$$\frac{a}{p}-\frac{b}{q}+\frac{c}{r}-\frac{d}{s}+\frac{e}{t}-\cdots,$$

so that

$$B=a\,,\quad D=b:a\,,\quad F=c:b\,,\quad H=d:c\,,\quad K=e:d\,,\quad \cdots,$$

$$P=p\,,\quad Q=q:p\,,\quad R=pr:q\,,\quad S=qs:pr\,,\quad T=prt:qs\,,\quad \cdots,$$

then the sum of the series

$$\frac{a}{p}-\frac{b}{q}+\frac{c}{r}-\frac{d}{s}+\frac{e}{t}-\cdots,$$

equals the following continued fraction [4]

$$\frac{a}{p+\frac{b:a}{\frac{aq-bp}{app}+\frac{c:b}{\frac{p^2(br-cq)}{bqq}+\frac{d:c}{\frac{q^2(cs-dr)}{cp^2r^2}+\frac{e:d}{\frac{p^2r^2(dt-es)}{dq^2s^2}+\cdots}}}}}$$

$$=\frac{a}{p+\frac{bp^2}{aq-bp+\frac{acqq}{br-cq+\frac{bdrr}{cs-dr+\frac{cess}{dt-es+\cdots}}}}}\,.$$

3. To illustrate this with some examples let us consider the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots,$$

the sum of which is $l2$[5] or $\int dx/(1+x)$ if after integration $x=1$; then

$$a=b=c=d=\cdots=1\,,\quad p=1\,,\quad q=2\,,\quad r=3\,,\quad s=4,\quad \cdots$$

and also $p=1$, $aq-bp=1$, $br-cq=1$, $cs-dr=1$, ... It follows that

$$\int\frac{dx}{1+x}=\frac{1}{1+\frac{1}{1+\frac{4}{1+\frac{9}{1+\frac{16}{1+}}}}}$$

and therefore the value of the continued fraction is $l2$.

4. Let us now consider the series

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\cdots,$$

[4] See §**70** in Section 4.1, in particular Theorem 4.5.

[5] $l2=\ln 2$, and the integration is taken from 0 to 1.

whose sum equals the area of a disc of diameter 1 or $\int dx/(1+x^2)$ with $x=1$ after integration. Then $a=b=c=d=\cdots=1$ and $p=1$, $q=3$, $r=5$, $s=7$, ... Hence we obtain Brouncker's continued fraction,

$$\int \frac{dx}{1+xx} = \frac{1}{1+}\,\frac{1}{2+}\,\frac{9}{2+}\,\frac{25}{2+}\,\frac{49}{2+\cdots},$$

discovered in relation with the quadrature of the circle.

5. Similarly, using general series one can obtain formulas converting integrals into continued fractions, where clearly $x=1$ after integration:

$$\int \frac{dx}{1+x^3} = \frac{1}{1+}\,\frac{1^2}{3+}\,\frac{4^2}{3+}\,\frac{7^2}{3+}\,\frac{10^2}{3+\cdots},$$

$$\int \frac{dx}{1+x^4} = \frac{1}{1+}\,\frac{1^2}{4+}\,\frac{5^2}{4+}\,\frac{9^2}{4+}\,\frac{13^2}{4+\cdots},$$

$$\int \frac{dx}{1+x^5} = \frac{1}{1+}\,\frac{1^2}{5+}\,\frac{6^2}{5+}\,\frac{11^2}{5+}\,\frac{16^2}{5+\cdots},$$

$$\int \frac{dx}{1+x^6} = \frac{1}{1+}\,\frac{1^2}{6+}\,\frac{7^2}{6+}\,\frac{13^2}{6+}\,\frac{19^2}{6+\cdots}.$$

6. This implies the general formula

$$\int \frac{dx}{1+x^m} = \frac{1}{1+}\,\frac{1^2}{m+}\,\frac{(m+1)^2}{m+}\,\frac{(2m+1)^2}{m+}\,\frac{(3m+1)^2}{m+\cdots},$$

with $x=1$ after integration. And if m is fractional then

$$\int \frac{dx}{1+x^{m/n}} = \frac{1}{1+}\,\frac{n}{m+}\,\frac{(m+n)^2}{m+}\,\frac{(2m+n)^2}{m+}\,\frac{(3m+n)^2}{m+\cdots}.$$

7.[6] Let us consider now the formula

$$\int \frac{x^{n-1}dx}{1+x^m},$$

which after integration and substitution of $x=1$ gives the following series:

$$\frac{1}{n} - \frac{1}{m+n} + \frac{1}{2m+n} - \frac{1}{3m+n} + \cdots$$

It follows that

$$\int \frac{x^{n-1}dx}{1+x^m} = \frac{1}{n+}\,\frac{n^2}{m+}\,\frac{(m+n)^2}{m+}\,\frac{(2m+n)^2}{m+}\,\frac{(3m+n)^2}{m+\cdots},$$

and this continued fraction is proportional to the last found.

8. Next I present the formula

$$\int \frac{x^{n-1}dx}{(1+x^m)^{\mu/\nu}},$$

[6] Compare §§3–7 in this appendix with (3.19), (3.25).

which after integration and substitution of $x = 1$ gives the following series:

$$\frac{1}{n} - \frac{1\mu}{\nu(m+n)} + \frac{\mu(\mu+\nu)}{1\times 2\nu^2(2m+n)} - \frac{\mu(\mu+\nu)(\mu+2\nu)}{1\times 2\times 3\nu^3(3m+n)} + \cdots,$$

which in turn by comparison with the general formula gives

$$a = 1, \quad b = \mu, \quad c = \mu(\mu+\nu), \quad d = \mu(\mu+\nu)(\mu+2\nu), \quad \cdots,$$
$$p = n, \quad q = \nu(m+n), \quad r = 2\nu^2(2m+n), \quad s = 6\nu^3(3m+n),$$
$$t = 24\nu^4(4m+n), \ldots$$

and also

$$\begin{aligned}
aq - bp &= \nu m + (\nu-\mu)n, \\
br - cq &= \mu\nu(3\nu-\mu)m + \mu\nu(nu-\mu)n, \\
cs - dr &= 2\mu\nu^2(\mu+\nu)(m(5\nu-2\mu)+n(\nu-\mu)), \\
dt - es &= 6\mu\nu^3(\mu+\nu)(\mu+2\nu)(m(7\nu-3\mu)+n(\nu-\mu)), \\
&\ \vdots
\end{aligned}$$

resulting after substitution in

$$\int \frac{x^{n-1}dx}{(1+x^m)^{\mu/\nu}} = \frac{1}{n} + \frac{\mu n^2}{\nu m + (\nu-\mu)n} + \frac{\nu(\mu+\nu)(m+n)^2}{(3\nu-\mu)m+(\nu-\mu)n}$$
$$+ \frac{2\nu(\mu+2\nu)(2m+n)^2}{(5\nu-2\mu)m+(\nu-\mu)n} + \frac{3\nu(\mu+3\nu)(3m+n)^2}{(7\nu-3\mu)m+(\nu-\mu)n} + \cdots$$

Let $\mu = 1$ and $\nu = 2$; then we have

$$\int \frac{x^{n-1}dx}{\sqrt{1+x^m}} = \frac{1}{n} + \frac{n^2}{2m+n} + \frac{6(m+n)^2}{5m+n}$$
$$+ \frac{20(2m+n)^2}{8m+n} + \frac{42(3m+n)^2}{11m+n} + \frac{72(4m+n)^2}{14m+n} + \cdots.$$

9.[7] However, if ν were 1 and μ were integer then the following continued fractions would result:

$$\int \frac{x^{n-1}dx}{(1+x^m)^2} = \frac{1}{n} + \frac{2n^2}{m-n} + \frac{1\times 3(m+n)^2}{m-n} + \frac{2\times 4(2m+n)^2}{m-n}$$
$$+ \frac{3\times 5(3m+n)^2}{m-n} + \frac{4\times 6(4m+n)^2}{m-n} + \cdots,$$

$$\int \frac{x^{n-1}dx}{(1+x^m)^3} = \frac{1}{n} + \frac{3n^3}{m-2n} + \frac{1\times 4(m+n)^2}{-2n} + \frac{2\times 5(2m+n)^2}{-m-2n}$$
$$+ \frac{3\times 6(3m+n)^2}{-2m-2n} + \frac{4\times 7\times(4m+n)^2}{-3m-2n} + \cdots.$$

These do not converge in view of the negative quantities present, but diverge.

[7] Compare §§8–9 with Ex. 4.6.

10. All these observations follow from the conversion of the continued fraction in §1 to the infinite series

$$A+\frac{B}{1P}-\frac{BD}{PQ}+\frac{BDF}{QR}-\frac{BDFH}{RS}+\cdots$$

After adding pairs of terms of opposite signs, this series is transformed into

$$A+\frac{BE}{1Q}+\frac{BDFI}{QS}+\frac{BDFHKN}{SV}+\cdots$$

Indeed,

$$C=P=\frac{Q-D}{E}\,,\quad G=\frac{S-HQ}{IQ}-\frac{F(Q-D)}{EQ}\,,$$
$$L=\frac{V-MS}{NS}-\frac{K(S-HQ)}{IS}\,,\;\cdots$$

Hence the infinite series

$$A+\frac{BE}{Q}+\frac{BDFI}{QS}+\frac{BDFHKN}{SV}+\cdots$$

is transformed into the continued fraction

$$A+\cfrac{B}{\cfrac{Q-D}{E}}+\cfrac{D}{E}+\cfrac{F}{\cfrac{E(S-HQ)-FI(Q-D)}{EIQ}}$$
$$+\cfrac{H}{I}+\cfrac{K}{\cfrac{I(V-MS)-KN(S-HQ)}{INS}}+\cdots,$$

which after elimination of the denominator fractions transforms into

$$A+\frac{BE}{Q-D}+\frac{D}{1}+\frac{FIQ}{E(S-HQ)-FI(Q-D)}+\frac{EHQ}{1}$$
$$+\frac{KNS}{I(V-MS)-KN(S-HQ)}+\frac{IMS}{1}+\cdots.$$

11. If now in turn we consider the series

$$\frac{a}{p}+\frac{b}{q}+\frac{c}{r}+\frac{d}{s}+\frac{e}{t}+\cdots$$

then comparing it with the preceding one, we obtain

$$Q=p\,,\quad S=\frac{q}{p}\,,\quad V=\frac{pr}{q}\,,\quad X=\frac{qs}{pr}\,,\quad Z=\frac{prt}{qs}\,,\;\cdots$$

and also

$$E=\frac{a}{B}\,,\quad I=\frac{b}{BDF}\,,\quad N=\frac{c}{BDFHK}\,,\;\cdots$$

These values turn the given series into

$$\frac{a}{p-D} + \frac{D}{1} + \frac{bp:1}{Da(q/p-Hp)-b(p-D)} + \frac{DHap:1}{1}$$
$$+ \frac{cq:p}{Hb(pr/q-Mq/p)-c(p/q-Hp)} + \frac{HMbq:p}{1} + \frac{dp:q}{Mcqs/pr-\cdots},$$

which contains many quantities that do not enter the initial series.

12. Meanwhile, let the series from §2,

$$\frac{b}{p} - \frac{bd}{pq} + \frac{bdf}{qr} - \frac{bdfh}{rs} + \dots,$$

be equal to the continued fraction

$$\frac{b}{p} + \frac{dp}{q-d} + \frac{fpq}{r-fp} + \frac{hqr}{s-hq} + \frac{krs}{\dots}.$$

If we make this series equal to the preceding one then we obtain

$$b = BE, \quad d = \frac{-DFI}{E}, \quad f = \frac{-HKN}{I}, \dots,$$
$$p = Q, \quad q = S, \quad r = V, \quad s = X, \dots,$$

hence the continued fraction from the preceding paragraph can be transformed into the continued fraction

$$A + \frac{BE}{Q} - \frac{DFIQ}{ES+DFI} - \frac{EHKNQS}{IV+HKNQ} - \frac{IMORSV}{NX+MORS+\cdots},$$

in which a progression law can easily be detected.

13. The series

$$A + \frac{B}{P} - \frac{BD}{PQ} + \frac{BDF}{QR} - \cdots,$$

which we obtained first from a continued fraction, can be easily transformed to the form

$$A + \frac{B}{2P} + \frac{BE}{2Q} - \frac{BDG}{2PR} + \frac{BDFI}{2QS} - \frac{BDFHL}{2RT} + \cdots,$$

which in turn, if we express the quantities C, E, G, I, ... in terms of those remaining, with the help of the equations given can be transformed into

$$A + \frac{B}{2P} + \frac{B(Q-D)}{2PQ} - \frac{BD(R-FP)}{2PQR} + \frac{BDF(S-HQ)}{2QRS} - \cdots,$$

and therefore equals the continued fraction

$$A + \frac{B}{P} + \frac{DP}{Q-D} + \frac{FPQ}{R-FP} + \frac{HQR}{S-HQ+\cdots}.$$

14.[8] All these facts follow immediately from considerations of continued fractions, and many other similar observations are mentioned in my previous dissertation[9]. Thus, I now continue the application of these observations to other cases and also present several methods for evaluating continued fractions, which for such fractions assign values by integration. To begin with, since Brouncker's quadrature of the circle was not only proved but, as it would seem, was a priori invented, I will study other expressions of the type similar to that obtained either by Brouncker or by Wallis; such expressions were considered by Wallis but it was not clearly mentioned whether Brouncker did everything himself or only discovered the part related to the quadrature of the circle. Later in fact I prove how one can find these remaining fractions, which were discovered using a high degree of intuition, on the basis of quite different principles and also show how one can find many fractions of this type.

15. These formulas, which Wallis used, lead to the assumption that the product of the two continued fractions in the following expression must be equal to a^2:

$$a^2 = \left(a-1+\frac{1}{2(a-1)}+\frac{9}{3(a-1)}+\frac{25}{2(a-1)+\cdots}\right) \times \left(a+1+\frac{1}{2(a+1)}+\frac{9}{3(a+1)}+\frac{25}{2(a+1)+\cdots}\right).$$

Similarly

$$(a+2)^2 = \left(a+1+\frac{1}{2(a+1)}+\frac{9}{3(a+1)}+\frac{25}{2(a+1)+\cdots}\right) \times \left(a+3+\frac{1}{2(a+3)}+\frac{9}{3(a+3)}+\frac{25}{2(a+3)+\cdots}\right).$$

Repeating this method for an infinite product, we obtain

$$a\frac{a(a+4)(a+4)(a+8)(a+8)(a+12)(a+12)\cdots}{(a+2)(a+2)(a+6)(a+6)(a+10)(a+10)(a+14)\cdots} = a-1+\frac{1}{2(a-1)}+\frac{9}{3(a-1)}+\frac{25}{2(a-1)+\cdots}.$$

16. If we now investigate the constant obtained from the infinite product by the method given in the previous dissertation[10], then we find that

$$\frac{a(a+4)(a+4)(a+8)(a+8)\cdots}{(a+2)(a+2)(a+6)(a+6)\cdots} = \frac{\int x^{a+1}\,dx : \sqrt{1-x^4}}{\int x^{a-1}\,dx : \sqrt{1-x^4}}.$$

Therefore the value of the continued fraction

$$a-1+\frac{1}{2(a-1)}+\frac{9}{3(a-1)}+\frac{25}{2(a-1)+\cdots}$$

equals the expression

$$a\frac{\int x^{a+1}\,dx : \sqrt{1-x^4}}{\int x^{a-1}\,dx : \sqrt{1-x^4}},$$

where one must put $x = 1$ after integration.

8 For §§14–17 see §§**60** and **61** in Section 3.2.
9 Euler (1750a).
10 Euler (1750a).

17. This theorem, which rather explicitly presents values of the continued fraction as integral formulas, the more so deserves to be mentioned as it is less obvious. Namely, although the case $a = 2$ has been considered already in presenting the circle quadrature, other cases do not follow from it. Indeed, if by the method described above one converts this continued fraction into a series, then complicated formulas show that its sum can hardly be evaluated, except for $a = 2$. Therefore for quite a long time I have undertaken great efforts to prove this theorem, so that its proof a priori can be related to this function; this research is in my opinion more difficult, but I believe it could result in great benefits. While any such research has so far been condemned to failure, I regretted most of all that Brouncker's method has been nowhere present and most likely has sunk into oblivion.

18.[11] As far as it is known from Wallis' considerations, Brouncker arrived at his formulas by interpolating the series

$$\frac{1}{2} + \frac{1 \times 3}{2 \times 4} + \frac{1 \times 3 \times 5}{2 \times 4 \times 6} + \cdots ,$$

whose intermediate terms, as Wallis showed, are present in the quadrature of the circle. This even gives the beginning of Brouncker's interpolation. The single fractions $1/2$, $3/4$, $5/6$, $\cdots$ were factored into binary multipliers, which in their totality constituted a continued progression. Thus if the following is true,

$$AB = \frac{1}{2}, \quad CD = \frac{3}{4}, \quad EF = \frac{5}{6}, \quad GH = \frac{7}{8}, \ \cdots$$

and the numbers A, B, C, D, E, ... represent a continued progression then the series would take the form

$$AB + ABCD + ABCDEF + \cdots ,$$

which being reduced to this form would interpolate itself; the factor with index $1/2$ is A, the factor with index $3/2$ is ABC and so on[12]. In view of what has been said, this interpolation brings back single fractions into binary multipliers.

19. It follows from the continuity law that

$$BC = \frac{2}{3}, \quad DE = \frac{4}{5}, \quad FG = \frac{6}{7}, \ \ldots .$$

Then

$$A = \frac{1}{2B}, \quad B = \frac{2}{3C}, \quad C = \frac{3}{4BD}, \quad D = \frac{4}{5E}, \ \ldots ,$$

which immediately implies that

$$A = \frac{1 \times 3 \times 3 \times 5 \times 5 \times 7 \times \cdots}{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times \cdots} ,$$

which is nothing other than the very formula invented by Wallis, the formula which represents the quadrature of the circle and does not at all resemble Brouncker's expression. Because this formula can so easily be discovered by the method of interpolation, it is especially surprising that Brouncker moving in the same direction arrived at a completely different expression; it was not seen that a way remains leading to a continued fraction. It is hardly possible that Brouncker actually wanted to develop A into a continued fraction; rather, by some chance and almost against his wish he discovered this new and completely different method. In his time continued

[11] In §§18–19 Euler explains Wallis' interpolation; see §**59** in Section 3.2 and §**71** in Section 4.2.
[12] See the discussion relating to (3.14).

fractions were completely unknown, and owing to this lucky chance, to everybody's benefit, for the first time their study could be expanded. A new method could be developed from this, leading to similar continued fractions, although at present it is not clear how this will be done.

20. Although I made numerous unsuccessful attempts to find this method, I did however come across a different method of realizing the interpolation of such a series by continued fractions; a method that gave me expressions completely different from those of Brouncker. Incidentally, I hope that it is not without interest to justify this method; with its help one can find continued fractions whose values could also be found by quadrature. When, after that, using a different method I tried to find values of arbitrary continued fractions that could be expressed by quadrature, I obtained beautiful formulas relating integrals, at least for the case when after integration the variable is substituted by a definite value; similar equalities for infinite products of constant terms are presented in my previous dissertation[13].

21.[14] To explain my method of interpolation let us take a widely known series,

$$\frac{p}{p+2q}+\frac{p(p+2r)}{(p+2q)(p+2q+2r)}+\frac{p(p+2r)(p+4r)}{(p+2q)(p+2q+2r)(p+2q+4r)}+\cdots$$

where the term with index $1/2$ is A, the term with index $3/2$ is ABC, the term with index $5/2$ is $ABCDE$ etc. It follows that

$$AB=\frac{p}{p+2q}, \quad CD=\frac{p+2r}{p+2q+2r}, \quad EF=\frac{p+4r}{p+2q+4r}, \cdots$$

and by the law of continuity

$$BC=\frac{p+r}{p+2q+r}, \quad DE=\frac{p+3r}{p+2q+3r}, \quad FG=\frac{p+5r}{p+2q+5r}$$

and so on.

22. To eliminate fractions we put

$$A=\frac{a}{p+2q-r}, \qquad B=\frac{b}{p+2q},$$
$$C=\frac{c}{p+2q+r}, \quad CD=\frac{d}{p+2q+2r}\ \cdots,$$

which implies that

$$ab=(p+2q-r)p, \quad bc=(p+2q)(p+r),$$
$$cd=(p+2q+r)(p+2r), \quad cd=(p+2q+2r)(p+3r), \ \cdots$$

Now we have

$$a=m-r+\frac{1}{\alpha}, \quad b=m+\frac{1}{\beta}, \quad c=m+r+\frac{1}{\gamma},$$
$$d=m+2r+\frac{\delta}{\gamma}, \quad e=m+3r+\frac{1}{\varepsilon}\ \cdots,$$

where the integer parts of the substitutions make an arithmetic progression with a constant difference r, which is exactly what is required by the progression of these multipliers themselves.

13 Euler (1750a).
14 For §§21–26 see Ex. 4.12 and Ex. 4.19.

Consequently these relations are expressed by a sequence of equalities where for brevity we set

$$p^2+2pq-pr-m^2+mr=P\ ,$$
$$2r(p+q-m)=Q\ ,$$
$$P\alpha\beta-(m-r)\alpha=m\beta+1\ ,$$
$$(P+Q)\beta\gamma-m\beta=(m+r)\gamma+1\ ,$$
$$(P+2Q)\gamma\delta-(m+r)\gamma=(m+2r)\delta+1\ ,$$
$$(P+3Q)\delta\varepsilon-(m+2r)\delta=(m+3r)\varepsilon+1\ ,$$
$$\vdots$$

23. These equalities imply that for the quantities α, β, γ, δ etc. the following identities hold:

$$\alpha=\frac{m\beta+1}{P\beta-(m-r)}=\frac{m}{P}+\frac{p(p+2q-r)\ :\ P^2}{-(m-r)\ :\ P+\beta}\ ,$$
$$\beta=\frac{(m+r)\gamma+1}{(P+Q)\gamma-m}=\frac{m+r}{P+Q}+\frac{(p+r)(p+2q):(P+Q)^2}{-m\ :\ (P+Q)+\gamma}\ ,$$
$$\gamma=\frac{(m+2r)\delta+1}{(P+2Q)\delta-(m+r)}=\frac{m+2r}{P+2Q}+\frac{(p+2r)(p+2q+r)\ :\ (P+2Q)^2}{-(m+r)\ :\ (P+2Q)+\delta}\ ,$$
$$\vdots$$

Let us put for the sake of brevity

$$p^2+2pq-mp-mq+qr=R\ ,$$
$$pr+qr-mr=S$$

and express the values of under consideration in terms of these quantities. Then we obtain the continued fraction

$$\alpha=\frac{m}{P}+\frac{p(p+2q-r)\ :\ P^2}{2rR\ :\ P(P+Q)}{}_{+}\frac{(p+r)(p+2q)\ :\ (P+Q)^2}{2r(R+S)\ :\ (P+Q)(P+2Q)}$$
$${}_{+}\frac{(p+2r)(p+2q+r)\ :\ (P+2Q)^2}{2r(R+2S)\ :\ (P+2Q)(P+3Q)}{}_{+\cdots}\ .$$

24. Therefore let $a=m-r+1/\alpha$; then

$$a=m-r+\frac{P}{m}{}_{+}\frac{p(p+2q-r)(P+Q)}{2rR}{}_{+}\frac{(p+r)(p+2q)P(P+2Q)}{2r(R+S)}$$
$${}_{+}\frac{(p+2r)(p+2q+r)(P+Q)(P+3Q)}{2r(R+2S)}{}_{+\cdots}\ .$$

From here the term of the proposed series,

$$\frac{p}{p+2q}+\frac{p(p+2r)}{(p+2q)(p+2q+2r)}+\frac{p(p+2r)(p+4r)}{(p+2q)(p+2q+2r)(p+2q+4r)}+\cdots\ ,$$

with index $1/2$ is

$$A = \frac{a}{p+2q-r}\,.$$

Since the general term of this series, with index n, is

$$\frac{\int y^{p+2q-1}\,dy\,(1-y^{2r})^{n-1}}{\int y^{p-1}\,dy\,(1-y^{2r})^{n-1}}\,,$$

the continued fraction under consideration is given by $a = (p+2q-r)\dfrac{\int y^{p+2q-1}\,dy : \sqrt{1-y^{2r}}}{\int y^{p-1}\,dy : \sqrt{1-y^{2r}}}$, where $y=1$ after integration.

25. If in our continued fractions there is an arbitrary letter m, then we have infinitely many continued fractions with the same value, of which the most remarkable may be studied. First let $m-r=p$ or $m=p+r$; then $P=2p(q-r)$, $Q=2r(q-r)$, $R=p(q-r)$ and $S=r(q-r)$, which implies that

$$a = p + \frac{2p(q-r)}{p+r} {\atop +} \frac{(p+2q-r)(p+r)}{r} {\atop +} \frac{(p+2q)(p+2r)}{r} {\atop +} \frac{(p+2q+r)(p+3r)}{r} {\atop +\cdots}\,.$$

If we take $r>q$ to avoid negative fractions this can be written as

$$a = \frac{p}{1} {\atop +} \frac{2(r-q)}{p+2q-r} {\atop +} \frac{(p+2q-r)(p+r)}{r} {\atop +} \frac{(p+2q)(p+2r)}{r} {\atop +} \frac{(p+2q+r)(p+3r)}{r} {\atop +\cdots}\,.$$

26. Now let $m=p+2q$; then Q and S disappear and we have $P=q(r-q)$ and $R=q(r-q)$. This implies that

$$a = p+q-r + \frac{q(r-q)}{p+q} {\atop +} \frac{p(p+2q-r)}{2r} {\atop +} \frac{(p+r)(p+2q)}{2r} {\atop +} \frac{(p+2r)(p+2q+r)}{2r} {\atop +\cdots}\,.$$

It follows that this continued fraction exactly equals the previous ones although they are of different types.

27. Assuming that $m=p+2q$, we obtain

$$P = 2q(r-p-2q) = -2q(p+2q-r)\,,$$

$$Q=-2qr\,,\quad R=-q(p+2q-r)\,,\quad S=-qr\,.$$

This gives the following continued fraction:

$$a = p+2q-r - \frac{2q(p+2q-r)}{p+2q} {\atop +} \frac{p(p+2q)}{r} {\atop +} \frac{(p+r)(p+2q+r)}{r} {\atop +} \frac{(p+2r)(p+2q+2r)}{r} {\atop +\cdots}\,.$$

In this way infinitely many continued fractions appear, all having the same value a, which can be found from the formula[15]

$$a = (p+2q-r)\frac{\int y^{p+2q-1}\,dy : \sqrt{1-y^{2r}}}{\int y^{p-1}\,dy : \sqrt{1-y^{2r}}}$$
$$= (p+2q-2r)\frac{\int y^{p+2q-2r-1}\,dy : \sqrt{1-y^{2r}}}{\int y^{p-1}\,dy : \sqrt{1-y^{2r}}}\,.$$

28. Before going further let us consider a special case. Let $r = 2q$; then

$$a = p\frac{\int y^{p+2q-1}\,dy : \sqrt{1-y^{4q}}}{\int y^{p-1}\,dy : \sqrt{1-y^{4q}}}\,.$$

Then we obtain the quantities

$$P = p^2+2mq-m^2\,, \quad Q = 4q(p+q-m)\,,$$
$$R = p^2+2pq+2qq-mp-mq\,, \quad S = 2q(p+q-m)\,,$$

which generate the following formula:

$$a = m-2q+\frac{P}{m+}\,\frac{p^2(P+Q)}{4qR+}\,\frac{(p+2q)^2P(P+2Q)}{4q(R+S)}$$
$$+\frac{(p+4q)^2(P+Q)(P+3Q)}{4q(R+2S)+\cdots}\,.$$

29. If we replace m by other values then we obtain other continued fractions. First, we have

$$a = p-\frac{2pq}{p+2q+}\,\frac{p(p+2q)}{2q+}\,\frac{(p+2q)(p+4q)}{2q+}\,\frac{(p+4q)(p+6q)}{2q+\cdots}\,.$$

or, instead of this fraction, for $r > q$,

$$a = \frac{p}{1+}\,\frac{2q}{p+}\,\frac{p(p+2q)}{2q+}\,\frac{(p+2q)(p+4q)}{2q+}\,\frac{(p+4q)(p+6q)}{2q+\cdots}\,.$$

Next, using the data of §26 we obtain the following continued fraction:

$$a = p-q+\frac{qq}{p+q+}\,\frac{pp}{4q+}\,\frac{(p+2q)^2}{4q+}\,\frac{(p+4q)^2}{4q+\cdots}\,.$$

In the third instance, §27 represents the following continued fraction:

$$a = p-\frac{2pq}{p+2q+}\,\frac{p(p+2q)}{2q+}\,\frac{(p+2q)(p+4q)}{2q+}\,\frac{(p+4q)(p+6q)}{2q+\cdots}\,.$$

which coincides with the first fraction presented in this paragraph so that in the case $r = 2q$ there are two such simple fractions.

[15] By §21, the infinite product a converts to integrals by Euler (1750a); see (4.32).

30. Let us assume that $q = p = 1$, which implies that

$$a = \frac{\int yy\,dy : \sqrt{1-y^4}}{\int dy : \sqrt{1-y^4}} .$$

First we have

$$a = 1 - \frac{2}{3+}\,\frac{1\times 3}{2}\,\frac{3\times 5}{+\;2}\,\frac{5\times 7}{+\;2}\,{+\cdots} .$$

Next,

$$a = \frac{1}{2+}\,\frac{1}{4+}\,\frac{9}{4+}\,\frac{25}{4+}\,\frac{49}{4+\cdots} .$$

It follows that

$$a = \frac{\int dy : \sqrt{1-y^4}}{\int yy\,dy : \sqrt{1-y^4}} = 2 + \frac{1}{4+}\,\frac{9}{4+}\,\frac{25}{4+}\,\frac{49}{4+\cdots} ,$$

which is a special case of the expression obtained in §16; therefore this formula, although not completely proved, is confirmed in one more case. After substituting in it $a = 3$, as a corollary we obtain

$$3\frac{\int x^4\,dx : \sqrt{1-x^4}}{\int xx\,dx : \sqrt{1-x^4}} = \frac{\int dx : \sqrt{1-x^4}}{\int xx\,dx : \sqrt{1-x^4}} = 2 + \frac{1}{4+}\,\frac{9}{4+}\,\frac{25}{4+}\,\frac{49}{4+\cdots} ,$$

so that now there is no doubt that this formula from §16 may be assumed valid for the cases $a = 2$ and $a = 3$, but soon this fact will be established in the widest sense.

31. Let $q = 1/2$ and $p = 1$; under the condition $r = 2q = 1$ we obtain

$$a = \frac{\int y\,dy : \sqrt{1-y^2}}{\int dy : \sqrt{1-y^2}} = \frac{2}{\pi} ,$$

where π denotes the perimeter of a circle of diameter 1. In general, we obtain in consequence

$$P = 1 + m - m^2 , \quad Q = 3 - 2m ,$$

$$R = \frac{5-3m}{2} , \quad S = \frac{3-2m}{2} ,$$

so that

$$a = m - 1 + \frac{1+m-m^2}{m}\,\frac{1^2(4-m-m^2)}{+\;5-3m}$$
$$+\,\frac{2^2(1+m-m^2)(7-3m-m^2)}{8-5m}\,\frac{3^2(4-m-m^2)(10-5m-m^2)}{+\;11-7m}\,{+\cdots}$$

In two special cases we obtain

$$\frac{\pi}{2} = \frac{1}{1-}\,\frac{1}{2+}\,\frac{1\times 2}{1+}\,\frac{2\times 3}{1+}\,\frac{3\times 4}{1+\cdots} = 1 + \frac{1}{1+}\,\frac{1\times 2}{1+}\,\frac{2\times 3}{1+}\,\frac{3\times 4}{1+\cdots} ,$$

$$\frac{\pi}{2} = \frac{1}{1:2+}\,\frac{1:4}{3:2+}\,\frac{1^2}{2+}\,\frac{2^2}{2+}\,\frac{3^2}{2+\cdots} = 2 = \frac{1}{2+}\,\frac{1^2}{2+}\,\frac{2^2}{2+}\,\frac{3^2}{2+\cdots} .$$

32. To make clear applications of these formulas to interpolation [by continued fractions], let us consider the series

$$\frac{2}{1}+\frac{2\times 4}{1\times 3}+\frac{2\times 4\times 6}{1\times 3\times 5}+\cdots,$$

whose term with index $1/2$ is to be found; let it be A. Hence

$$p=2, \quad r=1, \quad q=-\frac{1}{2}.$$

we have assumed that

$$A=\frac{a}{p+2q-r},$$

which implies that

$$A=\frac{a}{0},$$

from which the fact that these formulas cannot be applied if $p+2q-r=0$ is evident. However, this process could be stopped at the term with index $3/2$, which if the latter were equal to Z would give $A=2Z/3$; but $Z/2$ is the term of index $1/2$ for the series

$$\frac{4}{3}+\frac{4\times 6}{3\times 5}+\frac{4\times 6\times 8}{3\times 5\times 7}+\cdots,$$

which when compared with the general series gives

$$p=4, \quad r=1, \quad q=-\frac{1}{2},$$

so that we obtain

$$Z=\frac{2\int y^2\,dy : \sqrt{1-y^2}}{\int y^3\,dy : \sqrt{1-y^2}}=\frac{3\int dy : \sqrt{1-y^2}}{2\int y\,dy : \sqrt{1-y^2}}=\frac{3}{4}\pi,$$

and $A=\pi/2$. Consequently, according to §24 let

$$Z=a \quad \text{and} \quad A=\frac{2}{3}Z=\frac{2}{3}a;$$

then we obtain finally that

$$P=8+m-m^2, \quad Q=7-2m,$$

$$R=\frac{23-7m}{2}, \quad S=\frac{7-2m}{2},$$

$$\begin{aligned}A&=\frac{2}{3}a=\frac{\pi}{2}\\&=\frac{2(m-1)}{3}+\cfrac{2(8+m-m^2)}{3m+\cfrac{2\times 4\times 3(15-m-m^2)}{23-7m+\cfrac{3\times 5(8+m-m^2)(22-3m-m^2)}{90-9m+\cfrac{4\times 6(15-m-m^2)(29-5m-m^2)}{37-11m+\cdots}}}}\end{aligned}$$

33. In the special cases under study we have

$$a = \frac{3\pi}{4} = 4 - \frac{12}{5+}\,\frac{2\times 5}{1+}\,\frac{3\times 6}{1+}\,\frac{4\times 7}{1+\cdots}$$
$$= \frac{4}{1+}\,\frac{1\times 3}{2+}\,\frac{2\times 5}{1+}\,\frac{3\times 6}{1+}\,\frac{4\times 7}{1+\cdots},$$

or

$$\frac{3}{4}\pi = 1 + \frac{3}{1+}\,\frac{1\times 4}{1+}\,\frac{2\times 5}{1+}\,\frac{3\times 6}{1+}\,\frac{4\times 7}{1+\cdots}.$$

Similarly, using what was said in §26, we obtain

$$a = \frac{3}{4}\pi = \frac{5}{2} - \frac{3:4}{7:2+}\,\frac{2\times 4}{2+}\,\frac{3\times 5}{2+}\,\frac{4\times 6}{2+}\,\frac{5\times 7}{2+\cdots}$$
$$= 2 + \frac{1}{2+}\,\frac{1\times 3}{2+}\,\frac{2\times 4}{1+}\,\frac{3\times 5}{1+}\,\frac{4\times 6}{1+\cdots}.$$

Finally, the case presented in §27 gives

$$a = \frac{3}{4}\pi = 2 + \frac{2}{3+}\,\frac{3\times 4}{1+}\,\frac{4\times 5}{1+}\,\frac{5\times 6}{1+\cdots}$$

or

$$\frac{\pi}{2} = 1 + \frac{1}{1+}\,\frac{1\times 2}{1+}\,\frac{2\times 3}{1+}\,\frac{3\times 4}{1+}\,\frac{4\times 5}{1+\cdots},$$

which coincides with the expression presented in §31.

34. This method of interpolation results in an infinite number of continued fractions, whose values can be given in terms of quadratures of curves or integral formulas. If these fractions are irregular at their beginning then the first terms responsible for this irregularity are dropped to obtain continued fractions whose values are given as described above. So, from §25 and the assumption that $p+2q-r=f$ and $p+r=h$ we obtain the formula

$$r + \frac{fh}{r+}\,\frac{(f+r)(h+r)}{r+}\,\frac{(f+2r)(h+2r)}{r+\cdots}$$
$$= \frac{h(f-r)\int y^{h+r-1}\,dy : \sqrt{1-y^{2r}} - f(h-r)\int y^{f+r-1}\,dy : \sqrt{1-y^{2r}}}{f\int y^{f+r-1}\,dy : \sqrt{1-y^{2r}} - h\int y^{h+r-1}\,dy : \sqrt{1-y^{2r}}},$$

and this equality is always valid except for $f=h$. However, in the case $f=h$ it can be assumed that $f=h+dw$ and we then obtain

$$\frac{\int y^{h+r+dw-1}\,dy : \sqrt{1-y^{2r}}}{\int y^{h+r-1}\,dy : \sqrt{1-y^{2r}}} = 1 - rdw\int \frac{dx}{x^{r+1}}\int \frac{x^{h+2r-1}\,dx}{1-x^{2r}}$$

on substituting $x=1$ after integration. It follows that

$$r+\cfrac{hh}{r+\cfrac{(h+r)^2}{r+\cfrac{(h+2r)^2}{r+\cdots}}}$$

$$=\frac{r+hr(h-r)\int dx\,:\,x^{r+1}\int x^{h+2r-1}\,dx\,:\,(1-x^{2r})}{1-hr\int dx\,:\,x^{r+1}\int x^{h+2r-1}\,dx\,:\,(1-x^{2r})}$$

$$=\frac{r(h-r)^2\int dx\,:\,x^{r+1}\int x^{h+2r-1}\,dx\,:\,(1-x^{2r})}{1-r(h-r)\int dx\,:\,x^{r+1}\int x^{h+2r-1}\,dx\,:\,(1-x^{2r})}\,.$$

The definition of an integral implies that

$$\int\frac{dx}{x^{r+1}}\int\frac{x^{h+2r-1}\,dx}{1-x^{2r}}=\frac{-1}{rx^r}\int\frac{x^{h+2r-1}\,dx}{1-x^{2r}}+\frac{1}{r}\int\frac{x^{h+r-1}\,dx}{1-x^{2r}}$$

$$=\frac{1}{r}\int\frac{x^{h+r-1}\,dx}{1+x^r}$$

after substituting $x=1$. Therefore the following holds:

$$r+\cfrac{hh}{r+\cfrac{h(h+r)^2}{r+\cdots}}=\frac{r+h(h-r)\int x^{h+r-1}\,dx\,:\,(1+x^r)}{1-h\int x^{h+r-1}\,dx\,:\,(1+x^r)}$$

$$=\frac{1-(h-r)\int x^{h-1}\,dx\,:\,(1+x^r)}{\int x^{h-1}\,dx\,:\,(1+x^r)}\,.$$

This is the form which coincides with that given in §7.

35. In the same way as §26, under the assumption $p=f$ and $p+2q-r=h$ we have

$$2r+\cfrac{fh}{2r+\cfrac{(f+r)(h+r)}{2r+\cfrac{(f+2r)(h+2r)}{2r+\cdots}}}$$

$$=\frac{2(r-f)(r-h)\int y^{f-1}\,dy\,:\,(\sqrt{1-y^{2r}})-h(f+h-3r)\int y^{h+r-1}\,dy\,:\,(\sqrt{1-y^{2r}})}{2h\int y^{h+r-1}\,dy\,:\,(\sqrt{1-y^{2r}})-(f+h-r)\int y^{f-1}\,dy\,:\,(\sqrt{1-y^{2r}})}\,.$$

Since this formula does not change on exchanging f and h, it is obvious that

$$\frac{h\int y^{h+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{f-1}\,dy\,:\,\sqrt{1-y^{2r}}}=\frac{f\int y^{f+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{h-1}\,dy\,:\,\sqrt{1-y^{2r}}},$$

substituting $y=1$ after integrating. In fact this theorem is already included in those presented in my previous dissertation[16], which deals with infinite products of constant terms; there I found and proved many other theorems of this type.

36. The case $f=h+r$ deserves equal attention; then both the numerator and denominator vanish. As before I put $f=h+r+dw$ and by calculation obtain

$$2r+\cfrac{h(h+r)}{2r+\cfrac{(h+r)(h+2r)}{2r+\cfrac{(h+2r)(h+3r)}{2r+\cdots}}}$$

$$=\frac{h+2h(r-h)\int x^{h-r}\,dx\,:\,(1+x^r)}{-1+2h\int x^{h-r}\,dx\,:\,(1+x^r)}\,.$$

[16] Euler (1750a).

Therefore, if one puts $h = r = 1$ then

$$2+\frac{1\times 2}{2}\,\frac{}{+}\,\frac{2\times 3}{2}\,\frac{}{+}\,\frac{3\times 4}{2}\,\frac{}{+}\,\frac{4\times 5}{2}\,\frac{}{+\cdots} = \frac{1}{2\,l2-1}\,.$$

However, if the equality of §27 is treated in the same way then exactly the same formula is obtained.

37. After the statement of these results, to which the interpolation of series by continued fractions is reduced, I returned to Brouncker's expressions as well as to the native method not only for obtaining them but also for obtaining others of this type, which, it appears, Brouncker also used. Most importantly, the continued fractions invented by Brouncker, for which the values of the quantities A, B, C, D, if one follows the method presented, depend on each other in such a way that they can easily be compared, by Brouncker's method become different from each other, so that their mutual relationship is not obvious. This very difference finally led me to the invention of another method worthy of discovery.

38. Before I explain the interpolation method, I will state the following well-known lemma. Let there exist infinitely many letters α, β, γ, δ, ε etc., which are related in such a way that the following identities hold:

$$\begin{aligned}
\alpha\beta - m\alpha - n\beta - \kappa &= 0\,,\\
\beta\gamma - (m+s)\beta - (n+s)\gamma - \kappa &= 0\,,\\
\gamma\delta - (m+2s)\gamma - (n+2s)\delta - \kappa &= 0\,,\\
\delta\varepsilon - (m+3s)\delta - (n+3s)\varepsilon - \kappa &= 0\,,\\
&\vdots
\end{aligned}$$

If the letters α, β, γ, δ, ε etc. take the values

$$\begin{aligned}
\alpha &= m+n-s+\frac{ss-ms+ns+\kappa}{a}\,,\\
\beta &= m+n+s+\frac{ss-ms+ns+\kappa}{b}\,,\\
\gamma &= m+n+3s+\frac{ss-ms+ns+\kappa}{c}\,,\\
\delta &= m+n+5s+\frac{ss-ms+ns+\kappa}{d}\,,\\
&\vdots
\end{aligned}$$

then the above-mentioned equalities turn into the identities

$$\begin{aligned}
ab-(m-s)a-(n+s)b-ss+ms-ns-\kappa &= 0\,,\\
bc-mb-(n+2s)c-ss+ms-ns-\kappa &= 0\,,\\
cd-(m+s)c-(n+3s)d-ss+ms-ns-\kappa &= 0\,,\\
de-(m+s)d-(n+4s)e-ss+ms-ns-\kappa &= 0\,,\\
&\vdots
\end{aligned}$$

These substitutions are allowable since under them the formulas remain invariant.

39. If now in the same way one transforms these last equations into similar ones by corresponding substitutions, then one obtains

$$a = m+n-s+\frac{4ss-2ms+2ns+\kappa}{a_1},$$
$$b = m+n+s+\frac{4ss-2ms+2ns+\kappa}{b_1},$$
$$c = m+n+3s+\frac{4ss-2ms+2ns+\kappa}{c_1},$$
$$d = m+n+5s+\frac{4ss-2ms+2ns+\kappa}{d_1},$$
$$\vdots$$

After substitution, the relations at the end of §38 are reduced to the equalities

$$a_1b_1-(m-2s)a_1-(n+2s)b_1-4ss+2ms-2ns-\kappa=0,$$
$$b_1c_1-(m-s)b_1-(n+3s)c_1-4ss+2ms-2ns-\kappa=0,$$
$$c_1d_1-mc_1-(n+4s)d_1-4ss+2ms-2ns-\kappa=0,$$
$$d_1e_1-(m+s)d_1-(n+5s)e_1-4ss+2ms-2ns-\kappa=0,$$
$$\vdots$$

40. Continuing this process, we obtain

$$a_1 = m+n-s+\frac{9ss-3ms+3ns+\kappa}{a_2},$$
$$b_1 = m+n+s+\frac{9ss-3ms+3ns+\kappa}{b_2},$$
$$c_1 = m+n+3s+\frac{9ss-3ms+3ns+\kappa}{c_2},$$
$$\vdots$$

From these substitutions we obtain the equalities[17]

$$a_2b_2-(m-3s)a_2-(n+3s)b_2-9ss+3ms-3ns-\kappa=0,$$
$$b_2c_2-(m-2s)b_2-(n+4s)c_1-9ss+3ms-3ns-\kappa=0,$$
$$c_2d_2-(m-s)c_2-(n+5s)d_2-9ss+3ms-3ns-\kappa=0,$$
$$\vdots$$

[17] Compare the calculations of §§38–40 with the calculations at the start of Section 4.5.

41. If now one continues these substitutions to infinity then the values of the quantities α, β, γ etc. are given by the continued fractions

$$\alpha = m+n-s+\frac{ss-ms+ns+\kappa}{m+n-s}+\frac{4ss-2ms+2ns+\kappa}{m+n-s}+\frac{9ss-3ms+3ns+\kappa}{m+n-s}+\frac{16ss-4ms+4ns+\kappa}{m+n-s}+\cdots,$$

$$\beta = m+n+s+\frac{ss-ms+ns+\kappa}{m+n+s}+\frac{4ss-2ms+2ns+\kappa}{m+n+s}+\frac{9ss-3ms+3ns+\kappa}{m+n+s}+\frac{16ss-4ms+4ns+\kappa}{m+n+s}+\cdots,$$

$$\gamma = m+n+3s+\frac{ss-ms+ns+\kappa}{m+n+3s}+\frac{4ss-2ms+2ns+\kappa}{m+n+3s}+\frac{9ss-3ms+3ns+\kappa}{m+n+3s}+\frac{16ss-4ms+4ns+\kappa}{m+n+3s}+\cdots,$$

and these continued fractions are very similar to those of Brouncker, although the following continued fractions do not occur in the latter[18].

42. To make evident the use of these formulas in interpolation let us consider the series

$$\frac{p}{p+2q}+\frac{p(p+2r)}{(p+2q)(p+2q+2r)}+\frac{p(p+2r)(p+4r)}{(p+2q)(p+2q+2r)(p+2q+4r)}+\cdots$$

where the term with index $1/2$ is A, the term with index $3/2$ is ABC, the term with index $5/2$ is $ABCDE$ etc[19]. It follows that

$$AB=\frac{p}{p+2q}, \quad CD=\frac{p+2r}{p+2q+2r}, \quad EF=\frac{p+4r}{p+2q+4r}\ \cdots,$$

Now we assume that

$$A=\frac{a}{p+2q-r}, \quad B=\frac{b}{p+2q},$$
$$C=\frac{c}{p+2q+r}, \quad CD=\frac{d}{p+2q+2r}\ \cdots,$$

which implies that

$$ab=p(p+2q-r), \quad bc=(p+r)(p+2q),$$
$$cd=(p+2r)(p+2q+r), \quad cd=(p+3r)(p+2q+2r),\cdots$$

[18] This shows how close Euler was to Brouncker's original proof; see Theorem, 4.16.
[19] See the discussion centred on (3.14).

From this one concludes that

$$
\begin{aligned}
a &= p+q-r \;+\; \frac{g}{\alpha}\,,\\
b &= p+q \;+\; \frac{g}{\beta}\,,\\
c &= p+q+r \;+\; \frac{g}{\gamma}\,,\\
d &= p+q+2r \;+\; \frac{g}{\delta}\,;\\
&\vdots
\end{aligned}
$$

substitution of these values into the formula at the end of §38 makes obvious the following identities:

$$
\begin{aligned}
\alpha\beta-(p+q-r)\alpha-\,(p+q)\beta-q(r-q)&=0\,,\\
\beta\gamma-(p+q)\beta-\,(p+q+r)\gamma-q(r-q)&=0\,,\\
\gamma\delta-(p+q+r)\gamma-\,(p+q+2r)\delta-q(r-q)&=0\,,\\
\delta\varepsilon-(p+q+2r)\delta-\,(p+q+3r)\varepsilon-q(r-q)&=0\,,\\
&\vdots
\end{aligned}
$$

43. Comparing these identities with those we found in §38, we obtain

$$
m=p+q-r\,,\quad n=p+q\,,\quad \kappa=qr-qq\,,\quad s=r\,,
$$

and these relations imply that

$$
\begin{aligned}
ss-ms+ns+\kappa &= 2rr+qr-qq\,,\\
4ss-2ms+2ns+\kappa &= 6rr+qr-qq\,,\\
9ss-3ms+3ns+\kappa &= 12rr+qr-qq\,,\\
&\vdots
\end{aligned}
$$

The following continued fractions for a, b, c, d etc. have the required values:

$$
a=p+q-r+\frac{qr-qq}{2(p+q-r)}{}_{+}\,\frac{2rr+qr-qq}{2(p+q-r)}{}_{+}\,\frac{6rr+qr-qq}{2(p+q-r)}{}_{+}\,\frac{12rr+qr-qq}{2(p+q-r)}{}_{+\cdots}\,,
$$

$$
b=p+q+\frac{qr-qq}{2(p+q)}{}_{+}\,\frac{2rr+qr-qq}{2(p+q)}{}_{+}\,\frac{6rr+qr-qq}{2(p+q)}{}_{+}\,\frac{12rr+qr-qq}{2(p+q)}{}_{+\cdots}\,,
$$

$$c = p+q+r+\frac{qr-qq}{2(p+q+r)} + \frac{2rr+qr-qq}{2(p+q+r)} + \frac{6rr+qr-qq}{2(p+q+r)} + \frac{12rr+qr-qq}{2(p+q+r)} + \cdots$$

44. If for a given series the term with index n is

$$\frac{\int y^{p+2q-1}\,dy\,(1-y^{2r})^{n-1}}{\int y^{p-1}\,dy\,(1-y^{2r})^{n-1}}\,,$$

then

$$A = \frac{a}{p+2q-r} = \frac{\int y^{p+2q-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{p-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,,$$

or

$$a = (p+2q-r)\frac{\int y^{p+2q-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{p-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,.$$

Next, because $ab = p(p+2q-r)$ we obtain

$$b = \frac{p\int y^{p-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{p+2q-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,.$$

Since by a theorem of the preceding dissertation[20]

$$\frac{p\int y^{p-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{f+r-1}\,dy\,:\,\sqrt{1-y^{2r}}} = \frac{f\int y^{f-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{p+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}$$
$$= \frac{(f+r)\int y^{f+2r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{p+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,,$$

where $f = p+2q-r$, we obtain that

$$b = \frac{(p+2q)\int y^{p+2q+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{p+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,.$$

We obtain progressive terms similarly:

$$c = \frac{(p+2q+r)\int y^{p+2q+2r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{p+2r-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,,$$

$$d = \frac{(p+2q+2r)\int y^{p+2q+3r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{p+3r-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,,\cdots$$

20 Euler (1750a).

45. Consequently, in order that the law of progression be valid, the continued fraction

$$p+q+mr+\cfrac{qr-rr}{2(p+q+mr)}+\cfrac{2rr+qr-rr}{2(p+q+mr)}+\cfrac{6rr+qr-rr}{2(p+q+mr)}+\cdots$$

must be equal to

$$(p+2q+mr)\frac{\int y^{p+2q+(m+1)r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{p+(m+1)r-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,.$$

Therefore if $p+q+mr=s$, so that $p=s-q-mr$, we obtain that the value of the continued fraction

$$s+\cfrac{qr-rr}{2s}+\cfrac{2rr+qr-rr}{2s}+\cfrac{6rr+qr-rr}{2s}+\cfrac{12rr+qr-rr}{2s}+\cfrac{20rr+qr-rr}{2s}+\cdots$$

is given by

$$(q+s)\frac{\int y^{q+r+s-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{r+s-q-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,.$$

46. Similarly the value of the continued fraction

$$s+r+\cfrac{qr-rr}{2(s+r)}+\cfrac{2rr+qr-rr}{2(s+r)}+\cfrac{6rr+qr-rr}{2(s+r)}+\cdots$$

is given by

$$(q+r+s)\frac{\int y^{s+2r+q-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{s+2r-q-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,,$$

so that the product of these two continued fractions is $(s+q)(s+r-q)$, as the product of the integral formulas shows. By a theorem presented in the previous dissertation[21],

$$\frac{f}{a}=\frac{\int x^{a-1}\,dx\,:\,\sqrt{1-x^{2r}}\int x^{a+r-1}\,dx\,:\,\sqrt{1-x^{2r}}}{\int x^{f-1}\,dx\,:\,\sqrt{1-x^{2r}}\int x^{f+r-1}\,dx\,:\,\sqrt{1-x^{2r}}}\,,$$

so that the product of integral formulas may be reduced to this form.

47. The continued fraction obtained can be transformed to a more convenient form, since the individual numerators can be factored; thus we obtain

$$s+\cfrac{q(r-q)}{2s}+\cfrac{(r+q)(2r-q)}{2s}+\cfrac{(2r+q)(3r-q)}{2s}+\cfrac{(3r+q)(4r-q)}{2s}+\cdots\,,$$

the value of which is

$$(q+s)\frac{\int y^{r+s+q-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{r+s-q-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,.$$

21 Euler (1750a).

Hence if we add s to the continued fraction to keep to the law of progression then we obtain

$$\frac{(q+s)\int y^{r+s+q-1}\,dy : \sqrt{1-y^{2r}} + s\int y^{r+s-q-1}\,dy : \sqrt{1-y^{2r}}}{\int y^{r+s-q-1}\,dy : \sqrt{1-y^{2r}}}$$
$$= 2s + \frac{q(r-q)}{2s} + \frac{(r+q)(2r-q)}{2s} + \frac{(2r+q)(3r-q)}{2s} + \frac{(3r+q)(4r-q)}{2s} + \cdots$$

48. If now we put $r=2$ and $q=1$ then we obtain the continued fractions discovered by Brouncker, and they all can be found in the continued fraction

$$s + \frac{1}{2s} + \frac{9}{2s} + \frac{25}{2s} + \frac{49}{2s} + \frac{81}{2s} + \cdots$$

which equals

$$(s+1)\frac{\int y^{s+2}\,dy : \sqrt{1-y^4}}{\int y^s\,dy : \sqrt{1-y^4}},$$

and this expression coincides exactly with that considered in §16; therefore, as we mentioned, it must be in quite good agreement with the true form.

49. Although so far I have given many continued fractions whose values can be found from integral formulas, now I explain a method which in turn provides continued fractions from integrals. This method is based on a formula for an integral that reduces it to two other integrals, and this reduction is similar to a simple reduction transforming the integration of one differential to the integration of another differential. Let us consider the infinite sequence of integrals

$$\int P\,dx, \quad \int PR\,dx, \quad \int PR^2\,dx, \quad \int PR^3\,dx, \quad \int PR^4\,dx \ \ldots,$$

which, being compared after each has been integrated, assuming that they vanish at $x=0$, on substituting $x=1$, are found to satisfy

$$a\int P\,dx = b\int PR\,dx + c\int PR^2\,dx,$$

$$(a+\alpha)\int PR\,dx = (b+\beta)\int PR^2\,dx + (c+\gamma)\int PR^3\,dx,$$

$$(a+2\alpha)\int PR^2\,dx = (b+2\beta)\int PR^3\,dx + (c+2\gamma)\int PR^4\,dx,$$

$$(a+3\alpha)\int PR^3\,dx = (b+3\beta)\int PR^3\,dx + (c+3\gamma)\int PR^5\,dx,$$

and in general

$$(a+n\alpha)\int PR^n\,dx = (b+n\beta)\int PR^{n+1}\,dx + (c+n\gamma)\int PR^{n+2}\,dx.$$

50. Given these integral formulas, then by elementary transformations one can obtain continued fractions from them. We have

$$\frac{\int P\,dx}{\int PR\,dx} = \frac{b}{a} + \frac{c\int PR^2\,dx}{a\int PR\,dx}\,,$$

$$\frac{\int PR\,dx}{\int PR^2\,dx} = \frac{b+\beta}{a+\alpha} + \frac{(c+\gamma)\int PR^3\,dx}{(a+\alpha)\int PR^2\,dx}\,,$$

$$\frac{\int PR^2\,dx}{\int PR^3\,dx} = \frac{b+2\beta}{a+2\alpha} + \frac{(c+2\gamma)\int PR^4\,dx}{(a+2\alpha)\int PR^3\,dx}\,,$$

$$\frac{\int PR^3\,dx}{\int PR^4\,dx} = \frac{b+3\beta}{a+3\alpha} + \frac{(c+3\gamma)\int PR^5\,dx}{(a+3\alpha)\int PR^4\,dx}\,,$$

$$\vdots$$

Then expressing the preceding value in terms of the next, we obtain

$$\frac{\int P\,dx}{\int PR\,dx} = \frac{b}{a} + \frac{c:a}{(b+\beta)\,:\,(a+\alpha)} + \frac{(c+\gamma)\,:\,(a+\alpha)}{(b+2\beta)\,:\,(a+2\alpha)} + \frac{(c+2\gamma)\,:\,(a+2\alpha)}{(b+3\beta)\,:\,(a+3\alpha)} + \frac{(c+3\gamma)\,:\,(a+3\alpha)}{(b+4\beta)\,:\,(a+4\alpha)} + \cdots$$

The same expression inverted and with fractions rationalized becomes

$$\frac{\int PR\,dx}{\int P\,dx} = \frac{a}{b} + \frac{(a+\alpha)c}{b+\beta} + \frac{(a+2\alpha)(c+\gamma)}{b+2\beta} + \frac{(a+3\alpha)(c+2\gamma)}{b+3\beta} + \frac{(a+4\alpha)(c+3\gamma)}{b+4\beta} + \cdots\,.$$

51. If n is negative in

$$(a+n\alpha)\int PR^n\,dx = (b+n\beta)\int PR^{n+1}\,dx + (c+n\gamma)\int PR^{n+2}\,dx$$

then we obtain the following formulas:

$$(a-\alpha)\int \frac{P\,dx}{R} = (b-\beta)\int P\,dx + (c-\gamma)\int PR\,dx\,,$$

$$(a-2\alpha)\int \frac{P\,dx}{R^2} = (b-2\beta)\int \frac{P\,dx}{R} + (c-2\gamma)\int PR\,dx\,,$$

$$(a-3\alpha)\int \frac{P\,dx}{R^3} = (b-3\beta)\int \frac{P\,dx}{R^2} + (c-3\gamma)\int \frac{P\,dx}{R}\,,$$

$$(a-4\alpha)\int \frac{P\,dx}{R^4} = (b-4\beta)\int \frac{P\,dx}{R^3} + (c-4\gamma)\int \frac{P\,dx}{R^2}\,,$$

$$\vdots$$

They imply

$$\frac{\int PR\,dx}{\int P\,dx} = \frac{-(b-\beta)}{c-\gamma} + \frac{(a-\alpha)\int P\,dx\,:\,R}{(c-\gamma)\int P\,dx}\,,$$

$$\frac{\int P\,dx}{\int P\,dx\,:\,R} = \frac{-(b-2\beta)}{c-2\gamma} + \frac{(a-2\alpha)\int P\,dx\,:\,R^2}{(c-2\gamma)\int P\,dx : R}\,,$$

$$\frac{\int P\,dx : R}{\int P\,dx\,:\,R^2} = \frac{-(b-3\beta)}{c-3\gamma} + \frac{(a-3\alpha)\int P\,dx\,:\,R^3}{(c-3\gamma)\int P\,dx : R^2}\,,$$

$$\vdots$$

From these equalities we obtain

$$\frac{\int PR\,dx}{\int P\,dx} = \frac{-(b-\beta)}{c-\gamma} + \cfrac{(a-\alpha)\,:\,(c-\gamma)}{-(b-2\beta):(c-2\gamma) + \cfrac{(a-2\alpha)\,:\,(c-2\gamma)}{-(b-3\beta):(c-3\gamma) + \cfrac{(a-3\alpha)\,:\,(c-3\gamma)}{-(b-4\beta):(c-4\gamma)+\cdots}}}$$

or

$$\frac{(c-\gamma)\int PR\,dx}{\int P\,dx} = -(b-\beta) + \cfrac{(a-\alpha)(c-2\gamma)}{-(b-2\beta) + \cfrac{(a-2\alpha)(c-3\gamma)}{-(b-3\beta) + \cfrac{(a-3\alpha)(c-4\gamma)}{-(b-4\beta)+\cdots}}}$$

This gives a second continued fraction for the same quotient $\int PR\,dx/\int P\,dx$.

52. A special feature of this procedure is that one determines suitable functions P and R satisfying

$$(a+n\alpha)\int PR^n\,dx = (b+n\beta)\int PR^{n+1}\,dx + (c+n\gamma)\int PR^{n+2}\,dx\,,$$

at least in the case when after integration one puts $x=1$. In general one assumes that

$$(a+n\alpha)\int PR^n\,dx + R^{n+1}S = (b+n\beta)\int PR^{n+1}\,dx + (c+n\gamma)\int PR^{n+2}\,dx\,,$$

where the functions $R^{n+1}S$ vanish at $x=1$ and $x=0$. Passing to differentials and dividing by R^n, we obtain

$$(a+n\alpha)P\,dx + RdS + (n+1)SdR = (b+n\beta)PR\,dx + (c+n\gamma)PR^2\,dx\,;$$

this equality being valid for any n can be solved by the two equations

$$aPdx + RdS + SdR = bPRdx + cPR^2dx\,,$$

$$aPdx + SdR = \beta PRdx + \gamma PR^2dx\,.$$

These equations together imply

$$Pdx = \frac{RdS+SdR}{bR+cR^2-a} = \frac{SdR}{\beta R+\gamma R^2-\alpha}\,,$$

whence

$$\frac{dS}{S} = \frac{(b-\beta)RdR + (c-\gamma)R^2dR - (a-\alpha)dR}{\beta R^2 + \gamma R^3 - \alpha R}$$
$$= \frac{(a-\alpha)dR}{\alpha R} + \frac{(\alpha b - \beta a)dR + (\alpha c - \gamma a)RdR}{\alpha(\beta R + \gamma R^2 - \alpha)} .$$

From this equation one can find S as a function of R; then

$$P = \frac{SdR}{(\beta R + \gamma R^2 - \alpha)\,dx} ;$$

this determines the integrals $\int Pdx$ and $\int PRdx$ and finally the value of the corresponding continued fraction.

53. If the quantity R cannot be determined from x then one can find x from R. But when the condition of investigation requires that $R^{n+1}S$ must vanish at $x=0$ as well as at $x=1$ this determines the nature of the function R. Next, one should take care that the integrals $\int PR^n dx$ with $x=1$ after integration are finite, since if these integrals were equal either to 0 or infinity then it would be difficult to find the value $\int PRdx/\int Pdx$. In what follows it is mostly safe to separate the values of R so that PR^n is never negative when x varies between $x=0$ and $x=1$. It is often difficult to make $\int PR^n dx$ finite after the substitution $x=1$. There are cases when n is either a positive or a negative number.

54. Let us begin the development of this method for finding values of continued fractions from an example which we have already considered; thus we let the following continued fraction be given:

$$r + \frac{fh}{r} {}_{+} \frac{(f+r)(h+r)}{r} {}_{+} \frac{(f+2r)(h+2r)}{r} {}_{+\cdots} .$$

Its value, found in §34, is given by

$$\frac{h(f-r)\int y^{h+r-1}\,dy : \sqrt{1-y^{2r}} - f(h-r)\int y^{f+r-1}\,dy : \sqrt{1-y^{2r}}}{f\int y^{f+r-1}\,dy : \sqrt{1-y^{2r}} - h\int y^{h+r-1}\,dy : \sqrt{1-y^{2r}}} .$$

Let us make this continued fraction equal to the general continued fraction

$$\frac{a\int P\,dx}{\int PR\,dx} = b + \frac{(a+\alpha)c}{b+\beta} {}_{+} \frac{(a+2\alpha)(c+\gamma)}{b+2\beta} {}_{+} \frac{(a+3\alpha)(c+2\gamma)}{b+3\beta} {}_{+\cdots} .$$

Thus we obtain $b=r$, $\beta=0$, $\alpha=r$, $\gamma=r$, $a=f-r$, $c=h$. Substituting these values we get

$$\frac{dS}{S} = \frac{rRdR + (h-r)R^2dR - (f-2r)dR}{rR^3 - rR}$$
$$= \frac{(f-2r)dR}{rR} + \frac{rdR + (h-f+r)RdR}{r(R^2-1)}$$

and after integration we obtain

$$lS = \frac{f-2r}{r}lR + \frac{h-f}{2r}l(R+1) + \frac{h-f+2r}{2r}l(R-1) + lC$$

or

$$S = CR^{(f-2r)/r}(R^2-1)^{(h-f)/2r}(R-1) .$$

From this we have

$$R^{n+1}S = CR^{(f+(n-1)r)/r}(R^2-1)^{(h-f)/2r}(R-1)$$

and

$$Pdx = \frac{CR^{(f-2r)/r}(R^2-1)^{(h-f)/2r}dR}{r(R+1)}\ .$$

55. Since $R^{n+1}S$ must vanish in two cases, that is, after the substitutions $x=0$ or $x=1$, n being a positive number (one need not consider negative n here), we assume that f, h, r are positive numbers such that $h > f$, which is acceptable except for the case when $f = h$; next we let $f > r$. Under the assumptions made, the formula $R^{n+1}S$ vanishes in two cases, when $R = 0$ and $R = 1$, and the same occurs if $f = h$. Assuming that $f > r$ and putting $R = x$, we obtain

$$Pdx = \frac{x^{(f-2r)/r}(1-x^2)^{(h-f)/2r}dx}{1+x}\ ,$$

which is determined by the constant C. Hence for the continued fraction under consideration we obtain the value

$$(f-r)\frac{\int x^{(f-2r)/r}(1-x^2)^{(h-f)/2r}dx\ :\ (1+x)}{\int x^{(f-r)/r}(1-x^2)^{(h-f)/2r}dx\ :\ (1+x)}\ .$$

Substituting $x = y^r$, we obtain the unusual value

$$\frac{(f-r)\int y^{f-r-1}(1-y^{2r})^{(h-f)/2r}dy\ :\ (1+y^r)}{\int y^{f-1}(1-y^{2r})^{(h-f)/2r}dy\ :\ (1+y^r)}\ .$$

56. Thus we have obtained another value for the continued fraction

$$r+\frac{fh}{r}{}_{+}\frac{(f+r)(h+r)}{r}{}_{+\cdots}\ ,$$

which, although it contains integral formulas, does not however agree with the value already found. Next, this expression does not hold except for the case $f > r$; we denote by h the greater of two numbers f and h if there are not equal. If f is smaller than r then the value of the continued fraction could be recovered by consideration the of following one:

$$r+\frac{(f+r)(h+r)}{r}{}_{+}\frac{(f+2r)(h+2r)}{r}{}_{+\cdots}\ .$$

The value of this continued fraction is

$$\frac{f\int y^{f-1}(1-y^{2r})^{(h-f)/2r}dy\ :\ (1+y^r)}{\int y^{f+r-1}(1-y^{2r})^{(h-f)/2r}dy\ :\ (1+y^r)}$$

and has no restrictions. After putting the value equal to V the value of the continued fraction in question must be $r+fh/V$.

57. The [continued fraction in the] case $f = h$, which in §34 was considered using a special method and whose value equals

$$\frac{1-(h-r)\int x^{h-1}dx\ :\ (1+x^r)}{\int x^{h-1}dx\ :\ (1+x^r)} = \frac{(h-r)\int x^{h-r-1}dx\ :\ (1+x^r)}{\int x^{h-1}dx\ :\ (1+x^r)}\ ,$$

is exhausted by this very expression; for $f = h$ the expression found in §55 leads to

$$\frac{(h-r)\int y^{h-r-1}dy : (1+y^r)}{\int y^{h-1}dy : (1+y^r)} ,$$

an identical expression, and taking this into account the agreement of these two general expressions is clearly seen.

58. To see the equality of these expressions in every case we will prove the following lemma, which has already been proved on another occasion. Given the series

$$1+\frac{p}{q+s}+\frac{p(p+s)}{(q+s)(q+2s)}+\frac{p(p+s)(p+2s)}{(q+s)(q+2s)(q+3s)}+\cdots ,$$

where the numbers p, q, s are positive and $q > p$, then the sum of this series extended to infinity is $q/(q-p)$. The validity of this lemma can be established by my general method for the summation of series as follows. Let us consider the series

$$x^q+\frac{p}{q+s}x^{q+s}+\frac{p(p+s)}{(q+s)(q+2s)}x^{q+2s}+\cdots ,$$

whose sum on differentiation satisfies

$$\frac{dz}{dx}=qx^{q-1}+px^{q+s-1}+\frac{p(p+s)}{(q+s)}x^{q+2s-1}+\cdots ,$$

$$x^{p-q-s}\,dz=qx^{p-s-1}\,dx+px^{p-1}\,dx+\frac{p(p+s)}{(q+s)}x^{p+s-1}\,dx+\cdots .$$

After integration this implies that

$$\int x^{p-q-s}\,dz=\frac{qx^{p-s}}{p-s}+x^p+\frac{px^{p+s}}{(q+s)}+\cdots=\frac{qx^{p-s}}{p-s}+x^{p-q}z .$$

Differentiating this equality, we obtain

$$x^{p-q-s}\,dz=qx^{p-s-1}\,dx+x^{p-q}\,dz+(p-q)x^{p-q-1}z\,dx ,$$

$$dz(1-x^s)+(q-p)x^{s-1}z\,dx=qx^{q-1}\,dx ,$$

$$dz+\frac{(q-p)x^{s-1}z\,dx}{1-x^s}=\frac{qx^{q-1}\,dx}{1-x^s} .$$

The integral of the last equation is given by

$$\frac{z}{(1-x^s)^{(q-p)/s}}=q\int\frac{x^{q-1}\,dx}{(1-x^s)^{(q-p+s)/s}}$$

$$=\frac{qx^q}{(q-p)(1-x^s)^{(q-p)/s}}-\frac{pq}{q-p}\int\frac{x^{q-1}\,dx}{(1-x^s)^{(q-p)/s}} .$$

It follows that

$$z=\frac{qx^q}{q-p}-\frac{pq(1-x^s)^{(q-p)/s}}{q-p}\int\frac{x^{q-1}\,dx}{(1-x^s)^{(q-p)/s}} .$$

Therefore after substituting $x=1$ we obtain

$$z=\frac{q}{q-p}=1+\frac{p}{q+s}+\frac{p(p+s)}{(q+s)(q+2s)}+\cdots ,$$

which proves the lemma and at the same time clarifies that the lemma cannot be proved for cases other than $q > p$[22].

59. The value of the continued fraction

$$r+\frac{fh}{r}{}_{+}\frac{(f+r)(h+r)}{r}{}_{+}\frac{(f+2r)(h+2r)}{r}{}_{+\cdots}$$

can be expressed in two ways; one is

$$\frac{h(f-r)\int y^{h+r-1}\,dy : \sqrt{1-y^{2r}} - f(h-r)\int y^{f+r-1}\,dy : \sqrt{1-y^{2r}}}{f\int y^{f+r-1}\,dy : \sqrt{1-y^{2r}} - h\int y^{h+r-1}\,dy : \sqrt{1-y^{2r}}},$$

and another, found in §56, is

$$r+\frac{h\int y^{f+r-1}(1-y^{2r})^{(h-f)/2r}dy : (1+y^r)}{\int y^{f-1}(1-y^{2r})^{(h-f)/2r}dy : (1+y^r)}.$$

it showing the equality of these expressions deserves our attention. Since $1/(1+y^r) = (1-y^r)/(1-y^{2r})$, we obtain

$$\int y^{f-1}\,dy\,(1-y^{2r})^{(h-f)/2r} : (1+y^r)$$
$$= \int y^{f-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r} - \int y^{f+r-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r}$$

and similarly [23]

$$\int y^{f+r-1}\,dy\,(1-y^{2r})^{(h-f)/2r} : (1+y^r)$$
$$= \int y^{f+r-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r} - \int y^{f+2r-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r}$$
$$= \int y^{f+r-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r} - \frac{f}{h}\int y^{f-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r}.$$

Let us write

$$\frac{\int y^{f+r-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r}}{\int y^{f-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r}} = V;$$

then the second value of the continued fraction under consideration is given by

$$r+\frac{hV-f}{1-V}.$$

In addition, Let

$$\frac{\int y^{h+r-1}\,dy : \sqrt{1-y^{2r}}}{\int y^{f+r-1}\,dy : \sqrt{1-y^{2r}}} = W;$$

then the first value will be

$$\frac{h(f-r)W - f(h-r)}{f-hW},$$

22 See Exs. 4.41 and 4.42.
23 Apply Lemma 4.11 with $m = f$, $n = 2r$, $k = h - f$.

which implies that $V = f/hW$, and hence

$$\frac{\int y^{f+r-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r}}{\int y^{f-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r}} = \frac{f\int y^{h+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{h\int y^{f+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,;$$

the assumption behind this the equality is based on a theorem of the preceding dissertation [24] and on another theorem: [25]

$$\frac{\int y^{f+r-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r}}{\int y^{f+2r-1}\,dy\,(1-y^{2r})^{(h-f-2r)/2r}} = \frac{\int y^{h+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{\int y^{f+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,.$$

60. Let us now consider the following continued fraction,

$$2r + \frac{fh}{2r}{}_{+}\,\frac{(f+r)(h+r)}{2r}{}_{+}\,\frac{(f+2r)(h+2r)}{2r}{}_{+\cdots}\,,$$

whose value was found in §35 to be

$$\frac{2(r-f)(r-h)\int y^{f-1}\,dy\,:\,\sqrt{1-y^{2r}}}{2h\int y^{h+r-1}\,dy\,:\,\sqrt{1-y^{2r}}-(f+h-r)\int y^{f-1}\,dy : \sqrt{1-y^{2r}}}$$
$$-\frac{h(f+h-3r)\int y^{h+r-1}\,dy\,:\,\sqrt{1-y^{2r}}}{2h\int y^{h+r-1}\,dy\,:\,\sqrt{1-y^{2r}}-(f+h-r)\int y^{f-1}\,dy\,:\,\sqrt{1-y^{2r}}}\,.$$

If now we compare this continued fraction with

$$\frac{a\int P\,dx}{\int PR\,dx} = b + \frac{(a+\alpha)c}{b+\beta}{}_{+}\,\frac{(a+2\alpha)(c+\gamma)}{b+2\beta}{}_{+}\,\frac{(a+3\alpha)(c+2\gamma)}{b+3\beta}{}_{+\cdots}$$

then we obtain $b = 2r$, $\beta = 0$, $\alpha = r$, $\gamma = r$, $a = f-r$, $c = h$. It follows by §52 that

$$\frac{dS}{S} = \frac{(f-2r)dR}{rR} + \frac{2rdR+(h-f+r)RdR}{r(R^2-1)}$$

and after integration that

$$S = CR^{(f-2r)/2r}(R^2-1)^{(h-f-r)/2r}(R-1)^2\,,$$

which implies that

$$P\,dx = \frac{C}{r}R^{(f-2r)/2r}(R^2-1)^{(h-f-3r)/2r}(R-1)^2\,dR$$

and

$$R^{n+1}S = CR^{(f+(n-1)r)/2r}(R^2-1)^{(h-f-r)/2r}(R-1)^2\,.$$

This expression vanishes in two cases, $R = 0$ or $R = 1$; the only requirements are $f > r$ and $h-3r > f$, which can always be attained.

61. Let $R = x$ and the constant C be defined is such a way that

$$P\,dx = x^{(f-2r)/2r}\,dx(1-x^2)^{(h-f-3r)/2r}(1-x^2)\,,$$

[24] See Lemma 4.11.
[25] Corollary 4.13.

which after substituting $R = x = y^r$ becomes

$$P\,dx = y^{f-r-1}\,dy(1-y^{2r})^{(h-f-3r)/2r}(1-y^{2r})^2\,.$$

Hence the value of the proposed continued fraction will be

$$\frac{a\int P\,dx}{\int PR\,dx} = \frac{(f-r)\int y^{f-r-1}dy\,(1-y^{2r})^{(h-f-3r)/2r}(1-y^r)^2}{\int y^{f-1}dy\,(1-y^{2r})^{(h-f-3r)/2r}(1-y^r)^2}\,.$$

By a theorem from the preceding dissertation[26] it reduces to the first type[27] if one opens the parentheses in $(1-y^r)^2$. After that the formula reduces to two simpler integrals. I will demonstrate this for a more general example.

62. If an integral formula is given,

$$\int y^{m-1}\,dy\,(1-y^{2r})^{\kappa}(1-y^r)^n,$$

and $(1-y^r)^n$ is developed into the series

$$1 - ny^r + \frac{n(n-1)}{1\times 2}y^{2r} - \cdots\,,$$

and alternate terms are summed up, then the given form of the integral is reduced to the two following expressions:

$$\int y^{m-1}\,dy\,(1-y^{2r})^{\kappa}\left(1+\frac{n(n-1)}{1\times 2}\,\frac{m}{p}\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{1\times 2\times 3\times 4}\,\frac{m(m+2r)}{p(p+2r)}+\cdots\right)$$

$$-\int y^{m+r-1}\,dy\,(1-y^{2r})^{\kappa}\left(n+\frac{n(n-1)(n-2)}{1\times 2\times 3}\,\frac{m+r}{p+r}\right.$$
$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{1\times 2\times 3\times 4\times 5}$$
$$\left.\times\frac{(m+r)(m+3r)}{(p+r)(p+3r)}+\cdots\right),$$

where we put for brevity $m+2\kappa r+2r = p$. Therefore, in the case $n=2$ considered above,

$$\int y^{m-1}\,dy\,(1-y^{2r})^{\kappa}(1-y^r)^2$$
$$= \frac{m+p}{p}\int y^{m-1}\,dy\,(1-y^{2r})^{\kappa} - 2\int y^{m+r-1}\,dy\,(1-y^{2r})^{\kappa}\,.$$

[26] Euler (1750a).
[27] See §59.

This implies that, setting $r = (h-f-r)/2r$,

$$\frac{a\int P\,dx}{\int PR\,dx}$$
$$= \frac{(f-r)(f+h-3r) : (h-2r)\int y^{f-r-1}dy(1-y^{2r})^{r-1}}{(f+h-r) : (h-r)\int y^{f-1}dy(1-y^{2r})^{r-1} - 2\int y^{f+r-1}dy(1-y^{2r})^{r-1}}$$
$$- \frac{2(f-r)\int y^{f-1}dy(1-y^{2r})^{r-1}}{(f+h-r) : (h-r)\int y^{f-1}dy(1-y^{2r})^{r-1} - 2\int y^{f+r-1}dy(1-y^{2r})^{r-1}}$$
$$= \frac{h(f+h-r)\int y^{f+r-1}dy(1-y^{2r})^{r}}{(f+h-r)\int y^{f-1}dy(1-y^{2r})^{r} - 2h\int y^{f+r-1}dy(1-y^{2r})^{r}}$$
$$- \frac{2(f-r)(h-r)\int y^{f-1}dy(1-y^{2r})^{r}}{(f+h-r)\int y^{f-1}dy(1-y^{2r})^{r} - 2h\int y^{f+r-1}dy(1-y^{2r})^{r}} .$$

This expression, which must be equal to that found in §35, gives the following equality on substitution of the expression for r [28]:

$$\frac{\int y^{f+r-1}dy(1-y^{2r})^{(h-f-r)/2r}}{\int y^{f-1}dy(1-y^{2r})^{(h-f-r)/2r}} = \frac{\int y^{h+r-1}dy : \sqrt{1-y^{2r}}}{\int y^{f-1}dy : \sqrt{1-y^{2r}}} .$$

This ratio is already contained in theorems of the preceding dissertation [29].

63. Let us now sum up [what we know about] continued fractions by putting

$$P = x^{m-1}(1-x^r)^n(p+qx^r)^\kappa \quad \text{and} \quad R = x^r .$$

We must have

$$(a+\nu\alpha)\int PR^\nu\,dx = (b+\nu\beta)\int PR^{\nu+1}\,dx + (c+\nu\gamma)\int PR^{\nu+2}\,dx$$

and this implies by §52 that

$$S = \frac{1}{r}x^{m-r}(1-x^r)^n(p+qx^r)^\kappa(\gamma x^{2r}+\beta x^r-\alpha)$$

and

$$\frac{dS}{S} = \frac{(m-r)dx}{x} + \frac{nrx^{r-1}dx}{-1+x^r} + \frac{\kappa qrx^{r-1}dx}{p+qx^r} + \frac{2\gamma rx^{2r-1}dx+\beta rx^{r-1}dx}{-1+x^r}$$
$$= \frac{(a-\alpha)rdx}{\alpha x} + \frac{(\alpha b-\beta a)rx^{r-1}dx+(\alpha c-\gamma a)rx^{2r-1}dx}{\alpha(\gamma x^{2r}+\beta x^r-\alpha)} .$$

Now let $(p+qx^r)(x^r-1) = \gamma x^{2r}+\beta x^r-\alpha$, where $\gamma = q$, $\beta = p-q$, $\alpha = p$. In addition let $(a-\alpha)r/\alpha = m-r$, where $a = mp/r$. It follows then that

$$nqr+\kappa qr+2qr = \frac{cpr-mpq}{p}$$

or

$$c = \frac{mq}{r}+nq+(\kappa+2)q,$$

[28] Corollary 4.13.
[29] Euler (1750a), in which $a = f$, $b = 2r$, $c = -1/2$, $\gamma = h-f-r/2r$.

and finally that

$$b = \frac{m(p-q)}{r} + (n+1)p - (\kappa+1)q\,.$$

Assuming m and $n+1$ be positive numbers, so that $R^{\nu+1}S$ vanishes at $x=0$ and $x=1$, we obtain that the formula

$$\frac{\int x^{m+r-1}dx(1-x^r)^n(p+qx^r)^\kappa}{\int x^{m-1}dx(1-x^r)^n(p+qx^r)^\kappa} = \frac{\int PR\,dx}{\int P\,dx}\,,$$

equals the following continued fraction:

$$\frac{mp}{m(p-q)+(n+1)pr-(\kappa+1)qr} + \frac{pq(m+r)(m+nr+(\kappa+2)r)}{m(p-q)+(n+2)pr-(\kappa+2)qr} + \frac{pq(m+2r)(m+nr+(\kappa+2)r)}{m(p-q)+(n+3)pr-(\kappa+3)qr} + \cdots$$

64. To simplify this continued fraction we put

$$m+nr+r=a\,,\ m+\kappa r+r=b\,,\ m+nr+\kappa r=c\,;$$

which implies that $\kappa=(c-a)/r$, $n=(c-b)/r$, $m=a+b-c-r$ and therefore

$$\frac{p(a+b-c+r)}{ap-bq} + \frac{pq(a+b-c)(c+r)}{(a+r)p-(b+r)q} + \frac{pq(a+b-c+r)(c+2r)}{(a+2r)p-(b+2r)q} + \frac{pq(a+b-c+2r)(c+3r)}{(a+3r)p-(b+3r)q} + \cdots$$
$$= \frac{\int x^{a+b-c-1}dx(1-x^r)^{(c-b)/r}(p+qx^r)^{(c-a)/r}}{\int x^{a+b-c-r-1}dx(1-x^r)^{(c-b)/r}(p+qx^r)^{(c-a)/r}}\,,$$

where $x=1$ after integration. It is necessary that the numbers $a+b-c-r$ and $c-b+r$ be positive. If for brevity we put $a+b-c-r=g$ then

$$\frac{\int x^{g+r-1}dx(1-x^r)^{(c-b)/r}(p+qx^r)^{(c-a)/r}}{\int x^{g-1}dx(1-x^r)^{(c-b)/r}(p+qx^r)^{(c-a)/r}}$$
$$= \frac{pg}{ap-bq} + \frac{pq(c+r)(g+r)}{(a+r)p-(b+r)q} + \frac{pq(c+2r)(g+2r)}{(a+2r)p-(b+2r)q} + \cdots\,,$$

and this equality includes all the continued fractions discovered so far.

65. If of c and g are interchanged then the following continued fraction is obtained:

$$\frac{pc}{ap-bq} + \frac{pq(c+r)(g+r)}{(a+r)p-(b+r)q} + \frac{pq(c+2r)(g+2r)}{(a+2r)p-(b+2r)q} + \cdots$$

Its value is

$$= \frac{\int x^{c+r-1}dx(1-x^r)^{(g-b)/r}(p+qx^r)^{(g-a)/r}}{\int x^{c-1}dx(1-x^r)^{(g-b)/r}(p+qx^r)^{(g-a)/r}}\,.$$

Then

$$\frac{c\int x^{a+b-c-1}dx(1-x^r)^{(c-b)/r}(p+qx^r)^{(c-a)/r}}{\int x^{a+b-c-r-1}dx(1-x^r)^{(c-b)/r}(p+qx^r)^{(c-a)/r}}$$

$$=\frac{(a+b-c-r)\int x^{c+r-1}dx(1-x^r)^{(a-c-r)/r}(p+qx^r)^{(b-c-r)/r}}{\int x^{c-1}dx(1-x^r)^{(a-c-b)/r}(p+qx^r)^{(b-c-a)/r}}\,.$$

This, the most general form, contains most of the partial reductions. For instance, let $b=c+r$; then

$$\frac{c\int x^{a+r-1}dx(p+qx^r)^{(c-a)/r}\ :\ (1-x^r)}{\int x^{a-1}dx(p+qx^r)^{(c-a)/r}\ :\ (1-x^r)}=\frac{a\int x^{c+r-1}dx(1-x^r)^{(a-c-r)/r}}{\int x^{c-1}dx(1-x^r)^{(a-c-r)/r}}=c\,,$$

which implies that

$$\int\frac{x^{a+r-1}dx(p+qx^r)^{(c-a)/r}}{1-x^r}=\int\frac{x^{a-1}dx(p+qx^r)^{(c-a)/r}}{1-x^r}\,.$$

This consequently implies the following well-known theorem:

$$\int\frac{x^{m-1}dx(p+qx^r)^{\kappa}}{1-x^r}=\int\frac{x^{n-1}dx(p+qx^r)^{\kappa}}{1-x^r}\,,$$

where the integration is taken from $x=0$ to $x=1$. An exceptional case is the inconvenient case where $q+p=0$.

66. The continued fractions which are obtained by interpolation may be arranged so that their partial denominators are constant. To reduce them to a general form let us put $p=q=1$. Then we obtain the following continued fraction:

$$\frac{cg}{a-b+}\,\frac{(c+r)(g+r)}{a-b}\,{}_{+}\,\frac{(c+2r)(g+2r)}{a-b}\,{}_{+}\,\frac{(c+3r)(g+3r)}{a-b}\,{}_{+\cdots}$$

$$=\frac{c\int x^{g+r-1}dx(1-x^r)^{(c-b)/r}(1+x^r)^{(c-a)/r}}{\int x^{g-1}dx(1-x^r)^{(c-b)/r}(1+x^r)^{(c-a)/r}}\,;$$

its value is also given by

$$\frac{g\int x^{c+r-1}dx(1-x^r)^{(g-b)/r}(1+x^r)^{(g-a)/r}}{\int x^{c-1}dx(1-x^r)^{(g-b)/r}(1+x^r)^{(g-a)/r}}\,,$$

where $g=a+b-c-r$. Hence $a+b=c+g+r$ and therefore if $a-b=s$ then

$$a=\frac{c+g+r+s}{2}\qquad\text{and}\qquad b=\frac{c+g+r-s}{2}\,,$$

and we obtain

$$\frac{cg}{s+}\,\frac{(c+r)(g+r)}{s}\,{}_{+}\,\frac{(c+2r)(g+2r)}{s}\,{}_{+\cdots}$$

$$=\frac{c\int x^{g+r-1}dx(1-x^{2r})^{(c-g-r-s)/2r}(1-x^r)^{s/r}}{\int x^{g-1}dx(1-x^{2r})^{(c-g-r-s)/2r}(1-x^r)^{s/r}}$$

$$=\frac{g\int x^{c+r-1}dx(1-x^{2r})^{(g-c-r-s)/2r}(1-x^r)^{s/r}}{\int x^{c-1}dx(1-x^{2r})^{(g-c-r-s)/2r}(1-x^r)^{s/r}}\,.$$

67. Let us replace s by $2s$ in the last formula of §66 and put $c = q$, $g = r - q$. Then we obtain the continued fraction

$$\frac{q(r-q)}{2s} + \frac{(q+r)(2r-q)}{2s} + \frac{(q+2r)(3r-q)}{2s} + \cdots,$$

whose value can be written as either

$$\frac{q\int x^{2r-q-1}dx(1-x^{2r})^{(q-r-s)/r}(1-x^r)^{2s/r}}{\int x^{r-q-1}dx(1-x^{2r})^{(q-r-s)/2r}(1-x^r)^{2s/r}},$$

or

$$\frac{(r-q)\int x^{q+r-1}dx(1-x^{2r})^{(-q-s)/r}(1-x^r)^{2s/r}}{\int x^{q-1}dx(1-x^{2r})^{(-q-s)/2r}(1-x^r)^{2s/r}}.$$

The value of such a continued fraction has already been found and it is equal to

$$\frac{(q+s)\int y^{r+s+q-1}\,dy \,:\, \sqrt{1-y^{2r}}}{\int y^{r+s-q-1}\,dy \,:\, \sqrt{1-y^{2r}}} - s\,.$$

It follows that these formulas with integrals are equal, and this is an essential theorem.

68. If as in §48 $r = 2$ and $q = 1$ then

$$\frac{(1+s)\int y^{s+2}dy \,:\, \sqrt{1-y^4}}{\int y^s dy \,:\, \sqrt{1-y^4}} - s = \frac{\int x^2dx(1-x^4)^{(-s-1)/2}(1-x^2)^s}{\int dx(1-x^4)^{(-s-1)/2}(1-x^2)^s},$$

which is obvious if $s = 0$. In the case when s is an odd integer, the equality can easily be derived: if $s = 1$ then

$$\frac{\int xxdx : (1+xx)}{\int dx \,:\, (1+xx)} = \frac{x - \int dx \,:\, (1+xx)}{\int dx : (1+xx)} = \frac{4-\pi}{\pi},$$

after substituting $x = 1$. The previous formula says that

$$\frac{2\int y^3dy \,:\, \sqrt{1-y^4}}{\int ydy \,:\, \sqrt{1-y^4}} - 1 = \frac{4}{\pi} - 1 = \frac{4-\pi}{\pi}$$

as does the preceding one. If s is even then the development of $(1-xx)^s$ into monomials easily shows the equivalence of these two expressions.

69. Besides the continued fractions already found, the general form of the continued fractions presented above contains numerous other special cases as its corollary. Let $g = c$; then we obtain the continued fraction

$$\frac{c^2}{s} + \frac{(c+r)^2}{s} + \frac{(c+2r)^2}{s} + \cdots$$

with value

$$\frac{c\int x^{c+r-1}dx(1-x^r)^{s/r} \,:\, (1-x^{2r})^{(r+s)/2r}}{\int x^{c-1}dx(1-x^r)^{s/r} \,:\, (1-x^{2r})^{(r+s)/2r}}.$$

Substituting $c = 1$ and $r = 1$, we obtain

$$\frac{1}{s} + \frac{4}{s} + \frac{9}{s} + \frac{16}{s} + \cdots = \frac{\int xdx(1-x^s) \,:\, (1-xx)^{(s+1)/2}}{\int dx(1-x^s) \,:\, (1-xx)^{(s+1)/2}},$$

which allows us to compute the value of the continued fraction for several important values of s: for $s=0$,

$$V = \frac{\int xdx : \sqrt{1-xx}}{\int dx : \sqrt{1-xx}} = \frac{1}{2\int dy : (1+yy)} ;$$

for $s=2$,

$$V = \frac{2\int dx : \sqrt{1-xx} - 3\int xdx : \sqrt{1-xx}}{2\int xdx : \sqrt{1-xx} - \int dx : \sqrt{1-xx}} = \frac{1}{2\int y^2 dy : (1+yy)} - 2 ;$$

for $s=4$,

$$V = \frac{19\int xdx : \sqrt{1-xx} - 12\int dx : \sqrt{1-xx}}{3\int dx : \sqrt{1-xx} - 4\int xdx : \sqrt{1-xx}} = \frac{1}{2\int y^4 dy : (1+yy)} - 4 .$$

In general we have

$$V = \frac{1}{2\int y^s dy : (1+yy)} - s .$$

It is clear from this formula that even-s values are related to the quadrature of the unit circle whereas odd-s values are related to logarithms.

70. Let us consider now the continued fraction

$$1 + \frac{1}{2} {}_{+} \frac{4}{3} {}_{+} \frac{9}{4} {}_{+} \frac{16}{5} {}_{+} \frac{25}{6} {}_{+ \cdots}$$

and compare it with with the continued fraction in §64. This gives

$$pqcg = 1 , \quad pq(c+r)(g+r) = 4 , \quad pq(c+2r)(g+2r) = 9 ,$$
$$ap - bq = 2 , \quad (p-q)r = 1 ,$$

whence

$$c = g = r , \quad p = \frac{\sqrt{5}+1}{2r} , \quad p = \frac{\sqrt{5}-1}{2r} ,$$
$$a = \frac{r(1+3\sqrt{5})}{2\sqrt{5}} , \quad b = \frac{r(3\sqrt{5}-1)}{2\sqrt{5}} .$$

These substitutions into the formulas of §64 give the following formula:

$$1 + \frac{(\sqrt{5}-1)\int x^{2r-1} dx (1-x^r)^{(1-\sqrt{5})/2\sqrt{5}} (1+\sqrt{5}+(\sqrt{5}-1)x^r)^{-(\sqrt{5}-1)/2\sqrt{5}}}{\int x^{r-1} dx (1-x^r)^{(1-\sqrt{5})/2\sqrt{5}} (1+\sqrt{5}+(\sqrt{5}-1)x^r)^{-\sqrt{5}-1/2\sqrt{5}}} .$$

Because of the complexity of the exponents one cannot derive from this anything deserving any further attention [30].

71. At the same time [it may be noted that the] partial numerators in these continued fractions are products of two multipliers; thus I consider now a type of continued fraction of the class in

[30] See however (4.66) and Corollary 4.22.

which these numerators make an arithmetic progression. Let us put in the last formula of §50 $\gamma = 0$ and $c = 1$. Then

$$\frac{\int PRdx}{\int Pdx} = \frac{a}{b} + \frac{a+\alpha}{b+\beta} + \frac{a+2\alpha}{b+2\beta} + \frac{a+3\alpha}{b+3\beta} + \cdots .$$

Summarizing, we have

$$\begin{aligned}\frac{dS}{S} &= \frac{(a-\alpha)dR}{\alpha R} + \frac{(\alpha b - \beta a)dR + \alpha R dR}{\alpha(\beta R - \alpha)} \\ &= \frac{(a-\alpha)dR}{\alpha R} + \frac{dR}{\beta} + \frac{(\alpha^2 + \alpha\beta b - \beta^2 a)dR}{\alpha\beta(\beta R - \alpha)} ,\end{aligned}$$

which implies that

$$S = Ce^{R/\beta} R^{(a-\alpha)/\alpha} (\beta R - \alpha)^{(\alpha^2+\alpha\beta b-\beta^2 a)/\alpha\beta\beta} .$$

If $R = \alpha x/\beta$ then

$$S = Ce^{\alpha x/\beta\beta} x^{(a-\alpha)/\alpha} (1-x)^{(\alpha^2+\alpha\beta b-\beta^2 a)/\alpha\beta\beta}$$

and $R^{n+1}S$ vanishes in both the cases $x = 0$ and $x = 1$, provided that $\alpha^2 + \alpha\beta b > \beta^2 a$. Hence

$$Pdx = e^{\alpha x/\beta\beta} x^{a-\alpha/\alpha} dx (1-x)^{(\alpha^2+\alpha\beta b-\beta^2 a)/\alpha\beta\beta}$$

and the value of the continued fraction under consideration is

$$\frac{\int PRdx}{\int Pdx} = \frac{\alpha \int e^{\alpha x/\beta\beta} x^{a/\alpha} dx (1-x)^{(\alpha^2+\alpha\beta b-\beta^2 a)/\alpha\beta\beta}}{\beta \int e^{\alpha x/\beta\beta} x^{(a-\alpha)/\alpha} dx (1-x)^{(\alpha^2+\alpha\beta b-\beta^2 a)/\alpha\beta\beta}} ,$$

where $x = 1$ after integration[31].

72. To illustrate this case with an example, let $a = 1$, $\alpha = 1$ and $\beta = 1$. Then we obtain the continued fraction

$$\frac{1}{1} + \frac{2}{2} + \frac{3}{3} + \frac{4}{4} + \cdots ,$$

the value of which is

$$\frac{\int e^x x\,dx}{\int e^x x\,dx} = \frac{e^x x - e^x + 1}{e^x - 1} = \frac{1}{e-1}$$

after substituting $x = 1$. It follows that

$$e = 2 + \frac{2}{2} + \frac{3}{3} + \frac{4}{4} + \frac{5}{5} + \cdots$$

and this expression soon converges to the number e whose logarithm equals 1.

73. Let us assume now that $\beta = 0$ in the continued fraction of §71, so that

$$\frac{\int PRdx}{\int Pdx} = \frac{a}{b} + \frac{a+\alpha}{b} + \frac{a+2\alpha}{b} + \frac{a+3\alpha}{b} + \cdots$$

whence

$$\frac{dS}{S} = \frac{(a-\alpha)dR}{\alpha R} - \frac{bdR}{\alpha} - \frac{RdR}{\alpha}$$

[31] See §**87** in Section 4.7.

and therefore

$$S = CR^{(a-\alpha)/\alpha}e^{(-2bR-RR)/2\alpha}\ .$$

There are two cases when $R^{n+1}S$ vanishes, $R=0$ or $R=\infty$, provided that both a and α are positive numbers. Let $R = x/(1-x)$; then

$$S = Cx^{(\alpha-a)/\alpha} : (1-x)^{(\alpha-a)/\alpha} \exp\left\{\frac{2bx-(2b-1)xx}{2\alpha(1-x)^2}\right\}.$$

Since $dR = dx/(1-x)^2$, we obtain

$$\int P dx = \int \frac{x^{(a-\alpha)/\alpha}\, dx}{(1-x)^{(a+\alpha)/\alpha} \exp\left\{\dfrac{2bx-(2b-1)xx}{2\alpha(1-x)^2}\right\}}$$

and

$$\int PR dx = \int \frac{x^{a/\alpha}\, dx}{(1-x)^{(a+2\alpha)/\alpha} \exp\left\{\dfrac{2bx-(2b-1)xx}{2\alpha(1-x)^2}\right\}}\ .$$

74. Let us finally put in the last formula in §50 $a=1$, $c=1$, $\alpha=0$, $\gamma=0$; then

$$\frac{\int PR\, dx}{\int P\, dx} = \frac{1}{b} + \frac{1}{b+\beta} + \frac{1}{b+2\beta} + \frac{1}{b+3\beta} + \cdots$$

and

$$\frac{dS}{S} = \frac{R^2 dR + (b-\beta)R dR - dR}{\beta R^2}\ ,$$

whence

$$S = \exp\left\{\frac{RR+1}{\beta R}\right\} R^{(b-\beta)/\beta} \quad \text{and} \quad P dx = \exp\left\{\frac{RR+1}{\beta R}\right\} R^{(b-2\beta)/\beta} dR,$$

implying that

$$PR dx = \exp\left\{\frac{RR+1}{\beta R}\right\} R^{(b-\beta)/\beta} dR\ .$$

It is necessary that R be a function of x such that R^{n+1} vanishes at $x=0$ and $x=1$. It is more difficult to find such a function than in previous cases. I am not going to resolve this case with the same method but consider it with a different one, which will now be presented.

75. I have already mentioned this method of finding continued fractions[32], but since then I considered a special case, now I am going to present this method more generally. Its essence is not expressed in integral formulas as in the preceding cases but in finding solutions to differential equations in a way similar to that proposed some time ago by Earl Riccati. I consider the following equation:

$$ax^m\, dx + bx^{m-1} y dx + cy^2 dx + dy = 0\ .$$

By the substitution $x^{m+3} = t$ and $y = 1/cx + 1/xxz$ it transforms into

$$\frac{-c}{m+3} t^{(-m-4)/(m+3)} dt - \frac{b}{m+3} t^{-1/(m+3)} z dt - \frac{ac+b}{(m+3)c} z^2 dt + dz = 0\ ,$$

[32] See Euler (1744, §§21–35) and also §4.10 of the present book.

which is similar to the initial equation. Therefore if the value of z can be found from t then equally y can be found from x. Similarly, this equation can be reduced to another by the substitution

$$t^{(2m+5)/(m+3)} = u \quad \text{and} \quad z = \frac{-(m+3)c}{(ac+b)t} + \frac{1}{ttv}\,,$$

and such reductions can be continued up to infinity; when this is done, if each subsequent value is expressed by the preceding value then y can be expressed as

$$y = Ax^{-1} + \frac{1}{-Bx^{-m-1}} + \frac{1}{Cx^{-1}} + \frac{1}{-Dx^{-m-1}} + \frac{1}{Ex^{-1}} + \frac{1}{-Fx^{-m-1}} + \cdots\,,$$

where the quantities A, B, C, D , $\cdots$ have the following values:

$$A = \frac{1}{c}\,, \quad B = \frac{(m+3)c}{ac+b}\,, \quad C = \frac{(2m+5)(ac+b)}{c(ac-(m+2)b)}\,,$$

$$D = \frac{(3m+7)c(ac-(m+2)b)}{(ac+b)(ac+(m+3)b)}\,, \quad E = \frac{(4m+9)(ac+b)(ac+(m+3)b)}{c(ac-(m+2)b)(ac-(2m+4)b)}\,,$$

$$F = \frac{(5m+11)c(ac-(m+2)b)(ac-(2m+34)b)}{(ac+b)(ac+(m+3)b)(ac+(2m+5)b)}\,, \cdots$$

This law of composition is easier to understand using the following identities:

$$AB = \frac{m+3}{ac+b}\,, \quad BC = \frac{(m+3)(2m+b)}{ac-(m+2)b}\,, \quad CD = \frac{(2m+5)(3m+7)}{ac+(m+3)b}\,,$$

$$DE = \frac{(3m+7)(4m+9)}{ac-(2m+4)b}\,, \quad EF = \frac{(4m+9)(5m+11)}{ac+(2m+5)b}\,,$$

$$FG = \frac{(5m+11)(6m+13)}{ac-(3m+46)b}\,, \quad \cdots$$

76. If now one substitutes the values found into the continued fraction, then

$$cxy = 1 + \frac{(ac+b)x^{m+2}}{-(m+3)} + \frac{(ac-(m+2)b)x^{m+2}}{(2m+5)} + \frac{(ac+(m+3)b)x^{m+2}}{-(3m+7)} + \frac{(ac-(2m+4)b)x^{m+2}}{(4m+9)} + \cdots\,.$$

It is clear from this expression that the equation is integrable in the case where b equals one of the terms of the arithmetic progression

$$-ac\,, \quad \frac{-ac}{m+3}\,, \quad \frac{-ac}{2m+5}\,, \quad \frac{-ac}{3m+7}\,, \quad \ldots\,, \quad \frac{-ac}{im+2i+1}\,,$$

and also in the case where b equals one of the terms of the progression

$$\frac{ac}{m+2}\,, \quad \frac{ac}{2(m+2)}\,, \quad \frac{ac}{3(m+2)}\,, \quad \ldots\,, \quad \frac{ac}{im+2i}\,.$$

The above continued fraction represents the integral of the differential equation satisfying $cxy = 1$ for $x = 0$, provided that $m+2 > 0$; if $m+2 < 0$ then the integral is controlled by setting $cxy = 1$ for $x = \infty$.

77. Suppose now that $b = 0$ and $a = nc$, and substitute $x = 1$ after integration; then the equation

$$ncx^m dx + cy^2 dx + dy = 0$$

gives the following continued fraction, which determines the value of y at $x = 1$: by

$$y = \frac{1}{c} + \frac{n}{\frac{-(m+3)}{c}} + \frac{n}{\frac{2m+5}{c}} + \frac{n}{\frac{-(3m+7)}{c}} + \frac{n}{\frac{4m+9}{c}} + \cdots$$

If $c = \kappa$ then the equation

$$nx^m dx + y^2 dx + \kappa dy = 0$$

gives the value of y at $x = 1$ as

$$y = \kappa + \frac{n}{-(m\kappa+3\kappa)} + \frac{n}{(2m\kappa+5\kappa)} + \frac{n}{-(3m\kappa+7\kappa)} + \cdots$$

or equivalently

$$y = \kappa - \frac{n}{m\kappa+3\kappa} - \frac{n}{2m\kappa+5\kappa} - \frac{n}{3m\kappa+7\kappa} - \frac{n}{4m\kappa+9\kappa} - \cdots$$

78. Let us consider the following continued fraction:

$$b + \frac{1}{b+\beta} + \frac{1}{b+2\beta} + \frac{1}{b+3\beta} + \frac{1}{b+4\beta} + \cdots .$$

Then $\kappa = b$, $n = -1$, $(m+2)b = \beta$ or $m = \beta/b - 2$. Hence the value of this continued fraction equals the value of y at $x = 1$, satisfying

$$x^{(\beta-2b)/b} dx = y^2 dx + b dy ,$$

which at $x = 0$ satisfies $xy = b$ provided that $m+2 > 0$ and β/b is positive.

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† This refers to the Eneström Index in the Euler Archive: http://www.eulerarchive.com

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Index

Abelian integrals, 271
Abel's theorem, 265, 272
absolutely convergent continued fractions, 131, 234
algebraic function, 272
algebraic irrationals, 95
algebraic numbers, 95
algebraically conjugate element, 76
algorithm of regular fractions, 17
aliquot fractions, 5
angle trisection, 268
ascending continued fractions, 3, 66
associated fractions, 252
asymptotic expansion, 143

Bary's theorem, 355
Bauer–Muir transforms, 230
Bauer–Muir–Perron theory, 229
Beatty sequence, 38
Beatty's theorem, 70
Bernoulli numbers, 149
beta function, 173
Blaschke product, 329
Bombelli's method, 3, 146
Brouncker's continued fraction, 134
Brouncker's functional equation, 144
Brouncker's orthogonal polynomials, 316
Brouncker's program, 145
Budan's theorem, 303

calendar problem, 5
cardioid, 174
Carleman's criterion, 314
cattle problem, 88
Cauchy index, 309
ceiling sequence, 41
Cesàro–Nevai measure, 385
Cesàro sequence, 385
Chebyshev–Markoff theorem, 253, 254
Chebyshev polynomials, 315
Chebyshev polynomials of the second kind, 315
Chebyshev rational functions, 308
Chebyshev's example, 268
complementary error function, 198
complex Markoff test, 285
conjugate Markoff sequence, 45
consonant notes, 7
constructible field, 268
continuant, 17
continued fraction, 11
  and sums, 162
convergence, 123
convergent continued fraction, 123
convergents, 5, 12, 253
convex sequence, 147
correspondence, 36
cube duplication, 268

D. Bernoulli's inverse problem, 158
degree of algebraic number, 96
Descartes' theorem, 303
determinant identity, 328
discriminants, 95
dissonant, 7

equilibrium measure, 396
equivalence transform, 127
equivalent continued fractions, 127
equivalent irrationals, 92, 95
Erdös measures, 356
error function, 198
Euclidean algorithm, 1
Euclidean domain, 248
Euclidean rings, 247
Euler continued fraction, 160
Euler–Mascheroni constant, 151
Euler–Mindingen formulas, 15

Euler numbers, 192
Euler–Wallis formulas, 12
Euler's algorithm, 71, 72
Euler's algorithm for square surds, 259
Euler's form of Wallis' product, 169
Euler's gamma function, 146
Euler's multiplicative function, 120
Euler's quadrature formulas, 163
Euler's substitutions, 258
even part of a continued fraction, 240
evolution equation, 180
exposed point, 360
extreme point, 353

Farey's sequences, 28
Fatou's theorem, 324
Favard's theorem, 344
Fermat's question, 84
first Markoff period, 55
floor function, 17
Ford circles, 33
formal Laurent series, 249
C-fractions, 252
P-fractions, 248
fundamental inequalities, 237

Galois dual function, 410
Galois' theorem, 83
Gauss's continued fractions for ${}_2F_1$, 281
Gaussian distribution, 288
Gaussian quadrature, 299
general continued fractions, 130
Geronimus continued fraction, 332
Geronimus' theorems, 335
golden ratio, 18
Gram–Schmidt algorithm, 336
Green's function, 397
Gregorian calendar, 5

Hölder class, 388
Hamburger moment problem, 292
Hankel matrix, 256
Hardy spaces, 326
Helly's theorems, 325
Herglotz theorem, 323
Hermite polynomials, 315
Hermite–Stieltjes formula, 194
Hippasus of Metapontum, 2
Holland's theorem, 380
Huygens approximation, 21
Huygens' method, 4
Huygens' theorem, 252
Huygens' theory of real numbers, 19

hypergeometric function ${}_0F_1$, 278
hypergeometric function ${}_1F_1$, 279

integer part, 17
integrable by quadrature equation, 207
integration in finite terms, 270
intermediate convergents, 24
irrationality of π, 238
irreducible polynomial, 96

Jacobi formulas, 256
Jacobi matrix, 317
Jacobi's theorem, 256
Jean Bernoulli algorithm, 38
Jean Bernoulli period, 55
Jean Bernoulli sequences, 36, 38
Jean Bernoulli's theorem, 37
Jensen's inequality, 423

Khinchin–Ostrovskii theorem, 326
Kiepert's curve, 175
Koch and Seidel theorems, 125
Kronecker's theorem, 257

López condition, 391
Lagrange approximation, 22
Lagrange spectrum, 101, 102
Lagrange's identity, 67
Lagrange's theorem, 81
Lagrange's theory, 22
Laguerre polynomials, 320
Laguerre's theorem, 284
Lambert's theorem, 239
Laurent series, 247
Lebesgue derivative, 324
Legendre polynomials, 301
Legendre's theorem, 31, 239
Leibnitz' series, 193
lemniscate identity, 173
lemniscate of Bernoulli, 173, 174
Liouville's theorem, 272
logarithmic capacity, 396
logarithmic concave sequence, 308
logarithmic convex function, 148

Möbius transformation, 92
Markoff conditions, 44
Markoff sequence, 45, 49
  derivatives of, 53
  integrals of, 53
Markoff series, 45
Markoff's algorithm, 41
  oscillating, 58

Markoff's periods, 55
Markoff's theory, 98
Máté–Nevai condition, 366
T-matrix, 378
mean value, 37
mediant, 29
Metius' approximation, 10
minimal polynomial, 96
modified Euler continued fraction, 233
moment of inertia, 173
monic polynomial, 257

Nevai's class, 371
Nevai's theorems, 373
Newman–Schlömilch formula, 151
Newton–Cotes formulas, 298
nonextreme points, 353
nonprincipal convergents, 23
normal family, 285
normal pair, 30

octave, 7
odd part of a continued fraction, 240
orthogonal matrices, 296
orthogonal polynomials, 310
  in $L^2(d\sigma)$, 338
oscillating Markoff sequence, 46
outer function, 352

Padé approximants, 255
Padé pair, 255
Padé problem, 255
parabola theorem, 240, 243
paradox of quadratic equations, 124
paradox of Sofronov, 123
parameterization of $\mathbb{R}$, 34
partial denominators, 12
partial numerators, 12
partial sums, 158
Pell's equation, 84
perfect fifth, 7
periodic continued fraction, 71
periodic Jean Bernoulli sequences, 36
periodic measures, 405
Pochhammer symbol, 275
Poincaré metric, 354
pre-compact family, 285
Pringsheim's test, 238
Pringsheim's theorem, 238
pseudo-hyperbolic distance, 354
pure periodic continued fraction, 71
Pythagorean triples, 137

quadratic irrationals, 258
quadratic surds, 76
quadratic theory, 30
quadrature problem, 9
quadrature formulas, 298

racing algorithm, 50
Rakhmanov measures, 364
Rakhmanov's theorem, 358
Ramanujan's formula, 152, 153
ratio-asymptotic measure, 390
rational compact, 417
reduced quadratic irrational, 81
regular continued fraction, 2
regular Jean Bernoulli sequence, 48
Riccati's equation, 202, 206
Riccati's generalized equation, 206
Riesz product, 381

Schur functions, 327
Schur parameters, 327
Schur's algorithm, 327
Schur's theorem, 331
Scott–Wall inequalities, 240
second Markoff period, 55
separable polynomial, 268
Serret's theorem, 95
singular function, 380
sinnlos continued fraction, 130
sinusoidal spiral, 174
Smirnov's theorem, 326
Stieltjes continued fraction, 194
Stieltjes moment problem, 292
Stieltjes' theory, 285
Stirling's formula, 148
Stolz's theorem, 68, 229
Sturm's series, 305
Sturm's theorem, 304
Szegő function, 352
Szegő measure, 346
Szegő's entropy theorem, 347

three-term recurrence, 340
Totik's theorem, 378
transcendental numbers, 101
triangle sequences, 46

unconditionally convergent continued fractions, 130
universal measures, 386

Vahlen's theorem, 31
Van Vleck's theorem, 286

Verblunsky parameters, 344
Viète's formula, 131
Vincent's theorem, 307

Wall continued fraction, 327
Wall pair, 403
Wall polynomials, 328
Wallis' formula, 132
Wallis' hypergeometric
function, 275
Wallis' product, 131
Watson's lemma, 150
$*$-weak topology, 322
Weber–Fechner law, 7
well-tempered clavier, 7
witch of Agnesi, 136
Worpitsky's test, 237
Wronskian, 320

Zygmund class, 388