On Higher Dimensional Knots

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Which groups \( \pi \) can be fundamental group of the complement of an imbedded \( n \)-sphere in \((n + 2)\)-space?

A well-known necessary condition is for instance that \( \pi/\pi' \) be isomorphic to the group of integers. (\( \pi' \) denotes the commutator subgroup of \( \pi \).)

In the case \( n = 1 \), further necessary conditions have been given by various authors. (For instance, the first elementary ideal of \( \pi \) in the integral group ring of \( \pi/\pi' \) must be principal, generated by the “Alexander polynomial” \( \Delta_{\pi}(t) \), where \( t \) denotes a generator of \( \pi/\pi' \). Moreover, \( \Delta_{\pi}(t) \) must satisfy an equation \( \Delta_{\pi}(t^{-1}) = t^m \cdot \Delta_{\pi}(t) \) with \( m \) even. Compare [12], [4], [6].) However, the problem of characterizing knot groups by algebraic conditions remains unsolved.

Let us define a (differential) \( n \)-knot to be a differential imbedding \( f: S^n \to S^{n+2} \), and the group of the \( n \)-knot \( f: S^n \to S^{n+2} \) to be \( \pi_1(S^{n+2} - f(S^n)) \).

For \( n \geq 3 \), we get a complete algebraic characterization of the groups of \( n \)-knots. (See Theorem 1 below.)

For \( n = 2 \), we only give a set of sufficient conditions on the group \( \pi \) for the existence of an imbedding \( f: S^2 \to \Sigma^4 \) into a homotopy 4-sphere with \( \pi \cong \pi_1(\Sigma^4 - f(S^2)) \). A homotopy \( k \)-sphere is a closed differential \( k \)-manifold with the homotopy type of \( S^k \). (See Theorem 2.) A “good” set of algebraic conditions for a group \( \pi \) to be the group of a 2-knot is unknown to me. Recall that the first elementary ideal of \( \pi_1(S^4 - f(S^2)) \) need not be principal. Compare [5], Example 12. On the other hand, if the ideal is principal, generated by \( P(t) \), S. Kinoshita has proved that one can prescribe \( P(t) \) arbitrarily, subject to the only, obviously necessary condition, that \( P(1) = \pm 1 \). See [11]. I do not know whether in Theorem 2 the manifold \( \Sigma^4 \) can be taken to be \( S^4 \). This question may of course be related to the 4-dimensional Poincaré problem.

The case \( n = 1 \) will not be considered. All groups in this paper are assumed to be finitely presentable. The proofs will rely on the technique of spherical modifications as expounded in [10], §5.

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1. Statement of results

Define the weight \( w(\pi) \) of a group \( \pi \neq \{1\} \) to be the smallest integer \( k \) with the property that there exists a set of \( k \) elements \( x_1, \ldots, x_k \in \pi \) whose normal closure equals \( \pi \). By convention the trivial group \( \{1\} \) has weight 0.

It is easy to prove that for any differential imbedding \( f: S^n \to S^{n+2} \) the fundamental group \( \pi_1(S^{n+2} - f(S^n)) \) is of weight 1. (See Lemma 2 below.)

**Theorem 1.** Given \( n \geq 3 \). The group \( \pi \) is isomorphic to \( \pi_1(S^{n+2} - f(S^n)) \) for some differential imbedding \( f: S^n \to S^{n+2} \) if and only if \( \pi/\pi' \cong \mathbb{Z} \), the weight of \( \pi \) is 1, and \( H_3(\pi) = 0 \).

(Here \( H_3(\pi) \) denotes the second homology group of \( \pi \) with integral coefficients and trivial action of \( \pi \) on \( \mathbb{Z} \).)

Actually, the "only if" part of this theorem holds for \( n \geq 1 \):

**Lemma 1.** Let \( \Sigma^{n+2} \) be a homotopy \( (n + 2) \)-sphere, where \( n \geq 1 \). If

\[
\pi \cong \pi_1(\Sigma - f(S^n))
\]

for some differential imbedding \( f: S^n \to \Sigma^{n+2} \), then \( \pi/\pi' \cong \mathbb{Z} \), \( w(\pi) = 1 \), and \( H_3(\pi) = 0 \).

For \( n = 2 \), the argument in the proof of Theorem 1 breaks down. We shall get a partial result by strengthening the algebraic conditions. Define the *deficiency* of a presentation \((x_1, \ldots, x_k; R_1, \ldots, R_s)\) of a group to be the integer \( q - r \). Clearly, a group \( \pi \) with a presentation of deficiency 1 and such that \( \pi/\pi' \cong \mathbb{Z} \) satisfies the condition \( H_3(\pi) = 0 \). (Let \( K(1) \) be the \( q \)-fold wedge of \( S^2 \), and attach \( q - 1 \) two-dimensional cells to \( K(1) \) using \( R_1, \ldots, R_s \).)

The resulting 2-complex \( K \) satisfies \( \pi_1 K \cong \pi \) and \( H_2 K = 0 \). Hence \( H_3(\pi) = H_2 K/\rho(\pi_2 K) = 0 \), where \( \rho: \pi_2 K \to H_2 K \) is the Hurewicz homomorphism. Compare Hopf [7].

**Theorem 2.** Given a group \( \pi \) of weight 1, with a presentation of deficiency 1, and such that \( \pi/\pi' \cong \mathbb{Z} \). There exist a homotopy 4-sphere \( \Sigma^4 \) and a differential imbedding \( f: S^3 \to \Sigma^4 \) such that \( \pi \cong \pi_1(\Sigma^4 - f(S^3)) \).

The condition of deficiency 1 is definitely stronger (for groups satisfying \( \pi/\pi' \cong \mathbb{Z} \)) than the condition \( H_3(\pi) = 0 \). For example, the group \( G \) with the presentation \((x, a; a^2 = 1, x = axa)\) considered by R. Fox in [5], Example 12, has vanishing second homology group because it is the group of some 2-knot. (Compare Lemma 1.) However, this group does not have deficiency 1 (it has therefore deficiency 0) because its Alexander ideal, which is easily computed to be \( \mathfrak{a}_1 = (3, 1 + t) \) fails to be principal.

This shows that the group with presentation \((x, a; a^2 = 1, x = axa)\) is not "efficient" in the sense of D. Epstein [8].
(For the notion of the Alexander ideal, compare [4]. If the group \( \pi \), satisfying \( \pi/\pi' \cong \mathbb{Z} \), has deficiency 1, one can find a presentation of \( \pi \) of the form \( \langle x_1, \ldots, x_n; x_1C_1 = 1, \ldots, x_{n-1}C_{n-1} = 1 \rangle \), where \( C_1, \ldots, C_{n-1} \) belong to the commutator subgroup of the free group on \( x_1, \ldots, x_n \). It follows that an Alexander matrix of \( \pi \) has its last column consisting of zeros only. Hence this matrix has only one non-zero minor of order \( q - 1 \), i.e., the Alexander ideal \( \delta_1 \) is generated by a single element.)

The proof of Theorem 1 generalizes trivially to the case of links, i.e., unions of \( k \) disjointly imbedded \( n \)-spheres in \( S^{n+2} \).

**Theorem 3.** Given \( n \geq 3 \). The group \( \pi \) is isomorphic to \( \pi_1(S^{n+2} - L) \), where \( L \) is a union of \( k \) disjointly imbedded \( n \)-spheres if and only if \( \pi/\pi' \) is free abelian of rank \( k \), the weight of \( \pi \) is equal to \( k \), and \( H_2(\pi) = 0 \).

The proof follows closely the pattern of the proof of Theorem 1 and will be left to the reader.

Similarly:

**Theorem 4.** If the group \( \pi \) of weight \( k \) and such that \( \pi/\pi' \) is free abelian of rank \( k \) has a presentation of deficiency \( k \), then there exists a link \( L \) of \( k \) disjointly imbedded 2-spheres in some homotopy 4-sphere \( \Sigma^4 \), such that \( \pi \cong \pi_1(\Sigma^4 - L) \).

2. Proofs

We begin with the proof of a slight generalization of part of Lemma 1.

**Lemma 2.** Let \( M^{n+2} \) be a simply connected differential manifold, and \( V^n \) a connected submanifold. (Either or both manifolds possibly with boundary.) Then the weight of \( \pi_1(M - V) \) is at most 1.

Let \( a: S^1 \to M - V \) be an imbedding such that \( a(S^1) \) bounds a small 2-disc \( \tau \) in \( M \) which intersects \( V \) transversally at exactly one point. Taking a base point \( x_0 \in S^1 \), let \( a(x_0) = x_0 \) be the base point in \( M - V \). Let \( x \) be the homotopy class of \( a: (S^1, x_0) \to (M - V, x_0) \). I claim that the normal closure of \( x \) in \( \pi_1(M - V, x_0) \) is equal to \( \pi_1(M - V, x_0) \). Let \( z \in \pi_1(M - V, x_0) \) be an arbitrary element, and let \( f: (S^1, x_0) \to (M - V, x_0) \) be a differential mapping representing \( z \). Since \( M \) is simply connected, we can extend \( f \) to a map \( F: D^2 \to M \) which may be assumed to be differential and transversal to \( V \). (Compare [14].) Then \( F^{-1}(V) \) is a finite set of points \( u_1, \ldots, u_r \in D^2 - S^1 \). Let \( D_1, \ldots, D_r \) be small disjoint discs around \( u_1, \ldots, u_r \) contained in \( D^2 - S^1 \). Joining \( F(u_i) \) to \( y_0 = \tau \cap V \) by a path on \( V \), we can deform \( F \) in \( D_i \), keeping it fixed in \( D^2 - D_i \), so that the deformed map \( F \) satisfies \( F(D^2_i) = \tau \), and \( F(D_i - D^2_i) \cap V = \emptyset \), where \( D^2_i \) is concentric with \( D_i \) and of smaller positive radius. We do this for all \( i = 1, \ldots, r \). Now the new map \( F \) restricted to the boundary of \( D^2_i \)
represents \( \alpha' \), where \( \epsilon_i = +1 \) or \(-1\). Let \( z_i \) be a base point on the boundary of \( D^2 \), i.e., a point such that \( F(z_i) = x_0 \). A path on \( D^2 \) from \( z_i \) to the base point \( x_0 \) represents \( \alpha' \). The weight of \( \pi_1(M - V, x_0) \) is at most 1.

To complete the proof of Lemma 1 it remains to show that if \( f : S^n \to \Sigma^{n+2} \) is a differentiable imbedding, where \( \Sigma^{n+2} \) is a homotopy \((n + 2)\)-sphere, then \( H_*(\pi) = 0 \), where \( \pi = \pi_1(\Sigma^{n+2} - f(S^n)) \). This follows from the theorem of Hopf in [7] since \( H_q(\Sigma^{n+2} - f(S^n)) = 0 \) by Alexander duality.

The proofs of Theorem 2 and of the “if” part of Theorem 1 are based on the same construction. Thus we start proving them simultaneously. Eventually the proof will split into the two cases \( n = 2 \) and \( n \geq 3 \).

Let \( \pi \) be a given group of weight 1 and such that \( \pi/\pi' \cong \mathbb{Z} \).

Suppose we have succeeded constructing a manifold \( M^{n+2} \) such that \( \pi(M) \cong \pi \), and \( H_q(M) = 0 \) for \( 2 \leq q \leq n \), where \( n \geq 2 \). Let \( \alpha \in \pi \) be an element whose normal closure is equal to \( \pi \), and let \( \psi : S^2 \to M^{n+2} \) be a differential imbedding representing \( \alpha \), under some isomorphism \( \pi \cong \pi_1(M, x_0) \), where \( x_0 = \psi(x_0) \). Extending \( \psi \) to an imbedding \( \varphi : S^2 \times D^{n+1} \to M^{n+2} \), and performing the spherical modifification \( \chi(\varphi^2) \), we obtain a \( \Sigma^{n+2} \) which is easily seen to be a homotopy sphere. We have

\[
\Sigma^{n+2} = (M - \varphi(S^2 \times D^4)) \cup D^4 \times S^2,
\]

where \( D^{n+1} \) denotes the interior of \( D^{n+2} \), and \( \Sigma^{n+2} \) is simply connected by the theorem of van Kampen. Using \( \pi/\pi' \cong \mathbb{Z} \) and the fact that the homology class of \( \alpha \) must be a generator of \( H_q(M) \), it follows that

\[
H_q(\Sigma) = H_q(M) = 0.
\]

Compare [10], §5, Lemma 5.6. The vanishing of \( H_q(\Sigma) \) for \( 2 < q < n \) follows readily using a Mayer–Vietoris sequence.

Let \( \varphi' : D^4 \times S^2 \to \Sigma^{n+2} \) be the inclusion map, and \( f : S^n \to \Sigma^{n+2} \) the imbedding defined by \( f(u) = \varphi'(0, u) \). Clearly, \( \Sigma^{n+2} - f(S^n) \) and \( \Sigma^{n+2} - \varphi'(D^4 \times S^2) \) have the same homotopy type. Hence

\[
\pi_1(\Sigma^{n+2} - f(S^n)) \cong \pi_1(\Sigma^{n+2} - \varphi'(D^4 \times S^2)) \cong \pi_1(M - \varphi(S^2 \times D^{n+2})) \cong \pi_1(M) \cong \pi,
\]

since \( \dim M \geq 4 \).

If \( n = 2 \), I do not know how to go any further. For \( n \geq 3 \) the connected sum \( \Sigma \# (-\Sigma) \) is \( h \)-cobordant, and hence diffeomorphic to \( S^{n+2} \). (Compare [13].) Considering \( S^n \) as imbedded in the first summand, we get an imbedding \( f : S^n \to S^{n+2} \) such that \( \pi_1(S^{n+2} - f(S^n)) \cong \pi \). (If \( n = 3 \), then \( \Sigma^5 \)
is diffeomorphic to $S^3$ and we do not have to appeal to the connected sum $\Sigma \neq (-\Sigma)$.

It remains to construct the manifold $M^{n+2}$ such that $\pi_1 M \cong \pi$ and $H_* M = 0$ for $2 \leq k \leq n$.

Let $\pi = (\pi_1, \ldots, \pi_n; R_1, \ldots, R_n)$ be a finite presentation of $\pi$ (not necessarily of deficiency 1). We start with the manifold

$$M^{n+2}_0 = S^1 \times S^n \# S^1 \times S^n \# \cdots \# S^1 \times S^n,$$

i.e., the connected sum of $q$ copies of $S^1 \times S^n$. Observe that $M^{n+2}_0$ is $S$-parallelizable, i.e., the stabilized tangent bundle of $M^{n+2}_0$ is trivial. (Compare [10].) It is convenient to choose as base "point" on $M^{n+2}_0$ some contractible open subset $U \subset M_0$. The fundamental group $\pi_1(M_0, U)$ is a free group on $q$ generators which we identify with $x_1, \ldots, x_q$. Let $f_j : (S^1, x_j) \to (M_0, U)$, where $j = 1, \ldots, r$ be $r$ mutually disjoint differential imbeddings representing $R_j(x_1, \ldots, x_q)$. Since $M_0$ is orientable, we can extend $f_j$ to disjoint differential imbeddings $\varphi_j : S^1 \times D^{n+1} \to M_0$, and then perform a polyhedral framed modification $\chi(\varphi_1, \ldots, \varphi_r)$. In other words, we consider the manifold $M_1$ obtained from the disjoint union

$(M_0 - \bigcup \varphi_j(S^1 \times B^{n+1})) \cup (D^n \times S^1) \cup (D^n \times S^1) \cup \cdots \cup (D^n \times S^1)$,

by identifying $\varphi_j(u, v)$ for $u \in S^1$, $v \in S^n$ with $(u, v) \in (D^n \times S^n)$, $j = 1, \ldots, r$. We denote by $\varphi'_j$ the inclusion of $(D^n \times S^n)$ into $M_1$. It is known that $M_1$ has a natural differential structure, and that the extensions $\varphi'_j$ of $f_j$ can be chosen so that $M_1$ is $S$-parallelizable. (Compare [10], p. 521 and Lemma 5.4.) By the theorem of van Kampen, we have $\pi_1(M_1) \cong \pi$.

If the presentation $(\pi_1, \ldots, \pi_n, R_1, \ldots, R_n)$ of $\pi$ has deficiency 1, i.e., if $r = q - 1$, then $\varphi_q : H_1(\bigcup (S^1 \times D^{n+1})) \to H_1(M_0)$ must be injective since $H_1(M_0) \cong \mathbb{Z}$. Then $H_* M_0 = 0$ implies

$$H_4(M_0, \bigcup \varphi'_j(S^1 \times D^{n+1})) = 0,$$

and by excision we also have

$$H_4(M_1, \bigcup \varphi'_j(D^n \times S^n)) = 0.$$
If the presentation of \( \pi \) is not necessarily of deficiency 1, and all we know is that \( H_2(\pi) = 0 \), we have to assume \( n \geq 3 \) in order to be able to kill \( H_2 M_1 \) by spherical modifications.

Let \( \rho: \pi \to H_2 M_1 \) be the Hurewicz homomorphism. Then \( \rho(\pi M_1) \) is the subgroup of spherical homology classes, and according to H. Hopf \( H_2(\pi) = H_2 M_1 / \rho(\pi M_1) \), where the homology of \( \pi \) is understood with integer coefficients and trivial action of \( \pi \). (Compare [7].) Hence the assumption \( H_2(\pi) = 0 \) implies that \( \rho: \pi M_1 \to H_2 M_1 \) is surjective. The assumption \( n \geq 3 \) guarantees that every homology class \( \xi \in H_2 M_1 \) is representable by a differentiably imbedded 2-sphere. (Recall that according to [9] this is definitely false for \( n = 2 \).) Since \( M_1 \) is \( S \)-parallelizable, every differentiably imbedded 2-sphere in \( M_1 \) has a trivial normal bundle.

Following the discussion in \$6\$ of [10], we construct a finite sequence \( M_1, M_2, \ldots, M_n, \ldots \) of \((n + 2)\)-manifolds. \( M_{k+1} \) is obtained from \( M_k \) by a spherical modification \( \chi(\phi_k) \) were \( \phi_k: S^k \times D^{n} \to M_k \) represents a suitable class \( \xi_k \in H_2 M_k \).

Since \( M_{k+1} = (M_k - \phi_k(S^k \times B^{n})) \cup D^k \times S^{n-1} \), it follows from \( n \geq 3 \), using the theorem of van Kampen, that \( \pi_1 M_{k+1} \cong \pi_1 M_k \cong \pi \). Hence, at every stage of the construction, \( \rho: \pi_2 M_k \to H_2 M_k \) is surjective (by Hopf’s theorem [7]) and every class in \( H_2 M_k \) is representable by a differentiably imbedded 2-sphere. Taking care of performing only framed spherical modifications (see \$6\$ of [10]), all the manifolds \( M_k \) will be \( S \)-parallelizable, and every differentiably imbedded 2-sphere in \( M_k \) has a trivial normal bundle.

The choice of \( \xi_k \in H_2 M_k \) is made as follows: (1) If \( H_2 M_k \) is infinite, then \( H_2 M_k \) contains a primitive element \( \xi_k \). This means that there exists an element \( \eta \in H_2 M_k \) such that \( \xi_k \cdot \eta = 1 \). Then, killing \( \xi_k \) by a spherical modification also kills \( \eta \) and \( H_2 M_k \cong H_2 M_k / (\xi_k) \). For the proof, see Lemma 5.6 of [10]. (2) If \( H_2 M_k \) is finite (and non-zero) then \( n \) must equal 3. Killing an arbitrary element \( \xi_k \in H_2 M_k \) by spherical modification introduces a new class \( \xi' \in H_2 M_{k+1} \) necessarily of infinite order. More precisely, the sequence

\[
0 \to Z \to H_2 M_{k+1} \to H_2 M_k / (\xi_k) \to 0
\]

is exact in this case, as proved in [10], p. 16. The next modification will be taken to kill a primitive element \( \xi_{k+1} \) in the newly introduced infinite cyclic summand of \( H_2 M_{k+1} \). (In general \( \xi_{k+1} \neq \xi' \).)

It is left to the reader to convince himself that the non-vanishing of the fundamental groups of \( M_k \) in the present situation does not invalidate Lemma 5.6 and 5.8 of [10]. The proofs given in [10] apply word-for-word to the present case.

Hence, choosing \( \xi_k \in H_2 M_k \) to be primitive if \( H_2 M_k \) is infinite, and an
arbitrary generator if $H_qM$ is finite, the sequence $M_1, \ldots, M_n, \ldots$ leads in a finite number of steps to a manifold $M_i = M$ such that $\pi_iM \cong \pi$ and $H_qM = 0$.

If $n = 3$, then $H_qM = 0$ implies $H_qM = 0$ by Poincaré duality since $H_1M \cong \pi/\pi \cong \mathbb{Z}$ is free abelian. If $n > 3$, then $H_qM = \cdots = H_{n-1}M$ because then $H_qM$ is torsion free, and killing a primitive class of $H_qM$ leaves the $q$-dimensional homology group unchanged for $3 \leq q \leq n - 1$. Moreover, $H_qM = 0$ again by Poincaré duality.

Thus the manifold $M$ satisfies the requirements $\pi_iM \cong \pi$ and $H_qM = 0$ for $q = 2, \ldots, n$. This completes the proof of Theorem 1.

REMARK. Of course, having an imbedding $f : S^n \rightarrow S^q$ such that $\pi_i(S^n - f(S^n)) \cong \pi$ is easy to obtain an imbedding $f_\pi : S^n \rightarrow S^{m+2}$ with $\pi_i(S^{m+2} - f_\pi(S^n)) \cong \pi$ for all $m \geq 3$ by the Artin “spinning” construction. (Compare Artin [1].)

3. Appendix

This last section contains some remarks which came up in discussions during the Symposium. They are due for the most part to M. Hirsch, J. Milnor, and J. Stallings. (However, except for Theorem III and Milnor’s example under (3) below, the proofs given here are my own.)

The first question was: What is the situation in the case of a differentiably imbedded homotopy $n$-sphere $\Sigma^n$ in $S^{m+2}$?

THEOREM I. A homotopy $n$-sphere $\Sigma^n$ can be imbedded in $S^{m+2}$ (by a differentiably imbedding) if and only if $\Sigma^n$ is the boundary of a parallelizable $(n + 1)$-manifold.

PROOF. Suppose $\Sigma^n$ is imbeddable in $S^{m+2}$, then $\Sigma^n$ bounds a parallelizable $(n + 1)$-manifold. In fact, every $\Sigma^n \subset S^{m+2}$ bounds a parallelizable $V_{m+1} \subset S^{m+2}$. Let $U$ be a tubular neighborhood of $\Sigma^n \subset S^{m+2}$ and $T = \partial U = \partial U - U$.

Since $\Sigma^n$ has a trivial normal bundle in $S^{m+2}$, we have a diffeomorphism $\Sigma^n \times D^q \cong \partial U$. Let $\Sigma^n \subset T$ be the submanifold corresponding to $\Sigma^n \times \{x_1\}$, where $x_1 \in S^1 \subset D^q$. It is enough to show that $\Sigma^n$ bounds a parallelizable $(n + 1)$-manifold $V$ in $S^{m+2} - U$. The Pontrjagin-Thom construction applied to $\Sigma^n \subset T^{m+1}$ and a normal vector-field to $\Sigma^n \subset T$ yields a map $\Phi : T \rightarrow S^1$. We extend this map to a map $\Phi : S^{m+2} - U \rightarrow S^1$. The only obstruction to the extension is a cohomology class $\gamma \in H^1(S^{m+2} - U, T)$. (Coefficients in $\pi_1(S^n) \cong \mathbb{Z}$.) By the Hopf theorem [Proposition 13.1 on p. 189 of S. Hu, Homotopy Theory, Academic Press, 1959.], we have $\gamma = \delta \sigma$, where $\sigma \in H^2T$ is the Poincaré dual of the homology class of $\Sigma^n$. 

in \( T \). By the Alexander duality theorem, \( \Sigma_1 \) is homologous to zero in \( \mathbb{S}^{n+1} - U \), hence there exists a class \( \xi \in H_{n+1}(\mathbb{S}^{n+1} - U, T) \) such that \( \partial \xi = (\Sigma_1) \), the homology class of \( \Sigma_1 \) (in \( H_n(T) \)). Using the diagram

\[
\begin{array}{ccc}
H^i(\mathbb{S}^{n+1} - U) & \xrightarrow{i^*} & H^i(T) \\
\uparrow & & \uparrow \\
H_{n+1}(\mathbb{S}^{n+1} - U, T) & \xrightarrow{\partial} & H_n(T)
\end{array}
\]

where the vertical isomorphisms are given by Poincaré duality, we see that the dual of \( \xi \) maps by \( r \) onto \( \pm \sigma \). (The diagram is commutative up to sign.) Hence, \( \partial \sigma = \gamma = 0 \). Keeping \( \Phi|T = \Phi \) fixed, we can approximate \( \Phi : \mathbb{S}^{n+1} - U \to \Sigma^1 \) by a differential map, and assume that

\[
\Phi(\Sigma_1) = a \in \mathbb{S}^1
\]

is a regular value of \( \Phi \). Then \( \Phi^{-1}(a) = V_{n-1} \subset \mathbb{S}^{n+1} \) or at least the component of \( \Sigma_1 \) in \( \Phi^{-1}(a) \) is the desired manifold.

In particular, this shows that every higher knot has some sort of "generalized genus."

The converse is clear for \( n = 1 \) or \( 2 \). For \( n = 3 \), every compact orientable 3-manifold in imbeddable in \( \mathbb{S}^5 \). [M. Hirsch, \textit{Ann. of Math.}, 74 (1961), 494–497. Theorem 3.] If \( n = 4 \) we argue as follows. Every homotopy 4-sphere \( \Sigma^4 \) is the boundary of a contractible 6-manifold \( V \). (Compare [10], Theorem 6.6.) Let \( \Sigma^5 \) be the double of \( V \). The homotopy 5-sphere \( \Sigma^5 \) is the boundary of a contractible 6-manifold \( W \), and according to Smale [13] \( W \) is diffeomorphic to the 6-disc \( D^6 \). Hence, \( \Sigma^5 \) is diffeomorphic to \( \mathbb{S}^5 \), and so every homotopy 4-sphere is imbeddable in \( \mathbb{S}^5 \). Let \( n \geq 5 \), and suppose that the homotopy \( n \)-sphere \( \Sigma^5 \) is the boundary of a parallelizable \((n+1)\)-manifold. If \( n \) is even, then \( \Sigma^5 \) is \( h \)-cobordant to \( \mathbb{S}^5 \). (See [10], Theorem 6.6.) Hence, according to Smale [13], \( \Sigma^5 \) is diffeomorphic to \( \mathbb{S}^5 \). If \( n \) is odd, \( \Sigma^5 \) is \( h \)-cobordant and hence diffeomorphic to the boundary of a \( \frac{1}{2}(n-1) \)-connected manifold \( W^{n+1} \), where \( W \) is the connected sum along the boundary of a finite number of copies of some manifold \( W \). In Part II of [10] we give an explicit construction of \( W \) as a "thickened" wedge of \( k \)-spheres, where \( 2k = n + 1 \). Each \( S^k \) imbedded in \( W \) has a stably trivial normal bundle. Now any wedge of \( k \)-spheres can be imbedded in \( S^{n+1} \), and the imbeddability of \( W \) in \( S^{n+1} \) follows if we prove that every stably trivial \( SO_k \)-bundle over \( S^k \) can be realized by a normal field of \( k \)-planes over \( S^k \) in \( S^{n+1} \). Let \( f^{k+1} \) be a trivialization of the normal bundle of \( S^k \subset S^{k+1} \). If \( v \) is the field of normal vectors over \( S^k \) corresponding to a given map \( \alpha : S^k \to S^k \), i.e., \( v(x) = \sum_0^k \alpha(x)f_i(x) \) for all \( x \in S^k \), where

\[
f^{k+1}(x) = \{f_0(x), \ldots, f_k(x)\},
\]
then the $SO_k$-bundle over $S^k$ given by the field of normal $k$-planes orthogonal to $v$ has a characteristic element $\chi \in \pi_{k-1}(SO_k)$ given by $\chi = \tau(\alpha)$, where $\tau: \pi_n(S^k) \to \pi_{n-1}(SO_k)$ is the transgression homomorphism of the fibration $SO_k \to SO_{k+1} \to S^k$ and $(\alpha)$ is the homotopy class of $\alpha$. Stably trivial $SO_k$-bundles over $S^k$ are exactly those whose characteristic element is in the image of $\tau$. Hence, by a proper choice of $(\alpha)$, one can realize any stably trivial $SO_k$-bundle over $S^k$ by a field of normal $k$-planes in $S^{2k+1}$. This completes the proof of Theorem 1.

**Theorem II.** (J. Milnor.) If $\Sigma^n$ is the boundary of a parallelizable manifold, there exists a differential imbedding $f_\Sigma: \Sigma^n \to S^{n+2}$ such that $\pi_1(S^{n+2} - f_\Sigma(\Sigma^n)) \cong \mathbb{Z}$.

**Proof.** The statement is trivial for $n = 1, 2$ and $n$ even $\geq 6$ since then $\Sigma^n$ is diffeomorphic to $S^n$ as observed before. For $n = 4$ it follows since every homotopy 4-sphere is imbeddable in $S^6$. (Compare the proof of Theorem I.) If $n$ is odd, let $W^{2k} \subset S^{2k+1}$, where $2k = n + 1$, be a connected submanifold of $S^{2k+1}$ such that $bW = \Sigma^n$ and $S^{2k+1} - W$ is simply connected. We shall prove that for such an imbedding $\Sigma^n \subset S^{n+2}$, we have $\pi_1(S^{n+2} - \Sigma^n) \cong \mathbb{Z}$. The existence of a submanifold $W^{2k} \subset S^{2k+1}$ of the required kind will be shown at the end of the proof. Let $\ell$ be a loop in the complement of $\Sigma^n$. We can assume that $\ell$ is a differentiable curve, transversal to $W$. Let $A(\ell)$ be the algebraic intersection number $I(\ell, W) \in \mathbb{Z}$. Clearly, $A(\ell)$ depends only on the homotopy class of $\ell$ in $\pi_1(S^{n+2} - \Sigma^n, x_0)$, where $x_0 = \ell(0)$, and provides a homomorphism $\overline{A}: \pi_1(S^{n+2} - \Sigma^n, x_0) \to \mathbb{Z}$. It is obvious that $\overline{A}$ is surjective. Suppose that $A(\ell) = 0$. We show that $\ell$ is homotopic in $S^{n+2} - \Sigma^n$ in a loop in $S^{n+2} - W$. Unless $\ell$ is already a loop in $S^{n+2} - W$, there exist a pair $t < t'$ of values of $t$ and small positive numbers $\epsilon, \epsilon'$ such that $\ell([t + \epsilon, t' - \epsilon])$ is contained in $S^{n+2} - W$, and

$$I(\ell([t - \epsilon, t + \epsilon]), W) = I(\ell([t' - \epsilon', t' + \epsilon']), W) \neq 0,$$

where the sets $\ell([t - \epsilon, t + \epsilon]) \cap W$ and $\ell([t' - \epsilon', t' + \epsilon']) \cap W$ consist of a single point. Since $S^{n+2} - W$ is simply connected, $\ell([t + \epsilon, t' - \epsilon'])$ is homotopic in $S^{n+2} - W$ (and a fortiori in $S^{n+2} - \Sigma^n$) to a path $\ell'$ contained in a neighborhood of $W$. (Use a path on $W$ connecting $\ell(t)$ and $\ell(t')$.) Replacing the portion of $\ell$ between $t + \epsilon$ and $t' - \epsilon'$ by $\ell'$ and using a field of normal vectors to $W$ in $S^{n+2}$ we can "push" the path

$$\ell([t - \epsilon, t + \epsilon]) \cdot \ell'([t + \epsilon, t' - \epsilon']) \cdot \ell([t' - \epsilon', t' + \epsilon'])$$

across $W$ (keeping its endpoints fixed) without intersecting $\Sigma^n$, thus removing the two intersection points $\ell(t)$ and $\ell(t')$ of $\ell$ and $W$. The statement that $\ell'$ is homotopic in $S^{n+2} - \Sigma^n$ to a loop in $S^{n+2} - W$ then
follows by induction on the cardinality of the finite set \( A(I) \cap W \). Since \( S^{n+2} - W \) is simply connected, \( A(I) = 0 \) implies \( I \cong 1 \), hence \( A \) is an isomorphism and \( \pi_1(S^{n+2} - \Sigma^n) \cong \mathbb{Z} \).

If \( n \geq 5 \), we take for \( W \) a finite connected sum along the boundary of thickened wedge of \( k \)-spheres, and the imbedding \( W^{2k} \subset S^{2k+1} \) whose existence has been shown in the proof of Theorem I. Since \( W \) retracts by deformation on a wedge of \( k \)-spheres, we clearly have

\[
\pi_1(S^{2k+1} - W^{2k}) = \{ 1 \}.
\]

If \( n = 3 \), take an imbedding \( \Sigma^3 \subset S^4 \) which bounds a simply connected \( W^4 \subset S^4 \). (Compare Hirsch, Ann. of Math., 74 (1961), 494–497, Theorem 3.) I claim that \( S^5 - W \) is simply connected. Let \( f: S^5 \to S^5 - W \) be a given differential imbedding and denote again by \( f \) an extension \( f: (D^4, S^3) \to (S^5, S^5 - W) \). First, we can push \( f(D^4) \) away from \( S^5 = bW^4 \). To do this assume that \( f \) is transversal to \( W \) and \( \Sigma^3 \). If \( x, x' \) are two intersection points of \( f(D^4) \) and \( \Sigma^3 \) with opposite intersection numbers, then \( x, x' \) are the endpoints of a path \( w \) in \( f(D^4) \cap W \) and also endpoints of a path \( w' \) on \( \Sigma^3 \). Let \( \varphi: \varphi(Q) \to \varphi(W) \) be an immersion of the 2-disc with two corners \( Q \) into \( W \) such that \( \varphi|bQ = w' \cdot w^{-1} \). (\( W \) is simply connected.) Keeping \( \varphi|bQ \) fixed we can approximate \( \varphi \) by an imbedding into \( S^5 \) such that \( \varphi(\text{int } Q) \cap \Sigma^3 = \emptyset \). Using a field of normal 3-frames on \( \varphi(Q) \) we can push \( f \) along \( \varphi(Q) \) using the method of H. Whitney [Ann. of Math., 45 (1944), 220–246]. The effect of this operation is to remove the intersection points \( x, x' \) from \( f(D^4) \cap \Sigma^3 \). After a finite number of such operations we obtain a new mapping \( f: (D^4, S^3) \to (S^5 - \Sigma^3, S^4 - W^4) \) with \( f|S^3 \) unchanged.

Now, assuming \( f \) still transversal to \( W \), the intersection \( f(D^4) \cap W \) consists of a finite number of disjoint closed curves. Let \( C_1, \ldots, C_n \) be the corresponding curves on \( D^4 \). Let \( C_1 \) bound a disc \( D_1 \) on \( D^4 \). Since \( f(C_1) \) is homotopic to a point in \( W \), we can replace \( f|D_1 \) by a mapping into \( W \). Using a normal vector-field on \( W \), \( f \) can be pushed away from \( W \) in a neighborhood of \( D_1 \), thus reducing the number of components of \( f(D^4) \cap W \) by at least 1. Hence the intersection \( f(D^4) \cap W \) can also be removed in a finite number of steps, proving that \( f|S^3 \) is homotopic to a point in \( S^5 - W \). Thus \( \pi_1(S^5 - W) = \{ 1 \} \). This completes the proof of Theorem II.

**Theorem III.** (M. Hirsch.) Let \( f: \Sigma^n \to S^{n+2} \) be a differential imbedding with \( n \geq 5 \). If \( S^{n+2} - f(\Sigma^n) \) has the homotopy type of \( S^3 \), then \( \Sigma^n \) is diffeomorphic to \( S^3 \).

**Proof.** Let \( g: S^1 \to S^{n+2} - f(\Sigma^n) \) be a differential imbedding which is a homotopy equivalence. Let \( U \) be a tubular neighborhood of \( f(\Sigma^n) \) and
set \( V^{n+4} = S^{n+2} - U \). Extending \( g \) to an imbedding \( g: S^2 \times D^{n+1} \to V^{n+3} \) using a trivialization of the normal bundle of \( g(S^1) \). Then

\[
V' = V - g(S^1 \times \text{int} D^{n+1})
\]

provides a simple \( h \)-cobordism between \( g(S^1 \times S^n) \) and \( bU = bV \). It is easily seen that inclusions \( g(S^1 \times S^n) \subset V' \) and \( bU \subset V' \) are homotopy equivalences, and since the fundamental group is infinite cyclic, they are simple homotopy equivalences (compare J. H. C. Whitehead, \textit{Proc. London Math. Soc.}, 45 (1939), 243–327). It follows by a theorem of B. Masur [\textit{Ann. of Math.}, 77 (1963), 232–249] that \( bU \) is diffeomorphic to \( S^1 \times S^n \). Since \( \Sigma^1 \) has a trivial normal bundle, \( bU \) is diffeomorphic to \( S^1 \times \Sigma^n \). Hence there exists a diffeomorphism \( h: S^1 \times \Sigma^n \to S^1 \times S^n \).

Lifting \( h \) to the universal covering, we get a diffeomorphism \( H: \tilde{R} \times \Sigma^n \to \tilde{R} \times S^n \subset \tilde{R}^{n+1} \). Hence \( \Sigma^n \) can be differentially imbedded into \( \tilde{R}^{n+1} \). The compact region \( W \) bounded by \( H(0 \times \Sigma^n) \) in \( \tilde{R}^{n+1} \) is a contractible manifold with boundary \( \Sigma^n \). Using the assumption \( n \geq 5 \), it follows by Smale [13] that \( W \) is diffeomorphic to \( D^{n+1} \). In particular, \( \Sigma^n = bW \) is diffeomorphic to \( S^n \).

\textbf{Theorem IV.} Given a homotopy \( n \)-sphere \( \Sigma^n \) which is the boundary of a parallelizable manifold, \( n \geq 3 \), and a finitely presentable group \( \pi \) of weight 1, such that \( \pi/\pi \cong \mathbb{Z} \) and \( H_1(\pi) = 0 \), there exists a differential imbedding \( f: \Sigma^n \to S^{n+2} \) with \( \pi \cong \pi_1(S^{n+2} - f(\Sigma^n)) \).

\textbf{Proof.} Let \( \Sigma^n \subset S^{n+2} \) be an imbedding such that \( \pi_1(S^{n+2} - \Sigma^n) \cong \mathbb{Z} \) (Theorem II). Let \( f_\alpha: S^n \to S^{n+2} \) be an \( n \)-knot with \( \pi_1(S^{n+2} - f_\alpha(S^n)) \cong \pi \) (Theorem I). Take the relative connected sum \( (S^{n+2}, \Sigma^n) \# (S^{n+2}, f_\alpha(S^n)) \), and let \( f: \Sigma^n \to S^{n+2} \) be the resulting differential imbedding. Then \( \pi_1(S^{n+2} - f(\Sigma^n)) \cong \pi \mathbb{Z}(\alpha^{-1}) \), where \( \gamma \) generates \( \mathbb{Z} \), and \( \alpha \) is some element of \( \pi \). Hence, \( \pi_1(S^{n+2} - f(\Sigma^n)) \cong \pi \).

If the complement of a 1-knot \( f: S^1 \to S^n \) has an infinite cyclic fundamental group, then \( \mathbb{Z} - f(S^1) \) and \( S^n \) have the same homotopy type. This is a simple consequence of the Lemma of Dehn-Papakyriakopoulos [\textit{Ann. of Math.}, 66 (1957), 1–20].

For which values of \( n \) can one imbed \( S^n \) into \( S^{n+2} \) so that \( \pi_1(S^{n+2} - f(S^n)) \) is infinite cyclic but \( \pi_1(S^{n+2} - f(S^n)) \neq \pi_1(S^{n+2}) \)?

\textbf{Theorem V.} (J. Stallings) For \( n \geq 3 \) there exist smooth imbeddings \( f: S^n \to S^{n+2} \) such that \( \pi_1(S^{n+2} - f(S^n)) \cong \mathbb{Z} \) and \( \pi_1(S^{n+2} - f(S^n)) \neq 0 \).

For \( n = 2 \) the question remains open.

\textbf{Proof.} Let \( M_\theta = S^1 \times S^{n+1} \neq S^2 \times S^n \), and let \( J \) denote the multiplicative infinite cyclic group generated by the symbol \( t \). Choosing some base point \( x_0 \in M_\theta \) and a generator of \( \pi_1(M_\theta, x_0) \), we identify \( \pi_1(M_\theta, x_0) \).
with $J$. Then, $\pi_0(M_0, x_0)$ is a free $\mathbb{Z}[J]$-module on one generator, where $\mathbb{Z}[J]$ is the integral group ring of $J$. Choosing a generator of $\pi_0(M_0, x_0)$, we can identify $\pi_0(M_0, x_0)$ with $\mathbb{Z}[J]$. Let $\varphi_0 : S^3 \to M_0$ be a differential imbedding representing the element $(2 - t) \in \pi_0(M_0, x_0) = \mathbb{Z}[J]$. Clearly, $\varphi_0$ has a trivial normal bundle and we can extend $\varphi_0$ to an imbedding $\varphi : S^3 \times D^n \to M_0$. Let $M = \chi(M_0, \varphi)$ be the manifold obtained from $M_0$ by spherical modification. $M = (M_0 - \varphi(S^3 \times \text{int } D^n)) \cup D^3 \times S^{n-1}$.

Since the homology class of $\varphi_0(S^3)$ is a generator of $H_4M_0$, it follows that $H_iM = \cdots = H_nM = 0$. Since $n \geq 3$, $\pi_iM \cong \mathbb{Z}$ and it is easily seen that $\pi_2M$ is non-trivial. Indeed, $M_0$ and $M$ are the two components of the boundary of a manifold $W$ which is homotopy equivalent to $M_0 \cup e^3$ and $M \cup e^n$. Hence, the inclusions $M_0 \to W$ and $M \to W$ induce epimorphisms $i_* : \pi_2M_0 \to \pi_2W$ and $\pi_2M \to \pi_2W$. The kernel of $i_*$ is the ideal of $\mathbb{Z}[J]$ generated by $2 - t$. Hence, $\pi_2W = \text{Im } i_* \neq 0$, and therefore $\pi_2M \neq 0$.

Let $\psi : S^1 \times D^{n+1} \to M$ be an imbedding representing a generator of $\pi_1M \cong \mathbb{Z}$. Then, the manifold

$$\chi(M, \psi) = (M - \psi(S^1 \times \text{int } D^{n+1})) \cup D^3 \times S^n$$

is a homotopy $(n + 2)$-sphere $\Sigma^{n+2}$. Let $f : S^n \to \Sigma^{n+2}$ be the imbedding $S^n \to 0 \times S^n \subset D^3 \times S^n \subset \Sigma^{n+2}$. We have

$$\pi_1(\Sigma^{n+2} - f(S^n)) \cong \pi_1M \cong \mathbb{Z},$$

and

$$\pi_2(\Sigma^{n+2} - f(S^n)) \cong \pi_2(\Sigma^{n+2} - D^3 \times S^n) \cong \pi_2M \neq 0.$$ 

By suitably changing the differentiable structure of $\Sigma^{n+2}$ in the complement of $f(S^n)$ we obtain a differential imbedding $f : S^n \to S^{n+2}$ with

$$\pi_1(S^{n+2} - f(S^n)) \cong \mathbb{Z} \quad \text{and} \quad \pi_2(S^{n+2} - f(S^n)) \neq 0.$$ 

Finally, we discuss the independence of the algebraic conditions:

(i) $\pi/n' \cong \mathbb{Z};$

(ii) There exists an element $x \in \pi$ whose normal closure is $\pi$;

(iii) $H_4(\pi) = 0$.

1. Any finite cyclic group satisfies (ii) and (iii) and does not satisfy (i).

2. Let $G$ be a non-trivial group such that $G = G'$, and suppose that $G$ has a non-trivial finite dimensional unitary representation. We also assume that $G$ has a presentation of deficiency 0. For instance, let $G$ be
the group with the presentation \( G = \langle \alpha, \beta ; \alpha^2 = \beta^3 = (\alpha^{-1} \beta)^4 \rangle \). A representation \( \rho : G \to U(5) \) into the group of \( 5 \times 5 \) unitary matrices is given by

\[
\rho(\alpha) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad \rho(\beta) = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Under the above assumptions on \( G \), the group \( \pi = G * \mathbb{Z} \) satisfies (i) and (iii). It even satisfies the requirement stronger than (iii) of having deficiency 1. I claim that the weight of \( \pi \) is bigger than 1. We prove this for the above group \( G = \langle \alpha, \beta ; \alpha^2 = \beta^3 = (\alpha^{-1} \beta)^4 \rangle \). In the general case the proof is quite similar. Let \( W(\alpha, \beta, \gamma) \) be a word representing an element \( \xi \) in \( \pi = G * \mathbb{Z} \). (\( \gamma \) denotes a generator of \( \mathbb{Z} \).) Claim: \( \pi(\xi) \neq \{1\} \).

The statement is trivial unless the sum of exponents of \( \gamma \) in \( W(\alpha, \beta, \gamma) \) is equal to \( \pm 1 \). (Look at the abelianized group of \( \pi(\xi) \).) We can then assume that the exponent-sum of \( \gamma \) in \( W(\alpha, \beta, \gamma) \) is \( \pm 1 \). Regard \( W(A, B, X) = E \), where \( A = \rho(\alpha) \) and \( B = \rho(\beta) \) as an equation in \( U(5) \) for the unknown matrix \( X \). (\( E \) denotes the unit matrix of \( U(5) \).)

Following M. Gerstenhaber and O. Rothaus, consider the mapping \( w : U(5) \to U(5) \) defined by \( w(X) = W(A, B, X) \). (Compare Proc. Nat. Acad. Sci., 48 (1962), 1531–1533. I am very grateful to W. Magnus and G. Baumslag for pointing out this paper to me.) Since \( U(5) \) is connected, \( w \) is homotopic to the mapping \( w_{\rho} : U(5) \to U(5) \) defined by

\[
w_{\rho}(X) = W(E, E, X).
\]

Since \( X \) enters into \( W(A, B, X) \) with exponent-sum 1, we have

\[
W(E, E, X) = X.
\]

Hence, \( w_{\rho} \) is the identity mapping, and \( w \) must be a surjective map since \( U(5) \) is a finite dimensional manifold. In other words, the equation \( W(A, B, X) = E \) has a solution \( C \in U(5) \), i.e., \( W(A, B, C) = E \). Now, let \( \pi_{\alpha} \subset U(5) \) be the subgroup of \( U(5) \) generated by \( A, B, C \). Then \( \pi_{\alpha} \neq \{1\} \) and \( \pi(\xi) \) maps surjectively onto \( \pi_{\alpha} \) by \( \rho_{\alpha} : \pi(\xi) \to U(5) \) defined by \( \rho_{\alpha}(\alpha) = A, \rho_{\alpha}(\beta) = B, \rho_{\alpha}(\gamma) = C \). It follows that \( \pi(\xi) \neq \{1\} \).

Since this argument applies to any group element \( \xi \in \pi \), the group \( \pi \) does not satisfy (ii).

(3) J. Milnor gave the following example of a group \( \pi \) satisfying (i) and (ii) but not (iii). Let \( A \) be a unimodular \( 2 \times 2 \) matrix of determinant \( +1 \),
and let \( \alpha : S^1 \times S^1 \to S^1 \times S^1 \) be the corresponding automorphism of the torus:

\[
\alpha(z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d),
\]

where

\[
A = \begin{vmatrix}
a & b \\
c & d
\end{vmatrix}.
\]

\((a, b, c, d \in \mathbb{Z})\). Let \( E_4 \) be the closed orientable 3-manifold obtained from \( S^1 \times S^1 \times I \) by identifying \( S^1 \times S^1 \times \{0\} \) with \( S^1 \times S^1 \times \{1\} \) using \( \alpha \).

Then \( E_4 \) is the total space of a (locally trivial) fibration over \( S^1 \) with fibre \( S^1 \times S^1 \). It follows that \( \pi_1(E_4) \neq 0 \). Since \( \alpha(1, 1) = (1, 1) \), we can define a section \( S^1 \to E_4 \) by mapping each \( z \in S^1 \) into the point \((1, 1)\) of the fibre over \( z \). This defines an element \( \tau \in \pi_1(E_4, x_0) \), where \( x_0 \) is the point \((1, 1) \in S^1 \times S^1 \) of the fibre over \( 1 \in S^1 \).

Let \( \xi, \eta \) be the generators of the fundamental group of the fibre over \( 1 \in S^1 \) represented by \( \xi(t) = (e^{2\pi i t}, 1) \) and \( \eta(t) = (1, e^{2\pi i t}) \). The group \( \pi_1(E_4, x_0) \) is generated by \( \tau, \xi, \eta \) and has the presentation \( \langle \tau, \xi, \eta; [\xi, \eta] = 1, \tau^{-1}\xi\tau = \xi^a \eta^b, \tau^{-1}\eta\tau = \xi^c \eta^d \rangle \).

Hence, denoting \( \pi_1(E_4, x_0) \) by \( \pi \), we have \( \pi/\pi' \cong \mathbb{Z} \) if and only if \((a - 1)(d - 1) - bc = \pm 1\). Then, \( H_4(E_4) \cong \mathbb{Z} \) by Poincaré duality, and since \( \pi_1(E_4) = 0 \), it follows that

\[
H_4(\pi) = H_4(E_4)/\pi\pi_4(E_4) \cong \mathbb{Z}.
\]

Moreover, if \((a - 1)(d - 1) - bc = \pm 1\), the normal closure of \( \tau \) in \( \pi \) is equal to \( \pi \).

A simple example of a unimodular matrix satisfying the above requirements is

\[
\begin{vmatrix}
1 & 1 \\
-1 & 0
\end{vmatrix}
\]

The corresponding group, satisfying (i), (ii) but not (iii), has the presentation \( \langle \tau, \xi, \eta; \xi^{-a} \eta^{-b} \xi \eta = 1, \tau^{-1} \xi \tau = \xi \eta, \tau^{-1} \eta \tau = \xi^{-1} \rangle \).

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References

ON HIGHER DIMENSIONAL KNOTS