Lectures on the Theorem of Browder and Novikov

and

Siebenmann's Thesis

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PART I

THEOREM OF BROWDER AND NOVIKOV

§ 1. PRELIMINARIES.

1.1. THE CAP-PRODUCT.

The homology and the cohomology groups we use are the singular ones. Let $\mathbb{Z}$ denote the ring of integers and Λ an arbitrary commutative ring with $1 \neq 0$. For any topological space X and any integer $n \geqslant 0$ the set of singular n-simplices of X is denoted by $S_n(X)$. For any $s \in S_n(X)$ and any integer i satisfying $0 \leq i \leq n$ let $s(0,..i)$(resp. $s(i,..,n)$) denote the element of $S_i(X)$(resp. $S_{n-i}(X)$) got by restricting s to the front i-dimensional (resp. the rear (n-i)-dimensional) face of the standard n-simplex Δ_n. Let $C(X)$ denote the singular chain complex of X over $\mathbb{Z}$ and $C = C(X) \otimes_{\mathbb{Z}} \Lambda$ the chain complex of X over Λ. The cochain complex of X over Λ which is defined as $\mathrm{Hom}_{\mathbb{Z}}(C(X), \Lambda)$ is canonically isomorphic to $\mathrm{Hom}_{\Lambda}(C(X) \otimes_{\mathbb{Z}} \Lambda, \Lambda)$. The boundary homomorphism δ in $C^* = \mathrm{Hom}_{\Lambda}(C,\Lambda)$ is given by $f = (-1)^{n-1} f \circ \partial$ for every $f \in C^n(X, \Lambda) = \mathrm{Hom}\,(C_n, \Lambda)$ where $\partial : C_n \rightarrow C_{n-1}$ is the boundary homomorphism in C. As usual C^* is considered as a chain complex with $C^*_{-n} = C^n(X, \Lambda)$. The evaluation map $e : C^* \otimes_{\Lambda} C \rightarrow \Lambda$ is defined by $e(f \otimes c) = f(c) \; \forall f \in C^*_{-n}$ and $c \in C_n$ and $e \mid C^*_{-p} \otimes C_q = 0$ whenever $p \neq q$. Considering Λ as a chain complex (with all its elements of degree zero) it is easily seen that $e : C^* \otimes_{\Lambda} C \rightarrow \Lambda$ is a chain homomorphism.

For any two chain complexes A and B over Λ let $\alpha : H(A)\underset{\Lambda}{\otimes}H(B) \longrightarrow H(A\otimes B)$ be the natural map. If $x \in H_p(A)$ and $y \in H_q(B)$ and if z and z' are respectively cycles of A and B representing x and y, then $z \otimes z'$ is a cycle of $A \otimes B$ and the homology class of $z \otimes z'$ is by definition $\alpha(x \otimes y)$. Let $T : A \underset{\Lambda}{\otimes} B \longrightarrow B \underset{\Lambda}{\otimes} A$ be the chain isomorphism given by $T(a \otimes b) = (-1)^{pq}\, b \otimes a \;\forall\, a \in A_p,\ b \in B_q$.

The Alexander-Whitney diagonal map $m_o : C \longrightarrow C \underset{\Lambda}{\otimes} C$ is defined to be the unique Λ-homomorphism satisfying

$$m_o(s) = \sum_{i=0}^{n} s(0,..,i) \underset{\Lambda}{\otimes} s(i,..,n) \quad \forall\, s \in S_n(X).$$

It is well-known and is not hard to check that m_o is a chain map. We denote the composition of the chain homomorphisms indicated in the following diagram

$$C^* \underset{\Lambda}{\otimes} C \xrightarrow{\ Id_{C^*} \otimes m_o\ } C^* \underset{\Lambda}{\otimes} C \underset{\Lambda}{\otimes} C \xrightarrow{\ T \otimes Id_C\ } C \underset{\Lambda}{\otimes} C^* \underset{\Lambda}{\otimes} C \xrightarrow{\ Id_C \otimes e\ } C \underset{\Lambda}{\otimes} \Lambda = C$$

by $\cap : C^* \underset{\Lambda}{\otimes} C \longrightarrow C$. More explicitly this map is given by

$$\cap(f \otimes s) = f \cap s = \begin{cases} (-1)^{q(n-q)} f\,(s(n-q,\dots,n)).s(o,..,n-q) & \text{if } n \geq q \\ o & \text{if } n \quad q \end{cases}$$

for every $f \in C^q(X, \Lambda)$ and $s \in S_n(X)$. Let $H(\cap) : H(C^* \underset{\Lambda}{\otimes} C) \longrightarrow H(C)$ be the homomorphism induced by '$\cap$'.

For any $a \in H^q(C^*) = H_{-q}(C^*) = H^q(X, \Lambda)$ and $u \in H_n(C) = H_n(X, \Lambda)$ the element $H(\cap) \circ \alpha(a \otimes u)$ is called the cap-product of a by u and is denoted by $a \cap u$.

The chain map $e : C^* \underset{\Lambda}{\otimes} C \longrightarrow \Lambda$ induces a homomorphism $H(e) : H(C^* \underset{\Lambda}{\otimes} C) \longrightarrow \Lambda$. For any $a \in H^q(X, \Lambda)$ and $u \in H_q(X, \Lambda)$ the image $H(e) \circ \alpha(a \otimes u)$ is known as the value of the cohomology class a on the homology class u and is denoted by $a(u)$.

<u>1.2</u>. The following properties of the cap-product will be needed later.

① $(a \cup b) \cap u = a \cap (b \cap u) \ \forall a \in H^p(X, \Lambda)$, $b \in H^q(X, \Lambda)$ and $u \in H_n(X, \Lambda)$ with p, q, n arbitrary integers. Here $a \cup b$ denotes the Cup product of a and b.

② For any continuous map $f : Y \longrightarrow X$, if the induced homomorphisms in homology and cohomology are denoted by $f_* : H(Y, \Lambda) \longrightarrow H(X, \Lambda)$ and $f^* : H^*(X, \Lambda) \longrightarrow H^*(Y, \Lambda)$, then for any $a \in H^q(X, \Lambda)$ and $v \in H_n(Y, \Lambda)$

$$f_*(f^* a \cap v) = a \cap f_*(v).$$

<u>1.3</u>. <u>POINCARÉ DUALITY</u>.

When we refer to homology and cohomology groups without mentioning the coefficients we mean integer coefficients. Let M be a compact, connected, orientable manifold (without boundary) of dimension n. Then it is known that $H_n(M) \simeq \mathbb{Z}$. A choice of a generator u for $H_n(M)$ is known as an orientation for M. M together with a chosen orientation is called an oriented manifold and the distinguished element of $H_n(M)$ is called the fundamental class of M and is denoted by $[M]$.

Let $h : \mathbb{Z} \to \Lambda$ be the obvious ring homomorphism (which sends 1 of $\mathbb{Z}$ into 1 of Λ). Let $v = h_*([M])$ where $h_* : H_n(M) \longrightarrow H_n(M, \Lambda)$ is the homomorphism induced by h. Then Poincaré duality can be stated as follows:

The map $\Delta : H^q(M,\Lambda) \to H_{n-q}(M, \Lambda)$ given by $\Delta(x) = x \cap v$ is an isomorphism for all q.

In case M is not necessarily orientable it is true that $H_n(M; \mathbb{Z}_2) \simeq \mathbb{Z}_2$ and if v denotes the non zero element of $H_n(M; \mathbb{Z}_2)$ then $\cap v : H^q(M; \mathbb{Z}_2) \to H_{n-q}(M; \mathbb{Z}_2)$ is an isomorphism for all q.

When M is compact and not necessarily connected M is orientable if and only if each of its connected components is orientable. M being compact, the number of connected components is finite and denoting them by $\{M_j\}_{j=1}^{r}$ we have $H_n(M) \simeq \bigoplus_{j=1}^{r} H_n(M_j)$. If each M_j is oriented and if $[M_j]$ is the fundamental class of M_j then $[M] = \sum_{j=1}^{r} [M_j] \in H_n(M) = \bigoplus_{j=1}^{r} H_n(M_j)$ is defined to be the fundamental class of M.

1.4. All the vector bundles we consider are real vector bundles. For any X the trivial vector bundle of rank l over X will be denoted by θ^{l}_{X}. The total space and the base space of any vector bundle ξ will be denoted by $E(\xi)$ and B_{ξ} respectively. To denote that ξ is of rank k we just write ξ^{k}. If $f : Y \longrightarrow X$ is a continuous map and ξ any vector bundle over X the pull back bundle on Y is denoted by $f^{!}(\xi)$. If ξ carries a Riemannian metric, for any $\varepsilon > 0$ the subspace of $E(\xi)$ consisting of vectors of length $\leqslant \varepsilon$ is denoted by $E_{\varepsilon}(\xi)$ and the boundary consisting of vectors of length ε is denoted by $\dot{E}_{\varepsilon}(\xi)$. When B_{ξ} is compact the Thom space of ξ denoted by $T(\xi)$ is defined to be the one point compactification of $E(\xi)$. Let '∞' denote the point at infinity of $T(\xi)$. When ξ carries a Riemannian metric we can describe the Thom space alternatively as follows. Let $T_{\varepsilon}(\xi)$ be the quotient space got from $E_{\varepsilon}(\xi)$ by collapsing $\dot{E}_{\varepsilon}(\xi)$ to a point. The map $\beta : E_{\varepsilon}(\xi) \longrightarrow T(\xi)$ defined by $\beta(\vec{v}) = \dfrac{\vec{v}}{\varepsilon - \|\vec{v}\|}$ for $\vec{v} \in E_{\varepsilon}(\xi) - \dot{E}_{\varepsilon}(\xi)$ and $\beta(\vec{v}) = \infty$ for $\vec{v} \in \dot{E}_{\varepsilon}(\xi)$ passes down to a homeomorphism $\theta : T_{\varepsilon}(\xi) \longrightarrow T(\xi)$. Compactness of B_{ξ} is essential for θ to be a homeomorphism.

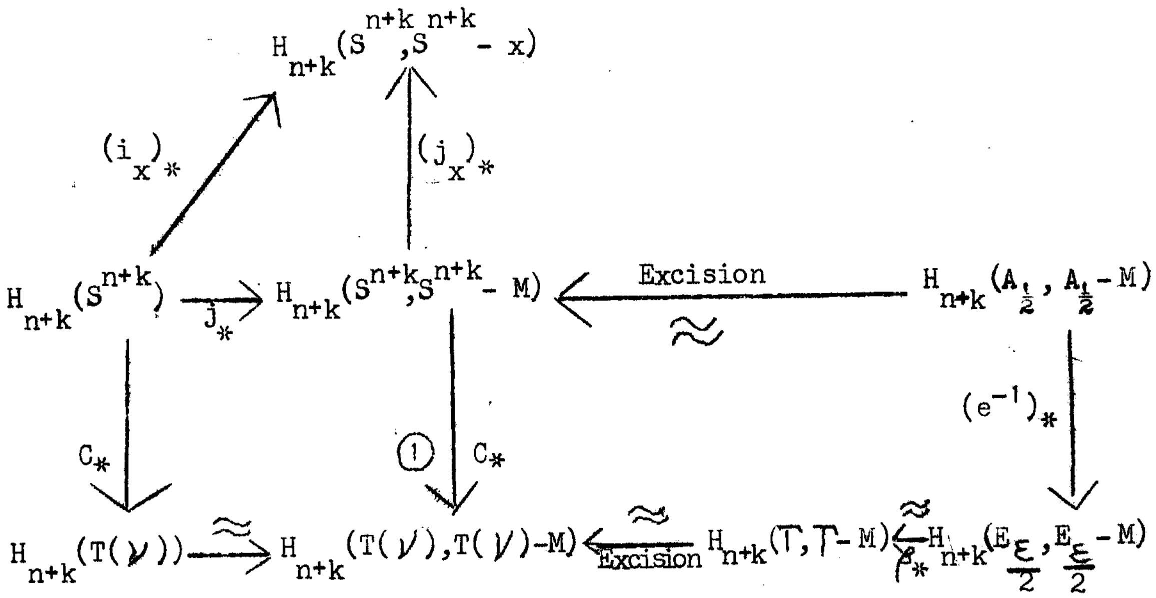
For any differential ($=C^{\infty}$) manifold M the tangent bundle of M will be denoted by τM. The word differentiable will always mean differentiable of class C^{∞} for us. For the rest of this section M denotes a compact, connected, oriented differential manifold of dimension $n \geq 0$ with $[M]$ as the fundamental class. By Whitney's imbedding theorem M can be differentiably imbedded in $\mathbb{R}^{n+k}$. Except when $n = 0$ the compactness of M automatically implies that $k \geq 1$. Even when $n = 0$ we can assume $k \geq 1$. Let ν be the normal bundle of this imbedding. Then $\tau_M \oplus \nu \simeq \theta_M^{n+k}$. Since τ_M and θ_M^{n+k} are both orientable it follows that ν is an orientable vector bundle. Identifying the tangent space to $\mathbb{R}^{n+k}$ at any point with $\mathbb{R}^{n+k}$ in the usual way and taking the usual Riemannian metric on $\tau_{\mathbb{R}^{n+k}} \simeq \theta^{2n+2k}_{\mathbb{R}^{n+k}}$ any element of $E(\nu)$ can be thought of as a pair $(x, \vec{v})$ with $x \in M$ and $\vec{v} \in \mathbb{R}^{n+k}$ in a direction normal to M at x. Let $e : E(\nu) \longrightarrow \mathbb{R}^{n+k}$ be defined by $e(x,v) = x+v$. $\exists$ an $\varepsilon > 0$ such that e is a diffeomorphism of the set $E_\varepsilon(\nu)$ on to a neighbourhood A of M. A is called a closed tubular neighbourhood of M. Let $\dot{A} = e(\dot{E}_\varepsilon(\nu))$. Considering S^{n+k} as the one point compactification of $\mathbb{R}^{n+k}$ we can define a map $C : S^{n+k} \longrightarrow T(\nu)$. This is the map got by collapsing the complement of $A - \dot{A}$ in S^{n+k} to a point. More precisely, $C \mid A = \beta \circ e^{-1}$ and $C \mid (S^{n+k} - A) = \infty$.

Let $\Phi : H_n(M) \longrightarrow H_{n+k}(T(\nu))$ be the Thom isomorphism [5].

<u>Proposition 1.5</u>. $\Phi([M]) = C_*(\zeta)$ for a generator ζ of $H_{n+k}(S^{n+k})$.

<u>Proof</u>. We have only to show that $C_* : H_{n+k}(S^{n+k}) \longrightarrow H_{n+k}(T(\nu))$ is an isomorphism. We abbreviate $E_\varepsilon(\nu)$ by E_ε etc. Let $A_{\frac{1}{2}} = e(E_{\varepsilon/2})$. Clearly $\beta \mid E_{\frac{\varepsilon}{2}}$ is a homeomorphism of $E_{\frac{\varepsilon}{2}}$ onto

the image Γ (say). Let x be any point in M (such a point exists because $\dim M > 0$ by assumption) and $i_x : S^{n+k} \longrightarrow (S^{n+k}, S^{n+k} - x)$ and $j_x : (S^{n+k}, S^{n+k} - M) \longrightarrow (S^{n+k}, S^{n+k} - x)$ the respective inclusions. Consider the following commutative diagram.



<u>Diagram 1</u>

The homomorphism indicated as β_* is an isomorphism since $\beta : E_{\frac{\epsilon}{2}} \rightarrow \Gamma$ is a homeomorphism. It follows that the homomorphism numbered (1) is an isomorphism. The space $T(\nu) - M$ is contractible in itself to ∞. Hence the map $H_{n+k}(T(\nu)) \to H_{n+k}(T(\nu), T(\nu)-M)$ is an isomorphism. (The assumption $k \geq 1$ is used here). Since $H_{n+k}(T(\nu)) \approx H_n(M) \approx \mathbb{Z}$ we have $H_{n+k}(S^{n+k}, S^{n+k}-M)$. Since $(i_x)_*$ is an isomorphism it follows that j_* is a monomorphism and that image of j_* is a direct summand of $H_{n+k}(S^{n+k}, S^{n+k}-M)$. The groups $H_{n+k}(S^{n+k})$ and $H_{n+k}(S^{n+k}, S^{n+k}-M)$ being both isomorphic to $\mathbb{Z}$ it follows that j_* is an isomorphism. It now follows that $C_* : H_{n+k}(S^{n+k}) \longrightarrow H_{n+k}(T(\nu))$ is an isomorphism.

1.6. THE INDEX OF A 4d-DIMENSIONAL MANIFOLD.

Let M be a compact, connected, oriented manifold of dimension 4d with d an integer ≥ 0 and let $[M]$ be the fundamental class of M. The image $h_*([M])$ of the fundamental class of M under the inclusion $h : \mathbb{Z} \to \mathbb{Q}$ is called the fundamental class with coefficients in $\mathbb{Q}$ and is also denoted by $[M]$. The map $(x,y) \rightsquigarrow (x \cup y)[M]$ of $H^{2d}(M, \mathbb{Q}) \times H^{2d}(M, \mathbb{Q}) \to \mathbb{Q}$ gives a symmetric, non degenerate bilinear form $H^{2d}(M, \mathbb{Q})$. Symmetry is clear from $x \cup y = (-1)^{2d.2d} y \cup x = y \cup x$. That it is non degenerate is a consequence of Poincaré duality together with the fact that $(a, u) \rightsquigarrow a(u)$ is a bilinear non degenerate pairing of $H^{2d}(M, \mathbb{Q}) \times H_{2d}(M, \mathbb{Q}) \to \mathbb{Q}$. This latter fact is embodied in the