ON THE S-EQUIVALENCE OF SOME GENERAL SETS OF MATRICES

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ABSTRACT. To help classify the set of square matrices over a ring $R$ under the relation of $S$-equivalence there is defined a module $A_V$ together with a pairing on its torsion submodule, which is referred to as the Seifert system of $V$. It is shown that if $R$ is a field, or $R$ is a PID and $\det(tV - V')$ has content 1, then the Seifert system characterizes an $S$-equivalence class. Furthermore, over a field $S$-equivalence is reducible to the notion of congruence.

1. Introduction. Two square matrices over a ring $R$ are called $S$-equivalent if one can be derived from the other by a sequence of the following operations (or their inverses);

(1.1) Congruences, i.e., replacing $V$ by $PVP'$, with $P$ unimodular over $R$,

(1.2) Row and column enlargements, i.e., replacing $V$ by,

\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & a & b \\
0 & c & V
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & a & b \\
0 & c & V
\end{pmatrix}
\]

To help classify matrices under this relation, we define a module $A_V$ over the ring $R[t, t^{-1}]$, together with a pairing on its torsion submodule, which will be an invariant of the $S$-equivalence class of $V$. We refer to this as the Seifert system for $V$.

The geometric aspects of the study of $S$-equivalence have principally been developed in the work of Levine [5, 6, 7]. If $K \subseteq S^{2n+1}$ is an odd dimensional knot, then any Seifert surface for $K$ determines an integral matrix, called a Seifert matrix. $S$-equivalence can in this case be interpreted as the matrix theoretic analogue of adding or subtracting handles to these surfaces. $S$-equivalence actually characterizes the so-called simple embeddings (see Kearton [3]). The module $A_V$ then corresponds to the integral homology of the universal abelian cover of $S^{2n+1} - K$, whose pairing is defined geometrically in Blanchfield [1].

Seifert matrices for knots can algebraically be characterized by the condition $\det(V - eV') = \pm 1$, where $e$ is either +1 or -1. These matrices have been classified algebraically by Trotter [10, 11]. The results of
this paper are generalizations of theorems of his. The present treatment has several advantages, though. It applies also to Seifert matrices for links in $S^3$ (see Keef [4]). In addition, the methods used here are considerably more elementary, although the general outline remains much the same. Finally, in the knot case it can be shown that multiplication by $1-t$ is an automorphism of $A_V$, a fact which has a central position in previous studies. The theorems in this paper will be proven without reference to this map, which is not in general either one-to-one or onto.

2. The Seifert system. We will assume all rings are integral domains. If $R$ is a ring, we let $R^m$ denote the set of all $m$ by 1 matrices (column vectors) over $R$, and let $R'$ be the field of fractions of $R$. We write $RC$ for the group ring over $R$ of the infinite cyclic group generated by $t$, written multiplicatively. So $RC \cong R[t, t^{-1}]$, the ring of Laurent polynomials over $R$. Clearly $RC' = R'(t)$. Let $-$ denote the conjugation on $RC$ which interchanges $t$ and $t^{-1}$.

If $V$ is a square matrix over $R$, we let $D(V) = \det(tV - V')$ and $E(V) = tV - V'$. It is easy to verify that $D(V)$ is, up to multiplication by units of $RC$, an invariant of the $S$-equivalence class of $V$. The relation $E(V) = -tE(V')$ is also easily checked.

In order to construct some algebraic invariants of an $S$-equivalence class, we begin with some general considerations. Suppose $S$ is a ring with a conjugation $\bar{}$, and $u \in S$ is a unit. A matrix $M$ over $S$ will be called $u$-Hermitian if $uM' = M$. Clearly $E(V)$ is $(-t)$-Hermitian over $RC$. We let $A_M$ denote the module $S^m/M^m$, and $TA_M$ denote its submodule of $S$-torsion. We define a pairing on $TA_M$, which takes its values in $S'/S$ as follows: if $x, y \in S^m$ project into $TA_M$, then there exists $a, b \in S^m$ satisfying $Ma = rx$, $Mb = sy$. Let $[x, y] = \bar{b}'Ma/r\bar{s}u \in S'$. To show this is independent of the choices of $a, b, r$ and $s$, we note that,

$$\bar{b}'x/\bar{s}u = \bar{b}'Ma/\bar{r}s\bar{u} = \bar{b}'M'a/r\bar{s}$$

(2.1)

Observe further that if $x = Ma$ (so $r = 1$) or $y = Mb$ (so $s = 1$), then $[x, y] \in S$. This implies that we may view $[,]$ as a pairing on $TA_M$ with values in $S'/S$.

DEFINITION 2.2. By the Seifert system of $M$ we will mean the module $A_M$ together with this pairing on $TA_M$. The Seifert systems of $M$ and $N$ are isomorphic if there is an isomorphism of $A_M$ and $A_N$ which restricts to an isometry of their $S$-torsion submodules.

We next consider the behavior of the Seifert system under a change of ground rings. Suppose $S$ and $S_0$ are rings with conjugations, and $f: S \to S_0$ is a homomorphism which preserves these conjugations. If $M$ is a $u$-Hermi-
tian matrix over $S$, then $f(M)$ (the matrix obtained by mapping all $M$'s entries into $S_0$) is clearly $f(u)$-Hermitian over $S_0$. Furthermore, $f$ induces a map $F: A_M \to A_{f(M)}$ by mapping $x \in S^m$ to $f(x) \in S_0^m$. If $f$ is also injective, then it determines a map $f: S'/S \to S_0'/S_0$. Using this map an easy verification shows that the pairing $[ , ]$ is preserved by $F$.

One particularly nice situation occurs when $M$ is non-singular. In this case $TA_M = A_M$, and the following gives us a useful expression for $[ , ]$.

**Lemma 2.3** If $M$ is a non-singular $u$-Hermitian matrix, then $[x, y] = y'M^{-1}x$.

**Proof.** Suppose $Ma = rx$. Therefore $a/r = M^{-1}x \in S'^m$, and so by (2.1), $[x, y] = y'a/r = y'M^{-1}x$.

If $V$ is a square matrix over a ring $R$, we denote $A_{E(V)}$ by the simpler $A_V$, which we refer to as the Seifert system determined by $V$.

**Proposition 2.4.** If $V$ and $W$ are $S$-equivalent matrices over $R$, then the Seifert systems determined by $V$ and $W$ are isomorphic.

**Proof.** Since the argument varies only in minor detail from that in Trotter [11, p. 177–179], we will be content with an outline. It is easily verified that if $Q$ is a unimodular matrix over $RC$, then $E(V)$ and $QE(V)\tilde{Q}'$ give isomorphic Seifert systems. So if $P$ is a unimodular matrix over $R$, and $W = PVP'$ then $E(W) = PE(V)\tilde{P}'$, so their Seifert systems are isomorphic. If $W$ is the row enlargement of 1.2 i, and, 

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tc - b' & 0 & I \end{bmatrix}$$

then it is easy to verify that $E(V)$ and $QE(W)\tilde{Q}'$, and hence $E(V)$ and $E(W)$, determine isomorphic Seifert systems. A similar argument applies to column enlargements.

The remainder of the paper will be an investigation of the converse of (2.4). Specifically, it will be shown that if $R$ is a field, or if $R$ is a PID and the content of $D(V)$ (i.e., the gcd of its coefficients) is a unit in $R$, then the Seifert system completely characterizes the $S$-equivalence class of $V$.

3. **S-equivalence with field coefficients.** Throughout this section we let $F$ be a field. $S$-equivalence over $F$ will be shown to be equivalent to the notion of congruence.

**Proposition 3.1.** Any square matrix over $F$ is $S$-equivalent to a matrix of the form,
Matrices in this form will be called \textit{reduced}.

\textbf{Proof.} The following can actually be viewed as an algorithm for putting a matrix into reduced form. Let $V$ be a square matrix over $F$. If $V$ is non-singular we are done. If not, there exists a non-singular matrix $P$ such that the top row of $PV$ is identically zero. So $PVP'$ has the form,

\[
\begin{bmatrix}
W & 0 \\
0 & 0
\end{bmatrix}
\text{ with } W \text{ non-singular.}
\]

Note if $a = 0$, then $V$ is also congruent to,

\[
\begin{bmatrix}
V_0 & 0 \\
0 & 0
\end{bmatrix}
\]

and we can start our process over with $V = V_0$. If $a \neq 0$, then $V$ is congruent to a matrix of the form,

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & d & b \\
0 & c & V_1
\end{bmatrix}
\]

and once again we can start our process over with $V = V_1$. Continuing as long as possible yields the result.

The reduced form can be used to analyse the algebraic structure of $A_V$. Clearly if $V$ is the reduced matrix in (3.1), then $A_V \cong A_W \oplus FC^k$ (where $k$ is the number of rows and columns of zeros). To help determine the structure of $A_W$, we let $R^m \subseteq RC^m$ denote the $R$ submodule consisting of vectors whose entries are elements of $R$.

\textbf{Lemma 3.2.} If $M$ and $N$ are unimodular matrices over $R$ of the same size, then $RC^m/(tM - N)RC^m$ is isomorphic as an $R$ module to $R^m$, where the $t$ automorphism is given by multiplication by $NM^{-1}$.

\textbf{Proof.} Clearly, mapping the standard basis for $RC^m$ to the standard basis for $R^m$ produces a map $f$: $RC^m \rightarrow R^m$, which is clearly an $R$-linear isomorphism when restricted to $R^m_\epsilon$. Furthermore, if $s \in R^m_\epsilon$, then $f((tM - N)s) = NM^{-1}Ms - Ns = 0$, and so since $R^m_\epsilon$ generates $RC^m$ as an $RC$ module, we can define, $\hat{f}$: $RC^m/(tM - N)RC^m \rightarrow R^m$. Clearly $\hat{f}$ is an isomorphism if we can show that $RC^m$ splits as an $R$ module into $(tM - N)RC^m \oplus R^m_\epsilon$. Since $f$ is identically zero on the first summand and is an isomorphism on the second, their intersection is zero. An easy compu-
tation shows that \((tM - N)RC^m + R_c^m\) is an \(RC\) submodule of \(RC^m\), and since it contains \(R_c^m\), which generates \(RC^m\), the proof is complete.

We call attention specifically to the fact contained in the proof of (3.2) that \(R_c^m \subseteq RC^m\) is mapped isomorphically onto \(RC^m/(tM - N)RC^m\) under the natural map. The standard basis for \(R_c^m\) therefore gives a basis for this module which we will often use (without specifically mentioning it) to determine matrix representations of bilinear forms or linear functions.

**Corollary 3.3.** If \(V\) is a non-singular matrix over \(F\), then \(A_V\) is isomorphic as an \(F\) vector space to \(F^m\), where the \(t\)-automorphism corresponds to multiplication by \(V'V^{-1}\).

**Corollary 3.4.** If \(V\) is a square matrix over \(F\), then \(D(V) \neq 0\) if and only if \(\dim_F A_V\) is finite if and only if \(V\) is \(S\)-equivalent to a non-singular matrix. In this case, \(\deg(D(V)) = \dim_F A_V\), which equals the size of any non-singular matrix \(S\)-equivalent to \(V\).

**Theorem 3.5.** Let

\[
V_0 = \begin{bmatrix} W_0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V_1 = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix}
\]

be matrices in reduced form. The following are then equivalent.

1. \(V_0\) is \(S\)-equivalent to \(V_1\).
2. \(V_0\) is congruent to \(V_1\).

Further, if these two matrices have the same size, then (1) and (2) are equivalent to

3. \(W_0\) is \(S\)-equivalent to \(W_1\), and
4. \(W_0\) is congruent to \(W_1\).

**Proof.** Clearly (4) implies (1), (2) and (3). Furthermore (2) and (3) imply (1). We now show (1) implies (4), which will conclude the proof. Note if

\[
V = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}
\]

is in reduced form, then clearly \(A_W\) is isometric to \(TA_V\). By (2.4) there is an isometry of \(TA_{V_0}\) and \(TA_{V_1}\), so the proof will be complete once it is shown that an isometry of \(A_{W_0}\) and \(A_{W_1}\) implies that \(W_0\) and \(W_1\) are congruent.

Using the assumed isometry identify \(A_{W_0}\) and \(A_{W_1}\), and call the resulting module \(A\). Note the two interpretations of \(A\) give two representations of \(A\) as a set of column vectors using (3.2). Observe further that if \(i = 0\) or \(1\), \(D(W_i) = \det(W_i) \det(tI - W_i'W_i^{-1})\). This implies that up to a constant \(D(W_i)\) is the characteristic polynomial of the automorphism of \(A\) given by
multiplication by \( t \), which is independent of the basis used to compute it. We call this common polynomial \( D(A) \).

Let \( a \in F \). By the theory of partial fractions \( F(t) \) is isomorphic as an \( F \) vector space to \( F[t, t^{-1}, (t - a)^{-1}] \oplus L_a \), where \( L_a \) can be described as the set of all rational polynomial expressions \( h/g \) such that \( \text{deg}(h) < \text{deg}(g) \), \( g(0) \neq 0 \), \( g(a) \neq 0 \). Define an \( F \) linear map \( f_a: F(t)/FC \to F \) by setting it equal to zero on \( F[t, t^{-1}, (t - a)^{-1}] \), and letting it equal \( h(a)/g(a) \) for \( h/g \in L_a \).

Let \( A^* \) be the dual space of \( F \) linear functionals on \( A \). Note maps \( h: A \to A^* \) correspond to bilinear forms on \( A \), and maps \( h: A^* \to A \) correspond to bilinear forms on \( A^* \) (since \( A^{**} \) can be identified with \( A \)).

Suppose \( a \in F \) satisfies \( D(A)(a) \neq 0 \). We define a bilinear form on \( A \) with values in \( F \) by combining the pairing \([,]\) into \( F(t)/FC \) with the map \( f_a \) into \( F \), i.e., \((x, y)_a = f_a([x, y])\). If \( W = W_0 \) or \( W_1 \), and we consider the vector representation of \( A \) given by (3.2), we claim \((,)_a \) has matrix \((aW - W')^{-1}D(A)(a) \). To see this observe that if \( b, c \in F^m_n \) represent \( x, y \in A \), then by (2.3), \([x, y] = b'E(W)^{-1}c \in F(t)/FC \).

Note \( E(W)^{-1} = \text{adj}(E(W))/D(W) \). This implies that all the entries of \( E(W)^{-1} \) are in \( L_a \), since \( D(W)(a) = D(A)(a) \neq 0 \) by supposition, \( D(W)(0) = \text{det}(-W') \neq 0 \) since \( W' \) is non-singular, and \( D(W) \) has a larger degree than any entry of \( \text{adj}(E(W)) \). Since \( f_a \) when restricted to \( L_a \) is merely substitution by \( a \), we have, \( f_a([x, y]) = b'(aW - W')^{-1}c \) which establishes the claim.

Suppose there actually exists a pair of distinct non-roots of \( D(A) \), \( r, s \in F \). We then have a pair of bilinear forms on \( A \), and hence a pair of adjoint maps \( h_r, h_s: A \to A^* \). Let \( g: A^* \to A \) be given by \((r - s)^{-1}(h_r^{-1} - h_s^{-1}) \). So \( g \) has matrix representation \((r - s)^{-1}((rW - W') - (sW - W')) = W \). \( g \), in turn defines a bilinear form on \( A^* \) which also has matrix \( W \).

We summarize this construction by noting that there is a bilinear form on \( A^* \) completely determined by the pairing \([,]\), and with respect to one basis it has matrix \( W_0 \) and with respect to another basis it has matrix \( W_1 \), and so \( W_0 \) and \( W_1 \) are congruent.

Assume now that \( D(A) \) does not have two non-roots. Embed \( F \) in a field \( F' \) where \( D(A) \) does have two non-roots (say by adjoining an indeterminant). Consider the diagram:

\[
\begin{array}{ccc}
A^* & \xrightarrow{g} & A \\
\downarrow & & \downarrow \\
(F'A)^* & \xrightarrow{g'} & F'A
\end{array}
\]

By the naturality of the Seifert system under an extension of the ground ring, \( g' \) exists as above. In fact, since \( g' \) has matrix \( W \), all of whose entries are in \( F \), \( g' \) can easily be seen to restrict to \( g \) as shown. \( g \) once again de-
terminates a congruence class of matrices to which $W_0$ and $W_1$ must both belong, which therefore completes the proof.

Note that we only used $S$-equivalence in the above proof to establish an isometry between $A_{W_0}$ and $A_{W_1}$. Since the number of zero rows and columns in a reduced matrix $V$ equals the rank of $A_V/TA_V$ as an $FC$ module, we have actually shown the following result.

**Theorem 3.6.** If $V_0$ and $V_1$ are square matrices over a field, then they are $S$-equivalent if and only if their Seifert systems are isomorphic.

We single out one fact established in the proof of (3.5).

**Corollary 3.7.** If $V$ is a non-singular matrix over a field $F$, then there exists an $F$ linear map $g: A_V^F 	o A_V$, whose matrix with respect to the basis for $A_V$ given by the isomorphism $F^m \subseteq FC^m \to A_V$ and its dual basis in $A_V^F$ is $V$. Furthermore, $g$ does not depend on the way $A_V$ is presented as the Seifert system of some matrix.

**4. $S$-equivalence of knot-like matrices.** Throughout this section we assume $R$ is a PID. If $V$ is a square matrix over $R$, then if we view it as a matrix over $R'$, its Seifert system is given by the $R'$ vector space $R'A_V$, for which all the results of the previous section apply.

**Definition 4.1.** A matrix $V$ over $R$ is called knot-like if the content of $D(V)$ (i.e., the gcd of its coefficients) is a unit in $R$.

Any Seifert matrix for a knot is knot-like over the integers. This can be seen by the relation $D(V)(e) = \pm 1$ where $e$ is $+1$ or $-1$, which is true for these matrices (see Trotter [11]).

**Proposition 4.2.** $V$ is knot-like if and only if $A_V$ is a torsion free $R$ module of finite rank.

**Proof.** By (3.4), $\text{rank}(A_V) = \dim_R(R'A_V)$ is finite if and only if $D(V) \neq 0$. So if $D(V) \neq 0$ we have an exact sequence,

$$0 \longrightarrow RC^m \xrightarrow{E(V)} RC^m \longrightarrow A_V \longrightarrow 0.$$ 

If we tensor this with $R_p (= R/pR$, where $p \in R$ is a prime) we get

$$0 \longrightarrow \text{Tor}_R(R_p, A_V) \longrightarrow R_pC^m \xrightarrow{E(V)} R_pC^m \longrightarrow R_p \otimes A_V \longrightarrow 0.$$ 

So $A_V$ has no $p$-torsion if and only if $E(V)$ is non-singular over $R_pC$ if and only if $p \nmid D(V)$. Letting $p$ vary over all primes in $R$ gives the result.

**Proposition 4.3.** If $V$ is a square matrix over $R$, and the content of $D(V)$ is square-free (e.g., if $V$ is knot-like), then $V$ is $S$-equivalent to a non-singular matrix.
PROOF. Assume $V$ is singular. Then $V$ is congruent to a matrix $V_0$ whose top row is zero. If $p \in R$ divides the first column of $V_0$, then clearly $p^2 | D(V)$, which cannot happen. So $V_0$ is in turn congruent to a matrix $V_1$ which can be row reduced. Continuing as long as possible yields the result.

COROLLARY 4.4. $V$ is $S$-equivalent to a unimodular matrix if and only if $A$ is a finitely generated free $R$ module.

PROOF. If $V$ is $S$-equivalent to a unimodular matrix, then by (3.2), $A_V$ has the stated form. Conversely suppose $A_V \cong R^m$. By (4.2) $V$ is knot-like, so by (4.3) we may assume it is non-singular. If $p \in R$ is a prime, then $R_p \otimes A_V \cong R_p^m$, so by (2.4) (with $F = R_p$), $V$ is non-singular mod $p$, i.e. $p \not| \det(V)$. Letting $p$ vary over all primes gives the result.

We are heading towards the following result on knot-like matrices.

THEOREM 4.5. Two knot-like matrices over a PID are $S$-equivalent if and only if their Seifert systems are isomorphic.

Before we can enter into its proof we will need some auxiliary concepts and Lemmas.

Assume $M$ is a finite dimensional $R'$ vector space. A free $R$ module $N \subseteq M$ is called a lattice if $R'N = M$. Let $N^* \subseteq M^*$ be the set of all $f \in M^*$ satisfying $f(N) \subseteq R$. $N^*$ is called the dual lattice of $N$. If $\{a_i\}$ is a basis for $N$ over $R$ (which clearly also must be a basis for $M$ over $R'$), then the dual basis $\{a^*_i\}$ for $M^*$ must clearly also be a basis for $N^*$ over $R$.

Suppose $g : M^* \to M$ is some fixed homomorphism. We call a lattice $N \subseteq M$ integral if $g(N^*) \subseteq N$. An integral lattice $N$ determines a congruence class of matrices over $R$ as follows: if $\{a_i\}$ and $\{a^*_i\}$ are dual basis for $N$ and $N^*$ respectively, then the matrix for $g$ with respect to these basis has all of its entries in $R$, since $g(N^*) \subseteq N$, and is clearly well defined up to a congruence over $R$. We call a representative of this congruence class "the" matrix generated by $N$ and denote it by $V_N$. The ambiguity in this terminology will be offset by the fact that a basis for $N$ will usually be implied.

Assume $V$ is a non-singular knot-like matrix over $R$. By (3.7) there is a homomorphism $g : (R'A_V)^* \to R'A_V$. Furthermore, if we consider the lattice $N_V \subseteq R'A_V$ given by the image of the maps $R^m_c \subseteq R^m_v \cong R'A_V$, we note that $N_V$ is integral and generates the matrix $V$. The above discussion now makes the following obvious.

PROPOSITION. 4.6. Suppose $V$ and $W$ are non-singular knot-like matrices whose Seifert systems are isomorphic. We identify $A_V$ and $A_W$ using this isomorphism and call the resulting module $A$. If $N_V = N_W$, then $V$ and $W$ are congruent.
4.4 and 4.6. now imply the following result.

**Corollary 4.7.** If $V$ and $W$ are unimodular matrices over $R$, then they are S-equivalent if and only if they are congruent.

The strategy of the proof of (4.5) will be to identify $A_V$ and $A_W$ as above, then to augment $N_V$ and $N_W$ in some reasonable fashion until they agree and then invoke (4.6). So we assume we have an RC module $A \subseteq R'A$ and a homomorphism $g: R'A^* \to R'A$. We call an integral lattice $N \subseteq A$ admissible if and only if it generates $A$ as an RC module, and $V_N$ in knot-like.

Suppose $N, N' \subseteq R'A$ are lattices. We choose bases $\{a_i\}$ and $\{b_i\}$ for $N$ and $N'$, and let $d(N, N')$ equal the determinant of a change of basis matrix from $\{b_i\}$ to $\{a_i\}$. Note $d(N, N')$ is only determined up to multiplication by units of $R$. If $p \in R$ is a prime, and $R_{(p)}$ is the local ring at $p$, then $R_{(p)}N$ and $R_{(p)}N'$ can be viewed as lattices over $R_{(p)}$. If $o(\ )$ is the valuation determined by $p$, then $o(d(R_{(p)}N, R_{(p)}N')) = o(d(N, N'))$.

**Lemma 4.8.** Let $N$ be an admissible lattice. An integral lattice $N'$ which generates $A$ as an RC module is admissible if and only if $d(N, N')$ is a unit of $R$.

**Proof.** Let $P$ be a change of basis matrix from a basis for $N'$ to one for $N$. Clearly $D(V_{N'}) = det(P)^2D(V_N)$, so the content of $D(V_{N'})$ is 1 if and only if $det(P)$ is a unit.

The augmentation of our admissible lattices is based on the following operation called *transferral of factors*.

**Lemma 4.9.** If $a$ and $k$ are relatively prime in $R$, then the matrices,

$$
V_0 = \begin{bmatrix}
x & a & q' \\
k w & k^2 y & k s' \\
p & kr & V
\end{bmatrix}
\quad \text{and} \quad
V_1 = \begin{bmatrix}
k^2 x & a & k q' \\
k w & y & s' \\
kp & r & V
\end{bmatrix}
$$

are S-equivalent.

This is proven in Trotter [11]. In going from $V_0$ to $V_1$ we say we are transferring a factor from the second row to the first column. We would like to relate this to our admissible lattices. Suppose $N \subseteq R'A$ is an integral lattice which has as a basis $\{c_i\}$, which generates the matrix $V_0$ above. If we let $N'$ be the lattice generated by $k^{-1}c_1, kc_2, c_3, \ldots, c_m$, then clearly $V_1 = V_{N'}$.

**Lemma 4.10.** $N$ is admissible if and only if $N'$ is.

**Proof.** By (4.8) $V_0$ is knot-like if and only if $V_1$ is. The lemma therefore reduces to showing that $A_{V_0} = A$ if and only if $A_{V_1} = A$. We show
\[ N' \subseteq A_{V_0}, N \subseteq A_V \] being handled similarly. \( A_{V_0} \) is presented by \( E(V_0) \), and examining its second column (and using \((a, k) = 1\)), we see that \( tc_1 \in kA_{V_0} \), and so \( k^{-1}c_1 \in A_{V_0} \), which implies the result.

**PROOF OF 4.5.** We assume \( V \) and \( W \) are non-singular knot-like matrices, and \( N_V, N_W \subseteq A \). Let \( d = d(N_V, N_V \cap N_W) \). Since \( N_V \cap N_W \subseteq N_Y \), \( d \) is an element of \( R \). Note \( d(N_W, N_V \cap N_V) = d(N_W, N_V) \cdot d(N_V, N_V \cap N_W) = d \) by (4.8). We induct on the sum of the exponents in a prime factorization of \( d \). Clearly if \( d \) is a unit \( N_Y = N_W \), and so by (4.6) \( V \) and \( W \) are congruent. So assume \( p \) is a prime which divides \( d \).

By the invariant factor theorem, we can select bases \( b_1, \ldots, b_m \) and \( c_1, \ldots, c_m \) for \( N_V \) and \( N_W \) respectively satisfying \( c_i = r_i b_i \) for some \( r_i \in R \). Assume these are ordered so that \( o(r_i) > 0 \) for \( i \leq s \), \( o(r_i) = 0 \) for \( s < i \leq q \), and \( o(r_i) < 0 \) for \( i > q \). We let \( S_V \) and \( S_W \) be the free \( R(\alpha) \) modules generated by the \( b_i \) and \( c_i \) respectively for \( i \leq s \), \( U \) be the \( R(\alpha) \) module generated by the \( b_i \) and \( c_i \) for \( s < i \leq q \), and \( T_V \) and \( T_W \) be the \( R(\alpha) \) modules generated by the remaining \( b \)'s and \( c \)'s. So \( S_W \subseteq pS_V \) and \( T_V \subseteq pT_W \). Let \( S^*, S^*, U^* \) and \( T^* \) be the \( R(\alpha) \) modules generated by the corresponding elements of the dual bases \( \{b_i^*\} \) and \( \{c_i^*\} \) for \( R'A^* \) (e.g., \( S^* \) is generated by \( b_i^*, i < s \)). So \( S^* \subseteq pS^* \) and \( T^* \subseteq pT^* \).

Assume now that \( s \geq m - q \). If \( m - q > s \), the same proof applies, reversing the roles of \( V \) and \( W \).

Since \( g(N^*_V) \subseteq N_V \) and \( g(N^*_W) \subseteq N_W \), \( V \mod p \) must have the form,

\[
\begin{bmatrix}
0 & 0 & C \\
0 & B & Z \\
D & X & Y
\end{bmatrix}
\]

where the three blocks of rows (respectively columns) correspond to \( S_V \), \( U \) and \( T_V \) (respectively \( S^*, U^* \) and \( T^* \)). Observe the upper left corner must actually be divisible by \( p^2 \).

We claim that \( C \) and \( D \) are not both square and non-singular. Assume they are. \( R_p \otimes A \) is then presented by the matrix

\[
\begin{bmatrix}
0 & 0 & tC-D' \\
0 & tB-B' & tZ-X' \\
tD-C' & tX-Z' & tY-Y'
\end{bmatrix}
\]

The image of \( T_V \) under the natural map \( R(\alpha) \cdot A \to R_p \otimes A \) is evidently isomorphic to the \( R_p C \) module presented by \( tD-C' \) which is non zero by (3.2). However \( T_V \subseteq pT_W \) implies that it must in fact be zero.

We now assume \( D \) is not a square non-singular matrix. If \( C \) is the one which is not non-singular we apply a similar proof, switching rows and columns.
Since $D$ is non-singular and $s \geq m - q$, we may perform column operations on the first $s$ columns of $V$ so that the resulting matrix has a first column divisible by $p$. If we apply the corresponding row operations to $V$ the result is a matrix $V_1$ congruent over $R$ to $V$. The first row of $V_1$ cannot be divisible by $p$, since $p \nmid D(V)$, so clearly we can apply row and column operations to the last $m - q$ rows and columns of $V_1$ to produce a matrix $V_2$, congruent to $V$, whose first column is divisible by $p$, whose first diagonal entry is divisible by $p^2$ and whose first row has only one entry not divisible by $p$. Therefore we see that it is possible to transfer a factor out of a column of $V$ corresponding to some element of $S^*$ and into a row corresponding to some element of $T_Y$.

Consider the lattice $N'$ determined as in (4.10), where $V_{N'}$ is the result of transferring the factor. $N'$ is evidently admissible, and it is easy to see that $o(d(N', N' \cap N_w)) < o(d)$, while all the other primes in $d(N', N' \cap N_w)$ and $d$ occur to equal powers.

This completes the proof of (4.5).

The parallel between Seifert matrices for knots and knot-like matrices can be extended (see Keef [4]). For instance it can be shown that any pair of $S$-equivalent knot-like matrices whose determinants are a prime are in fact congruent. Further extensions are limited by the fact that $1-t$ is not an automorphism of $A_Y$ for a general knot-like matrix.

REFERENCES


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