to indicate the direction, as well as the magnitude, of the most rapid decrease of Ψ.

18.] There are cases, however, in which the conditions

$$\frac{dZ}{dy} - \frac{dY}{dz} = 0, \quad \frac{dX}{dz} - \frac{dZ}{dx} = 0, \quad \text{and} \quad \frac{dY}{dx} - \frac{dX}{dy} = 0,$$

which are those of $Xdx + Ydy + Zdz$ being a complete differential, are fulfilled throughout a certain region of space, and yet the line-integral from A to P may be different for two lines, each of which lies wholly within that region. This may be the case if the region is in the form of a ring, and if the two lines from A to P pass through opposite segments of the ring. In this case, the one path cannot be transformed into the other by continuous motion without passing out of the region.

We are here led to considerations belonging to the Geometry of Position, a subject which, though its importance was pointed out by Leibnitz and illustrated by Gauss, has been little studied. The most complete treatment of this subject has been given by J. B. Listing *.

Let there be p points in space, and let l lines of any form be drawn joining these points so that no two lines intersect each other, and no point is left isolated. We shall call a figure composed of lines in this way a Diagram. Of these lines, $p-1$ are sufficient to join the p points so as to form a connected system. Every new line completes a loop or closed path, or, as we shall call it, a Cycle. The number of independent cycles in the diagram is therefore $\kappa = l - p + 1$.

Any closed path drawn along the lines of the diagram is composed of these independent cycles, each being taken any number of times and in either direction.

The existence of cycles is called Cyclosis, and the number of cycles in a diagram is called its Cyclomatic number.

Cyclosis in Surfaces and Regions.

Surfaces are either complete or bounded. Complete surfaces are either infinite or closed. Bounded surfaces are limited by one or more closed lines, which may in the limiting cases become finite lines or points.

A finite region of space is bounded by one or more closed surfaces. Of these one is the external surface, the others are

* *Der Census Raümlicher Complexe*, Gött. Abh., Bd. x. S. 97 (1861).

included in it and exclude each other, and are called internal surfaces.

If the region has one bounding surface, we may suppose that surface to contract inwards without breaking its continuity or cutting itself. If the region is one of simple continuity, such as a sphere, this process may be continued till it is reduced to a point; but if the region is like a ring, the result will be a closed curve; and if the region has multiple connexions, the result will be a diagram of lines, and the cyclomatic number of the diagram will be that of the region. The space outside the region has the same cyclomatic number as the region itself. Hence, if the region is bounded by internal as well as external surfaces, its cyclomatic number is the sum of those due to all the surfaces.

When a region encloses within itself other regions, it is called a Periphractic region.

The number of internal bounding surfaces of a region is called its periphractic number. A closed surface is also periphractic, its number being unity.

The cyclomatic number of a closed surface is twice that of the region which it bounds. To find the cyclomatic number of a bounded surface, suppose all the boundaries to contract inwards, without breaking continuity, till they meet. The surface will then be reduced to a point in the case of an acyclic surface, or to a linear diagram in the case of cyclic surfaces. The cyclomatic number of the diagram is that of the surface.

19.] THEOREM I. *If throughout any acyclic region*

$$X\,dx + Y\,dy + Z\,dz = -D\Psi,$$

the value of the line-integral from a point A to a point P taken along any path within the region will be the same.

We shall first shew that the line-integral taken round any closed path within the region is zero.

Suppose the equipotential surfaces drawn. They are all either closed surfaces or are bounded entirely by the surface of the region, so that a closed line within the region, if it cuts any of the surfaces at one part of its path, must cut the same surface in the opposite direction at some other part of its path, and the corresponding portions of the line-integral being equal and opposite, the total value is zero.

Hence if AQP and $AQ'P$ are two paths from A to P, the line-integral for $AQ'P$ is the sum of that for AQP and the closed path

$AQ'PQA$. But the line-integral of the closed path is zero, therefore those of the two paths are equal.

Hence if the potential is given at any one point of such a region, that at any other point is determinate.

20.] THEOREM II. *In a cyclic region in which the equation*

$$Xdx + Ydy + Zdz = -D\Psi$$

is everywhere fulfilled, the line-integral from A to P, along a line drawn within the region, will not in general be determinate unless the channel of communication between A and P be specified.

Let K be the cyclomatic number of the region, then K sections of the region may be made by surfaces which we may call Diaphragms, so as to close up K of the channels of communication, and reduce the region to an acyclic condition without destroying its continuity.

The line-integral from A to any point P taken along a line which does not cut any of these diaphragms will be, by the last theorem, determinate in value.

Now let A and P be taken indefinitely near to each other, but on opposite sides of a diaphragm, and let K be the line-integral from A to P.

Let A' and P' be two other points on opposite sides of the same diaphragm and indefinitely near to each other, and let K' be the line-integral from A' to P'. Then $K' = K$.

For if we draw AA' and PP', nearly coincident, but on opposite sides of the diaphragm, the line-integrals along these lines will be equal. Suppose each equal to L, then the line-integral of $A'P'$ is equal to that of $A'A + AP + PP' = -L + K + L = K =$ that of AP.

Hence the line-integral round a closed curve which passes through one diaphragm of the system in a given direction is a constant quantity K. This quantity is called the Cyclic constant corresponding to the given cycle.

Let any closed curve be drawn within the region, and let it cut the diaphragm of the first cycle p times in the positive direction and p' times in the negative direction, and let $p - p' = n_1$. Then the line-integral of the closed curve will be $n_1 K_1$.

Similarly the line-integral of any closed curve will be

$$n_1 K_1 + n_2 K_2 + \ldots + n_K K_K;$$

where n_K represents the excess of the number of positive passages of the curve through the diaphragm of the cycle K over the number of negative passages.

If two curves are such that one of them may be transformed into the other by continuous motion without at any time passing through any part of space for which the condition of having a potential is not fulfilled, these two curves are called Reconcileable curves. Curves for which this transformation cannot be effected are called Irreconcileable curves *.

The condition that $Xdx + Ydy + Zdz$ is a complete differential of some function Ψ for all points within a certain region, occurs in several physical investigations in which the directed quantity and the potential have different physical interpretations.

In pure kinematics we may suppose X, Y, Z to be the components of the displacement of a point of a continuous body whose original coordinates are x, y, z, then the condition expresses that these displacements constitute a *non-rotational strain* †.

If X, Y, Z represent the components of the velocity of a fluid at the point x, y, z, then the condition expresses that the motion of the fluid is irrotational.

If X, Y, Z represent the components of the force at the point x, y, z, then the condition expresses that the work done on a particle passing from one point to another is the difference of the potentials at these points, and the value of this difference is the same for all reconcileable paths between the two points.

On Surface-Integrals.

21.] Let dS be the element of a surface, and ϵ the angle which a normal to the surface drawn towards the positive side of the surface makes with the direction of the vector quantity R, then $\iint R \cos \epsilon dS$ is called the *surface-integral of R over the surface S.*

THEOREM III. *The surface-integral of the flux through a closed surface may be expressed as the volume-integral of its convergence taken within the surface.* (See Art. 25.)

Let X, Y, Z be the components of R, and let l, m, n be the direction-cosines of the normal to S measured outwards. Then the surface-integral of R over S is

$$\begin{aligned}\iint R \cos \epsilon\, dS &= \iint X l\, dS + \iint Y m\, dS + \iint Z n\, dS \\ &= \iint X dy\, dz + \iint Y dz\, dx + \iint Z dx\, dy; \qquad (1)\end{aligned}$$

* See Sir W. Thomson 'On Vortex Motion,' *Trans. R. S. Edin.*, 1869.

† See Thomson and Tait's *Natural Philosophy*, § 190 (*i*).

the values of X, Y, Z being those at a point in the surface, and the integrations being extended over the whole surface.

If the surface is a closed one, then, when y and z are given, the coordinate x must have an even number of values, since a line parallel to x must enter and leave the enclosed space an equal number of times provided it meets the surface at all.

Let a point travelling from $x=-\infty$ to $x=+\infty$ first enter the space when $x=x_1$, then leave it when $x=x_2$, and so on; and let the values of X at these points be X_1, X_2, &c., then

$$\iint X\,dy\,dz = \iint \{(X_2-X_1)+(X_4-X_3)+\&c.+(X_{2n}-X_{2n-1})\}\,dy\,dz. \quad (2)$$

If X is a quantity which is continuous, and has no infinite values between x_1 and x_2, then

$$X_2-X_1=\int_{x_1}^{x_2}\frac{dX}{dx}\,dx; \quad (3)$$

where the integration is extended from the first to the second intersection, that is, along the first segment of x which is within the closed surface. Taking into account all the segments which lie within the closed surface, we find

$$\iint X\,dy\,dz = \iiint \frac{dX}{dx}\,dx\,dy\,dz, \quad (4)$$

the double integration being confined to the closed surface, but the triple integration being extended to the whole enclosed space. Hence, if X, Y, Z are continuous and finite within a closed surface S, the total surface-integral of R over that surface will be

$$\iint R\cos\epsilon\,dS = \iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)dx\,dy\,dz, \quad (5)$$

the triple integration being extended over the whole space within S.

Let us next suppose that X, Y, Z are not continuous within the closed surface, but that at a certain surface $F(x, y, z)=0$ the values of X, Y, Z alter abruptly from X, Y, Z on the negative side of the surface to X', Y', Z' on the positive side.

If this discontinuity occurs, say, between x_1 and x_2, the value of X_2-X_1 will be

$$\int_{x_1}^{x_2}\frac{dX}{dx}\,dx+(X'-X), \quad (6)$$

where in the expression under the integral sign only the finite values of the derivative of X are to be considered.

In this case therefore the total surface-integral of R over the closed surface will be expressed by

$$\iint R\cos\epsilon\, dS = \iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)dx\,dy\,dz + \iint (X'-X)\,dy\,dz$$
$$+\iint (Y'-Y)\,dz\,dx + \iint (Z'-Z)\,dx\,dy\,; \quad (7)$$

or, if l', m', n' are the direction-cosines of the normal to the surface of discontinuity, and dS' an element of that surface,

$$\iint R\cos\epsilon\, dS = \iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)dx\,dy\,dz$$
$$+\iint \{l'(X'-X)+m'(Y'-Y)+n'(Z'-Z)\}\,dS', \quad (8)$$

where the integration of the last term is to be extended over the surface of discontinuity.

If at every point where X, Y, Z are continuous

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz} = 0, \quad (9)$$

and at every surface where they are discontinuous

$$l'X'+m'Y'+n'Z' = l'X+m'Y+n'Z, \quad (10)$$

then the surface-integral over every closed surface is zero, and the distribution of the vector quantity is said to be Solenoidal.

We shall refer to equation (9) as the General solenoidal condition, and to equation (10) as the Superficial solenoidal condition.

22.] Let us now consider the case in which at every point within the surface S the equation

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz} = 0 \quad (11)$$

is fulfilled. We have as a consequence of this the surface-integral over the closed surface equal to zero.

Now let the closed surface S consist of three parts S_1, S_0, and S_2. Let S_1 be a surface of any form bounded by a closed line L_1. Let S_0 be formed by drawing lines from every point of L_1 always coinciding with the direction of R. If l, m, n are the direction-cosines of the normal at any point of the surface S_0, we have

$$R\cos\epsilon = Xl+Ym+Zn = 0. \quad (12)$$

Hence this part of the surface contributes nothing towards the value of the surface-integral.

Let S_2 be another surface of any form bounded by the closed curve L_2 in which it meets the surface S_0.

Let Q_1, Q_0, Q_2 be the surface-integrals of the surfaces S_1, S_0, S_2, and let Q be the surface-integral of the closed surface S. Then

$$Q = Q_1+Q_0+Q_2 = 0\,; \quad (13)$$

and we know that $$Q_0 = 0\,; \tag{14}$$
therefore $$Q_2 = -Q_1\,; \tag{15}$$
or, in other words, the surface-integral over the surface S_2 is equal and opposite to that over S_1 whatever be the form and position of S_2, provided that the intermediate surface S_0 is one for which R is always tangential.

If we suppose L_1 a closed curve of small area, S_0 will be a tubular surface having the property that the surface-integral over every complete section of the tube is the same.

Since the whole space can be divided into tubes of this kind provided
$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0, \tag{16}$$
a distribution of a vector quantity consistent with this equation is called a Solenoidal Distribution.

On Tubes and Lines of Flow.

If the space is so divided into tubes that the surface-integral for every tube is unity, the tubes are called Unit tubes, and the surface-integral over any finite surface S bounded by a closed curve L is equal to the *number* of such tubes which pass through S in the positive direction, or, what is the same thing, the number which pass through the closed curve L.

Hence the surface-integral of S depends only on the form of its boundary L, and not on the form of the surface within its boundary.

On Periphractic Regions.

If, throughout the whole region bounded externally by the single closed surface S_1, the solenoidal condition
$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0$$
is fulfilled, then the surface-integral taken over any closed surface drawn within this region will be zero, and the surface-integral taken over a bounded surface within the region will depend only on the form of the closed curve which forms its boundary.

It is not, however, generally true that the same results follow if the region within which the solenoidal condition is fulfilled is bounded otherwise than by a single surface.

For if it is bounded by more than one continuous surface, one of these is the external surface and the others are internal surfaces, and the region S is a periphractic region, having within it other regions which it completely encloses.

If within any of these enclosed regions, S_1, the solenoidal condition is not fulfilled, let

$$Q_1 = \iint R \cos \epsilon \, dS_1$$

be the surface-integral for the surface enclosing this region, and let Q_2, Q_3, &c. be the corresponding quantities for the other enclosed regions.

Then, if a closed surface S' is drawn within the region S, the value of its surface-integral will be zero only when this surface S' does not include any of the enclosed regions S_1, S_2, &c. If it includes any of these, the surface-integral is the sum of the surface-integrals of the different enclosed regions which lie within it.

For the same reason, the surface-integral taken over a surface bounded by a closed curve is the same for such surfaces only bounded by the closed curve as are reconcileable with the given surface by continuous motion of the surface within the region S.

When we have to deal with a periphractic region, the first thing to be done is to reduce it to an aperiphractic region by drawing lines joining the different bounding surfaces. Each of these lines, provided it joins surfaces which were not already in continuous connexion, reduces the periphractic number by unity, so that the whole number of lines to be drawn to remove the periphraxy is equal to the periphractic number, or the number of internal surfaces. When these lines have been drawn we may assert that if the solenoidal condition is fulfilled in the region S, any closed surface drawn entirely within S, and not cutting any of the lines, has its surface-integral zero.

In drawing these lines we must remember that any line joining surfaces which are already connected does not diminish the periphraxy, but introduces cyclosis.

The most familiar example of a periphractic region within which the solenoidal condition is fulfilled is the region surrounding a mass attracting or repelling inversely as the square of the distance.

In this case we have

$$X = m\frac{x}{r^3}, \quad Y = m\frac{y}{r^3}, \quad Z = m\frac{z}{r^3};$$

where m is the mass supposed to be at the origin of coordinates.

At any point where r is finite

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0,$$

but at the origin these quantities become infinite. For any closed surface not including the origin, the surface-integral is zero. If a closed surface includes the origin, its surface-integral is $4\pi m$.

If, for any reason, we wish to treat the region round m as if it were not periphractic, we must draw a line from m to an infinite distance, and in taking surface-integrals we must remember to add $4\pi m$ whenever this line crosses from the negative to the positive side of the surface.

On Right-handed and Left-handed Relations in Space.

23.] In this treatise the motions of translation along any axis and of rotation about that axis, will be assumed to be of the same sign when their directions correspond to those of the translation and rotation of an ordinary or right-handed screw *.

For instance, if the actual rotation of the earth from west to east is taken positive, the direction of the earth's axis from south to north will be taken positive, and if a man walks forward in the positive direction, the positive rotation is in the order, head, right-hand, feet, left-hand.

If we place ourselves on the positive side of a surface, the positive direction along its bounding curve will be opposite to the motion of the hands of a watch with its face towards us.

This is the right-handed system which is adopted in Thomson and Tait's *Natural Philosophy*, § 243. The opposite, or left-handed system, is adopted in Hamilton's and Tait's *Quaternions*. The operation of passing from the one system to the other is called, by Listing, *Perversion*.

The reflexion of an object in a mirror is a perverted image of the object.

When we use the Cartesian axes of x, y, z, we shall draw them

* The combined action of the muscles of the arm when we turn the upper side of the right-hand outwards, and at the same time thrust the hand forwards, will impress the right-handed screw motion on the memory more firmly than any verbal definition. A common corkscrew may be used as a material symbol of the same relation.

Professor W. H. Miller has suggested to me that as the tendrils of the vine are right-handed screws and those of the hop left-handed, the two systems of relations in space might be called those of the vine and the hop respectively.

The system of the vine, which we adopt, is that of Linnæus, and of screw-makers in all civilized countries except Japan. De Candolle was the first who called the hop-tendril right-handed, and in this he is followed by Listing, and by most writers on the rotatory polarization of light. Screws like the hop-tendril are made for the couplings of railway-carriages, and for the fittings of wheels on the left side of ordinary carriages, but they are always called left-handed screws by those who use them.

so that the ordinary conventions about the cyclic order of the symbols lead to a right-handed system of directions in space. Thus, if x is drawn eastward and y northward, z must be drawn upward.

The areas of surfaces will be taken positive when the order of integration coincides with the cyclic order of the symbols. Thus, the area of a closed curve in the plane of xy may be written either

$$\int x\,dy \quad \text{or} \quad -\int y\,dx\,;$$

the order of integration being x, y in the first expression, and y, x in the second.

This relation between the two products $dx\,dy$ and $dy\,dx$ may be compared with that between the products of two perpendicular vectors in the doctrine of Quaternions, the sign of which depends on the order of multiplication, and with the reversal of the sign of a determinant when the adjoining rows or columns are exchanged.

For similar reasons a volume-integral is to be taken positive when the order of integration is in the cyclic order of the variables x, y, z, and negative when the cyclic order is reversed.

We now proceed to prove a theorem which is useful as establishing a connexion between the surface-integral taken over a finite surface and a line-integral taken round its boundary.

24.] THEOREM IV. *A line-integral taken round a closed curve may be expressed in terms of a surface-integral taken over a surface bounded by the curve.*

Let X, Y, Z be the components of a vector quantity $\mathfrak{A}$ whose line-integral is to be taken round a closed curve s.

Let S be any continuous finite surface bounded entirely by the closed curve s, and let ξ, η, ζ be the components of another vector quantity $\mathfrak{B}$, related to X, Y, Z by the equations

$$\xi = \frac{dZ}{dy} - \frac{dY}{dz}, \quad \eta = \frac{dX}{dz} - \frac{dZ}{dx}, \quad \zeta = \frac{dY}{dx} - \frac{dX}{dy}. \tag{1}$$

Then the surface-integral of $\mathfrak{B}$ taken over the surface S is equal to the line-integral of $\mathfrak{A}$ taken round the curve s. It is manifest that ξ, η, ζ fulfil of themselves the solenoidal condition

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0.$$

Let l, m, n be the direction-cosines of the normal to an element of the surface dS, reckoned in the positive direction. Then the value of the surface-integral of $\mathfrak{B}$ may be written

$$\iint (l\xi + m\eta + n\zeta)\, dS. \tag{2}$$

In order to form a definite idea of the meaning of the element dS, we shall suppose that the values of the coordinates x, y, z for every point of the surface are given as functions of two independent variables α and β. If β is constant and α varies, the point (x, y, z) will describe a curve on the surface, and if a series of values is given to β, a series of such curves will be traced, all lying on the surface S. In the same way, by giving a series of constant values to α, a second series of curves may be traced, cutting the first series, and dividing the whole surface into elementary portions, any one of which may be taken as the element dS.

The projection of this element on the plane of y, z is, by the ordinary formula,

$$l\, dS = \left(\frac{dy}{d\alpha}\frac{dz}{d\beta} - \frac{dy}{d\beta}\frac{dz}{d\alpha}\right) d\beta\, d\alpha. \tag{3}$$

The expressions for $m\,dS$ and $n\,dS$ are obtained from this by substituting x, y, z in cyclic order.

The surface-integral which we have to find is

$$\iint (l\xi + m\eta + n\zeta)\, dS; \tag{4}$$

or, substituting the values of ξ, η, ζ in terms of X, Y, Z,

$$\iint \left(m\frac{dX}{dz} - n\frac{dX}{dy} + n\frac{dY}{dx} - l\frac{dY}{dz} + l\frac{dZ}{dy} - m\frac{dZ}{dx}\right) dS. \tag{5}$$

The part of this which depends on X may be written

$$\iint \left\{ \frac{dX}{dz}\left(\frac{dz}{d\alpha}\frac{dx}{d\beta} - \frac{dz}{d\beta}\frac{dx}{d\alpha}\right) - \frac{dX}{dy}\left(\frac{dx}{d\alpha}\frac{dy}{d\beta} - \frac{dx}{d\beta}\frac{dy}{d\alpha}\right) \right\} d\beta\, d\alpha; \tag{6}$$

adding and subtracting $\frac{dX}{dx}\frac{dx}{d\alpha}\frac{dx}{d\beta}$, this becomes

$$\iint \left\{ \frac{dx}{d\beta}\left(\frac{dX}{dx}\frac{dx}{d\alpha} + \frac{dX}{dy}\frac{dy}{d\alpha} + \frac{dX}{dz}\frac{dz}{d\alpha}\right) - \frac{dx}{d\alpha}\left(\frac{dX}{dx}\frac{dx}{d\beta} + \frac{dX}{dy}\frac{dy}{d\beta} + \frac{dX}{dz}\frac{dz}{d\beta}\right) \right\} d\beta\, d\alpha; \tag{7}$$

$$= \iint \left(\frac{dX}{d\alpha}\frac{dx}{d\beta} - \frac{dX}{d\beta}\frac{dx}{d\alpha}\right) d\beta\, d\alpha. \tag{8}$$

As we have made no assumption as to the form of the functions α and β, we may assume that α is a function of X, or, in other words, that the curves for which α is constant are those for which

X is constant. In this case $\frac{dX}{d\beta} = 0$, and the expression becomes by integration with respect to α,

$$\iint \frac{dX}{d\alpha}\frac{dx}{d\beta}\,d\beta\,d\alpha = \int X\frac{dx}{d\beta}\,d\beta\,; \tag{9}$$

where the integration is now to be performed round the closed curve. Since all the quantities are now expressed in terms of one variable β, we may make s, the length of the bounding curve, the independent variable, and the expression may then be written

$$\int X\frac{dx}{ds}\,ds, \tag{10}$$

where the integration is to be performed round the curve s. We may treat in the same way the parts of the surface-integral which depend upon Y and Z, so that we get finally,

$$\iint (l\xi + m\eta + n\zeta)\,dS = \int \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right) ds\,; \tag{11}$$

where the first integral is extended over the surface S, and the second round the bounding curve s*.

On the effect of the operator ∇ on a vector function.

25.] We have seen that the operation denoted by ∇ is that by which a vector quantity is deduced from its potential. The same operation, however, when applied to a vector function, produces results which enter into the two theorems we have just proved (III and IV). The extension of this operator to vector displacements, and most of its further development, is due to Professor Tait †.

Let σ be a vector function of ρ, the vector of a variable point. Let us suppose, as usual, that

$$\rho = ix + jy + kz,$$

and $$\sigma = iX + jY + kZ;$$

where X, Y, Z are the components of σ in the directions of the axes.

We have to perform on σ the operation

$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}.$$

Performing this operation, and remembering the rules for the

* This theorem was given by Professor Stokes. *Smith's Prize Examination*, 1854, question 8. It is proved in Thomson and Tait's *Natural Philosophy*, § 190 (*j*).

† See *Proc. R. S. Edin.*, April 28, 1862. 'On Green's and other allied Theorems,' *Trans. R. S. Edin.*, 1869–70, a very valuable paper; and 'On some Quaternion Integrals,' *Proc. R. S. Edin.*, 1870–71.