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Cohomology
of Sheaves

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Introduction

   This text exposes the basic features of cohomology of
sheaves and its applications. The general theory of sheaves is
very limited and no essential result is obtainable without turn-
ing to particular classes of topological spaces. The most satis-
factory general class is that of locally compact spaces and it
is the study of such spaces which occupies the central part of
this text.

   The fundamental concepts in the study of locally compact
spaces is cohomology with compact support and a particular class
of sheaves, the so-called soft sheaves. This class plays a double
role as the basic vehicle for the internal theory and is the key
to applications in analysis. The basic example of a soft sheaf
is the sheaf of smooth functions on $\mathbb{R}^n$ or more generally on any
smooth manifold. A rather large effort has been made to demon-
strate the relevance of sheaf theory in even the most elementary
analysis. This process has been reversed in order to base the
fundamental calculations in sheaf theory on elementary analysis.

   The central theme of the text is Poincaré duality or rather
its generalizations by Borel and Verdier. In its first form this
appears as a duality between cohomology and cohomology with com-

pact support. A more general Poincaré duality theory is developed for a continuous map between locally compact spaces. The important special case of a closed imbedding admits generalization to arbitrary topological spaces and is best understood in the framework of local cohomology. This theory is used for construction of characteristic classes of all sorts: Chern classes, Stiefel-Whitney classes, ...

For further applications to algebraic topology, a homology theory is developed for locally compact spaces and proper maps. This allows one to express Poincaré duality as an isomorphism between homology and cohomology. Applications are given to the classical theory of topological manifolds: fundamental class, diagonal class, Lefschetz fixed point formula ...

This homology theory is particularly suited for the study of algebraic varieties and a detailed introduction to (co)homology classes of algebraic cycles is given, including a topological definition of the local intersection symbol. It is a rather remarkable feature that this homology theory more or less automatically grinds out algebraic cycles.

A word about homological algebra. The first chapter of the text gives an introduction to homological algebra sufficient for most of the text. The last chapter, or appendix if you wish, gives an introduction to derived categories used in the more advanced parts of the text and in the proofs of the basic cup product formulas. It is my hope that this will give some readers motivation for Verdier's rather difficult text (1) on triangulated categories.

It remains for me to thank W. Fulton and R. MacPherson for their encouragement to publish the text, to thank a number of colleagues, who read part of the manuscript, H.H. Andersen, J.P. Hansen, A. Kock, O. Kroll, O.A. Laudal and H.A. Nielsen and to thank Else Yndgaard for excellent typing and cooperation.

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I. Homological Algebra

I.1 Exact categories

Consider a category with <u>zero object</u> 0, that is for every object A there is precisely one morphism $A \to 0$ and precisely one $0 \to A$.

A <u>zero morphism</u> $A \to B$ is one which can be factored $A \to 0 \to B$.

A <u>kernel</u>, Ker f for a morphism $f: A \to B$ is a pair (K,i) where $i: K \to A$ is a monomorphism with $f\,i = 0$ and such that any morphism $g: X \to A$ with $f\,g = 0$ factors through $i: K \to A$.

A <u>cokernel</u>, Cok f for f is a pair (C,p) where $p: B \to C$ is an epimorphism with $p\,f = 0$ such that any morphism $h: B \to Y$ with $h\,f = 0$ factors through p.

We shall assume that every morphism has kernel and cokernel.

An <u>image</u>, Im f is a kernel for a cokernel.

A <u>coimage</u>, Coim f is a cokernel for a kernel.

Every morphism f has a <u>canonical factorization</u>

$$A \to \operatorname{Coim} f \overset{f'}{\to} \operatorname{Im} f \to B$$

Definition 1.1. An <u>exact category</u> is a category with zero objects, kernels, cokernels and such that $\text{Coim} f \xrightarrow{f'} \text{Im} f$ always is an isomorphism.

In the remaining part of this section we shall work in an exact category.

Definition 1.2. A sequence of morphisms

$$\cdots A^{n-1} \xrightarrow{f^{n-1}} A^n \xrightarrow{f^n} A^{n+1} \xrightarrow{f^{n+1}} \cdots$$

is called <u>exact</u> if $\text{Im}(f^{n-1}) = \text{Ker}(f^n)$, for all n.

Proposition 1.3. Consider the exact, commutative diagram

$$
\begin{array}{cccc}
A \to & B \to & C \to & D \\
\downarrow a & \downarrow b & \downarrow c & \downarrow d \\
A' \to & B' \to & C' \to & D' \\
\downarrow & & & \\
0 & & &
\end{array}
$$

The induced sequence $\text{Ker} b \to \text{Ker} c \to \text{Ker} d$ is exact.

Proof. Break the diagram into two pieces

$$
\begin{array}{ccccc}
0 \to & E \to & C \to & D \\
& \downarrow e & \downarrow c & \downarrow d \\
0 \to & E' \to & C' \to & D'
\end{array}
\qquad
\begin{array}{cccc}
A \to & B \to & E \to & 0 \\
\downarrow a & \downarrow b & \downarrow e & \\
A' \to & B' \to & E' \to & 0 \\
& & \downarrow & \\
& & 0 &
\end{array}
$$

We have to prove that

α) $0 \to \text{Ker} e \to \text{Ker} c \to \text{Ker} d$ is exact

β) $\text{Ker} b \to \text{Ker} e$ is surjective

α) Check that $\text{Ker} e \to \text{Ker} c$ is a kernel for $\text{Ker} c \to D$.

β.1) Check that $\text{cok} b \to \text{Cok} e$ is an isomorphism (use the dual statement to α , if necessary).

β.2) The exact commutative diagram

$$
\begin{array}{cccc}
A' \to & B' \to & E' \to & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 \to & \text{Cok} b \to & \text{Cok} e \to & 0
\end{array}
$$

shows that $A' \to \text{Im} b \to \text{Im} e$ is exact: replace A' by a kernel for $B' \to E'$ and use α).

β.3) This gives an exact commutative diagram

$$
\begin{array}{cccc}
A \to & B \to & E \to & 0 \\
\downarrow a' & \downarrow b' & \downarrow e' & \\
A' \to & \text{Im} b \to & \text{Im} e \to & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 &
\end{array}
$$

Check that e' is a cokernel for $\text{Ker} b' \to E$.

The dual statement is

Q.E.D.

Proposition 1.4. Consider the exact, commutative diagram

$$
\begin{array}{cccc}
& & & 0 \\
& & & \downarrow \\
A \to & B \to & C \to & D \\
\downarrow a & \downarrow b & \downarrow c & \downarrow d \\
A' \to & B' \to & C' \to & D'
\end{array}
$$

The induced sequence $\text{Cok} a \to \text{Cok} b \to \text{Cok} c$ is exact.

__Corollary 1.5.__ Consider morphisms $f: X \to Y$ and $g: Y \to Z$.
The following sequence is exact

$$0 \to \operatorname{Ker} f \to \operatorname{Ker} gf \to \operatorname{Ker} g \to \operatorname{Cok} f \to \operatorname{Cok} gf \to \operatorname{Cok} g \to 0.$$

__Proof.__ Apply 1.3 and 1.4 to the two diagrams

$$Q.E.D.$$

__Snake Lemma 1.6.__ Consider the exact commutative diagram

There is an exact sequence

$$\operatorname{Ker} b \to \operatorname{Ker} c \to \operatorname{Ker} d \overset{\partial}{\to} \operatorname{Cok} b \to \operatorname{Cok} c \to \operatorname{Cok} d.$$

More precisely

1) Put $K = \operatorname{Ker}(C \to D')$. $K \to \operatorname{Ker} d$ is an epimorphism.

2) Put $K' = \operatorname{Cok}(B \to C')$. $\operatorname{Cok} b \to K'$ is a monomorphism.

3) There exists a unique map

$$\partial: \operatorname{Ker} d \to \operatorname{Cok} b$$

such that the two composites

$$K \to C \overset{c}{\to} C' \to K'$$
$$K \to \operatorname{Ker} d \overset{\partial}{\to} \operatorname{Cok} b \to K'$$

are the same.

4) The six-term sequence above is exact.

__Proof.__ Let f denote the morphism $C' \to \operatorname{Ker}(D' \to E')$.
Consider the exact commutative diagram

It follows from 1.3 applied twice that

$$B \longrightarrow K \longrightarrow \operatorname{Ker} d \longrightarrow 0$$

is exact, and similar, that

$$0 \longrightarrow \operatorname{Cok} b \longrightarrow K' \longrightarrow D'$$

is exact. This proves 1), 2), 3). By 1.3, 1.4 and duality it suffices to prove that

$$\text{Ker } c \longrightarrow \text{Ker } d \longrightarrow \text{Cok } b$$

is exact. It suffices to prove exactness of

$$\text{Ker } c \longrightarrow \text{Ker } d \longrightarrow K'.$$

Consider the diagram

$$
\begin{array}{ccccccc}
B & \longrightarrow & K & \longrightarrow & \text{Ker } d & \longrightarrow & 0 \\
\downarrow{\scriptstyle \text{id}} & & \downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & C' & \longrightarrow & K' & \longrightarrow & 0 \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}
$$

Conclusion by 1.3.

$$Q.E.D.$$

Let us record a much used special case

Five lemma 1.7. Given an exact commutative diagram

$$
\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}
$$

If a,b,d,e are isomorphism, then c is an isomorphism.

I.2 Homology of complexes

We shall discuss the concept, homology in the framework of an exact category.

By a complex we understand a sequence $C^{\cdot} = (C^n, \partial^n)_{n \in \mathbb{Z}}$ of objects and morphisms

$$\dots \longrightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \xrightarrow{\partial^{n+1}} C^{n+2} \longrightarrow \dots$$

with $\partial^{n+1}\partial^n = 0$ for all $n \in \mathbb{Z}$. The ∂'s are called differentials or boundary operators.

A morphism of complexes $f: C^{\cdot} \longrightarrow D^{\cdot}$ is a sequence $f = (f^n)_{n \in \mathbb{Z}}$ of morphisms $f^n: C^n \dashrightarrow D^n$ with

$$f^{n+1}\partial^n = \partial^n f^n \quad \text{for all} \quad n \in \mathbb{Z}.$$

For a complex $C^{\cdot}$ we define for $n \in \mathbb{Z}$ the n'th homology object

2.1 $$H^n(C^{\cdot}) = \text{Ker } \partial^n/\text{Im } \partial^{n-1}$$

A morphism $f: C^{\cdot} \longrightarrow D^{\cdot}$ of complexes will induce a morphism on homology

2.2 $$H^n(f) = H^n(C^{\cdot}) \longrightarrow H^n(D^{\cdot})$$

Consider a sequence of complexes

$$0 \longrightarrow P^{\cdot} \xrightarrow{f} Q^{\cdot} \xrightarrow{g} R^{\cdot} \longrightarrow 0$$

which is a chainwise exact, i.e. with

$$0 \longrightarrow P^n \longrightarrow Q^n \longrightarrow R^n \longrightarrow 0$$

exact for all $n \in \mathbb{Z}$. We shall construct the so called connecting morphism

2.3
$$c^n: H^n(R^\cdot) \to H^{n+1}(P^\cdot)$$

and derive a long exact sequence

2.4
$$H^n(P^\cdot) \xrightarrow{H^n(f)} H^n(Q^\cdot) \xrightarrow{H^n(g)} H^n(R^\cdot) \xrightarrow{c^n} H^{n+1}(P^\cdot) \xrightarrow{H^{n+1}(f)} H^{n+1}(Q^\cdot)$$

Construction. For a complex $C^\cdot$ we put

2.5
$$Z^{n+1}(C^\cdot) = \mathrm{Ker}\, \partial^{n+1}, \qquad 'Z^n(C^\cdot) = \mathrm{Cok}\, \partial^{n-1}$$

The boundary $\partial^n: C^n \to C^{n+1}$ induces

$$d^n: {}'Z^n(C^\cdot) \to Z^{n+1}(C^\cdot)$$

As is easily seen we have an exact sequence

2.6
$$0 \to H^n(C^\cdot) \to {}'Z^n(C^\cdot) \xrightarrow{d^n} Z^{n+1}(C^\cdot) \to H^{n+1}(C^\cdot) \to 0$$

We can now derive a commutative diagram

$$
\begin{array}{ccccccc}
'Z^n(P^\cdot) & \longrightarrow & 'Z^n(Q^\cdot) & \longrightarrow & 'Z^n(R^\cdot) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow Z^{n+1}(P^\cdot) & \longrightarrow & Z^{n+1}(Q^\cdot) & \longrightarrow & Z^{n+1}(R^\cdot) & &
\end{array}
$$

whose rows are exact as one easily derives from 1.3 and 1.4. We can now conclude the construction by appealing to the snake lemma 1.6.

The connecting morphism 2.3 has the following functorial property. Given

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P^\cdot & \longrightarrow & Q^\cdot & \longrightarrow & R^\cdot & \longrightarrow & 0 \\
& & \downarrow u & & \downarrow v & & \downarrow w & & \\
0 & \longrightarrow & U^\cdot & \longrightarrow & V^\cdot & \longrightarrow & W^\cdot & \longrightarrow & 0
\end{array}
$$

a commutative diagram of complexes whose rows are chainwise exact.

Then the following diagram is <u>commutative</u>

2.7
$$
\begin{array}{ccc}
H^n(R^\cdot) & \xrightarrow{\quad c^n \quad} & H^{n+1}(P^\cdot) \\
\downarrow{\scriptstyle H^n(w)} & & \downarrow{\scriptstyle H^{n+1}(u)} \\
H^n(W^\cdot) & \xrightarrow{\quad c^n \quad} & H^{n+1}(U^\cdot)
\end{array}
$$

The commutativity results from the origin of the connecting morphism. We shall at once take a step towards a realization of the connecting morphism.

Lemma 2.8. Given an exact sequence

$$0 \longrightarrow P^\cdot \xrightarrow{\ f\ } Q^\cdot \xrightarrow{\ g\ } R^\cdot \longrightarrow 0$$

of complexes and a morphism of complexes $h: R^\cdot \to P^\cdot[1]$. — For $n \in \mathbb{Z}$ let $i^n: L^n \to Q^n$ be a kernel for $\partial^n g^n: Q^n \to R^{n+1}$ and let $p^{n+1}: Q^{n+1} \to {}'L^{n+1}$ be a cokernel for $\partial^n f^n: P^n \to Q^{n+1}$. Then

$$H^n(h): H^n(R^\cdot) \to H^{n+1}(P^\cdot)$$

is equal to the connecting morphism if and only if

$$p^{n+1}\, f^{n+1}\, h^n\, g^n\, i^n = p^{n+1}\, \partial^n\, i^n$$

where ∂^n is the differential from $Q^\cdot$.

Proof. Use I.1.5 to see that the sequences

$$L^n \to {}'Z^n(Q^\cdot) \to Z^{n+1}(R^\cdot)$$

$$'Z^n(P^\cdot) \to Z^{n+1}(Q^\cdot) \to {}'L^{n+1}$$

are exact. It follows from 1.6 and the diagram below 2.6, that

$$Im(L^n \to 'Z^n(R\,")) = H^n(R^{\cdot})$$

$$Coim(Z^{n+1}(P^{\cdot}) \to 'L^{n+1}) = H^{n+1}(P^{\cdot})$$

We shall first show that the composite

$$L^n \longrightarrow H^n(R\,") \xrightarrow{H^n(h)} H^{n+1}(P\,") \longrightarrow 'L^{n+1}$$

equals $p^{n+1}f^{n+1}h^ng^ni^n$. To see this bring three commutative diagrams together

$$L^n \longrightarrow H^n(R\,") \xrightarrow{H^n(h)} H^{n+1}(P\,")$$

(commutative diagram with vertices L^n, $H^n(R\,")$, $H^{n+1}(P\,")$, $'Z^n(R\,")$, $'Z^n(h)$, $'Z^{n+1}(P\,")$, $'L^{n+1}$)
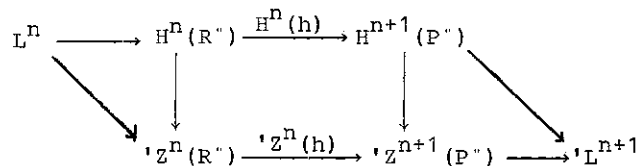
Next bring three more commutative diagrams together

(commutative diagram with $L^n \xrightarrow{i^ng^n} R^n \xrightarrow{h^n} P^{n+1} \xrightarrow{f^{n+1}} Q^{n+1}$ and $'Z^n(R\,") \xrightarrow{'Z^n(h)} 'Z^{n+1}(P\,") \longrightarrow 'L^{n+1}$)
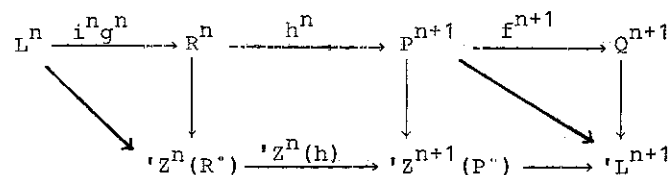
This proves the promised formula. - The result now follows from 1.5.

Q.E.D.

I.3 Additive categories

By an additive category we understand a category in which the Hom-sets come equipped with abelian group structures making all compositions bilinear. Besides we require the presence of a zero object and <u>direct sums</u>, i.e. for objects P and Q, there exists an object $P \oplus Q$ and morphisms i: $P \to P \oplus Q$ and j: $Q \to P \oplus Q$, such that whenever given

(commutative diagram with P, Q, i, j, f, g, $P \oplus Q$, X)
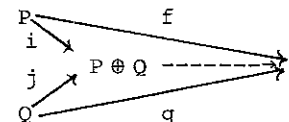
morphisms f and g as above we can fill in the dotted arrow making the diagram commutative, and this in a unique way. - In particular, we can find p: $P \oplus Q \to P$ with pi = 1 and pj = 0 and in the same way q: $P \oplus Q \to Q$ with qi = 0 and qj = 1. Let us prove that $P \oplus Q$, p,q makes up a <u>direct product</u>, i.e. given a diagram

(commutative diagram with Y, f, g, P, q, Q, $P \oplus Q$)
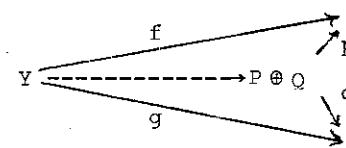
we can fill in the dotted arrow in a unique way making the diagram commute. - Notice the relation

3.1   1 = ip + jq

which we can check by composing both sides with i and j using the relations

3.2   pi = 1, pj = 0, qi = 0, qj = 1

The result will follow from the following

Lemma 3.3. Consider a diagram
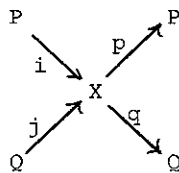
which satisfies relation 3.1 and 3.2. Then the arrows i,j form a direct sum and the arrows p,q form a direct product.

Proof. Let us prove the second statement. Given f: Y → P and g: Y → Q. Then

$$F = if + jg: Y \to X$$

satisfies pF = f and qF = g as it follows from 3.2. - Conversely, given F with pF = f and qF = g. Then using 3.1 we get

$$F = ipF + jqF = if + jg$$

from which uniqueness follows.

Q.E.D.

Definition 3.4. A sequence of morphisms

$$C \xrightarrow{f} D \xrightarrow{g} E$$

is called a split exact sequence, if for every object X, the following sequence of abelian groups is exact

3.5 $0 \to \text{Hom}(X,C) \to \text{Hom}(X,D) \to \text{Hom}(X,E) \to 0$

Let us notice that, gf = 0 as it follows from 3.5 with X = C.

Example 3.6. With the notation of 3.3

$$P \xrightarrow{i} X \xrightarrow{p} Q$$

is a split exact sequence. - It will follow from the proof of 3.7, that all split exact sequences have this form

Proposition 3.7. Given a split exact sequence

$$C \xrightarrow{f} D \xrightarrow{g} E$$

Then for every object Y, the induced sequence of abelian groups

$$0 \to \text{Hom}(E,Y) \to \text{Hom}(D,Y) \to \text{Hom}(C,Y) \to 0$$

is exact.

Proof. Let us first notice that the exact sequence 3.5 with E = X shows that we can find a section to g i.e. a morphism s: E → D such that gs = 1. Let s be any such section. Notice that

$$g(1-sg) = 0$$

By the exactness of 3.5 with X = D we can find r: D → C such that 1-sg = fr or

$$1 = sg + fr$$

Let us prove that

$$gf = 0, \quad gs = 1, \quad rf = 1, \quad rs = 0 \quad .$$

We have already accounted for the two first. To prove the third relation, remark

$$f(rf) = (1-sq)f = f-sqf = f = f1$$

whence $rf = 1$ since f is a monomorphism. To prove the fourth relation notice that

$$r = r(sg + fr) = rsg + rfr = rsg + r$$

which shows that $rsg = 0$ and whence

$$rs = rs(gs) = (rsg)s = 0$$

It is now easy to conclude by Lemma 3.3.   Q.E.D.

Definition 3.8. A morphism $f: C \to D$ is called a split monomorphism if there exists a morphism $g: D \to E$ such that f,g form a split exact sequence. The dual notion is split epimorphism.

Additive Functors 3.9. Let A and B be additive categories. A functor

$$F: A \to B$$

is called additive, if for every pair of objects X and Y in A, F induces linear maps

$$Hom(X,Y) \to Hom(F(X),F(Y))$$

Let us notice that the additive functor F preserves split exact sequences as it follows from 3.3.

Matrix notation

A family of objects $(X_t)_{t\in T}$ indexed by a finite set T gives rise to an object X and a family of morphisms

$$X_t \xrightarrow{i_t} X \xrightarrow{p_t} X_t \qquad ; \quad t \in T$$

such that for all u and v in T

3.10
$$\sum_{t\in T} i_t p_t = 1 \qquad p_u i_v = \begin{cases} 1 & \text{for } u = v \\ 0 & \text{for } u \neq v \end{cases}$$

These data can be constructed by iterating the previous process. The object X is often written $\oplus_{t\in T} X_t$ and is called the direct sum of the X_t's. - Given a second finite family of objects $(Y_s)_{s\in S}$ with $Y = \oplus_{s\in S} Y_s$. To a morphism $f: X \to Y$ we associate the matrix

$$\{f_{st}\}_{(s,t)\in S\times T} \qquad ; \quad f_{st} = p_s f i_t$$

Conversely to such a matrix with

$$f_{st} \in Hom(X_t, Y_s) \qquad ; \quad (s,t) \in S\times T$$

corresponds a uniquely determined morphism $f: X \to Y$. - Let there be given still another finite family $(Z_r)_{r\in R}$ with $Z = \oplus_{r\in R} Z_r$. For morphisms $f: X \to Y$ and $g: Y \to Z$ we have

3.11
$$(gf)_{rt} = \sum_s g_{rs} f_{st}$$

as it follows from the following calculation

$$(gf)_{rt} = p_r g f i_t = p_r g (\Sigma i_s p_s) f i_t =$$

$$\sum_s p_r g i_s p_s f i_t = \sum_s g_{rs} f_{st}$$

I.4 Homotopy theory of complexes

Throughout this section we shall work in a fixed additive category. A complex $C^{\cdot} = (C^n, \partial^n)$ is a sequence of objects and morphisms

$$\ldots \to C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \to \ldots$$

with $\partial^n \partial^{n-1} = 0$ for all $n \in \mathbb{Z}$.

A morphism of complexes $f: C^{\cdot} \to D^{\cdot}$ $f = (f^n)_{n \in \mathbb{N}}$, is a sequence of morphisms $f^n : C^n \to D^n$ with

$$f^{n+1} \partial^n = \partial^n f^n \quad ; \quad n \in \mathbb{Z}$$

Morphisms of complexes are composed in the obvious way, i.e. componentwise.

Definition 4.1. For a complex $C^{\cdot}$ and $p \in \mathbb{Z}$ we find a new complex $C^{\cdot}[p]$ given by

$$(C^{\cdot}[p])^n = C^{n+p}$$

with boundary operator $(-1)^p \partial^p$. - Given a morphism of complexes $f: C^{\cdot} \to D^{\cdot}$. Define $f[n]: C[n] \to D[n]$ by the formula $f[n]^p = f^{n+p}$.

Definition 4.2. Let $f^{\cdot}, g^{\cdot}: C^{\cdot} \to D^{\cdot}$ be morphisms of complexes. A homotopy $s^{\cdot}$ from $f^{\cdot}$ to $g^{\cdot}$ is a sequence $s^{\cdot} = (s^n)_{n \in \mathbb{Z}}$, $s^n: C^n \to D^{n-1}$ of morphisms, such that

$$f^n - g^n = \partial^{n-1} s^n + s^{n+1} \partial^n \quad ; n \in \mathbb{Z}$$

Morphisms f and g are said to be homotopic if there exists a homotopy from f to g, we write $f \sim g$. Homotopy is an equivalence relation compatible with composition. The additive group of homotopy classes of morphisms from $C^{\cdot}$ to $D^{\cdot}$ is denoted $[C^{\cdot}, D^{\cdot}]$.

We can now form a new additive category by considering the complexes $P^{\cdot}, Q^{\cdot}, \ldots$ in our category as objects and $[P^{\cdot}, Q^{\cdot}]$ as the group of morphisms from $P^{\cdot}$ to $Q^{\cdot}$. The new category is called the homotopy category.

A morphism of complexes $f: P^{\cdot} \to Q^{\cdot}$ is called a homotopy equivalence if it represents an isomorphism in the homotopy category, i.e. if there exists a morphism of complexes $g: Q^{\cdot} \to P^{\cdot}$ such that

$$fg \sim 1 \quad \text{and} \quad gf \sim 1$$

Definition 4.3. For complexes $X^{\cdot}$ and $Y^{\cdot}$ let $\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot})$ denote the complex of abelian groups given by

$$\mathrm{Hom}^n(X^{\cdot}, Y^{\cdot}) = \prod_{p \in \mathbb{Z}} \mathrm{Hom}(X^p, Y^{p+n})$$

and boundary operator

$$[\partial^n f]^p = \partial^{n+p} f^p + (-1)^{n+1} f^{p+1} \partial^p$$

Remark the key formula, $n \in \mathbb{Z}$

4.4
$$[X^{\cdot}, Y^{\cdot}[n]] = H^n \mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot})$$

The complex $\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot})$ is obviously functorial, covariant in $Y^{\cdot}$ and contravariant in $X^{\cdot}$.

We shall now introduce the main theme of a homotopy theory of complexes: Consider a chainwise split exact sequence of complexes

$$P^{\cdot} \xrightarrow{f} Q^{\cdot} \xrightarrow{g} R^{\cdot}$$

i.e. two morphisms of complexes such that for each $n \in \mathbb{Z}$, the sequence

$$P^n \xrightarrow{f^n} Q^n \xrightarrow{g^n} R^n$$

is a split exact sequence. Choose for each $n \in \mathbb{Z}$ a section $s^n : R^n \to Q^n$ to g^n, i.e. $g^n s^n = 1$ or just $gs = 1$, where $s: R \to Q$. We use the notation that R, which is $R^{\cdot}$ with the dot removed, denotes the graded object underlying the complex $R^{\cdot}$.

$$g(\partial s - s \partial) = \partial(gs) - g(s\partial) = 0$$

Thus we can find $h: R \to P[1]$ such that

4.5 $$fh = \partial s - s \partial$$

Using this relation twice, we find

$$f(h\partial + \partial h) = (fh)\partial + \partial(fh) =$$

$$(\partial s - s \partial)\partial + \partial(\partial s - s \partial) = \partial s \partial - \partial s \partial = 0$$

Using that f is a monomorphism we find

$$h\partial + \partial h = 0$$

which shows that $h: R \to P[1]$ is in fact a morphism of complexes

4.6 $$h : R^{\cdot} \to P^{\cdot}[1]$$

The morphism h is unique up to homotopy:

Let $\sigma : R \to Q$ be an alternative section of g and let $k : R^{\cdot} \to P^{\cdot}[1]$ be given by

$$fk = \partial \sigma - \sigma \partial$$

Since $g(s-\sigma) = 0$ we can find $t : Q \to P$ such that $s - \sigma = ft$. We find that

$$f(h-k) = \partial(s-\sigma) - (s-\sigma)\partial = \partial ft - ft\partial = f(\partial t - t \partial)$$

Using that f is a monomorphism we find

$$h - k = \partial t - t \partial = \partial t + t(-\partial)$$

which shows that $h \sim k$.

Definition 4.7. With the notation above, the class of h in $[R^{\cdot}, P^{\cdot}[1]]$ is called the homotopy invariant of the chainwise split exact sequence $P^{\cdot} \to Q^{\cdot} \to R^{\cdot}$.

For a given complex $X^{\cdot}$ our chainwise split exact sequence yields a short exact sequence of complexes of abelian groups

$$0 \to \operatorname{Hom}^{\cdot}(X^{\cdot}, P^{\cdot}) \to \operatorname{Hom}^{\cdot}(X^{\cdot}, Q^{\cdot}) \to \operatorname{Hom}^{\cdot}(X^{\cdot}, R^{\cdot}) \to 0$$

which in turn gives rise to a long exact homology sequence with connecting morphism

4.8 $$[1, h[n]] : [X^{\cdot}, R^{\cdot}[n]] \to [X^{\cdot}, P^{\cdot}[n+1]]$$

Proof. With the notation of 4.5 given morphism of complexes $x^n: X^{..} \to R^{..}[n]$. Lift this to $sx^n: X \to Q[n]$. The result of applying the boundary operator from the complex $\mathrm{Hom}^{.}(X^{..}, Q^{..}[n])$ is

$$\partial sx^n + (-1)^{n+1} sx^n \partial = (\partial s - s\partial) x^n = f h x^n$$

where we have used the formula 4.5.

Q.E.D.

For a complex $Y^{..}$ our split exact sequence yields a short exact sequence of complexes of abelian groups

$$0 \to \mathrm{Hom}^{..}(R^{..}, Y^{..}) \to \mathrm{Hom}^{..}(Q^{..}, Y^{..}) \to \mathrm{Hom}^{..}(P^{..}, Y^{..}) \to 0$$

which gives rise to a long exact homology sequence with <u>connecting morphism</u>

4.9 $\qquad (-1)^{n+1}[h,1]: [P^{..}, Y^{..}[n]] \to [R^{..}, Y^{..}[n+1]]$

Proof. With the notation of 4.5 let $r: Q \to P$ denote the retraction satisfying

$$rf = 1, \quad rs = 0, \quad 1 = sg + fr$$

Let us first prove formula

4.10 $\qquad\qquad hg = r\partial - \partial r$

It suffices to prove the formula after we have applied the monomorphism f on both sides

$$fgh = (\partial s - s\partial)g = \partial sg - sg\partial$$
$$= \partial(1 - fr) - (1 - fr)\partial = f(r\partial - \partial r)$$

Let us now consider an element of $[P^{..}, Y^{..}[n]]$ represented by a morphism of complexes $y^n: P^{..} \to Y^{..}[n]$. Lift this to $y^n r: Q \to Y[n]$ and apply the boundary operator from $\mathrm{Hom}^{..}(Q^{..}, Y^{..})$

$$\partial y^n r + (-1)^{n+1} y^n r\partial = (-1)^n y^n \partial r + (-1)^{n+1} y^n r\partial =$$
$$(-1)^{n+1} y^n (r\partial - \partial r) = (-1)^{n+1} y^n hg$$

which proves the formula 4.9

Q.E.D.

Let us consider a <u>commutative</u> diagram

$$
\begin{array}{ccccc}
P^{..} & \xrightarrow{\ f\ } & Q^{..} & \xrightarrow{\ g^{.}\ } & R^{..} \\
\downarrow{u} & & \downarrow{v} & & \downarrow{w} \\
U^{..} & \xrightarrow{\ a\ } & V^{..} & \xrightarrow{\ b\ } & W^{..}
\end{array}
$$

of complexes, involving chainwise split exact sequences f,g and a,b. The resulting homotopy invariants k and c makes the following diagram <u>homotopy commutative</u>

4.11
$$
\begin{array}{ccc}
R^{..} & \xrightarrow{\ k\ } & P^{..}[1] \\
\downarrow{w} & & \downarrow{u[1]} \\
W^{..} & \xrightarrow{\ c\ } & U^{..}[1]
\end{array}
$$

Proof. Let $X^{..}$ be an arbitrary complex. The diagram above give rise to a commutative diagram of complexes of abelian groups

$$0 \to \mathrm{Hom}^{..}(X^{..}, P^{..}) \to \mathrm{Hom}^{..}(X^{..}, Q^{..}) \to \mathrm{Hom}^{..}(X^{..}, R^{..}) \to 0$$
$$\downarrow\qquad\qquad\quad \downarrow\qquad\qquad\quad \downarrow$$

It follows from 2.7 and 4.8 that the following diagram is commutative

$$
\begin{array}{ccc}
[X^{''},R^{''}] & \xrightarrow{\;[1,k]\;} & [X^{''},P^{''}[1]] \\
\downarrow {\scriptstyle [1,w]} & & \downarrow {\scriptstyle [1,u[1]]} \\
[X^{''},W^{''}] & \xrightarrow[\;[1,c]\;]{} & [X^{''},U^{''}[1]]
\end{array}
$$

Now specify to $X^{''} = R^{''}$ and evaluate the maps on the identity of $R^{''}$.

<div align="right">Q.E.D.</div>

We shall now introduce a very fundamental terminology, that of

Triangles

A triangle diagram:

$$
\begin{array}{ccc}
 & C^{''} & \\
{\scriptstyle w}\nwarrow & & \nearrow{\scriptstyle v} \\
A^{''} & \xrightarrow{\;u\;} & B^{''}
\end{array}
$$
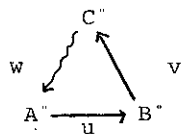
By a <u>triangle</u> is understood three complexes $A^{''},B^{''},C^{''}$ and three morphisms of complexes

$$
A^{''} \xrightarrow{\;u\;} B^{''} \xrightarrow{\;v\;} C^{''} \xrightarrow{\;w\;} A^{''}[1]
$$

such that there exists a chainwise split exact sequence of complexes

$$
P^{''} \xrightarrow{\;f\;} Q^{''} \xrightarrow{\;g\;} R^{''}
$$

with homotopy invariant h and homotopy equivalences a,b,c making the following diagram homotopy commutative

$$
\begin{array}{ccccccc}
A^{''} & \xrightarrow{u} & B^{''} & \xrightarrow{v} & C^{''} & \xrightarrow{w} & A^{''}[1] \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\
P^{''} & \xrightarrow{f} & Q^{''} & \xrightarrow{g} & R^{''} & \xrightarrow{h} & P^{''}[1]
\end{array}
$$

For complexes $X^{''}$ and $Y^{''}$ the triangle above gives rise to long exact sequences

$$
\to [X^{''},A^{''}] \xrightarrow{[1,u]} [X^{''},B^{''}] \xrightarrow{[1,v]} [X^{''},C^{''}] \xrightarrow{[1,w]} [X^{''},A^{''}[1]] \xrightarrow{[1,u[1]]} [X^{''},B^{''}[1]] -
$$

4.12

$$
\to [B^{''}[1],Y^{''}] \xrightarrow{[u[1],1]} [A^{''}[1],Y^{''}] \xrightarrow{[w,1]} [C^{''},Y^{''}] \xrightarrow{[v,1]} [B^{''},Y^{''}] \xrightarrow{[u,1]} [A^{''},Y^{''}]
$$

The <u>mapping cone</u>. Let $u:P^{''} \to Q^{''}$ denote a morphism of complexes we define the mapping cone $Con^{''}(u)$ by

$$
Con^{''}(u) = P^{''}[1]\oplus Q^{''}, \qquad \partial = \begin{pmatrix} -\partial & 0 \\ -u & \partial \end{pmatrix}
$$

Notice that the sequence

$$
Q^{''} \xrightarrow{\;\binom{0}{1}\;} P^{''}[1]\oplus Q^{''} \xrightarrow{\;(1,0)\;} P^{''}[1]
$$

is chainwise split exact. Using the splitting $s = \binom{1}{0}$ we get

$$
\partial s - s\partial = \begin{pmatrix} -\partial & 0 \\ -u & \partial \end{pmatrix}\binom{1}{0} - \binom{1}{0}(-\partial) = \binom{0}{-u} = \binom{0}{1}(-u)
$$

which shows that the homotopy invariant is

$$
-u[1]: \; P^{''}[1] \longrightarrow Q^{''}[1]
$$

as a consequence we have a <u>triangle</u>

4.13

$$
\begin{array}{ccc}
 & P^{''}[1] & \\
{\scriptstyle -u[1]}\nwarrow & & \nearrow{\scriptstyle (1,0)} \\
Q^{''} & \xrightarrow[\;\binom{0}{1}\;]{} & Con^{''}(u)
\end{array}
$$
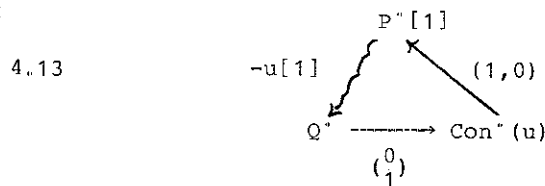
The mapping cylinder

Given a morphism $u: P^{\cdot} \to Q^{\cdot}$ of complexes. From this we derive the morphism

$$\begin{pmatrix} 1 \\ -u \end{pmatrix} : P^{\cdot} \to P^{\cdot} \oplus Q^{\cdot}$$

The cone over this morphism is the <u>mapping cylinder of</u> u, $\text{Cyl}^{\cdot}(u)$. Explicitly

$$\text{Cyl}^{\cdot}(u) = P[1] \oplus P \oplus Q \quad ; \quad \begin{pmatrix} -\partial & 0 & 0 \\ -1 & \partial & 0 \\ u & 0 & \partial \end{pmatrix}$$

The mapping cylinder $\text{Cyl}^{\cdot}(u)$ is homotopy equivalent to $Q^{\cdot}$ in a canonical way: the two morphisms of complexes

$$Q^{\cdot} \xrightarrow{\hspace{2cm}} \text{Cyl}^{\cdot}(u) \xrightarrow{\hspace{2cm}} Q^{\cdot}$$
$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \hspace{3cm} (0,u,1)$$

composes to the identity of $Q^{\cdot}$. The composite of these two morphisms in the opposite order is homotopic to the identity on $\text{Cyl}^{\cdot}(u)$ as it follows from the formula

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}(0,u,1) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\partial & 0 & 0 \\ -1 & \partial & 0 \\ u & 0 & \partial \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\partial & 0 & 0 \\ -1 & \partial & 0 \\ u & 0 & \partial \end{pmatrix}$$

Consider the chainwise split exact sequence

$$P^{\cdot} \xrightarrow{\hspace{2cm}} \text{Cyl}^{\cdot}(u) \xrightarrow{\hspace{2cm}} \text{Con}^{\cdot}(u)$$

explicitly given by

$$\partial \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -\partial & 0 & 0 \\ -1 & \partial & 0 \\ u & 0 & \partial \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -\partial & 0 \\ -u & \partial \end{pmatrix}$$

$$P \xrightarrow{\hspace{2cm}} P[1] \oplus P \oplus Q \xrightarrow{\hspace{2cm}} P[1] \oplus Q$$

The second of the two morphisms has the section $\begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$, which we will use to calculate the homotopy invariant of the sequence. We have

$$\begin{pmatrix} -\partial & 0 & 0 \\ -1 & \partial & 0 \\ u & 0 & \partial \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\partial & 0 \\ -u & \partial \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(1,0)$$

Thus we have constructed a <u>triangle</u>

$$P^{\cdot} \longrightarrow \text{Cyl}^{\cdot}(u) \longrightarrow \text{Con}^{\cdot}(u) \xrightarrow{(1,0)} P[1]$$

Consider the following diagram

$$
\begin{array}{ccccccc}
P^{\cdot} & \xrightarrow{u} & Q^{\cdot} & \xrightarrow{\binom{0}{1}} & \text{Con}^{\cdot}(u) & \xrightarrow{(1,0)} & P^{\cdot}[1] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle\binom{0}{0}{1}} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
P^{\cdot} & \xrightarrow{\binom{0}{1}{0}} & \text{Cyl}^{\cdot}(u) & \xrightarrow{\binom{-1\,0\,0}{0\,0\,1}} & \text{Con}^{\cdot}(u) & \xrightarrow{(1,0)} & P^{\cdot}[1]
\end{array}
$$

The first square is homotopy commutative, since $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has the homotopy inverse $(0,u,1)$ and

$$(0,u,1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = u$$

From this we conclude that a morphism $u: P^{\cdot} \to Q^{\cdot}$ of complexes give rise to a <u>triangle</u>

4.14

$$\begin{array}{ccc} & \text{Con}^{\cdot}(u) & \\ (1,0) \swarrow & & \nwarrow \binom{0}{1} \\ P^{\cdot} \xrightarrow{\quad u \quad} & & Q^{\cdot} \end{array}$$
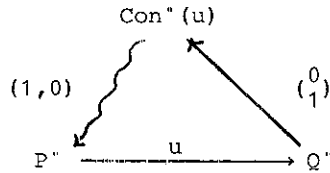
In fact, any triangle has this form.

Proposition 4.15. Given a chainwise split exact sequence of complexes

$$P^{\cdot} \xrightarrow{\quad u \quad} Q^{\cdot} \xrightarrow{\quad v \quad} R^{\cdot}$$

with homotopy invariant $w: R^{\cdot} \to P^{\cdot}[1]$. Then the following diagram[*]

$$\begin{array}{ccccccc} P^{\cdot} & \xrightarrow{u} & Q^{\cdot} & \xrightarrow{\binom{0}{1}} & \text{Con}^{\cdot}(u) & \xrightarrow{(1,0)} & P^{\cdot}[1] \\ \downarrow{1} & & \downarrow{1} & & \downarrow{(0,v)} & & \downarrow{1} \\ P^{\cdot} & \xrightarrow{u} & Q^{\cdot} & \xrightarrow{v} & R^{\cdot} & \xrightarrow{w} & P[1] \end{array}$$
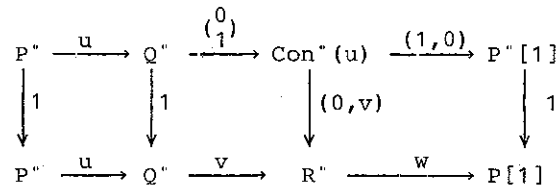
is homotopy commutative and the columns are homotopy equivalences.

Proof. According to the results preceding 4.14 it suffices to prove a similar result for the diagram

$$\begin{array}{ccccccc} P^{\cdot} & \xrightarrow{\binom{0}{1}{0}} & \text{Cyl}^{\cdot}(u) & \xrightarrow{\left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & \text{Con}^{\cdot}(u) & \xrightarrow{(1,0)} & P^{\cdot}[1] \\ \downarrow{1} & & \downarrow{(0,u,1)} & & \downarrow{(0,v)} & & \downarrow{1} \\ P^{\cdot} & \xrightarrow{u} & Q^{\cdot} & \xrightarrow{v} & R^{\cdot} & \xrightarrow{w} & P^{\cdot}[1] \end{array}$$

___
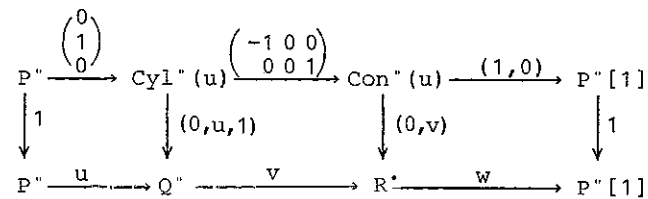
[*] It is the relation $vu = 0$ which ensures that $(0,v): \text{Con}^{\cdot}(u) \to R^{\cdot}$ is a morphism of complexes.

The first square is obviously commutative. The second square is commutative since $vu = 0$. From the functoriality of the homotopy invariant, 4.11 follows that the third square is homotopy commutative.

It remains to prove that $(0,v)$ is a homotopy equivalence. It suffices to prove that $(0,v)$ induces an isomorphism after transformation by any of the functors of the form $[X^{\cdot},-]$, where $X^{\cdot}$ is a complex. We can apply the five lemma 1.7 to the diagram

$$\begin{array}{ccccccccc} [X^{\cdot},P^{\cdot}] & \to & [X^{\cdot},Q^{\cdot}] & \to & [X^{\cdot},\text{Con}(u)] & \to & [X^{\cdot},P^{\cdot}[1]] & \to & [X^{\cdot},Q^{\cdot}[1]] \\ \downarrow{1} & & \downarrow{1} & & \downarrow{[1,(0,v)]} & & \downarrow{1} & & \downarrow{1} \\ [X^{\cdot},P^{\cdot}] & \to & [X^{\cdot},Q^{\cdot}] & \to & [X^{\cdot},R^{\cdot}] & \to & [X^{\cdot},P^{\cdot}] & \to & [X^{\cdot},Q^{\cdot}[1]] \end{array}$$
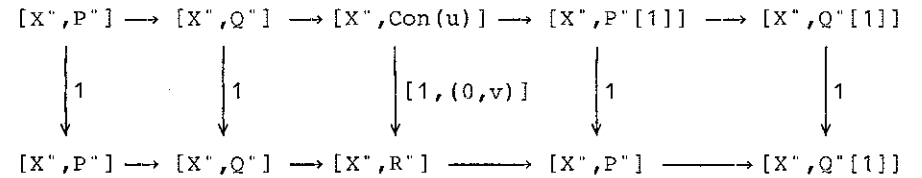
to conclude the proof.

Q.E.D.

Turning triangles 4.16. Given a triangle

$$\begin{array}{ccc} & C^{\cdot} & \\ h \nwarrow & & \nearrow g \\ A^{\cdot} & \xrightarrow{\quad f \quad} & B^{\cdot} \end{array}$$
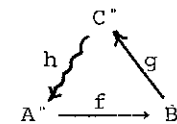
then the following diagram is a triangle

$$
\begin{array}{ccc}
& A^{\cdot\cdot}[1] & \\
{}^{-f[1]}\nearrow\kern-1.5em\swarrow & & \nwarrow{}^{h} \\
B^{\cdot\cdot} & \xrightarrow{\quad g \quad} & C^{\cdot\cdot}
\end{array}
$$
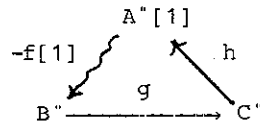
Proof. According to Lemma 4.15 and 4.14 we can replace the triangle above with a mapping cone triangle. The result of turning the mapping cone triangle 4.14 is

$$
\begin{array}{ccc}
& P^{\cdot\cdot}[1] & \\
{}^{-u[1]}\nearrow\kern-1.5em\swarrow & & \nwarrow{}^{(1,0)} \\
Q^{\cdot\cdot} & \xrightarrow{\quad} & Con^{\cdot\cdot}(u) \\
& \binom{0}{1} &
\end{array}
$$
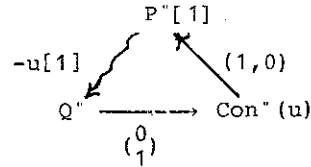
which is a triangle, see 4.13.

Q.E.D.

Corollary 4.17. For any morphism $u: P^{\cdot\cdot} \to Q^{\cdot}$ of complexes there exists a homotopy commutative diagram

$$
\begin{array}{ccc}
P^{\cdot\cdot} & \xrightarrow{\quad u \quad} & Q^{\cdot} \\
\Big\downarrow & & \Big\downarrow \\
C^{\cdot} & \xrightarrow{\quad f \quad} & D^{\cdot\cdot}
\end{array}
$$

where f is a chainwise split epimorphism and the vertical arrows are homotopy equivalences.

Proof. Turn the mapping cone 4.14 twice.

Q.E.D.

Twisting triangles 4.18. Given a triangle

$$
\begin{array}{ccc}
& C^{\cdot} & \\
{}^{h}\nearrow\kern-1.5em\swarrow & & \nwarrow{}^{g} \\
A^{\cdot} & \xrightarrow{\quad f \quad} & B^{\cdot\cdot}
\end{array}
$$
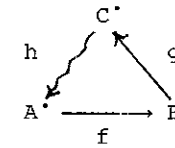
Then for any $n \in \mathbb{Z}$, the diagram

$$
\begin{array}{ccc}
& C^{\cdot}[n] & \\
{}^{(-1)^{n}h[n]}\nearrow\kern-1.5em\swarrow & & \nwarrow{}^{(-1)^{n}g[n]} \\
A^{\cdot\cdot}[n] & \xrightarrow{\quad} & B^{\cdot}[n] \\
& (-1)^{n}f[n] &
\end{array}
$$
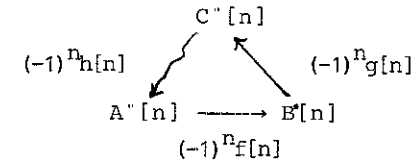
is a triangle.

Proof. Note first, that the second triangle above is isomorphic to

$$
\begin{array}{ccc}
& C^{\cdot}[n] & \\
{}^{(-1)^{n}h[n]}\nearrow\kern-1.5em\swarrow & & \nwarrow{}^{g[n]} \\
A^{\cdot}[n] & \xrightarrow{\quad} & B^{\cdot}[n] \\
& f[n] &
\end{array}
$$
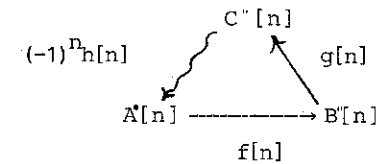
via $(-1)^{n}, 1, (-1)^{n}$. The result now follows rather immediately from formula 4.5.

Q.E.D.

Filling in the third arrow 4.19. Consider the diagram
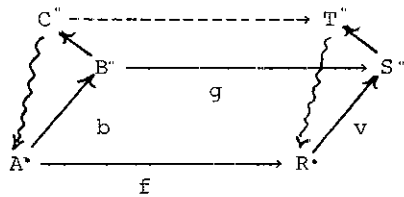
consisting of triangles ABC and RST and arrows f and g making the bottom square homotopy commutative. Then the dotted arrow may be filled in making the remaining two squares homotopy commutative.

Proof. We may assume that the triangles are mapping cones, 4.14. Consider a homotopy, realizing that the bottom square is homotopy commutative

$$gu - vf = k\partial + \partial k$$

Define h: Con"(u) → Con"(v) to be

$$\begin{pmatrix} f & 0 \\ -k & g \end{pmatrix} \quad : A[1] \oplus B \to R[1] \oplus S$$

Notice that

$$\begin{pmatrix} -\partial & 0 \\ -v & \partial \end{pmatrix}\begin{pmatrix} f & 0 \\ -k & g \end{pmatrix} = \begin{pmatrix} f & 0 \\ -k & g \end{pmatrix}\begin{pmatrix} -\partial & 0 \\ -u & \partial \end{pmatrix}$$

$$\begin{pmatrix} f & 0 \\ -k & g \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}g \ ; \quad f(1,0) = (1,0)\begin{pmatrix} f & 0 \\ -k & g \end{pmatrix}$$

Q.E.D.

Signs

Let there be given a triangle
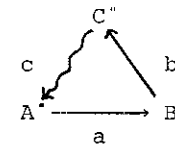
If we replace a by −a and b by −b we still have a triangle as it follows by using an isomorphism of the form (−1,1,−1). More generally we can change sign on any two of the arrows and we will still have a triangle. However, if we replace c by −c we will in general not have a triangle. An example is given on the basis of the following lemma.

Lemma 4.20. Given a triangle
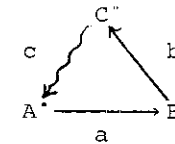
If [A"[1],C"] = 0, then c is the only morphism from C' to A'[1] in the homotopy category that makes a,b,c into a triangle.

Proof. Let x ∈ [C",A"[1]] be such that a,b,x form a triangle. Then according to 4.19 we can find θ: C" → C" such that the following diagram is commutative
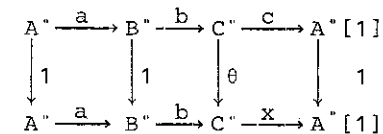
Notice that $(1-\theta) b = 0$ whence we can find $h \in [A^{\cdot \cdot}[1], C^{\cdot \cdot}]$, such that $hc = 1-\theta$. Since $[A^{\cdot \cdot}[1], C^{\cdot \cdot}] = 0$ we must have $\theta = 1$ and from the equation $x\theta = 1c$, we conclude that $x = c$.

$$Q.E.D.$$

Example 4.21. Consider the following triangle of complexes of abelian groups

$$
\begin{array}{ccccccc}
\mathbb{Z} & \xrightarrow{\binom{1}{0}} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(0,1)} & \mathbb{Z} & \xrightarrow{\quad 1 \quad} & \mathbb{Z} \\
\downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 3 & -1 \\ 0 & 3 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle -3} \\
\mathbb{Z} & \xrightarrow{\binom{1}{0}} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\quad} & \mathbb{Z} & \xrightarrow{\quad} & 0 \\
& & & (0,1) & & &
\end{array}
$$

$$A^{\cdot \cdot} \xrightarrow{\ a\ } B^{\cdot \cdot} \xrightarrow{\ b\ } C^{\cdot \cdot} \xrightarrow{\ c\ } A^{\cdot \cdot}[1]$$

Notice that $[A^{\cdot \cdot}[1], B^{\cdot \cdot}] = 0$ and $[B^{\cdot \cdot}, A^{\cdot \cdot}[1]] = \mathbb{Z}/(3)$ generated by c. In particular c is not homotopic to $-c$ and we conclude from Lemma 4.20 that $a, b, -c$ is not a triangle.

The triangle of two composable morphisms

Starting from two composable morphisms of complexes

$$u: X^{\cdot \cdot} \to Y^{\cdot \cdot}, \qquad v: Y^{\cdot \cdot} \to Z^{\cdot \cdot}$$

we can build a new triangle

4.22

$$
\begin{array}{c}
\text{Con}^{\cdot \cdot}(v) \\
\begin{smallmatrix} \binom{0\ 0}{1\ 0} \end{smallmatrix} \ \rotatebox{-30}{\leadsto} \quad \nwarrow \ \begin{smallmatrix} \binom{u[1]\ 0}{0\ \ 1} \end{smallmatrix} \\
\text{Con}^{\cdot \cdot}(u) \xrightarrow[\ \binom{1\ 0}{0\ v}\ ]{} \text{Con}^{\cdot \cdot}(vu)
\end{array}
$$

We shall derive this triangle from the mapping cone triangle

$$
\begin{array}{c}
\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right) \quad \text{Con}^{\cdot \cdot}(V) \quad \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\
\rotatebox{-30}{\leadsto} \qquad \nwarrow \\
\left(\begin{smallmatrix} -\partial & 0 \\ -u & \partial \end{smallmatrix}\right); \ \text{Con}^{\cdot \cdot}(u) \xrightarrow{\hspace{2cm}} \text{Con}^{\cdot \cdot}(vu); \ \left(\begin{smallmatrix} -\partial & 0 \\ -vu & \partial \end{smallmatrix}\right) \\
V = \left(\begin{smallmatrix} 1 & 0 \\ 0 & v \end{smallmatrix}\right)
\end{array}
$$

and the following identfication $\text{Con}^{\cdot \cdot}(v)$ and $\text{Con}^{\cdot \cdot}(V)$

$$
\begin{array}{c}
\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\
\left(\begin{smallmatrix} -\partial & 0 \\ -v & \partial \end{smallmatrix}\right); \quad \text{Con}^{\cdot \cdot}(v) \rightleftarrows \text{Con}^{\cdot \cdot}(V) \quad ; \quad \left(\begin{smallmatrix} \partial & 0 & 0 & 0 \\ u & -\partial & 0 & 0 \\ -1 & 0 & -\partial & 0 \\ 0 & -v & -vu & \partial \end{smallmatrix}\right) \\
\left(\begin{smallmatrix} 0 & 1 & u & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)
\end{array}
$$

using the explicit homotopy

$$
\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) =
$$

$$
\left(\begin{smallmatrix} \partial & 0 & 0 & 0 \\ u & -\partial & 0 & 0 \\ -1 & 0 & -\partial & 0 \\ 0 & -v & -vu & \partial \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) \dotplus \left(\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} \partial & 0 & 0 & 0 \\ u & -\partial & 0 & 0 \\ -1 & 0 & -\partial & 0 \\ 0 & -v & -vu & \partial \end{smallmatrix}\right)
$$

I.5 Abelian categories

A category which is _additive_ and _exact_ is called an abelian category. Throughout this section we shall work in a fixed abelian category. A close analysis of the cocartesian diagrams will be of great use when it comes to construction of injective resolutions.

Definition 5.1. A commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\
C & \xrightarrow{\ g\ } & D
\end{array}
$$

is called _cocartesian_ if whenever given a pair of morphisms

$p: B \to X$ and $q: C \to X$ with $pf = qa$ there exists a unique

morphism $r: D \to X$ with $rb = p$ and $rg = q$.

Given morphisms a and f as above we can always find b and g making the diagram cocartesian: For D take a co-kernel for the morphism

$$(a,-f): A \longrightarrow C \oplus B$$

The dual notion is _cartesian diagram_.

Proposition 5.2. Consider a _cocartesian_ diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\
C & \xrightarrow{\ g\ } & D
\end{array}
$$

1) Cok $a \cong$ Cok b,   Cok $f \cong$ Cok g.

2) The morphisms Ker $a \to$ Ker b, Ker $f \to$ Ker g are epimorphisms. - These two epimorphisms have isomorphic kernels.

3) Our cocartesian diagram is in addition _cartesian_ if and only if

Ker $a \to$ Ker b and Ker $f \to$ Ker g

are isomorphisms.

Proof. 1) Follows by direct verification. 2) We can express the assumption that the diagram is cocartesian, by saying that the sequence

$$A \xrightarrow{\ \binom{a}{-f}\ } C \oplus B \xrightarrow{\ (g,b)\ } D \longrightarrow 0$$

is exact. Let $\widetilde{A}$ denote the coimage for $\binom{a}{-f}$. There results an exact sequence

$$0 \longrightarrow \widetilde{A} \xrightarrow[\binom{\widetilde{a}}{-\widetilde{f}}]{} C \oplus B \xrightarrow[(g,b)]{} D \longrightarrow 0$$

which proves that the diagram

$$
\begin{array}{ccc}
\widetilde{A} & \xrightarrow{\ \widetilde{f}\ } & B \\
\downarrow{\scriptstyle \widetilde{a}} & & \downarrow{\scriptstyle b} \\
C & \xrightarrow{\ g\ } & D
\end{array}
$$

is cartesian. By the statement dual to 1) we find isomorphisms

Ker $\widetilde{a} \cong$ Ker b, Ker $\widetilde{f} \cong$ Ker g.

It is easy to verify that the morphism

Ker $a \to$ Ker $\widetilde{a}$, Ker $f \to$ Ker $\widetilde{f}$

are epimorphisms, 1.5. The last statement of 2) follows from 1.3. - 3) If the diagram is cartesian, it follows by a statement dual to 1), that the kernels are isomorphic. - To prove the converse, one checks that the morphism $\begin{pmatrix} a \\ -f \end{pmatrix}: A \to C \oplus B$ is a monomorphism.

Q.E.D.

Homology and homotopy 5.3. Let $P^{\cdot}$ and $Q^{\cdot}$ be chain complexes. Homotopic chain map $f, g: P^{\cdot} \to Q^{\cdot}$ will induce the same map on homology, i.e.

$$H^n(f) = H^n(g) \quad ; \quad n \in \mathbb{N}.$$

Proof. Let $s = (s^n)_{n \in \mathbb{Z}}$ be a homotopy from f to g, i.e.

$$g^n - f^n = s^{n+1}\partial^n + \partial^{n-1}s^n ; \quad n \in \mathbb{Z}$$

Notice that the following diagram is commutative

$$
\begin{array}{ccc}
 & & Z^n(P^{\cdot}) \\
 & \overset{\sigma^n}{\nearrow} & \downarrow Z^n(g-f) \\
Q^{n-1} & \xrightarrow{d^{n-1}} & Z^n(Q^{\cdot})
\end{array}
$$

where σ^n is the restriction of s^n to $Z^n(P^{\cdot})$ and d^{n-1} is the result of factoring ∂^{n-1} through the canonical monomorphism $Z^n(Q^{\cdot}) \to Q^n$. Notice that

$$\mathrm{Cok}(d^{n-1}) = H^n(Q^{\cdot})$$

from which the result follows.

Q.E.D.

The connecting morphism 5.4. Let there be given a chainwise split exact sequence of complexes

$$0 \longrightarrow P^{\cdot} \xrightarrow{f} Q^{\cdot} \xrightarrow{g} R^{\cdot} \longrightarrow 0$$

The homotopy invariant $h: R^{\cdot} \to P^{\cdot}[1]$ from 4.7 will induce a morphism in homology

$$H^n(h): H^n(R^{\cdot}) \to H^{n+1}(P^{\cdot})$$

which is identical with the connecting morphism introduced in 2.3.

Proof. Let us introduce the objects

$$K^n = \mathrm{Ker}(\partial^n g^n) \qquad C^{n+1} = \mathrm{Cok}(\partial^n f^n)$$

According to Lemma 2.8 it must be proved that

$$p \, f \, h \, g \, i = p \, \partial \, i$$

Choose a splitting $s: R \to Q$ of g. We can write $fh = \partial s - s\partial$ and whence

$$p \, f \, h \, g \, i = p(\partial s - s \partial)g \, i = p \partial s g \, i$$

let now $r: Q \to P$ be a retraction of f such that $1 = fr+sg$. Then

$$p\,f\,h\,g\,i = p\partial(1-fr)i = p\,\partial\,i - p\,\partial\,f\,r\,i$$

Notice that $p\,\partial\,f = 0$ by the definition of p.

$$Q.E.D.$$

<u>Corollary 5.5</u>. A triangle of complexes
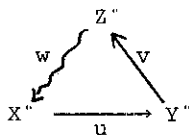
will induce a long <u>exact</u> homology sequence
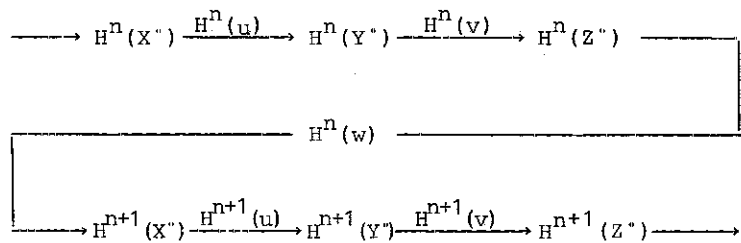
<u>Definition 5.6</u>. A morphism of complexes $f: X" \to Y"$ is called a <u>quasi-isomorphism</u> if $H^n(f): H^n(X") \to H(Y")$ is an isomorphism for all $n \in \mathbb{Z}$.

Notice that $f: X" \to Y"$ is a quasi-isomorphism if and only if $H"(Con"(f)) = 0$ as it follows by considering the mapping cone triangle 4.14
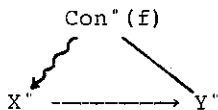
<u>Truncation</u>. Let $K"$ denote a complex and $n \in \mathbb{Z}$. We form a new complex

$$\tau_{\leq n}K" \quad \ldots \to K^{n-2} \to K^{n-1} \to \mathrm{Ker}\,\partial^n \to 0 \ldots$$

Notice the canonical inclusion

5.7
$$\tau_{\leq n}K" \longrightarrow K"$$

induces an isomorphism on homology in levels $\leq n$. Notice also that homotopic morphisms $f,g: K" \to L"$ will induce <u>homotopic</u> morphisms

$$f,g: \tau_{\leq n}K" \longrightarrow \tau_{\leq n}L"$$

Similarly we define the complex

$$\tau_{\geq n}K": \qquad 0 \to \mathrm{Cok}\,\partial^{n-1} \to K^{n+1} \to K^{n+2} \to \ldots$$

which comes equipped with a canonical morphism

5.8
$$K" \longrightarrow \tau_{\geq n}K"$$

which induces an isomorphism on homology in levels $\geq n$. Notice again that formation of $\tau_{\geq n}$ carries over to the homotopy category. - We can establish a relationship between these by noticing that the composite

$$\tau_{\leq n}K" \longrightarrow K" \longrightarrow \tau_{\geq n+1}K"$$

is zero and induces a <u>quasi-isomorphism</u>

5.9
$$K"/\tau_{\leq n}K" \longrightarrow \tau_{\geq n+1}K"$$

I.6 Injective resolutions

In this section we shall work in a fixed abelian category A. We say that an object I of A is <u>injective</u> if for any monomorphism $f: X \to Y$ and any morphism $x: X \to I$, there exists a morphism $y: Y \to I$ with $x = yf$.

$$0 \longrightarrow X \xrightarrow{\quad f \quad} Y$$
$$x \downarrow \swarrow y$$
$$I$$
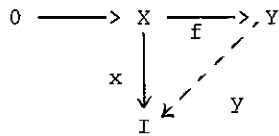
Let I be an injective object of A. Any short exact sequence in A

$$0 \to X \to Y \to Z \to 0$$

give rise to an exact sequence

$$0 \to \mathrm{Hom}(Z,I) \to \mathrm{Hom}(Y,I) \to \mathrm{Hom}(X,I) \to 0$$

as it follows from the definition.

In the rest of this section we shall assume that A has <u>enough injectives</u>, i.e. that any object of A admits a monomorphism into an injective object. - We shall mostly be concerned with <u>bounded below complexes</u>, i.e. complexes $K^{\cdot}$ with $K^n = 0$ for $n << 0$.

<u>Theorem 6.1</u>. Any bounded below complex $X^{\cdot}$ admits a quasi-isomorphism $f: X^{\cdot} \to I^{\cdot}$ into a bounded below complex of injective objects (<u>an injective resolution of $X^{\cdot}$</u>)

<u>Proof</u>. We shall proceed by increasing induction on $n \in \mathbb{Z}$. Suppose f has already been constructed up to level $n \in \mathbb{Z}$. That is we have the solid diagram

$$\longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow {}^{\prime}Z^n(X^{\cdot}) \xrightarrow{\bar{\partial}^n} Z^{n+1}(X^{\cdot}) \longrightarrow X^{n+1}$$
$$f^{n-1} \downarrow \qquad f^n \downarrow \qquad \downarrow \qquad \bigcirc \qquad \downarrow \qquad \downarrow$$
$$\longrightarrow I^{n-1} \xrightarrow{\partial^{n-1}} I^n \longrightarrow \mathrm{Cok}(\partial^{n-1}) \dashrightarrow Y^{n+1} \dashrightarrow I^{n+1}$$
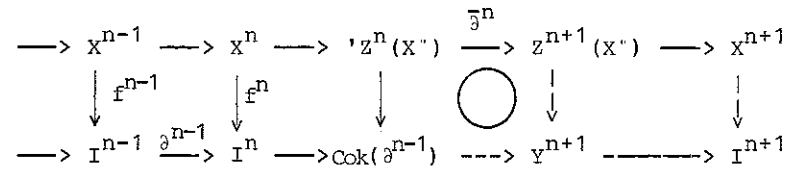
We shall assume this done in such a way that

$$0 \to H^n(X^{\cdot}) \to \mathrm{Cok}(\partial^{n-1})$$

is exact, here we identify $H^n(X^{\cdot})$ with the kernel of $\bar{\partial}^n$, 2.6. Now fill in the two dotted arrows to make the circled square <u>cocartesian</u>. It follows from 5.2 that the circled square gives rise to an exact sequence

$$0 \to H^n(X^{\cdot}) \to \mathrm{Cok}(\partial^{n-1}) \to Y^{n+1} \to H^{n+1}(X^{\cdot}) \to 0$$

Next, choose a monomorphism $Y^{n+1} \to I^{n+1}$ with I^{n+1} injective and fill in the last dotted arrow at random. Notice that

$$0 \to \mathrm{Cok}(I^n \to Y^{n+1}) \to \mathrm{Cok}(I^n \to I^{n+1})$$

is exact, which concludes the inductive step.

Q.E.D.

<u>Theorem 6.2</u>. Let $I^{\cdot}$ be a bounded below complex of injective objects. Any quasi-isomorphism $c: X^{\cdot} \to Y^{\cdot}$ induces an isomorphism

$$[c,1]: [Y^{\cdot},I^{\cdot}] \xrightarrow{\sim} [X^{\cdot},I^{\cdot}]$$

<u>Proof</u>. Consider the mapping cone triangle 4.14

$$
\begin{array}{ccc}
 & C^{\cdot} & \\
 \swarrow & & \nwarrow \\
X^{\cdot} & \longrightarrow & Y^{\cdot}
\end{array}
$$

It follows from the long homology sequence I.5.5

$$\to H^n(X^{\cdot}) \to H^n(Y^{\cdot}) \to H^n(C^{\cdot}) \to H^{n+1}(X^{\cdot}) \to H^{n+1}(Y^{\cdot})$$

that $H^n(C^{\cdot}) = 0$ for all $n \in \mathbb{Z}$. Next consider the homotopy sequence induced by $[-,I^{\cdot}]$

$$\to [C^{\cdot},I^{\cdot}] \to [Y^{\cdot},I^{\cdot}] \to [X^{\cdot},I^{\cdot}] \to [C^{\cdot}[-1],I^{\cdot}] \to$$

to see that it suffices to prove that

6.3 $[C^{\cdot},I^{\cdot}] = 0$ <u>whenever</u> $C^{\cdot}$ <u>is a complex with</u>
          $H^{\cdot}(C^{\cdot}) = 0$, <u>and</u> $I^{\cdot}$ <u>is a bounded</u>
          <u>below complex of injectives</u>.

We must prove that any morphism of complexes $f: C^{\cdot} \to I^{\cdot}$ is homotopic to zero. We are going to construct a homotopy by induction on $n \in \mathbb{Z}$. Suppose we have already constructed $s^p: C^p \to I^{p-1}$ for all $p \le n$ with

$$s^p \partial^{p-1} + \partial^{p-2} s^{p-1} = f^{p-1} \qquad ; \quad p \le n$$

We shall proceed to construct $s^{n+1}: C^{n+1} \to I^n$ such that the relation above holds with $p = n+1$.
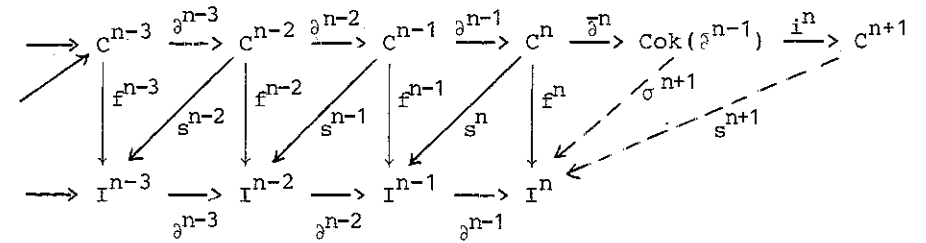
The following calculation shows that $(f^n - \partial^{n-1} s^n)\partial^{n-1} = 0$

$$(f^n - \partial^{n-1} s^n)\partial^{n-1} = \partial^{n-1} f^{n-1} - \partial^{n-1} s^n \partial^{n-1} =$$

$$= \partial^{n-1} s^n \partial^{n-1} + \partial^{n-1} \partial^{n-2} s^{n-1} - \partial^{n-1} s^n \partial^{n-1} = 0$$

Let $\partial^n: C^n \to C^{n-1}$ be factorized

$$C^n \xrightarrow{\bar{\partial}^n} \mathrm{Cok}(\partial^{n-1}) \xrightarrow{i^n} C^{n+1}$$

We can find $\sigma^{n+1}: \mathrm{Cok}(\partial^{n-1}) \longrightarrow I^n$ such that

$$f^n - \partial^{n-1} s^n = \sigma^{n+1} \bar{\partial}^n$$

Note, that $i^n: \mathrm{Cok}(\partial^{n-1}) \to C^{n+1}$ is a monomorphism. Any extension s^{n+1} of σ^{n+1} to C^{n+1} will do.

$$Q.E.D.$$

We shall now derive some consequences of 6.1 and 6.2 in terms of the following two categories

6.4

     $K^+(A)$ the homotopy category of bounded
                  below complexes in A,

     $D^+(A)$ the homotopy category of bounded
                  below complexes of injectives in A

Let there for each X in $K^+(A)$ be <u>chosen</u> a quasi-isomorphism

6.5
$$c: X" \longrightarrow \rho X"$$

where $\rho X"$ is an object of $D^+(A)$. For a morphism $f: X" \to Y^\bullet$ in $K^+(A)$, there exists one and only one morphism $\rho f: \rho X" \to \rho Y"$ making the following diagram commutative

$$
\begin{array}{ccc}
X" & \xrightarrow{c} & \rho X" \\
\downarrow f & & \downarrow \rho f \\
Y" & \xrightarrow{c} & \rho Y"
\end{array}
$$

It follows easily from 6.2, that this construction yields a canonical isomorphism

6.6
$$[X", I"] = [\rho X", I"]$$

where X varies through $K^+(A)$ and $I"$ varies through $D^+(A)$. Otherwise expressed the functor

6.7
$$\rho: K^+(A) \to D^+(A)$$

is a <u>left adjoint</u> to the inclusion $i: D^+(A) \to K^+(A)$. In this interpretation the resolution morphism c from 6.5 plays the role of <u>adjunction</u> morphism.

<u>Theorem 6.8</u>. The additive functor

$$\rho: K^+(A) \to D^+(A)$$

transforms triangles into triangles.

<u>Proof</u>. Let us start with a triangle in $K^+(A)$

$$X" \xrightarrow{f} Y" \xrightarrow{g} Z" \xrightarrow{h} X"[1]$$

We can form the mapping cone triangle in $D^+(A)$ based on $\rho f: \rho X" \to \rho Y"$
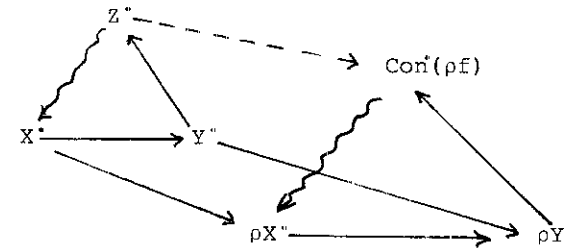
and fill in the dotted arrow in $K^+(A)$, making the diagram commutative. Writing out the long homology ladder 2.7 we conclude from the five Lemma 1.7, that the dotted arrow is a quasi-isomorphism. It is now easy to conclude the proof by means of 6.2.

Q.E.D.

<u>Construction of triangles from short exact sequences of complexes</u>.

Let there be given a short exact sequence of bounded below complexes in A

$$0 \longrightarrow P^\bullet \xrightarrow{f} Q" \xrightarrow{g} R" \longrightarrow 0$$

By an <u>injective resolution</u> of this sequence we understand a <u>commutative</u>, not merely homotopy commutative diagram

6.9
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P" & \xrightarrow{f} & Q" & \xrightarrow{g} & R" & \longrightarrow & 0 \\
& & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & I" & \xrightarrow{u} & J" & \xrightarrow{v} & K" & \longrightarrow & 0
\end{array}
$$

where the bottom sequence is exact and consists of bounded below complexes of injectives, and the vertical arrows are quasi-isomorphisms.

Proposition 6.10. Any short exact sequence of bounded below complexes has an injective resolution.

Proof. Let us first choose quasi-isomorphisms $b: Q^{\cdot} \to J^{\cdot}$ and $c: R^{\cdot} \to K^{\cdot}$ where $J^{\cdot}$ and $K^{\cdot}$ are bounded below complexes of injectives, 6.1. Next, choose $v: J^{\cdot} \to K^{\cdot}$ such that the following diagram is homotopy commutative, 6.2

$$
\begin{array}{ccc}
Q^{\cdot} & \xrightarrow{g} & R^{\cdot} \\
\downarrow{b} & & \downarrow{c} \\
J^{\cdot} & \xrightarrow{v} & K^{\cdot}
\end{array}
$$

By 4.17 we may assume that v is a chainwise split epimorphism. The diagram is homotopy commutative, so choose a homotopy $s: Q \to K[-1]$ from cg to vb

$$cg - vb = s\partial + \partial s$$

Next, choose $t: Q \to J[-1]$ such that $s = vt$. The formula above may be rewritten

$$cg = vb + vt\partial + \partial vt = v(b + t\partial + \partial t)$$

which shows that we can modify b within its homotopy class making the diagram commutative. We can now quite simply put $I^{\cdot} = \text{Ker}(v)$ to obtain a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P^{\cdot} & \xrightarrow{f} & Q^{\cdot} & \xrightarrow{g} & R^{\cdot} & \longrightarrow & 0 \\
& & a\downarrow & & b\downarrow & & c\downarrow & & \\
0 & \longrightarrow & I^{\cdot} & \xrightarrow{u} & J^{\cdot} & \xrightarrow{v} & K^{\cdot} & \longrightarrow & 0
\end{array}
$$

It remains to prove that a is a quasi-isomorphism. To see this write out the long homology sequences and apply the five lemma.

Q.E.D.

A resolution as the one in 6.7 gives rise to a triangle in $D^{+}(A)$

6.11
$$
\begin{array}{ccc}
& K^{\cdot} & \\
w\nearrow & & \searrow v \\
I^{\cdot} \xrightarrow{\ \ u\ \ } & & J^{\cdot}
\end{array}
$$

this triangle is unique up to homotopy.

Proof. Let $X^{\cdot}$ be an arbitrary bounded below complex of injective objects. A diagram like 6.9 will give rise to two exact sequences of complexes of abelian groups and a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & \text{Hom}^{\cdot}(K^{\cdot},X^{\cdot}) & \to & \text{Hom}^{\cdot}(J^{\cdot},X^{\cdot}) & \to & \text{Hom}^{\cdot}(I^{\cdot},X^{\cdot}) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Hom}^{\cdot}(R^{\cdot},X^{\cdot}) & \to & \text{Hom}^{\cdot}(Q^{\cdot},X^{\cdot}) & \to & \text{Hom}^{\cdot}(P^{\cdot},X^{\cdot}) & \to & 0
\end{array}
$$

By taking homology we obtain two sets of connecting morphism, making the following diagram commute

$$
\begin{array}{ccc}
[I^{\cdot},X^{\cdot}] & \longrightarrow & [K^{\cdot},X^{\cdot}[1]] \\
\downarrow[c,1] & & \downarrow[a,1] \\
[P^{\cdot},X^{\cdot}] & \longrightarrow & [R^{\cdot},X^{\cdot}[1]]
\end{array}
$$

According to formula 4.9 the bottom homomorphism is $-[w,1]$, where w is the third side in the triangle. This description fixes w and thereby shows that the triangle is unique.

Q.E.D.

In terms of the resolution functor ρ we have assigned to the short exact sequence of complexes a definite triangle

6.12

$$
\begin{array}{ccc}
 & \rho R^{\cdot} & \\
\theta(f,g) \nwarrow\!\!\!\rightsquigarrow & & \nearrow \rho g \\
\rho P^{\cdot} & \longrightarrow & \rho Q^{\cdot}
\end{array}
$$

Given a commutative diagram of complexes in A

$$
\begin{array}{ccccccccc}
0 & \to & P^{\cdot} & \overset{f}{\to} & Q^{\cdot} & \overset{g}{\to} & R^{\cdot} & \to & 0 \\
 & & \downarrow u & & \downarrow v & & \downarrow w & & \\
0 & \to & U^{\cdot} & \overset{a}{\to} & V^{\cdot} & \overset{b}{\to} & W^{\cdot} & \to & 0
\end{array}
$$

with exact rows. Then the following diagram is commutative

6.13

$$
\begin{array}{ccc}
\rho R^{\cdot} & \overset{\theta(f,g)}{\longrightarrow} & \rho P^{\cdot}[1] \\
\rho w \downarrow & & \downarrow \rho u[1] \\
\rho W^{\cdot} & \underset{\theta(a,b)}{\longrightarrow} & \rho U^{\cdot}[1]
\end{array}
$$

as it follows by a consideration similar to the one made in the proof of 6.11.

Truncation in the derived category

For an integer n consider the two endofunctors $\tau_{\leq n}$ and $\tau_{\geq n+1}$ of $K^{+}(A)$. These give rise to two endofunctors $\rho\tau_{\leq n}$ and $\rho\tau_{\geq n+1}$ of $D^{+}(A)$ which we shall still denote by $\tau_{\leq n}$ and $\tau_{\geq n+1}$ when no confusion is possible. For $X^{\cdot}$ in $D^{+}(A)$ we

shall construct a triangle in $D^{+}(A)$

6.14

$$
\begin{array}{ccc}
 & \tau_{\geq n+1}X^{\cdot} & \\
\rightsquigarrow\!\!\!\swarrow & & \nwarrow \\
\tau_{\leq n}X^{\cdot} & \longrightarrow & X^{\cdot}
\end{array}
$$

which depends functorially on $X^{\cdot}$. To do so we start with the short exact sequence of complexes in A

$$0 \to \tau_{\leq n}X^{\cdot} \to X^{\cdot} \to X^{\cdot}/\tau_{\leq n}X^{\cdot} \to 0$$

this we can combine with the quasi-isomorphism 5.9

$$X^{\cdot}/\tau_{\leq n}X^{\cdot} \longrightarrow \tau_{\geq n+1}X^{\cdot}$$

and apply the previous procedure.

Let us analyse this a bit closer: Let $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n+1}$ denote the full subcategories of $\mathcal{D} = D^{+}(A)$ whose objects are complexes $X^{\cdot}$ with

6.15
$$
\begin{array}{lll}
\mathcal{D}^{\leq n}: & H^{i}(X^{\cdot}) = 0 ; & i > n \\
\mathcal{D}^{\geq n+1}: & H^{i}(X^{\cdot}) = 0 ; & i < n+1
\end{array}
$$

These two categories satisfies

6.16 $\qquad [I^{\cdot}, J^{\cdot}] = 0 \qquad$ for $I^{\cdot} \in \mathcal{D}^{\leq n}$ and $J^{\cdot} \in \mathcal{D}^{\geq n+1}$

Proof. The complex $J^{\cdot}$ is a bounded below complex of injectives with $H^{i}(J^{\cdot}) = 0$ for $i \leq n$. It follows easily, that $\text{Cok}(\partial^{i})$ is an injective object for $i \leq n$, whence the complex $\tau_{\geq n+1}J^{\cdot}$ is a complex of injectives, homotopy equivalent to $J^{\cdot}$. The assumption on $I^{\cdot}$ implies that $\tau_{\leq n}I^{\cdot} \to I^{\cdot}$

is a quasi-isomorphism. Whence by 6.2

$$[I^{\cdot},J^{\cdot}] = [\tau_{\leq n}I^{\cdot}, \tau_{\geq n+1}J^{\cdot}]$$

the last homotopy group is zero for trivial reasons.

Q.E.D.

With $J^{\cdot} \in \mathcal{D}^{\geq n+1}$ we deduce from 6.14 an exact sequence

$$[\tau_{\leq n}X^{\cdot}[1],J^{\cdot}] \to [\tau_{\geq n+1}X^{\cdot},J^{\cdot}] \to [X^{\cdot},J^{\cdot}] \to [\tau_{\leq n}X^{\cdot},J^{\cdot}] \to$$

Let us notice that quite generally, $n \in \mathbb{Z}$

6.17
$$\mathcal{D}^{\leq n}[+1] = \mathcal{D}^{\leq n-1} \subseteq \mathcal{D}^{\leq n}$$

$$\mathcal{D}^{\geq n+1}[-1] = \mathcal{D}^{\geq n+2} \subseteq \mathcal{D}^{\geq n+1}$$

Combining this with the sequence above and 6.16 we get

6.18
$$[\tau_{\leq n+1}X^{\cdot},J^{\cdot}] = [X^{\cdot},J^{\cdot}] \quad ; \quad J \in \mathcal{D}^{\geq n+1}$$

which shows that $\tau_{\leq n+1}$ is <u>left adjoint</u> to the inclusion of $\mathcal{D}^{\geq n+1}$ in $\mathcal{D}$. In a similar way we deduce from 6.14, 6.16, 6.17 that

6.19
$$[I^{\cdot}, \tau_{\leq n}X^{\cdot}] = [I^{\cdot},X^{\cdot}] \quad ; \quad I^{\cdot} \in \mathcal{D}^{\leq n}$$

which shows that $\tau_{\leq n}$ is a <u>right adjoint</u> to the inclusion of $\mathcal{D}^{\leq n}$ into $\mathcal{D}$, . Let us finally remark that the <u>curly arrow</u> in 6.14 <u>is unique</u>, see 4.20.

<u>Remark 6.20</u>. Given $K^{\cdot}$ in $\mathcal{D}$, such that

$$[K^{\cdot},J^{\cdot}] = 0 \quad \text{for all} \quad J^{\cdot} \quad \text{in} \quad \mathcal{D}^{\geq n+1}$$

then $K^{\cdot}$ is in $\mathcal{D}^{\leq n}$ as it follows from 6.18 and 6.16.

I.7   Right derived functors

We shall first develop the classical part of the theory of derived functors.

Let $T: A \to \mathcal{B}$ be a covariant additive functor between abelian categories A and $\mathcal{B}$ of which A is assumed to have enough injectives.

For an object X of A choose an <u>injective resolution of</u> X, i.e. a long exact sequence with $I^0, I^1, \ldots$ injective

$$0 \to X \xrightarrow{\varepsilon} I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} I^2 \ldots$$

Consider X as a complex concentrated in degree zero, and let $I^{\cdot}$ denote the complex

$$0 \to I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} I^2 \ldots$$

We can then interprete the resolution above as a <u>quasi-isomorphism</u> of complexes $\varepsilon: X \to I^{\cdot}$. Given a second resolution $X \to J^{\cdot}$, according to 6.2 there exists one and up to homotopy only one $\theta: I^{\cdot} \to J^{\cdot}$ making the following diagram homotopy commutative [*]
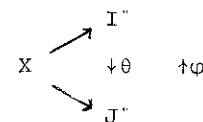
In the same way we construct $\varphi: J^{\cdot} \to I^{\cdot}$. It follows from 6.2, that $\theta\varphi \sim 1$ and $\varphi\theta \sim 1$. In view of this uniqueness it makes sense to define for $n \in \mathbb{Z}$

7.1
$$R^n T(X) = H^n T(I^{\cdot})$$

[*] Since the complexes $I^{\cdot}$ and $J^{\cdot}$ are positive and the complex X

where $X \to I^{..}$ is an injective resolution. - Given a morphism
$f: X \to Y$ in A we shall construct a morphism

$$R^n T(f): R^n T(X) \to R^n T(Y)$$

Let $X \to I^{..}$ and $Y \to J^{..}$ be injective resolutions. According to 6.2
we can choose $\psi: I^{..} \to J^{..}$, unique up to homotopy, making the follow-
ling diagram homotopy[*)] commutative:

$$
\begin{array}{ccc}
X^{.} & \longrightarrow & I^{..} \\
\downarrow f & & \downarrow \psi \\
Y^{.} & \longrightarrow & J^{..}
\end{array}
$$

In this way we have for each $n \in \mathbb{Z}$ constructed an additive
covariant functor

$$R^n T: A \to B$$

Notice, that $R^n T = 0$ for $n < 0$

Connecting morphisms. Let there be given a short exact
sequence in $\underline{A}$

$$0 \to X \to Y \to Z \to 0$$

Then according to 6.10 we can find a commutative diagram with
exact rows

$$
\begin{array}{ccccccc}
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I^{..} & \to & J^{..} & \to & K^{..} & \to & 0
\end{array}
$$

where the vertical arrows are injective resolutions. Remark that
the bottom sequence is chainwise split, and derive a short exact
sequence of complexes in B

$$0 \to T(I^{..}) \to T(J^{..}) \to T(K^{..}) \to 0$$

By taking homology we obtain a long exact sequence

7.2 $\quad \to R^n T(X) \to R^n T(Y) \to R^n T(Z) \xrightarrow{\partial^n} R^{n+1} T(X) \to R^{n+1} T(Y)$

Given a commutative diagram in A with exact rows

$$
\begin{array}{ccccccc}
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & U & \to & V & \to & W & \to & 0
\end{array}
$$

then there results a commutative ladder

7.3 $\quad
\begin{array}{ccccccccc}
\to & R^n T(X) & \to & R^n T(Y) & \to & R^n T(Z) & \xrightarrow{\partial^n} & R^{n+1} T(X) & \to & R^{n+1}(Y) & \to \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to & R^n T(U) & \to & R^n T(V) & \to & R^n T(W) & \xrightarrow{\partial^n} & R^{n+1} T(U) & \to & R^{n+1} T(W)
\end{array}
$$

this follows from 6.13.

Left exact functors. Let us now assume that our additive
functor $T: A \to B$ is left exact, i.e. preserve kernels. This means
that we may identify T and $R^0 T$.

Definition 7.4. Let $T: A \to B$ be a left exact functor. An
object X of A is called T-acyclic, if

$$R^i T(X) = 0 \quad \text{for all} \quad i \geq 1$$

Acyclicity theorem 7.5. Let $T: A \to B$ be a left exact functor. A quasi-isomorphism $f: X^{\cdot} \to Y^{\cdot}$ between bounded below complexes of T-acyclic objects is transformed into a quasi-isomorphism $T(f): T(X^{\cdot}) \to T(Y^{\cdot})$.

Proof. Consider the mapping cone 4.14 to see that it suffices to treat the case where $Y^{\cdot} = 0$. Let us prove that $\mathrm{Ker}(\partial^m)$, the kernel of the m'th differential in $X^{\cdot}$, is T-acyclic. By a translation argument it suffices to prove that $\mathrm{Ker}(\partial^0)$ is T-acyclic. Let us prove that

$$R^i T(\mathrm{Ker}\,\partial^0) = R^{i+p}(\mathrm{Ker}(\partial^{-p}) \quad ; \quad i \geq 1, \quad p \in \mathbb{N}$$

This is done by induction on $p \in \mathbb{N}$ using the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\partial^{-p-1}) \to X^{-p-1} \to \mathrm{Ker}(\partial^{-p}) \longrightarrow 0$$

The formula with p large shows that $\mathrm{Ker}(\partial^0)$ is T-acyclic. Let us now prove the following statement for all $n \in \mathbb{Z}$ "any complex $X^{\cdot}$ of T-acyclic objects with $H^{\cdot}(X^{\cdot}) = 0$ and $X^i = 0$ for all $i < n$ has $H^0(X^{\cdot}) = 0$". This is done by decreasing induction on $n \in \mathbb{Z}$. To see that "n $\Rightarrow$ n-1" consider the short exact sequence

$$0 \to \tau_{\leq n} X^{\cdot} \to X^{\cdot} \to \tau_{\geq n} X^{\cdot} \to 0$$

Transform this by T to get an exact sequence

$$0 \to T(\tau_{\leq n} X^{\cdot}) \to T(X^{\cdot}) \to T(\tau_{\geq n} X^{\cdot}) \to 0$$

The inductive step can now be deduced from the sequence

$$H^0(T(\tau_{\leq n} X^{\cdot})) \to H^0(T(X^{\cdot})) \to H^0(T(\tau_{\geq n} X^{\cdot}))$$

We can now conclude the proof by a translation argument.

$$Q.E.D.$$

Theorem on unbounded complexes 7.6. Let $T: A \to B$ be a left exact functor with $R^n T = 0$ for some $n \in \mathbb{N}$. Then any complex $E^{\cdot}$ in A admits quasi-isomorphism $E^{\cdot} \to X^{\cdot}$ into a complex of T-acyclic objects.- Moreover, any quasi-isomorphism $f: X^{\cdot} \to Y^{\cdot}$ between complexes of T-acyclic objects is transformed into a quasi-isomorphism $T(f): T(X^{\cdot}) \to T(Y^{\cdot})$.

Proof. Choose a short exact sequence of complexes

$$0 \longrightarrow E^{\cdot} \xrightarrow{v} F^{\cdot} \xrightarrow{u} G^{\cdot} \longrightarrow 0$$

where $F^{\cdot}$ is a complex of T-acyclic objects. This can for example be done by choosing for each $n \in \mathbb{N}$ a monomorphism $i^n: E^n \to I^n$ into an injective object. Put $F^n = I^n \oplus I^{n+1}$, $\partial^n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $v^n = \begin{pmatrix} i^n \\ i^{n+1}\partial^n \end{pmatrix}$ and let $(u,G^{\cdot})$ be a cokernel for v. The morphism

$$\begin{pmatrix} v \\ 0 \end{pmatrix}: E^{\cdot} \longrightarrow \mathrm{Con}^{\cdot}(u)[-1]$$

is a quasi-isomorphism as it follows from the proof of 4.15, where the dual statement is treated, see also XI.3.5. Put $X^{(1)} = \mathrm{Con}^{\cdot}(u)[-1]$ and notice that this is a complex of objects with vanishing $R^{n-1}T$. Iterate this construction to obtain a sequence of quasi-isomorphisms

$$E^{\cdot} \to X^{(1)} \to X^{(2)} \to \ldots \to X^{(n)}$$

The composite of these is a resolution of $E^{\cdot}$ by T-acyclic objects.

To prove the second part of the theorem we may assume that $Y^{\cdot} = 0$. It follows from the proof of 7.5, that any truncation $\tau_{\geq p} X^{\cdot}$ consists of T-acyclic objects. Notice that

$$H^p(T(X^{\cdot})) = H^p(T(\tau_{\geq p-2} X^{\cdot}))$$

and deduce the result from 7.5.

<div align="right">Q.E.D.</div>

Corollary 7.7. Suppose every object of the abelian category A admits an injective resolution of length n, where n is a fixed integer. Then any complex $E^{\cdot}$ admits a quasi-isomorphism $E^{\cdot} \to I^{\cdot}$ into a complex of injective objects.

Proof. We shall use the notation of the proof above and the theory of Ext from I.8. - For any objects R of A of the functor $Ext^{n+1}(R,-)$ is zero by assumption. It follows that all functors of the form $Ext^n(R,-)$ vanish on the chain objects of $X^{(1)}$. More generally, all functors of the form $Ext^{n+1-p}(R,-)$ vanish on the chain objects of $X^{(p)}$. Take $p = n$ to see that $X^{(n)}$ is a complex of injectives.

<div align="right">Q.E.D.</div>

Corollary 7.8. Let $T = A \to B$ and $U: B \to C$ be left exact additive functors between abelian categories with enough injectives. If T is exact and transforms injectives into U-acyclics

$$R^n(U \circ T) = (R^n U) \circ T$$

Proof. Given an object X of A with injective resolution $X \to I^{\cdot}$. Choose an injective resolution $T(X) \to J^{\cdot}$ in B. According to 6.2, we can find $f: T(I^{\cdot}) \to J^{\cdot}$ making the following deagram commutative

$$
T(X) \begin{array}{c} \nearrow T(I^{\cdot}) \\ \ \ \ \ \downarrow f \\ \searrow J^{\cdot} \end{array}
$$

According to 7.5, $U(f)$ is a quasi-isomorphism.

<div align="right">Q.E.D.</div>

We shall now introduce the derived functor in the sense of Verdier. - We can first extend our functor T in the obvious way to a functor

$$T^+: K^+(A) \to K^+(B)$$

We define the derived functor $\qquad RT: D^+(A) \to D^+(B)$

by $RT = \rho T^+ i$ as displayed in the diagram, i denote the inclusion.

7.9
$$
\begin{array}{ccc}
D^+(A) & \xrightarrow{i} & K^+(A) \\
\downarrow RT & & \downarrow T^+ \\
D^+(B) & \xleftarrow{\rho} & K^+(B)
\end{array}
$$

Theorem 7.10. The derived functor

$$RT : D^+(A) \to D^+(B)$$

transforms triangles into triangles.

Proof. By the formula $RT = \rho T^+ i$ we need to check that each of these three functors transforms triangles into triangles. This is trivial for i, the reference for T^+ is 3.9 and the reference for ρ is 6.8.

Q.E.D.

Given additive functors $U, V : A \to B$ and a natural transformation $\gamma : U \to V$ of additive functors. We can in an obvious way extend γ to a transformation $\gamma^+ : U^+ \to V^+$ and finally to a natural transformation, compare 7.9

7.11 $$R\gamma : RU \longrightarrow RV$$

Consider additive functors $T : A \to B$ and $U : B \to C$. Let us establish a natural transformation

7.12 $$R(U \circ T) \to RU \circ RT$$

To do so consider the commutative diagram

$$
\begin{array}{ccccc}
D^+(A) & \xrightarrow{RT} & D^+(B) & \xrightarrow{RU} & D^+(C) \\
\rho \Big\updownarrow i & & \rho \Big\updownarrow i & & \rho \Big\updownarrow i \\
K^+(A) & \xrightarrow{T^+} & K^+(B) & \xrightarrow{U^+} & K^+(C)
\end{array}
$$

Notice that $(U \circ T)^+ = U^+ \circ T^+$ and consequently

$$R(U \circ T) = \rho U^+ T^+ i , \qquad RU \circ RT = \rho U^+ i \rho T^+ i$$

The adjunction morphism $1 \to i\rho$ relative to B will induce the natural transformation 7.12.

The adjunction $1 \to i\rho$ is always a quasi-isomorphism, from which we conclude

7.13 $$R(U \circ T) \xrightarrow{\sim} RU \circ RT \qquad \text{if } U \text{ is exact}$$

Another simple case is the following

7.14 $$R(U \circ T) \xrightarrow{\sim} RU \circ RT \qquad \text{if } T \text{ preserves injectives}$$

The all important case is the folllowing

Theorem 7.15. Let $T : A \to B$ and $U : B \to C$ be additive functors between abelian categories with enough injectives. If T transforms injectives into U-acyclics and U is left exact, then

$$R(U \circ T) \xrightarrow{\sim} RU \circ RT$$

Proof. Consider an object $I^{\cdot}$ of $D^+(A)$ and an injective resolution $c : T(I^{\cdot}) \to J^{\cdot}$. We must prove that

$$U(c) : UT(I^{\cdot}) \to U(J^{\cdot})$$

is a quasi-isomorphism. This is a consequence of 7.5.

Q.E.D.

I.8 Composition products

Let A be an abelian category with enough injective objects. For objects M and N in A, choose an injective resolution $J^{\cdot}$ of N and define for each $q \in \mathbb{Z}$

8.1 $$\text{Ext}^q(M,N) = H^q(\text{Hom}(M,J^{\cdot}))$$

This amounts to consider $\text{Ext}^q(M,-)$ as the q'th derived functor of the Hom-functor

$$\text{Hom}(M,-): A \to Ab$$

If $I^{\cdot}$ is an injective resolution of M we can use Theorem 6.2 to write Ext as a homotopy group in $D^+(A)$

8.2 $$\text{Ext}^q(M,N) = [I^{\cdot},J^{\cdot}[q]]$$

This representation reveals that composition in the derived category gives rise to a pairing

$$\text{Ext}^p(N,P) \times \text{Ext}^q(M,N) \to \text{Ext}^{p+q}(M,P)$$

Let namely $K^{\cdot}$ be an injective resolution of P and let $\circ$ denote composition in the derived category.

$$[J^{\cdot},K^{\cdot}[p]] \times [I^{\cdot},J^{\cdot}[q]] \to [I^{\cdot},K^{\cdot}[p+q]]$$

8.3 $$(\alpha,\beta) \to \alpha[q] \circ \beta$$

The result $\underline{\alpha \cup \beta}$ of pairing α with β is called the composition product. From the categorical origin of the product follows immediately that it is associative and bilinear, and that the pairing for $p = q = 0$ is composition in the category A.

We shall now relate the composition product to the connecting morphism arizing from a short exact sequence in A

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

Consider a resolution of this sequence by a triangle 6.11 in $D^+(A)$. The element

8.4 $$\theta \in [I_3^{\cdot},I_1^{\cdot}[1]] = \text{Ext}^1(M_3,M_1)$$

is called the $\underline{\text{characteristic class}}$ of the short exact sequence above. - Let $J^{\cdot}$ be an injective resolution of N. The short exact sequence of complexes

$$0 \to \text{Hom}(M_3,J^{\cdot}) \to \text{Hom}(M_2,J^{\cdot}) \to \text{Hom}(M_1,J^{\cdot}) \to 0$$

give rise to a connecting morphism, $n \in \mathbb{N}$

$$\partial^n: \text{Ext}^n(M_1,N) \to \text{Ext}^{n+1}(M_3,N)$$

which is related to θ by the formula

8.5 $$\partial^n(\alpha) = (-1)^{n+1} \alpha \cup \theta \quad ; \quad \alpha \in \text{Ext}^n(M_1,N)$$

$\underline{\text{Proof.}}$ Consider the commutative, exact diagram

$$
\begin{array}{ccccccccc}
0 & \to & \text{Hom}^{\cdot}(I_3^{\cdot},J^{\cdot}) & \to & \text{Hom}^{\cdot}(I_2^{\cdot},J^{\cdot}) & \to & \text{Hom}^{\cdot}(I_1^{\cdot},J^{\cdot}) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Hom}^{\cdot}(M_3,J^{\cdot}) & \to & \text{Hom}^{\cdot}(M_2,J^{\cdot}) & \to & \text{Hom}^{\cdot}(M_1,J^{\cdot}) & \to & 0
\end{array}
$$

recall that the vertical arrows are quasi-isomorphisms. The result follows from 2.7 and 4.9.

Q.E.D.

Let us now consider a <u>covariant</u> additive functor $T: A \to B$ from the abelian category A to the abelian category B. Given objects M and N in A and $p,q \in \mathbb{Z}$, let us introduce a pairing

$$\text{Ext}^p(M,N) \times R^qT(M) \to R^{p+q}(N)$$

With the notation from 8.2 we can represent $\alpha \in \text{Ext}^p(M,N)$ as a morphism of complexes $\alpha: I^{\cdot} \to J^{\cdot}[p]$. This will induce a morphism of complexes

$$T(\alpha): T(I^{\cdot}) \to T(J^{\cdot})[p]$$

and consequently a morphism on homology

$$H^qT(\alpha): H^qT(I^{\cdot}) \to H^{p+q}T(R^{\cdot})$$

which we can evaluate on $\beta \in H^qT(I^{\cdot})$:

8.6 $\qquad \alpha \cup \beta = H^qT(\alpha)\beta$

From the categorical origin follows that the pairing is associative

8.7 $\qquad \alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$

$\qquad \gamma \in R^pT(M), \beta \in \text{Ext}^q(M,N), \alpha \in \text{Ext}^p(N,P)$

In particular this makes $R^{\cdot}T(M)$ a graded left $\text{Ext}^{\cdot}(M,M)$-module.

We shall now proceede to investigate the relationship between the product and the connecting morphism arizing from an exact sequence in A

$$0 \to D \to E \to F \to 0$$

The connecting homomorphism, 7.3 $\qquad \partial^n: R^nT(F) \to R^{n+1}T(D)$

is related to the characteristic class $\qquad \theta \in \text{Ext}^1(F,D)$

of the sequence through the formula

8.8 $\qquad \partial^n(\alpha) = \theta \cup \alpha \qquad ; \alpha \in R^nT(F)$

<u>Proof</u>. Consider a resolution of the sequence above by a triangle in $D^+(A)$.
It follows from 4.8 that $\partial^n = H^nT(\theta)$ which proves the result.

Let us assume that the category A has enough <u>injectives</u> and enough <u>projectives</u>[*]. With the notation of 8.2 let $K^{\cdot} \to M$ and $L^{\cdot} \to N$ be projective resolutions. We can represent $\text{Ext}^q(M,N)$ by

8.9 $\qquad \text{Ext}^q(M,N) = [K^{\cdot}, L^{\cdot}[q]]$

<u>Proof</u>. According to 8.1 we can write

$$\text{Ext}^q(M,N) \xrightarrow{\sim} [M, J^{\cdot}[q]]$$

By Theorem 6.2, the resolution $K^{\cdot} \to M$ will induce an isomorphism

$$[M, J^{\cdot}[q]] \xrightarrow{\sim} [K^{\cdot}, J^{\cdot}[q]]$$

By the dual of 6.2 the resolutions $L^{\cdot} \to N$ and $N \to J^{\cdot}$ will induce isomorphisms

$$[K^{\cdot}, L^{\cdot}[q]] \xrightarrow[\sim]{} [K^{\cdot}, N[q]] \xrightarrow[\sim]{} [K^{\cdot}, J^{\cdot}[q]]$$

Compose these isomorphisms to get the result.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Q.E.D.

---

[*] Compare the following section I.9.

I.9   Resumé of the projective case

Throughout this chapter we shall work in an abelian

category A. We say that an object P is <u>projective</u>, if for

any epimorphism $f: X \to Y$ and any morphism $y: P \to Y$, there

exists a morphism $x: P \to X$, such that $y = fx$

We shall assume that the category A has enough <u>projectives</u>, i.e.

that there for every object X of A exists an epimorphism

$P \to X$ where P is projective.

The notion of projective is <u>dual</u> to that of injective in

the sense of category theory. We shall list a number of propo-

sitions obtained by applying duality to previous results. The

duality is maintained in the notation. Thus for example Theorem

6.2° is dual to Theorem 6.2.

Let us now turn to complexes. In this context it is standard

to change the notation such that a complex $C^{\cdot} = (C^n, \partial^n)_{n \in \mathbb{Z}}$ is

rewritten $C_{\cdot} = (C_n, \partial_n)_{n \in \mathbb{Z}}$ according to the convention

$$C_n = C^{-n}, \qquad \partial_n = \partial^{-n}$$

Thus a complex is a sequence of objects and morphisms

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}$$

subjected to $\partial_{n+1}\partial_n = 0$ for all $n \in \mathbb{Z}$. The same convention

applies to morphisms of complexes: $f_n = f^{-n}$.

We shall mostly be concerned with <u>bounded above</u> complexes,

i.e. complexes $C_{\cdot}$ with $C_n = 0$ for $n \ll 0$.

<u>Theorem 6.1°</u>. For any bounded above complex $X_{\cdot}$, there

exists a bounded above complex $P_{\cdot}$ of projectives and a quasi-

isomorphism $P_{\cdot} \to X_{\cdot}$.

<u>Theorem 6.2°</u>. Let $P_{\cdot}$ be a bounded above complex of

projectives. Any quasi-isomorphism $c: X_{\cdot} \to Y_{\cdot}$ induces an iso-

morphism

$$[1,c]: [P_{\cdot},X_{\cdot}] \xrightarrow{\sim} [P_{\cdot},Y_{\cdot}]$$

<u>The derived category</u>. On the basis of 6.1° and 6.2° we shall

introduce the categories

6.4°

$K^{-}(A)$	the homotopy category of bounded above complexes in A
$D^{-}(A)$	the homotopy category of bounded above complexes of projectives in A

The inclusion of the subcategory $D^{-}(A)$ in $K^{-}(A)$ is denoted

$$i: D^{-}(A) \longrightarrow K^{-}(A)$$

For each $X_{\cdot}$ in $K(A)$ choose a quasi-isomorphism

6.5°
$$p: \lambda X_{\cdot} \longrightarrow X_{\cdot}$$

where $\lambda X_{\cdot}$ is an object of $D^{-}(A)$. For a morphism $f: X_{\cdot} \to Y_{\cdot}$

in $K^{-}(A)$ there exists one and only one morphism $\lambda f: \lambda X_{\cdot} \to \lambda Y_{\cdot}$

making the following diagram commutative

$$\begin{array}{ccc}
\lambda X_{..} & \xrightarrow{\;\;p\;\;} & X_{..} \\
\downarrow{\scriptstyle \lambda f} & & \downarrow{\scriptstyle f} \\
\lambda Y_{..} & \xrightarrow{\;\;p\;\;} & Y_{..}
\end{array}$$

This construction yields a canonical isomorphism

$6..6^{O}$ $\qquad\qquad [P_{..}, X_{..}] = [P_{..}, \lambda X_{..}]$

where $X_{..}$ varies through $K^{-}(A)$ and $P_{..}$ varies through $D^{-}(A)$. Otherwise expressed the functor

$6..7^{O}$ $\qquad\qquad \lambda: K^{-}(A) \to D^{-}(A)$

is a right adjoint to the inclusion $i: D^{-}(A) \to K^{-}(A)$. In this interpretation the resolution morphism p from $6..5^{O}$ plays the role of adjunction morphism.

Theorem 6.8^{O}. The resolution functor

$$\lambda: K^{-}(A) \to D^{-}(A)$$

transforms triangles into triangles.

We shall next refine theorem $6..8^{O}$ by performing a construction which assigns a triangle in $D^{-}(A)$ to a short exact sequence of bounded above complexes in A.

$$0 \to C_{..} \to D_{..} \to E_{..} \to 0$$

The key result is

Proposition 6.10^{O}. There exists a commutative, not merely homotopy commutative diagram of complexes in A

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_{..} & \longrightarrow & Q_{..} & \longrightarrow & R_{..} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{..} & \longrightarrow & D_{..} & \longrightarrow & E_{..} & \longrightarrow & 0
\end{array}$$

where the top row is a short exact sequence of bounded above complexes of projectives and the vertical arrows are quasi-isomorphism.

The short exact sequence of projectives constructed in 6.10^{O} gives rise to a triangle

$6..11^{O}$

$$\begin{array}{ccc}
 & & R^{\cdot} \\
 & \nearrow & \uparrow \\
P^{\cdot} & \longrightarrow & Q^{\cdot}
\end{array}$$

which is unique up to homotopy.

I.10 Complexes of free abelian groups

We shall be concerned with complexes $L_.$ of free abelian groups.

10.1. Structure theorem. Any complex of free abelian groups is isomorphic to a complex of the form

$$\bigoplus_{i \in \mathbb{Z}} L_.^{(i)}$$

where for each $i \in \mathbb{Z}$, the complex $L_.^{(i)}$ can be written

$$L_.^{(i)}: \quad 0 \to L_{i+1}^{(i)} \to L_i^{(i)} \to 0 \qquad \text{with } H_{i+1}(L_.^{(i)}) = 0.$$

Proof. Let $L_.$ be a complex of free abelian groups. For i fixed let us notice that $\text{Ker } \partial_i$ is a direct summand of L_i as it follows from the fact that $\text{Im } \partial_i$ is a free abelian group, in fact any subgroup of a free abelian group is free. Choose for each $i \in \mathbb{Z}$ a direct summand N_i to $\text{Ker } \partial_i$ in L_i and let $L_.^{(i)}$ denote the complex $d_{i+1}: N_{i+1} \to \text{Ker } \partial_i$ where d_{i+1} is the restriction of ∂_{i+1} to N_{i+1}. Quite obviously $L_. \overset{\sim}{\to} \oplus L_.^{(i)}$.

Q.E.D.

Corollary 10.2. For any complex $L_.$ of free abelian groups there is a morphism of complexes $\qquad L_. \to H_.(L_.)$ inducing the indentity in homology.

Let us remark that neither this morphism nor its homotopy class is canonical as one sees from example 10.6 below.

Corollary 10.3. For a complex $L_.$ of free abelian groups and a complex $X_.$ of abelian groups $X_.$ with $H_.(X_.) = 0$ we have

$$[L_., X_.] = 0$$

Proof. According to 10.1 it suffices to treat the case where $L_.$ has only two non vanishing terms. This case follows from 6.2°.

Q.E.D.

Corollary 10.4. Two complexes $L_.$ and $N_.$ of free abelian groups are homotopy equivalent if and only if $H_.(N_.) \overset{\sim}{\to} H_.(L_.)$.

Proof. Choose morphisms as in 10.2

$$b: N_. \to H_.(N_.) \qquad a: L_. \to H_.(L_.)$$

and let $\varphi: H_.(L_.) \to H_.(N_.)$ be an isomorphism. Consider the triangle

and the resulting long exact sequence

$$[L_., N_.] \to [L_., H_.(N_.)] \to [L_., \text{Con}_.(b)]$$

Since $H_.(\text{Con}_.(b)) = 0$ it follows from 10.3 that $[L_., \text{Con}_.(b)] = 0$. Consequently we can find $f: L_. \to N_.$ such that $bf \sim \varphi a$. Notice that f is a quasi-isomorphism and consequently has a contractible mapping cone by 10.3 It follows that f is a homotopy equivalence.

Q.E.D.

Left column:

Corollary 10.5. Let $f: L_. \to N_.$ be a morphism of complexes. If all homology groups of $L_.$ are free, then

$$H_.(Con_.(f)) \overset{\sim}{\to} H_.(Con_.(H(f)))$$

Proof. Choose morphisms of complexes

$$a: L_. \to H_.(L_.) \qquad b: N_. \to H_.(N_.)$$

It follows from 10.3 that a is a homotopy equivalence. Thus we can find a homotopy commutative diagram

$$
\begin{array}{ccc}
L_. & \xrightarrow{\ f\ } & N_. \\
a \downarrow & & \downarrow b \\
H_.(L_.) & \xrightarrow{\ g\ } & H_.(N_.)
\end{array}
$$

Since $H(a) = 1$ and $H(b) = 1$ we find $g = H(f)$. According to 4.19 we can choose a morphism $c: Con_.(f) \to Con_.(H(f))$ such that a,b,c is a morphism of triangles. It follows from the long exact homology ladder and the five lemma 1.7, that c is a quasi-isomorphism.

Q.E.D.

Example 10.6. Consider the two complexes

$$L_.: \mathbb{Z} \xrightarrow{\ 2\ } \mathbb{Z} \qquad N_.: \mathbb{Z} \to 0$$

and the morphism $f: L_. \to N_.$ induced by multiplication by 3. Notice that

$$H_.(Con_.(f)) = \mathbb{Z} \qquad H_.(Con_.(H(f))) = \mathbb{Z} \oplus \mathbb{Z}/2$$

which shows that the assumption "$H_.(L_.)$ free" is important in 10.5.

Right column:

I.11 Sign rules

Let us consider a fixed additive category $C_.$. By a double complex we understand data $C^{..} = (C, \partial_1, \partial_2)$ where

$$C = (C^{p,q})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}} \quad \partial_1 = (\partial_1^{p,q})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}} \quad \partial_2 = (\partial_2^{p,q})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$$

is a family of objects and two families of morphisms

$$\partial_1^{p,q}: C^{p,q} \to C^{p+1,q} \qquad \partial_2^{p,q}: C^{p,q} \to C^{p,q+1}$$

subjected to the following conditions

11.1 $$\partial_1 \partial_1 = 0 \qquad \partial_2 \partial_2 = 0 \qquad \partial_1 \partial_2 = \partial_2 \partial_1$$

Let us assume that for all $n \in \mathbb{Z}$ the direct sum

11.2 $$Tot^n C^{..} = \bigoplus_{p+q=n} C^{p,q}$$

is finite. Then we define $\partial^n: Tot^n C^{..} \to Tot^{n+1} C^{..}$ by the formula

11.3 $$\partial^n = \bigoplus_{p+q=n} \partial_1^{p,q} + (-1)^p \partial_2^{p,q}$$

This defines the total complex associated to the double complex

$$Tot^. C^{..} = ((Tot^n C^{..})_{n \in \mathbb{Z}}, (\partial^n)_{n \in \mathbb{Z}})$$

By a morphism $f: C^{..} \to D^{..}$ of couble complexes we understand an indexed family of morphisms $f^{p,q}: C^{p,q} \to D^{p,q}$ which commutes with ∂_1 and ∂_2. Such a morphism will in the obvious manner induce a morphism $Tot f: Tot^. C^{..} \to Tot^. D^{..}$

Given a second morphism of double complexes $g: C^{..} \to D^{..}$. By a homotopy from f to g we understand a pair (s_1, s_2) where s_1 is a family of morphisms $s_1^{p,q}: C^{p,q} \to C^{p-1,q}$ and s_2 a family of morphism $s_2^{p,q}: C^{p,q} \to C^{p,q-1}$ such that

11.4 $$\partial_2 s_1 = s_1 \partial_2 \qquad \partial_1 s_2 = s_2 \partial_1$$

11.5 $$f - g = s_1 \partial_1 + \partial_1 s_1 + \partial_2 s_2 + s_2 \partial_2$$

To these data we can associate $s^n: \mathrm{Tot}^n C^{..} \to \mathrm{Tot}^{n-1} D^{..}$ by

11.6 $$s^n = \bigoplus_{p+q=n} s_1^{p,q} + (-1)^p s_2^{p,q}$$

We leave it to the reader to show that

11.7 $$\mathrm{Tot}\, f - \mathrm{Tot}\, g = \partial s + s \partial$$

Bivariant functor 11.8. Let $T: A \times B \to C$ be a biadditive functor contravariant in the first variable and covariant in the second variable. To a complex $A^{.}$ in A and a complex $B^{.}$ in B we associate a double complex $T(A^{.}, B^{.})$ in C given by

$$T(A^{.}, B^{.}) = (T(A^{-q}, B^p), T(1, \partial^p), (-1)^{q+1} T(\partial^{-q-1}, 1))_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$$

$$
\begin{array}{ccc}
T(A^{-q-1}, B^p) & \xrightarrow{\ T(1, \partial^p)\ } & T(A^{-q-1}, B^{p+1}) \\
\Big\uparrow {\scriptstyle (-1)^{q+1} T(\partial^{-q-1}, 1)} & & \Big\uparrow {\scriptstyle (-1)^{q+1} T(\partial^{-q-1}, 1)} \\
T(A^{-q}, B^p) & \xrightarrow{\ T(1, \partial^p)\ } & T(A^{-q}, B^{p+1})
\end{array}
$$

The total complex associated to this double complex is denoted $T^{.}(B^{.}, A^{.})$. If we apply these conventions to the Hom-functor we recover the complex 4.3.

Contravariant functor 11.9. Let $T: A \to C$ be a contravariant additive functor. To a complex $C^{.}$ we associate the complex $T(C^{.})$ given by

$$T(C^{.}) = (T(C^{-q}), (-1)^{q+1} T(\partial^{-q-1}))$$

$$
\begin{array}{c}
T(C^{-q-1}) \\
\Big\uparrow {\scriptstyle (-1)^{q+1} T(\partial^{-q-1})} \\
T(C^{-q})
\end{array}
$$

Translation with $m \in \mathbb{Z}$ gives rise to an isomorphism whose effect on $T(C^q)$ is $(-1)^{qm}$

11.10 $$\nu_m: T(C^{.}[-m]) \xrightarrow{\sim} T(C^{.})[m]$$

11.11 A triangle in the category A is transformed by the contravariant functor $T: A \to C$ into a triangle in C

$$
\begin{array}{ccc}
& G & \\
{\scriptstyle f}\nearrow & & \nwarrow {\scriptstyle e} \\
E & \xrightarrow{\ g\ } & F
\end{array}
$$

$$
\begin{array}{ccc}
& T(E) & \\
{\scriptstyle T(f[-1]) \circ \nu_1}\swarrow & & \nwarrow {\scriptstyle T(g)} \\
T(G) & \xrightarrow{\ T(e)\ } & T(F)
\end{array}
$$

If $U: C \to D$ is a second covariant functor then the automorphism of $U(T(C^p))$ given by $(-1)^p$ gives a canonical isomorphism

11.12 $$\rho: U(T(C^{.})) \xrightarrow{\sim} U \circ T(C^{.})$$

II. Sheaf Theory

II.0.   Direct limits of abelian groups

By a directed set we understand a non-empty set I equipped with a preorder, i.e. a relation satisfying

$$a \leq a \quad \text{for all} \quad a \in I$$

$$a \leq b \quad \text{and} \quad b \leq c \quad \text{implies} \quad a \leq c$$

subject to the condition that for any $a \in I$ and $b \in I$ there exists $c \in I$ with $a \leq c$ and $b \leq c$.

By a direct system of abelian groups over I we understand data (F_a, f_{ba}) consisting of an abelian group F_a for each $a \in I$ and a linear map $f_{ba}: F_a \to F_b$ for each pair a,b with $a \leq b$ subject to the conditions

$$f_{cb}f_{ba} = f_{ca} \qquad ; \quad a \leq b, \quad b \leq c$$

$$f_{aa} = 1, \quad \text{the identity on} \quad F_a \quad ; \quad a \in I$$

We shall now construct the direct limit $\varinjlim F_a$ of our direct system: We form the direct sum of the groups F_a,

$$\bigoplus_{a \in I} F_a$$

For each $a \in I$ we let i_a denote the canonical map from F_a to the direct sum. We put

0.1
$$\varinjlim F_a = \bigoplus_{a \in I} F_a \Bigg/ {i_b f_{ba} x_a - i_a x_a \atop ; \ a \leq b, \ x_a \in F_a}$$

i.e. the quotient group of the direct sum with respect to the subgroup generated by the elements written under the bar. - The composite of i_a and the projection of $\oplus F_a$ onto $\varinjlim F_a$ is denoted

$$f_a: F_a \to \varinjlim F_a$$

It follows at once from the construction that

$$f_b f_{ba} = f_a \qquad ; \quad a \leq b$$

and that we have the following universal

0.2 Mapping property of $\varinjlim F_a$. Let there be given an abelian group G and for each $a \in I$ a linear map $g_a: F_a \to G$ subject to the conditions

$$g_b \ f_{ba} = g_a \qquad ; \quad a \leq b$$

then there exists a unique $g: \varinjlim F_a \to G$ with

$$g f_a = g_a \qquad ; \quad a \in I .$$

Proposition 0.3. 1) Any element of $\varinjlim F_a$ has the form $f_u(x_u)$ for some $u \in I$ and some $x_u \in F_u$.

2) For $u \in I$ and $x_u \in F_u$ we have $f_u(x_u) = 0$ if and only if there exists $v \geq u$ with $f_{vu}(x_u) = 0$.

<u>Proof</u>. 1) According to the construction of $\varinjlim F_a$, its typical element is a finite sum

$$\sum f_a(x_a) \qquad ; \quad x_a \in F_a$$

Choose $u \in I$ larger than all indices occurring in this sum. Then the following relation will justify 1)

$$\overset{\rightarrow}{\underset{a}{\sum}} f_a(x_a) = \underset{a}{\sum} f_u f_{ua}(x_a) = f_u (\underset{a}{\sum} f_{ua} x_a)$$

2) Let us assume that $f_u(x_u) = 0$. Then we can write $i_u(x_u)$ as a finite sum of the form

$$i_u x_u = \sum i_b f_{ba} x_a - i_a x_a$$

Choose $v \in I$ larger than any of the indices occurring in the sum above. We can write

$$i_v f_{vu} x_u = i_v f_{vu} x_u - i_u x_u + \sum i_b f_{ba} x_a - i_a x_a$$

The typical element under Σ can be rewritten

$$i_b f_{ba} x_a - i_a x_a = i_v f_{va} x_a - i_a x_a + i_v f_{vb}(-f_{ba} x_a) - i_b(-f_{ba} x_a)$$

A moment's reflection shows that we can write

$$i_v f_{vu} x_u = \underset{a; a<v}{\sum} i_v f_{va} z_a - i_a z_a$$

with $z_a \in F_a$ zero except for finitely many $a \in I$. According to the basic character of $\oplus F_u$ this implies $z_a = 0$ for all $a \in I$ and consequently $f_{vu} x_u = 0$.

$$Q.E.D.$$

The following Corollary is the <u>key to practical evaluation</u> of the symbol $\varinjlim F_a$.

<u>Corollary 0.4</u>. With the notation from 0.2 suppose that

1) every element of G has the form $g_u(x_u)$ for some $u \in I$ and some $x_u \in F_u$

2) $g_u(x_u) = 0 \Rightarrow \exists\, v \geq u \quad f_{vu} x_u = 0$.

Then $\varinjlim F_a$ is isomorphic to G, through g.

<u>Excactness</u>. Given direct systems of abelian groups (E_a, e_{ba}) and (F_a, f_{ba}) over I. By a morphism from the first system to the second we understand a family of linear maps (φ_a) where for each $a \in I$ $\varphi_a \colon E_a \to F_a$, subject to the condition that

$$f_{ba} \varphi_a = \varphi_b\, e_{ba} \qquad ; \quad a \leq b$$

By the mapping property of direct limits 0.2, we can find one and only one $\varphi \colon \varinjlim E_a \to \varinjlim F_a$ such that

$$\varphi \circ e_a = f_a \circ \varphi_a \qquad ; \quad a \in I \ .$$

This map is the <u>direct limit</u> of the morphism and is denoted

$$\varinjlim \varphi_a : \varinjlim E_a \to \varinjlim F_a \, .$$

<u>Proposition 0.5</u>. Given morphisms of direct systems over I

$$(E_a, E_{ba}) \xrightarrow{(\varphi_a)} (F_a, F_{ba}) \xrightarrow{(\psi_a)} (G_a, G_{ba})$$

such that the sequence

$$E_a \xrightarrow{\varphi_a} F_a \xrightarrow{\psi_a} G_a$$

is exact for all $a \in I$. Then the following sequence is exact

$$\varinjlim E_a \xrightarrow{\varinjlim \varphi_a} \varinjlim F_a \xrightarrow{\varinjlim \psi_a} \varinjlim G_a$$

<u>Proof</u>. Straightforward application of the previous proposition.

<div align="right">Q.E.D.</div>

<u>General categories 0.6</u>. Let C be an arbitrary category. By a direct system in C we understand data (F_a, f_{ba}) consisting of an object F_a of C for each $a \in I$ and morphism $f_{ba} : F_a \to F_b$ for each pair $a \leq b$ subjected to the conditions

$$f_{cb} f_{ba} = f_{ca} \qquad ; \quad a \leq b \leq c$$

$$f_{aa} = 1, \quad \text{the identity on } F_a \qquad ; \quad a \in I.$$

By a <u>direct limit</u> of the system we understand data (f_a, F) consisting of an object F and morphisms $f_a : F_a \to F$ such that

$$f_b F_{ba} = f_a \qquad ; \quad a \leq b$$

and which has a universal mapping property similar to 0.2.

<u>Proposition 0.7</u> Let C be an additive category in which arbitrary direct limits exist. Then arbitrary <u>direct sums</u> exist, i.e. systems with a mapping property generalizing I.3.

<u>Proof</u>. Let $(C_s)_{s \in S}$ be a family of objects in C. For any finite subset A of S we put $C_A = \oplus_{s \in A} C_s$, and for finite subsets $A \subseteq B$ we define $f_{BA} : C_A \to C_B$ in a rather obvious way. In this manner we have created a direct system in C over the directed set of finite subsets of S. It is easy to see that the direct limit of this system is a direct sum.

<div align="right">Q.E.D.</div>

<u>Remark 0.8</u>. To a preorered set I we can associate the category I whose objects are the points of I. The morphisms from a to b are given by

$$\text{Hom}(a,b) = \begin{cases} \{(b,a)\} & \text{for } a \leq b \\ \emptyset & \text{otherwise} \end{cases}$$

With this notation a direct system in C over I may be interpreted as a covariant functor $F : I \to C$, and a morphism $F \to G$ of direct systems as a natural transformation of functors.

II.1   Presheaves and sheaves

Let   X   denote a topological space. By a _presheaf_   F   on

X   we understand a collection of data as follows

1) For each open subset   U   of   X   an abelian group $F(U)$,

the (group) of sections of   F   _over_   U.

2) For each pair $U \supseteq V$ of open subsets of   X   a linear

map $r_{VU}: F(U) \to F(V)$,   the restriction from   U   to   V.

These data are subject to the conditions that for open

sets $U \supseteq V \supseteq W$ we have

1.1 $$r_{WV} r_{VU} = r_{WU} \qquad ; r_{UU} = 1$$

This definition can be rephrased if we organize the ordered set

of open subsets of   X   into a category. A presheaf is contravariant

functor   F   from this category to the category of abelian groups.

A morphism $f: F \to G$ of _presheaves_   on   X   is a natural

transformation of functors, i.e. for each open set   U   we have

given a linear map $f(U): F(U) \to G(U)$ such that whenever

$U \supseteq V$ are open subsets, the following diagram is commutative

1.2
$$\begin{array}{ccc}
F(U) & \xrightarrow{r_{VU}} & F(V) \\
\downarrow{\scriptstyle f(U)} & & \downarrow{\scriptstyle f(V)} \\
G(U) & \xrightarrow{r_{VU}} & G(V)
\end{array}$$

Two morphisms of presheaves $f: F \to G$ and $g: G \to K$ can be

composed to $g \circ f: F \to K$ according to the convention

1.3 $$(g \circ f)(U) = g(U) \circ f(U)$$

Let us define the _sum_ $f + g$ of two morphisms $f, g: F \to G$

by the convention

1.4 $$(f+g)(U) = f(U) + g(U)$$

With this convention the category of presheaves on   X   is an

additive category, as the reader easily verifies by construct-

ing the direct sum $F \oplus G$ of two presheaves by the formula

1.5 $$(F \oplus G)(U) = F(U) \oplus G(U)$$

In fact the category of presheaves on   X   is an abelian cate-

gory as one sees by constructing kernel and cokernel for a

morphism of sheaves $f: F \to G$ by the rules

1.6
$$Ker(f)(U) = Ker(f(U))$$

$$cok(f)(U) = Cok(f(U))$$

The section functor. Given a presheaf   F   on   X   and an

open set   U   of   X. Put

1.7 $$\Gamma(U,F) = F(U)$$

the (group of) sections of   F   _over_   U. - The elements of

$\Gamma(X,F)$ are called the global sections of   F.

Definition 1.8. A presheaf   F   on   X   is called a _sheaf_

if for any family $(U_i)_{i \in I}$ of open subsets of   X   and any

family of sections $(s_i)_{i \in I}$, $s_i \in F(U_i)$, with

$$r_{U_i \cap U_j, U_i} (s_i) = r_{U_i \cap U_j, U_j} (s_j) \qquad ; \; i,j \in I$$

there exists one and only one $s \in F(U)$, where $U = \bigcup_{i \in I} U_i$, with

$$r_{U_i, U} (s) = s_i \qquad ; \quad i \in I$$

Example 1.9. For an open set U of X let $C(U)$ denote the set of $\mathbb{R}$-valued continuous functions on X. For open subsets $U \supseteq V$, restriction of functions in the ordinary sense of the word defines a linear map $r_{VU}: C(U) \to C(V)$. In fact we have defined a sheaf C on X.

Example 1.10. An abelian group D gives rise to a sheaf $\underset{\sim}{D}$ on the topological space X: For an open set U we let $\underset{\sim}{D}(U)$ denote the group of locally constant functions on U with values in D. Restriction maps in the presheaf $\underset{\sim}{D}$ is ordinary restriction of functions. $\underset{\sim}{D}$ is a sheaf on X.

Example 1.11. Let us describe the sheaf 0 of analytic functions of the complex plane $\mathbb{C}$: $\Gamma(U,0)$ is the group of analytic functions $f: U \to \mathbb{C}$. Restriction in the presheaf 0 is that of restriction of functions. This defines the sheaf of analytic functions. Consider the morphisms of sheaves

$$0 \longrightarrow \underset{\sim}{\mathbb{C}} \overset{j}{\longrightarrow} 0 \overset{D}{\longrightarrow} 0 \longrightarrow 0$$

where j is the inclusion of the sheaf of locally constant functions into the sheaf of analytic functions. The morphism D assigns to an open set U and $s \in \Gamma(U,0)$ the complex derivative of s, $Ds \in \Gamma(U,0)$.

II.2   Localization

In this section we shall study the process of localizing a problem formulated in terms of sheaves. This technique will be applied to show that the category of sheaves on a topological space X is an abelian category.

Let us fix a point x of X. The set of open neighbourhoods U of x in X ordered by inclusion form a directed set. A presheaf F on X will induce a direct system of abelian groups on this set. We define the stalk of F at x by

$$2.1 \qquad\qquad F_x = \varinjlim F(U)$$

A morphism of $f: F \to G$ of presheaves on X will induce a morphism of direct systems and consequently give rise to a map

$$f_x: F_x \longrightarrow G_x$$

In this way we have for each $x \in X$ constructed a functor from the category of presheaves to the category of abelian groups. The following lemma allows us to pass information from local to global.

Lemma 2.2. Let F and G be sheaves on X.

   i) If two morphisms of sheaves $f,g: F \to G$ are such that $f_x = g_x$ for all $x \in X$, then $f = g$.

   ii) A morphism of sheaves $f: F \to G$ satisfies the condition

$$f_x: F_x \to G_x \text{ is injective for all } x \in X$$

if and only if, the map $f(U): F(U) \to G(U)$ is injective for all open subsets U of X.

iii) A morphism of sheaves $f: F \to G$ such that

$$f_x: F_x \longrightarrow G_x \text{ is an isomorphism for all } x \in X$$

is an isomorphism of sheaves on X.

Proof. i) We shall prove that $f(U) = g(U)$ for any open subset of X. Consider the commutative diagram

$$
\begin{array}{ccc}
F(U) & \longrightarrow & \prod_{x \in U} F_x \\
\downarrow f(U) & & \downarrow \prod_{x \in U} f_x \\
G(U) & \longrightarrow & \prod_{x \in U} G_x
\end{array}
$$

Note that the horizontal arrows are injective by the fact that F and G are sheaves. Since $f_x = g_x$ for all $x \in U$, the diagram remains commutative if we replace $f(U)$ with $g(U)$. Thus $f(U) = g(U)$.

ii) Assume f_x injective for all $x \in X$. From the commutative diagram above we conclude that $f(U)$ is injective. The opposite implication follows from exactness of $\varinjlim$ II.0.5.

iii) Let us first prove that for a given open subset U of X, $f(U): F(U) \to G(U)$ is an isomorphism. It follows from ii that $f(U)$ is injective. So let us prove that any $t \in G(U)$ belongs to the image of $f(U): F(U) \to G(U)$. For any $x \in X$ choose an open subset U^x containing $x \in X$ and a section

$r^x \in F(U^x)$ such that $f(r^x)$ and t has the same stalk of x. Next, choose an open neighbourhood V^x of x in U^x such that if we let s^x denote the restriction of r^x to V^x, then

$$f(V^x)(s^x) = r_{V^x, U}(t)$$

Notice, that for $x, y \in U$, s^x and s^y have the same restriction to $V^x \cap V^y$ as it follows from the formula above and the injectivity of $f(V^x \cap V^y)$. Thus we can find $s \in F(U)$ with restriction s^x to V^x. Clearly, $f(U)(s) = t$.

Let us finally notice that $f: F \to G$ is an isomorphism: The inverse is given by $f(U)^{-1}$, as U varies through the open subsets of X.

Q.E.D.

Sheafification of a presheaf. Let F be a presheaf on X. We shall associate a sheaf $\tilde{F}$ to the presheaf F. The sections of $\tilde{F}$ over the open set U is the set of those

$$s \in \prod_{z \in U} F_z$$

which for every $x \in U$ satisfies the following condition: "there exists an open neighbourhood W of x in U and a section t of F over W such that $pr_w s = t_{\overline{w}}$ for all $w \in W$".

In order to define restriction maps in the presheaf $\tilde{F}$ notice that for open subsets $U \supseteq V$ of X the natural maps

$$\prod_{x \in U} F_x \longrightarrow \prod_{x \in V} F_x$$

will map $\Gamma(U,\tilde{F})$ into $\Gamma(V,\tilde{F})$. It is left to the reader

to show that $\tilde{F}$ is a sheaf. Given a section s of F over

an open set U of X, the collection of its stalks at the

points of U will define a section of $\tilde{F}$ over U. We have

in fact established a morphism of presheaves

$$2.3 \qquad\qquad i_F: F \longrightarrow \tilde{F}$$

Proposition 2.4. Let F be a presheaf on the space X.

i) The canonical morphism $i_F: F \to \tilde{F}$ induces an iso-

morphism on all stalks.

ii) Given a morphism $f: F \to G$ from F into a sheaf G.

Then there exists one and only one morphism $\varphi: \tilde{F} \to G$, such

that $f = \varphi \circ i_F$.

Proof. i) Given $x \in X$ and an open neighbourhood U of

x. Projection induces a map

$$\Gamma(U,\tilde{F}) \longrightarrow F_x$$

compatible with restriction maps in $\tilde{F}$. Passing to the direct

limit we define a map $\tilde{F}_x \to F_x$ which is easily seen to be a

left inverse to $(i_F)_x$. Thus it remains to prove that $(i_F)_x$

is surjective, which is left to the reader.

ii) Let us first notice that $f: F \to G$ will induce a

morphism $\tilde{f}: \tilde{F} \to \tilde{G}$ which for each open subset U of X will

yield a commutative diagram

$$
\begin{array}{ccc}
F(U) & \xrightarrow{\ i_F(U)\ } & \tilde{F}(U) \\
{\scriptstyle f(U)}\downarrow & & \downarrow{\scriptstyle \tilde{f}(U)} \\
G(U) & \xrightarrow[\ i_G(U)\ ]{} & \tilde{G}(U)
\end{array}
$$

It follows from i and 2.2.iii that $i_G(U)$ is an isomorphism.

Thus we can define $\varphi(U): \tilde{F}(U) \to G(U)$ by putting

$$\varphi(U) = i_G(U)^{-1} \tilde{f}(U)$$

This defines a morphism $\varphi: \tilde{F} \to G$ with $f = \varphi \circ i_F$. Uniqueness

of φ follows from 2.2.

$$\text{Q.E.D.}$$

The category $Sh(X)$ of sheaves on X is the full sub-

category of presheaves whose objects are the sheaves on X.

Notice that the direct sum of two sheaves, calculated in the

category of presheaves, is a sheaf. Thus $Sh(X)$ is an additive

category.

Theorem 2.5. The category $Sh(X)$ of sheaves on the topo-

logical space X is an abelian category.

Proof. Given a morphism $f: F \to G$ of sheaves. Let $Ker(f)$

and $cok(f)$ denote kernel and cokernel presheaf as calculated

by the formula 1.6. It is easily seen that $Ker(f)$ is in fact

a sheaf, and that the inclusion of $Ker(f)$ in F is a kernel

in the sense of I.1 for the morphism f in the category $Sh(X)$.

Put

$$Cok(f) = \widetilde{cok(f)}$$

The composite of the two canonical morphisms

$$G \to \text{cok}(f) \to \widetilde{\text{cok}}(f) = \text{Cok}(f)$$

is easily seen to be a cokernel in the sense of I.1 for the morphism f in Sh(X). With the terminology of I.1 consider the canonical factorization of f: F → G in the category Sh(X)

$$F \longrightarrow \text{Coim}(f) \overset{\tilde{f}}{\to} \text{Im}(f) \to G$$

We will prove by localization that $\tilde{f}$ is an isomorphism in Sh(X), so let there be given $x \in X$. Notice that the localization functor preserves kernels and cokernels 2.4. From this we conclude that $(\tilde{f})_x$ is an isomorphism. From 2.2 we conclude that $\tilde{f}$ is an isomorphism.

Q.E.D.

It is now time to adopt the general notions from the theory of abelian categories I.5.

Theorem 2.6. Given a sequence of morphisms of sheaves on X

$$E \overset{f}{\longrightarrow} F \overset{g}{\longrightarrow} G$$

This sequence is exact in Sh(X) if and only if

$$E_x \overset{f_x}{\longrightarrow} F_x \overset{g_x}{\longrightarrow} G_x \qquad \text{is exact for all } x \in X.$$

Proof. Let us first fix an $x \in X$. Localization at x is an additive functor $Sh(X) \longrightarrow Ab$

which preserves Kernels and Cokernels as it follows from the proof of 2.5. Thus the localisation functor is exact.

To prove the opposite inclusion let us make the general remark that the sequence f,g is exact if and only if

the two compositions along the diagonals in the diagram above are zero. Now, suppose the local sequences f_x, g_x are exact for all $x \in X$, then f,g is exact as it follows from the first part of 2.2 and the previous remark.

Q.E.D.

Complexes of sheaves

Let F' denote a complex of sheaves on X. For $n \in \mathbb{Z}$ let $H^n F'$ denote the n'th cohomology sheaf, defined according to general principles for abelian categories.

To describe this let us exhibit a presheaf on X: Given an open set U of X we can form the complex of abelian groups $\Gamma(U,F')$. For open sets $U \supseteq V$, restriction from U to V will induce a chain map $\Gamma(U,F') \longrightarrow \Gamma(V,F')$ Thus we have established a presheaf on X

2.7 $$U \longmapsto H^n \Gamma(U,F')$$

$H^n F'$ is the sheaf associated to the presheaf 2.7 as it follows from the fact that the functor $G \longmapsto \tilde{G}$ is an exact functor from the category of presheaves to that of sheaves.

Direct limit of sheaves

Let $(F_\lambda, F_{\mu\lambda})$ denote a direct system of sheaves on X over the directed set I, compare II.0.4. Let us prove that such a system has a direct limit $(f_\lambda, \varinjlim F_\lambda)$.

To do so we shall first construct a presheaf F on X. An open subset U gives rise to a direct system of abelian groups $(F_\lambda(U), f_{\mu\lambda}(U))$ and we put $F(U) = \varinjlim F_\lambda(U)$ For open sets $U \supseteq V$ restriction will define a morphism of directed systems

$$r_{VU} : (F_\lambda(U), f_{\mu\lambda}(U)) \longrightarrow (F_\lambda(V), f_{\mu\lambda}(V))$$

whose direct limit is restriction from U to V in the presheaf F. Our construction provides morphisms $f_\lambda : F_\lambda \to F$, which makes (f_λ, F) a direct limit of the system $(F_\lambda, f_{\mu\lambda})$ in the category of presheaves on X.

Let $\varinjlim F_\lambda$ denote the sheaf associated the presheaf F above. With the notion of 2.3 the data $(\varinjlim F_\lambda, i_F \circ f_\lambda)$ is a direct limit in $Sh(X)$ of the direct system $(F_\lambda, f_{\mu\lambda})$ as it follows from 2.4.

Let us describe the stalk of $\varinjlim F_\lambda$ at a given point $x \in X$. We get a canonical isomorphism

2.8

$$\boxed{\varinjlim F_{\lambda,x} \xrightarrow{\sim} (\varinjlim F_\lambda)_x}$$

Proof. According to 2.4 it suffices to prove a similar statement with $\varinjlim F_\lambda$ replaced by the presheaf F, constructed above. The verification is straightforward and left to the reader.

Q.E.D.

II.3 Cohomology of sheaves

In this section we shall introduce cohomology groups for sheaves on a fixed topological space X. This is based on homological algebra in the category $Sh(X)$ of sheaves on X.

Theorem 3.1. The abelian category $Sh(X)$ has enough injectives.

Proof. We shall use that the category Ab of abelian groups has enough injectives: an abelian group D is injective in Ab if and only if it is a divisible group, - any abelian group can be imbedded into a divisible group.

We shall describe an auxiliary construction: To a family $D = (D_x)_{x \in X}$ of abelian groups we can associate a sheaf D_* on X by the formula

$$\Gamma(U, D_*) = \prod_{x \in U} D_x$$

The restriction maps in the sheaf D_* are induced by projections. We leave it to the reader to establish a natural isomorphism

$$\operatorname{Hom}(F, D_*) \xrightarrow{\sim} \prod_{x \in X} \operatorname{Hom}(F_x, D_x)$$

as F varies through $Sh(X)$. This shows that a family $D = (D_x)_{x \in X}$ of divisible groups gives rise to an injective sheaf D_* on X.

Consider a fixed sheaf E on X. For each $x \in X$ choose an embedding $E_x \to D_x$ of the stalk of E at x into a

divisible group D_x and put $D = (D_x)_{x \in X}$. The construction
above provides an imbedding $E \to D_*$ of E into an injective
sheaf D_*.

Q.E.D.

The basic functor for homological algebra on $Sh(X)$ is
the global section functor

$$\Gamma(X,-): Sh(X) \to Ab$$

This is a left exact functor. We shall now apply the general
notions of I.7. - The i'th derived functor of $\Gamma(X,-)$ evaluated
on a sheaf F will be denoted $H^i(X,F)$ and is called the
i'th cohomology group of the space X with coefficients in
the sheaf F. To calculate this we must take an injective
resolution $F \to I^{\cdot}$ of F to get

3.2 $$H^i(X,F) = H^i \Gamma(X,I^{\cdot})$$

Example 3.3. On an open subset X of the complex plane $\mathbb{C}$
we have a fundamental exact sequence of sheaves, compare 1.11

$$0 \longrightarrow \underset{\sim}{\mathbb{C}} \longrightarrow 0 \overset{D}{\longrightarrow} 0 \longrightarrow 0$$

This gives rise to a long exact sequence

$$0 \to \Gamma(X,\underset{\sim}{\mathbb{C}}) \longrightarrow \Gamma(X,0) \overset{D}{\longrightarrow} \Gamma(X,0) \longrightarrow$$
$$\longrightarrow H^1(X,\underset{\sim}{\mathbb{C}}) \longrightarrow H^1(X,0) \longrightarrow H^1(X,0) \longrightarrow$$

In particular, we see that if $H^1(X,\underset{\sim}{\mathbb{C}}) = 0$ then for any
$f \in \Gamma(X,0)$ we can find $g \in \Gamma(X,0)$ with $Dg = f$.

Definition 3.4. A sheaf F on X is called flabby, if
for any open set U of X, $r_{UX}: F(X) \to F(U)$ is surjective.

Theorem 3.5. A flabby sheaf F on X is $\Gamma(X,-)$-acyclic.
Any sheaf I on X which is injective in $Sh(X)$ is flabby.

Proof. Let I be a sheaf on X which is injective in
$Sh(X)$. According to the proof of 3.1 we can find a monomor-
phism $i: I \to E$ from I into a flabby sheaf E. Let $f: E \to I$
denote a retraction of i, i.e. $f \circ i = 1$. For an open set
U of X consider the commutative diagram

$$
\begin{array}{ccc}
E(X) & \overset{r_{UX}}{\longrightarrow} & E(U) \\
\downarrow {\scriptstyle f(X)} & & \downarrow {\scriptstyle f(U)} \\
I(X) & \overset{r_{UX}}{\longrightarrow} & I(U)
\end{array}
$$

Notice, that the vertical arrows and the upper horizontal
arrow are epimorphisms. It follows that the lower horizontal
arrow is an epimorphism.

Let us now consider a short exact sequence of sheaves
on X with E flabby

$$0 \longrightarrow E \overset{e}{\longrightarrow} F \overset{f}{\longrightarrow} G \longrightarrow 0$$

Let us prove that f induces a surjective map on the global
sections. - So let there be given a section s of G over
X. Let us introduce the ordered set I of pairs (U,u)
where U is an open subset of X and u a section of F
over U with $f(u) = r_{UX}(s)$. The symbol $(U,u) \leq (V,v)$ means
that $U \subseteq V$ and that $u = r_{UV}(v)$.

The set I has the property that any totally ordered sub-set J (a $\leq$ b or b $\leq$ a for any pair of elements of J) has an upper bound. The content of Zorn's lemma is that ordered sets of this type have maximal elements.

Let (U,u) be a maximal element of I. Suppose U $\neq$ X, and pick x $\in$ X-U. Choose an open neighbourhood V of x and a section v of F over V with f(v) = r_{VX}(s). The difference $r_{V\cap U, U}$(u) - $r_{V\cap U, V}$(v) represents a section of E over V $\cap$ U. Extend this to a section of E over V and add the result to v. After this modification, u and v will have the same restriction to U $\cap$ V and thereby define a section of F over U $\cup$ V contradicting the maximality of (u,U).

Let us return to the exact sequence above. We leave it to the reader to prove "E and F flabby implies G flabby".

We can now prove that $H^1(X,E) = 0$ for a flabby sheaf E: Choose an exact sequence as above with F injective to get

$$H^0(X,F) \to H^0(X,G) \to H^1(X,E) \to H^1(X,F)$$

from which we can draw the conclusion, since the first map is surjective and the last group is zero.

Let us finally prove by induction on n > 1 that "$H^n(X,E) = 0$ for all flabby sheaves E on X". Choose an exact sequence as above with F injective. This makes G flabby by our earlier work. Consider the cohomology sequence

$$\to H^n(X,G) \to H^{n+1}(X,E) \to H^{n+1}(X,F) \to$$

to conclude the inductive step of the proof.

Q.E.D.

The Godement resolution 3.6. Let F be a sheaf on the topological space X. We have a natural imbedding $F \to C^0F$, described in the proof of 3.1:

3.7
$$\Gamma(U, C^0F) = \prod_{x \in U} F_x$$

Let F^1 denote the cokernel of $F \to C^0F$ and put $C^1 = C^0F^1$. Iterate this process to get a flabby resolution $F \to C^\cdot F$, where $C^\cdot F$ is given by

3.8
$$0 \to C^0F \to C^1F \to C^2F \to \dots \to C^nF \to \dots$$

The construction is functorial: a morphism of sheaves f: F $\to$ G will induce a commutative diagram of complexes of sheaves

3.9
$$\begin{array}{ccc} F & \longrightarrow & C^\cdot F \\ \downarrow f & & \downarrow C^\cdot f \\ G & \longrightarrow & C^\cdot G \end{array}$$

Another virtue of the resolution is that for each x $\in$ X the localization

3.10
$$F_x \longrightarrow (C^\cdot F)_x$$

is a homotopy equivalence in Ab. This is a consequence of the fact that the imbedding $F \to C^0F$ comes equipped with canonical local retractions, namely the projections of $\prod F_x$ onto its factors.

II.4   Direct and inverse image of sheaves. f_*, f^*.

In this section we consider a fixed continuous map $f: X \to Y$. We shall first describe a procedure pull back or inverse image, which to a sheaf G on Y associates a sheaf f^*G on X. The sections of f^*G over an open subset U of X is the set of those

$$s \in \prod_{z \in U} G_{f(z)}$$

which for each $x \in U$ satisfies the following condition: "there exists an open neighbourhood W of x in U, an open set V containing $f(W)$ and a section t of G over V, such that

4.1 $$pr_w s = t_{f(w)} \qquad ; \quad w \in W$$

where $pr_w: \prod_{z \in U} G_{f(z)} \to G_{f(w)}$ is the projection".

For open sets $U \supseteq V$ in X the natural projection

$$\prod_{z \in U} G_{f(z)} \longrightarrow \prod_{z \in V} G_{f(z)}$$

will transform $\Gamma(U, f^*G)$ into $\Gamma(V, f^*G)$. It is left to the reader to verify that in fact we have constructed a sheaf f^*G on X.

Given an open subset V of Y and a section t of G over V. The stalks of t will define a section of f^*G over $f^{-1}(V)$, i.e. we have the socalled adjunction map

4.2 $$a(V): \Gamma(V, G) \longrightarrow \Gamma(f^{-1}(V), f^*G)$$

Given a point $x \in X$. As V varies through the open neighbourhoods of $f(x)$ the adjunction map induces an isomorphism

4.3 $$G_{f(x)} \overset{\sim}{\to} (f^*G)_x \qquad ; \quad x \in X$$

Proof. Let $a_x: G_{f(x)} \to (f^*G)_x$ denote the map obtained by the procedure above. A look at the definition of f^*G shows that a_x is surjective. To prove that a_x is injective let us construct a map $p_x: (f^*G)_x \to G_{f(x)}$ with $p_x a_x = 1$: For an open neighbourhood U of x we can consider the composite

$$\Gamma(U, f^*G) \longrightarrow \prod_{z \in U} G_{f(z)} \overset{pr_x}{\longrightarrow} G_{f(x)}$$

and pass to the limit over all such U's to obtain p_x.

Q.E.D.

The construction of f^*G from G is easily seen to define an additive functor which is exact, as it follows from 4.3

4.4 $$f^*: Sh(Y) \longrightarrow Sh(X)$$

Starting from a sheaf F on X we shall construct a sheaf f_*F on Y, the direct image of F by f. The sections over the open subset V of Y are given by

4.5 $$\Gamma(V, f_*F) = \Gamma(f^{-1}(V), F)$$

the restriction maps are those induced from F. - This construction is seen to define a left exact additive functor

4.6 $$f_*: Sh(X) \longrightarrow Sh(Y)$$

Let us now prove the fundamental formula

4.7 $$\mathrm{Hom}(f^*G,F) = \mathrm{Hom}(G,f_*F)$$

for all sheaves G on Y and all sheaves F on X, i.e.

Theorem 4.8. The functor f^* is left adjoint to the functor f_*.

Proof. Let us first record that the adjunction maps 4.2 fits together to form the adjunction morphism

4.9 $$a: G \longrightarrow f_*f^*G$$

Starting with $\varphi: f^*G \to F$ we can construct a morphism $\psi: G \to f_*F$ given by $\psi = f_*\varphi \circ a$. Notice that for an open set V of Y we have the commutative diagram

4.10
$$
\begin{array}{ccc}
\Gamma(V,G) & \xrightarrow{\psi(V)} & \Gamma(V,f_*F) \\
\downarrow{\scriptstyle a(V)} & & \downarrow{\scriptstyle =} \\
\Gamma(f^{-1}(V),f^*G) & \xrightarrow{\varphi(f^{-1}(V))} & \Gamma(f^{-1}(V),f)
\end{array}
$$

Remark that the diagram 4.10 can serve us as definition of the transform ψ of φ. Let us remark that the map we have constructed is injective

4.11 $$\mathrm{Hom}(f^*G,F) \to \mathrm{Hom}(G,f_*F) \qquad ; \varphi \mapsto f_*\varphi \circ a$$

To see this let us fix a point $x \in X$ and take the direct limit over all open neighbourhoods V of $f(x)$ to obtain from 4.10 a commutative diagram

4.12
$$
\begin{array}{ccc}
G_{f(x)} & \xrightarrow{\psi_{f(x)}} & (f_*F)_{f(x)} \\
\downarrow{\scriptstyle a_x} & & \downarrow{\scriptstyle b_x} \\
(f_*G)_x & \xrightarrow{\varphi_x} & F_x
\end{array}
$$

Recalling from 4.3 that a_x is an isomorphism, we see how the stalks of φ are determined by those of ψ. Injectivity of 4.11 thus follows from 2.2.i.

To prove surjectivity of 4.11 let $\psi: G \to f_*F$ be given. For an open set U of X form the product

$$\prod_{x \in U} b_x \psi_{f(x)} : \prod_{x \in U} G_{f(x)} \longrightarrow \prod_{x \in U} F_x$$

It is left to the reader to see that this map will transform $\Gamma(U,f^*G)$ into $\Gamma(U,\widetilde{F})$ where $\widetilde{F}$ denotes the sheaf associated the presheaf F. In this way we obtain a morphism $f^*G \to \widetilde{F}$ which we compose with the inverse of the isomorphism $F \to \widetilde{F}$, 2.4 to obtain a morphism $\varphi: f^*G \to F$. It is now easy to see that $\psi = f_*\varphi \circ a$.

Q.E.D.

Corollary 4.13. The additive functor $f_*: \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ transforms injectives into injectives.

Proof. Let I be an injective sheaf on X. The formula $\mathrm{Hom}(G,f_*I) = \mathrm{Hom}(f^*G,I)$ and the fact that f^* is exact, show that $G \mapsto \mathrm{Hom}(G,f_*I)$ is an exact functor on $\mathrm{Sh}(Y)$.

Q.E.D.

II.5   Continuous maps and cohomology

In this section we shall discuss the action of a continuous
map $f: X \to Y$ on cohomology.

Let there be given a sheaf G on Y. We shall introduce
a natural map

5.1 $f*: H^{"}(Y,G) \longrightarrow H^{"}(X,f^{*}G)$

To describe this choose an injective resolution $G \to J^{"}$ on
Y and an injective resolution $f*J^{"} \to I^{"}$ on X, I.6.1.
Since f* is exact this makes $I^{"}$ an injective resolution
of f*G. The composite

$$\Gamma(Y,J) \xrightarrow{a} \Gamma(X,f*J^{"}) \longrightarrow \Gamma(X,I^{"})$$

represents 5.1 on the chain level by definition.

Scolium 5.2. Given a $\Gamma(Y,-)$-acyclic resolution $t: G \to T$
and a $\Gamma(X,-)$-acyclic resolution $s: f*G \to S^{"}$ and a morphism
of complexes $\psi: T^{"} \to f_*S^{"}$ making the following diagram
commutative

$$
\begin{array}{ccc}
G & \xrightarrow{t} & T^{"} \\
\downarrow{a} & & \downarrow{\psi} \\
f_*f^*G & \xrightarrow{f_*s} & f_*S^{"}
\end{array}
$$

then $f*: H^{"}(Y,G) \to H^{"}(X,f*G)$ is represented by the chain map

$$\Gamma(Y,T^{"}) \xrightarrow{\psi} \Gamma(Y,f_*S^{"}) = \Gamma(X,S^{"})$$

Proof. Choose an injective resolution $T^{"} \to J^{*}$ on Y
and an injective resolution $f*J^{"} \to I^{"}$ on X, I.6.1.
Consider the following commutative diagram

$$
\begin{array}{ccc}
f*T^{"} & \xrightarrow{\varphi} & S^{"} \\
\downarrow & & \downarrow \\
f*J^{"} & \dashrightarrow & I^{"}
\end{array}
$$

where φ corresponds to ψ by adjunction 4.7. Fill in the
dotted arrow to make the square homotopy commutative. Finally
consider the homotopy commutative diagram

$$
\begin{array}{ccccc}
\Gamma(Y,T^{\cdot}) & \xrightarrow{a} & \Gamma(X,f*T^{"}) & \xrightarrow{\varphi} & \Gamma(X,S^{"}) \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma(Y,J^{"}) & \xrightarrow{a} & \Gamma(X,f*J^{"}) & \longrightarrow & \Gamma(X,I^{"})
\end{array}
$$

and use the commutative diagram 4.10. Let us finally remark that
the two extreme vertical arrows are quasi-isomorphisms I.7.5.

Q.E.D.

Closed subspaces

Let $i: Z \to X$ denote the inclusion of a closed subspace
Z of the topological space X. For a sheaf E on Z and
a point x of X we have

$$(i_*E)_x = \varinjlim_{x \in U} \Gamma(U \cap Z, E)$$

From this results immediately that

5.3
$$(i_*E)_x = \begin{cases} E_x & \text{for } x \in Z \\ 0 & \text{for } x \in X-Z \end{cases}$$

As a consequence we see that the functor

$$i_*: Sh(Z) \to Sh(X) \quad \text{is } \underline{exact}$$

From 5.3 and 4.12 follows that adjunction yields an isomorphism $i^*i_*E \cong E$. Moreover,

5.4
$$H^{\cdot}(X,i_*E) \cong H^{\cdot}(Z,E)$$

Proof. An injective resolution $E \to I^{\cdot}$ on Z transforms into an injective resolution $i_*E \to i_*I^{\cdot}$ on X. The composite

$$I^{\cdot}(X,i_*I^{\cdot}) \xrightarrow{a} \Gamma(Z,i^*i_*I^{\cdot}) \longrightarrow \Gamma(Z,I^{\cdot})$$

yields the desired isomorphism.

Q.E.D.

Lemma 5.5. Let $i: Z \to X$ denote the inclusion of a closed subspace and F a sheaf on X. The adjunction morphism $a: F \to i_*i^*F$ has the following mapping property: Any morphism $F \to G$, into a sheaf G, whose stalks are zero outside Z, admits a unique factorization through $a: F \to i_*i^*F$.

Proof. Let us first remark that the adjunction morphism $a: F \to i_*i^*F$ is an epimorphism in $Sh(X)$ as it follows by localization using 5.3. Let us consider a morphism $f: F \to G$. This gives rise to a commutative diagram in $Sh(X)$ where a and b are adjunction morphisms.

$$\begin{array}{ccc} E & \xrightarrow{a} & i_*i^*F \\ \downarrow f & & \downarrow i_*i^*f \\ G & \xrightarrow{b} & i_*i^*G \end{array}$$

It follows by localization that b is an isomorphism in $Sh(X)$.

For a sheaf F on X we shall often use the abuse of notation $H^{\cdot}(Z,F)$ for $H^{\cdot}(Z,i^*F)$, where $i: Z \to X$ is the inclusion of the subspace Z.

Mayer-Vietoris sequence 5.6. Let A and B denote closed subspaces of the topological space X. A sheaf F on X gives rise to a long exact sequence

$$\to H^p(A \cup B,F) \to H^p(A,F) \oplus H^p(B,F) \to H^p(A \cap B,F) \to$$

Proof. Let us put names to the inclusions $h: A \cup B \to X$, $i: A \to X$, $j: B \to X$, $k: A \cup B \to X$. Consider the exact sequence

5.7
$$0 \longrightarrow h_*h^*F \longrightarrow i_*i^*F \oplus j_*j^*F \longrightarrow k_*k^*F \longrightarrow 0$$

where the morphisms are sum and difference of adjunction morphisms, taking into account the mapping principle 5.5. This sequence is exact as one sees by localization. The long cohomology sequence can be identified by means of 5.4.

Q.E.D.

Open subspaces

Let $j: U \to X$ denote the inclusion of an open subspace and F a sheaf on X. Let us remark that $j*F$ has a particular simple description in this case: the sections over an open set $V \subseteq U$ is given by

$$5.8 \qquad \Gamma(V,j*F) = \Gamma(V,F)$$

Let us consider a flabby resolution $F \to D^{\cdot}$ on X. This is transformed into a flabby resolution $j*F \to j*D^{\cdot}$ on U, which fits into a commutative diagram

$$
\begin{array}{ccc}
F & \longrightarrow & D^{\cdot} \\
\downarrow \text{adj} & & \downarrow \text{adj} \\
j_*j*F & \longrightarrow & j_*j*D^{\cdot}
\end{array}
$$

As a consequence of Scolium 5.2 we can represent

$$j^*: H^{\cdot}(X,F) \longrightarrow H^{\cdot}(U,j*F)$$

by the chain map

$$5.9 \qquad r_{UX}: \Gamma(X,D^{\cdot}) \longrightarrow \Gamma(U,D^{\cdot})$$

Mayer Vietoris sequence 5.10. Let U and V be open subsets of X. Any sheaf F on X induces a long exact sequence

$$\to H^n(U \cup V,F) \to H^n(U,F) \oplus H^n(V,F) \to H^n(U \cap V,F) \to$$

Proof. Consider the sequence

$$0 \to \Gamma(U \cup V,F) \to \Gamma(U,F) \oplus \Gamma(V,F) \to \Gamma(U \cap V,F) \to 0$$

where the two maps are sum and difference of restriction maps. Note that this sequence is exact whenever F is a flabby sheaf on X. In general replace F by a flabby resolution $D^{\cdot}$ and pass to cohomology using 5.9.

Q.E.D.

Application to $R^n f_*$

Let us return to a general continuous map $f: X \to Y$. The additive functor

$$f_*: Sh(X) \to Sh(Y)$$

is <u>left exact and transforms injectives into injectives</u> as it follows from the fact that f_* has a left adjoint which is exact by 4.4 and 4.8. The n'th derived functor of this is denoted $R^n f_*$, $n \in \mathbb{Z}$.

Proposition 5.11. Let F be a sheaf on X and $n \in \mathbb{Z}$. The cohomology sheaf $R^n f_* F$ is the sheaf associated to the presheaf

$$V \longmapsto H^n(f^{-1}(V),F) \ .$$

Proof. Let $I^{\cdot}$ be an injective resolution of F on X. We have

$$R^n f_* F = H^n f_* I^{\cdot}$$

It follows from 2.7 that $R^n f_* F$ is the sheaf associated to the presheaf $V \longmapsto H^n \Gamma(V,f_* I^{\cdot})$. Using 5.9 we find that

$$H^n \Gamma(V,f_* I^{\cdot}) = H^n \Gamma(f^{-1}(V),I^{\cdot}) = H^n(f^{-1}(V),F)$$

which is the desired result.

Q.E.D.

II.6   Locally closed subspaces. $h_!, h^!$.

Given a sheaf F on a topological space X and a section s of F over the open subset U of X. We define

Supp(s), the support of s

$$\text{Supp}(s) = \{x \in U \mid s_x \neq 0\}$$

Notice that the subset Supp(s) is closed relative to U.

We shall be concerned with a locally closed subspace W of X. This means that any point $v \in W$ has an open neighbourhood V in X such that $W \cap V$ is closed relative to V. The inclusion of W in X will be denoted $h: W \to X$.

Definition 6.1. For a sheaf E on W we let $h_!E$ denote the sheaf on X whose sections over the open set U of X is given by

$$\Gamma(U, h_!E) = \{s \in \Gamma(W \cap U, E) \mid \text{Supp}(s) \text{ is closed rel. to } U\}$$

The restriction maps of $h_!E$ are induced from h_*E, of which it is a subsheaf. We record this as a canonical monomorphism

6.2
$$h_!E \longrightarrow h_*E$$

From the description of the sections of $h_!E$ follows that

6.3
$$(h_!E)_x = \begin{cases} E_x & \text{for } x \in W \\ 0 & \text{for } x \in X-W \end{cases}$$

which in particular shows that $h_!$ is exact.

Proposition 6.4. The functor $h_!: \text{Sh}(W) \to \text{Sh}(X)$ is an equivalence between the category of sheaves on W and the full subcategory of $\text{Sh}(X)$ made up of the sheaves for which

$$F_x = 0 \quad \text{for all} \quad x \in X-W$$

The inverse functor is induced by $f*$.

Proof. For a sheaf E on W let us record the following simple identity among sheaves on W

6.5
$$h*h_!E = E$$

Next, consider a sheaf F on X whose stalks vanish outside W. A close examination reveals that the adjunction morphism $F \to h_*h*F$ may be factored through the monomorphism 6.2 as follows

$$F \to h_!h*F \to h_*h*F$$

to yield an isomorphism $F \cong h_!h*F$.

Q.E.D.

We shall construct a functor $h^!$

$$\text{Sh}(W) \xrightarrow[\ h_!\ ]{\ \overset{h^!}{\longleftarrow}\ } \text{Sh}(X)$$

which satisfies the following identity

$$\text{Hom}(h_!E, F) = \text{Hom}(E, h^!F)$$

for sheaves E on W and sheaves F on X. Otherwise expressed

<u>Proposition 6.6</u>. The functor $h_!: Sh(W) \to Sh(X)$ has a right adjoint $h^!: Sh(X) \to Sh(W)$.

<u>Proof</u>. For a sheaf F on X we let F^W denote the sheaf on X whose sections over the open subset U are given by

$$\Gamma(U,F^W) = \{s \in \Gamma(U,F) \mid Supp(s) \subseteq W\}$$

The stalks of F^W are zero at all points outside W, so if we put $h^!F = h*F^W$ we get according to the last line of 6.4 applied to F^W

6.7 $$\Gamma(U,h_!h^!F) = \{s \in \Gamma(U,F) \mid Supp(s) \subseteq W\}$$

This formula provides a monomorphism $h_!h^!F \to F$. It follows from 6.7 that any morphism $G \to F$ where G is a sheaf on X whose stalks are zero outside W may be factored through the monomorphism $h_!h^!F \to F$. For a sheaf E on W we get accordingly

$$Hom(h_!E,F) = Hom(h_!E,h_!h^!F)$$

By the first part of 6.4 we have an isomorphism

$$h_!: Hom(E,h^!F) \longrightarrow Hom(h_!E,h_!h^!F)$$

We can now compose the two isomorphisms. Q.E.D.

<u>Proposition 6.8</u>. The functor $h^!: Sh(X) \to Sh(W)$ is left exact and transforms injective sheaves into injective sheaves.

<u>Proof</u>. The left exactness is a formal consequence of the presence of a left adjoint $h_!$ to $h^!$. The second statement is a formal consequence of the exactness of $h_!$.

Q.E.D.

Open and closed subspaces

We shall make a close investigation of the special case of open and closed subsets.

<u>Proposition 6.9</u>. i) For the inclusion $i: Z \to X$ of a closed subspace we have $i_* = i_!$.

ii) For the inclusion $j: U \to X$ of an open subspace of X, we have $j* = j^!$.

<u>Proof</u>. i) For a sheaf E on Z we have a natural map 6.2 $i_!E \to i_*E$ which is an isomorphism as one checks by localization using 6.3 and 5.3.

ii) For a sheaf F on X we have according to the proof of 6.6 that $j^!F = j*F^W$. Combined with the inclusion $F^W \to F$ this yields a <u>monomorphism $j^!F \to j*F$</u>. One checks by localization that this is an isomorphism.

Q.E.D.

<u>Corollary 6.10</u>. The inclusion $j: U \to X$ of an open subspace transforms an injective sheaf I on X into an injective sheaf $j*I$ on U.

Proof. Combine 6.9.ii and 6.8.

<div align="right">Q.E.D.</div>

In the case where U and Z are complementary subspaces of X, a sheaf F on X give rise to an exact sequence

6.11 $$0 \to j_! j^* F \to F \to i_* i^* F \to 0$$

of sheaves on X, where the two morphism are adjunction morphisms. The exactness is seen by localization.

For the benefit of the reader we shall make a small tableaux over the basic functors

6.12

$$
\begin{array}{ccc}
 & \mathrm{Sh}(U) & \\
\downarrow j_! \quad & \uparrow j^* & \quad \downarrow j_* \\
 & \mathrm{Sh}(X) & \\
\downarrow i^* \quad & \uparrow i_* & \quad \downarrow i^! \\
 & \mathrm{Sh}(Z) &
\end{array}
$$

exact	exact	left exact
	preserves injectives	preserves injectives

The adjoint relationship between the various functions can be read off the table. Identities:

$$j^* j_* = 1 \quad j^* j_! = 1 \quad i^! i_* = 1 \quad i^* i_* = 1$$

Inverse image of locally closed subspaces

Let us consider a continuous map f: X → Y and a locally closed subspace B of Y. Let us notice that $A = f^{-1}(B)$ is a locally closed subspace of X. This can be seen by remarking that a subspace is locally closed if and only if it is the intersection between an open and a closed subspace. - Let h: B → Y and g: A → X denote the inclusions. We shall establish a natural isomorphism

6.13 $$f^* h_! G = g_! f^* G$$
$$
\begin{array}{ccc}
X & \xleftarrow{\ g\ } & A \\
f \downarrow & & \downarrow f \\
Y & \xleftarrow{\ h\ } & B
\end{array}
$$

for all sheaves G on B.

Construction. To be precise, let φ: A → B denote the map satisfying hφ = fg. We start with the adjunction morphism $G \to \varphi_* \varphi^* G$ from which we deduce a morphism $h_* G \to f_* g_* \varphi^* G$ or by adjunction a morphism

$$\theta: f^* h_* G \to g_* \varphi^* G$$

We have monomorphisms $f^* h_! G \to f^* h_* G$ and $g_! \varphi^* G \to g_* \varphi^* G$, compare 6.2. One can now check by localization that θ will induce on isomorphisms between these two subsheaves.

<div align="right">Q.E.D.</div>

From 6.13 we deduce by adjunction a natural isomorphism

6.14 $$h^! f^* F = f_* g^! F$$

for all sheaves F on X.

Fundamental diagrams

For a locally closed subset W of X and a sheaf F on X we put

6.15
$$F_W = h_! h^* F$$

where $h: W \to X$ denotes the inclusion of W in X.

Closed subsets $S \subseteq Z$ of X with complements $O \supseteq U$ give rise to an exact commutative diagram

6.16
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_U & \longrightarrow & F & \longrightarrow & F_Z & \longrightarrow & 0 \\
& & \downarrow & & \downarrow 1 & & \downarrow & & \\
0 & \longrightarrow & F_O & \longrightarrow & F & \longrightarrow & F_S & \longrightarrow & 0
\end{array}
$$

For closed subsets A and B of X with complements U and V respectively we have exact commutative diagrams

6.17
$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{U \cap V} & \longrightarrow & F_U & \longrightarrow & F_{U \cap B} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_V & \longrightarrow & F_X & \longrightarrow & F_B & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{A \cap V} & \longrightarrow & F_A & \longrightarrow & F_{A \cap B} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

6.18
$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{U \cap V} & \longrightarrow & F_U \oplus F_V & \longrightarrow & F_{U \cup V} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \xrightarrow{\binom{1}{1}} & F \oplus F & \xrightarrow{(1,-1)} & F & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_{A \cup B} & \longrightarrow & F_A \oplus F_B & \longrightarrow & F_{A \cap B} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

II.7 Cup product

Throughout this section we shall work with a fixed commutative ring k. By a k-sheaf F on the topoloigcal space X we understand a sheaf F together with a k-module structure on $F(U)$ for each open set U of X, making the restriction maps k-linear. By a morphism $f: F \to G$ of k-sheaves we understand a morphism of sheaves such that $f(U): F(U) \to G(U)$ is k-linear for all open subsets U of X. The category of k-sheaves on X will be denoted

7.1
$$Sh(X,k)$$

The Hom functor in this category is denoted Hom_k.

Let D be a k-module. The sheaf $\underset{\sim}{D}$ introduced in 1.10 is in a natural way a k-sheaf. For any k-sheaf F on X we have a natural isomorphism

7.2
$$\text{Hom}_k(\underset{\sim}{D}, F) = \text{Hom}_k(D, \Gamma(X,F))$$

Proof. Let $p: X \to Pt$ denote the projection of X onto a point. The formula 7.2 may be interpreted as expressing that p_* is a right adjoint to p^*, II.4.7.

Q.E.D.

Proposition 7.3. The category $Sh(X,k)$ is an abelian category with enough injectives. Any injective object in $Sh(X,k)$ is a flabby sheaf.

Proof. The previous chapters II.1-6 could very well
have been worked out for k-sheaves. We shall freely refer
to these sections as if this had been done.

In the present context we need to supplement 3.1 with
the fact that "the category of k-modules has enough injectives"
in order to prove the first part of the proposition.

The second part follows by a straightforward modification
of the proof of 3.5. Alternatively we can consider the in-
clusion j: U → X of an open subspace and the corresponding
adjunction morphism $j_! j^* \underset{\sim}{k} \to \underset{\sim}{k}$, which is a monomorphism. An
injective object E in Sh(X,k) will transform this into a
surjective map

$$\mathrm{Hom}_k(\underset{\sim}{k}, E) \longrightarrow \mathrm{Hom}_k(j_! j^* \underset{\sim}{k}, E)$$

The first term is $\Gamma(X,E)$ as it follows from 7.2. The second
term can be evaluated by means of 6.6 and 6.9.

$$\mathrm{Hom}(j_! j^* \underset{\sim}{k}, E) = \mathrm{Hom}(j^* \underset{\sim}{k}, j^* E) = \mathrm{Hom}(\underset{\sim}{k}, j^* E) = \Gamma(U, E)$$

Looking a bit closer one sees that the surjective map above may
be identified with $r_{UX} \colon \Gamma(X,E) \to \Gamma(U,E)$.

Q.E.D.

The same ideas will prove that

Proposition 7.4. Any k-sheaf F has a resolution of the
form

$$\to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to F \to 0$$

where P_s is a direct sum of sheaves of the form $j_! \underset{\sim}{k}$
where j: U → X is the inclusion of an open subspace of X.

Proof. It suffices to establish the existence of an epi-
morphism P → F where P is a sheaf of type described anove.
For the inclusion j: U → X of an open subset we have the
adjunction formula 6.6 and 6.9

$$\mathrm{Hom}(j_! \underset{\sim}{k}, F) = \mathrm{Hom}(\underset{\sim}{k}, j^* F) = \Gamma(U,F)$$

It follows that $s \in \Gamma(U,F)$ determines a morphism of sheaves
$j_! \underset{\sim}{k} \to F$, which transforms $1 \in \Gamma(U, j_! \underset{\sim}{k})$ into $s \in \Gamma(U,F)$. The
result follows by varying s and U.

Q.E.D.

For any k-sheaf F on X and $p \in \mathbb{Z}$ we have

7.5
$$\boxed{\mathrm{Ext}_k^{\cdot}(\underset{\sim}{k}, F) = H^{\cdot}(X,F)}$$

Proof. Let F → I˙ denote an injetive resolution of F
in Sh(X,k). Thus

$$\mathrm{Ext}_k^{\cdot}(\underset{\sim}{k}, F) = H^{\cdot} \mathrm{Hom}_k(\underset{\sim}{k}, I^{\cdot})$$

By 7.3 and 3.5 F → I˙ is a $\Gamma(X,-)$-acyclic resolution of
F in Sh(X) thus by the acyclicity theorem I.7.5

$$H^{\cdot}(X,F) = H^{\cdot} \Gamma(X, I^{\cdot})$$

From formula 7.2 we get with D = k

$$\Gamma(X, I^{\cdot}) = \mathrm{Hom}_k(\underset{\sim}{k}, I^{\cdot})$$

and the result follows.

Q.E.D.

The formula 7.5 constitutes the basis for introducing cup products in sheaf cohomology. Let us first notice that this in case $F = \underset{\sim}{k}$ gives

7.6 $$\text{Ext}_k(\underset{\sim}{k},\underset{\sim}{k}) = H^{''}(X,\underset{\sim}{k})$$

which makes $H^{''}(X,\underset{\sim}{k})$ into an associative ring and provides $H^{''}(X,F)$ with the structure of a right $H^{''}(X,\underset{\sim}{k})$-module, I.8. This is called the cup product and is denoted

7.7 $$\xi \cup \alpha \in H^{p+q}(X,F) \qquad ; \quad \xi \in H^p(X,F), \ \alpha \in H^q(X,\underset{\sim}{k})$$

Let $f: X \to Y$ be a continuous map and G a sheaf on Y. We have the formula

7.8 $$f^*(\eta \cup \beta) = f^*\eta \cup f^*\beta \quad ; \quad \eta \in H^p(Y,G), \beta \in H^q(Y,k)$$

Verification. This formula is best described in terms of derived categories. Let us start with three functors

$$\begin{array}{ccc} \text{Sh}(Y,k) & \xrightarrow{\ f^*\ } & \text{Sh}(X,k) \\ {\scriptstyle \Gamma(Y,-)} \searrow & & \swarrow {\scriptstyle \Gamma(X,-)} \\ & k\text{-}mod & \end{array}$$

and a natural transformation of additive functors 4.2

$$a: \Gamma(Y,-) \longrightarrow \Gamma(X,-) \circ f^*$$

This gives us three derived functors I.7.9

7.9 $$\begin{array}{ccc} D^+(Y,k) & \xrightarrow{\ Rf^*\ } & D^+(X,k) \\ {\scriptstyle R\Gamma(Y,-)} \searrow & & \swarrow {\scriptstyle R\Gamma(X,-)} \\ & D^+(k) & \end{array}$$

and a natural transformation (compose I.7.11 and I.7.12)

7.10 $$R\Gamma(Y,-) \longrightarrow R\Gamma(X,-) \circ Rf^*$$

In this picture the formula 7.8 becomes clear recalling that f^* is exact.

Once we have put up this machinery we might as well apply it to triangles to get by means of I.7.10

Proposition 7.11. A short exact sequence of sheaves on Y

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

will induce a commutative ladder

$$\begin{array}{ccccccccc}
\longrightarrow & H^p(Y,E) & \longrightarrow & H^p(Y,F) & \longrightarrow & H^p(Y,G) & \longrightarrow & H^{p+1}(Y,E) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^p(X,f^*E) & \longrightarrow & H^p(X,f^*F) & \longrightarrow & H^p(X,f^*G) & \longrightarrow & H^{p+1}(X,f^*E) & \longrightarrow
\end{array}$$

II.8 Tensor product of sheaves

Throughout this section k denotes a fixed commutative ring. In this context it is important to notice that tensor-product commutes with direct limits.

Lemma 8.1. Let (F_a, f_{ba}) denote direct system of k-modules over the directed set I. For any k-module E there is a canonical isomorphism

$$\varinjlim E \otimes_k F_a \overset{\sim}{\to} E \otimes_k \varinjlim F_a$$

Proof. Let $f_a: F_a \to \varinjlim F_a$ denote the canonical map. Consider the system of maps

$$1 \otimes f_a: E \otimes_k F_a \to E \otimes_k \varinjlim F_a \quad ; \ a \in I$$

of abelian groups. We shall check that this satisfies the universal mapping property II.0.2. - To do so we shall appeal to the description of $\text{Hom}_{\mathbb{Z}} (E \otimes_k F, G)$ as the set of bilinear maps $m: E \times F \to G$ which satisfies

$$m(re, f) = m(e, rf) \quad ; \ r \in k, \ e \in E, \ f \in F$$

It is now easy to conlude by means of II.0.3.

Q.E.D.

Definition 8.2. Let F and G be k-sheaves on the topological space X. The tensor product $F \otimes_k G$ is the sheaf associated to the presheaf

$$U \longmapsto F(U) \otimes_k G(U)$$

As a consequence of 2.4 and 8.1 we find that

8.3
$$(F \otimes_k G)_x = F_x \otimes_k G_x \quad ; \ x \in X$$

from which we conclude that tensor product of sheaves has exactness properties similar to those of ordinary tensor product.

Tensor product of complexes

Given two complexes $R^{\cdot}$ and $S^{\cdot}$ of k-sheaves. The tensor product $R^{\cdot} \otimes_k S^{\cdot}$ is the complex given by

$$[R^{\cdot} \otimes_k S^{\cdot}]^n = \bigoplus_{p+q=n} R^p \otimes S^q$$

and differential is given by

8.4
$$\partial^{p+q}(r^p \otimes s^q) = \partial^p(r^p) \otimes s^q + (-1)^p r^p \otimes \partial^q(s^q)$$

This is in accordance with the general rules of I.11.

Commutativity 8.5. Let $R^{\cdot}$ and $S^{\cdot}$ be complexes of k-sheaves on X. The formula

$$\sigma(r^p \otimes s^q) = (-1)^{pq} r^q \otimes s^p$$

defines a natural commutativity constraint

$$\sigma: R^{\cdot} \otimes_k S^{\cdot} \to S^{\cdot} \otimes_k R^{\cdot}$$

Translation 8.6. Let $A^{\cdot}$ and $B^{\cdot}$ denote complexes of k-sheaves, $m, n \in \mathbb{Z}$. The map

$$a^{m+p} \otimes b^{n+q} \longmapsto (-1)^{pn} a^{m+p} \otimes b^{n+q}$$

induces a natural isomorphism

$$\tau_{m,n}: A^{\cdot}[m] \otimes B^{\cdot}[n] \overset{\sim}{\to} A^{\cdot} \otimes B^{\cdot}[m+n]$$

which is related to the commutatitivy constraint by the following commutative diagram

8.7

$$
\begin{array}{ccc}
A^{\cdot}[m] \otimes B^{\cdot}[n] & \xrightarrow{\tau_{m,n}} & A^{\cdot} \otimes B^{\cdot}[m+n] \\
\downarrow \sigma & & \downarrow (-1)^{mn}\sigma[m+n] \\
B^{\cdot}[n] \otimes A^{\cdot}[m] & \xrightarrow{\tau_{n,m}} & B^{\cdot} \otimes A^{\cdot}[m+n]
\end{array}
$$

Associativity. The usual associatitivy constraint for the tensor product extends without difficulties to complexes. Notice the following commutative diagram

8.8

$$
\begin{array}{ccc}
A^{\cdot}[m] \otimes B^{\cdot}[n] \otimes C^{\cdot}[r] & \xrightarrow{1 \otimes \tau_{n,r}} & A^{\cdot}[m] \otimes (B^{\cdot} \otimes C^{\cdot})[n+r] \\
\downarrow \tau_{m,n} \otimes 1 & & \downarrow \tau_{m,n+r} \\
(A^{\cdot} \otimes B^{\cdot})[m+n] \otimes C^{\cdot}[r] & \xrightarrow{\tau_{m+n,r}} & (A^{\cdot} \otimes B^{\cdot} \otimes C^{\cdot})[m+n+r]
\end{array}
$$

Homotopy 8.9. Given homotopic morphisms $f, g: A^{\cdot} \to B^{\cdot}$ and homotopic morphisms $u, v: P^{\cdot} \to Q^{\cdot}$. Then

$$f \otimes u, \quad g \otimes v \quad : A^{\cdot} \otimes P^{\cdot} \to B^{\cdot} \otimes Q^{\cdot}$$

are homotopic morphisms.

Proof. Use the general construction I.11.6.

Q.E.D.

Triangles 8.10. Given a triangle in $K^{+}(X,k)$

Then for any complex of sheaves $P^{\cdot}$,

is a triangle.

Proof. We may assume that the sequence

$$0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0$$

is a chainwise split exact sequence which comes equipped with a chainwise splitting $s: C \to B$ such that $fh = \partial s - s\partial$. This

gives rise to a chainwise splitting $S = s \otimes 1$ of the sequence

$$0 \longrightarrow A^{\cdot \cdot} \otimes P^{\cdot} \xrightarrow{f \otimes 1} B^{\cdot \cdot} \otimes P^{\cdot} \xrightarrow{g \otimes 1} C^{\cdot \cdot} \otimes P^{\cdot} \longrightarrow 0$$

$$(\partial S - S \partial)(x^p \otimes y^q) = \partial(sx^p \otimes y^q) - S(\partial x^p \otimes y^q + (-1)^p x^p \otimes \partial y^q) =$$

$$\partial s x^p \otimes y^q + (-1)^p s x^p \otimes \partial y^q - s \partial x^p \otimes y^q - (-1)^p s x^p \otimes y^q =$$

$$((\partial s - s \partial) x^p) \otimes y^q = (f \otimes 1)(h \otimes 1)(x^p \otimes y^q)$$

Q.E.D.

Let us call special attention to the isomorphism

8.11 $\qquad \tau_{0,n}: M^{\cdot \cdot} \otimes \underset{\sim}{k}[n] \longrightarrow M^{\cdot}[n]$

explicitly given by $\tau_{0,n}(x^p \otimes 1[n]) = (-1)^{np} x^p$. The effect of tensoring the triangle 8.10 from the right with $\underset{\sim}{k}[n]$ is the triangle

The tensor product makes $K^+(X,k)$ into a category with an internal tensor product satisfying the pentagon and hexagon axiom of Mac Lane (1) 158(5), 180(3).

II.9 Local cohomology

In this section we consider the inclusion $i: A \to X$ of closed sets of an arbitrary topological space. The object of study is the behaviour of a sheaf in a neighbourhood of A.

Definition 9.1. For a sheaf F on X we define

$$\Gamma_A(X,F) = \{s \in \Gamma(X,F) \mid \text{Supp}(s) \subseteq A\}$$

the sections of F with support in A.

This is a left exact functor. The p'th derived functor of $\Gamma_A(X,-)$ evaluated on a sheaf F is denoted $H_A^p(X,F)$ and is called the p'th local cohomology group with support in A and coefficients F.

Proposition 9.2. Let U denote the complement of the closed subset A of X. A sheaf F on X gives rise to a long exact sequence

$$\to H_A^p(X,F) \to H^p(X,F) \to H^p(U,F) \to$$

Proof. Consider the tautological exact sequence

$$0 \to \Gamma_A(X,F) \to \Gamma(X,F) \to \Gamma(U,F)$$

and note that r_{UX} is surjective in case F is a flabby sheaf. Thus an injective resolution $F \to I^{\cdot}$ gives rise to a short exact sequence of complexes

$$0 \longrightarrow \Gamma_A(X,I\text{''}) \longrightarrow \Gamma(X,I\text{''}) \longrightarrow \Gamma(U,I\text{''}) \longrightarrow 0$$

which gives rise to a cohomology sequence.

<div align="right">Q.E.D.</div>

Corollary 9.3. For a flabby sheaf F on X

$$H_A^p(X,F) = 0 \qquad ; p > 0$$

Suppose we have given two closed subsets A and B of X. For a sheaf F on X we deduce the following commutative exact diagram where U and V denotes the complements of A and B

9.4

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{A \cap B}(X,F) & \longrightarrow & \Gamma_A(X,F) & \longrightarrow & \Gamma_{A \cap V}(V,F) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_B(X,F) & \longrightarrow & \Gamma(X,F) & \longrightarrow & \Gamma(V,F) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_{U \cap B}(U,F) & \longrightarrow & \Gamma(U,F) & \longrightarrow & \Gamma(U \cap V,F) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

In case F is flabby this diagram gives a 3×3-diagram consisting of 6-exact sequences as it follows by inspection. If we apply the sequence above to a flabby resolution of F we get six long exact sequences related in various ways. Let us record the one resulting from the 1. row

Excision exact sequence 9.5.

$$\rightarrow H_{A-V}^p(X,F) \rightarrow H_A^p(X,F) \rightarrow H_{A \cap V}^p(V,F) \rightarrow$$

here A is a closed subset of X and V is an open subset. In case V contains A this degenerates into an

Excision isomorphism 9.6.

$$H_A^p(X,F) \overset{\sim}{\rightarrow} H_A^p(V,F) \qquad ; \qquad p \in \mathbb{Z}$$

Functoriality 9.7. Let A be a closed subset of X and B a closed subset of Y. A continuous map $f\colon X \rightarrow Y$ with $f(X-A) \subseteq Y-B$ will for any sheaf G on Y induce a commutative ladder, where $V = Y - B$ and $U = X - A$

$$
\begin{array}{ccccccccc}
\longrightarrow & H_B^p(Y,G) & \longrightarrow & H^p(Y,G) & \longrightarrow & H^p(V,G) & \longrightarrow & H_B^{p+1}(Y,G) & \rightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_A^p(X,f*G) & \rightarrow & H^p(X,f*G) & \rightarrow & H^p(U,f*G) & \rightarrow & H_A^{p+1}(X,f*G) & \rightarrow
\end{array}
$$

Proof. Let $G \rightarrow J\text{''}$ and $f*J\text{''} \rightarrow I\text{''}$ be injective resolutions. This gives rise to a commutative exact diagram of complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_B(Y,J\text{''}) & \longrightarrow & \Gamma(Y,J\text{''}) & \longrightarrow & \Gamma(V,J\text{''}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_A(X,I\text{''}) & \longrightarrow & \Gamma(X,I\text{''}) & \longrightarrow & \Gamma(U,I\text{''}) & \longrightarrow & 0
\end{array}
$$

from which the result follows.

<div align="right">Q.E.D.</div>

Interpretation in terms of Ext

Let us now pass to the framwork of k-sheaves, where k is a fixed commutative ring. As before, we consider the inclusion $i: A \to X$ of a closed subspace and the inclusion $j: U \to X$ of the complement. Recall the exact sequence 6.11

$$0 \longrightarrow j_! \underset{\sim}{k} \longrightarrow \underset{\sim}{k} \longrightarrow i_* \underset{\sim}{k} \longrightarrow 0$$

For a k-sheaf F this gives an exact sequence

9.8 $$\to \text{Ext}^p(i_* \underset{\sim}{k}, F) \to \text{Ext}^p(\underset{\sim}{k}, F) \to \text{Ext}^p(j_! \underset{\sim}{k}, F) \to$$

which we may identify with the sequence 9.2 as it follows from 7.5 and the proof of 7.3. - The formula

$$H^{\cdot}(X,k) = \text{Ext}^{\cdot}(\underset{\sim}{k}, \underset{\sim}{k})$$

allows us to introduce cup product in local cohomology according to the general principles of I.8

9.9 $$\alpha \cup \beta \in H_A^p(X, \underset{\sim}{k}) \quad ; \quad \alpha \in H^p(X, \underset{\sim}{k}), \beta \in H_A^q(X, \underset{\sim}{k})$$

Let us notice that the boudary operator of the sequence

$$\to H^q(X, \underset{\sim}{k}) \to H^q(U, \underset{\sim}{k}) \overset{\partial}{\to} H_A^{q+1}(X, \underset{\sim}{k}) \to$$

satisfies the formula, compare I.8.5

9.10 $$\partial(\alpha \cup \beta) = (-1)^p \alpha \cup \partial \beta \quad ; \alpha \in H^p(X, \underset{\sim}{k}), \quad \beta \in H^q(U, \underset{\sim}{k})$$

The same type of formula is valid for the excision sequence 9.5. To see this, remark that the diagram 9.4 can be derived from 6.17. - Let us take the opportunity to mention that the Mayer-Vietoris sequence 5.10 can be derived from 6.18 again with a formula like 9.10 as a consequence. In a similar way we derive from 6.18 a

Mayer-Vietoris sequence 9.11.

$$\longrightarrow H_A^q(X,k) \oplus H_B^q(X,k) \longrightarrow H_{A \cup B}^q(X,k) \overset{\partial}{\longrightarrow} H_{A \cap B}^{q+1}(X,k) \longrightarrow$$

again satisfying a relation like 9.10.

The extraordinary cup product

Let $i: Z \to X$ denote the inclusion of a closed subspace. The adjunction morphism $\underset{\sim}{k} \to i_* \underset{\sim}{k}$ induces an isomorphism

9.12 $$\text{Ext}^{\cdot}(i_* \underset{\sim}{k}, i_* \underset{\sim}{k}) \overset{\approx}{\to} H^{\cdot}(Z, \underset{\sim}{k})$$

Proof. Let $\underset{\sim}{k} \to I^{\cdot}$ denote an injective resolution in $\text{Sh}(Z,k)$. We have by adjunction 4.8

$$I(Z, I^{\cdot}) = \text{Hom}(i^* \underset{\sim}{k}, I^{\cdot}) = \text{Hom}(\underset{\sim}{k}, i_* I^{\cdot}) = \text{Hom}(i_* \underset{\sim}{k}, i_* I^{\cdot})$$

where we have used Lemma 5.5. Conclusion by the fact that $i_* I^{\cdot}$ is an injective resolution of $i_* \underset{\sim}{k}$ in $\text{Sh}(X,k)$.

Q.E.D.

If we combine formula 9.12 with the fact that

9.13 $$\mathrm{Ext}^{\cdot}(i_*\underset{\sim}{k}, F) = H_Z^{\cdot}(X, F)$$

we can introduce the <u>extraordinary cup product</u>

9.14 $$\beta \cup \gamma \in H_Z^{q+r}(X, F) \qquad ; \ \beta \in H_Z^{q}(X, F), \ \gamma \in H^r(Z, k)$$

The two cup products are interrelated by the following formulas.
We let $r: H_Z^{\cdot}(X, k) \to H^{\cdot}(X, k)$ denote the restriction map

9.15 $$\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$$

$$\alpha \in H^p(X, k), \ \beta \in H_Z^{q}(X, k), \ \gamma \in H^r(Z, k)$$

9.16 $$r(\alpha) \cup \beta = \alpha \cup i^*r(\beta)$$

$$\alpha \in H_Z^p(X, k), \ \beta \in H_Z^{q}(X, k)$$

9.17 $$\alpha \cup \beta = (-1)^{pq}\beta \cup i^*(\alpha)$$

$$\alpha \in H^p(X, k), \ \beta \in H_Z^{q}(X, k)$$

The first formula is obvious, the second formula will be veri-
fied below while the third formula will be proved in the next
section.

<u>Verification of 9.16</u>. We shall work in the category $D^+(X, k)$.
Let $I^{\cdot}$ be an injective resolution of $\underset{\sim}{k}$, $J^{\cdot}$ an injective re-
solution of $i_*\underset{\sim}{k}$ and let $\pi: I^{\cdot} \to J^{\cdot}$ represent the adjunction
morphism $\underset{\sim}{k} \to i_*\underset{\sim}{k}$. According to 9.12 we have that

$$[\pi, 1]: [J^{\cdot}, J^{\cdot}[p]] \longrightarrow [I^{\cdot}, J[p]] \qquad ; \ p \in \mathbb{Z}$$

is an isomorphism. The map $i^*: H^p(X, k) \to H^p(Z, k)$ can be described
as the composite

$$[I^{\cdot}, I^{\cdot}[p]] \xrightarrow{\ [1, \pi]\ } [I^{\cdot}, J^{\cdot}[p]] \xrightarrow{\ [\pi, 1]^{-1}\ } [J^{\cdot}, J^{\cdot}[p]]$$

while the restriction $r: H_Z^p(X, k) \to H^p(X, k)$ is represented

$$[\pi, 1]: [J^{\cdot}, I^{\cdot}[p]] \longrightarrow [I^{\cdot}, I^{\cdot}[p]]$$

Consider $\alpha \in [J^{\cdot}, I^{\cdot}[p]]$ and $\beta \in [J^{\cdot}, I^{\cdot}[q]]$. We have $r(\alpha) = \alpha\pi$,
$r(\beta) = \beta\pi$ and $i^*r(\alpha) = \pi\alpha$. This gives

$$\beta \cup i^*r(\alpha) = \beta[p](\pi\alpha) = (\beta\pi[p])\alpha = r(\beta) \cup \alpha$$

which is the desired result.

$$\text{Q.E.D.}$$

II.10 Cross products

In this section we shall perform a considerable extension of the cup product introduced in the previous section. We shall work with a fixed commutative ring k. Given a topological space X, two closed subspaces A and B and k-sheaves E and F on X. We shall construct a <u>cup product</u>

10.1 $\qquad \alpha \cup \beta \in H_{A \cap B}^{p+q}(X, E \otimes F) \qquad ; \alpha \in H_A^p(X,E), \beta \in H_B^q(X,F)$

Let us list some of its properties. First of all the following diagram is <u>commutative</u>, here U denotes the complement of A

10.2
$$
\begin{array}{ccccccc}
H_A^p(X,E) & \longrightarrow & H^p(X,E) & \longrightarrow & H^p(U,E) & \xrightarrow{\ \partial\ } & H_A^{p+1}(X,E) \\
\downarrow \cup\beta & & \downarrow \cup\beta & & \downarrow \cup\beta & & \downarrow \cup\beta \\
H_{A \cap B}^p(X,E \otimes F) & \to & H_B^p(X,E \otimes F) & \to & H_{U \cap B}^p(X,E \otimes F) & \xrightarrow{\partial} & H_{A \cap B}^{p+1}(X,E \otimes F)
\end{array}
$$

The cup product is <u>anticommutative</u>

10.3 $\qquad \alpha \cup \beta = (-1)^{pq} \beta \cup \alpha \qquad ; \quad \alpha \in H_A^p(X,E), \quad \beta \in H_B^q(X,F)$

In the sense that the canonical symmetry $E \otimes F \xrightarrow{\sim} F \otimes E$ transforms $\alpha \cup \beta$ into $(-1)^{pq} \beta \cup \alpha$. - Given a continuous map $f: W \to X$ and closed subsets C and D of W with $f(W-C) \subseteq X-A$ and $f(W-D) \subseteq X-B$, then

10.4 $\qquad f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta \qquad ; \quad \alpha \in H_A^p(X,k), \beta \in H_B^q(B,k)$

10.5 $\qquad \alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$

$$\alpha \in H_A^p(X,E), \ \beta \in H_B^q(X,F), \ \gamma \in H_C^r(X,G)$$

Given a sequence of k-sheaves

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

which is <u>pointwise split exact</u>, i.e. whose localization at each point of X is a split exact sequence of k-modules. For any k-sheaf D on X we have

10.6 $\qquad \partial(\alpha \cup \beta) = (\partial \alpha) \cup \beta \qquad ; \ \alpha \in H_A^p(X,G), \ \beta \in H_B^q(X,D)$

Cross products

Let A be a closed subset of the topolocial space X and B a closed subset of the topological space Y. For a sheaf F on X and a sheaf G on Y we consider

$$
\begin{array}{ccc}
 & X \times Y & \\
 \swarrow^{p} & & \searrow^{q} \\
X & & Y
\end{array}
$$

$$p^*: H_A^\cdot(X,F) \to H_{A \times Y}^\cdot(X \times Y, p^*F) \quad q^*: H_B^\cdot(Y,G) \to H_{X \times B}^\cdot(X \times Y, q^*G)$$

We can now define a cohomology class, the <u>cross product</u>

$$\alpha \times \beta \in H_{A \times B}^{p+q}(X \times Y, p^*F \otimes q^*G) \quad ; \quad \alpha \in H_A^p(X,F), \quad \beta \in H_B^q(Y,G)$$

by the following formula involving the cup product

10.7 $\qquad \alpha \times \beta = p^*\alpha \cup q^*\beta$

In case Y = X we let $\Delta: X \to X \times X$ denote the diagonal map, and

we deduce from 10.4 that

10.8 $\qquad \alpha \cup \beta = \Delta^*(\alpha \times \beta) \qquad ; \qquad \alpha \in H_A^p(X,F), \ \beta \in H_B^q(Y,G)$

We shall now construct the cup product. The basic idea is to construct a category with an internal tensor product in which we can represent cohomology classes as arrows and cup product as tensor product of arrows.

An auxilary category

Let X denote a toplogical space and k a commutative ring. The category $K^+(X,k)$ comes equipped with an internal tensor product obeying certain rules II.8. This is certainly not the case for $D^+(X,k)$ being the homotopy category of injective sheaves. Alternatively $D^+(X,k)$ can be obtained from $K^+(X,k)$ by inverting all quasi-isomorphisms XI.2.6. This is a perfect description when k is a field since the system of quasi-isomorphisms is stable under $\otimes$. In general we shall invert a smaller class of quasi-isomorphisms namely the pointwise homotopy equivalences, i.e. morphisms $f: R^\cdot \to S^\cdot$ in $K^+(X,k)$ such that for each point $x \in X$, $f_x: R_x^\cdot \to S_x^\cdot$ is a homotopy equivalence of complexes of k-modules. The class of pointwise homotopy equivalences is stable under tensor product, moreover

Lemma 10.9. For any bounded below complex of k-sheaves $F^\cdot$ there exists a pointwise homotopy equivalence $F^\cdot \to S^\cdot$ where $S^\cdot$ is bounded below complex of flabby k-sheaves.

proof. Recall from 3.6 the Godement construction which to a sheaf F assigns the Godement resolution $C^\cdot F$. The construction is functorial, thus we may consider $C^\cdot F^\cdot$ as a double complex I.11. Let us consider the double complex $F^{\cdot\cdot}$ given by

$$F^{pq} = \begin{cases} F^q & \text{for} \quad p = 0 \\ 0 & \text{for} \quad p \neq 0 \end{cases}$$

The Godement construction furnishes a morphism $F^{\cdot\cdot} \to C^\cdot(F^\cdot)$ of double complexes which induces a morphism of single complexes

$$F^\cdot \longrightarrow \text{Tot}^\cdot(C^\cdot(F^\cdot))$$

This is a pointwise homotopy equivalence: In fact for $x \in X$ a homotopy inverse $C^\cdot(F^\cdot)_x \to F^{\cdot\cdot}_x$ is furnished by the canonical local retractions, compare the remarks after 3.10. Finally use the formula I.11.6 to construct the needed homotopies.

$$\text{Q.E.D.}$$

The class P of pointwise homotopy equivalences satisfies the condition FR 1-5 of Verdier compare XI.1,2,6. The localized category will be denoted

$$G^+(X,k) = P^{-1}K^+(X,k)$$

By the fact that P is stable under tensor product, the category $G^+(X,k)$ comes equipped with an internal tensor product. The following two lemmas reveal the role $G^+(X,k)$ is going to play.

Lemma 10.10. Let D be a k-sheaf with the property that $Ext_k^p(D,S) = 0$ for all $p \geq 1$ and all flabby sheaves S. Then for any k-sheaf F and any $p \in \mathbb{Z}$

$$Ext_k^p(D,F) = Hom(D,F[p])$$

where Hom is calculated in $G^+(X,k)$.

Proof. Throughout the proof the functor Hom is calculated in $G^+(X,k)$. Let us first prove that for a complex of flabby sheaves S" in $K^+(X,k)$ then

$$[D,S"] = Hom(D,S")$$

Suppose first that $f \in [D,S"]$ represents the morphism zero $D \to S"$ in $G^+(X,k)$. This means that we can find $s: S" \to T"$ in P such that $sf = 0$ in $K^+(X,k)$. By Lemma 10.9 we may assume that T" is a complex of flabby sheaves. The quasi-isomorphism $s: S" \to T"$ is a morphism between complexes of Hom(D,-) acyclic objects. Thus we conclude from the acyclicity theorem I.7.5 that f is zero.

To see that our map is surjective notice that the general element of Hom(D,S") is a fraction $D \overset{f}{\to} T \overset{s}{\leftarrow} S$ where s is in P. Again by 10.9 we may assume that T consists of flabby sheaves. We conclude from the acylicity theorem that f may be factored through s.

To prove the stated result choose $s: F \to S"$ in P where S" is a complex of flabby sheaves and choose an injective resolution $S" \to I"$. We have

$$Hom(D,F[p]) \overset{\sim}{\to} Hom(D,S"[p]) = [D,S"[p]] = [D,I"[p]]$$

The last group is $Ext_k^p(D,F)$.

Q.E.D.

Lemma 10.11. Let $h: W \to X$ denote the inclusion of a locally closed subspace. Then

$$Ext^p(h_*\underset{\sim}{k},S) = 0 \qquad , p \geq 1$$

for any flabby k-sheaf S.

Proof. Let us write $W = A \cap V$ where A is a closed subset of and V an open subset of X. If we let U denote the complement of A in X we have an exact sequence of k-sheaves, compare the first column of 6.17.

$$0 \longrightarrow \underset{\sim}{k}_{U \cap V} \longrightarrow \underset{\sim}{k}_V \longrightarrow \underset{\sim}{k}_W \longrightarrow 0$$

from which we derive the isomorphism, compare 9.4

$$Ext_k"(h_*\underset{\sim}{k},F) = H_{A \cap V}"(V,F)$$

The result follows from 9.3.

Q.E.D.

Let us record an important canonical isomorphism. For locally closed subsets A and B of X we have

10.12 $$\underset{\sim}{k}_{A \cap B} = \underset{\sim}{k}_A \otimes \underset{\sim}{k}_A$$

Let us now consider closed subsets A and B of X and k-sheaves E and F. Cohomology classes $\alpha \in H_A^p(X,E)$ and $\beta \in H_B^q(X,F)$ may be viewed as morphisms in $G^+(X,k)$

$$\alpha: \underset{\sim}{k}_A \to E[p] \qquad \beta: \underset{\sim}{k}_B \to F[q]$$

We then define the <u>cup product</u> by the formula

10.13 $\qquad\qquad \alpha \cup \beta = \tau_{p,q} \circ \alpha \otimes \beta$

taking the isomorphism 10.12 into account.

We shall now proceed to establish the formulas 10.1-6. The basic principle is simply to write up a suitable commutative diagram in the derived category $G^+(X,k)$.

Proof of 10.3. According to 8.7 we have the following commutative diagram

$$
\begin{array}{ccccccc}
\underset{\sim}{k}_{A\cap B} & \xrightarrow{\ \sim\ } & \underset{\sim}{k}_A \otimes \underset{\sim}{k}_B & \dashrightarrow & E[p]\otimes F[q] & \xrightarrow{\tau_{p,q}} & E\otimes F[p+q] \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \quad (-1)^{pq} & & \downarrow{\scriptstyle\sigma[p+q]} \\
\underset{\sim}{k}_{B\cap A} & \longrightarrow & \underset{\sim}{k}_B \otimes \underset{\sim}{k}_A & \longrightarrow & F[q]\otimes E[p] & \xrightarrow{\tau_{q,p}} & F\otimes E[p+q]
\end{array}
$$

The <u>cup product</u> extends 9.9 as it follows from the following commutative diagram

$$
\begin{array}{ccccc}
\underset{\sim}{k}\otimes \underset{\sim}{k}_A & \xrightarrow{1\otimes\beta} & \underset{\sim}{k}\otimes \underset{\sim}{k}[q] & \xrightarrow{\alpha\otimes 1} & \underset{\sim}{k}[p]\otimes \underset{\sim}{k}[q] \\
\downarrow & & \downarrow{\scriptstyle\tau_{0,q}} & & \downarrow{\scriptstyle\tau_{p,q}} \\
\underset{\sim}{k}_A & \xrightarrow{\ \beta\ } & \underset{\sim}{k}[q] & \xrightarrow{\alpha[q]} & \underset{\sim}{k}[p+q]
\end{array}
$$

Proof of formula 9.17. Let us first establish the following formula, where $i: A \to X$ denotes the inclusion

$$\beta \cup i^*\alpha = \beta \cup \alpha \quad ; \alpha \in H^p(X,k), \beta \in H_A^q(X,k)$$

here the left hand side is defined in 9.14 and the right hand side is defined in 10.13. This can be read off the commutative diagram

$$
\begin{array}{ccccc}
\underset{\sim}{k}_A\otimes \underset{\sim}{k} & \xrightarrow{1\otimes\alpha} & \underset{\sim}{k}_A\otimes k[p] & \xrightarrow{\beta\otimes 1} & \underset{\sim}{k}[q]\otimes \underset{\sim}{k}[p] \\
\downarrow & & \downarrow{\scriptstyle\tau_{0,p}} & & \downarrow{\scriptstyle\tau_{p,q}} \\
\underset{\sim}{k}_A & \xrightarrow{i_*i^*\alpha} & \underset{\sim}{k}_A[p] & \xrightarrow{\beta[p]} & \underset{\sim}{k}[p+q]
\end{array}
$$

Formula 9.17 is now a consequence of 10.3.

Proof of 10.6. The pointwise, split exact sequence

$$0 \to D \to F \to G \to 0$$

gives rise to a triangle in $G^+(X,k)$. Let $\partial: G \to E[1]$ denote the third side of the triangle. We leave it to the reader to check that the accompanying diagram is commutative.

$$
\begin{array}{ccc}
\underset{\sim}{k}_A \otimes \underset{\sim}{k}_B & \xrightarrow{\ 1\ } & \underset{\sim}{k}_A\otimes \underset{\sim}{k}_B \\
\downarrow{\scriptstyle\alpha\otimes 1} & \quad \partial[p]\otimes 1 & \downarrow{\scriptstyle\partial[p]\alpha\otimes 1} \\
G[p]\otimes \underset{\sim}{k}_B & \longrightarrow & E[p+1]\otimes \underset{\sim}{k}_B \\
\downarrow{\scriptstyle 1\otimes\beta} & & \downarrow{\scriptstyle 1\otimes\beta} \\
G[p]\otimes D[q] & \xrightarrow{\partial[p]\otimes 1} & E[p+1]\otimes D[q] \\
\downarrow{\scriptstyle\tau_{p,q}} & & \downarrow{\scriptstyle\tau_{p+1,q}} \\
G\otimes D[p+q] & \xrightarrow{\partial\otimes 1[p+q]} & E\otimes D[p+q+1]
\end{array}
$$

<u>Proof of 10.2.</u> The short exact sequence

$$0 \longrightarrow \underset{\sim}{k}_U \longrightarrow \underset{\sim}{k} \longrightarrow \underset{\sim}{k}_A \longrightarrow 0$$

is pointwise split exact. Let $\partial: \underset{\sim}{k}_A \to \underset{\sim}{k}_U[1]$ denote the characteristic arrow. If we tensor the sequence above with $\underset{\sim}{k}_B$ we get the sequence

$$a \longrightarrow \underset{\sim}{k}_{U\cap B} \longrightarrow \underset{\sim}{k}_B \longrightarrow \underset{\sim}{k}_{A\cap B} \longrightarrow 0$$

The characteristic arrow of this sequence may be identified with
$\partial \otimes 1: \underset{\sim}{k}_A \otimes k_B \to \underset{\sim}{k}_U[1] \otimes k_B.$

$$
\begin{array}{ccccc}
\underset{\sim}{k}_A \otimes \underset{\sim}{k}_B & \xrightarrow{\partial \otimes 1} & \underset{\sim}{k}_U[1] \otimes \underset{\sim}{k}_B & \xrightarrow{\ulcorner 1,0} & \underset{\sim}{k}_U \otimes \underset{\sim}{k}_B[1] \\
\downarrow{\scriptstyle 1 \otimes \beta} & & \downarrow{\scriptstyle 1 \otimes \beta} & & \downarrow{\scriptstyle 1 \otimes \beta[1]} \\
\underset{\sim}{k}_A \otimes F[q] & \xrightarrow{\partial \otimes 1} & \underset{\sim}{k}_U[1] \otimes F[q] & \xrightarrow{\ulcorner 1,0} & \underset{\sim}{k}_U \otimes F[q][1] \\
\downarrow{\scriptstyle \alpha[1] \partial \otimes 1} & & \downarrow{\scriptstyle \alpha[1] \otimes 1} & & \downarrow{\scriptstyle \alpha \otimes 1[1]} \\
F[p+1] \otimes F[q] & \xrightarrow{1} & E[p+1] \otimes F[q] & \xrightarrow{\ulcorner 1,0} & E[p] \otimes F[q][1] \\
\downarrow{\scriptstyle \ulcorner p+1,q} & & \downarrow{\scriptstyle \ulcorner p+1,q} & & \downarrow{\scriptstyle \ulcorner p,q[1]} \\
E[p+q+1] & \xrightarrow{1} & E[p+q+1] & \xrightarrow{(-1)^q} & E[p+q+1]
\end{array}
$$

From the commutative diagram above we deduce

$$(\partial \alpha) \cup \beta = (-1)^q \partial (\alpha \cup \beta) \quad ; \quad \alpha \in H_A{}^p(X,E), \ \beta \in H_B{}^q(Y,F)$$

Multiply this formula with $(-1)^{p+1}$ to get the desired result, compare I.8.5.

<div align="right">Q.E.D.</div>

<u>Proposition 10.14.</u> Let $\underset{\sim}{k} \to C^{\cdot}$ be an acyclic resolution in $Sh(X,k)$ which is a pointwise homotopy equivalence. Given a morphism of complexes $\wedge: C^{\cdot} \otimes C^{\cdot} \to C^{\cdot}$ which makes the accompanying diagram commutative. Given

$$
\begin{array}{ccc}
\underset{\sim}{k} \otimes \underset{\sim}{k} & \longrightarrow & \underset{\sim}{k} \\
\downarrow & & \downarrow \\
C^{\cdot} \otimes C^{\cdot} & \xrightarrow{\wedge} & C^{\cdot}
\end{array}
$$

$$a \in \Gamma(X,C^p) \quad \text{with} \quad \partial a = 0 \quad \text{representing} \quad \alpha \in H^p(X,k)$$
$$b \in \Gamma(X,C^q) \quad \text{with} \quad \partial b = 0 \quad \text{representing} \quad \beta \in H^q(X,k)$$

then $a \wedge b$ represents $\alpha \cup \beta$

<u>Proof.</u> Consider the following commutative diagram in $G^+(X,k)$

$$
\begin{array}{ccccccc}
\underset{\sim}{k} & \xrightarrow{\alpha \otimes \beta} & \underset{\sim}{k}[p] \otimes \underset{\sim}{k}[q] & \xrightarrow{\ulcorner p,q} & \underset{\sim}{k} \otimes \underset{\sim}{k}[p+q] & \longrightarrow & \underset{\sim}{k}[p+q] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\underset{\sim}{k} & \xrightarrow{a \otimes b} & C^{\cdot}[p] \otimes C^{\cdot}[q] & \xrightarrow{\ulcorner p,q} & C^{\cdot} \otimes C^{\cdot}[p+q] & \xrightarrow{\wedge[p+q]} & C^{\cdot}[p+q]
\end{array}
$$

Notice $\ulcorner_{p,q}(a \otimes b) = (-1)^{0q} a \otimes b = a \otimes b$ and the result follows.

<div align="right">Q.E.D.</div>

II.11 Flat sheaves

Let k denote a fixed commutative ring. A k-module F is called _flat_ if any exact sequence

$$0 \to R \to S \to T \to 0$$

of k-modules is transformed into an exact sequence

$$0 \longrightarrow R \otimes F \longrightarrow S \otimes F \longrightarrow T \otimes F \longrightarrow 0$$

Let us notice that a projective k-module is flat, and that any direct limit of flat modules is flat as it follows from 8.1. Let us mention a theorem of D. Lazard to effect that any flat module can be realized as a direct limit of finitely generated free modules, Bourbaki (2) A.X.14. The reader may use this to simplify some of the proofs in these notes.

Lemma 11.1. Given an exact sequence

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots P_0 \longrightarrow F \longrightarrow 0$$

of k-modules. If $F, P_0, \ldots, P_{n-1}$ are flat modules then P_n is a flat module.

Proof. By splitting the sequence into short exact sequences one sees that it suffices to treat the case $n = 1$. This case is easy to treat by means of I.1.3.

Q.E.D.

Theorem 11.2. Let $F.$ be a bounded above complex of flat k-modules and $R.$ a bounded above complex of k-modules with $H.(R.) = 0.$ Then $H.(F. \otimes R.) = 0.$

Proof. We shall change to the "upper dot" notation in order not to mess up the truncation notation I.5.7. - It suffices to prove the following statement for all $n \in \mathbb{Z}$:

"Any complex $R^{\cdot}$ with $H^{\cdot}(R^{\cdot}) = 0$ and $R^i = 0$ for $i > n$ has $H^0(R^{\cdot} \otimes F^{\cdot}) = 0$". - This is done by increasing induction on $n,$ noticing that the statement is obvious for $n \ll 0.$ In order to prove "$n \Rightarrow n+1$" consider the exact sequence

$$0 \longrightarrow \tau_{\leq n} R^{\cdot} \longrightarrow R^{\cdot} \longrightarrow \tau_{\geq n} R^{\cdot} \longrightarrow 0$$

The resulting short exact sequence of complexes

$$0 \longrightarrow (\tau_{\leq n} R^{\cdot}) \otimes F^{\cdot} \longrightarrow R^{\cdot} \otimes F^{\cdot} \longrightarrow (\tau_{\geq n} R^{\cdot}) \otimes F^{\cdot} \longrightarrow 0$$

gives an exact sequence

$$H^0((\tau_{\leq n} R^{\cdot}) \otimes F^{\cdot}) \longrightarrow H^0(R^{\cdot} \otimes F^{\cdot}) \longrightarrow H^0(\tau_{\geq n} R^{\cdot}) \otimes F^{\cdot})$$

from which the result follows.

Q.E.D.

Corollary 11.3. Let $R. \to S.$ be a quasi-isomorphism of bounded above complexes of k-modules and $F.$ a bounded above complex of flat k-modules. Then

$$R. \otimes F. \longrightarrow S. \otimes F.$$

is a quasi-isomorphism.

Proof. Consider the mapping cone of $R. \to S.$ and combine
11.2 with 8.10.

Q.E.D.

Corollary 11.4. Let $F.$ be a bounded above complex of flat
modules with $H.(F.) = 0.$ Then for any bounded above complex $R.$
we have $H.(F.\otimes R.) = 0.$

Proof. Choose a projective resolution $P. \to R.$ and notice
that $P.\otimes F. \to R.\otimes F.$ is a quasi-isomorphism 11.3 and apply 11.2
to the complex $R.\otimes F.$

Q.E.D.

Let us briefly discuss the functors $Tor_i.$ Given k-modules
M and $N.$ Choose projective resolutions $P. \to M$ and $Q. \to N$
and consider the following two quasi-isomorphisms, 11.3

$$P.\otimes N \Leftarrow P.\otimes Q. \Rightarrow M\otimes Q.$$

This makes sense to the following definition

11.5 $\qquad Tor_i(M,N) = H_i(P.\otimes N) = H_i(M\otimes Q.) = H_i(P.\otimes Q.)$

It follows from 11.4 that we can use resolutions $P.$
and $Q.$ of flat modules in the definition 11.5.

Let us now consider a topological space $X.$ We shall
generalize the discussion above to $K^-(X,k),$ the homotopy cate-
gory of bounded below complexes of k-sheaves on $X.$

Definition 11.6. A k-sheaf F on X is called flat if
for every point x of X the stalk F_x is a flat k-module.

Proposition 11.7. Let $R. \to S.$ be a quasi-isomorphism in
$K^-(X,k).$ For any complex $F.$ of flat k-sheaves in $K^-(X,k)$

$$R.\otimes F. \longrightarrow S.\otimes F.$$

is a quasi-isomorphism.

Proof. Follows from 11.3 by localization 8.3.

Q.E.D.

Theorem 11.8. For any $R.$ in $K^-(X,k)$ there exists a quasi-
isomorphism $F. \to R.$ where $F.$ is a bounded above complex
of flat k-sheaves (a flat resolution of $R.$).

Proof. For any k-sheaf R there exists an epimorphism $F \to R$
where F is a flat k-sheaf, 7.4. In general, follow a procedure
dual to the one used in the proof of I.6.1 see also I.9.

Q.E.D.

Proposition 11.9. Given quasi-isomorphisms $E_\cdot \to R_\cdot$ and
$F_\cdot \to R_\cdot$ in $K^-(X,k)$ where $P_\cdot$ and $Q_\cdot$ are complexes of
flat sheaves. Then there exists a
commutative diagram consisting of
quasi-isomorphisms in $K^-(X,k)$
where $D_\cdot$ is a complex of flat
sheaves.

$$
\begin{array}{ccc}
D_\cdot & \longrightarrow & E_\cdot \\
\downarrow & & \downarrow \\
F_\cdot & \longrightarrow & P_\cdot
\end{array}
$$

Proof. According to the proof of XI.2.4 we can construct
such a diagram consisting of quasi-isomorphisms, but no con-
ditions on $D_\cdot$. Combine this with 11.8 to get the result.

$$Q.E.D.$$

Definition 11.10. Let $P_\cdot$ and $Q_\cdot$ be complexes in $K^-(X,k)$.
Choose flat resolutions $E_\cdot \to P_\cdot$ and $F_\cdot \to G_\cdot$. We put

$$P_\cdot \overset{L}{\otimes} G_\cdot = E_\cdot \otimes F_\cdot$$

and define the Tor-sheaves

$$Tor_i^k(P_\cdot,Q_\cdot) = H_i(E_\cdot \otimes_k F_\cdot)$$

The constructions above may be interpreted in terms of the
derived category $D^-(X,k)$ introduced in XI.2.6°. According to
11.8 the derived category $D^-(X,k)$ is equivalent to the category
defined from the homotopy category of bounded above complexes of
flat sheaves by inverting all quasi-isomorphisms, see XI.6. -
This makes it evident that $D^-(X,k)$ comes equipped with an internal
tensor product, which allow us to interpret the formulas from
II.8 in $D^-(X,k)$.

II.12 $Hom(E,F)$

Let X denote a topological space and k a commutative
ring. We shall work with the category $Sh(X,k)$. - Two sheaves
E and F give rise to a presheaf

$$U \mapsto Hom(j^*E,j^*F) \quad ; \; j: U \to X$$

which is easily seen to be a sheaf. This will be denoted
$Hom(E,F)$. Thus with the notation above

12.1 $$\Gamma(U,Hom(E,F)) = Hom(j^*E,j^*F)$$

Given sheaves E,F,G on X we have a fundamental isomorphism

12.2 $$Hom(E \otimes F,G) = Hom(E,Hom(F,G))$$

Proof. It suffices to prove a formula of the form

$$Hom(E \otimes F,G) = Hom(E,Hom(F,G))$$

where E,F and G are presheaves on X. This follows from

$$Hom(\Gamma(U,E) \otimes \Gamma(U,F),\Gamma(U,G)) = Hom(\Gamma(U,E),Hom(\Gamma(U,F),\Gamma(U,G)))$$

which we consider well known from algebra.

$$Q.E.D.$$

Corollary 12.3. If F is a flat k-sheaf and G an inject-
ive k-sheaf, then $Hom(F,G)$ is an injective k-sheaf.

In case $F^\cdot$ and $G^\cdot$ are complexes we define $Hom^\cdot(F^\cdot,G^\cdot)$ by

12.4 $$Hom^n(F^\cdot,G^\cdot) = \prod_{p \in \mathbb{Z}} Hom(F^n,G^{n+p})$$

the differential being similar to that of I.4.3.

III. Cohomology with Compact Support

III.1   Locally compact spaces

Let X denote a locally compact space, i.e. a Hausdorff topological space in which every point has a compact neighbourhood. Let us prove two simple facts about locally compact spaces.

1.1   Any neighbourhood of a point $x \in X$ contains a compact neighbourhood of x.

Proof. It suffices to prove this for an open neighbourhood V of x which is contained in a compact subset K. Let us consider the compact set K-V. For each point $y \in K-V$ choose disjoint open neighbourhoods V_y of y and U^y of x. The covering (V_y) of K-V can be refined to a finite covering $(V_y)_{y \in S}$ of K-V. The compact set $K - \bigcup_{x \in S} V_y$ is contained in V and contains $\bigcap_{y \in S} U^y \cap V$.

Q.E.D.

1.2   For any open set U containing a compact set K, there exists a compact set $L \subseteq U$ with $K \subseteq \mathring{L}$.

Proof. For each $x \in K$ choose a compact neighbourhood N_x of x contained in U. The covering $(\mathring{N}_x)_{x \in K}$ of K can be refined to a finite covering $(\mathring{N}_x)_{x \in S}$ of K. Put $L = \bigcup_{x \in S} N_x$.

Q.E.D.

Definition 1.3. For a sheaf F on the locally compact space X put

$$\Gamma_c(X,F) = \{s \in \Gamma(X,F) \mid \text{Supp}(s) \text{ is compact}\}$$

the sections of F over X with compact support.

This defines a left exact functor

$$\Gamma_c(X,-): \quad \text{Sh}(X) \longrightarrow Ab$$

The i'th derived functor of this evaluated on the sheaf F will be denoted $H_c^i(X,F)$ and is called the i'th cohomology group with compact support with coefficients in F.

Consider a continuous map f: $X \longrightarrow Y$ between locally compact spaces which is proper, i.e. such that the inverse image of any compact subset of Y is a compact subset of X. For a sheaf G on Y, the adjunction map $a: \Gamma(Y,G) \to \Gamma(X,f*G)$ will transform a section of G with compact support into a section of f*G with compact support and thus induce a natural transformation

1.4 $$a: \Gamma_c(Y,G) \longrightarrow \Gamma_c(X,f*G)$$

Since f* is exact we can extend this to

1.5 $$f*: H_c^{"}(Y,G) \longrightarrow H_c^{"}(X,f*G)$$

by choosing an injective resolution $G \to J^{\cdot}$ and an injective resolution $f^*J^{\cdot} \to I^{\cdot}$. The composite

$$I_c(Y,J^{\cdot}) \overset{a}{\to} I_c(X,f^*J^{\cdot}) \to I_c(X,I^{\cdot})$$

represents 1.5 on the chain level, noticing that $I^{\cdot}$ is an injective resolution of f^*G.

Proceeding as in II.7.9 we can derive from an exact sequence of sheaves on Y

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

a commutative ladder of abelian groups

1.6
$$\longrightarrow H_c^P(Y,E) \longrightarrow H_c^P(Y,F) \longrightarrow H_c^P(Y,G) \longrightarrow H_c^{P+1}(Y,E) \longrightarrow$$
$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
$$\longrightarrow H_c^P(X,f^*E) \longrightarrow H_c^P(X,f^*F) \longrightarrow H_c^P(X,f^*G) \to H_c^{P+1}(X,f^*E) \to$$

Closed subspaces

Consider in particular the inclusion $i: Z \to X$ of a closed subspace. A sheaf E on Z gives rise to a canonical isomorphism, compare II.5.4

1.7
$$H_c^{\cdot}(Z,E) = H_c^{\cdot}(X,i_*E)$$

For closed subspaces A and B of X the exact sequence II.5.7 yields a

Mayer-Vietoris sequence 1.8

$$\longrightarrow H_c^P(A \cup B,F) \longrightarrow H_c^P(A,F) \oplus H_c^P(B,F) \longrightarrow H_c^P(A \cap B,F)$$

III.2 Soft sheaves

Let X denote a locally compact space. We shall introduce an important class of $I_c(X,-)$-acyclic sheaves on X.

Definition 2.1. A sheaf S on X is called soft if whenever there is given data

1)   a compact subset K of X

2)   an open subset U of X containing K

3)   a section s of S over U

then there exists a section t of S over X, such that s and t have the same restriction to some open neighbourhood of K contained in U.

The following proposition gives rise to an alternative formulation of softness.

Theorem 2.2. Let $i: Z \to X$ denote the inclusion of a compact subset into a locally compact space X. For any sheaf F on X, there is a canonical isomorphism

$$\varinjlim I(U,F) \overset{\simeq}{\to} I(Z,i^*F)$$
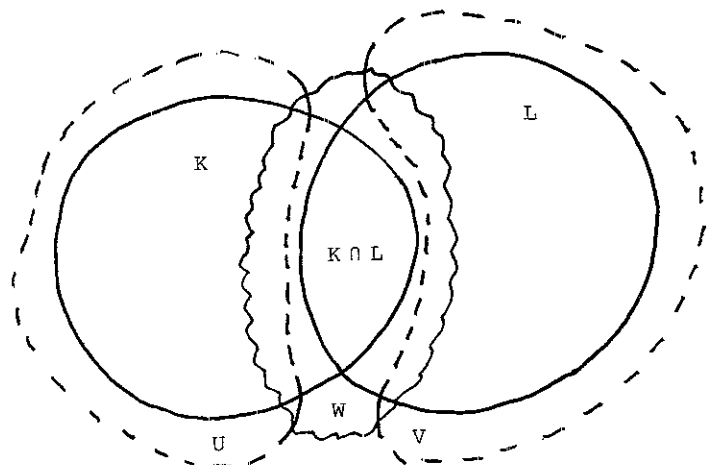
the limit is taken over all open $U \supseteq Z$.

Proof. Notice that the canonical map

$$\varinjlim I(U,F) \to I(Z,i^*F)$$

is injective for trivial reasons. To prove that the map is surjective consider $s \in I(Z,i^*F)$. For each point z of Z choose

a compact neighbourhood K^Z of z in Z and an open subset U^Z of X containing K^Z and $s^Z \in I(U^Z,F)$ with $(s^Z)_x = s_x$ for all $x \in K^Z$. Since Z is compact, finitely many K^Z's will cover Z. This leaves us with the following problem:

Given compact subsets K and L of X, sections s and t of F over open subsets containing K and L respectively, whose stalks agree on $K \cap L$. Find a section of F over an open subset of X containing $K \cup L$ which agrees with s in a neighbourhood of K and t in a neighbourhood of L.

Choose an open neighbourhood W of $K \cap L$ in X such that s and t have the same restriction to W. Choose disjoint open subsets U and V of X containing $L-W$ and $K-W$ respectively, 1.2. Finally use the sheaf axiom to construct a section of F over $V \cup U \cup W$ which extends s and t.

Q.E.D.

This theorem shows that <u>a sheaf S on X is soft if and only if for any compact subset K of X, the restriction map $I(X,S) \to I(K,S)$ is surjective</u>. In the rest of this section we shall utilize this formulation which is technically easier to work with once we make two remarks on sections over closed subsets.

<u>Gluing sections 2.3</u>. Let Z and W be two closed subsets of a topolocial space X and F a sheaf on X. Given $s \in I(Z,F)$ and $t \in I(W,F)$ which have the same restrictions to $W \cap Z$ then there exists one and only one section of F over $Z \cup W$ with restriction s to Z and restriction t to W, as it follows from II.5.6.

<u>Extension by zero 2.4</u>. Let Z be a closed subset of the topological space X and F a sheaf on X. Any section s of F over Z with restriction 0 to ∂Z can be extended to a section of F over X with restriction 0 to the complement of Z in X.

To see this let us notice that $\dot{Z}$ and X-Supp(s) form an open covering of X, which we can use to glue the restriction of s to $\dot{Z}$ and the zero section over X-Supp(s) to a section of F over X.

Let us mention that a subspace W of a locally compact space X is locally compact if and only if W is a <u>locally closed</u> subspace of X, compare II.6.

Corollary 2.5. Let h: W → X denote the inclusion of a locally closed subspace W of X. A soft sheaf S on X will induce a soft sheaf h*S on W.

Corollary 2.6. Let i: Z → X denote the inclusion of a closed subspace. A soft sheaf S on X has surjective restriction map

$$\Gamma_c(X,S) \longrightarrow \Gamma_c(Z,S)$$

Proof. Given $s \in \Gamma_c(Z,i*S)$. Choose a compact subset L of X whose interior $\dot{L}$ contains Supp(s)

Let $\bar{s}$ denote the section of S over $\partial L \cup (Z \cap L)$ with restriction 0 to ∂L and whose restriction to $L \cap Z$ coincides with the restriction of s to $L \cup Z$, 2.3. Extend $\bar{s}$ to a section t of S over L, and finally extend t by zero outside L, 2.4.

Q.E.D.

Theorem 2.7. Any soft sheaf S on the locally compact space X is $\Gamma_c(X,-)$-acyclic, i.e.

$$H_c^n(X,S) = 0 \quad \text{for} \quad n \geq 1$$

Proof. Let there be given a short exact sequence of sheaves on X

$$0 \longrightarrow E \xrightarrow{e} F \xrightarrow{f} G \longrightarrow 0$$

with E soft. Let us prove that the projection

$$\Gamma_c(X,F) \longrightarrow \Gamma_c(X,G)$$

is surjective. - Assume first that X is compact. So, let there be given a section s of G over X. For each point x of X we can choose a compact neighbourhood K_x of x and a section t_x of F over K_x which projects onto the restriction of s to K_x. We can refine the covering $(\dot{K}_x)_{x \in X}$ to a finite covering. - This leads us to the following construction:

Given compact subsets K and L of X, $u \in \Gamma(K,F)$ and $v \in \Gamma(L,F)$ which projects onto s. The section

$$r_{L \cap K, K}(u) - r_{L \cap K, L}(v)$$

represents a section of E over $L \cap K$ and can as such be extended to a section $w \in \Gamma(L,E)$. We can now consider the sections $u \in \Gamma(K,F)$ and $v+e(w) \in \Gamma(L,F)$ and glue these together to get a section of F over $K \cup L$ which projects onto s. - We can now apply this construction to our finite covering to obtain a section of F over X which projects onto $s \in \Gamma(X,G)$.

In case X is no longer compact we can choose a compact set K such that $\dot{K}$ contains Supp(s). Choose a section t of F over K which projects onto the restriction of s to K.

The restriction of t to ∂K represents a section of E over ∂K which we extend to a section of E over K, and subtract from t. Thus we may assume that the restriction of t to ∂K is zero. We can now extend t by zero outside K, to obtain a global section of F which projects onto s.

Let us remark that these considerations proves that if we add the assumption that F is soft, then G is soft.

Let us now return to our soft sheaf S and choose an exact sequence

$$0 \longrightarrow S \longrightarrow I \longrightarrow T \longrightarrow 0$$

with I injective. Using that I is flabby we conclude first that I is soft and next that T is soft. We can now conclude from the first part that $H_c^1(X,S) = 0$. The general case follows by an induction similar to the one applied in the proof of II.3.5.

$$Q.E.D.$$

Proposition 2.8. Let S denote a sheaf on a locally compact space X. If every point of X has an open neighbourhood U such that the restriction of S to U is a soft sheaf on U, then S is soft.

Proof. Let us first treat the case where X is compact. By assumption we can find a finite number of compact subsets $X_1, \ldots, X_n$ of X such that the restriction of S to each of these is soft. To prove that S is soft consider a compact subset K of X and $s \in \Gamma(K,S)$. Put $X^p = K \cup X_1 \cup X_p$, $p = 0, \ldots, n$. Suppose we have already extended $s = s^0$ for a section s^p of S over X^p. By assumption we can extend the restriction of s^p

to $X^p \cap X_{p+1}$ to a section σ^{p+1} of S over X_{p+1}. By 2.3 we can find a section s^{p+1} of S over X^{p+1} which extends s^p and σ^{p+1}. We can now proceed by induction

Let us now consider the general case. We know that the restriction of S to any compact subset of X is soft. Let us attempt to extend a section s of S over the compact subset K to a global section. Choose a compact subset L of X with $K \subseteq \overset{\cdot}{L}$. Extend the section over $K \cup \partial L$ which is s on K and zero on ∂L to a section of S over L. Use 2.4 to extend that section to a global section which is zero outside L.

$$Q.E.D.$$

Sheaves of modules

Let X denote a locally compact space equipped with a sheaf of commutative rings A. It is understood that the sections over any open set is a commutative ring with 1 and that the restriction maps respect the ring structure and maps 1 to 1. - Let E be a sheaf of A-modules on X, i.e. $\Gamma(U,E)$ carries a structure of unitary $\Gamma(U,A)$-modules and the restriction maps satisfy the following condition

$$r_{VU}(ae) = r_{VU}(a)r_{VU}(e) \quad ; \ a \in \Gamma(V,A), \ e \in \Gamma(V,E)$$

Proposition 2.9. Let A be a soft sheaf of commutative rings. Any sheaf E of A-modules is soft.

Proof. Let K be a compact subset of X and L a compact neighbourhood of K. There exists $\sigma \in \Gamma(X,A)$ such that $\sigma = 1$ in an open neighbourhood of K and such that the restriction of σ to the complements of $\overset{\cdot}{L}$ is zero.

To see this let us consider the section on K ∪ ∂L which is 1 on K and 0 on ∂L. Extend this to a section τ of A over L and let σ be the extension of τ by zero outside L, 2.4.

Let us now show that E is soft; so let s be a section of E over an open neighbourhood U of K. Let L be a compact neighbourhood of K contained in U and σ a global section of A as above. If we let τ denote the restriction of σ to L, then the section τs of E over L is zero on ∂L, and therefore extendable to all of X.

$$Q.E.D.$$

Local operators 2.10. Let S and T be sheaves on the locally compact space X and D: Γ(X,S) → Γ(X,T) a linear map which satisfies the following condition

$$\text{Supp } D(f) \subseteq \text{Supp}(f) \quad \text{for all} \quad f \in \Gamma_c(X,S).$$

If f is soft then there exists a unique morphism of sheaves d: S → T which induces D on the level of global sections.

Proof. For any point of X the restriction maps Γ_c(X,S) → S_x is surjective. Thus the uniqueness follows from II.2.2. - Given an open subset U of F and s ∈ Γ(U,S). Cover U with open subsets U_i such that there exists t_i ∈ Γ_c(X,S) whose restriction to U_i equals that of s. Notice that D(s_j) is independent of s_j since D is a local operator, i.e. satisfies the support condition. Using that T is a sheaf we can glue the D(s_j)'s together to get a section of T over U. The remaining details are left to the reader.

$$Q.E.D.$$

III.3 Soft sheaves on $\mathbb{R}^n$

A basic sheaf on $\mathbb{R}^n$ is the sheaf C^∞ of smooth, i.e. infinitely often differentiable, $\mathbb{R}$-valued functions. We shall prove that this sheaf is soft on the basis of the following classical.

Lemma 3.1. There exists a smooth, positive real function β on $\mathbb{R}$ with compact support and β(0) ≠ 0.

Proof. For t > 0 consider the function exp(-1/t). By a simple induction we find that

$$\frac{d^k}{dt^k} \exp(-1/t) = P_k(1/t) \exp(-1/t) \quad ; \; k \in \mathbb{N}$$

where P_k is a polynomial of degree 2k. Whence

$$\lim_{t \downarrow 0} \frac{d^k}{dt^k} \exp(-1/t) = 0$$

As a consequence we can define a smooth function α by

$$\alpha(t) = \begin{cases} \exp(-1/t) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

For β take the function β(t) = α(1-t)α(1+t).

$$Q.E.D.$$

Theorem 3.2. The sheaf C^∞ of smooth functions on $\mathbb{R}^n$ is soft.

<u>Proof</u>. Let us first remark that for any $a \in \mathbb{R}^n$ and any neighbourhood U of a there exists a positive smooth function b on $\mathbb{R}^n$ with support in U and $b(a) \neq 0$. With the notation of 3.1, such functions can be supplied by the formula

$$b(x) = \prod_{i=1}^{n} \beta(m(x_i - a_i))$$

where m is a sufficiently large integer.

For a compact set K and a compact neighbourhood L of K there exists a positive smooth function with support in L and with strictly positive values on K: For each $a \in K$ choose a positive smooth function φ_a on $\mathbb{R}^n$ with support in L and $\varphi_a(a) \neq 0$. Let U_a be the set of points $x \in \mathbb{R}^n$ where $\varphi_a(x) > 0$. Cover K with finitely many U_a's and use the corresponding sum $\Sigma \varphi_a$.

For a compact set K and a compact neighbourhood N of K there exists a smooth function on $\mathbb{R}^n$ which is zero outside N and which is constant 1 in a neighbourhood of K: Choose a compact neighbourhood L of K with $L \subseteq \dot{N}$ and choose a positive smooth function φ on $\mathbb{R}^n$ which is strictly positive on L and which is zero outside N. Choose another positive smooth function ψ which is strictly positive on $N-\dot{L}$ and which is zero in a neighbourhood of K. This can be done since there exists a compact neighbourhood of $N-\dot{L}$ which does not meet K, K and $L-\dot{N}$ being disjoint compacts. The function $\varphi + \psi$ is strictly positive on N, thus the function $\varphi(\varphi+\psi)^{-1}$ is defined in an open neighbourhood of N, is zero outside N and takes the value 1 in a neighbourhood of K.

The sheaf C^∞ is soft: Let s be a section of C^∞ over an open neighbourhood U of the compact set K. Choose a com-

pact neighbourhood N of K contained in U and a smooth function on $\mathbb{R}^n$ which is zero outside N and which is 1 in a neighbourhood of K. The function φs can be extended to all of $\mathbb{R}^n$.

Q.E.D.

<u>Corollary 3.3</u>. The sheaf C of continuous real functions on $\mathbb{R}^n$ is soft. More generally the sheaf $C^{(p)}$ of p-times continuously differentiable real functions on $\mathbb{R}^n$ is soft.

<u>Proof</u>. Notice that $C^{(p)}$ is a sheaf of modules over the soft sheaf C^∞ and apply 2.9.

Q.E.D.

<u>Example 3.4</u>. The fundamental example of a locally compact space is the real line $\mathbb{R}$. The constant sheaf $\underset{\sim}{\mathbb{R}}$ has the "Calculus resolution" which is soft

$$0 \longrightarrow \underset{\sim}{\mathbb{R}} \longrightarrow C^{(1)} \xrightarrow{D} C \longrightarrow 0$$

where D is ordinary differentiation. To calculate cohomology we consider the exact sequence

$$0 \longrightarrow \Gamma_c(\mathbb{R}, C^{(1)}) \xrightarrow{D} \Gamma_c(\mathbb{R}, C) \xrightarrow{\int} \mathbb{R} \longrightarrow 0$$

where $\int$ is given by the formula

$$\int f = \int_{-\infty}^{+\infty} f(x)\,dx$$

Consequently we find

$$3.5 \qquad H_c{}^i(\mathbb{R},\underset{\sim}{\mathbb{R}}) = \begin{cases} 0 & \text{for } i \neq 1 \\ \mathbb{R} & \text{for } i = 1 \end{cases}$$

Let us notice another corollary to 3.2.

Corollary 3.6. Let W be a locally closed subspace of $\mathbb{R}^n$. The sheaf of continuous real valued functions on W is a soft sheaf on W.

Proof. Let $h: W \to \mathbb{R}^n$ denote the inclusion of W in $\mathbb{R}^n$ and let C resp. $\mathcal{D}$ be the sheaf of continuous functions on $\mathbb{R}^n$ resp. W. Notice that we have a canonical morphism

$$C \longrightarrow h_* \mathcal{D}$$

which we can describe as follows: Let U be an open subset of $\mathbb{R}^n$ and $f \in \Gamma(U,C)$. We can compose f with the restriction of h to $h^{-1}(U) = W \cap U$ to obtain a continuous function on $W \cap U$ i.e. an element of $\Gamma(U, h_* \mathcal{D}) = \Gamma(h^{-1}(U), \mathcal{D})$. By adjunction we derive a morphism of sheaves of rings $h^* C \to \mathcal{D}$. Thus we have equipped $\mathcal{D}$ with the structure of a $h^* C$-module. The sheaf $h^* C$ is soft according to 2.5. Conclusion by 2.9.

$$\text{Q.E.D.}$$

Example 3.7. On the unit interval $[0,1]$ we have an exact sequence of sheaves analogous to the one in 3.4

$$0 \longrightarrow \underset{\sim}{\mathbb{R}} \longrightarrow C^{(1)} \xrightarrow{D} C \longrightarrow 0$$

where $C^{(1)}$ and C respectively denotes the sheaf continuously differentiable functions on $[0,1]$ resp. continuous functions. The sheaf $C^{(1)}$ is soft since it may be viewed as a sheaf of modules over the restriction of C^∞ to $[0,1]$, compare the proof of 3.6. The sequence above furnishes a soft resolution of $\mathbb{R}$ and we find

$$3.8 \qquad H^i([0,1],\underset{\sim}{\mathbb{R}}) = 0 \qquad ; \quad i \geq 1.$$

Using general topology we can generalize 3.6 to an arbitrary locally compact space.

Urysohn 3.9. The sheaf of $\mathbb{R}$-valued continuous functions on a locally compact space X is soft.

Proof. Let us first prove that given a compact subset K of X and a compact neighbourhood N of K then there exists a continuous function on X, which is constant 1 in a neighbourhood of K and constant 0 outside N. - For this, choose compact sets L and M with

$$K \subseteq \dot{L} \subseteq L \subseteq \dot{M} \subseteq M \subseteq \dot{N} \subseteq N$$

In the compact space N we have to disjoint compact subsets L and N-$\dot{M}$. According to Urysohn's lemma we can find a continuous function f on N with value 1 on L and value zero on N-$\dot{M}$. We can now use the open covering X-M,$\dot{N}$ to construct a continuous function which agrees with f on $\dot{N}$ and is 0 outside M. We can now conclude the proof as in 3.2 or 2.9.
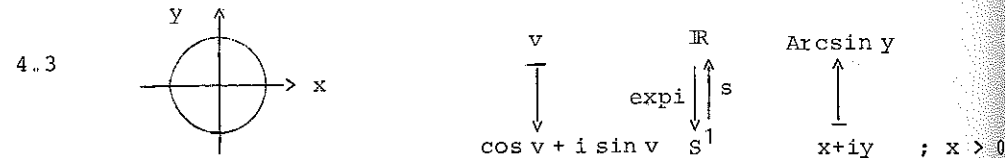
$$\text{Q.E.D.}$$

III.4   The exponential sequence

The multiplicative group of complex numbers with absolute value 1 is denoted S^1. This group fits into an exact sequence

4.1 $$0 \longrightarrow \mathbb{Z} \xrightarrow{\ 2\pi\ } \mathbb{R} \xrightarrow{\ \text{expi}\ } S^1 \longrightarrow 0$$

where the map expi is given by

4.2 $$\text{expi } v = \cos v + i \sin v$$

The map expi has <u>local sections</u>, in the sense that for every $z \in S^1$ there exists an open neighbourhood U of z in S^1 and a continuous map $s: U \to \mathbb{R}$, such that $\text{expi} \circ s = 1$: Over the part of S^1 which lies in the open right halfplane we have the following section

4.3

By translation in the group S^1 we see that expi has local sections everywhere. From the presence of local sections we conclude that expi is an <u>open map</u>, i.e. transforms open sets into open sets.

Let X be an arbitrary topological space. The exponential sequence above gives rise to a short exact sequence of sheaves

4.4 $$0 \longrightarrow \mathbb{Z} \longrightarrow C(\mathbb{R}) \xrightarrow{\ \text{expi}\ } C(S^1) \longrightarrow 0$$

where $C(\mathbb{R})$ and $C(S^1)$ denotes the sheaf of continuous functions on X with value in $\mathbb{R}$ and S^1 respectively. In this section we shall exploit this sequence in three simple cases.

The real line $\mathbb{R}$

We shall base our investigations on the following classical result.

<u>Lemma 4.5</u>. Any continuous map $f: \mathbb{R} \to S^1$ can be factored $f = \text{expi} \circ F$, where $F: \mathbb{R} \to \mathbb{R}$ is a continuous map.

<u>Proof</u>. Let us first solve the analogous problem for $f: [0,1] \to S^1$. For $t \in [0,1]$ choose an open interval I_t with center in t such that the restriction of f to $I_t \cap [0,1]$ admits a factorization through expi. Let J_t denote the open interval which makes up the middle third of I_t. According to Borel-Heine we can find a finite subset S of $[0,1]$ such that $(J_t)_{t \in S}$ covers $[0,1]$. Choose an integer n such that $1/n$ is smaller than the length of each of the intervals J_t, $t \in S$. The restriction of f to an interval of the form $[p/n, p+1/n]$ can be written $f = \text{expi} \circ F_p$, where $F_p: [p/n, p+1/n] \to \mathbb{R}$ is continuous. Notice that $F_p(p/n) - F_{p-1}(p/n)$ is an integral mul-

tiple of 2π. We can now successively fit the F_p's together by adding integral multiples of 2π to get the desired factorization of f.

Returning to the original problem, we can for each $n \in \mathbb{Z}$ choose a lifting F^n: $[n,n+1] \to S^1$ of the restriction of f to that interval and finally fit these together.

Q.E.D.

Theorem 4.6. The exponential sequence 4.4 is a soft resolution of the constant sheaf $\underset{\sim}{\mathbb{Z}}$ on $\mathbb{R}$.

Proof. Let us prove that for any open subset U of $\mathbb{R}$ the following sequence

$$\Gamma(U,C(\mathbb{R})) \longrightarrow \Gamma(U,C(S^1)) \to 0$$

is exact. Remark first that this is true in case U is an open interval as it follows from 4.5. The general case follows from the fact that U is union of disjoint open intervals.

Let us finally prove that $C(S^1)$ is soft. So let there be given a compact subset K of $\mathbb{R}$ and a section t of $C(S^1)$ over an open neighbourhood U of K. Lift t to a section s of $C(\mathbb{R})$ over U. After shrinking U we may extend s to a global section r of $C(\mathbb{R})$, since $C(\mathbb{R})$ is soft. The section $\exp i \circ r$ will deliver the looked for global section of $C(S^1)$.

Q.E.D.

On the basis of the resolution 4.4 we will show that

4.7
$$H_c^{\cdot}(\mathbb{R}; \mathbb{Z}) = 0, \mathbb{Z}, 0, \ldots$$

To do so we introduce the <u>winding number</u>, $I(s)$ of $s \in \Gamma_c(\mathbb{R},C(S^1))$ To this end choose a continuous map $t: \mathbb{R} \to \mathbb{R}$ such that $s = \exp i \circ t$, 4.5. Since s has compact support it follows that $t(x)$ is constant for large values of x, resp. small values of x. Put

4.8
$$I(s) = \frac{1}{2\pi} (t(+\infty) - t(-\infty))$$

It is left to the reader to verify exactness of the sequence

$$0 \longrightarrow \Gamma_c(\mathbb{R},C(\mathbb{R})) \xrightarrow{\exp i} \Gamma_c(\mathbb{R},C(S^1)) \xrightarrow{I} \mathbb{Z} \to 0$$

Let us calculate the action of a <u>proper</u> continuous map $f: \mathbb{R} \to \mathbb{R}$ on $H_c^1(\mathbb{R};\mathbb{Z})$. Remark first that proper in this context means that $|f(x)| \longrightarrow +\infty$ for $x \to +\infty$ and $x \to -\infty$. Taking into account that $f^*\underset{\sim}{\mathbb{Z}} = \underset{\sim}{\mathbb{Z}}$ we can identify the adjunction morphism with the morphism

$$\underset{\sim}{\mathbb{Z}} \xrightarrow{\quad \circ f \quad} f_*\underset{\sim}{\mathbb{Z}}$$

which to an open set V and a locally constant function $s: V \to \mathbb{Z}$ associates the locally constant function $s \circ f: f^{-1}(V) \to \mathbb{Z}$. The morphism of can be lifted to the exponential resolution as follows, compare Scolium II.5.2

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi} C(\mathbb{R}) \xrightarrow{\text{expi}} C(S^1) \longrightarrow 0$$

$$\downarrow \circ f \qquad \downarrow \circ f \qquad \downarrow \circ f$$

$$0 \longrightarrow f_*\mathbb{Z} \xrightarrow{2\pi} f_*C(\mathbb{R}) \xrightarrow{\text{expi}} f_*C(S^1) \rightarrow 0$$

We shall evaluate the action f^* of f on $H_c^1(\mathbb{R},\mathbb{Z})$ in terms of the symbol $[f]$, which we now define

4.9

$f(+\infty)$	$+\infty$	$+\infty$	$-\infty$	$-\infty$
$f(-\infty)$	$-\infty$	$+\infty$	$+\infty$	$-\infty$
$[f]$	1	0	-1	0

With the notation of 4.8 let us prove the formula

$$I(s \circ f) = [f] \, I(s) \qquad ; \quad s \in I_c(\mathbb{R}, C(S^1))$$

Choose $t: \mathbb{R} \longrightarrow \mathbb{R}$ such that $s = \text{expi} t$. This gives $s \circ f = \text{expi} t \circ f$ and consequently, 4.8

$$I(s \circ f) = \frac{1}{2\pi} \left(t(f(+\infty)) - t(f(-\infty)) \right)$$

which ensures us the validity of the formula, i.e. that

$$\begin{array}{ccc} I_c(\mathbb{R}, C(S^1)) & \xrightarrow{\quad I \quad} & \mathbb{Z} \\ \downarrow \circ f & & \downarrow \cdot [f] \\ I_c(\mathbb{R}, C(S^1)) & \xrightarrow{\quad I \quad} & \mathbb{Z} \end{array}$$

is a commutative diagram. Thus we have proved that

4.10 $$f^*\alpha = [f]\alpha \qquad ; \quad \alpha \in H_c^1(\mathbb{R},\mathbb{Z})$$

for a proper continuous map $f: \mathbb{R} \to \mathbb{R}$.

We shall now use the exponential resolution to make a closer study of the sequence 3.4

$$0 \to I_c(\mathbb{R}, C^{(1)}) \to I_c(\mathbb{R}, C) \to H_c^1(\mathbb{R}, \mathbb{R}) \longrightarrow 0$$

which tells us that a continuous function $f: \mathbb{R} \longrightarrow \mathbb{R}$ with compact support represents a cohomology class.

Proposition 4.11. A continuous real function φ on $\mathbb{R}$ with compact support represents an integral cohomology class[*] if and only if

$$\int_{-\infty}^{+\infty} \varphi(x) \, dx \quad \text{is an integer.}$$

Proof. Consider the following exact, commutative diagram of sheaves on $\mathbb{R}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R} & \longrightarrow & C^{(1)}(\mathbb{R}) & \xrightarrow{D} & C(\mathbb{R}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & C^{(1)}(\mathbb{C}) & \xrightarrow{D} & C(\mathbb{C}) & \longrightarrow & 0 \end{array}$$

where $C(\mathbb{C})$ resp. $C^{(1)}(\mathbb{C})$ denotes the sheaf of $\mathbb{C}$-valued continuous functions resp. continuously differentiable functions.

---

[*] Quite generally let X denote a locally compact space. A cohomology class $\theta \in H_c^n(X,\mathbb{R})$ is called integral, if it belongs to the image of

$$H_c^n(X,\mathbb{Z}) \longrightarrow H_c^n(X,\mathbb{R})$$

induced from the inclusion of sheaves $\mathbb{Z} \to \mathbb{R}$.

The unlabelled arrows are inclusions. It follows that it

suffices to prove a result similar to 4.11 with $\mathbb{R}$ replaced

by $\mathbb{C}$. - Next consider the following diagram of sheaves on $\mathbb{R}$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \underset{\sim}{\mathbb{Z}} & \xrightarrow{2\pi} & C^{(1)}(\mathbb{R}) & \xrightarrow{\text{expi}} & C(S^1) & \longrightarrow & 0 \\
& & \Big\downarrow & & {\scriptstyle\frac{1}{2\pi}}\Big\downarrow & & {\scriptstyle\frac{1}{2\pi i}}\text{Dlog}\Big\downarrow & & \quad {}^{*)} \\
0 & \longrightarrow & \underset{\sim}{\mathbb{C}} & \longrightarrow & C^{(1)}(\mathbb{C}) & \xrightarrow{D} & C(\mathbb{C}) & \longrightarrow & 0
\end{array}
$$

where the unlabelled arrows are simple inclusions. This induces

a commutative diagram

$$
\begin{array}{ccc}
\Gamma_c(\mathbb{R},\, C(S^1)) & \xrightarrow{I} & \mathbb{Z} \\
\Big\downarrow {\scriptstyle\frac{1}{2\pi i}\text{Dlog}} & & \Big\downarrow \\
\Gamma_c(\mathbb{R},\, C(\mathbb{C})) & \xrightarrow{\int} & \mathbb{C}
\end{array}
$$

as the following computation shows: An element f of

$\Gamma_c(\mathbb{R},\, C(S^1))$ may be represented by $\text{expi} \circ g$ where $g \in \Gamma(\mathbb{R},\, C(\mathbb{R}))$,

$$
\int_{-\infty}^{+\infty} \text{Dlog}(\text{expi } g\,(x))\,dx = i \int_{-\infty}^{+\infty} g'(x)\,dx =
$$

$$
i(\lim_{x \to +\infty} g(x) - \lim_{x \to -\infty} g(x)) = 2\pi i\, I(f)
$$

where we have used the formula 4.8.

<div align="right">Q.E.D.</div>

---

*) The evaluation of Dlog on f is $\frac{f'}{f}$.

The unit interval [0,1]

Let us consider the exponential sequence of sheaves on the

unit interval [0,1]

$$
0 \longrightarrow \underset{\sim}{\mathbb{Z}} \xrightarrow{2\pi} C(\mathbb{R}) \xrightarrow{\text{expi}} C(S^1) \longrightarrow 0
$$

This is a <u>soft resolution</u> of the constant sheaf $\underset{\sim}{\mathbb{Z}}$ on [0,1].

To prove that $C(S^1)$ is soft we can use the remark that a con-

tinuous map $V \to S^1$, where V is a relative open subset of

[0,1], can be extended to a continuous map on an open neighbour-

hood of V in $\mathbb{R}$. - From the proof of 4.5 follows that the

sequence

$$
0 \to \mathbb{Z} \xrightarrow{2\pi} \Gamma([0,1],C(\mathbb{R})) \xrightarrow{\text{expi}} \Gamma([0,1],C(S^1)) \longrightarrow 0
$$

is exact. Thus we have found that

4.12 $\qquad\qquad H^i([0,1],\underset{\sim}{\mathbb{Z}}) = 0 \qquad ; i \geq 1$

A result of fundamental importance.

The 1-sphere S^1

Let us once more consider the exponential sequence of

sheaves

$$
0 \longrightarrow \underset{\sim}{\mathbb{Z}} \xrightarrow{2\pi} C(\mathbb{R}) \xrightarrow{\text{expi}} C(S^1) \longrightarrow 0
$$

This is a <u>soft resolution</u> of the constant sheaf $\underset{\sim}{\mathbb{Z}}$ on S^1 as

it follows from the local character of softness, 2.8.

In order to calculate cohomology by means of the exponential sequence we shall introduce the notion of <u>degree</u> of a continuous map $f: S^1 \to S^1$. To do so let us remark that it follows from 4.5 that we can find a commutative diagram of continuous maps

4.13

$$
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\ F\ } & \mathbb{R} \\
\downarrow{\scriptstyle \text{expi}} & & \downarrow{\scriptstyle \text{expi}} \\
S^1 & \xrightarrow{\ f\ } & S^1
\end{array}
$$

The map F is unique up to an integral multiple of 2π. Thus it makes sense to put

4.14
$$
\boxed{\deg(f) = \frac{1}{2\pi}(F(2\pi) - F(0))}
$$

The following sequence of abelian groups is exact

$$
0 \longrightarrow \mathbb{Z} \xrightarrow{\ 2\pi\ } \Gamma(S^1, C(\mathbb{R})) \xrightarrow{\text{expi}} \Gamma(S^1, C(S^1)) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0
$$

<u>Proof</u>. To see that the degree map is surjective, let us notice that $m \in \mathbb{Z}$ gives rise to a map $z \longmapsto z^m$ of degree m. Conversely, let $f: S^1 \to S^1$ be a map of degree zero with the notation from 4.13, remark that the expression $F(x+2\pi) - F(x)$ is independent of x. Thus we find that F is periodic with period 2π, which shows that f can be factored through expi.

Q.E.D.

The product hk of two continuous maps $h, k: S^1 \to S^1$ is given by $z \longmapsto h(z)k(z)$. We have

4.15
$$
\deg(hk) = \deg(h) + \deg(k)
$$

as it follows rather easily from the definition.

The notion of degree introduced in 4.14 is multiplicative in the sense that two continuous maps $f, g: S^1 \to S^1$ satisfy

4.16
$$
\deg(g \circ f) = \deg(g)\deg(f)
$$

<u>Proof</u>. Choose continuous maps $F, G: \mathbb{R} \to \mathbb{R}$ such that

$$
f \circ \text{expi} = \text{expi} \circ F, \quad g \circ \text{expi} = \text{expi} \circ G
$$

From these formulas follows $\quad g \circ f \circ \text{expi} = \text{expi} \circ G \circ F$

According to the definition 4.14 we find

$$
\deg(g \circ f) = \frac{1}{2\pi}[G(F(2\pi)) - G(F(0))] =
$$

$$
\frac{1}{2\pi}[G(F(0) + 2\pi \deg(f)) - G(F(0))]
$$

We leave it to the reader to prove the formula

$$
G(x+2\pi n) - G(x) = 2\pi n \deg(g) \quad ; x \in \mathbb{R}, \quad n \in \mathbb{Z}
$$

Finally, use this with $x = F(0)$ and $n = \deg(f)$ to get 4.16.

Q.E.D.

<u>Proposition 4.17</u>. The action of a continuous map $f: S^1 \longrightarrow S^1$ on $H^1(S^1, \mathbb{Z})$ is given by

$$
f^*\alpha = \deg(f)\alpha \quad ; \quad \alpha \in H^1(S^1, \mathbb{Z})
$$

<u>Proof</u>. Lift f to the exponential resolution as follows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underset{\sim}{\mathbb{Z}} & \xrightarrow{2\pi} & C(\mathbb{R}) & \xrightarrow{\text{expi}} & C(S^1) & \longrightarrow & 0 \\ & & \downarrow{\text{of}} & & \downarrow{\text{of}} & & \downarrow{\text{of}} & & \\ 0 & \longrightarrow & f_*\underset{\sim}{\mathbb{Z}} & \xrightarrow{2\pi} & f_*C(\mathbb{R}) & \xrightarrow{\text{expi}} & f_*C(S^1) & \longrightarrow & 0 \end{array}$$

It follows from 4.16 that the following diagram

$$\begin{array}{ccc} \Gamma(S^1,C(S^1)) & \xrightarrow{\deg} & \mathbb{Z} \\ \downarrow{\text{of}} & & \downarrow{\deg(f)} \\ \Gamma(S^1,C(S^1)) & \xrightarrow{\deg} & \mathbb{Z} \end{array}$$

is commutative, and the result follows.

$$\text{Q.E.D.}$$

<u>Proposition 4.18</u>. Any continuous endomorphism of the topological group S^1 has the form $z \longmapsto z^n$; $n \in \mathbb{Z}$.

<u>Proof</u>. Let $f: S^1 \to S^1$ be an endomorphism and choose $F: \mathbb{R} \to \mathbb{R}$ as in 4.13. By adding a constant to F we may assume that $F(0) = 0$. Since f is an endomorphism we have

$$\text{expi}(F(x+y) - F(x) - F(y)) = 1 \quad ; x,y \in \mathbb{R}$$

or otherwise expressed $F(x+y) - F(x) - F(y) \in 2\pi\mathbb{Z}$ Using that $\mathbb{R}$ is connected we get that F is a continuous endomorphism of $\mathbb{R}$. Consequently F must have the form $x \longmapsto ax$. It follows that a is an integral multiple of 2π.

$$\text{Q.E.D.}$$

III.5   Cohomology of direct limits

Let us consider a locally compact space X and a direct system of sheaves $(F_\lambda, f_{\mu\lambda})$ over a directed set I.

<u>Theorem 5.1</u>. The canonical map, $p \in \mathbb{N}$

$$\varinjlim H_c^p(X,F_\lambda) \underset{\sim}{\to} H_c^p(X,\varinjlim F_\lambda)$$

is an isomorphism.

<u>Proof</u>. Put $F = \lim F_\lambda$ and let $f_\lambda: F_\lambda \to F$ denote the canonical map. Let us at once notice that for the inclusion $i: Z \to X$ of a (closed) subspace we have

$$i^*F = \varinjlim i^*F_\lambda$$

as it follows by localization I.2.8. - Let us prove that the canonical map

$$\text{5.2} \qquad \varinjlim \Gamma_c(X,F_\lambda) \to \Gamma_c(X,F)$$

is an isomorphism. Let us first prove injectivity in case X is compact. So let there be given $s \in \Gamma(X,F_\lambda)$ which maps to zero in $\Gamma(X,F)$. For each $x \in X$ we can find $\lambda(x) \geq \lambda$ and an open neighbourhood U_x of x such that $f_{\lambda(x),\lambda}(s)$ has restriction zero to U_x. Let us choose a finite set of points S of X such that $(U_x)_{x \in S}$ covers X and choose $\mu \in I$, such that $\lambda(x) \leq \mu$ for all $x \in S$. It follows that $f_{\mu\lambda}(s) = 0$.

Let us prove surjectivity of 5.2 in case X is compact. So let there be given $s \in \Gamma(X,F)$. For each $x \in X$ choose a compact neighbourhood K_x of x in X, $\lambda(x) \in I$ and $s^x \in \Gamma(K_x, F_{\lambda(x)})$ such that $f_{\lambda(x)}(s^x)$ equals the restriction of s to K_x. Since X is compact we can choose a finite subset T of X such that $(K_x)_{x \in T}$ covers X. Consider two points, $x, y \in T$: according to our first result, the canonical map

$$\varinjlim \Gamma(K_x \cap K_y, F_\lambda) \to \Gamma(K_x \cap K_y, F)$$

is injective. It follows that we can choose $\mu \in I$ with $\mu \geq \lambda(x)$ for all $x \in T$, and such that $f_{\mu, \lambda(x)}(s^x)$ and $f_{\mu, \lambda(y)}(s^y)$ have the same restriction to $K_x \cap K_y$ for all $(x,y) \in T \times T$. The section $t \in \Gamma(X, F_\mu)$ with restriction $f_{\mu, \lambda(x)}(s^x)$ to K_x for all $x \in T$ will now be mapped onto $s \in \Gamma(X,F)$.

Let us now prove injectivity of 5.2 in general. Given $s \in \Gamma_c(X, F_\lambda)$ which is mapped to zero in $\Gamma_c(X,F)$. Using the previous result for $K = \mathrm{Supp}(s)$ we find $\mu \geq \lambda$ such that $f_{\mu\lambda}(s) = 0$ in $\Gamma(K, F_\mu)$. From this we conclude by localization that $f_{\mu\lambda}(s) = 0$ in $\Gamma(X, F_\mu)$.

Let us prove that 5.2 is surjective. So let there be given $s \in \Gamma_c(X,F)$. Choose a compact set K containing $\mathrm{Supp}(s)$ in its interior. Choose $\lambda \in I$ and $t \in \Gamma(K, F_\lambda)$ such that $f_\lambda(t)$ equals the restriction of s to K. Since the restriction of $f_\lambda(t)$ to ∂K is zero, we can find $\mu \geq \lambda$ such that $f_{\mu\lambda}(t)$ has restriction zero to ∂K. We can now extend $f_{\mu\lambda}(t)$ by zero outside K to obtain a global section of F with compact support which is transformed into $s \in \Gamma(X,F)$ by f_λ.

Let us remark that if F_λ is soft for all $\lambda \in I$, then $\varinjlim F_\lambda$ is soft. This follows from the first part and the remarks following 2.4.

Let us finally consider the Godement resolution $C^{\cdot}F_\lambda$ of F_λ, II.3.6. This gives us a direct system of complexes of flabby sheaves on X. The direct limit $\varinjlim C^{\cdot}F_\lambda$ is a soft resolution of F by our previous remark. Thus

$$H_c^p(X,F) = H^p \Gamma_c(X, \varinjlim C^{\cdot}F_\lambda) = H^p \varinjlim \Gamma_c(X, C^{\cdot}F_\lambda)$$

$$= \varinjlim H^p \Gamma_c(X, C^{\cdot}F_\lambda) = \varinjlim H_c^p(X, F_\lambda)$$

and our journey ends.

Q.E.D.

Corollary 5.3. Let X denote a compact space which is acyclic, i.e. for which

$$\mathbb{Z} = H^0(X, \mathbb{Z}) \quad \text{and} \quad H^i(X, \mathbb{Z}) = 0 \quad \text{for} \quad i \geq 1$$

Then for any abelian group k

$$k \cong H^0(X, k) \quad \text{and} \quad H^i(X, k) = 0 \quad \text{for} \quad i \geq 1$$

Proof. The abelian group k is direct limit of its finitely generated subgroups. Thus by 5.1 it suffices to treat the case where k is finitely generated. - Choose a finitely generated free resolution $0 \longrightarrow \mathbb{Z}^m \longrightarrow \mathbb{Z}^n \longrightarrow k \longrightarrow 0$ The long exact sequence on cohomology gives at once $H^i(X,k) = 0$ for $i \geq 1$. The assumption $\mathbb{Z} = H^0(X, \mathbb{Z})$ implies that X is connected, and consequently that any locally constant function with values in k is constant.

Q.E.D.

III.6   Proper base change and proper homotopy

Let $f: X \to Y$ be a <u>proper</u> map between locally compact spaces. Recall that this means that the inverse image of a compact subset of Y is a compact subset of X. In particular the <u>fiber</u> $f^{-1}(y)$ over a point $y \in Y$ is compact.

6.1   A proper continuous map $f: X \to Y$ between
         locally compact spaces is <u>closed</u>, i.e. a closed
         subset Z of X is transformed into a closed
         subset $f(Z)$ of Y.

<u>Proof</u>. Let y be a point of Y which does not belong to $f(Z)$. Choose a compact neighbourhood N of y in Y and consider the compact subset $Z \cap f^{-1}(N)$ of X. This is mapped onto

$$f(Z \cap f^{-1}(N)) = f(Z) \cap N$$

This shows that $f(Z) \cap N$ is compact. Thus $N - f(Z)$ is a neighbourhood of y in Y which does not meet $f(Z)$.

$$Q.E.D.$$

<u>Theorem 6.2</u>. Let $f: X \to Y$ be a proper map between locally compact spaces, and let F be a sheaf on X. For $y \in Y$ and $n \in \mathbb{N}$ the restriction map

$$(R^n f_* F)_y \to H^n(f^{-1}(y), F)$$

is an isomorphism.

<u>proof</u>. Recall from II.5.11 that $R^n f_* F$ is the sheaf on Y associated to the presheaf

$$V \longmapsto H^n(f^{-1}(V), F)$$

As a consequence we have

$$(R^n f_* F)_y = \lim_{\substack{\to \\ V}} H^n(f^{-1}(V), F)$$

where the limit is taken over all open neighbourhoods V of y in Y. By lemma 6.3 below we have an isomorphism

$$\lim_{\to} H^n(U, F) \overset{\sim}{\to} H^n(f^{-1}(y), F)$$

where the limit is taken over all open neighbourhoods U of $f^{-1}(y)$ in X. From the fact that f is closed 6.1, follows that any such U contains an open subset of the form $f^{-1}(V)$ where V is an open neighbourhood of y in Y:

$$U \supseteq f^{-1}(Y - f(X-U))$$

Thus we may identify the two direct limits above and the result follows.

$$Q.E.D.$$

<u>Lemma 6.3</u>. Let K be a compact subset of the locally compact space X. For any sheaf F on X there is a canonical isomorphism

$$\lim_{\to} H^n(U, F) \overset{\sim}{\to} H^n(K, F)$$

where the limit is taken over all open neighbourhoods U of K in X.

Proof. Let i: K → X denote the inclusion. Recall from 2.2 that the formula is correct for n = 0, i.e. that

$$\varinjlim \Gamma(U,F) \underset{\sim}{\to} \Gamma(K,i^*F)$$

Let us remark that i* transforms soft sheaves into soft sheaves. In particular an injective resolution I˙ of F transforms into a soft resolution i*I˙ of i*F. Thus

$$H^n(K,i^*F) = H^n(K,i^*I^{\cdot}) = H^n(\varinjlim \Gamma(U,I^{\cdot})) =$$

$$\varinjlim H^n\Gamma(U,I^{\cdot}) = \varinjlim H^n(U,F)$$

where the last step is justified by II.5.9.

Q.E.D.

Vietoris-Begle mapping theorem 6.4. Let f: X → Y be a proper map between locally compact spaces with acyclic fibers, 5.3. For any sheaf G on Y the induced map

$$f^*: H_c^{\cdot}(Y,G) \longrightarrow H_c^{\cdot}(X,f^*G)$$

is an isomorphism.

Proof. Let us first prove that the adjunction map a: $G \to f_*f^*G$ is an isomorphism. This will be done by localization, so consider

a point y of Y. Let us first remark that the canonical map

$$(f_*f^*G)_y \to H^0(f^{-1}(y),G_y)$$

is an isomorphism since f is proper 6.2. Thus we may identify a_y with the canonical map

$$G_y \longrightarrow H^0(f^{-1}(y),G_y)$$

which is an isomorphism according to 5.3.

Let $f^*G \to I^{\cdot}$ be an injective resolution. Let us prove that $H^i f_*I^{\cdot} = 0$ for $i \geq 1$. This is done by localization so consider a point y of Y. According to 6.1 we have

$$(H^i f_*I^{\cdot})_y = R^i f_*(f^*G)_y = H^i(f^{-1}(y),G_y)$$

The last group is zero by 5.3. - Using that f_* is left exact and transforms injectives to injectives we conclude that $f_*f^*G \to f_*I^{\cdot}$ is an injective resolution of f_*f^*G. Combine this with the fact that the adjunction morphism $G \to f_*f^*G$ is an isomorphism and conclude by an adaption of Scolium II.5.2 to cohomology with compact support.

Q.E.D.

Example 6.5. For the n-cube $[0,1]^n$ we have

$$H^i([0,1]^n,\underset{\sim}{k}) = 0 \qquad ; \quad i \geq 1$$

for any constant sheaf $\underset{\sim}{k}$. This follows from 4.12 and 6.4 using

the projection

$$[0,1]^n \to [0,1]^{n-1} \quad ; \quad (x_1, x_2, \ldots, x_n) \mapsto (x_2, \ldots, x_n)$$

Let $f, g: X \to Y$ be proper continuous maps between locally compact spaces. A _proper homotopy_ from f to g is a proper continuous map $F: X \times [0,1] \to Y$ with

$$F(x,0) = f(x), \quad F(x,1) = g(x) \quad ; \quad x \in X$$

Proposition 6.6. Two properly homotopic maps $f, g: X \to Y$ between locally compact spaces induces the same map

$$f^*, g^*: \; H_c^\cdot(Y, \underset{\sim}{k}) \to H_c^\cdot(X, \underset{\sim}{k})$$

for any abelian group k.

Proof. Let $F: X \times [0,1] \to Y$ be a proper homotopy from f to g. Let $p: X \times [0,1] \to X$ denote the projection and for $t \in [0,1]$ let $i_t: X \to X \times [0,1]$ be given by $i_t(x) = (x,t), \; x \in X$. By Vietoris-Begle, $p^*: H_c^\cdot(X, \underset{\sim}{k}) \to H_c^\cdot(X \times [0,1], \underset{\sim}{k})$ is an isomorphism. From the formula $p \circ i_t = 1$ we deduce $i_t^* \circ p^* = 1$. This shows that

$$i_t^{\;*}: \; H_c^\cdot(X \times [0,1], \underset{\sim}{k}) \to H_c^\cdot(X, \underset{\sim}{k})$$

is independent of $t \in [0,1]$. Notice that $f = F \circ i_0$ and $g = F \circ i_1$ consequently

$$F^* = i_0^{\;*} \circ F^* = i_1^{\;*} \circ F^* = g^*$$

as required.

The Vietoris-Begle theorem can be refined in various ways. Here is one such refinement.

Proposition 6.7. Let $f: X \to Y$ be a proper continuous map between locally compact spaces. Assume that the fibers all satisfy the condition

$$H^0(f^{-1}(y), \underset{\sim}{\mathbb{Z}}) = \mathbb{Z} \quad \text{and} \quad H^i(f^{-1}(y), \underset{\sim}{\mathbb{Z}}) = 0 \quad \text{for} \quad i = 1, \ldots, n.$$

Then for any sheaf G on Y the induced map

$$f^*: \; H_c^{\;p}(Y, G) \to H_c^{\;p}(X, f^*G)$$

is an isomorphism for $p = 0, \ldots, n-1$ and a monomorphism for $p = n$.

Proof. Let $f^*G \to I^\cdot$ and $G \to J^\cdot$ be injective resolutions on X and Y respectively. Choose a morphism of complexes $\psi: J^\cdot \to f_* I^\cdot$ making the following diagram commutative

$$
\begin{array}{ccc}
G & \longrightarrow & J^\cdot \\
\downarrow a & & \downarrow \psi \\
f_* f^* G & \longrightarrow & f_* I^\cdot
\end{array}
$$

It follows from the proof of 6.4 that $H^p \psi$ is an isomorphism for $p = 0, \ldots, n$. Consider the mapping cone triangle

We can conclude that $H^p C^{\cdot} = 0$ for $p = 0, \ldots, n-1$. Since $C^{\cdot}$ is a bounded below complex of injectives this implies that $C^{\cdot} \to \tau_{\geq n} C^{\cdot}$ is a homotopy equivalence, I.5.8. – The transformed triangle

$$\Gamma_c(Y,C^{\cdot})$$

$$\Gamma_c(Y,J^{\cdot}) \longrightarrow \Gamma_c(X,I^{\cdot})$$

and the resulting long exact sequence

$$\to H^{n-1}(\Gamma_c(Y,C^{\cdot})) \to H_c^{\,n}(Y,G) \overset{f*}{\to} H_c^{\,n}(X,f*G) \to$$

allows us to make the conclusion.

Q.E.D.

III.7   Locally closed subspaces

Let $h: W \to X$ denote the inclusion of the locally closed subspace W of the locally compact space X. For a sheaf E on W we have the fundamental formula

7.1 $$\Gamma_c(X,h_! E) = \Gamma_c(W,E)$$

as it follows from the definition II.6.1.

Proposition 7.2. The inclusion of a locally closed subspace $h: W \to X$ will transform a soft sheaf S on W into a soft sheaf $h_! S$ on X.

Proof. Let K be a compact subset of X. We are going to prove that $\Gamma_c(X,h_! S) \to \Gamma(K,h_! S)$ is surjective. Let us introduce some notation and recall the formula II.6.13

$$\begin{array}{ccc} W \cap K & \overset{q}{\longrightarrow} & K \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ W & \overset{h}{\longrightarrow} & X \end{array}$$

II.6.13

$$i*h_! S = q_! j*S$$

$$\Gamma(K,h_! S) = \Gamma(K,q_! j*S) = \Gamma_c(W \cap K,S)$$

Again by 7.1 $\Gamma_c(X,h_! S) = \Gamma_c(W,S)$. From 2.6 we conclude that $\Gamma_c(W,S) \to \Gamma_c(W \cap K,S)$ is surjective.

Q.E.D.

Corollary 7.3. For any sheaf E on W we have a canonical isomorphism

$$H_c^{"}(W,E) = H_c^{"}(X,h_!E)$$

Proof. Consider an injective resolution $E \to I^{"}$ on W. Recall that an injective sheaf is soft, since it is a flabby sheaf. Accordingly $h_!E \to h_!I^{"}$ is a $\Gamma_c(X,-)$-acyclic resolution. Using the acyclicity Theorem I.7.5 and formula 7.1 we get

$$H_c^{"}(X,h_!S) = H^{"}\Gamma_c(X,h_!I^{"}) = H^{"}\Gamma_c(W,I^{"}) = H_c^{'}(W,F)$$

<div align="right">Q.E.D.</div>

Open and closed subspaces

Let us now turn to the case where $j: U \to X$ is the inclusion of an open subset of the locally compact space X. For a sheaf F on X, the adjunction morphism $j_!j^*F \to F$ will induce a morphism

$$H_c^{"}(X,j_!j^*F) \to H_c^{"}(X,F)$$

which we can compose with the isomorphism 7.3 to get a natural map of fundamental importance "extension by zero"

7.4 $$j_!: H_c^{"}(U,F) \to H_c^{"}(X,F)$$

Meyer Vietoris sequence 7.5. Two open subsets U and V of X give rise to a long exact sequence

$$\to H_c^p(U \cap V,F) \to H_c^p(U,F) \oplus H_c^p(V,F) \to H_c^p(U \cup V,F)$$

for any sheaf F on X.

Proof. Let us put names to the inclusions

$$h: U \cap V \to X, \quad i: U \to X, \quad j: V \to X, \quad k: U \cup V \to X.$$

Consider the exact sequence of sheaves on X

$$0 \to h_!h^*F \to i_!i^*F \oplus j_!j^*F \to k_!k^*F \to 0$$

where the morphisms are sum and difference between adjunction morphisms. The exactness is established by localization. The resulting cohomology sequence can be identified by means of 7.3.

<div align="right">Q.E.D.</div>

Let $i: Z \to X$ denote the inclusion of the complement $Z = X-U$ in X. We can now use the fundamental sequence II 6.11

$$0 \longrightarrow j_!j^*F \longrightarrow F \longrightarrow i_*i^*F \longrightarrow 0$$

to derive a long exact sequence

7.6 $$\longrightarrow H_c^p(U,F) \longrightarrow H_c^p(X,F) \longrightarrow H_c^p(Z,F) \longrightarrow$$

Variations over 7.6. From the diagrams at the end of section II.6 can be deduced a number of useful exact sequences. We give some examples.

Given closed subspaces $S \subseteq Z$ of X with complements O and U respectively. Then we deduce from II.6.16 an exact commutative ladder of abelian groups

7.7

$$\longrightarrow H_c^{p-1}(Z,F) \longrightarrow H_c^p(U,F) \longrightarrow H_c^p(X,F) \longrightarrow H_c \longrightarrow$$
$$\downarrow i^* \qquad \downarrow j_! \qquad \downarrow 1 \qquad \downarrow i^*$$
$$\longrightarrow H_c^{p-1}(S,F) \longrightarrow H_c^p(O,F) \longrightarrow H_c^p(X,F) \longrightarrow H_c^p(S,F) \longrightarrow$$

where $i: S \to Z$ and $j: U \to O$ denotes the inclusions.

Let there be given a proper, continuous map $f: X \to Y$, a closed subset B on Y with morphism $V = Y-B$. We put $A = f^{-1}B$ and $U = f^{-1}(V)$. A sheaf G on Y will give rise to a commutative ladder

7.8

$$\longrightarrow H_c^p(V,G) \longrightarrow H_c^p(X,G) \longrightarrow H_c^p(B,G) \longrightarrow H_c^{p+1}(V,G) \longrightarrow 0$$
$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$
$$\longrightarrow H_c^p(U,f^*G) \longrightarrow H_c^p(X,f^*G) \longrightarrow H_c^p(A,f^*G) \longrightarrow H_c^{p+1}(U,f^*G) \longrightarrow$$

Proof. The exact sequence of sheaves on Y, in notation of II.6.15

$$0 \longrightarrow G_V \longrightarrow G \longrightarrow G_B \longrightarrow 0$$

transforms under f^* into the sequence

$$0 \longrightarrow f^*G_U \longrightarrow f^*G \longrightarrow f^*G_A \longrightarrow 0$$

as one sees by II.6.13. Conclusion by 1.6.

Q.E.D.

III.8   Cohomology of the n-sphere

In this section we shall make some geometric applications of cohomology with constant coefficients. We consider a fixed commutative ring k. The n-sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$$

has cohomology as follows

8.1
$$H^p(S^n,k) = \begin{cases} k & \text{for } p = 0,n \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let us introduce the half spheres

$$S_+^n = \{x \in S^n \mid x_{n+1} \geq 0\} \qquad S_-^n = \{x \in S^n \mid x_{n+1} \leq 0\}$$

Each of these spaces are homeomorphic to $[0,1]^n$, thus they have vanishing higher cohomology groups by 6.5. Notice that $S_+^n \cap S_-^n = S^{n-1}$ and deduce the result from the Mayer-Vietoris sequence 1.8

$$\longrightarrow H^p(S^n,k) \longrightarrow H^p(S_+^n,k) \oplus H^p(S_-^n,k) \longrightarrow H^p(S_+^n \cap S_-^n,k) \longrightarrow$$

Q.E.D.

Brouwer's fixpoint theorem 8.2. Any continuous selfmap of the unitball $B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ has a fixpoint.

<u>Proof.</u> Let $f: B^n \to B^n$ be a continuous map without fix-points. We can then construct a continuous map $r: B^n \to S^{n-1}$ as follows

$r(x)$ denotes the point where the oriented line from $f(x)$ to x intersects S^{n-1}.

Let $i: S^{n-1} \to B^n$ denote the inclusion. Note, that $r \circ i = 1$. This implies that

$$i^*: H^n(B^n, \mathbb{Z}) \to H^n(S^{n-1}, \mathbb{Z})$$

is surjective, contradicting the calculation 8.2.

<div align="right">Q.E.D.</div>

8.3
$$H_c^p(\mathbb{R}^n, k) = \begin{cases} k & \text{for } p = n \\ 0 & \text{otherwise} \end{cases}$$

<u>Proof.</u> Let $N = (0, \ldots, 0, 1)$ denote the northpole of S^{n+1}. Stereographic projection from N will map $S^{n+1} - \{N\}$ homeomorphically onto $\mathbb{R}^n$. The result follows from the exact sequence

$$\to H_c^p(S^{n+1} - \{N\}, k) \to H_c^p(S^{n+1}, k) \to H_c^p(\{N\}, k) \to$$

and the formula 8.1.

<div align="right">Q.E.D.</div>

<u>Cohomology of a closed halfspace</u> $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$

8.4
$$H_c^p(\mathbb{R}_+^n, k) = 0 \qquad \text{for all } p \in \mathbb{N}$$

<u>Proof.</u> Use stereographic projection to see that $\mathbb{R}_+^n$ is homeomorphic to one half of the sphere S^{n+1} minus N, i.e. homeomorphic to $[0,1]^n - \{0\}$. The long exact sequence

$$H_c^p([0,1]^n - \{0\}, k) \to H^p([0,1]^n, k) \to H^p(\{0\}, k) \to$$

allow us to make the conclusion.

<div align="right">Q.E.D.</div>

<u>The inclusion</u> $j: D \to \mathbb{R}^n$ <u>of an open disc</u> in $\mathbb{R}^n$ induces an isomorphism

8.5
$$j_!: H_c^{\cdot}(D, k) \xrightarrow{\sim} H_c^{\cdot}(\mathbb{R}^n, k)$$

<u>Proof.</u> Consider the exact sequence

$$\to H_c^p(D, k) \to H_c^p(\mathbb{R}^n, k) \to H_c^p(\mathbb{R}^n - D, k) \to$$

to see that it suffices to prove that $H_c^{\cdot}(\mathbb{R}^n - D, k) = 0$. This can be seen by the method used in the proof of 8.4.

<div align="right">Q.E.D.</div>

<u>The degree of a proper map</u> $f: \mathbb{R}^n \to \mathbb{R}^n$ is by definition the integer <u>deg(f)</u> characterized by the formula

8.6
$$f^*\alpha = \deg(f)\alpha \qquad ; \alpha \in H_c^n(\mathbb{R}^n, \mathbb{Z}) .$$

<u>Proposition 8.7</u>. A proper map $f: \mathbb{R}^n \to \mathbb{R}^n$ whose degree is different from zero is surjective.

<u>Proof</u>. The image $Z = f(\mathbb{R}^n)$ is a closed subset of $\mathbb{R}^n$. Assume $Z \neq \mathbb{R}^n$ and let $i: Z \to \mathbb{R}^n$ denote the inclusion. Consider the exact sequence where j denotes the inclusion of $U = \mathbb{R}^n - Z$ in $\mathbb{R}^n$

$$H_c^n(U, \mathbb{Z}) \xrightarrow{j_!} H_c^n(\mathbb{R}^n, \mathbb{Z}) \xrightarrow{i^*} H_c^n(Z, \mathbb{Z})$$

The map $j_!$ is surjective as one sees by considering an open disc D contained in U and noticing that the inclusion $k: D \to \mathbb{R}^n$ induces an isomorphism

$$k_!: H_c^n(D, \mathbb{Z}) \longrightarrow H_c^n(\mathbb{R}^n, \mathbb{Z})$$

It follows that i^* is zero. Since $f: \mathbb{R}^n \to \mathbb{R}^n$ can be factored through i it follows that $f^* = 0$.

$$Q.E.D.$$

<u>Proposition 8.8</u>. A complex polynomial, $n \geq 1$

$$f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$$

defines a proper map $f: \mathbb{C} \to \mathbb{C}$ of degree n.

<u>Proof</u>. Let us first show that f is properly homotopic to the map $z \mapsto z^n$. The formula

$$F(z,t) = z^n + t a_1 z^{n-1} + \ldots + t a_n$$

defines a continuous map $F: \mathbb{C} \times [0,1] \to \mathbb{C}$. To see that F is proper, notice the inequality

$$|F(z,t)| \geq |z^n| \; (1 - |a_1 z^{-1}| \ldots - |a_n z^{-n}|)$$

and use this to find a constant r such that

$$|F(z,t)| \geq \tfrac{1}{2}|z^n| \qquad ; \quad t \in [0,1], \; |z| \geq r.$$

From this follows that $F: \mathbb{C} \times [0,1] \to \mathbb{C}$ is proper.

To see that $z \mapsto z^n$ has degree n it suffices to calculate the action on the open unit disc D^2. The exact sequence

$$H_c^1(B^2, \mathbb{Z}) \to H_c^1(S^1, \mathbb{Z}) \to H_c^2(D^2, \mathbb{Z}) \to H_c^2(B^2, \mathbb{Z})$$

shows that $H^1(S^1, \mathbb{Z}) \xrightarrow{\sim} H_c^2(D^2, \mathbb{Z})$. The action of $z \mapsto z^n$ on $H^1(S^1, \mathbb{Z})$ is multiplication by n as it follows from 4.15.

$$Q.E.D.$$

<u>Fundamental theorem of algebra 8.9</u>. Any non constant complex polynomial has a root in $\mathbb{C}$.

<u>Proof</u>. Combine 8.7. and 8.8.

$$Q.E.D.$$

<u>Proposition 8.10</u>. A transformation $\sigma \in Gl_n(\mathbb{R})$ has degree $\text{sign}(\det(\sigma))$, the sign of the determinant.

<u>Proof</u>. The automorphism group of $H_c^n(\mathbb{R}^n, \mathbb{Z})$ is cyclic of order 2: multiplication with $+1$ or multiplication with -1. Using the fact that the commutator group of $Gl_n(\mathbb{R})$ is $Sl_n(\mathbb{R})$ we find that $\deg(\sigma)$ depends only on $\det(\sigma)$. Thus it suffices to treat the case of a diagonal matrix of the form

$$\sigma = \begin{Bmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & r \end{Bmatrix}$$

This will be done by induction on the dimension. The case $n = 1$ has been treated in 4.10. The inductive step is accomplished by the Mayer-Vietoris sequence 1.8

$$\to H_c^p(\mathbb{R}^n, \mathbb{Z}) \to H_c^p(\mathbb{R}_+^n, \mathbb{Z}) \oplus H_c^p(\mathbb{R}_-^n, \mathbb{Z}) \to H_c^p(\mathbb{R}_+^n \cap \mathbb{R}_-^n, \mathbb{Z}) \to$$

and the result 8.4.

$$Q.E.D.$$

<u>Corollary 8.11</u>. An orthogonal transformation $\sigma \in O_{n+1}(\mathbb{R})$ acts on $H^n(S^n, \mathbb{Z})$ as multiplication by $\det(\sigma)$, the determinant of σ.

<u>Proof</u>. Let D^{n+1} denote the open unit disc in $\mathbb{R}^{n+1}$. It follows from 8.5 that $O_{n+1}(\mathbb{R})$ acts on $H_c^{n+1}(D^{n+1}, \mathbb{Z})$ through the determinant. The exact sequence

$$\to H_c^n(B^{n+1}, \mathbb{Z}) \to H_c^n(S^n, \mathbb{Z}) \to H_c^{n+1}(D^{n+1}, \mathbb{Z}) \to H_c^{n+1}(B^{n+1}, \mathbb{Z})$$

allows us to make the final conclusion.

$$Q.E.D.$$

By a <u>tangent vector field</u> on a sphere S^p we understand a continuous map $v: S^p \to \mathbb{R}^{p+1}$ such that x and $v(x)$ are orthogonal for all $x \in S^p$. As an example, the sphere S^{2n-1} admits the tangent vector field

$$(-x_2, x_1, -x_4, x_3, \ldots, -x_{2n}, x_{2n-1}) \quad ; \ x \in S^{2n-1}$$

Notice that each tangent vector is a unit vector.

<u>The hairy ball theorem 8.12</u>. Any continuous tangent vector field on an even dimensional sphere S^{2n} has a zero.

<u>Proof</u>. Assume to the contrary that $v: S^{2n} \to \mathbb{R}^{2n+1}$ represents a zero free tangent-vectorfield. By normalization we can assume that $v(x)$ is a unit vector for all $x \in S^{2n}$. Consider the homotopy $f: S^{2n} \times [0,1] \to S^{2n}$ given by

$$f(x,t) = \cos(\pi t)x + \sin(\pi t)v(x)$$

Notice, that $f(x,0) = x$ and $f(x,1) = -x, \ x \in S^{2n}$. This shows that the <u>antipodal</u> map is homotopic to the identity, contradicting 8.11.

$$Q.E.D.$$

Topological manifolds

   A topological space X is called a <u>topological n-manifold</u>,
if it is a Hausdorff space, and each point x of X has an
open neighbourhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. In the first
case x is called an <u>interior</u> point, otherwise a <u>boundary point</u>.
The set of boundary points forms the boundary ∂X of X.

   In order to distinguish the two sets of points we shall
introduce a presheaf $o\hbar$ on X. For the inclusion j: V → U
of two open subsets of X consider

$$j_!: H_c^n(V,k) \longrightarrow H_c^n(U,k)$$

and the linear transposed map

8.13 $\mathrm{Hom}_k(H_c^n(U,k),k) \to \mathrm{Hom}_k(H_c^n(V,k),k)$

The presheaf $o\hbar$ on X defined by 8.13 is called the <u>orientation</u>
<u>presheaf on</u> X relative to k. From 8.4 and 8.5 we find

8.14 $o\hbar_x \;\widetilde{\approx}\; \begin{cases} 0 & \text{if } x \in \partial X \\ k & \text{if } x \in X-\partial X \end{cases}$

   <u>Proposition 8.15</u>. A homemorphism between topological mani-
folds will transform boundary points into boundary points.

   <u>Proof</u>. Follows from 8.14.

                                                          Q.E.D.

   <u>Example 8.16</u>. A Möbius band M is not homeomorphic to an
ordinary band $S^1 \times [0,1]$ since ∂M is connected while the
ordinary band has a disconnected boundary.

III.9   Dimension of locally compact spaces

   In this section we shall prove a number of vanishing theorems
for cohomology with compact support. A basic such result is

9.1 $H_c^2(\mathbb{R},F) = 0$ for all sheaves F on $\mathbb{R}$

We shall derive this from the following general principle.

   <u>Minimality principle 9.2</u>. Let F be a sheaf on the locally
compact space X and let Z denote a family of closed subsets
of X stable under arbitrary intersections, in particular $X \in Z$.
Let $\alpha \in H_c^p(X,F)$ be a non vanishing cohomology class. For
$Z \in Z$ let $\alpha|_Z \in H_c^p(Z,F)$ denote the restriction of α to Z.
The ordered set

$$\{Z \in Z \mid \alpha|_Z \neq 0\}$$

contains minimal elements.

   <u>Proof</u>. Follows immediately from Zorn's lemma and lemma 9.3
below.

                                                          Q.E.D.

   <u>Lemma 9.3</u>. Let F denote a sheaf on the locally compact
space X and Z a downward directed family of subsets of X
with intersection W. We have

$$\varinjlim H_c^p(Z,F) \;\widetilde{\to}\; H_c^p(W,F)$$

the limit taken over $Z \in Z$.

Proof. For a closed subset Y of X, let $i: Y \longrightarrow X$ denote the inclusion and put $F_Y = i_* i^* F$. For closed sets $Z \supseteq Y$, the adjunction morphism, II.5.5, will induce a morphism

$$F_Z \longrightarrow F_Y$$

and thereby create a direct system of sheaves on the set of closed subsets of X. With the notation above, we find by localization, that

$$\varinjlim F_Z \xrightarrow{\sim} F_W$$

Conclusion by 5.1 and 7.3.

$$Q.E.D.$$

Proof of 9.1. Given $\alpha \in H_c^2(\mathbb{R}, F)$ with $\alpha \neq 0$. Choose a closed subset Z of $\mathbb{R}$ such that $\alpha|_Z \neq 0$ and which is minimal with that property. Let z be a point of Z which separates two points of Z and put

$$Z_- =]-\infty, z] \cap Z, \quad Z_+ = [z, +\infty[\cap Z$$

Use the Mayer-Vietoris sequence 1.8

$$\rightarrow H_c^1(\{z\}, F) \rightarrow H_c^2(Z, F) \rightarrow H_c^2(Z_-, F) \oplus H_c^2(Z_+, F) \rightarrow$$

to obtain a contradiction.

$$Q.E.D.$$

Definition 9.4. Let X be a locally compact space. The dimension of X, dim X, is the smallest integer n for which

$$H_c^{n+1}(X, F) = 0 \quad \text{for all sheaves } F \text{ on } X$$

Using imbedding into injective sheaves it is easy to deduce that for any sheaf F on X

$$H_c^i(X, F) = 0 \quad \text{for } i > \dim X.$$

Proposition 9.5. For a locally compact space X we have

$$\dim X \times \mathbb{R} \leq \dim X + 1$$

Proof. Let X have finite dimension n, and let Z denote the set of closed subsets of $X \times \mathbb{R}$ of the form $X \times Z$, where Z is a closed subset of $\mathbb{R}$. For a sheaf F on $X \times \mathbb{R}$ and a non trivial cohomology class

$$\alpha \in H_c^{n+2}(X \times \mathbb{R}, F) \qquad ; \alpha \neq 0$$

use the minimality principle 9.2 and a Mayer-Vietoris sequence similar to the one used in the proof of 9.1 to deduce a contradiction.

$$Q.E.D.$$

Example 9.6. dim $\mathbb{R}^n = n$.

Proposition 9.7. A locally closed subspace W of the locally compact space X satisfies $\dim W \leq \dim X$

Proof. This is a consequence of 7.3.

$$Q.E.D.$$

Example 9.8. For a proper closed subset Z of $\mathbb{R}^n$

$$H_c^n(Z,k) = 0$$

for any constant sheaf k. - Let $j: U \longrightarrow \mathbb{R}^n$ denote the inclusion of the complement and consider the exact sequence.

$$H_c^n(U,k) \xrightarrow{j_!} H_c^n(\mathbb{R}^n,k) \longrightarrow H_c^n(Z,k) \longrightarrow H_c^{n+1}(U,k)$$

The last group is zero by 9.7. The map $j_!$ is surjective as it follows by considering an open disc $D \subseteq U$ and using 8.5.

Proposition 9.9. Let X be a locally compact space of dimension $\leq n$. Given an exact sequence

$$0 \to F \to S^0 \to S^1 \to \ldots \to S^{n-1} \to S^n \to 0$$

If $S^0, S^1, \ldots, S^{n-1}$ are soft sheaves, then S^n is soft.

Proof. It follows from the long exact sequence III.7 and the material in III.2, that

A sheaf G on X is soft if and only if $H_c^1(U,G) = 0$ for all open subsets U of X.

By splitting the complex into short exact sequences one sees that for any open set U

$$H_c^1(U,S^n) = H_c^{n+1}(U,F).$$

The last group is zero by 9.7.

$$Q.E.D.$$

Definition 9.10. For a point x of the locally compact space X, we put

$$\dim_x X = \inf \dim U$$

where U varies over open neighbourhoods of x in X.

The concept of dimension is local in the sense made precise by the formula

9.11 $$\dim X = \sup_{x \in X} \dim_x X$$

Proof. It follows from 9.7 that

$$\sup_x \dim_x X \leq \dim X$$

Put $n = \sup \dim_x X$. For a given sheaf F on X choose a resolution

$$0 \to F \to S^0 \to S^1 \to \ldots \to S^{n-1} \to S^n \to 0$$

where $S^0, \ldots, S^{n-1}$ are soft sheaves. For each point x of X choose an open neighbourhood U_x of x with $\dim U_x = \dim_x X$. According to 9.9 the restriction of S^n to U_x is soft. It follows from 2.8 that S^n is a soft sheaf on X. This proves that $H_c^{n+1}(X,F) = 0$.

$$Q.E.D.$$

III.10   Wilder's finiteness theorem

In this section we shall use a Mayer-Vietoris technique to establish a basic finiteness result.

<u>Theorem 10.1</u>. Let X be a compact topological manifold with boundary and k a noetherian ring. Each cohomology module $H^i(X,k)$ is finitely generated over k.

We shall prove a theorem which is both more general and contains more information in the concrete case.

<u>Theorem 10.2</u>. Let X be a locally compact space, k a noetherian ring and F a k-sheaf on X. Suppose that for any $i \in \mathbb{N}$,   any $x \in X$ and any compact neighbourhood N of x, there exists a compact neighbourhood $M \subseteq N$ of x such that the map $H^i(N,F) \to H^i(M,F)$ has finitely generated image. Then for any pair of compact subsets L and K of X with $\overset{.}{L} \supseteq K$ the restriction map

$$H^i(L,F) \longrightarrow H^i(K,F) \qquad ;\ i \in \mathbb{Z}$$

has finitely generated image.

<u>Proof</u>. This is done by increasing induction on $i \in \mathbb{Z}$.   To accomplish the inductive step, let us fix L and let L denote the set of compact subsets A for which there exists a compact subset C with $A \subseteq \overset{.}{C} \subseteq C \subseteq \overset{.}{L}$ such that $H^i(L,F) \to H^i(C,F)$ has finitely generated image. The set L has the following properties.

1) Every point x has a compact neighbourhood $M \in L$.

2) If $B \in L$ then any compact subset of B belongs to L.

3) If $A \in L$ and $R \in L$, then $A \cup R \in L$. Only 3) needs a proof which will be given below. It follows from Borel-Heine and 1), 2), 3) that every compact subset K of L belongs to L.

3) To see this choose for A compact sets B and C with $A \subseteq \overset{.}{B} \subseteq B \subseteq \overset{.}{C} \subseteq C \subseteq \overset{.}{L}$ such that $H^i(L,F) \to H^i(C,F)$ has finite generated image. Choose similar data S and T for R. Consider the following commutative diagram constructed on the basis of Mayer-Vietoris sequences II.5.6

$$
\begin{array}{ccc}
 & & H^i(L,F) \longrightarrow H^i(L,F) \oplus H^i(L,F) \\
 & & \downarrow \qquad \binom{1}{1} \qquad \downarrow \\
H^{i-1}(C \cap T,F) \longrightarrow & H^i(C \cup T,F) \longrightarrow & H^i(C,F) \oplus H^i(T,F) \\
\downarrow & \downarrow & \binom{1}{1} \\
H^{i-1}(B \cap S,F) \longrightarrow & H^i(B \cup S,F) &
\end{array}
$$

and conclude on the basis of the lemma below that the map $H^i(L,F) \to H^i(B \cup S,F)$ has finitely generated image.

Q.E.D.

<u>Lemma 10.3</u>. Given a commutative diagram of modules over a noetherian ring. Suppose that CDE is exact and that CE and BE have finitely generated image. Then the composite AE of AD and DE has finitely generated image.

$$
\begin{array}{ccc}
 & A \longrightarrow B \\
 & \downarrow \qquad \downarrow \\
C \longrightarrow & D \longrightarrow E \\
\downarrow & \downarrow \\
E \longrightarrow & E
\end{array}
$$

<u>Proof</u>. Left to the reader with the recommendation to reduce to the case where A = B and B = E.

Q.E.D.

IV. Cohomology and Analysis

IV.1 Homotopy invariance of sheaf cohomology

In this section we shall prove that sheaf cohomology with constant coefficient is a homotopy invariant of the space. Recall that continuous maps $f,g: X \to Y$ are said to be homotopic if there exists a continuous map $F: X \times [0,1] \to Y$ with

$$F(x,0) = f(x), \quad F(x,1) = g(x) \quad ; \quad x \in X$$

Theorem 1.1. Homotopic continuous maps $f,g: X \to Y$ induce the same map

$$f^*, g^*: H^{\cdot}(Y,k) \longrightarrow H^{\cdot}(X,k)$$

for any abelian group k.

Let us first indicate how this is done in case X and Y are locally compact. The first thing to do is to prove the Vietoris-Begle mapping theorem 1.2 below for ordinary cohomology as opposed to cohomology with compact support, III.6.4. Next, adapt the proof of III.6.6 to ordinary cohomology. None of this offers any difficulties and is left to the reader.

Vietoris-Begle mapping theorem 1.2. Let $f: X \to Y$ be a proper map between locally compact spaces with acyclic fibers. For any sheaf G on Y the induced map

$$f^*: H^{\cdot}(Y,G) \longrightarrow H^{\cdot}(X,f^*G)$$

is an isomorphism.

Proof of 1.1. Let us use an ad hoc terminology and say that a continuous map $f: X \to Y$ between topological spaces is proper if

1.3
i. f is closed, i.e. the image of a closed set is closed.

ii. Any two distinct points of X in the same fiber has disjoint neighbourhoods in X.

iii. The fibers of f are compact.

For a proper map $f: X \to Y$ we have an isomorphism

1.4
$$(R^p f_* F)_y = H^p(f^{-1}(y), F)$$

for $y \in Y$ and F a sheaf on X. – This statement is proved exactly as III.6.2 with the following modification of III.6.3:

Let K be a compact subspace of X with the
property that any two distinct points of K
have disjoint neighbourhoods in X. Then for
any sheaf F on X

$$H^n(K,F) = \varinjlim H^n(U,F)$$

as U varies over all open neighbourhoods of K
in X.

To prove this, notice first that the case $n = 0$ is covered
by the proof of III.2.2. This can be used to prove that a flabby
sheaf on X restricts to a soft sheaf on K. The proof can be
completed as the proof of III.6.3.

We can now generalize 1.2 to a continuous map $f: X \to Y$,
proper in the sense introduced above. Finally, copy the proof
of III.6.6, using Lemma 1.5 below.

Q.E.D.

Lemma 1.5. Let X be topological space and C a compact
topological space. The projection $p: X \times C \to X$ is closed, i.e.
the image of a closed set is a closed set.

Proof. Let Z denote a closed subset of $X \times C$ and x a
point of X outside $p(Z)$. For each point $c \in C$ choose open
neighbourhoods V_c of c and U^c of x such that $U^c \times V_c \cap Z = \emptyset$.
Let S be a finite subset of C such that $(V_c)_{c \in S}$ cover C.
The set $\underset{c \in S}{\cap} U^c$ is an open neighbourhood of x in X which
does not meet $p(Z)$.

Q.E.D.

Application to fibre bundles

Let F denote a topological space. By a **fibre bundle with**
fiber F we understand a continuous map $f: X \to Y$ such that
each point y of Y admits a neighbourhood V and a homeo-
morphism $g: f^{-1}(V) \overset{\sim}{\to} V \times F$ such that the following diagram is
commutative

$$
\begin{array}{ccc}
f^{-1}(V) & \xrightarrow[\quad g \quad]{\sim} & V \times F \\
& \searrow^{f} \quad \swarrow^{pr_1} & \\
& V &
\end{array}
$$

Theorem 1.6. Let $f: X \to Y$ be a fibre-bundle whose fibre
F is homotopic to a compact space and let k denote an abelian
group. For each point y of Y the canonical map

$$(R^p f_* \underset{\sim}{k})_y \longrightarrow H^p(f^{-1}(y), \underset{\sim}{k})$$

is an isomorphism.

Proof. The problem is local on Y so we may assume that
$f: Y \times F \to Y$ is the projection onto the first factor. Let
$g: F \to C$ be a homotopy equivalence, where C is compact and
let $p: Y \times C \to Y$ denote the projection. According to 1.5, the
map p satisfies the three conditions 1.3. Thus we can make the
conclusion 1.4 for p. It remains to involve Theorem 1.1 in
order to make the correct identifications.

Q.E.D.

IV.2 Locally compact space, countable at infinity

In this section we consider a locally compact space X which is countable at infinity, i.e. which is union of a countable family $(K_n)_{n \in \mathbb{N}}$ of compact subset. From this we can construct a filtration $(X^n)_{n \in \mathbb{N}}$ of X by compact subsets. We shall proceed inductively: $X^0 = K_0$, X^{n+1} is a compact neighbourhood of $X^n \cup K_n$. Notice,

$$2.1 \qquad\qquad X^n \subseteq \overset{\cdot}{X}^{n+1} \qquad ; \quad n \in \mathbb{N}$$

Theorem 2.2. Let X be a locally compact space, countable at infinity. A soft sheaf E on X satisfies

$$H^i(X,E) = 0 \qquad ; \; i \geq 1$$

Proof. Consider a short exact sequence of sheaves on X

$$0 \longrightarrow E \overset{c}{\longrightarrow} F \overset{f}{\longrightarrow} G \longrightarrow 0$$

where E is soft. Let us prove that $\Gamma(X,F) \to \Gamma(X,G)$ is surjective. We shall use a filtration $(X^n)_{n \in \mathbb{N}}$ of X by compact subsets satisfying 2.1. Let there be given a section t of G over X. We shall proceed inductively: Suppose we have constructed a section s^n of F over X^n which projects onto the restriction t^n of t to X^n. Choose a lifting $r^{n+1} \in \Gamma(X^{n+1},F)$ of t^{n+1}. The difference between s^n and the restriction of r^{n+1} to X^n defines a section of E over X^n. Extend this to a section of E over X^{n+1} and add the result to r^{n+1}. This gives a section

s^{n+1} of F over X^{n+1} which extends s^n and projects onto t^{n+1}. – From the family of sections $(s^n)_{n \in \mathbb{N}}$ we can construct a section of F over X, namely the section s whose restriction to $\overset{\cdot}{X}^n$ agrees with the restriction of s^n to $\overset{\cdot}{X}^n$. It is clear that s projects onto t.

If we apply this result to an exact sequence with F injective, we get $H^1(X,E) = 0$. Let us notice that the sheaf G is soft by the proof of III.2.7. The cohomology sequence

$$\longrightarrow H^{p-1}(X,G) \longrightarrow H^p(X,E) \longrightarrow H^p(X,F)$$

will now allow us to conclude the proof by induction.

$$Q.E.D.$$

Corollary 2.3. Let X denote a locally compact space countable at infinity and E a soft sheaf on X. For any closed subspace Z of X the restriction

$$\Gamma(X,E) \longrightarrow \Gamma(Z,E)$$

is surjective.

Proof. Put $U = X-Z$ and let $i: Z \to X$ and $j: U \to X$ denote the inclusions. The exact sequence

$$0 \longrightarrow j_! j^*E \longrightarrow E \longrightarrow i_* i^*E \longrightarrow 0$$

consists of soft sheaves, III.2.5 and III.7.2. The exact sequence

$$0 \to \Gamma(X, j_! j^*E) \to \Gamma(X, E) \to \Gamma(Z, i^*E) \to H^1(X, j_! j^*E)$$

and 2.2. allows us to make the conclusion.

$$Q.E.D.$$

Proposition 2.4. Let X denote a locally compact space countable at infinity. For any sheaf F on X and any closed subspace Z of X we have

$$\varinjlim H^p(U, F) \overset{\sim}{\to} H^p(Z, F) \qquad ; \quad p \in \mathbb{N}$$

Proof. Let us consider the case $p = 0$. The map

$$\varinjlim \Gamma(U, F) \to \Gamma(Z, F)$$

is injective for general reasons and surjective by 2.3. Consider an exact sequence

$$0 \longrightarrow F \longrightarrow I \longrightarrow J$$

where I and J are injective sheaves. This induces a commutative exact diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \varinjlim \Gamma(U, F) & \longrightarrow & \varinjlim \Gamma(U, I) & \longrightarrow & \varinjlim \Gamma(U, J) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(Z, F) & \longrightarrow & \Gamma(Z, I) & \longrightarrow & \Gamma(Z, J)
\end{array}
$$

The two vertical arrows to the right are isomorphisms by our previous remarks. From this we conclude that the last vertical arrow is an isomorphism.

In the general case, choose an injective resolution $F \to I^{\cdot}$. The resolution $i^*F \to i^*I^{\cdot}$ is a soft resolution and consequently a $\Gamma(Z, -)$-acyclic resolution as it follows from the fact that Z is countable at infinity and 2.2. Consequently

$$\varinjlim H^p(U, F) = H^p \varinjlim \Gamma(U, I^{\cdot}) = H^p \Gamma(Z, i^*I^{\cdot}) = H^p(Z, F)$$

$$Q.E.D.$$

Let me list three useful remarks from pointset topology, which will facilitate the applications of the material of this section.

2.5   A locally compact space with a countable basis is countable at infinity.

2.6   A subspace of a space with a countable basis has itself a countable basis.

2.7   A metric space has a countable basis if and only if it contains a countable dense subset.

Remark 2.8. The results of this section can be generalized to a locally compact space which is paracompact. This can be done by using the fact that such a space is the topological sum of locally compact spaces each of which is countable at infinity, Bourbaki (2) §9, no. 10, Thm. 5. The general reference for sheaves on paracompact spaces is Godement (1).

IV.3 Complex Logarithms

In this section we shall make some sheaf theoretical interpretations of a number of elementary problems in ordinary complex analysis.

Argument of complex number

Let U denote a connected open subset of the complex plane. The exponential sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi} C(\mathbb{R}) \xrightarrow{\text{expi}} C(S^1) \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi} \Gamma(U, C(\mathbb{R})) \xrightarrow{\text{expi}} \Gamma(U, C(S^1)) \longrightarrow H^1(U, \mathbb{Z}) \longrightarrow 0$$

where we have used that $H^1(U, C(\mathbb{R})) = 0$, 2.2 and III.3.3.

Let us suppose that U does not contain the origin, in which case $C(S^1)$ has a canonical section over U, namely $z \longmapsto z|z|^{-1}$. Any lifting of this section to $C(\mathbb{R})$ is called an <u>argument</u> and is denoted <u>arg</u>. Thus

$$z = |z| \exp(i \arg z) \qquad ; \; z \in U$$

Collecting this together we have proved that

3.1

If $H^1(U, \mathbb{Z}) = 0$ then we can find a continuous argument function on U.

Logarithms of complex numbers

The multiplicative group of complex numbers $\mathbb{C}^*$ fits into an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathbb{C} \xrightarrow{\text{exp}} \mathbb{C}^* \longrightarrow 0$$

of topological groups. The exponential map has local sections in virtue of the decomposition

$$
\begin{array}{ccccc}
x+iy & \mathbb{C} & \longrightarrow & \mathbb{C}^* & z \\
& \downarrow & & \downarrow & \\
(x,y) & \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R}^+ \oplus S^1 & (|z|, z|z|^{-1}) \\
& (x,y) & \longmapsto & (e^x, \exp i\, y) &
\end{array}
$$

and the fact that $\text{expi}: \mathbb{R} \longrightarrow S^1$ has local sections. There results an exact sequence of sheaves on $\mathbb{C}$

3.2
$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} C(\mathbb{C}) \xrightarrow{\text{exp}} C(\mathbb{C}^*) \longrightarrow 0$$

For a connected open subset U of $\mathbb{C}$ we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \Gamma(U, C(\mathbb{C})) \xrightarrow{\text{exp}} \Gamma(U, C(\mathbb{C}^*)) \longrightarrow H^1(U, \mathbb{Z}) \longrightarrow 0$$

where we have used $H^1(U, C(\mathbb{C})) = 0$.

If U does not contain the origin, the inclusion of U in $\mathbb{C}^*$ represents an element of $\Gamma(U, C(\mathbb{C}^*))$. Any lifting of this to a section of $C(\mathbb{C})$ over U will be called a <u>logarithm</u> on U. Thus a logarithm on U is a continuous map $\log: U \to \mathbb{C}$, such

that

$$z = \exp(\log z) \qquad ; \ z \in U$$

Using the commutative diagram above we see that logarithm and argument are related by the formula

3.3 $$\log z = \log |z| + i \arg z \qquad ; \ z \in U$$

It follows from the lemma below that <u>log z is an analytic function on U.</u> From the relation $\exp(\log z) = z$ we find at once that

3.4 $$\frac{d}{dz} \log z = \frac{1}{z} \qquad ; \ z \in U$$

<u>Lemma 3.5.</u> Any continuous section of

$$\exp: \mathbb{C} \to \mathbb{C}^*$$

over any open subset U of $\mathbb{C}^*$ is analytic.

<u>Proof.</u> By a translation argument in the group $\mathbb{C}^*$ it suffices to construct one analytic section over some open neighbourhood of 1 in $\mathbb{C}^*$. - The power series

$$\ell(z) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{z^n}{n} \qquad ; \ |z| < 1$$

has radius of convergence 1 and satisfies

$$\ell'(z) = \frac{1}{z+1} \qquad ; \ |z| < 1$$

Note, $\ell(0) = 0$. Let us prove that

$$\exp(\ell(z-1)) = z \qquad ; \ |z-1| < 1$$

Notice that the two functions have the same logarithmic derivative and both takes the value 1 for $z = 1$.

Q.E.D.

<u>Corollary 3.6.</u> Let $\mathcal{O}$ denote the sheaf of analytic functions on $\mathbb{C}$ and $\mathcal{O}^*$ the sheaf of zerofree analytic functions on $\mathbb{C}$. Then we have an exact sequence of sheaves

$$0 \longrightarrow \underset{\sim}{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

<u>n'th roots of unity 3.7.</u> For an integer $n \geq 2$ let μ_n denote the n'th roots of unity. Raising to the n'th power gives a sequence of multiplicative sheaves on the complex plane

$$0 \longrightarrow \mu_n \longrightarrow \mathcal{O}^* \xrightarrow{n} \mathcal{O}^* \longrightarrow 0$$

which is exact as it follows from 3.6. For a connected open set U of $\mathbb{C}$ we get the exact sequence

$$0 \longrightarrow \mu_n \longrightarrow \Gamma(U, \mathcal{O}^*) \xrightarrow{n} \Gamma(U, \mathcal{O}^*) \longrightarrow H^1(U, \mu_n)$$

Let there be given a fixed $f \in \Gamma(U, \mathcal{O}^*)$ where U is an open subset of $\mathbb{C}$ with $H^1(U, \mu_n) = 0$. We want to analyse the space

$$Z = \{ (x,y) \in U \times \mathbb{C} \mid y^n = f(x) \}$$

Let us prove that this is homeomorphic to $U \times \mu_n$: Choose $g \in \Gamma(U, O^*)$ with

$$g(z)^n = f(z) \qquad ; \quad z \in U$$

We can construct two continuous maps

$$
\begin{array}{ccc}
(x,y) & z & (x, \mu g(x)) \\
\downarrow & \updownarrow & \uparrow \\
(x, yg(x)^{-1}) & U \times \mu_n & (x, \mu)
\end{array}
$$

which are easily seen to be inverse to each other.

Let us drop the assumptions on U. The above result shows that $Z \to U$ is a covering space with μ_n as covering group.

The pointed complex plane $\mathbb{C}^*$

Let us recall that an analytic function f on $\mathbb{C}^*$ can be expanded in a Laurent series

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n \qquad ; \quad z \in \mathbb{C}^*$$

and that the residue of f is defined by

3.8 $$\mathrm{Res}(f,0) = a_{-1}$$

From this follows easily that we have an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \Gamma(\mathbb{C}^*, O) \xrightarrow{D} \Gamma(\mathbb{C}^*, O) \xrightarrow{\mathrm{Res}} \mathbb{C} \longrightarrow 0$$

Another basic feature is that a continuous map $f: \mathbb{C}^* \to \mathbb{C}^*$ has a <u>degree</u>, $\deg(f)$ given by

$$f^*\alpha = \deg(f)\alpha \qquad ; \quad \alpha \in H^1(\mathbb{C}^*, \mathbb{Z})$$

Using the material from III.4 we deduce an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \Gamma(\mathbb{C}^*, C(\mathbb{C})) \xrightarrow{\exp} \Gamma(\mathbb{C}^*, C(\mathbb{C}^*)) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

and as a consequence of Proposition 3.6

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \Gamma(\mathbb{C}^*, O) \xrightarrow{\exp} \Gamma(\mathbb{C}^*, O^*) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

The degree can be calculated by the formula

3.9
$$\boxed{\deg(f) = \mathrm{Res}\left(\frac{f'}{f}; 0\right)}$$

Proof. We can write f in the form

$$f(z) = z^n \exp(g(z)) \qquad ; \qquad z \in \mathbb{C}^*$$

where g is analytic in $\mathbb{C}^*$. This gives

$$\frac{f'(z)}{f(z)} = n \frac{1}{z} + g'(z)$$

Note that $\mathrm{Res}(g',0) = 0$, and the result follows.

Q.E.D.

IV.4 Complex curve integrals. The monodromy theorem.

The complex plane comes equipped with the following
sequence of sheaves

4.1
$$0 \to \underset{\sim}{\mathbb{C}} \to 0 \overset{D}{\to} 0 \to 0$$

where $D = d/dz$ is the complex differentiation. This sequence
is exact as it follows from the theory of complex power series.
Let there be given an open subset U of $\mathbb{C}$, a continuous curve
$\gamma: [a,b] \to U$ and an analytic function $f(z)$ on U. We are going
to make sense to the symbol

$$\int_\gamma f(z)\,dz$$

Consider the pull back along γ of the sequence above

$$0 \to \underset{\sim}{\mathbb{C}} \to \gamma^*0 \overset{D}{\to} \gamma^*0 \to 0$$

Since $H^1([a,b],\mathbb{C}) = 0$ we get an exact sequence

$$0 \to \mathbb{C} \to \Gamma([a,b],\gamma^*0) \to \Gamma([a,b],\gamma^*0) \to 0$$

The adjunction morphism $0 \to \gamma_*\gamma^*0$ induces a map

$$\Gamma(U,0) \to \Gamma([a,b],\gamma^*0)$$

Let γ^*f denote the image of f under this map, choose
$g \in \Gamma([a,b],\gamma^*0)$ with $Dg = \gamma^*f$ and put

$$\int_\gamma f(z)\,dz = g(b) - g(a)$$

where $g(t)$ denotes evaluation of the stalk $g_t \in 0_{\gamma(t)}$ at
$\gamma(t)$.

Let us establish the basic rules for this symbol. For an
analytic function $h(z)$ on U we have

4.2
$$\int_\gamma Dh(z)\,dz = h(\gamma(b)) - h(\gamma(a))$$

as it follows by considering $\gamma^*h \in \Gamma([a,b],\gamma^*0)$. -

Subdivide $[a,b]$ into two intervals $[a,c]$ and $[c,b]$
and let γ_1,γ_2 denote the respective restrictions. We leave
it to the reader to show that

4.3
$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

In case $\gamma: [a,b] \to U$ is continuously differentiable, our symbol
can be evaluated as follows

4.4
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

Proof. Suppose first that $f = Dh$. In that case

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt} h(\gamma(t))\,dt = h(\gamma(b)) - h(\gamma(a))$$
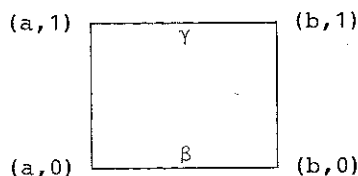
and the result follows from 4.2. In the general case we can use
4.3 to make a reduction to this case by a finite subdivision,
using Borel-Heine.

Q.E.D.

<u>Proposition 4.5</u>. Let $\beta, \gamma: [a,b] \to U$ be continuous curves which are homotopic with fixed endpoints. Then for any analytic function $f(z)$ on U

$$\int_\gamma f(z)\,dz = \int_\beta f(z)\,dz$$

<u>Proof</u>. Put $K = [a,b]$ and $I = [0,1]$ and let $\Pi: K \times I \to U$ be the required homotopy i.e. a continuous map

which is constant on the two vertical lines, and such that $\Pi(s,0) = \beta(s)$, $\Pi(s,1) = \gamma(s)$ for $s \in K$. Consider the pull back of the exact sequence 4.1 along $\Pi: K \times I \to U$

$$0 \longrightarrow \underset{\sim}{\mathbb{C}} \longrightarrow \Pi^* 0 \longrightarrow \Pi^* 0 \longrightarrow 0$$

Using $H^1(K \times I, \mathbb{C}) = 0$ we derive an exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow I(K \times I, \Pi^* 0) \xrightarrow{D} I(K \times I, \Pi^* 0) \longrightarrow 0$$

Let $\Pi^* f \in I(K \times I, \Pi^* 0)$ denote the transform of f by the adjunction morphism $0 \to \Pi_* \Pi^* 0$, and choose $G \in I(K \times I, \Pi^* 0)$ with $DG = \Pi^* f$. For the curve $\gamma_t: K \to U$, $\gamma_t(s) = \Pi(s,t)$, we have

$$\int_{\gamma_t} f(z)\,dz = G(b,t) - G(a,t) \qquad ; \quad t \in [0,1]$$

Notice that the stalks $G_{(b,t)} \in 0_{\Pi(b,0)}$ and $G_{(a,t)} \in 0_{\Pi(a,0)}$ are independent of t since $[b] \times I$ and $[a] \times I$ are connected. In particular $G(b,t) - G(a,t)$ is independent of $t \in I$.

Q.E.D.

Let $\gamma: [a,b] \to \mathbb{C} - \{0\}$ denote a continuous curve. Choose a continuous function $\theta: [a,b] \to \mathbb{R}$ such that, III.4.5

$$\gamma(t) = |\gamma(t)| \exp i\,\theta(t) \qquad ; \quad t \in [a,b]$$

We shall establish the formula

4.6
$$\int_\gamma \frac{dz}{z} = \log|\gamma(b)\,\gamma(a)^{-1}| + i(\theta(b) - \theta(a))$$

<u>Proof</u>. According to our discussion of complex logarithms IV.3 we can find $\log \in I([a,b], \gamma^* 0)$ with $D\log = \gamma^*(z \mapsto z^{-1})$ and

$$\log(t) = \log|\gamma(t)| + i\theta(t) \qquad ; \quad t \in [a,b]$$

from which the result follows.

Q.E.D.

In particular, for a closed curve $\gamma: [a,b] \to \mathbb{C} - \{0\}$, we can put

4.7
$$I(\gamma, 0) = \frac{1}{2\Pi}(\theta(b) - \theta(a))$$

and we find from 4.6

4.8
$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z} = I(\gamma;0)$$

Definition 4.9. Let $s \in \mathbb{C}$ and $\gamma: [a,b] \to \mathbb{C} - \{s\}$ be a continuous curve. The winding number of γ with respect to s is given by

$$I(\gamma;s) = I(\gamma-s;0)$$

By a simple translation argument, we find from 4.8, that

4.10
$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-s} = I(\gamma;s)$$

Logarithmic derivatives

A zerofree analytic function g on an open set U of $\mathbb{C}$ has a logarithmic derivative $D\log g = g'/g$. Notice, that

$$D\log gh = D\log g + D\log h \qquad ; \; g,h \in I(U,\mathcal{O}^*)$$

This gives a sequence of sheaves on $\mathbb{C}$

4.11
$$0 \longrightarrow \underset{\sim}{\mathbb{C}}^* \longrightarrow \mathcal{O}^* \xrightarrow{D\log} \mathcal{O} \longrightarrow 0$$

which is exact: Given an open subset U of $\mathbb{C}$, a point x of U and $f \in I(U,\mathcal{O})$. Choose a smaller neighbourhood V of x and $F \in I(V,\mathcal{O})$ with $F' = f$; this gives $D\log \exp F = f$ by direct calculation. - From 4.11 we deduce an exact sequence

$$I(U,\mathcal{O}^*) \xrightarrow{D\log} I(U,\mathcal{O}) \longrightarrow H^1(U,\mathbb{C}^*)$$

1. Order differential equation

Let X denote an open subset of $\mathbb{C}$ and f an analytic function on X. We shall analyse the first order differential equation

$$Dv = fv$$

According to our analysis of the logarithmic derivative any point of X has a neighbourhood U in which the equation admits a zerofree analytic solution v, say. - For a second solution w on U we get

$$D(wv^{-1}) = (Dw)v^{-1} - wv^{-2}Dv = fwv^{-1}-wv^{-2}fv = 0$$

In particular, if U is connected we find, that wv^{-1} is constant. In conclusion we have shown

Proposition 4.12. The solution sheaf to the 1. order equation $Dv = fv$ is a locally constant $\mathbb{C}$-sheaf of rank 1. - The equation admits a zerofree solution on any contractible open subset U of $\mathbb{C}$.

Let us specialize to the case $X = \mathbb{C} - \{0\}$. Let U_+ be the complement to $i\,]-\infty,0]$ in $\mathbb{C}$ and U_- the complement to $i[0,+\infty[$. Choose a solution $v_+ \neq 0$ on U_+ and a solution $v_- \neq 0$ on U_- such that

$$v_+(z) = v_-(z) \quad ; \quad \text{Re } z < 0, \quad z \in U_+ \cap U_-$$

Consider finally the number $m \in \mathbb{C}^*$ given by

$$v_-(z) = m\,v_+(z) \quad ; \quad \text{Re } z > 0, \quad z \in U_+ \cap U_-$$

The number $m \in \mathbb{C}^*$ is called the <u>monodromy</u> of the differential equation. The monodromy depends in fact only on the solution sheaf. In particular, the solution sheaf is constant if and only if $m = 1$.

The monodromy of the equation is

4.13
$$\boxed{m = \exp(2\pi i \, \mathrm{Res}(f;0))}$$

Proof. Let us make the two choices

$$F_+ \in \Gamma(U_+, 0) \quad \text{with} \quad DF_+ = f$$

$$F_- \in \Gamma(U_-, 0) \quad \text{with} \quad DF_- = f$$

We shall assume this done in such a way that

$$F_+(z) = F_-(z) \quad ; \quad z \in U_+ \cap U_-, \ \mathrm{Re}\, z < 0$$

To calculate the monodromy we can use the solutions $v_+ = \exp F_+$ and $v_- = \exp F_-$. We have

$$m = v_-(1) v_+(1)^{-1} = \exp(F_-(1) - F_+(1))$$

$$= \exp(F_-(1) - F_-(-1) + F_+(-1) - F_+(1))$$

$$= \exp \int_{S^1} f(z)\, dz$$

It remains to quote the formula

$$\int_{S^1} f(z)\, dx = 2\pi i \, \mathrm{Res}(f;0)$$

<u>The principle of unique continuation 4.14.</u> Let E be a sheaf on the topological space X which satisfies the principle of unique continuation i.e. <u>any section of E over any open set has open support.</u> Given an open connected subset U of E and two sections $s, t \in \Gamma(U, E)$. If $s_x = t_x$ for one point x of U, then $s = t$.

Given a continuous map $f: W \to X$ then the sheaf f^*E on W will likewise satisfy the principle of unique continuation. In particular, constant sheaves or more generally <u>locally constant sheaves</u> satisfy the principle of unique continuation.

We are especially interested in the fact that the <u>sheaf 0 of analytic functions</u> on the complex plane satisfies the principle of unique continuation.

<u>Analytic continuation 4.15.</u> Let c be a point of the complex plane and $s \in 0_c$ an analytic germ, i.e. a convergent power series. Given a continuous curve $\gamma: [a,b] \to \mathbb{C}$ with $\gamma(a) = c$. By an analytic continuation of s along γ we understand a section

$$g \in \Gamma([a,b], \gamma^*0) \quad ; \quad g_a = s$$

An analytic continuation of s along γ is unique if it exists by the principle of unique continuation.

<u>Monodromy theorem 4.16.</u> Let X denote a connected open subset of the complex plane, c a point of X and $s \in 0_c$ an analytic germ, which admits analytic continuation along any continuous curve $\theta: [a,b] \longrightarrow X$ with $\theta(a) = c$. - Given continuous curves

$\beta, \gamma: [a,b] \to X$ with $\beta(a) = \gamma(a) = c$ and $\beta(b) = \gamma(b)$, which are homotopic in X with fixed end points. Analytic continuations of s along β and γ respectively have the same stalk at the other endpoint b.

Proof. Let $\Pi: [a,b] \times [0,1] \to X$ denote a homotopy as in the proof of 4.5. The sheaf $A = \Pi^* 0$ has the section s over $\{a\} \times [0,1]$ which can be extended to a section over $[a,b] \times \{t\}$ for all $t \in [0,1]$. By proper base change 1.4, such a section can be extended to a section s_t of A over $[a,b] \times V_t$ where V_t is an open neighbourhood of t in $[0,1]$. By the principle of unique continuation, these sections will determine a global section of A. Conclusion by the fact that the restriction of A to $\{b\} \times [0,1]$ is constant.

Q.E.D.

A wide class of examples, where the monodromy theorem applies, is the case where s satisfies a linear differential equation

$$4.17 \qquad s^{(n)} + a_1 s^{(n-1)} + \ldots + a_{n-1} s + a_n = 0$$

where $a_1, \ldots, a_n$ are analytic functions on X.

To see this let us remark that the solution sheaf E to the equation 4.17 is a locally constant $\mathbb{C}$-sheaf of rank n as it follows from the theory of differential equations over an open disc. Given a continuous curve $\gamma: [a,b] \to X$. Then $\gamma^* E$ is a constant sheaf on $[a,b]$, as it follows from the following lemma, and consequently any germ over a can be continued over the whole interval.

Lemma 4.18. Any locally constant sheaf on $[0,1]$ is constant.

Proof. Use the procedure from the proof of III.4.5 to find a finite subdivision of $[0,1]$ such that the sheaf is constant on each subinterval and proceed by a simple induction.

Q.E.D.

Proposition 4.20. Let X be a simply connected and locally path-connected topological space. Any locally constant sheaf on X is constant.

Proof. Let E be a locally constant sheaf on X and c a point of X. The restriction map $\Gamma(X,E) \to E_c$ is injective by the principle of unique continuation. Let us prove that a given $s \in E_c$ extends to a global section of E.

For $x \in X$ choose a path γ_x from c to x and choose an open, path connected neighbourhood U_x of x such that the restriction of E to U_x is constant. Let $s_x \in \Gamma(U_x, E)$ be the continuation of s along γ_x.

We must show that given x and y in X, then s_x and s_y have the same stalk at any point z of $U_x \cap U_y$. To this end connect c and z in two ways: Follow γ_x from c to x and pass from x to z by a path in U_x. Similarly, follow γ_y from c to y and pass from y to z by a path in U_y. - The two paths from c to z are homotopic and the result follows from the monodromy principle.

Q.E.D.

<u>Definition 4.21</u>. Let E be a locally constant sheaf on the topological space X and $\gamma: [0,1] \to X$ a loop with $\gamma(0) = \gamma(1) = c$. The linear transformation

$$E_c \xrightarrow{\sim} \gamma^*E_0 \xrightarrow{r_0^{-1}} \Gamma([0,1],\gamma^*E) \xrightarrow{r_1} \gamma^*E_1 \xrightarrow{\sim} E_c$$

where r_0 and r_1 denote restriction to the stalks, is called the <u>monodromy</u> along γ.

<u>Euler's equation 4.22</u>. The differential equation

$$z^2 w'' + azw' + bw = 0 \qquad ; \quad a,b \in \mathbb{C}$$

has a solution sheaf on $\mathbb{C}-\{0\}$ which is a locally constant $\mathbb{C}$-sheaf of rank two. In order to calculate the monodromy matrix we introduce the operator

$$L(w) = z^2 w'' + azw' + bw$$

and the polynomial

$$q(\nu) = \nu(\nu-1) + a\nu + b$$

By explicit calculation we find that

$$L(z^\nu) = q(\nu)z^\nu$$

and by applying $\frac{\partial}{\partial \nu}$ to this formula

$$L(z^\nu \log z) = q'(\nu)z^\nu + q(\nu)z^\nu \log z$$

If the indicial equation $q(\nu) = 0$ has two solutions r and s, the Euler equation has solutions z^r and z^s. If r is a double root of the indicial equation, then we have solutions z^r and $z^r \log z$. By analytic continuation and counter clockwise

$$z^r \longmapsto e^{2\pi i r} z^r$$

$$\log z \longmapsto \log z + 2\pi i$$

Consequently we find the monodromy matrix

$r \neq s$	$r = s$
$\begin{Bmatrix} e^{2\pi i r} & 0 \\ 0 & e^{2\pi i s} \end{Bmatrix}$	$e^{2\pi i r} \begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}$ *)

*) Notice that $\begin{Bmatrix} 1 & 2\pi i \\ 0 & 1 \end{Bmatrix}$ is conjugated to $\begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}$.

<u>Bessel's equation 4.23</u>.

$$z^2 w'' + zw' + (z^2 - n^2)w = 0$$

has monodromy matrix

$n \notin \mathbb{Z}$	$n \in \mathbb{Z}$
$\begin{Bmatrix} e^{2\pi i n} & 0 \\ 0 & e^{-2\pi i n} \end{Bmatrix}$	$\begin{Bmatrix} 1 & 1 \\ 0 & 1 \end{Bmatrix}$

In case $n = 0$ this can be seen from the explicit solutions

$$J_0(z) = \sum_{m=0}^{+\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m}$$

$$K_0(z) = -\sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\left(\frac{z}{2}\right)^{2m} + \log z J_0(z)$$

IV.5 The inhomogeneous Cauchy-Riemann equations

A smooth $\mathbb{C}$-valued function f on an open subset of the complex plane is known to be analytic if and only if f = u+iv satisfies the Cauchy-Riemann equations

5.1
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ , \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

Introducing the operator $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ this can be expressed by $\frac{\partial}{\partial \bar{z}}$ f = 0.

Theorem 5.2. Let g be a smooth $\mathbb{C}$-valued function on the open subset X of $\mathbb{C}$. Then the partial differential equation

$$\frac{\partial}{\partial \bar{z}} f = g$$

can be solved with a smooth function f on X.

Proof. Let us introduce the sheaf C of $\mathbb{C}$-valued smooth functions on the complex plane. We shall be concerned with the morphism $\frac{\partial}{\partial \bar{z}}$: $C \to C$.

Functions with compact support

The point of departure is the existence of an operator

$$T: \Gamma_c(\mathbb{C}, C) \longrightarrow \Gamma(\mathbb{C}, C)$$

with $(\frac{\partial}{\partial \bar{z}} \circ T)(f) = f$ for all $f \in \Gamma_c(\mathbb{C}, C)$. Such an operator is given by

$$Tf(w) = \frac{1}{2\pi i} \iint \frac{f(z)}{z-w} dz \wedge d\bar{z} \qquad ; \ w \in \mathbb{C}$$
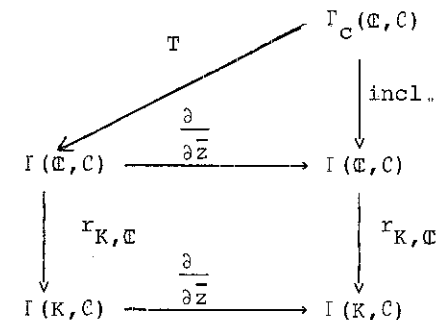
See this chapter 7.20 for an elementary discussion.

A compact subset K of $\mathbb{C}$

The induced operator

$$\frac{\partial}{\partial \bar{z}}: \Gamma(K, C) \longrightarrow \Gamma(K, C)$$

is surjective as it follows from the accompanying commutative diagram, noticing that the composite of the two vertical arrows to the extreme right is surjective, as it follows from the fact that C is a soft sheaf.

The case $X = \mathbb{C}$

Given $g \in \Gamma(\mathbb{C}, C)$ we are looking for $f \in \Gamma(\mathbb{C}, C)$ with $\frac{\partial}{\partial \bar{z}} f = g$. Choose a sequence of compact discs $K^0, \ldots, K^n, \ldots$ with center in the origin and union $\mathbb{C}$. We are going to construct a sequence $f_0, f_n, \ldots$ where $f_n \in \Gamma(K^n, C)$ with $\frac{\partial}{\partial \bar{z}} f_n = g$ and

$$\sup_{K^n} |f^{n+1} - f^n| \leq 2^{-n}$$

Suppose we have already constructed $f_0, \ldots, f_n$. Choose $\varphi_{n+1} \in \Gamma(K^{n+1}, C)$ with $\frac{\partial}{\partial \bar{z}} \varphi_{n+1} = g$. Notice that $\frac{\partial}{\partial \bar{z}} (\varphi_{n+1} - f_n) = 0$ which proves that $\varphi_{n+1} - f_n$ is analytic on K^n. Expand this function in a power series around the origin to see that there exists a polynomial p_{n+1} such that

$$\sup_{K^n} |\varphi_{n+1} - f - p_{n+1}| \leq 2^{-n}$$

and put $f_{n+1} = \varphi_{n+1} - p_{n+1}$. Define $f(z) = \lim\limits_{n\to\infty} f_n(z)$. Notice

for $z \in K^n$ we have

$$f(z) = \lim\limits_{p\to\infty}(f_p(z) - f_n(z)) + f_n(z)$$

which shows that $f - f_n$ is analytic on K^n and consequently

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{\partial}{\partial \bar{z}} f_n(z) = g(z).$$

The general open subset X of $\mathbb{C}$

We are going to follow the proof scheme of the previous case. Thus we must show the existence of an increasing sequence $K^0, \ldots, K^n$ of compact subsets of X, such that every compact subset of X is contained in one of the K^n's and such that any $f \in \Gamma(K^n, 0)$ can be uniformly approximated by analytic functions on X.

We shall use Runge's approximation theorem 5.3 below. For $n \in \mathbb{N}$ let K^n be the complement in $\mathbb{C}$ to the set

$$\bigcup_{a\in\mathbb{C}-X} D(a;\tfrac{1}{n}) \cup \{x \in \mathbb{C} \mid |x| > n\}$$

Let us prove that any bounded connected component V of $\mathbb{C} - K^n$ meets $\mathbb{C} - X$: A point x of V must be contained in one of the discs $D(a, \tfrac{1}{n})$. This disc must be contained in V since V is a connected component, while $a \in V$. Conlusion by Runge's theorem.

Q.E.D.

Runge's approximation theorem 5.3. Let K be a compact subset of the complex plane, and P a subset of $\mathbb{C} - K$ consisting in one point from each bounded, connected component of $\mathbb{C} - K$. Any analytic function defined in a neighbourhood of K can be uniformly approximated on K by rational functions with poles in P.

An elementary reference is Burckel (1) VIII.§2.8.11.

IV.6 Existence theorems for analytic functions

We shall analyse a number of classical existence theorems in complex analysis by means of the following basic

Theorem 6.1. The sheaf 0 of analytic functions on the complex plane satisfies

$$H^i(X, 0) = 0 \qquad ; \; i \geq 1$$

for any open subset X of $\mathbb{C}$.

Proof. Let C denote the sheaf of $\mathbb{C}$-valued smooth functions on the complex plane. The sequence

$$0 \longrightarrow 0 \longrightarrow C \xrightarrow{\frac{\partial}{\partial \bar{z}}} C \longrightarrow$$

is exact as it follows from the local part of 5.2. We are in fact dealing with a soft resolution of the sheaf 0. Thus $H^{\cdot}(X, 0)$ can be calculated as the cohomology groups of the complex

$$0 \longrightarrow \Gamma(X, C) \xrightarrow{\frac{\partial}{\partial \bar{z}}} \Gamma(X, C) \longrightarrow 0$$

Conclusion by 5.2.

Q.E.D.

Primitives

The exact sequence 4.1

$$0 \longrightarrow \underset{\sim}{\mathbb{C}} \longrightarrow 0 \xrightarrow{D} 0 \longrightarrow 0$$

gives rise to a cohomology sequence

6.2
$$\Gamma(X,\mathcal{O}) \xrightarrow{D} \Gamma(X,\mathcal{O}) \to H^1(X,\mathbb{C}) \to 0$$

which shows how the problem of primitives depends on the topology of X.

Logarithms

The exponential sequence 3.6

$$0 \to \underset{\sim}{\mathbb{Z}} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

gives rise to an exact sequence

6.3
$$\Gamma(X,\mathcal{O}) \to \Gamma(X,\mathcal{O}^*) \to H^1(X,\mathbb{Z}) \to 0$$

and an isomorphism

6.4
$$H^1(X,\mathcal{O}^*) \xrightarrow{\sim} H^2(X,\mathbb{Z})$$

We shall prove in VI.6.8 that $H^2(X,\mathbb{Z}) = 0$.

Example 6.5. Let us consider the case where X is a disc with its center s removed. An analytic function in X can be expanded in a Laurent series

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-s)^n \qquad ; z \in X$$

where $\sum_{n=0}^{+\infty} a_n(z-s)^n$ converges for all $z \in X$ while the principal part $\sum_{m=1}^{+\infty} a_{-m}(z-s)^{-m}$ converges for all $z \in \mathbb{C}-\{s\}$. The residue of f at s is

$$\text{Res}(f;s) = a_{-1}$$

By inspection we see that f has a primitive in X if and only if $\text{Res}(f;s) = 0$. Thus the residue will induce an isomorphism

$$\text{Res}(-;s): H^1(X,\mathbb{C}) \xrightarrow{\sim} \mathbb{C}$$

Recall that f is said to be meromorphic at s if the principal part of the Laurent expansion of f at s is finite, i.e. the expansion has the form

$$f(z) = \sum_{n=m}^{+\infty} a_n (z-s)^n ; \qquad a_m \neq 0$$

The integer m is called the degree (or multiplicity) of f at s and is denoted $\text{div}(f;s)$

Mittag-Leffler theorem 6.6. Given a discrete subset S of X which is closed relative to X, and let there for each $s \in S$ be given a finite series

$$\frac{a_1}{(z-s)^1} + \frac{a_2}{(z-s)^2} + \dots$$

Then there exists an analytic function on U-S which for each $s \in S$ has the above series as the principal part of the Laurent expansion at s.

Proof. Let us describe a general construction departing from a locally compact space X and an abelian group D. Let $\mathcal{D}$ denote the sheaf of D-valued functions with discrete support: for an open subset U of X we can describe $\Gamma(U,\mathcal{D})$ as the set of functions $s: U \to D$ for which the set $\{x \in U \mid s(x) \neq 0\}$ is discrete and closed relative to U. Remark that the sheaf $\mathcal{D}$

is <u>soft</u> (not flabby in general).

Let $\mathbb{C}^{\oplus \mathbb{N}}$ denote the direct sum of countably many copies of $\mathbb{C}$. We can think of an element of $\mathbb{C}^{\oplus \mathbb{N}}$ as a sequence $a_1, a_2, \ldots$ of complex numbers of which only finitely many are different from zero. By the <u>sheaf of Laurent expansions, Lau</u> we understand the sheaf of $\mathbb{C}^{\oplus \mathbb{N}}$-valued functions on $\mathbb{C}$ with discrete support. Thus our initial data is precisely a section of Lau over X.

The sheaf $\underline{M \text{ of meromorphic functions on } \mathbb{C}}$ is the sheaf associated the presheaf whose sections over the open set U is

$$\{f/g \mid f,g \in \Gamma(U), \ g_x \neq 0 \ \text{for all} \ x\}$$

Given $m \in \Gamma(U,M)$ and $s \in U$. In a neighbourhood of s we can represent $m = f/g$ where f and g are analytic functions with $g_x \neq 0$ for all $x \in U$. It follows that m can be represented by an analytic function in a punctured neighbourhood of s which is meromorphic at s in the sense of 6.5. The sheaf M is a sheaf of rings on $\mathbb{C}$. Here we shall only be concerned with the underlying additive sheaf. Taking the principal part of the Laurent expansion defines a morphism of sheaves $pp: M \to Lau$, which fits into an exact sequence of sheaves

$$0 \longrightarrow 0 \longrightarrow M \xrightarrow{pp} Lau \longrightarrow 0$$

Theorem 6.1 provides us with an exact sequence

$$0 \longrightarrow \Gamma(X,0) \longrightarrow \Gamma(X,M) \xrightarrow{pp} \Gamma(X,Lau) \longrightarrow 0$$

from which the result can be read off.

Q.E.D.

We shall now present a theorem of Weierstrass which antedates the theorem of Mittag-Leffler.

<u>Weierstrass' theorem 6.7</u>. Given a discrete subset S of X closed relative to X and let there for each $s \in S$ be given an integer $d(s) \in \mathbb{Z}$. - There exists an analytic function f on $X-S$ which has no zeroes on $X-S$ and which is meromorphic at each $s \in S$ of degree $d(s)$.

<u>Proof</u>. The sheaf of divisors, Div is the sheaf of $\mathbb{Z}$-valued functions on the complex plane with discrete support.

Let $M*$, the <u>multiplicative sheaf of moromorphic functions</u> denote the sheaf associated to the presheaf whose sections over the open set U is

$$\{f/g \mid f,g \in \Gamma(U,0), \ f_x \neq 0 \ \text{and} \ g_x \neq 0 \ \text{for all} \ x \in U\}$$

Alternatively $\Gamma(U,M*)$ can be described at the group of multiplicative units in $\Gamma(U,M)$, where M is the sheaf of meromorphic functions. We can fit $M*$ into an exact sequence

$$0 \longrightarrow 0* \longrightarrow M* \xrightarrow{div} Div \longrightarrow 0$$

where div is defined at the end of 6.5. From this we deduce the cohomology sequence

$$0 \longrightarrow \Gamma(X,0*) \longrightarrow \Gamma(X,M*) \xrightarrow{div} \Gamma(X,Div) \longrightarrow H^1(X,0*)$$

Recall from the discussion of logarithms 6.4 that

$$H^1(X, O^*) \overset{\sim}{\to} H^2(X, \mathbb{Z})$$

Thus the result follows from VI.6.8.

<div align="right">Q.E.D.</div>

Corollary 6.8. Any meromorphic function on a connected open subset X of the complex plane is a quotient of two analytic functions on X.

Proof. Let m be a non zero meromorphic function on X. Write the divisor of m as

$$\text{div}(m) = F - G$$

where F and G are positive divisors on X, i.e. assume positive values only. According to 6.7 we can find non zero meromorphic functions f and g on X, with $\text{div}(f) = F$ and $\text{div}(g) = G$. Since F and G are positive, it follows that f and g are analytic functions on X, moreover

$$\text{div}(mf^{-1}g) = 0$$

i.e. $mf^{-1}g$ is a zerofree analytic function on X.

<div align="right">Q.E.D.</div>

IV.7 De Rham theorem

Let us describe the basic sheaves of differential geometry. To begin consider the sheaf C of smooth $\mathbb{R}$-valued functions on a finite dimensional real vector space E. Let $C(E)$ denote the sheaf of vector fields on E: the sections over an open subset U of E is the set of smooth maps from U to E. Given $f \in \Gamma(U, C)$ and $v \in \Gamma(U, C(E))$ define $D_v f \in \Gamma(U, C)$ to be the directional derivative, i.e.

$$7.1 \qquad D_v f(x) = \lim_{t \to 0} \frac{f(x + tv(x)) - f(x)}{t - 0} \qquad ; \quad x \in U$$

Let $\mathcal{D}er_{\mathbb{R}}(C, C)$ denote the sheaf of $\mathbb{R}$-derivations in C: Let j: U → E denote the inclusion of an open set, a section of $\mathcal{D}er_{\mathbb{R}}(C, C)$ over U is a morphism of $\mathbb{R}$-sheaves

$$D: j^*C \longrightarrow j^*C$$

such that for any open set V of U

$$7.2 \qquad D(fg) = gD(f) + fD(g) \qquad ; \quad f, g \in \Gamma(V, C).$$

Proposition 7.3. The directional derivative induces an isomorphism of sheaves

$$C(E) \longrightarrow \mathcal{D}er_{\mathbb{R}}(C, C)$$

Proof. Let us assume that $E = \mathbb{R}^n$. Let U be an open subset of $\mathbb{R}^n$ and $v = (v_1, \ldots, v_n)$ a smooth vector field on U.

We have

$$D_v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}$$

as it follows from the chain rule. This proves that our map is injective. Consider an arbitary section D of $\mathrm{Der}_{\mathbb{R}}(C,C)$ over U. Put

$$P = D - \Sigma D(x_i) \frac{\partial}{\partial x_i}$$

Notice, that P is section of $\mathcal{D}er_{\mathbb{R}}(C,C)$ over U with $P(x_i) = 0$, $i = 1,\ldots,n$. Let us prove that $P = 0$. So let there be given a smooth function f on an open subset V of U. We must show that $Pf(a) = 0$ for any $a \in V$. According to Lemma 7.4 below we can write

$$f(x) = f(a) + \Sigma(x_i - a_i)h_i(x)$$

where $h_1,\ldots,h_n$ are smooth functions defined in a neighbourhood of a. If we confuse f with its restriction to that neighbourhood we get

$$Pf(x) = \Sigma(x_i - a_i)Ph_i(x)$$

which shows that $Pf(a) = 0$.

$$\text{Q.E.D.}$$

Lemma 7.4. Let D be an open disc in $\mathbb{R}^n$ with center a. Any smooth function f on D can be written

$$f(x) = f(a) + \Sigma(x_i - a_i)h_i(x)$$

where $h_1,\ldots,h_n$ are smooth functions on D.

Proof. By means of the chain rule we get

$$f(x) - f(a) = \int_0^1 \frac{d}{dt} f(t(x-a) + a)dt = \Sigma(x_i - a_i) \int_0^1 \frac{\partial f}{\partial x_i}(t(x-a)+a)dt$$

$$\text{Q.E.D.}$$

Let X denote a differentiable manifold and C the sheaf of $\mathbb{R}$-valued smooth functions on X. The tangent sheaf of X is by definition the sheaf $T = \mathcal{D}er_{\mathbb{R}}(C,C)$. The sections of T are called tangent vector fields.

Given two tangent vector fields v and w over the same open subset U of X. We define the bracket $[v,w] \in \Gamma(U,T)$ by the formula

$$7.5 \qquad\qquad [v,w] = v \circ w - w \circ v$$

The De Rham complex

For $p \in \mathbb{N}$ we put

$$\Omega^p = \mathcal{H}om_C^{(p)}(T,C)$$

the sheaf of alternating p-forms on T. To describe this let us first remark that $\Omega^0 = C$. For $p \geq 1$ consider the inclusion $j: U \to X$ of an open set. The sections of Ω^p over U is the set of morphisms of sheaves

$$\omega: j^*T \times \cdots \times j^*T \to j^*C \qquad\qquad (\text{p-factors})$$

such that for each open set V of U, $\omega(v)$ is an alternating

p-form on the $C(V)$-module $T(V)$. - Recall the presence of a pairing exterior product

$$\Gamma(U,\Omega^p) \times \Gamma(U,\Omega^q) \to \Gamma(U,\Omega^{p+q})$$

given by the formula

7.6 $\quad \alpha \wedge \beta(v_1,\dots,v_{p+q}) = \frac{1}{p!q!} \sum_\sigma \text{sign}(\sigma)\alpha(v_{\sigma(1)},\dots,v_{\sigma(p)})\beta(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)})$

the sum being over all permutations σ of $[1,\dots,p+q]$. This makes $\Omega^{\cdot}$ into a sheaf of graded algebras which is anti commutative:

7.7 $\qquad\qquad\qquad \alpha^p \wedge \beta^q = (-1)^{pq}\beta^q \wedge \alpha^p$

The representation $T = \mathcal{D}er_{\mathbb{R}}(C,C)$ and simple evaluation give rise to a morphism of sheaves

$$d: C \longrightarrow \Omega^1$$

in fact an $\mathbb{R}$-derivation. This can be extended to $d: \Omega^p \to \Omega^{p+1}$ by the formula

7.8 $\qquad d\omega(v_1,\dots,v_{p+1}) = \sum_i (-1)^{i+1} v_i\omega(v_1,\dots,\hat{v}_i,\dots,v_{p+1})$

$$+ \sum_{i<j} (-1)^{i+j} \omega([v_i,v_j],v_1,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_{p+1})$$

This satisfies $d \circ d = 0$ and

7.9 $\qquad\qquad d(\omega^p \wedge \omega^q) = d\omega^p \wedge \omega^q + (-1)^p \omega^p \wedge d\omega^q$

The exterior derivative d and $\Omega^{\cdot}$ constitutes the de Rham complex of X.

Theorem 7.10. Let X be a differentiable manifold. The de Rham complex $\Omega^{\cdot}$ is a soft resolution of the sheaf $\underset{\sim}{\mathbb{R}}$.

Proof. Recall from III.2.8 that soft is a local notion. Thus it follows from III.3.2 that C is a soft sheaf on X. This implies that any sheaf of C-modules is soft III.2.9.

Thus it remains to be shown that for any point $x \in X$, $\Omega^{\cdot}_x$ is a resolution of $\mathbb{R}$. By a suitable choice of coordinate system it suffices to treat the case where $X = \mathbb{R}^n$ and x is the origin. Conclusion by 7.11.

$$\text{Q.E.D.}$$

Poincaré lemma 7.11. Let $\Omega^{\cdot}$ denote the de Rham complex on $\mathbb{R}^n$. For an open disc D with center in 0 we have

$$H^i\Gamma(D,\Omega^{\cdot}) = 0 \qquad \text{for } i \geq 1$$

Proof. For $p \geq 1$ let k^p denote the operator

$$k^p: \Gamma(D,\Omega^p) \longrightarrow \Gamma(D,\Omega^{p-1})$$

which maps the differential

$$\omega = f(x_1,\dots,x_n)dx_{i_1} \wedge dx_{i_2} \dots \wedge dx_{i_p} \qquad ; \; i_1 < i_2 < \dots < i_p$$

into the differential form

$$k^P \omega = x_{i_1} \left(\int_0^1 f(0,\ldots,0,tx_{i_1},x_{i_1+1},\ldots,x_n)\,dt \right) dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

We must verify the formula $\qquad k^{p+1}\partial^p + \partial^{p-1}k^p = 1 \qquad ; \quad p \geq 1$

To this end we offer the following details:

$$\partial^{p-1}k^P \omega = \left(\int_0^1 \frac{d}{dt}(tf(\ldots,0,tx_{i_1},x_{i_1+1},\ldots))\,dt \right) dx_{i_1} \wedge dx_{i_2}\cdots$$

$$+ \sum_{j>i_1} x_{i_1} \left(\int_0^1 f'_j(\ldots,0,tx_{i_1},x_{i_1+1},\ldots)\,dt \right) dx_j \wedge dx_{i_2}\cdots$$

$$k^{p+1}\partial^P \omega = \sum_{j<i_1} \left(\int_0^1 x_j f'_j(\ldots,0,tx_j,x_{j+1},\ldots)\,dt \right) dx_{i_1} \wedge dx_{i_2}\cdots$$

$$- \sum_{j>i_1} x_{i_1} \left(\int_0^1 f'_j(\ldots,0,tx_{i_1},x_{i_1+1},\ldots)\,dt \right) dx_j \wedge dx_{i_2}\cdots$$

leaving it to the reader to evaluate the integrals in the first and the third line.

Q.E.D.

De Rham theorem 7.12. Let X denote a differentiable manifold which is countable at infinity. Then

$$H^\cdot(X,\mathbb{R}) = H^\cdot \Gamma(X,\Omega^\cdot)$$

Moreover, the cup product is induced by exterior product.

Proof. Combine 7.10 and 2.2 with I.7.5. The statement about the cup product follows from II.10.14.

Q.E.D.

Integration 7.13. Let Ω^n denote the sheaf of smooth n-forms on $\mathbb{R}^n$. We can define

$$\int : \Gamma_c(\mathbb{R}^n, \Omega^n) \longrightarrow \mathbb{R}$$

by means of the usual integral on $\mathbb{R}^n$

$$\int f(x_1,\ldots,x_n)\,dx_1 \wedge \cdots \wedge dx_n = \int f(x_1,\ldots,x_n)\,dx_1\cdots dx_n$$

Let us prove the fundamental formula

$$\int d\omega = 0 \qquad ; \quad \omega \in \Gamma_c(\mathbb{R}^n, \Omega^{n-1})$$

It suffices to treat the case $\omega = g(x_1,\ldots,x_n)\,dx_1\cdots\wedge\widehat{dx_i}\cdots\wedge dx_n$ where g is a smooth function on $\mathbb{R}^n$ with compact support

$$\int d\omega = (-1)^{i-1} \int \frac{\partial g}{\partial x_i}(x_1,\ldots,x_n)\,dx_1 \wedge \cdots \wedge dx_n$$

$$= (-1)^{i-1} \int_{-\infty}^{+\infty}dx_1\cdots \int_{-\infty}^{+\infty}\frac{\partial g}{\partial x_i}(x_1,\ldots,x_n)\,dx_i = 0$$

In conclusion the integral induces an isomorphism

$$\int : H_c^n(\mathbb{R}^n,\mathbb{R}) \longrightarrow \mathbb{R}$$

More generally let X be an oriented smooth n-manifold, i.e. X admits an atlas (U_α,ϕ_α) whose transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ have positive Jacobi determinants. Then we can construct integrals

$$\int : \Gamma_c(U_\alpha,\Omega^n) \to \mathbb{R}$$

which agrees on the overlaps. From the fact that Ω^n is soft

follows V.1.2 that the presheaf

$$U \mapsto \Gamma_c(U,\Omega^n)^\vee$$

is a sheaf. Thus we obtain an integral

$$\int : \Gamma_c(X,\Omega^n) \to \mathbb{R}$$

which satisfies the relation

$$\int d\omega = 0 \qquad ; \ \omega \in \Gamma_c(X,\Omega^{n-1})$$

using the fact that the presheaf $\quad U \mapsto \Gamma_c(U,\Omega^{n-1})^\vee$

is a sheaf, we can reduce the problem to $\mathbb{R}^n$, where it allready

has been delt with.

Example 7.14. Let V denote a finite dimensional real

vector space. The exact sequence in cohomology with compact

support gives rise to a boundary map

$$H_c^{\ 0}(\{0\},\mathbb{R}) \to H_c^{\ 1}(V,\mathbb{R})$$

Let us seek a de Rham representation of the image of the constant

function 1. To this end consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_c(V-\{0\},\Omega^0) & \longrightarrow & \Gamma_c(V,\Omega^0) & \longrightarrow & \Gamma_c(\{0\},\Omega^0) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \quad \rho \longmapsto 1 & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & \Gamma_c(V-\{0\},\Omega^1) & \longrightarrow & \Gamma_c(V,\Omega^1) & \longrightarrow & \Gamma_c(\{0\},\Omega^1) & \longrightarrow & 0
\end{array}
$$

Take any smooth function ρ on V with compact support, which

is constant 1 in a neighbourhood of 0 in V. The restriction of

$d\rho$ to $V-\{0\}$ represents the canonical cohomology class.

Example 7.15. Let X denote an open subset of $\mathbb{C}$ and $f(z)$

an analytic function on X. The exact sequence 4.1 gives rise to

a boundary map

$$H^0(X,0) \longrightarrow H^1(X,\mathbb{C})$$

The image of $f(z)$ by this map can be realized as a de Rham coho-

mology class by the differential form $\underline{-f(z)\,dz}$. To see this

consider the commutative diagram

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{C} & \longrightarrow & 0 & \xrightarrow{\ \frac{d}{dz}\ } & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \Omega^0 \otimes \mathbb{C} & \xrightarrow{\ d^0\otimes 1\ } & \Omega^1 \otimes \mathbb{C} & \xrightarrow{\ d^1\otimes 1\ } & \Omega^2 \otimes \mathbb{C} & \longrightarrow & 0
\end{array}
$$

where the second vertical arrow is separation in real and

imaginary part while the third vertical arrow can be described

by decomposing the analytic function f into

$$f(x+iy) = u(x,y) + iv(x,y) \qquad ; \ x+iy \in X$$

where u and v are smooth real functions on X. To $f(z)$ we

associate the differential form

$$f(z)\,dz = (u(x,y)\,dx - v(x,y)\,dy) + i(u(x,y)\,dy + v(x,y)\,dx)$$

The commutativity of the diagram follows from the Cauchy-Riemann

equations 5.1. Conclusion by XI.4.9.

<u>Example 7.16</u>. Let ρ be a smooth real function on $\mathbb{C}$ with compact support which is constant in a neighbourhood of 0. Let us show that

$$\int_{\mathbb{C}^*} \frac{1}{2\pi i} \frac{dz}{z} \wedge d\rho = \rho(0)$$

In case $\rho(0) = 0$ the function ρ has compact support on $\mathbb{C}^*$ and we find that

$$\int_{\mathbb{C}^*} \frac{1}{2\pi i} \frac{dz}{z} \wedge d\rho = -\frac{1}{2\pi i} \int_{\mathbb{C}^*} d(\rho\frac{dz}{z}) = 0$$

As a consequence the value of the integral depends only on $\rho(0)$. Consider a smooth function $\rho: [0, +\infty] \to \mathbb{R}$ with compact support which is constant in a neighbourhood of 0. The composite of ρ and $r = \sqrt{x^2 + y^2}$ defines a function which we also denote by ρ. We have

$$d\rho = \frac{d\rho}{dr}\left(\frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy\right) = \frac{1}{r} \frac{d\rho}{dr}(x dx + y dy)$$

$$\frac{1}{z} dz = \tfrac{1}{2} d \log (x^2+y^2) + \frac{1}{r^2} i(-y dx + x dy)$$

Using polar coordinates we get

$$\int \frac{1}{2\pi i} \frac{dz}{z} \wedge d\rho = -\frac{1}{2\pi} \frac{1}{r^3} \int \frac{d\rho}{dr} r^2 dx \wedge dy =$$

$$-\frac{1}{2\pi} \iint \frac{1}{r} \frac{d\rho}{dr} dx dy = -\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{+\infty} \frac{1}{r} \frac{d\rho}{dr} r dr = \rho(0)$$

<u>Example 7.17</u>. The previous example can be generalized to $\mathbb{R}^n$ as follows. Put

$$\theta(x) = r(x)^{-n} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \cdots \wedge \hat{dx_i} \cdots \wedge dx_n$$

where $r(x) = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$. This is a closed differential form on the space $\mathbb{R}^n - \{0\}$:

$$d\theta = dr^{-n} \wedge r^n\theta + r^{-n} dr^n\theta = -\frac{n}{2} r^{-n-2} dr^2 \wedge r^n\theta + nr^{-n} dx_1 \cdots dx_n$$

which is zero by direct calculation of the first term. For a smooth function $\rho: \mathbb{R}^n \to \mathbb{R}$ with compact support and constant in a neighbourhood of the origin we have

$$\sigma_{n-1} \rho(0) = (-1)^n \int_{\mathbb{R}^n - \{0\}} \theta \wedge d\rho$$

where $\sigma_{n-1} = n\mu_n$, i.e. n times the volume of the unit ball in $\mathbb{R}^n$. The formula can be proved in a way similar to the proof of the formula 7.16.

<u>Example 7.18</u>. The formula

$$\int_{\mathbb{C}^*} \frac{1}{2\pi i} \frac{dz}{z} \wedge d\rho = \rho(0)$$

is true for any smooth function ρ with compact support: According to 7.16 it suffices to treat the case where $\rho(0) = 0$. Using Stoke's theorem we get

$$\int_{|z| \geq \epsilon} \frac{1}{2\pi i} \frac{dz}{z} \wedge d\rho = \int_{|z| \geq \epsilon} \frac{1}{2\pi i} d(\rho \wedge \frac{dz}{z}) = \int_{|z| = \epsilon} \frac{1}{2\pi i} \rho \frac{dz}{z}$$

write out the last integral to get the   estimate

$$\left| \iint_{|z| \geq \varepsilon} \frac{1}{2\pi i} \frac{dz}{z} \wedge d\rho \right| \leq \sup_{|z| = \varepsilon} |\rho(z)|.$$

from which the result follows

Example 7.19. For a smooth function $f: \mathbb{C} \to \mathbb{C}$ with compact

support er get from 7.18

$$f(w) = \frac{1}{2\pi i} \int \frac{dz}{z-w} \wedge df(z)$$

or with $df(z) = \frac{\partial f}{\partial z}(z) dz + \frac{\partial f}{\partial \bar{z}}(z) d\bar{z}$

$$f(w) = \frac{1}{2\pi i} \int \frac{\partial f}{\partial \bar{z}}(z) \frac{1}{z-w} dz \wedge d\bar{z}.$$

Example 7.20. For a smooth function $f: \mathbb{C} \to \mathbb{C}$ with compact support we have

$$\frac{\partial}{\partial \bar{w}} \int f(z) \frac{1}{z-w} dz \wedge d\bar{z} = f(w).$$

According to 7.19 it suffices to prove that

$$\frac{\partial}{\partial \bar{w}} \int f(z) \frac{1}{z-w} dz \wedge d\bar{z} = \int \frac{\partial f}{\partial \bar{z}}(z) \frac{1}{z-w} dz \wedge d\bar{z}$$

which is easily checked by changing into polar coordinates

$$\int f(z) \frac{1}{z-w} dz \wedge d\bar{z} = -\frac{1}{\pi} \int_0^{2\pi} d\theta \int_0^{+\infty} f(w+re^{i\theta}) e^{i\theta} dr$$

IV.8   Relative cohomology

Let Z denote a closed subset of the topological space X and $U = X-Z$ the complement. The inclusions are denoted $i: Z \to X$ and $j: U \to X$ respectively.

Definition 8.1. For a sheaf F on X we put

$$H^p(X,Z;F) = H^p(X,j_! j^*F) \qquad ; \quad p \in \mathbb{Z}$$

From the short exact sequence II.6.11   we deduce by means of II.5.4 a long exact sequence

8.2 $\qquad \to H^{p-1}(Z,F) \to H^p(X \ Z,F) \to H^p(X,F) \to H^p(Z,F)$

For closed subsets A and B of X we derive from the first row of II.6.17 an exact sequence

8.3. $\qquad \to H^p(X,A \cup B;F) \to H^p(X,A;F) \to H^p(B,B \cap A;F) \to$

Proposition 8.4. Suppose X is a locally compact space, countable at infinity and Z a closed subspace of X. Then for any soft sheaf S on X, $\qquad H^p(X,Z;S) = 0 \qquad ; \ p \geq 1$

Proof. The restriction i^*F of F to Z is a soft sheaf III.2.5. It follows from 2.2 that $H^p(Z,S) = 0$; $p \geq 1$. The result follows from the long exact sequence 8.2 using that

$$H^0(X,S) \longrightarrow H^0(Z,S)$$

is surjective, 2.3.

Q.E.D.

IV.9 Classification of locally constant sheaves

Let us introduce the notion of equivariant k-sheaves. Given a commutative ring k and a topological space X on which the group G acts from the right. By a G-action on the sheaf E we understand data $(a_\sigma)_{\sigma \in G}$ where $a_\sigma: E \to \sigma_* E$ is a morphism of k-sheaves subject to the conditions

9.1
$$a_{\sigma\tau} = \tau_* a_\sigma \circ a_\tau \qquad ; \quad \sigma, \tau \in G$$

By a G-equivariant sheaf E we understand a sheaf equipped with a G-action. By a morphism h: E → F of G-equivariant sheaves we understand a morphism of k-sheaves which makes the following diagram commutative

$$
\begin{array}{ccc}
E & \longrightarrow & \sigma_* E \\
\downarrow h & & \downarrow \sigma_* h \\
F & \longrightarrow & \sigma_* F
\end{array}
\qquad ; \quad \sigma \in G
$$

The category of G-equivariant sheaves on X will be denoted $Sh^G(X,k)$. In the following we put B = X/G and let f: X → B denote the projection. We shall introduce two functors

9.2
$$Sh^G(X,k) \xLeftarrow[Inv^G]{f^*} Sh(B,K)$$

Let us first remark that a G-action on a k-sheaf E on X can be described in terms of isomorphisms $a^\sigma: \sigma^* E \to E$ as it follows by adjunction. Thus for a sheaf F on B the canonical isomorphism

$$\sigma^* f^* F \longrightarrow f^* F \qquad ; \quad \sigma \in G$$

defines a G-action on $p^* F$. — Given a G-sheaf E on X. In order to describe the sections of $Inv^G E$ over an open set V of B we can remark that the formula 9.1 defines a left action of G on $I(f^{-1}(V), E)$. We put

$$I(V, Inv^G E) = I(f^{-1}(V), E)^G$$

i.e. the G-invariant sections of E over $f^{-1}(V)$. We leave it to the reader to prove that f* is a left adjoint to Inv, i.e.

9.3
$$Hom_G(f^* F, E) = Hom(F, Inv^G E)$$

To a k-module N with a left G-action we can associate a G-action on N: To $\sigma \in G$ and the open set U of X we associate the map

9.4
$$Hom(U,N) \to Hom(\sigma^{-1}(U), N) \qquad ; \quad \varphi \longmapsto \sigma_N \circ \varphi \circ \sigma$$

Lemma 9.5. In case X is connected, the functor $E \to I(X,E)$ induces an equivalence between the category of G-sheaves on X, whose underlying k-sheaf is constant, and the category of k-modules with a left G-action.

Proof. The inverse functor is furnished by 9.4.

Q.E.D.

Lemma 9.6. Suppose every point $b \in B$ admits an open neighbourhood V such that $f^{-1}(V) \to V$ is G-equivariantly isomorphic to pr: V × G → V. Then f* and Inv^G induces an equivalence between the category of locally constant k-sheaves on B and G-sheaves whose underlying k-sheaf is locally constant.

<u>Proof</u>. It suffices to prove a similar statement for
$B \times G \to B$ with locally constant replaced by constant. This is
left to the reader.

<div align="right">Q.E.D.</div>

Let us turn to the main theme of this section. Given a
topological space B and a basepoint $b \in B$. Regarding the fun-
damental group $\pi_1(B,b)$ we use the convention proposed by Deligne
(1) that $\alpha * \beta$ denotes the loop obtained by first running through
the loop β. Given a locally constant k-sheaf E on B, then
monodromy 4.21 defines a <u>left</u> action of $\pi_1(B,b)$ on E_b, the
stalk of E at b.

<u>Theorem 9.7</u>. Let B be a connected, locally path-connected
space for which every point $y \in B$ admits an open neighbourhood V
such that $\pi_1(V,y) \to \pi_1(B,y)$ is trivial. Then, monodromy is an
equivalence between the category of locally constant k-sheaves on
B and the category of left $\pi_1(B,b)$-equivariant k-modules.

<u>Proof</u>. Put $G = \pi_1(B,b)$ and let $f: X \to B$ denote the uni-
versal covering space of X. The space X is simply connected,
Spanier (1) 2.5.15. Thus it follows from 4.20 that any locally
constant sheaf on X is constant. A combination of the two pre-
vious lemmas gives an equivalence between the locally constant
k-sheaves and the G-equivariant k-modules.

In order to identify this correspondence with the monodromy
correspondence, consider the base point x of X, $f(x) = b$.
For a locally constant sheaf E on B, we want to transport the

G-action on $\Gamma(X,f^*E)$ to the stalk E_b of E at b. To this
end introduce names to the canonical map

$$E_b \xrightarrow{f_x} f^*E_x \xleftarrow{r_x} \Gamma(X,f^*E)$$

The result is quite simply

9.8 $$\sigma \longmapsto f_x^{-1} \, r_x \, \sigma r_x^{-1} \, f_x$$

For a fixed $\sigma \in G$ the action on E_b may be calculated as the
composite of the maps

$$E_b \xrightarrow{f_x} f^*E_x \xrightarrow{r_x^{-1}} \Gamma(X,f^*E) \xrightarrow{r_{\sigma(x)}} f^*E_{\sigma(x)} \xrightarrow{f_{\sigma(x)}^{-1}} E_b$$

i.e. by the formula

9.9 $$\sigma \longmapsto f_{\sigma(x)}^{-1} r_{\sigma(x)} r_x^{-1} f_x \qquad ; \qquad \sigma \in G$$

The formula 9.9 follows from 9.8 using the following commutative
diagram

It remains to identify 9.8 with the recipe 4.21.

<div align="right">Q.E.D.</div>

V. Duality with Coefficient in a Field

V.1 Sheaves of linear forms

In this chapter we shall work with a fixed field k. For a k-vector space N we let N^V denote the dual vector space, i.e.
$N^V = \text{Hom}_k(N,k)$.

Let X be a locally compact space and S a k-sheaf on X. The inclusion $j: V \to U$ of two open subsets of X will induce a map

$$j_! : \Gamma_c(V,S) \to \Gamma_c(U,S)$$

"extension by zero". The k-linear dual of this

$$j_!^V : \Gamma_c(V,S)^V \to \Gamma_c(U,S)^V$$

gives rise to restriction maps in a presheaf S^V defined by

1.1 $$\Gamma(U,S^V) = \Gamma_c(U,S)^V$$

Proposition 1.2. For a soft k-sheaf S on the locally compact space X, the presheaf S^V is a sheaf.

Proof. Let us first prove that for any two open sets W and V the sequence

$$0 \to \Gamma_c(V \cup W,S^V) \to \Gamma_c(V,S^V) \oplus \Gamma_c(W,S^V) \to \Gamma_c(V \cap W,S^V)$$

formed by the sum and difference between two restriction maps is exact. - Consider the Mayer-Vietoris sequence III.7.5

$$0 \to \Gamma_c(V \cap W,S) \to \Gamma_c(V,S) \oplus \Gamma_c(W,S) \to \Gamma_c(V \cup W,S) \to H_c^1(V \cap W,S)$$

and notice that $H_c^1(V \cap W,S) = 0$, since the restriction of S to $V \cap W$ is soft. The result now follows by taking the k-linear dual of the Mayer-Vietoris sequence.

Let us next consider a set $\mathcal{U}$ of open subsets of X with the property, that for any pair $V,W \in \mathcal{U}$ there exists $U \in \mathcal{U}$ containing V and W. We shall prove that S^V has the sheaf property relative to such a directed covering: Let 0 denote the union of all the sets from $\mathcal{U}$. It follows rather directly from Borel-Heine that

$$\lim_{\substack{\longrightarrow \\ U \in \mathcal{U}}} \Gamma_c(U,S) = \Gamma_c(0,S)$$

Using the universal mapping property of $\lim_{\longrightarrow}$ it follows directly from this formula that S^V has the sheaf property relative to $\mathcal{U}$.

In order to show that S^V has the sheaf property relative to an arbitrary family V of open sets, we can introduce the family $\mathcal{U}$ of open sets which are unions of finitely many open sets from V, and then we use the previous result.

Q.E.D.

Proposition 1.3. Let S be a soft k-sheaf on the finite dimensional locally compact space X. For any k-sheaf F on X, the sheaf $S \otimes_k F$ is soft.

<u>Proof.</u> Let $j: U \to X$ be the inclusion of an open subset. Let us establish a canonical isomorphism for any k-sheaf S

$$1.4 \qquad S \otimes_k j_! \underline{k} = j_! j^* S$$

Consider an open subset V of X. Scalar multiplication

$$\Gamma(V,S) \otimes_k \Gamma(V \cap U, \underline{k}) \to \Gamma(V \cap U, S)$$

will induce a linear map II.6.1

$$\Gamma(V,S) \otimes_k \Gamma(V, j_! \underline{k}) \to \Gamma(V, j_! j^* S)$$

and by variation of V a morphism of sheaves

$$S \otimes_k j_! \underline{k} \to j_! j^* S$$

By localization this is seen to be an isomorphism, II.6.3.

Let the dimension of X be n and consider a resolution of F by k-sheaves II.7.4

$$P_{n-1} \xrightarrow{\partial_{n-1}} P_{n-2} \cdots P_1 \xrightarrow{\partial_1} P_0 \longrightarrow F \longrightarrow 0$$

where the P's are direct sums of k-sheaves of the form $j_! \underline{k}$ where $j: U \to X$ is the inclusion of an open subset. From the formula 1.4 and III.7.2 follows that $S \otimes_k j_! \underline{k}$ is soft. Let us remark that a direct sum of soft sheaves is soft as it follows from III.5. Using this we conclude that $S \otimes_k P_i$ is soft.

Let K denote the kernel of ∂_{n-1}. From the resolution above we obtain an exact sequence of sheaves

$$0 \to S \otimes K \to S \otimes P_{n-1} \to S \otimes P_{n-2} \to \cdots \to S \otimes P_0 \to S \otimes F \to 0$$

Using III.9.9. we conclude that $S \otimes F$ is soft.

<div align="center">Q.E.D.</div>

<u>Proposition 1.5.</u> Let X be a finite dimensional locally compact space and S a soft k-sheaf on X. There is a natural isomorphism

$$\Gamma_c(X, F \otimes S)^\vee \cong \mathrm{Hom}(F, S^\vee)$$

as F varies through $\mathrm{Sh}(X,k)$.

<u>Proof.</u> Let $j: U \to X$ denote the inclusion of an open subspace. Consider the natural maps

$$\Gamma(U,F) \otimes \Gamma_c(U,S) \to \Gamma_c(U, F \otimes S) \xrightarrow{j_!} \Gamma_c(X, F \otimes S)$$

The dual of the composite can be written

$$\Gamma_c(X, F \otimes S)^\vee \longrightarrow \mathrm{Hom}(\Gamma(U,F), \Gamma_c(U,S)^\vee)$$

By variation of U this defines a map

$$1.6 \qquad \Gamma_c(X, F \otimes S)^\vee \longrightarrow \mathrm{Hom}(F, S^\vee)$$

We shall prove that this is an isomorphism.

Let us first check the case $F = j_! \underline{k}$ where j is the inclusion of an open subspace. In order to evaluate the left hand side of 1.6 we use formula 1.4 and III.7.1 to get

$$\Gamma_c(X, j_! \underline{k} \otimes S)^\vee = \Gamma_c(X, j_! j^* S)^\vee = \Gamma_c(U,S)^\vee = \Gamma(U, S^\vee)$$

We can evaluate the right hand side of 1.6 by II.6. and II.7.2

$$\mathrm{Hom}(j_! \underline{k}, S^\vee) = \mathrm{Hom}(\underline{k}, j^* S^\vee) = \Gamma(U, j^* S^\vee) = \Gamma(U, S^\vee)$$

It is left to the reader to check that these identifications transform the map 1.6 into the identity.

The general case can be handled by choosing a presentation of F of the form

$$P \longrightarrow Q \longrightarrow F \longrightarrow 0$$

where P and Q are direct sums of sheaves of the form just described. The two functors involved in 1.6 transform direct sums into direct products. It follows that the two vertical maps to the right in the commutative diagram below are isomorphisms

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma_c(X,F{\otimes}S)^{\vee} & \longrightarrow & \Gamma_c(X,Q{\otimes}S)^{\vee} & \longrightarrow & \Gamma_c(X,P{\otimes}S)^{\vee} \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(F,S^{\vee}) & \longrightarrow & \mathrm{Hom}(Q,S^{\vee}) & \longrightarrow & \mathrm{Hom}(P,S^{\vee})
\end{array}
$$

It is easily seen that the diagram is exact. From this follows that the vertical arrow to the left is an isomorphism.

Q.E.D.

Corollary 1.7. The sheaf $S^{\vee}$ is injective in $Sh(X,k)$.

Proof. According to 1.5 we must show that

$$F \longmapsto \Gamma_c(X,F \otimes S)^{\vee}$$

is an exact functor. This follows from 1.3 and III.2.7.

Q.E.D.

V.2. Verdier duality

Let X denote a locally compact space of finite dimension and let k denote a fixed field. The category $Sh(X,k)$ of k-sheaves on X is an abelian category with enough injectives. We let $D^+(X,k)$ denote the homotopy category of bounded below complexes of injective k-sheaves.

To an object $I^{\cdot}$ of $D^+(X,k)$ we can associate $[\Gamma_c(X,I^{\cdot}),k]$ the set of homotopy classes of morphisms of complexes of k-vector-spaces from $\Gamma_c(X,I^{\cdot})$ to k. More naively

$$[\Gamma_c(X,I^{\cdot}),k] = H^0\Gamma_c(X,I^{\cdot})^{\vee}$$

The assignment $I^{\cdot} \longmapsto [\Gamma_c(X,I^{\cdot}),k]$ defines a contravariant functor from $D^+(X,k)$ to the category of k-vectorspaces which is representable:

Verdier duality 2.1. There exists $\mathcal{D}^{\cdot}$ in $D^+(X,k)$ and a natural isomorphism

$$[I^{\cdot},\mathcal{D}^{\cdot}] \xrightarrow{\sim} [\Gamma_c(X,I^{\cdot}),k]$$

as $I^{\cdot}$ varies through $D^+(X,k)$.

Proof. Let us first establish the proper sign conventions. For a complex $L^{\cdot}$ of k-vectorspaces we put $L^{\cdot\vee} = \mathrm{Hom}^{\cdot}(L^{\cdot},k)$ with the notation of I.4.3. Notice that $L^{\cdot\vee}$ is a complex of k-vectorspaces whose p'th differential is given by

$$(-1)^{p+1}(\partial^{-p-1})^{\vee}: (L^{-p})^{\vee} \rightarrow (L^{-p-1})^{\vee}$$

This formula will also be used to extend the functor $S \mapsto S^{\vee}$ on $Sh(X,k)$ given by 1.1 to complexes of k-sheaves, see I.11.9.

Let $S^{\cdot}$ denote a bounded soft resolution of the constant sheaf $\underline{k}$ on X. Recall from 1.7 that $S^{\cdot \vee}$ is a bounded sheaf of injective k-sheaves. For $I^{\cdot}$ in $D^{+}(X,k)$ and integers p and q we have a canonical isomorphism, 1.5

$$\Gamma_c(X,I^p \otimes S^q)^{\vee} = \text{Hom}(I^p, S^{q\vee})$$

Taking the direct sum over all p, q with $p+q = -n$ we deduce a canonical isomorphism of k-vector spaces

$$[\Gamma_c(X,I^{\cdot} \otimes S^{\cdot})^{\vee}]^n \xrightarrow{\approx} \text{Hom}^n(I^{\cdot}, S^{\cdot\vee})$$

It is left to the reader to check the signs to see that this in fact gives an isomorphism of complexes

2.2 $\qquad \Gamma_c(X,I^{\cdot} \otimes S^{\cdot})^{\vee} = \text{Hom}^{\cdot}(I^{\cdot}, S^{\cdot\vee})$

From the quasi-isomorphism $\underline{k} \to S^{\cdot}$ we deduce a quasi-isomorphism

$$I^{\cdot} \longrightarrow I^{\cdot} \otimes S^{\cdot}$$

Noticing that both complexes are soft 1.3, we deduce still another quasi-isomorphism I.7.5

$$\Gamma_c(X,I^{\cdot}) \longrightarrow \Gamma_c(X,I^{\cdot} \otimes S^{\cdot})$$

The k-linear dual of this

$$\Gamma_c(X,I^{\cdot} \otimes S^{\cdot})^{\vee} \longrightarrow \Gamma_c(X,I^{\cdot})^{\vee}$$

combined with the isomorphism 2.2 yields a final quasi-iso-morphism

2.3 $\qquad \text{Hom}^{\cdot}(I^{\cdot}, S^{\cdot\vee}) \to \Gamma_c(X,I^{\cdot})^{\vee}$

passing to H^0 this yields an isomorphism

$$[I^{\cdot}, S^{\cdot\vee}] \xrightarrow{\sim} H^0\Gamma_c(X,I^{\cdot})^{\vee}$$

Finally put $\mathcal{D}^{\cdot} = S^{\cdot\vee}$.

$$Q.E.D.$$

The complex $\mathcal{D}^{\cdot}$ from 2.1 is called <u>the dualizing complex</u>. This is a bounded below complex of injective k-sheaves uniquely determined up to homotopy, in particular the cohomology sheaves $H^p\mathcal{D}^{\cdot}$, $p \in \mathbb{Z}$, are uniquely determined up to isomorphism. We shall now calculate these: Recall that an inclusion of open subspaces $j : V \to U$ give rise to a map

$$j_! : H_c^p(V,\underline{k}) \to H_c^p(U,\underline{k})$$

"extension by zero". The k-linear dual of this

$$j_!^{\vee} : H_c^p(U,\underline{k})^{\vee} \to H_c^p(V,\underline{k})^{\vee} \qquad ; \quad p \in \mathbb{Z}$$

gives rise to a presheaf $U \longmapsto H_c^p(U,\underline{k})^{\vee}$.

Proposition 2.4. Let $\mathcal{D}^{\cdot}$ denote the dualizing complex for the locally compact space X. For any integer p, the cohomology sheaf $H^{-p}\mathcal{D}^{\cdot}$ is the sheaf associated to the pre-sheaf

$$U \longmapsto H_c^{\,p}(U,\underset{\sim}{k})^{\vee}$$

Proof. Let us first establish a canonical isomorphism for all integers $p \in \mathbb{Z}$

2.5
$$H_c^{\,p}(X,F)^{\vee} = H^{-p}\mathrm{Hom}(F,\mathcal{D}^{\cdot})$$

as F runs through $Sh(X,k)$. To do this let us choose an injective resolution $I^{\cdot}$ of F. Notice that

$$H_c^{\,p}(X,F) = H^p\Gamma_c(X,I^{\cdot}) = H^0\Gamma_c(X,I^{\cdot}[p])$$

From the duality formula 2.1 we get

$$H_c^{\,p}(X,F)^{\vee} = [\Gamma_c(X,I^{\cdot}[p]),k] = [I^{\cdot}[p],\mathcal{D}^{\cdot}] = [I^{\cdot},\mathcal{D}^{\cdot}[-p]]$$

The quasi-isomorphism $F \to I^{\cdot}$ allows us to make the identification

$$[I^{\cdot},\mathcal{D}^{\cdot}[-p]] = H^{-p}\mathrm{Hom}^{\cdot}(I^{\cdot},\mathcal{D}^{\cdot}) = H^{-p}\mathrm{Hom}(F,\mathcal{D}^{\cdot})$$

which proves 2.5. - Consider in particular $F = j_!\underset{\sim}{k}$ where $j: U \to X$ is the inclusion of an open subset U of X. This gives

$$H_c^{\,p}(U,\underset{\sim}{k})^{\vee} = H_c^{\,p}(X,j_!\underset{\sim}{k})^{\vee} = H^{-p}\mathrm{Hom}(j_!\underset{\sim}{k},\mathcal{D}^{\cdot})$$

Using the adjunction formula II.6.6 and II.7.2

$$\mathrm{Hom}(j_!\underset{\sim}{k},\mathcal{D}^{\cdot}) = \mathrm{Hom}(\underset{\sim}{k},j*\mathcal{D}^{\cdot}) = \Gamma(U,\mathcal{D}^{\cdot})$$

we get the formula

2.6
$$H^{-p}\Gamma(U,\mathcal{D}^{\cdot}) \overset{\sim}{\to} H_c^{\,p}(U,\underset{\sim}{k})^{\vee}$$

The formula 2.5 is functorial in F, which implies that 2.6 is compatible with restriction to an open subset of V.

Q.E.D.

The next result is already contained in the proof of 2.2 but I find it enlightening to deduce it as a corollary to 2.4.

Corollary 2.7. The dualizing complex for an n-dimensional locally compact space X may be represented by a complex $\mathcal{D}^{\cdot}$ of injective k-sheaves where

$$\mathcal{D}^i = 0 \quad \text{for} \quad i \notin [-n,0]$$

Proof. Let $\mathcal{D}^{\cdot}$ be a dualizing complex. It follows from 2.5 that

$$H^p\mathcal{D}^{\cdot} = 0 \quad \text{for} \quad p \notin [-n,0]$$

From this we conclude that the canonical morphism I.5.8

$$\mathcal{D}^{\cdot} \longrightarrow \tau_{\geq -n}\,\mathcal{D}^{\cdot}$$

is a quasi-isomorphism, and that the second of these complexes

consists of injectives. Thus we may assume that $\mathcal{D}^p = 0$ for $p < -n$. - Notice that the following sequence is exact

$$0 \longrightarrow \text{Ker } \partial^0 \longrightarrow \mathcal{D}^0 \xrightarrow{\partial^0} \mathcal{D}^1 \xrightarrow{\partial^1} \mathcal{D}^2 \longrightarrow \ldots$$

Interpreting this as an injective resolution of Ker ∂^0 we get for any k-sheaf F on X

$$\text{Ext}^1(F, \text{Ker } \partial^0) = H^1 \text{ Hom}(F, \mathcal{D}^{\cdot})$$

Using the formula 2.6 we get

$$H^1 \text{ Hom}(F, \mathcal{D}^{\cdot}) = H_c^{-1}(X, F)^{\vee} = 0$$

from which we conclude that Ker ∂^0 is injective. It follows that the canonical morphism

$$\tau_{\leq 0} \, \mathcal{D}^{\cdot} \longrightarrow \mathcal{D}^{\cdot}$$

is a homotopy equivalence.

Q.E.D.

Corollary 2.8. On a locally compact space X of dimension n, the presheaf

$$U \longmapsto H_c^n(U, \underset{\sim}{k})^{\vee}$$

is a sheaf.

Proof. Let the dualizing complex $\mathcal{D}^{\cdot}$ be represented as in 2.7. For an open subset U of X we have an exact sequence

$$0 \to \Gamma(U, H^{-n}\mathcal{D}^{\cdot}) \to \Gamma(U, \mathcal{D}^{-n}) \to \Gamma(U, \mathcal{D}^{-n+1})$$

By the formula 2.6 we have

$$H^{-n}\Gamma(U, \mathcal{D}^{\cdot}) = H_c^n(U, \underset{\sim}{k})^{\vee}$$

A combination of the two results realizes $H_c^n(U, \underset{\sim}{k})^{\vee}$ as sections in the sheaf $H^{-n}\mathcal{D}^{\cdot}$.

Q.E.D.

Example 2.9. Let X denote a locally closed subspace of $\mathbb{R}^n$ and $o\hbar$ the sheaf from 2.8. For a point $x \in X$ we have

$$o\hbar_x = \begin{cases} k & \text{for } x \in \dot{X} \\ 0 & \text{for } x \in \partial X \end{cases}$$

The first statement has already been noted in III.8.14. To prove the second statement choose an open disc D with center in x such that X ∩ D is closed relative to D. It follows that X ∩ D is homeomorphic to a proper closed subspace of $\mathbb{R}^n$ and as noted in III.9.8 $H_c^n(X \cap D, k) = 0$.

V.3 Orientation of topological manifolds

Let X be a topological manifold of dimension n, i.e.
a Hausdorff space locally homeomorphic to $\mathbb{R}^n$. Consider a
fixed field k and recall from III.8.13 that the corresponding
orientation sheaf $o\hbar$ has sections over the open subset U of
X given by

3.1 $$\Gamma(U, o\hbar) = H_c^n(X, \underline{k})^\vee$$

The k-sheaf $o\hbar$ is locally isomorphic to $\underline{k}$ loc.cit. - Let us
recall the formula II.7.5

$$H^p(X, o\hbar) = \text{Ext}^p(\underline{k}, o\hbar) \qquad ; \quad p \in \mathbb{Z}$$

where Ext is calculated in Sh(X,k). From general principles
I.8.6 we deduce for $p,q \in \mathbb{Z}$ a cupproduct

$$\alpha \cup \beta \in H_c^{p+q}(X, o\hbar) \qquad ; \quad \alpha \in H^p(X, o\hbar), \ \beta \in H_c^q(X, \underline{k})$$

subjected to Poincaré duality:

Poincaré duality 3.2. There exists a linear form

$$\int : H_c^n(X, o\hbar) \to k$$

such that for each $p \in \mathbb{Z}$ the bilinear form

$$\int \alpha \cup \beta \qquad ; \ (\alpha, \beta) \in H^p(X, o\hbar) \times H_c^{n-p}(X, k)$$

induces an isomorphism

$$H^p(X, o\hbar) \to H_c^{n-p}(X, k)^\vee$$

Proof. Let $\mathcal{D}^\cdot$ denote a dualizing complex represented as
in 2.7. For $x \in X$ we have according to 2.4

$$H^{-p}\mathcal{D}^\cdot_x = \varinjlim H_c^p(U, k) \qquad ; \quad p \in \mathbb{Z}$$

where the direct limit is taken over all open neighbourhoods U
of x in X. The point x has a fundamental system of open
neighbourhoods each of which is homeomorphic to $\mathbb{R}^n$. This im-
plies, III.8.14.

$$H^{-p}\mathcal{D}^\cdot = 0 \qquad ; \quad p \neq n$$

According to 2.8 we have $H^{-n}\mathcal{D}^\cdot = o\hbar$. We can record this as a
quasi-isomorphism

3.3 $$o\hbar[n] \xrightarrow{\ \sim\ } \mathcal{D}^\cdot$$

Otherwise expressed $o\hbar \to \mathcal{D}^\cdot[-n]$ is an injective resolution of the
sheaf $o\hbar$. For $p \in \mathbb{Z}$ we get accordingly

$$H^p(X, o\hbar) = H^p\Gamma(X, \mathcal{D}^\cdot[-n]) = H^0\Gamma(X, \mathcal{D}^\cdot[p-n]) = [\underline{k}, \mathcal{D}^\cdot[p-n]]$$

Choose an injective resolution $\underline{k} \to K^\cdot$ to get

$$H^p(X, o\hbar) = [K^\cdot, \mathcal{D}^\cdot[p-n]] = [K^\cdot[n-p], \mathcal{D}^\cdot]$$

Let us now reorganize the duality isomorphism 2.1

$$[I^\cdot, \mathcal{D}^\cdot] \xrightarrow{\sim} [\Gamma_c(X, I^\cdot), k]$$

Put $I^{.} = \mathcal{D}^{.}$ and let the image of $1 \in [\mathcal{D}^{.}, \mathcal{D}^{.}]$ be denoted

$$3.4 \qquad \int : \Gamma_c(X, \mathcal{D}^{.}) \to k$$

According to the Yoneda principle for representable functors the duality isomorphism is given by

$$3.5 \qquad \varphi \longmapsto \int \circ \Gamma_c(X, \varphi) \qquad ; \varphi \in [I^{.}, \mathcal{D}^{.}]$$

Let us apply this to $I^{.} = K^{.}[n-p]$ to get

$$[K^{.}[n-p], \mathcal{D}^{.}] = [\Gamma_c(X, K^{.}[n-p]), k] = H_c^{n-p}(X, k)$$

Finally use 3.5 to identify this isomorphism with the one described by the cup product formula.

$$Q.E.D.$$

By a k-orientation of a topological manifold X we understand an isomorphism

$$\underset{\sim}{k} \xrightarrow{\sim} o\hbar$$

of k-sheaves, where $o\hbar$ is the orientation sheaf. We shall say that X is orientable relative to k if a k-orientation exists and k-unorientable in the opposite case.

We shall now give a cohomological criterion for orientability of a manifold.

Proposition 3.6. Let X be a connected topological manifold of dimension n. Then

$$H_c^n(X, \underset{\sim}{k}) \xrightarrow{\sim} k \quad \text{if } X \text{ is orientable relative to } k.$$

$$H_c^n(X, \underset{\sim}{k}) \xrightarrow{\sim} 0 \quad \text{if } X \text{ is unorientable relative to } k.$$

Proof. Suppose X is orientable relative to k, and let $o\hbar \xrightarrow{\sim} k$ be an orientation of X. According to Poincaré duality 3.2 with $p = 0$ we get

$$H^0(X, \underset{\sim}{k}) \simeq H_c^n(X, \underset{\sim}{k})$$

Since X is connected we have $H^0(X, k) = k$ and it follows that $H_c^n(X, \underset{\sim}{k}) \xrightarrow{\sim} k$.

Let us next assume that $H_c^n(X, \underset{\sim}{k}) \neq 0$. Then we get from Poincaré duality 3.2 that

$$\Gamma(X, o\hbar) \neq 0$$

Let s be a non trivial section of $o\hbar$ over X. Recall that $\text{Supp}(s)$ is a closed subset of X for general reasons. The fact that $o\hbar$ is locally constant of rank 1 implies that $\text{Supp}(s)$ is open. Thus $\text{Supp}(s) = X$ since X is connected, and $\text{Supp}(S) \neq \emptyset$. In resumé: $o\hbar \xrightarrow{\sim} \underset{\sim}{k}$. This in turn implies $H_c^n(X, \underset{\sim}{k}) \xrightarrow{\sim} k$.

$$Q.E.D.$$

Case $k = \mathbb{F}_2$. Let us prove that in case the coefficient field $k = \mathbb{F}_2$, the field on two elements, then

$$3.7 \qquad o\hbar = \underset{\sim}{\mathbb{F}}_2$$

This is a consequence of

Lemma 3.8. Let X be an arbitrary topological space. Any $\mathbb{F}_2$-sheaf ω locally isomorphic to $\underset{\sim}{\mathbb{F}}_2$ is isomorphic to $\underset{\sim}{\mathbb{F}}_2$.

Proof. Recall the formula II.7.5

$$\text{Hom}(\underset{\sim}{\mathbb{F}}_2, \omega) = \Gamma(X, \omega)$$

This allows us to identify an isomorphism $s: \underset{\sim}{\mathbb{F}}_2 \overset{\sim}{\to} \omega$ with a global section

$$s \in \Gamma(X, \omega) \quad \text{with} \quad s_x \neq 0 \quad \text{for all} \quad x \in X.$$

Let us prove that there exists at most one such s: If t is another such section, we can find a section u of $\Gamma(X, \mathbb{F}_2^*)$ such that $t = us$, but $\mathbb{F}_2^* = \{1\}$. For each $x \in X$, choose an open neighbourhood U_x of x and an isomorphism s_x between $\underset{\sim}{\mathbb{F}}_2$ and the restriction of ω to U_x. By the previous result s_x and s_y have the same restriction to $U_x \cap U_y$. Thus we can glue the s_x's together to a global isomorphism.

Q.E.D.

V.4 Submanifolds of $\mathbb{R}^n$ of codimension 1

The origin of this topic is the celebrated theorem of Jordan: The complement of a simple closed plane curve has two connected components.

The basis for counting connected components of a topological space is the following

Proposition 4.1 [*]. Let X be a topological space and k a field. The set c.c X of connected components of X can be counted as follows

$$\#c.c\ X = \dim_k H^0(X, k)$$

Proof. Suppose $\#$ c.c X is finite. In that case each connected component is both open and closed. It follows that the characteristic functions for the connected components form a basis for $H^0(X, k)$.

Suppose conversely that $H^0(X, k)$ is finite dimensional. Assume $X \neq \emptyset$ and let $\mathcal{C}$ denote the set of non empty subsets of X which are both open and closed. Let $C \in \mathcal{C}$ be chosen such that $\dim H^0(C, k)$ is as small as possible. This implies that C is connected: otherwise we would have a partition $C = E \cup D$ of C into the non empty open subsets with $E \cap D = \emptyset$. From the formula

$$\dim H^0(C, k) = \dim H^0(E, k) + \dim H^0(D, k)$$

we derive at a contradiction. Thus we have proved that X has a connected component which is open (and closed). It is now

---

[*] For a set S, we let $\#S \in \overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$ denote the number of elements in S. For a k-vector space V we put $\dim_k V = \#B$ where B is a basis for V.

easy to proceed by induction on $\dim_k H^0(X,k)$.

<div align="right">Q.E.D.</div>

Corollary 4.2. Let X be a locally compact space and k a field. The set $c.c.c\,X$ of compact connected components of X can be counted as follows

$$\# c.c.c\,X = \dim_k {}^{_!}H_c^0(X,k)$$

Proof. In case X is compact the result follows from 4.1. In case X is non compact we let $\hat{X}$ denote the 1-point compactification of X. The space $\hat{X}$ is compact and contains a point ∞ such that $\hat{X} - \{\infty\}$ is homeomorphic to X. The exact sequence

$$0 \to H_c^0(X,k) \to H_c^0(\hat{X},k) \to H_c^0(\{\infty\},k) \to 0$$

shows that

$$\dim_k H^0(\hat{X},k) = 1+\dim_k H_c^0(X,k)$$

Let $C(\infty)$ denote the connected component of $\hat{X}$ which contains ∞. We leave it to the reader to identify the remaining connected components of $\hat{X}$ and $c.c.c\,X$. This gives

$$\# c.c\,\hat{X} = 1 + \#c.c.c\,X$$

The result follows from 4.1 by combining the two formulas.

<div align="right">Q.E.D.</div>

Let us now return to our main theme <u>a proper closed subset X of $\mathbb{R}^n$</u>. For an arbitrary field we have

4.3
$$\# c.c(\mathbb{R}^n - X) = 1 + \dim_k H_c^{n-1}(X,k)$$

Proof. Consider the exact sequence $(X \neq \mathbb{R}^n)$

$$0 \to H_c^{n-1}(X,k) \to H_c^n(\mathbb{R}^n-X,k) \to H_c^n(\mathbb{R}^n,k) \to 0$$

deduced from the III.8.3 and III.8.5 see the proof of III.8.7, and compare this with Poincaré duality 3.2 for $\mathbb{R}^n-X$

$$H^0(\mathbb{R}^n-X,k) \xrightarrow{\sim} H_c^n(\mathbb{R}^n-X,k)^\vee$$

to deduce the result.

<div align="right">Q.E.D.</div>

Jordan-Brouwer separation theorem 4.4. Let $\gamma: S^{n-1} \to \mathbb{R}^n$ be an injective continuous map. The complement of $\gamma(S^{n-1})$ in $\mathbb{R}^n$ has two connected components.

Proof. We have $H^{n-1}(S^{n-1},k) \xrightarrow{\sim} k$ according to III.8.8, and the result follows from 4.3.

<div align="right">Q.E.D.</div>

Let us turn to another important classical theorem

Invariance of domain 4.5. Any injective continuous map $f: \mathbb{R}^n \to \mathbb{R}^n$ is open, i.e. transforms open sets into open sets.

Proof. It remains the same to prove that any injective continuous map $f: B^n \to \mathbb{R}$ will transform the interior D^n of B^n into an open subset of $\mathbb{R}^n$.

Let us first remark that it follows from 4.3 that $\mathbb{R}^n - f(B^n)$ is connected, whereas $\mathbb{R}^n - f(S^{n-1})$ has precisely two connected components. Notice that this set admits the partition

$$\mathbb{R}^n - f(S^{n-1}) = \mathbb{R}^n - f(B^n) \cup f(D^n)$$

Thus the connected components of $\mathbb{R}^n - f(S^{n-1})$ are $\mathbb{R}^n - f(B^n)$ and $f(D^n)$. Quite generally the connected components of an open subset of $\mathbb{R}^n$ are open. Thus $f(D^n)$ is open.

Q.E.D.

Theorem 4.6. Let X be a closed, connected submanifold of $\mathbb{R}^n$ of codimension 1. The complement $\mathbb{R}^n - X$ has two connected components U and V and $X = \bar{U} \cap \bar{V}$.

Proof. Let us first remark that X is oriented relative to $k = \mathbb{F}_2$ according to 3.7. Thus we conclude from 3.6 and 4.3 that $\mathbb{R}^n - X$ has two connected components, U and V say. - We have $\bar{U} \cup \bar{V} = \mathbb{R}^n$ as it follows by remarking that the complement is an open subset of $\mathbb{R}^n$ contained in X, but X has no interior points as a subset of $\mathbb{R}^n$. - Consider the Mayer-Vietoris sequence

$$H_c^{n-1}(\bar{U} \cap \bar{V}, k) \to H_c^n(\bar{U} \cup \bar{V}, k) \to H_c^n(\bar{U}, k) \oplus H_c^n(\bar{V}, k)$$

Since $\bar{U}$ and $\bar{V}$ are proper closed subsets of $\mathbb{R}^n$ we find from III.9.8 that the two groups in the direct sum are zero. We con-

clude that $H_c^{n-1}(\bar{U} \cap \bar{V}, k) = 0$. We can now apply lemma 4.7 below to the closed subset $\bar{U} \cap \bar{V}$ of X to get the result.

Q.E.D.

Lemma 4.7. Let X be a connected n-dimensional manifold, oriented relative to k. For a proper closed subset Z of X we have

$$H_c^n(Z, k) = 0$$

Proof. Consider the long exact sequence

$$\to H_c^n(X-Z, k) \xrightarrow{e} H_c^n(X, k) \to H_c^n(Z, k) \to 0$$

The restriction map in the orientation sheaf $\Gamma(X, \mathit{or}) \to \Gamma(U, \mathit{or})$ may be identified with e^v, the linear dual of e, by 3.1. Since $\mathit{or} \underset{\sim}{\to} k$ and X is connected we conclude that e^v is injective. From this follows that e is surjective.

Q.E.D.

Theorem 4.8. A closed submanifold X of $\mathbb{R}^n$ of codimension $n-1$ is orientable relative to any field. Moreover

$$\# \, c.c(\mathbb{R}^n - X) = 1 + \# c.c \, X$$

Proof. We must prove that any connected component Y of X is orientable relative to k. According to 4.6 we find that $\mathbb{R}^n - Y$ has two connected components. Using 4.3 we conclude that $H_c^n(Y, k)$ is 1-dimensional, which implies that Y is orientable, 3.6. From Poincaré duality applied to the oriented manifold X we get that

$$H^0(X, k) \xrightarrow{\sim} H_c^n(X, k)^v$$

Conclusion by 4.1 and 4.3.

Q.E.D.

V.5 Duality for a subspace

Let X denote a locally compact space of finite dimension and k a fixed field. The corresponding dualizing complex will be denoted $\mathcal{D}^{\cdot}$. Verdier duality is a natural isomorphism

5.1 $$[I^{\cdot},\mathcal{D}^{\cdot}] \overset{\sim}{\to} [\Gamma_c(X,I^{\cdot}),k]$$

as I varies through $D^+(X,k)$. Put $I^{\cdot} = \mathcal{D}^{\cdot}$ and let the image of $1 \in [\mathcal{D}^{\cdot},\mathcal{D}^{\cdot}]$ be denoted

5.2 $$\int_X : \Gamma_c(X,\mathcal{D}^{\cdot}) \longrightarrow k$$

which we call the <u>trace map</u>. According to the Yoneda principle for representable functors the duality isomorphism can be recovered from the trace map by the formula

5.3 $$\varphi \longmapsto \int_X \circ \Gamma_c(X,\varphi) \qquad ;\varphi \in [I^{\cdot},\mathcal{D}^{\cdot}]$$

<u>Lemma 5.4</u>. Let $S^{\cdot}$ denote a bounded below complex of soft k-sheaves. The map 5.3 induces an isomorphism

$$[S^{\cdot},\mathcal{D}^{\cdot}] \overset{\sim}{\longrightarrow} [\Gamma_c(X,S^{\cdot}),k]$$

<u>Proof</u>. Let $S^{\cdot} \to I^{\cdot}$ be an injective resolution in Sh(X,k). This gives rise to a commutative diagram

$$\begin{array}{ccc} [I^{\cdot},\mathcal{D}^{\cdot}] & \longrightarrow & [\Gamma_c(X,I^{\cdot}),k] \\ \downarrow & & \downarrow \\ [S^{\cdot},\mathcal{D}^{\cdot}] & \longrightarrow & [\Gamma_c(X,S^{\cdot}),k] \end{array}$$

The vertical maps are isomorphisms according to I.6.2 and I.7.5. Conclusion by 5.1.

Q.E.D.

Let us now consider the inclusion h: W → X of a locally closed subspace W of X. The functors $h_!$ and $h^!$ of II.6 are adjoint:

$$\text{Hom}(h_! E,F) = \text{Hom}(E,h^! F)$$

for sheaves E on W and F on X. In particular we have an adjunction morphism

$$h_! h^! \mathcal{D}^{\cdot} \longrightarrow \mathcal{D}^{\cdot}$$

Applying $\Gamma_c(X,-)$ to this we get a map, III.7.1

$$\Gamma_c(W,h^!\mathcal{D}^{\cdot}) \to \Gamma_c(X,\mathcal{D}^{\cdot})$$

The composite of this with $\int_X$ will be denoted

5.5 $$\int_W : \Gamma_c(W,h^!\mathcal{D}^{\cdot}) \longrightarrow k$$

<u>Theorem 5.6</u>. The complex $h^!\mathcal{D}^{\cdot}$ is a dualizing complex for W. The corresponding trace map $\int_W$ is given by 5.5 above.

<u>Proof</u>. Consider the commutative diagram below. The morphisms involved are those discussed above.

$$
\begin{array}{ccc}
[J^{\cdot},h^{!}\mathcal{D}^{\cdot}] & \xrightarrow{\ \sim\ } & [h_{!}J^{\cdot},\mathcal{D}^{\cdot}] \\
\downarrow {\scriptstyle \Gamma_{c}(W,-)} & & \downarrow {\scriptstyle \Gamma_{c}(X,-)} \\
[\Gamma_{c}(W,J^{\cdot}),\Gamma_{c}(W,h^{!}\mathcal{D}^{\cdot})] & \xrightarrow{\ \sim\ } & [\Gamma_{c}(X,h_{!}J^{\cdot}),\Gamma_{c}(X,\mathcal{D}^{\cdot})] \\
\downarrow {\scriptstyle [-,\int_{W}]} & & \downarrow {\scriptstyle [-,\int_{X}]} \\
[\Gamma_{c}(W,J^{\cdot}),k] & \xrightarrow{\ \sim\ } & [\Gamma_{c}(X,h_{!}J^{\cdot}),k]
\end{array}
$$

The composite of the two vertical arrows to the right is an isomorphism by Lemma 5.4. It follows that the composite of the two vertical arrows to the left is an isomorphism.

<div align="right">Q.E.D.</div>

<u>Corollary 5.7</u>. Let X be a topological manifold with orientation sheaf $o\hbar_{X}$ and trace map $\int_{X}: H_{c}^{n}(X,o\hbar_{X}) \to k$. Let $j: U \to X$ be the inclusion of an open subspace. The orientation sheaf on U is $j^{*}o\hbar_{X}$ and the corresponding trace map $\int_{W}: H_{c}^{n}(U,j^{*}o\hbar_{X}) \to k$ is given by

$$\int_{W} \alpha = \int_{X} j_{!}\,\alpha \qquad ; \ \alpha \in H_{c}^{n}(U,j^{*}o\hbar_{X}).$$

<u>Proof</u>. Follows from the fact that $o\hbar\,[n] \xrightarrow{\sim} \mathcal{D}^{\cdot}$, 3.3.

<div align="right">Q.E.D.</div>

V.6 Alexander duality

Let us start with a discussion of cup products in cohomology with compact support. We consider a fixed commutative ring k.

Consider a closed subspace of a locally compact space X and a k-sheaf F on X. We shall construct a <u>cup product</u>

6.1 $$\alpha \cup \beta \in H_{c}^{p+q}(X,F) \qquad ; \ \alpha \in H_{Z}^{p}(X,F),\ \beta \in H_{c}^{q}(Z,k)$$

To this end we let $i: Z \to X$ denote the inclusion and use the representation II.9.8

$$H_{Z}^{p}(X,k) = \operatorname{Ext}_{k}^{p}(i_{*}k,F)$$

An element $\alpha \in \operatorname{Ext}_{k}^{p}(i_{*}k,F)$ will induce a linear map

$$\beta \longmapsto \alpha \cup \beta \quad ; \quad H_{c}^{q}(X,i_{*}k) \longrightarrow H_{c}^{p+q}(X,F)$$

The result follows from the identification III.1.7

$$H_{c}^{\cdot}(X,i_{*}k) = H_{c}^{\cdot}(Z,k)$$

Put $U = X-Z$ and let $i: Z \to X$ and $j: U \to X$ denote the inclusions. The cup product and the two basic long exact sequences

$$H^{p}(U,k) \xleftarrow{\ j^{*}\ } H^{p}(X,k) \xleftarrow{\ r\ } H_{Z}^{p}(X,k) \xleftarrow{\ \partial\ } H^{p-1}(U,k)$$
$$\qquad\qquad \gamma \qquad\qquad \beta \qquad\qquad \alpha$$

$$H_{c}^{n-p}(U,k) \xrightarrow[\ j_{!}\ ]{} H_{c}^{n-p}(X,k) \xrightarrow{} H_{c}^{n-p}(Z,k) \xrightarrow[\ \partial\ ]{} H_{c}^{n-p+1}(U,k)$$
$$\qquad\quad \zeta \qquad\qquad \eta \qquad \quad i^{*}\ \xi$$

are interrelated through the three formulas

6.2 $$(-1)^p \partial \alpha \cup \xi = j_!(\alpha \cup \partial \xi)$$

6.3 $$r \beta \cup \eta = \beta \cup i^* \eta$$

6.4 $$j_!(j^* \gamma \cup \zeta) = \gamma \cup j_! \zeta$$

Verification. The canonical exact sequence

$$0 \to j_! \underset{\sim}{k} \to \underset{\sim}{k} \to i_* \underset{\sim}{k} \to 0$$

gives rise to a canonical class, I.8.4

$$\theta \in \text{Ext}^1(i_* \underset{\sim}{k}, j_! \underset{\sim}{k})$$

The long exact sequence in local cohomology

$$\leftarrow H^p(U,k) \xleftarrow{j^*} H^p(X,k) \xleftarrow{r} H_Z^p(X,k) \xleftarrow{\partial} H^{p-1}(U,k) \leftarrow$$

may be identified with the Ext-sequence, I.8.5 and II.9.8

$$\longleftarrow \text{Ext}^p(j_! \underset{\sim}{k}, \underset{\sim}{k}) \longleftarrow \text{Ext}^p(\underset{\sim}{k}, \underset{\sim}{k}) \longleftarrow \text{Ext}^p(i_* \underset{\sim}{k}, \underset{\sim}{k}) \xleftarrow{\cup \theta (-1)^p} \text{Ext}^{p-1}(j_! \underset{\sim}{k}, \underset{\sim}{k})$$

while the long exact sequence in cohomology with compact support

$$H_c^{n-p}(U, \underset{\sim}{k}) \longrightarrow H_c^{n-p}(X, \underset{\sim}{k}) \longrightarrow H_c^{n-p}(Z, \underset{\sim}{k}) \longrightarrow H_c^{n-p+1}(U, \underset{\sim}{k})$$

may be identified with

$$H_c^{n-p}(X, j_! \underset{\sim}{k}) \longrightarrow H_c^{n-p}(X, \underset{\sim}{k}) \longrightarrow H_c^{n-p}(X, i_* \underset{\sim}{k}) \xrightarrow{\theta \cup} H_c^{n-p+1}(X, j_! \underset{\sim}{k})$$

With these identifications the formulas become more ore less obvious.

Excision formulas for cup product

Let Z denote a closed subset of the locally compact space X and O an open subspace. For any k-sheaf F we have

$$
\begin{array}{ccc}
Z \cap O & \xrightarrow{\;l\;} & O \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle j} \\
Z & \xrightarrow{\;i\;} & X
\end{array}
$$

6.5 $$\omega \cup h_! \alpha = j_!(j^* \omega \cup \alpha) \quad ; \quad \omega \in H_Z^p(X,F), \alpha \in H_c^q(Z \cap O)$$

where $j^* \omega \in H_{Z \cap O}^p(O,F)$ is the restriction of ω.

Proof. Let us recall II.6.13, that for a sheaf G on Z we have $j^* i_* G = l_* h^* G$. In particular if we take an injective resolution $k \to K^\cdot$ in $Sh(Z,k)$ we find that $j^* i_* K^\cdot$ is an injective resolution of $l_* \underset{\sim}{k}$, since

$$l_* \underset{\sim}{k} = l_* h^* \underset{\sim}{k} = j^* i_* \underset{\sim}{k}$$

Let $F \to J^\cdot$ be an injective resolution $Sh(X,k)$. Since

$$H_Z^p(X,F) = \text{Ext}^p(i_* k, F) = [i_* K^\cdot, I^\cdot[p]]$$

we can represent ω as a morphism of complexes $\omega: i_* K^\cdot \to I^\cdot[p]$. The commutative diagram

$$
\begin{array}{ccc}
j_! j^* i_* K^\cdot & \xrightarrow{\;j_! j^* \omega\;} & j_! j^* I^\cdot[p] \\
\downarrow & & \downarrow \\
i_* K^\cdot & \xrightarrow[\;\omega\;]{} & I^\cdot[p]
\end{array}
$$

transforms by $\Gamma_c(X,-)$ into the diagram, compare III.7.1.

$$I_c(O,j^*i_*K^\cdot) \xrightarrow{\quad I_c(O,j^*\omega) \quad} I_c(O,j^*I^\cdot[p])$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$I_c(X,i_*K^\cdot) \xrightarrow{\quad I_c(X,\omega) \quad} I_c(X,I^\cdot[p])$$

Applying H^q to this we get the commutative diagram

$$H_c^{\;q}(O \cap Z,k) \xrightarrow{\quad j^*\omega \cup \quad} H_c^{\;p-q}(O,j^*F)$$

$$\downarrow h_! \qquad\qquad\qquad\qquad\qquad \downarrow j_!$$

$$H_c^{\;q}(Z,k) \xrightarrow{\quad \omega \cup \quad} H_c^{\;p+q}(X,F)$$

from which the result follows.. $\qquad\qquad$ Q.E.D.

$\underline{\text{Alexander Duality 6.6.}}$ Let X denote an n-dimensional topo-logical manifold oriented relative to the field k, and let

$$\int_X : H_c^{\;n}(X,k) \longrightarrow k$$

denote the trace map. For a closed subset Z of X the bi-linear form

$$\int_X \alpha \cup \beta \qquad\qquad ; \quad \alpha \in H_Z^{\;p}(X,k), \beta \in H_c^{\;n-p}(Z,k)$$

induces an isomorphism $H_Z^{\;p}(X,k) \xrightarrow{\sim} H_c^{\;n-p}(Z,k)^\vee$.

$\underline{\text{Proof}}$. The duality isomorphism

$$\varphi \longmapsto \int_X \circ I(X,\varphi) ; \quad [I^\cdot, \mathcal{D}^\cdot] \xrightarrow{\sim} [I_c(X,I^\cdot),k]$$

can be rewritten $\qquad\qquad [I^\cdot, \mathcal{D}^\cdot[p-n]] \xrightarrow{\sim} (H^{n-p}I_c(X,I^\cdot))^\vee$

Applied to an injective resolution $i_*k \to I^\cdot$ this gives

$$[i_*k, \mathcal{D}^\cdot[-n][p]] \xrightarrow{\sim} H_c^{\;n-p}(Z,k)^\vee$$

The result follows by the fact that $\mathcal{D}^\cdot[-n]$ is an injective resolution of $\underset{\sim}{k}$, 3.3.

$$\text{Q.E.D.}$$

Alexander duality for $\mathbb{R}^n$

Let Z denote a proper closed subspace of $\mathbb{R}^n$. We have an isomorphism

$$H_Z^{\;p}(X,k) \xrightarrow{\sim} H_c^{\;n-p}(Z,k)^\vee \qquad ; \quad p \in \mathbb{N}$$

by Alexander duality. Using the long exact sequence with $U = \mathbb{R}^n - Z$

$$H^p(\mathbb{R}^n,k) \to H^p(U,k) \to H_Z^{\;p+1}(\mathbb{R}^n,k) \to H^{p+1}(\mathbb{R}^n,k) \to$$

we deduce the following formulas

$$\dim_k H^0(U,k) = 1 + \dim_k H_c^{\;n-1}(Z,k)$$

6.7

$$\dim_k H^p(U,k) = \dim_k H_c^{\;n-p-1}(Z,k) \qquad ; \quad p \geq 1$$

The first of these has been exploited in 4.3. Let us work out the second relation in case $p = n-1$. We get according to 4.2

6.8 $\qquad\qquad \dim_k H^{n-1}(U,k) = \# \text{c.c.c}(Z)$

In case $p = n$ we get

6.9 $\qquad\qquad H^n(U,k) = 0 \qquad\qquad ; \quad U \subseteq \mathbb{R}^n$

V.7 Residue theorem for n-1 forms on $\mathbb{R}^n$

In this section we shall generalize Cauchy's residue theorem to $\mathbb{R}^n$.

Local symbols

We consider a fixed orientation of $\mathbb{R}^n$ with trace map

$$\int_{\mathbb{R}^n} : \; H_c^n(\mathbb{R}^n, \mathbb{C}) \to \mathbb{C}$$

For a point $s \in \mathbb{R}^n$, the cup product gives a non degenerate pairing by Alexander duality 6.6

$$H_{\{s\}}^n(\mathbb{R}^n, \mathbb{C}) \times H_c^0(\{s\}, \mathbb{C}) \longrightarrow H_c^n(\mathbb{R}^n, \mathbb{C})$$

We let $1_s \in H_c^0(\{s\}, \mathbb{C})$ denote the constant 1 and denote by $\omega_s \in H^{n-1}(\mathbb{R}^n - \{s\}, \mathbb{C})$ the class such that

7.1
$$\int_{\mathbb{R}^n} \partial \, \omega_s \cup 1_s = 1$$

where $\partial : H^{n-1}(\mathbb{R}^n - \{s\}, \mathbb{C}) \to H_{\{s\}}^n(\mathbb{R}^n, \mathbb{C})$ denotes the boundary in the long exact sequence of local cohomology.

For an open neighbourhood U of s and $\omega \in H^{n-1}(U - \{s\}, \mathbb{C})$ we let $\partial \omega \in H_{\{s\}}^n(U, \mathbb{C})$ denote image under the boundary map from the long exact sequence in local cohomology. By excision II.9.6 we can identify $\partial \omega$ with an element of $H_{\{s\}}^n(\mathbb{R}^n, \mathbb{C})$ which we denote

$$\partial_s \omega \in H_{\{s\}}^n(\mathbb{R}^n, \mathbb{C})$$

It follows from II.9.7 that the symbol $\partial_s \omega$ is unchanged if we replace ω by the restriction of ω to $V - \{s\}$ where V is an open neighbourhood of s in U. Finally put

7.2
$$\mathrm{Tr}(\omega; s) \; = \; \int_{\mathbb{R}^n} \partial_s \, \omega \cup 1_s$$

Let there be given $\gamma : S^{n-1} \to \mathbb{R}^n$ a continuous map and an orientation of S^{n-1} with trace map

$$\int_{S^{n-1}} H^{n-1}(S^{n-1}, \mathbb{C}) \to \mathbb{C}$$

For $s \in \mathbb{R}^n - \gamma(S^{n-1})$ we define the index by

7.3
$$I(\gamma; s) \; = \; \int_{S^{n-1}} \gamma^* \omega_s$$

Theorem 7.4. Let U denote an open subset of $\mathbb{R}^n$ and $\gamma : S^{n-1} \to U$ a continuous map. The function $s \mapsto I(\gamma; s)$ is locally constant on the complement Z of U in $\mathbb{R}^n$. Let $I \in H_c^0(Z, \mathbb{C})$ be the corresponding cohomology class. We have

$$\int_{S^{n-1}} \gamma^* \omega = \int_{\mathbb{R}^n} \partial \omega \cup I \qquad ; \; \omega \in H^{n-1}(U, \mathbb{C})$$

where $\partial \omega \in H_Z^n(\mathbb{R}^n, \mathbb{C})$ denotes the image of ω by the boundary map of the long exact sequence of local cohomology.

Proof. Choose an open subset V of U containing $\gamma(S^{n-1})$ and such that $H_c^1(V, \mathbb{C})$ is a finite dimensional vector space. Take for example V to be a finite union of open cubes. - The linear form on $H^{n-1}(V, \mathbb{C})$

$$\omega \longmapsto \int_{S^{n-1}} \gamma^* \omega$$

can according to Poincaré duality 3.2 be represented

$$\int_{S^{n-1}} \gamma^* \omega = \int_V \omega \cup \varphi$$

where $\varphi \in H_c^1(V,\mathbb{C})$. Let $j: V \to U$ denote the inclusion and put $\theta = j_!\varphi$. We get for $\omega \in H^{n-1}(U,\mathbb{C})$ using 6.4 and 5.7

$$\int_U \omega \cup \theta = \int_U \omega \cup j_!\varphi = \int_U j_!(j^*\omega \cup \varphi) = \int_V j^*\omega \cup \varphi$$

which proves the formula

7.5
$$\int_{S^{n-1}} \gamma^*\omega = \int_U \omega \cup \theta \qquad ; \qquad \omega \in H^{n-1}(U,\mathbb{R})$$

Consider a point $s \notin U$ and let $j_s: U \to \mathbb{R}^n-[s]$ denote the inclusion. Proceeding as above we find

$$\int_{S^{n-1}} \gamma^*\omega = \int_{\mathbb{R}^n-\{s\}} \omega \cup j_{s!}\theta \qquad ; \qquad \omega \in H^{n-1}(\mathbb{R}^n-\{s\},\mathbb{C})$$

At this point let us introduce the commutative diagram III.7.7

$$
\begin{array}{ccccccc}
H_c^0(\mathbb{R}^n,\mathbb{C}) & \to & H_c^0(\mathbb{R}^n-U,\mathbb{C}) & \xrightarrow{\partial}_{\sim} & H_c^1(U,\mathbb{C}) & \to & H_c^1(\mathbb{R}^n,\mathbb{C}) \\
 & & \downarrow & & \downarrow{\scriptstyle j_{s!}} & & \\
 & & H_c^0(\{s\},\mathbb{C}) & \xrightarrow{\partial}_{\sim} & H_c^1(\mathbb{R}^n-\{s\},\mathbb{C}) & &
\end{array}
$$

Put $I = (-1)^n \partial^{-1}\theta$ to get $j_{s!}\theta = (-1)^n I(s)\partial 1_s$ and

$$\int_{S^{n-1}} \gamma^*\omega = (-1)^n I(s) \int_{\mathbb{R}^n-\{s\}} \omega \cup \partial 1_s$$

In particular with $\omega = \omega_s$ we get by 7.3

$$I(\gamma; s) = (-1)^n I(s) \int_{\mathbb{R}^n-\{s\}} \omega_s \cup \partial 1_s$$

From formula 5.7, 6.2 and 7.1 we deduce

$$(-1)^n \int_{\mathbb{R}^n-\{s\}} \omega_s \cup \partial 1_s = (-1)^n \int_{\mathbb{R}^n} j_{s!}(\omega_s \cup \partial 1_s) = \int_{\mathbb{R}^n} \partial \omega_s \cup 1_s = 1$$

and consequently $I(s) = I(\gamma;s)$. This proves the first part of the theorem. From 7.5 we get with $\theta = (-1)^n \partial I$

$$\int \gamma^*\omega = (-1)^n \int_U \omega \cap \partial I = \int_{\mathbb{R}^n} \partial \omega \cup I$$

where we have used 5.7 and 6.2.

$$Q.E.D.$$

Theorem 7.6. Let U be an open set of $\mathbb{R}^n$, and S a discrete subset of U, which is closed relative to U. Given a continuous map $\gamma: S^{n-1} \to U-S$ such that

$$I(\gamma;z) = 0 \quad \text{for all} \quad z \in \mathbb{R}^n-U$$

Then for any $\omega \in H^{n-1}(U-S,\mathbb{C})$

$$\boxed{\int_{S^{n-1}} \gamma^*\omega = \sum_{s\in S} \mathrm{Tr}(\omega;s)\, I(\gamma;s)}$$

Proof. Let P denote the support of the function $s \longmapsto I(\gamma;s)$ on the closed subset $S \cup (\mathbb{R}^n-U)$. We must have $P \subseteq S$: to exclude a point $z \in \mathbb{R}^n-U$ we can remark that $I(\gamma;z) = 0$ by assumption and that our function is locally constant. In conclusion P is a compact subset of the discrete set S, i.e. P is a finite subset of S. Let $I \in H_c^0(P,\mathbb{C})$ denote the restriction of the function $s \longmapsto I(\gamma;s)$ to P.

For each $s \in P$ choose an open disc D_s with center s contained in U such that $D_s \cap S = \{s\}$. Make the discs so small that they do not intersect and let $j: D \to U$ denote the inclusion of their union into U. We have according to 7.4, 6.5 that

$$\int_{S^{n-1}} \gamma^* \omega = \int_D j^* \partial \omega \cup I = \int_D \partial j^* \omega \cup I$$

In order to evaluate the right hand side we may replace ω by $\sum_{s \in P} \mathrm{Tr}(\omega;s) \omega_s$. This gives

$$\int_U \partial j^* \omega \cup I = \sum_{s \in P} \mathrm{Tr}(\omega;s) I(\gamma;s) \int_{\mathbb{R}^n} \partial \omega_s \cup 1_s$$

and the result follows from 7.1.

$$Q.E.D.$$

Example 7.7. For $s \in \mathbb{C}$ we find

$$\mathrm{Tr}\left(\frac{1}{2\pi i} \frac{1}{z-s}; s\right) = 1$$

as it follows from IV.7.16. From this we find

$$\mathrm{Tr}(f(z)dz) = 2\pi i \, \mathrm{Res}(f(z)dz;s)$$

for a meromorphic function f defined in a neighbourhood of s.

Example 7.8. The closed differential form

$$\theta = r^{-n} \sum_{j=1}^n (-1)^{i-1} x_i \, dx_1 \cdots \wedge \hat{dx}_i \cdots \wedge dx_n$$

discussed in IV.7.17 have local trace

$$\mathrm{Tr}(\theta;0) = \sigma_{n-1}$$

This follows from 6.4. and loc.cit.

VI. Poincare Duality with General Coefficients

VI.1 Verdier duality

In this section we consider a locally compact space X of finite dimension and a noetherian commutative ring k. Recall that $D^+(X,k)$ denotes the homotopy category of bounded below injective complexes of k-sheaves on X and $D^+(k)$ the homotopy category of bounded below complexes of injective k-modules. We are going to prove that the derived functor

$$R\Gamma_c(X,-): D^+(X,k) \to D^+(k)$$

has a right adjoint.

Verdier duality 1.1. There exists a functor

$$G^{\cdot} \longmapsto G^{\cdot !}, \quad D^+(k) \to D^+(X,k)$$

and a natural isomorphism

$$[R\Gamma_c(X,I^{\cdot}),G^{\cdot}] = [I^{\cdot},G^{\cdot !}]$$

as $I^{\cdot}$ varies through $D^+(X,k)$ and $G^{\cdot}$ varies through $D^+(k)$.

Proof. 1) For a soft, flat k-sheaf S and a k-module G we are going to define a k-sheaf $D(S,G)$ on X: For an open subset U of X we put

$$\Gamma(U,D(S,G)) = \operatorname{Hom}(\Gamma_c(U,S),G)$$

The inclusion $j: V \to U$ of two open subsets will induce

$$j_!: \Gamma_c(V,S) \to \Gamma_c(U,S)$$

The transform of this by the functor $\operatorname{Hom}_k(-,G)$ is restriction from U to V in the presheaf $D(S,G)$. The proof of V.1.2 shows that $D(S,G)$ is a sheaf.

2) The next thing to do is to notice that the tensor product $S \otimes F$ of a soft and flat k-sheaf S with an arbitary k-sheaf F, is a soft k-sheaf by the proof of V.1.3, see II.11.

3) For an open subset U of X, the composite

$$\Gamma(U,F) \otimes \Gamma_c(U,S) \to \Gamma_c(U,F \otimes S) \to \Gamma_c(X,F \otimes S)$$

transforms by $\operatorname{Hom}_k(-,G)$ into a map

$$\operatorname{Hom}(\Gamma_c(X,F \otimes S),G) \to \operatorname{Hom}(\Gamma(U,F),\operatorname{Hom}(\Gamma_c(U,S),G))$$

and by variation of U in fact a map

$$\operatorname{Hom}(\Gamma_c(X,F \otimes S),G) \to \operatorname{Hom}(F,D(S,G))$$

which is seen to be an isomorphism still provided S is soft and flat: use the proof of V.1.5.

4) If G is an injective k-module, then the sheaf $D(S,G)$ is injective in $Sh(X,k)$: According to the isomorphism above it suffices to prove that the functor

$$F \longmapsto \operatorname{Hom}(\Gamma_c(X,F \otimes S),G)$$

is exact on $Sh(X,k)$. This follows from 2).

5) Let $\underset{\sim}{k} \to S^{\cdot}$ be a bounded resolution of $\underset{\sim}{k}$ by soft and flat sheaves, see 1.3 below. An object $I^{\cdot}$ of $D^+(k)$ gives rise to a complex $D^{\cdot}(S^{\cdot},I^{\cdot})$ by general principles, I.11. In fact a bounded below complex of injective k-modules as it follows from 4). The isomorphism from 3) extends to an isomorphism

$$\operatorname{Hom}^{\cdot}(\Gamma_c(X,F^{\cdot} \otimes S^{\cdot}),G^{\cdot}) \overset{\sim}{\to} \operatorname{Hom}^{\cdot}(F^{\cdot},D^{\cdot}(S^{\cdot},G^{\cdot}))$$

6) In case $I^{\cdot}$ is an object of $D^+(X,k)$ we successively deduce quasi-isomorphisms

$$I^{\cdot} \to I^{\cdot} \otimes k \to I^{\cdot} \otimes S^{\cdot} \qquad \text{by II.11.7}$$

$$\Gamma_c(X,I^{\cdot}) \to \Gamma_c(X,I^{\cdot} \otimes S^{\cdot}) \quad \text{by 2) and I.7.5.}$$

$$\operatorname{Hom}^{\cdot}(\Gamma_c(X,I^{\cdot} \otimes S^{\cdot}),G^{\cdot}) \to \operatorname{Hom}^{\cdot}(\Gamma_c(X,I^{\cdot}),G^{\cdot}) \quad \text{by I.6.2}$$

7) A combination of 5) and 6) yields a quasi-isomorphism

1.2
$$\operatorname{Hom}^{\cdot}(I^{\cdot},D^{\cdot}(S^{\cdot},G^{\cdot})) \to \operatorname{Hom}^{\cdot}(\Gamma_c(X,I^{\cdot}),G^{\cdot})$$

Put $G^{\cdot !} = D^{\cdot}(S^{\cdot},G^{\cdot})$ and apply H^0 to 1.2 to get

$$[I^{\cdot},G^{\cdot !}] \underset{\sim}{\to} [\Gamma_c(X,I^{\cdot}),G^{\cdot}]$$

and our journey ends.

Q.E.D.

<u>Proposition 1.3</u>. Let X be a locally compact space of finite dimension and k a noetherian ring. There exists a bounded resolution $\underset{\sim}{k} \to S^{\cdot}$ of $\underset{\sim}{k}$ in $Sh(X,k)$ by soft and k-flat sheaves.

<u>Proof</u>. More generally let F be a flat k-sheaf and

$$0 \longrightarrow C^0 F \xrightarrow{\partial^0} C^1 F \xrightarrow{\partial^1} \dots \longrightarrow C^n F \xrightarrow{\partial^n} C^{n+1} F$$

the Godement resolution of F, II.3.6. Let us prove that $C^n F$ and $Im \partial^n$ are flat sheaves for all $n \in \mathbb{N}$. By the iterative nature of the Godement resolution it suffices to treat the case $n = 0$. For an open subset U of X we have

$$\Gamma(U, C^0 F) = \prod_{x \in U} F_x$$

and it follows from Lemma 1.4 below that $\Gamma(U, C^0 F)$ is a flat k-module: since a direct limit of flat k-modules is flat it follows that the stalks of $C^0 F$ are flat k-modules. Noticing that $F \to C^0 F$ has local retractions it follows that $Im \partial^0$ is a flat k-sheaf.

Let the dimension of X be n. The truncation $S^{\cdot} = \tau_{\leq n} C^{\cdot} F$ is flat and soft, III.9.9.

Q.E.D.

<u>Lemma 1.4</u>. Let A be a noetherian ring and $(F_i)_{i \in I}$ a family of flat A-modules. The product $F = \prod_{i \in I} F_i$ is a flat A-module.

<u>Proof</u>. Let M be a finitely generated A-module. Let us prove that the canonical map

$$M \otimes F \to \prod_{i \in I} M \otimes F_i$$

is an isomorphism. This is clear in the case where M is finitely generated and free: check the case $M = A$. In the general case choose an exact sequence of the form $A^p \to A^q \to M \to 0$ and consider the exact commutative diagram

$$
\begin{array}{ccccccc}
A^p \otimes F & \longrightarrow & A^q \otimes F & \longrightarrow & M \otimes F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\prod A^p \otimes F_i & \longrightarrow & \prod A^q \otimes F_i & \longrightarrow & \prod M \otimes F_i & & 0
\end{array}
$$

to conclude that the vertical arrow to the right is an isomorphism. As a consequence, the functor $\otimes F$ is exact on the category of finitely generated k-modules. Let us now prove that

$$Tor_1^A(M,F) = 0$$

for any A-module M. The case where M is finitely generated follows by considering an exact sequence of the form $0 \to N \to A^r \to M \to 0$. The general case follows by considering M as the direct limit of its finitely generated submodules: notice that it follows from II.8.1 that $Tor_A^p(-,F)$ commutes with direct limits.

Q.E.D.

VI.2 The dualizing complex $\mathcal{D}^{\cdot}$.

Let X denote a locally compact space of finite dimension and k a noetherian ring. We choose an injective resolution $k \to K^{\cdot}$ and define the dualizing complex $\mathcal{D}^{\cdot} = K^{\cdot !}$ with the notation of 1.1. Recall that we have a natural isomorphism

2.1
$$[I^{\cdot}, \mathcal{D}^{\cdot}] \stackrel{\sim}{\to} [\Gamma_c(X, I^{\cdot}), K^{\cdot}]$$

as $I^{\cdot}$ varies through $D^{+}(X,k)$. Put $I^{\cdot} = \mathcal{D}^{\cdot}$ and let the image of $1 \in [\mathcal{D}^{\cdot}, \mathcal{D}^{\cdot}]$ be denoted

2.2
$$\int_X : \Gamma_c(X, \mathcal{D}^{\cdot}) \longrightarrow K^{\cdot}$$

This is called the trace map. According to the Yoneda principle for representable functors the duality isomorphism can be recovered from the trace map by the formula

2.3
$$\varphi \longmapsto \int_X \circ \Gamma_c(X, \varphi) \qquad ; \quad \varphi \in [I^{\cdot}, \mathcal{D}^{\cdot}]$$

It is useful to give this a slightly broader formulation

Proposition 2.4. For any bounded below complex $S^{\cdot}$ of soft k-sheaves on X, the trace map induces an isomorphism

$$[S^{\cdot}, \mathcal{D}^{\cdot}] \stackrel{\sim}{\longrightarrow} [\Gamma_c(X, S^{\cdot}), K^{\cdot}]$$

Proof. Analogous to the proof of V.5.4.

Q.E.D.

Proposition 2.5. Let X be a locally compact space of dimension n. The dualizing complex may be represented by a bounded complex $\mathcal{D}^{\cdot}$ of injectives with $\mathcal{D}^p = 0$ for $p < -n$. Moreover

$$\Gamma(U, H^{-n}\mathcal{D}^{\cdot}) = \operatorname{Hom}(H_c^n(U,k), k)$$

for any open subset U of X.

Proof. Let $\underset{\sim}{k} \to S^{\cdot}$ be a soft resolution of $\underset{\sim}{k}$ of length n. Consider the inclusion $j : U \to X$ of an open subset. By 2.4 applied to the soft complex $j_! j^* S^{\cdot}[p]$, $p \in \mathbb{Z}$, we get

$$[j_! j^* S^{\cdot}, \mathcal{D}^{\cdot}[-p]] = [\Gamma_c(U, S^{\cdot}[p]), K^{\cdot}]$$

where we have used III.7.1. The left hand side can be identified by means of II.6.6 and II.7.2 obtaining

2.6
$$H^{-p}\Gamma(U, \mathcal{D}^{\cdot}) = [\Gamma_c(U, S^{\cdot}[p]), K^{\cdot}]$$

For $p > n$, the complex $\Gamma_c(U, S^{\cdot}[p])$ is concentrated in strictly negative degrees, thus the right hand side is 0. By II.2.7 we conclude that

$$H^{-p}\mathcal{D}^{\cdot} = 0 \qquad \text{for} \quad p > n$$

Thus we may replace $\mathcal{D}^{\cdot}$ by $\tau_{\geq -n}\mathcal{D}^{\cdot}$. In the sequel we shall assume that $\mathcal{D}^p = 0$ for $p < -n$. This gives rise to an exact sequence of sheaves on X

$$0 \rightarrow H^{-n}\mathcal{D}^{\cdot} \rightarrow \mathcal{D}^{-n} \rightarrow \mathcal{D}^{-n+1} \rightarrow \quad \ldots$$

and isomorphisms

$$\Gamma(U,H^{-n}\mathcal{D}^{\cdot}) = H^{-n}\Gamma(U,\mathcal{D}^{\cdot}) = [\Gamma_c(U,S^{\cdot}[n]),K^{\cdot}]$$

The right hand side can be evaluated by noticing that the first of the two complexes is located in negative degrees and the second complex is located in positive degrees. It follows that

$$[\Gamma_c(U,S^{\cdot}[n]),K^{\cdot}] = \mathrm{Hom}(H^0\Gamma_c(U,S^{\cdot}[n]),H^0(K^{\cdot})) = \mathrm{Hom}(H_c^n(U,k),k)$$

Q.E.D.

Let us consider the inclusion $h: W \rightarrow X$ of a locally closed subspace W. The adjunction morphism, II.6.6

$$h_! h^! \mathcal{D}^{\cdot} \rightarrow \mathcal{D}^{\cdot}$$

will induce a chain map, compare III.7.1

$$\Gamma_c(W,h^!\mathcal{D}^{\cdot}) \longrightarrow \Gamma_c(X,\mathcal{D}^{\cdot})$$

which we compose with $\int_X$ to obtain a chain map

2.7
$$\int_W : \Gamma_c(W,h^!\mathcal{D}^{\cdot}) \rightarrow K^{\cdot}$$

The proof of V.5.6 shows that $h^!\mathcal{D}^{\cdot}$ is a dualizing complex for W with trace map $\int_W$.

VI.3 Lefschetz duality

In this section we consider an n-dimensional topological manifold X with boundary ∂X, compare III.8.14. We let k denote a noetherian ring and or_X, the underlined orientation sheaf relative to k. The sections over an open subset U of X are given by

3.1
$$\Gamma(U,or_X) = \mathrm{Hom}_k(H_c^n(U,k),k)$$

The dualizing complex $\mathcal{D}^{\cdot}$ and the orientation sheaf are related through a quasi-isomorphism

3.2
$$or_X[n] \xrightarrow{\sim} \mathcal{D}^{\cdot}$$

Proof. Let $\underline{k} \rightarrow S^{\cdot}$ be a soft resolution of $\underline{k}$. Recall from 2.6 that for each open subset U of X we have an isomorphism

$$H^{-p}\Gamma(U,\mathcal{D}^{\cdot}) = [\Gamma_c(U,S^{\cdot}[p]),K^{\cdot}] \quad ; \; p \in \mathbb{Z}$$

In case U is isomorphic to $\mathbb{R}^n$ or $\mathbb{R}^{n-1} \times \mathbb{R}_+$, the complex $\Gamma_c(U,S^{\cdot}[p])$ is quasi-isomorphic to $H_c^n(U,k)[p-n]$ and we find that

$$H^{-p}\Gamma(U,\mathcal{D}^{\cdot}) = \mathrm{Ext}^{n-p}(H_c^n(U,k),k)$$

Using that $H_c^n(U,k)$ is a free module (in fact isomorphic to k or 0 respectively) we find

$$H^{-p}\Gamma(U,\mathcal{D}^{\cdot}) = 0 \quad ; \; p \neq n$$

From this we can calculate the stalks of $H^{-p}\mathcal{D}^{\cdot}$ to conclude that $H^{-p}\mathcal{D}^{\cdot} = 0$ for $p \neq n$. The result now follows from 2.5.

Q.E.D.

Proposition 3.3. Let X denote a manifold with boundary ∂X and $j: \dot{X} \to X$ the inclusion of the interior $\dot{X} = X - \partial X$. The orientation sheaf $o\hbar_X$ for X is given by $o\hbar_X = j_! o\hbar$ where $o\hbar$ is the orientation sheaf for $\dot{X}$, the interior of X.

Proof. Let us first remark that the stalk of $o\hbar_X$ at a boundary point is zero, as it follows from 3.1. Next remark that $o\hbar_X$ and $j_! o\hbar$ have the same restriction to $\dot{X}$. It follows from II.6.4 that these two sheaves are identical.

$$Q.E.D.$$

Lefschetz duality 3.4. Let X be a manifold with boundary ∂X. If $\dot{X} = X - \partial X$ is oriented relative to k, we have an isomorphism

$$H^{n-p}(X, \partial X; k) \xrightarrow{\sim} [R^{\cdot}\Gamma_c(X,k)[p], K^{\cdot}]$$

where $K^{\cdot}$ is an injective resolution of the k-module k.

Proof. Let $\underset{\sim}{k} \to I^{\cdot}$ be an injective resolution. Consider the duality isomorphism $\qquad [I^{\cdot}[p], \mathcal{D}^{\cdot}] = [\Gamma_c(X, I^{\cdot}[p]), K^{\cdot}]$

$$[I^{\cdot}[p], \mathcal{D}^{\cdot}] = [I^{\cdot}[p-n], \mathcal{D}^{\cdot}[-n] = [\underset{\sim}{k}[p-n], \mathcal{D}^{\cdot}[-n]] = H^{n-p}(X, \mathcal{D}^{\cdot}[-n])$$

Using the orientation $\mathcal{D}^{\cdot}[-n] \simeq o\hbar_X = j_! \underset{\sim}{k}$ we get

$$H^{n-p}(X, j_! \underset{\sim}{k}) \xrightarrow{\sim} [\Gamma_c(X, I^{\cdot})[p], K^{\cdot}]$$

Recall the definition of the relative group, IV.8.1:

$H^{\cdot}(X, \partial X; k) = H^{\cdot}(X, j_! \underset{\sim}{k})$ and the result follows.

$$Q.E.D.$$

VI.4 Algebraic duality

Let $X^{\cdot}$ and $E^{\cdot}$ be complexes over a fixed commutative ring A. For $n \in \mathbb{Z}$ consider the map

$$X^n \to \text{Hom}^n(\text{Hom}^{\cdot}(X^{\cdot}, E^{\cdot}), E^{\cdot}) = \prod_{i \in \mathbb{Z}} \text{Hom}^i(X^{\cdot}, E^{\cdot}), E^{i+n})$$

which to $x^n \in X^n$ assigns the product over $i \in \mathbb{Z}$ of the maps

$$\text{Hom}^i(X^{\cdot}, E^{\cdot}) \longrightarrow E^{i+n} ; \qquad f \longmapsto (-1)^{in} f(x_n)$$

It is left to the reader to show that this defines a chain map

4.1 $$\text{ev}: X^{\cdot} \to \text{Hom}^{\cdot}(\text{Hom}^{\cdot}(X^{\cdot}, E^{\cdot}), E^{\cdot})$$

which we call the _evaluation_ map.

Definition 4.2. Let A denote a noetherian ring. A _dualizing complex_ for A is a bounded complex $D^{\cdot}$ of injective modules with finitely generated cohomology modules, such that

$$\text{ev}: X^{\cdot} \to \text{Hom}^{\cdot}(\text{Hom}^{\cdot}(X^{\cdot}, D^{\cdot}), D^{\cdot})$$

is a quasi-isomorphism for any bounded complex $X^{\cdot}$ with finitely generated cohomology modules.

Proposition 4.3. A bounded complex $D^{\cdot}$ of injective modules with finitely generated cohomology modules is a dualizing complex for A if and only if

$$\text{ev}: A \longrightarrow \text{Hom}^{\cdot}(D^{\cdot}, D^{\cdot})$$

is a quasi-isomorphism.

<u>Proof.</u> Assume that the evaluation map is an isomorphism for $X^{\cdot} = A$. Consider a finitely generated module M. Recall that we consider M a complex, placing M in degree zero and placing zero's elsewhere. Let us prove that

$$\text{ev}: M \longrightarrow \text{Hom}^{\cdot}(\text{Hom}^{\cdot}(M,D^{\cdot}),D^{\cdot})$$

is a quasi-isomorphism. For $p \in \mathbb{Z}$ put

$$U^{p}(M) = H^{p} \text{Hom}^{\cdot}(\text{Hom}^{\cdot}(M,D^{\cdot}),D^{\cdot})$$

The evaluation map gives rise to morphisms

$$\text{ev}^{p}(M): H^{p}(M) \longrightarrow U^{p}(M) \qquad ; \quad p \in \mathbb{Z}$$

which we proceed to prove are isomorphisms. Notice that this is the case for $M = A^{m}$: by additivity of the functors involved it suffices to consider $M = A$ and use the assumption.

We will now prove the statement by decreasing induction on $p \in \mathbb{Z}$. Notice, that the result is true for large values of p since $D^{\cdot}$ is bounded. Assume the result is true for $p+1$. Choose a short exact sequence $\quad 0 \longrightarrow N \longrightarrow A^{m} \longrightarrow M \longrightarrow 0$ and consider the resulting long exact sequence

$$
\begin{array}{ccccccccc}
H^{p}(N) & \longrightarrow & H^{p}(A^{m}) & \longrightarrow & H^{p}(M) & \longrightarrow & H^{p+1}(N) & \longrightarrow & H^{p+1}(A^{m}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
U(N) & \longrightarrow & U^{p}(A^{m}) & \longrightarrow & U^{p}(M) & \longrightarrow & U^{p+1}(N) & \longrightarrow & U^{p+1}(A^{m})
\end{array}
$$

Using I.1.4 we deduce an exact sequence

$$\text{Cok ev}^{p}(A^{m}) \to \text{Cok ev}^{p}(M) \to \text{Cok ev}^{p+1}(N)$$

from which we deduce that $\text{ev}^{p}(M)$ is surjective for any finitely generated module. In particular we can insert the dotted arrow in the diagram and conclude by I.1.3.

We shall now treat the general case of a bounded complex $X^{\cdot}$ by induction on the "length" of $X^{\cdot}$. In case $X^{\cdot}$ is concentrated in a single degree the result follows by decalage from the first part of the proof. The general case follows by considering the exact sequence I.5.7

$$0 \to \tau_{\leq n}X^{\cdot} \to X^{\cdot} \to X^{\cdot}/\tau_{\leq n}X^{\cdot} \to 0$$

of complexes.

<div align="right">Q.E.D.</div>

<u>Corollary 4.4.</u> Let A be a noetherian ring with the property that A admits a bounded injective resolution $A \to D^{\cdot}$ (a <u>Gorenstein ring</u>). Then $D^{\cdot}$ is a dualizing complex.

<u>Proof.</u> The quasi-isomorphism $A \to D^{\cdot}$ gives rise to a quasi-isomorphism, compare I.6.2 $\quad \text{Hom}^{\cdot}(D^{\cdot},D^{\cdot}) \to \text{Hom}^{\cdot}(A,D^{\cdot}) = D^{\cdot}$

from which we deduce that $\quad \text{ev}: A \to \text{Hom}^{\cdot}(D^{\cdot},D^{\cdot}) \quad$ is a quasi-isomorphism.

<div align="right">Q.E.D.</div>

Example 4.5. The complex of abelian groups

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is a dualizing complex for the ring $\mathbb{Z}$.

Proposition 4.6. Let $C^{\cdot}$ denote a bounded complex of abelian groups. Put

$$DC^{\cdot} = \mathrm{Hom}^{\cdot}(C^{\cdot}, K^{\cdot})$$

where $K^{\cdot}$ is the complex 4.5. Then there is a natural exact sequence

$$0 \to \mathrm{Ext}(H^{i+1}C^{\cdot}; \mathbb{Z}) \to H^{-i}DC^{\cdot} \to \mathrm{Hom}(H^{i}C^{\cdot}; \mathbb{Z}) \to 0$$

Proof. A combination of the exact sequence

$$0 \to \tau_{\leq i}C^{\cdot} \to C^{\cdot} \to C^{\cdot}/\tau_{\leq i}C^{\cdot} \to 0$$

and the quasi-isomorphism I.5.9

$$C^{\cdot}/\tau_{\leq i}C^{\cdot} \longrightarrow \tau_{\geq i+1}C^{\cdot}$$

yields the exact sequence, see also I.6.14

$$H^{-i-1}D\tau_{\leq i}C^{\cdot} \to H^{-i}D\tau_{\geq i+1}C^{\cdot} \to H^{-i}DC^{\cdot} \to H^{-i}D\tau_{\leq i}C^{\cdot} \to H^{-i+1}D\tau_{\geq i+1}C^{\cdot}$$

The two extreme groups are zero for rather trivial reasons. We leave it to the reader to identify the resulting short exact sequence. Q.E.D.

VI.5 Universal coefficients

Let X denote a locally compact space and N an abelian group. We are going to establish a natural exact sequence for each $p \in \mathbb{Z}$

5.1 $$0 \to H_c^p(X, \mathbb{Z}) \otimes N \to H_c^p(X, N) \to \mathrm{Tor}_1(H_c^{p+1}(X, \mathbb{Z}), N) \to 0$$

The derivation is based on the following

Lemma 5.2. Let X be a locally compact space. There exists a positive complex $L^{\cdot}$ of torsion free abelian groups and a natural quasi-isomorphism

$$L^{\cdot} \otimes N \overset{\sim}{\to} R^{\cdot}\Gamma_c(X, N)$$

as N varies through the category of abelian groups.

Proof. Let S be a torsion free sheaf on X, i.e. a sheaf whose stalks are all torsion free abelian groups. An integer $m \neq 0$ will induce a monomorphism $0 \to S \overset{m}{\to} S$ which transforms into a monomorphism $\Gamma_c(X, S) \overset{m}{\to} \Gamma_c(X, S)$. This proves that $\Gamma_c(X, S)$ is a torsion free abelian group. Suppose in addition that S is a soft sheaf. Let us prove that for any abelian group N the canonical map

$$\Gamma_c(X, S) \otimes N \to \Gamma_c(X, S \otimes N)$$

is an isomorphism. Observe that N is direct limit of its finitely generated subgroups. Thus we can use III.5.1 and II.8.1

to reduce to the case where N is finitely generated. In that case we can choose a presentation $0 \to \mathbb{Z}^p \to \mathbb{Z}^q \to N \to 0$ and deduce an exact, commutative diagram

$$0 \longrightarrow I_c(X,S) \otimes \mathbb{Z}^p \longrightarrow I_c(X,S) \otimes \mathbb{Z}^q \longrightarrow I_c(X,S) \otimes N \longrightarrow 0$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$0 \longrightarrow I_c(X,S \otimes \mathbb{Z}^p) \longrightarrow I_c(X,S \otimes \mathbb{Z}^q) \longrightarrow I_c(X,S \otimes N) \longrightarrow 0$$

from which the result follows. - Let us now prove that $S \otimes N$ is a soft sheaf on X: Let $i: K \to X$ denote the inclusion of a compact subspace. We have

$$I(K,i^*(S \otimes N)) = I(K,i^*S \otimes N) = I(K,S) \otimes N$$

which implies that $I(X,S \otimes N) \to I(K,S \otimes N)$ is surjective.

Let us now consider the Godement resolution $\mathbb{Z} \to C^{\cdot}$, II.3.6. This is a resolution of $\mathbb{Z}$ by soft, torsion free sheaves. Moreover, at each stalk this is a homotopy equivalence, II.3.6. For an abelian group N we deduce the soft resolution $C^{\cdot} \otimes N$ of N. In conclusion

$$I_c(X,C^{\cdot}) \otimes N = I_c(X,C^{\cdot} \otimes N) = R^{\cdot}I_c(X,N)$$

i.e. we can use $L^{\cdot} = I_c(X,C^{\cdot})$.

Q.E.D.

Proof of 5.1. Consider the canonical exact sequence

$$0 \longrightarrow \tau_{\leq p}L^{\cdot} \longrightarrow L^{\cdot} \longrightarrow L^{\cdot}/\tau_{\leq p}L^{\cdot} \longrightarrow 0$$

and the quasi-isomorphism, I.5.9

$$L^{\cdot}/\tau_{\leq p}L^{\cdot} \longrightarrow \tau_{\geq p+1}L^{\cdot}$$

Consider at the same time a free resolution $F^{\cdot} \to N$ of length 1. From these data we deduce an exact sequence

$$H^{p-1}(\tau_{\geq p+1}L^{\cdot} \otimes F^{\cdot}) \to H^p(\tau_{\leq p}L^{\cdot} \otimes F^{\cdot}) \to H^p(L^{\cdot} \otimes F^{\cdot}) \to H^p(\tau_{\geq p+1}L^{\cdot} \otimes F^{\cdot}) \to H^{p+1}(\tau_{\leq p}L^{\cdot} \otimes F^{\cdot})$$

The two extreme groups are zero for rather simple reasons. We leave it to the reader to identify the resulting short exact sequence, using the following diagram

$$0 \longrightarrow H^{p+1} \otimes F_1 \longrightarrow \mathrm{Cok}\,\partial^p \otimes F_1 \longrightarrow L^{p+1} \otimes F_1$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$0 \longrightarrow H^{p+1} \otimes F_0 \longrightarrow \mathrm{Cok}\,\partial^p \otimes F_0 \longrightarrow L^{p+1} \otimes F_0$$

Q.E.D.

Let us add some information on the lowest cohomology groups with compact support.

Proposition 5.3. For a locally compact space X the group $H_c^0(X,\mathbb{Z})$ is a free abelian group and the group $H_c^1(X,\mathbb{Z})$ is torsion free.

<u>Proof.</u> Let F denote the group of all $\mathbb{Z}$-valued functions on the set X which are bounded. This is a free abelian group according to a theorem of Nöbeling (1). The group $H_c^0(X,\mathbb{Z})$ is a subgroup of F, which makes it a free group.

To prove that $H_c^1(X,\mathbb{Z})$ is torsion free we consider an integer $m \neq 0$ and the long exact sequence

$$\to H_c^0(X,\mathbb{Z}) \to H_c^0(X,\mathbb{Z}/m) \to H_c^1(X,\mathbb{Z}) \xrightarrow{m} H_c^1(X,\mathbb{Z}) \to$$

Choose a set theoretical section $s: \mathbb{Z}/m \to \mathbb{Z}$ to the projection $\mathbb{Z} \to \mathbb{Z}/m$ such that $s(0) = 0$. This induces a set theoretical section to the first arrow in the exact sequence above. This proves that multiplication with m on $H_c^1(X,\mathbb{Z})$ is injective.

Q.E.D.

VI.6 Alexander duality

Let X denote an n-dimensional oriented topological manifold. For a closed subset Z of X there is a quasi-isomorphism, Alexander duality

6.1 $$R^{\cdot}\Gamma_Z(X,\mathbb{Z}) \xrightarrow{\sim} DR^{\cdot}\Gamma_c(Z,\mathbb{Z})[-n]$$

where we have used the notation of 4.6.

<u>Proof.</u> As usual we let $\mathcal{D}^{\cdot}$ denote the dualizing complex and $K^{\cdot}$ denote the complex $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. For an object $I^{\cdot}$ of $D^+(X,\mathbb{Z})$ we have a quasi-isomorphism 2.1

6.2 $$[I^{\cdot}, \mathcal{D}^{\cdot}] \xrightarrow{\sim} D\Gamma_c(X,I^{\cdot})$$

Let $i: Z \to X$ denote the inclusion and $I^{\cdot}$ and injective resolution of $i_*\underset{\sim}{\mathbb{Z}}$. This gives a quasi-isomorphism, II.9.13

$$\Gamma_Z(X,\mathcal{D}^{\cdot}) \xrightarrow{\sim} DR^{\cdot}\Gamma_c(Z,\mathbb{Z})$$

The orientation of X provides us with a quasi-isomorphism $\underset{\sim}{\mathbb{Z}}[n] \xrightarrow{\sim} \mathcal{D}^{\cdot}$, and the result follows.

Q.E.D.

<u>Proposition 6.3.</u> Let X denote an n-dimensional topological manifold with orientation sheaf $o\hslash$ relative to $\mathbb{Z}$. For any noetherian ring k the orientation sheaf relative to k is $o\hslash \otimes_{\mathbb{Z}} k$.

Proof. The sheaf $o\hbar \otimes_{\mathbb{Z}} k$ is generated by the presheaf

$$U \longmapsto \text{Hom}_{\mathbb{Z}}(H_c^n(U,\mathbb{Z}),\mathbb{Z}) \otimes_{\mathbb{Z}} k$$

while the orientation sheaf relative to k is given by, 5.1

$$U \longmapsto \text{Hom}_{\mathbb{Z}}(H_c^n(U,\mathbb{Z}),k)$$

These explicit formulas define a morphism of sheaves from $o\hbar \otimes_{\mathbb{Z}} k$ to the orientation sheaf relative to k. This is an isomorphism as one checks by localization.

Q.E.D.

Theorem 6.4. Let X denote a connected n-dimensional manifold. The following conditions are equivalent

1) $H_c^n(X,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$

2) X is orientable relative to $\mathbb{Z}$.

3) X is orientable relative to $\mathbb{Z}/p$ for any prime p.

4) $H_c^n(X,\mathbb{Z}/p) \xrightarrow{\sim} \mathbb{Z}/p$ for any prime p.

Proof. 1) $\Rightarrow$ 4) Follows from universal coefficients 5.1.

4) $\Rightarrow$ 3) This has already been noted in V.3.6.

1) $\Rightarrow$ 2) Given $x \in X$ and D an open neighbourhood of x homeomorphic to a disc. Let us show that $r: H_c^n(D,\mathbb{Z}) \to H_c^n(X,\mathbb{Z})$ is an isomorphism. For any prime p consider the commuatative diagram

$$
\begin{array}{ccc}
H_c^n(D,\mathbb{Z}) \otimes \mathbb{Z}/p & \xrightarrow{\ r\otimes 1\ } & H_c^n(X,\mathbb{Z}) \otimes \mathbb{Z}/p \\
\downarrow & & \downarrow \\
H_c^n(D,\mathbb{Z}/p) & \xrightarrow{\hspace{2cm}} & H_c^n(X,\mathbb{Z}/p)
\end{array}
$$

The two vertical arrows are isomorphisms according to universal coefficient, 5.1. The horizontal map at the bottom is an isomorphism since X is oriented relative to $\mathbb{Z}/p$. Thus we have proved that r is an isomorphism mod p for all primes. Since $H_c^n(D,\mathbb{Z})$ and $H_c^n(X,\mathbb{Z})$ are finitely generated and free, it follows that r is an isomorphism.

2) $\Rightarrow$ 3) This is an immediate consequence of 6.3.

3) $\Rightarrow$ 1) Let U denote an open connected subset of X which is the union of finitely many discs, i.e. open subsets of X homeomorphic to $\mathbb{R}^n$. Let us remark that $H_c^n(U,\mathbb{Z})$ is finitely generated as it follows by a simple argument based on the Mayer-Vietoris sequence III.7.5

$$\to H_c^n(V,\mathbb{Z}) \oplus H_c^n(W,\mathbb{Z}) \to H_c^n(V \cup W,\mathbb{Z}) \to 0$$

From universal coefficients 5.1 and Poincaré duality III.3.2 and III.4.1 follows that for any finite field k

$$\dim_k H_c^n(U,\mathbb{Z}) \otimes k = \dim_k H_c^n(U,k) = \# \, c.c.(U) = 1$$

Using the structure theorem for finitely generated abelian groups we conclude that $H_c^n(U,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$, and thereby that U is oriented. - Let $\mathfrak{U}$ be the set of all non empty open subsets of the type just described. Let us notice that any two sets U and V from $\mathfrak{U}$ is contained in a third set W from $\mathfrak{U}$. This follows from the fact that for any two points x and y in X there exists a chain of discs $D_0,\ldots,D_s$, with $x \in D_0$ and $y \in D_s$ such that $D_{i-1} \cap D_i$ is non empty for $i = 1,\ldots,s$. We can now conclude from 6.5 below that

$$H_c{}^n(X,\mathbb{Z}) = \lim_{\substack{\to \\ u}} H_c{}^n(U,\mathbb{Z})$$

Let $j: U \to V$ denote the inclusion between two sets from u.
Let us prove that $j_!: H_c{}^n(U,\mathbb{Z}) \longrightarrow H_c{}^n(V,\mathbb{Z})$ is an isomorphism.
From the fact that V is oriented we conclude that

$$j_!^V: \mathrm{Hom}(H_c{}^n(V,\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(H_c{}^n(U,\mathbb{Z}),\mathbb{Z})$$

is an isomorphism. Since $H_c{}^n(U,\mathbb{Z})$ and $H_c{}^n(V,\mathbb{Z})$ are isomorphic
to $\mathbb{Z}$ we conclude that $j_!$ is an isomorphism. - The result now
follows by a simple argument involving the direct limit description
of $H_c{}^n(X,\mathbb{Z})$.

Q.E.D.

Lemma 6.5. Let X denote a locally compact space and
u a covering of X by open subsets such that any two sets U
and V from u is contained in a third set W from u.
Then for any sheaf F on X

$$H_c{}^p(X,F) = \lim_{\to} H_c{}^p(U,F)$$

where the limit is taken over all U in u.

Proof. For an open set U put $F_U = j_! j^* F$ where
$j: U \to X$ is the inclusion. It follows from II.2.8 that
$F = \lim_{\to} F_U$. Using III.5.1 we deduce that

$$H_c{}^p(X,F) = \lim_{\to} H_c{}^p(X,F_U)$$

and the result follows from III.7.3.

Q.E.D.

Remark 6.6. Let us notice an interesting consequence of 6.5
compare the proof of V.1.2: Let X denote a locally compact
space of dimension n. Consider a commutative ring k and a
k-sheaf F on X. The presheaf

$$U \longmapsto \mathrm{Hom}_k(H_c{}^n(U,F),k)$$

is a sheaf, proceeding as in the proof of V.1.2.

Proposition 6.7. Let X denote an oriented n-dimensional
connected manifold. For any point $x \in X$, the restriction map

$$H_{\{x\}}{}^n(X,\mathbb{Z}) \to H_c{}^n(X,\mathbb{Z})$$

is an isomorphism.

Proof. Let $j: U \to X$ denote the inclusion of an open
connected neighbourhood of x. The map

$$j_!^V: \mathrm{Hom}(H_c{}^n(X,\mathbb{Z}),\mathbb{Z}) \to \mathrm{Hom}(H_c{}^n(U,\mathbb{Z}),\mathbb{Z})$$

is an isomorphism according to the theory of orientation 3.1

It follows from 6.4 that

$$j_! : H_c^n(U, \mathbb{Z}) \to H_c^n(X, \mathbb{Z})$$

is an isomorphism. Thus by excision the question is local, and we may replace X by S^n. This case is left to the reader.

<div align="right">Q.E.D.</div>

Theorem 6.8. Let X denote an oriented compact manifold of dimension n. For any proper open subset U of X we have

$$H^n(U, \mathbb{Z}) = 0$$

Proof. Put $Z = X-U$ and consider the exact sequence

$$\to H^n(X, \mathbb{Z}) \to H^n(U, \mathbb{Z}) \to H_Z^{n+1}(X, \mathbb{Z}) \to$$

The first map is zero: Pick $x \in X-U$ and notice that the map can be factored through $H^n(X-\{x\}, \mathbb{Z})$ which is zero as it follows from 6.7. By Alexander duality 6.1

$$H_Z^{n+1}(X, \mathbb{Z}) = H^1 DR\ \Gamma_c(Z, \mathbb{Z})$$

From 4.6 we deduce an isomorphism

$$H_Z^{n+1}(X, \mathbb{Z}) = Ext(H^0(Z, \mathbb{Z}), \mathbb{Z})$$

From this the result follows since $H^0(Z, \mathbb{Z})$ is a free abelian group, 5.3.

<div align="right">Q.E.D.</div>

VII. Direct Image with Proper Support

VII.1 The functor $f_!$

The inclusion $h: W \to X$ of a locally subspace of a locally compact space gives rise to a very useful functor $h_!: Sh(W) \to Sh(X)$, compare II.6 and III.7. In order to generalize this notion let us observe that h is a proper map if and only if W is a closed subspace of X, III.6.

Definition 1.1. Let $f: X \to Y$ be a continuous map between locally compact spaces. For a sheaf F on X and an open subset V of Y put

$$\Gamma(V, f_! F) = \left\{ s \in \Gamma(f^{-1}(V), F) \ \middle| \ \begin{array}{l} Supp(s) \xrightarrow{f} Y \\ \text{is a proper map} \end{array} \right\}$$

We consider $f_! F$ as a subpresheaf of $f_* F$.

Proposition 1.2. The presheaf $f_! F$ is a sheaf on Y.

Proof. It suffices to prove that the map $f: X \to Y$ has the following property:

Given a family $(V_i)_{i \in I}$ of open subsets of Y with union V and a closed subset S of $f^{-1}(V)$. If the restriction $f: S \cap V_i \to V_i$ is proper for all i, then the restriction $f: S \to V$ is proper.

Let K be a compact subset of V. We are going to construct a family $(K_i)_{i \in I}$ of compact subsets of K, with $K_i \subseteq V_i$ for each $i \in I$, whose union is K and such that K_i is empty except for finitely many $i \in I$.

For the construction of the K_i's we may assume that I is a finite set by Borel-Heine. For $x \in K$ choose a compact neighbourhood K_x of x contained in some V_i, $i \in I$. Using Borel-Heine we obtain a finite covering of K by compact subsets of K each of which is contained in some V_i, $i \in I$. Let K_i be the union at those of the components of the covering which are contained in U_i. This gives the desired family $(K_i)_{i \in I}$. - To conclude the proof notice that

$$S \cap f^{-1}(K) = S \cap f^{-1}(\cup K_i) = \cup (S \cap f^{-1}(K_i))$$

which shows that $S \cap f^{-1}(K)$ is compact.

Q.E.D.

Let us consider the left exact functor

1.3
$$f_!: Sh(X) \longrightarrow Sh(Y)$$

The i'th derived functor evaluated on the sheaf F will be denoted $R^i f_! F$.

Theorem 1.4. Let $f: X \to Y$ denote a continuous map between locally compact spaces. For $y \in Y$ we have a natural isomorphism

$$(R^i f_! F)_y = H_c^i(f^{-1}(y), F) \qquad ; i \in \mathbb{Z}$$

as F varies through the category of sheaves on X.

Proof. Let us first treat the case $i = 0$. Consider an open neighbourhood V of y in Y and the restriction

$$\Gamma(V, f_! F) \longrightarrow \Gamma(f^{-1}(y), F)$$

Consider a $s \in \Gamma(V, f_! F)$ which maps to zero. This means that $Supp(s) \cap f^{-1}(y)$ is empty or otherwise expressed $y \notin f(Supp(s))$. Let W denote the complement of $Supp(s)$ in V, this is an open neighbourhood of y to which the restriction of s is zero. Thus we have proved that the restriction

$$(f_! F)_y \to \Gamma_c(f^{-1}(y), F)$$

is injective. In case F is soft the restriction map is surjective as it follows by remarking that $\Gamma_c(X, F)$ is a subgroup of $\Gamma(Y, f_! F)$ and that restriction from $\Gamma_c(X, F)$ to $\Gamma_c(f^{-1}(y), F)$ is surjective III.2.6. - In the general case consider an exact sequence $0 \to F \to S \to T$ with S and T soft and use the resulting diagram

$$0 \longrightarrow (f_!F)_y \longrightarrow (f_!S)_y \longrightarrow (f_!F)_y$$

$$0 \longrightarrow \Gamma_c(f^{-1}(y),F) \to \Gamma_c(f^{-1}(y),F) \to \Gamma_c(f^{-1}(y),F)$$

to conclude that restriction is an isomorphism. - Let us now choose an injective resolution $F \to I^\cdot$. We have

$$(R^i f_! F)_y = (H^i f_! I^\cdot)_y = H^i(f_! I^\cdot)_y = H^i \Gamma_c(f^{-1}(y),I^\cdot)$$

The result follows by noticing that the restriction of $I^\cdot$ to $f^{-1}(y)$ is a soft resolution of the restriction of F to $f^{-1}(y)$, III.2.5.

$$Q.E.D.$$

A diagram as below is called <u>cartesian</u> if $(q,h): A \to X \times B$ induces an isomorphism between A and the subspace $\{(x,b) \in X \times B \mid f(x) = p(b)\}$ of $X \times B$.

<u>Corollary 1.5.</u> Consider a cartesian square of locally compact spaces. For any sheaf F on X we have

$$p^* R^i f_! F \overset{\sim}{\to} R^i h_! q^* F \quad ; \ i \in \mathbb{Z}$$

$$\begin{array}{ccc} A & \xrightarrow{q} & X \\ \downarrow h & & \downarrow f \\ B & \xrightarrow{p} & Y \end{array}$$

<u>Proof.</u> The canonical morphism

$$p^* f_! F \longrightarrow h_! q^* F$$

<u>Corollary 1.6.</u> Let $f: X \to Y$ denote a continuous map between locally compact spaces. A soft sheaf S on X is transformed into a soft sheaf $f_! S$ on Y. Moreover $R^i f_! S = 0$ for $i > 0$.

<u>Proof.</u> Let us prove that for any compact subset K of Y and any sheaf S on X

$$1.7 \qquad \Gamma(K, f_! S) = \Gamma_c(f^{-1}(K),S)$$

To this end let $p: K \to Y$ and $q: f^{-1}(K) \to X$ denote the inclusions and $h: f^{-1}(K) \to K$ the restriction of f. According to 1.5 we have $p^* f_! S = h_! q^* S$. Apply $\Gamma(K,-)$ to this identity to get 1.7.

In case S is soft we shall prove that $f_! S$ is soft, i.e. that the restriction map

$$\Gamma(Y, f_! S) \longrightarrow \Gamma(K, f_! S)$$

is surjective. Notice that $\Gamma(Y, f_! S)$ contains $\Gamma_c(X,S)$ as a subgroup. Restriction from this group to $\Gamma_c(f^{-1}(K),S)$ is surjective according to III.2.6 and the result follows from 1.7.

The second part follows from 1.4 by localization using the fact that S induces soft sheaves on any of the fibres of f, III.2.5.

$$Q.E.D.$$

is an isomorphism as one sees by localization using 1.5 and
identification of the fibre of h over b ∈ B with the fiber
of f over p(b). Let us remark that in case S is soft,
then q*S is acyclic for $h_!$: Notice that q*S induces soft
sheaves on the fibres of h and apply 1.4. For a soft resolution
S˙ of F we have

$$p^* H^i f_! S^. = H^i p^* f_! S^. = H^i h_! q^* S^.$$

from which the result follows using I.7.5.

Q.E.D.

Let us now consider a commutative ring k and interprete
$f_!$ as a functor

$$f_!: Sh(X,k) \longrightarrow Sh(Y,k)$$

and similarly for the derived functor

1.8 $$Rf_!: D^+(X,k) \longrightarrow D^+(Y,k)$$

Given two continuous maps between locally compact spaces
f: X → Y and g: Y → Z then it is easily seen that

1.9 $$(f \circ g)_! = g_! \circ f_!$$

and as a consequence of 1.6 and I.7.15

1.10 $$R(g \circ f)_! = Rg_! \circ Rf_!$$

VII.2 The Künneth formula

Let k denote a noetherian ring and f: X → Y a con-
tinuous map of locally compact spaces both of finite dimension.
We are first going to construct with the notation of XI.2 a
functor

2.1 $$Rf_!: D^-(X,k) \longrightarrow D^-(Y,k)$$

on the basis of the following

Lemma 2.2. Let X denote a locally compact space of
finite dimension and k a commutative ring. Given a soft sheaf
S and a sheaf F on X. If either S or F is flat then
$S \otimes_k F$ is a soft sheaf.

Proof. In case S is flat we can use the proof of V.1.3.
In case F is flat, that proof needs a small modification:
notice that $K = Ker \partial_{n-1}$ is a flat sheaf II.11.1.

Q.E.D.

Lemma 2.3. Let X denote a locally compact space of
finite dimension and k a noetherian ring. For any F˙ in
$K^-(X,k)$ there exists a quasi-isomorphism F˙ → T˙ in $K^-(X,k)$
where T˙ is a complex of soft sheaves (a soft resulution of F')

Proof. Choose according to VI.1.3 a quasi-isomorphism
k → S˙ where S˙ is a bounded complex of soft and flat sheaves.

According to II.11.2 this gives a quasi-isomorphism $F^{\cdot} \to S^{\cdot} \otimes F^{\cdot}$. The complex $S^{\cdot} \otimes F^{\cdot}$ is a bounded above complex of soft sheaves by Lemma 2.2.

Q.E.D.

Construction of $Rf_!: D^-(X,k) \to D^-(Y,k)$

Let $S^-(X,k)$ denote the homotopy category of bounded above complexes of soft k-sheaves. It follows from 2.2 and XI.2 that we may identify $D^-(X,k)$ with the category obtained by inverting all quasi-isomorphisms in $S^-(X,k)$. - The functor $f_!$ extends to a functor

$$f_!: S^-(X,k) \longrightarrow S^-(Y,k)$$

according to 1.6. It remains to establish that $f_!$ transforms quasi-isomorphisms into quasi-isomorphisms. To see this remark that $R^n f_! = 0$ for $n > \dim X$ and use I.7.6.

Let us remark that the functor $f^*: Sh(Y,k) \to Sh(X,k)$ being exact extends immediately to a functor

$$f^*: D^-(Y,k) \longrightarrow D^-(X,k)$$

With the notations above and the notation of II.11.10 we have the fundamental

Projection formula 2.4.

$$Rf_!(F^{\cdot} \overset{L}{\otimes} f^*G^{\cdot}) = (Rf_!F^{\cdot}) \overset{L}{\otimes} G^{\cdot}$$

for $F^{\cdot}$ in $D^-(X,k)$ and G in $D^-(Y^{\cdot},k)$.

Proof. Let us first prove that for a sheaf F on X and a flat sheaf G on Y there is a canonical isomorphism

2.5 $$f_!(F) \otimes G \overset{\sim}{\to} f_!(F \otimes f^*G)$$

Let us first notice that there is a natural transformation from the left hand side of the formula to the right hand side. Thus it follows from 1.4 that we may assume that Y is a point. Next, let us remark that both sides of the formula are left exact functors in F. Thus we may assume that F is soft.

It is now time to vary the k-module G. The case where G is a free module follows from the fact that Γ_c preserves direct sums, III.5.1. In general consider an exact sequence

$$0 \to L_2 \to L_1 \to L_0 \to G \to 0$$

where L_1 and L_0 are free modules. Since G is flat we can conclude that L_2 is flat. There results an exact sequence of sheaves on X

$$0 \to S \otimes L_2 \to S \otimes L_1 \to S \otimes L_0 \to S \otimes G \to 0$$

This is a sequence of soft sheaves on X by 2.2. It follows from I.7.5 that the sequence

$$0 \to \Gamma_c(X, S \otimes L_2) \to \Gamma_c(X, S \otimes L_1) \to \Gamma_c(X, S \otimes L_0) \to \Gamma_c(X, S \otimes G) \to 0$$

is exact, in particular we deduce a commutative exact diagram

$$I_c(X,F) \otimes L_1 \longrightarrow I_c(X,F) \otimes L_0 \longrightarrow I_c(X,F) \otimes G \longrightarrow 0$$

$$I_c(X,F \otimes L_1) \longrightarrow I_c(X,F \otimes L_0) \longrightarrow I_c(X,F \otimes G) \longrightarrow 0$$

from which we conclude that the vertical arrow to the right is an isomorphism. This proves 2.5.

Let us notice that in case F is soft, then $F \otimes f^*G$ is soft as it follows from 2.2 and the fact that f^*G is flat. With this remark in hand it is a simple matter to extend the formula 2.5 to the derived categories.

Q.E.D.

Base change 2.6. Consider a cartesian square of locally compact spaces of finite dimension. Then

$$\begin{array}{ccc} A & \xrightarrow{q} & X \\ \downarrow h & & \downarrow f \\ B & \xrightarrow{p} & Y \end{array}$$

$$p^*Rf_! F^{\cdot} \xrightarrow{\sim} Rh_! q^*F^{\cdot}$$

for all $F^{\cdot}$ in $D^-(X,k)$.

Proof. Let us remark that a soft sheaf S on X transforms into a sheaf q^*S on A which is acyclic for $h_!$ as it follows from 1.5 and 1.6. For a complex $S^{\cdot}$ in $K^-(X,k)$ choose a soft resolution $q^*S^{\cdot} \to T^{\cdot}$ in $K^-(A,k)$. According to 1.6 we have

$$p^*f_! S^{\cdot} \xrightarrow{\sim} h_! q^*S^{\cdot} \to h_! T^{\cdot}$$

The second arrow is a quasi-isomorphism by I.7.7.

Q.E.D.

With the notation of 2.6 put $c = fq = ph$.

Künneth formula 2.7.

$$Rc_!(h^*E^{\cdot} \overset{L}{\otimes} q^*F^{\cdot}) = (Rp_!E^{\cdot}) \overset{L}{\otimes} (Rf_!F^{\cdot})$$

for $E^{\cdot}$ in $D^-(B,k)$ and $F^{\cdot}$ in $D^-(X,k)$.

Proof. Use first the projection formula 2.4 and next the base change formula 2.6 to get

$$Rq_!(h^*E^{\cdot} \overset{L}{\otimes} q^*F^{\cdot}) = Rq_!(h^*E^{\cdot}) \overset{L}{\otimes} F = f^*(Rp_!E^{\cdot}) \overset{L}{\otimes} F^{\cdot}$$

Apply $Rf_!$ to this and use $Rc_! = Rf_! \circ Rq_!$ to get

$$Rc_!(h^*E^{\cdot} \overset{L}{\otimes} q^*F) = Rf_!(f^*(Rp_!E^{\cdot}) \overset{L}{\otimes} F^{\cdot})$$

Apply the projection formula once more to get the result.

Q.E.D.

In particular for two locally compact spaces X and Y of finite dimension and a noetherian ring k

2.8
$$RI_c^{\cdot}(X \times Y, k) = RI_c^{\cdot}(X,k) \overset{L}{\otimes}_k RI_c^{\cdot}(Y,k)$$

VIII.3 Global form of Verdier duality

Let k be a noetherian ring and $f: X \to Y$ a continuous
map between locally compact spaces of finite dimension.

Theorem 3.1. There exists an additive functor

$$f^!: D^+(Y,k) \to D^+(X,k)$$

and a natural isomorphism

$$[Rf_!F^\cdot, G^\cdot] \xrightarrow{\sim} [F^\cdot, f^!G^\cdot]$$

as $F^\cdot$ varies through $D^+(X,k)$ and $G^\cdot$ varies through $D^+(Y,k)$.

Proof. We shall follow the proof of VI.1.1 closely and give
the needed modifications. For a soft and flat sheaf S on X and
a sheaf G on Y the functor

$$F \longrightarrow \text{Hom}(f_!(S \otimes F), G)$$

from $Sh(X,k)$ to the category of k-modules transforms kernels
into cokernels and direct sums into direct products: To see
this notice that the functor $F \mapsto f_!(S \otimes F)$ is exact as it follows
from 2.2 and 1.6. The same functor preserves direct sums or
more generally direct limits as it follows from 1.4 and III.5.

1) With S and G as above, the presehaf on X

$$U \longmapsto \text{Hom}(f_!(S_U), G)$$

is a sheaf on X,   here $S_U = j_!j^*S = S \otimes j_!k$ where $j: U \to X$
denotes the inclusion: For open subsets U and V of X
consider the transform of the exact sequence

$$0 \longrightarrow k_{U \cap V} \longrightarrow k_U \oplus k_V \longrightarrow k_{U \cup V} \longrightarrow 0$$

by the functor $F \longmapsto \text{Hom}(f_!(S \otimes F), G)$.

2) Let $f^!(S,G)$ denote the above sheaf on X. Here S
is a k-flat and soft sheaf on X and G is any sheaf on Y.

3) There is a canonical isomoprphism

3.2 $$\text{Hom}(f_!(F \otimes S), G) \xrightarrow{\sim} \text{Hom}(F, f^!(S,G))$$

as F varies through the category $Sh(X,k)$. - Let us first
establish the identification

3.3 $$f_*f^!(S,G) = Hom(f_!S,G)$$

To this end consider the inclusion $j: V \to Y$ of an open subset
of Y and notice that

$$\text{Hom}(j^*f_!S, j^*G) = \text{Hom}(j_!j^*f_!S, G) = \text{Hom}(f_!(S_{f^{-1}(V)}), G)$$

as it follows by the base change property 1.5. -
By adjunction we deduce from 3.3 a morphism of sheaves

$$f^*Hom(f_!S,G) \to f^!(S,G)$$

Let us now depart from the morphism $f_*F \otimes f_!S \to f_!(F \otimes S)$

and deduce first a morphism, II.12

$$\text{Hom}(f_!(F \otimes S), G) \to \text{Hom}(f_*F, Hom(f_!S, G))$$

and by adjunction a morphism

$$\text{Hom}(f_!(F \otimes S), G) \to \text{Hom}(F, f_*Hom(f_!S, G))$$

Combine this with the morphism above to get

$$\text{Hom}(f_!(F \otimes S), G) \to \text{Hom}(F, f^!(S, G))$$

To prove that this is an isomorphism we shall vary F. It suffices to check the case $F = j_!\tilde{k}$ where j is the inclusion of an open subset of X.

4-7) Needs no essential modifications.

$$\text{Q.E.D.}$$

Example 3.4. Let $h: W \to X$ denote the inclusion of a locally closed subspace. With the notation of II.6 the functor

$$h^!: \text{Sh}(X, k) \to \text{Sh}(W, k)$$

is left exact and transforms injectives into injectives. Thus it extends to a functor $\quad h^!: D^+(X, k) \to D^+(W, k)$ which is a right adjoint to the functor $h_!$ described in II.6. This provides an extension of Theorem 3.1 for this kind of immersions beyond the framwork of locally compact spaces. The case of a closed subspace will be explored in Chapter VIII.

VII.4   Covering spaces

Let $f: X \to Y$ be a finite covering space of degree n, i.e. for each point $y \in Y$ there exists an open neighbourhood V of y such that $f: f^{-1}(V) \to V$ is isomorphic to the projection $V \times [1, n] \to V$. For a sheaf F on X and $y \in Y$ we have a canonical isomorphism

4.1
$$(f_*F)_y \xrightarrow{\sim} \underset{x \in f^{-1}(y)}{\oplus} F_x$$

from which we conclude that the functor

4.2
$$f_*: \text{Sh}(X, k) \to \text{Sh}(Y, k) \quad \text{is exact}$$

For a sheaf G on Y we deduce from 4.1 an isomorphism on the stalks at $y \in Y$

$$(f_*f^*G)_y \xrightarrow{\sim} \underset{x \in f^{-1}(y)}{\oplus} G_y$$

Compose this with "summation" to get the trace map

4.3
$$tr_y: (f^*f_*G)_y \longrightarrow G_y \qquad ; \; y \in Y$$

There exists a unique morphism of sheaves

4.4
$$tr: f_*f^*G \longrightarrow G$$

whose stalks are those recorded in 4.3: Uniqueness follows from II.2.2.i. By II.12 the problem is local on Y. Thus it suffices to treat the case where the covering is trivial which

is left to the reader. - For a sheaf F on X the trace map
induces an isomorphism

$$\boxed{\mathrm{Hom}(F,f*G) \ \tilde{\to} \ \mathrm{Hom}(f_*F,G)}$$

4.5

Proof. The construction above yields more generally a
morphism of sheaves on Y

4.6
$$f_* Hom(F,f*G) \ \to \ Hom(f_*F,G)$$

We shall in fact prove that the morphism 4.6 is an isomorphism:
The new problem is local on Y. Thus it suffices to treat the
case of a trivial covering which is left to the reader.

Q.E.D.

Corollary 4.7. The pull back functor

$$f*: \ \mathrm{Sh}(Y,k) \ \to \ \mathrm{Sh}(X,k)$$

transforms injectives into injectives.

Proof. Follows formally from the presence of a left
adjoint which is exact by 4.2.

Q.E.D.

From our discussion follows that in the case where X
and Y are locally compact we have $\underline{f^! = f*}$ $\underline{\text{in}}$ $\underline{\text{this}}$
$\underline{\text{particular}}$ $\underline{\text{case of Verdier duality 3.1.}}$

For a sheaf G on Y let res: $G \to f_* f*G$ denote the stan-
dard adjunction morphism, II.4.9. By localization on Y follows
that

4.8
$$\boxed{\mathrm{tr} \ o \ \mathrm{res} = \mathrm{n}}$$

Let us consider an injective resolution $\underline{k} \to G^{\cdot}$. Apply the
functor $H^{\cdot}I(Y,-)$ to the morphisms

4.9
$$G^{\cdot} \xrightarrow{\mathrm{res}} f_* f*G^{\cdot} \xrightarrow{\mathrm{tr}} G^{\cdot}$$

to obtain morphisms on cohomology f* = res

4.10
$$H^{\cdot}(Y,k) \xrightarrow{\mathrm{res}} H^{\cdot}(X,k) \xrightarrow{\mathrm{tr}} H^{\cdot}(Y,k)$$

which satisfies the relation 4.8. Notice that

4.11
$$\mathrm{tr}(f*\eta \cup \xi) = \eta \cup \mathrm{tr}(\xi) \qquad ; \ \eta \in H^{\cdot}(Y,k), \ \xi \in H^{\cdot}(X,k)$$

as it follows from the construction above.

VII.5 Local form of Verdier duality

In this section we shall give a local version of Verdier
duality. To do so we shall use the general theory of derived
categories as exposed in Chapter XI.

Given a topological space and a commutative ring k. For
$F^{\cdot}$ in $D^{-}(X,k)$ and $F^{\cdot}$ in $D^{+}(X,k)$ we shall define $R\,Hom^{\cdot}(E,F)$
in $D^{+}(X,k)$: choose an injective resolution $F^{\cdot} \to J^{\cdot}$ and put

$$5.1 \qquad\qquad R\,Hom^{\cdot}(E^{\cdot},F^{\cdot}) = Hom^{\cdot}(E^{\cdot},J^{\cdot}).$$

This takes a particularly simple form if we represent the de-
rived categories as follows $D^{-}(X,k)$: The category obtained
from the homotopy category of bounded above complexes of flat
sheaves by inverting all quasi-isomorphisms. $D^{+}(X,k)$: The
homotopy category of bounded below complexes of injective sheaves.
In particular, with $E^{\cdot}$ and $F^{\cdot}$ such represented, the complex
$Hom^{\cdot}(E^{\cdot},F^{\cdot})$ is automatically a bounded below complex of injective
sheaves, II. 12.3.

Theorem 5.2. Let k be a noetherian ring and $f: X \to Y$ a
continuous map of locally compact spaces of finite dimension. There
is a natural isomorphism in $D^{+}(X,k)$

$$R\,Hom^{\cdot}(Rf_{!}E^{\cdot},F^{\cdot}) \overset{\sim}{\to} Rf_{*}\,R\,Hom^{\cdot}(E^{\cdot},f^{!}F^{\cdot})$$

as $E^{\cdot}$ varies through $D^{-}(X,k)$ and $F^{\cdot}$ through $D^{+}(X,k)$.

Proof. Let S be a soft and flat sheaf on X. The iso-
morphism 3.2 extends easily to an isomorphism

$$5.3 \qquad\qquad Hom(f_{!}(E \otimes S),F) = f_{*}Hom(E,f^{!}(S,F))$$

as E varies through $Sh(X,k)$ and F through $Sh(Y,k)$. Let
us choose a fixed bounded flat and soft resolution $\underline{k} \to S^{\cdot}$ on
X, VI.1.3. For a bounded above complex $E^{\cdot}$ of <u>flat</u> sheaves
on X and a bounded below complex $F^{\cdot}$ of <u>injective</u> sheaves
on Y we deduce from 5.3 an isomorphism.

$$5.4 \qquad\qquad Hom^{\cdot}(f_{!}(E^{\cdot} \otimes S^{\cdot}),F^{\cdot}) \overset{\sim}{\to} f_{*}Hom^{\cdot}(E^{\cdot},f^{!}(S^{\cdot},F^{\cdot}))$$

We can conclude the proof by the following four references: The
morphism $E^{\cdot} \to E^{\cdot} \otimes S^{\cdot}$ is a soft resolution of $E^{\cdot}$ by 2.2. The
complex $f^{!}(S^{\cdot},F^{\cdot})$ is a bounded below complex of injective
sheaves on X as it follows from the proof of 3.1. The complex
$Hom^{\cdot}(E^{\cdot},f^{!}(S^{\cdot},F^{\cdot}))$ is a bounded below complex of injective
sheaves as it follows from II.12.3. Any bounded above complex
admits a flat resolution by II.11.8.

$$Q.E.D.$$

Example 5.5. Consider a finite covering $f: X \to Y$. Verdier
duality is simply represented by the isomorphism 4.6.

VIII. Characteristic Classes

VIII.1 Local duality

In this section we consider a fixed commutative ring k. Let us recall that the inclusion $i: Z \to X$ of a closed subspace gives rise to two functors

$$\text{Sh}(Z,k) \underset{i_*}{\overset{i^!}{\underset{\longrightarrow}{\longleftarrow}}} \text{Sh}(X,k)$$

which are mutually adjoints

1.1 $$\text{Hom}(i_*E,F) = \text{Hom}(F,i^!F)$$

The functor $i^!$ is left exact and carries injectives into injectives. We shall study the derived functor $R^p i^!$, $p \in \mathbb{Z}$.

For a sheaf F on X we shall give an explicit description of the sheaf $i^! F$ on Z or rather the sheaf $i_* i^! F$ on X. It follows from the formula II.6.7 and II.9.1 that the sections of $i_* i^! F$ over the open subset V of X are

1.2 $$\Gamma(V,i_*i^!F) = \Gamma_{Z\cap V}(V,F)$$

For an integer $p \in \mathbb{Z}$ we shall describe a presheaf on X: An open subset U of the open subset V of X gives rise to a restriction map

$$H_{V\cap Z}{}^p(V,F) \longrightarrow H_{U\cap Z}{}^p(U,F)$$

and thereby a presheaf $U \mapsto H_{U\cap Z}{}^p(U,F)$.

Proposition 1.3. The sheaf associated to the presehaf

$$U \longmapsto H_{U\cap Z}{}^d(U,F) \qquad ; \ d \in \mathbb{Z}$$

is $i_* R^d i^! F$. If $R^p i^! F = 0$ for all $p < d$, then

$$\Gamma(U,i_*R^di^!F) = H_{Z\cap U}{}^d(U,F)$$

for any open subset U of X.

Proof. Let $I^{\cdot}$ denote an injective resolution of the sheaf F on W. We have

$$i_* R^p i^! F = i_* H^p i^! I^{\cdot} = H^p i_* i^! I^{\cdot}$$

Thus it follows from II.2.7 that $i_* R^p i^! F$ is the sheaf associated to the presheaf of

$$U \longmapsto H^p \Gamma(U,i_*i^!I^{\cdot})$$

From the description 1.2 we deduce that

$$H^p_\Gamma(U, i_* i^! I^\cdot) = H^p \Gamma_{U \cap Z}(U, I^\cdot) = H_{U \cap Z}{}^p(U, F)$$

which proves the first part of the statement.

If $R^p i^! F = 0$ for $p < d$ then the complex $i^! I^\cdot$ is homotopic to $\tau_{\geq d} i^! I^\cdot$ and we deduce from the formula above that

$$H^p_{U \cap Z}(U, F) = H^p \Gamma(U, i_* \tau_{\geq d} i^! I^\cdot) = \Gamma(U, i_* H^p i^! I^\cdot)$$

from which the result follows since $R^p i^! F = H^p i^! I^\cdot$.

$$Q.E.D.$$

Most of our calculations will be based on the following proposition

Proposition 1.4. Let $p \colon X \to Z$ be a retraction to the inclusion $i \colon Z \to X$ of a closed subspace. For a sheaf F on X, the sheaf associated to the presheaf

$$V \longmapsto H_V{}^n(p^{-1}(V), F) \qquad ; \ n \in \mathbb{Z}$$

is $R^n i^! F$. If $R^n i^! F = 0$ for all $n < d$, then

$$\Gamma(V, R^d i^! F) = H_{i(V)}{}^d(p^{-1}(V), F)$$

for any open subset V of Z.

proof. Let $I^\cdot$ denote an injective resolution of F. $R^n i^! F$ is the sheaf associated to the presheaf

$$V \longmapsto H^n \Gamma(V, i^! I^\cdot) \qquad ; \ V \subseteq Z$$

Using formula 1.2 we deduce that

$$H^n \Gamma(V, i^! I^\cdot) = H^n \Gamma(p^{-1}(V), i_* i^! I^\cdot) = H^n \Gamma_{i(V)}(p^{-1}(V), I^\cdot)$$

from which the first result follows.

If $R^n i^! F = 0$ for $n < d$ we deduce that $i^! I^\cdot$ is homotopic to $\tau_{\geq d} i^! I^\cdot$ and get that

$$H_{i(V)}{}^d(p^{-1}(V), F) = H^d \Gamma(V, \tau_{\geq d} i^! I^\cdot) = \Gamma(V, H^d i^! I^\cdot)$$

and the second result follows since $H^d i^! I^\cdot = R^d i^! F$.

$$Q.E.D.$$

Local duality theorem 1.5. Let $i \colon Z \to X$ denote the inclusion of a closed subspace. We have

$$[i_* I^\cdot, J^\cdot] = [I^\cdot, i^! J^\cdot]$$

for $I^\cdot$ in $D^+(Z, k)$ and $J^\cdot$ in $D^+(X, k)$.

Proof. The identity 1.1 gives an isomorphism

$$\mathrm{Hom}^\cdot(i_* I^\cdot, J) = \mathrm{Hom}^\cdot(I^\cdot, i^! J^\cdot)$$

from which the formula results by applying the functor H^0.

$$Q.E.D.$$

1.6
$$H_Z^d(X,k) = [\underline{k}[-d], R\,i^!\underline{k}]$$

Proof. Apply 1.5 with $I^{\cdot}$ equal to the constant sheaf $\underline{k}[p]$ on Z and $J^{\cdot}$ an injective resolution of $\underline{k}$ on X.

Q.E.D.

Proposition 1.7. Let X denote a locally compact space of finite dimension, Z a closed subspace of X and k a noetherian ring. If $\mathcal{D}_X^{\cdot}$ denotes the dualizing complex for X, then the dualizing complex $\mathcal{D}_Z^{\cdot}$ for Z is given by

$$\mathcal{D}_Z^{\cdot} = R\,i^!\mathcal{D}_X^{\cdot}$$

Proof. Let $k \to K^{\cdot}$ denote an injective resolution in the category of k-modules. For $I^{\cdot}$ in $D^+(Z,k)$ we have

$$[I^{\cdot}, i^!\mathcal{D}_X^{\cdot}] = [i_*I^{\cdot}, \mathcal{D}_X^{\cdot}] = [\Gamma_c(X, i_*I^{\cdot}), K^{\cdot}] = [\Gamma_c(Z, I^{\cdot}), K^{\cdot}]$$

where we have used Verdier duality VI.2.1.

Q.E.D.

Corollary 1.8. Let X denote an n-dimensional topological manifold and $i: Z \to X$ the inclusion of a closed submanifold of dimension m. If both manifolds are oriented relative to the noetherian ring k, then

$$R\,i^!\underline{k} \xrightarrow{\sim} \underline{k}[m-n]$$

Proof. We have $\mathcal{D}_X^{\cdot} \xrightarrow{\sim} \underline{k}[n]$ and $\mathcal{D}_Z^{\cdot} \xrightarrow{\sim} \underline{k}[m]$ by VI.3.2.

Q.E.D.

VIII.2   Thom Class

In this section we consider the inclusion $i: Z \to X$ of a closed subspace and a commutative ring k. We shall study a fixed quasi-isomorphism

2.1
$$\tau: \underline{k}[-d] \xrightarrow{\sim} R\,i^!\underline{k}$$

where $d \in \mathbb{N}$ is a fixed integer. The basic examples are the case where Z and X are oriented manifolds 1.8 and the case where i is the zero section in an oriented vector bundle or more generally an oriented microbundle VIII.3. Our basic objective is to establish some important cup product formulas arising in this situation.

By local duality, 1.6 we may interprete τ as a local cohomology class, the so called Thom class

2.2
$$\tau \in H_Z^d(X,k)$$

The extraordinary cup product II.9.14 gives rise to the

Thom isomorphism 2.3. The cup product

$$\alpha \to \tau \cup \alpha \qquad ; \quad H^p(Z,k) \to H_Z^{d+p}(X,k)$$

is an isomorphism for all $p \in \mathbb{Z}$.

Proof. Let $I^{\cdot}$ be an injective resolution of the constant sheaf $\underline{k}$ on Z and $J^{\cdot}$ an injective resolution of $\underline{k}$ on X. Using $\tau[d]^{-1}: i^!J^{\cdot}[d] \xrightarrow{\sim} I^{\cdot}$ we get from the local duality that

$$H_Z^{d+p}(X,k) = [i_*I^{\cdot},J^{\cdot}[d+p]] = [I^{\cdot},i^!J^{\cdot}[d+p]] \xrightarrow{\sim} [I^{\cdot},I^{\cdot}[p]]$$

from which the Thom isomorphism can be read off.

<div align="right">Q.E.D.</div>

The Thom isomorphism can be composed with restriction
$r: H_Z^{\cdot}(X,k) \to H^{\cdot}(X,k)$ to obtain the <u>Gysin map</u>

$$i_*: H^p(Z,k) \to H^{d+p}(X,k) \qquad ; \qquad p \in \mathbb{Z}$$

2.4

$$i_*(\alpha) = r(\tau \cup \alpha) \qquad ; \qquad \alpha \in H^p(Z,k)$$

The Gysin map satisfies the

Projection formula 2.5

$$i_*(\alpha \cup i^*\beta) = i_*(\alpha) \cup \beta \qquad ; \quad \alpha \in H^p(Z,k), \beta \in H^q(X,k)$$

<u>Proof.</u> Using the formula II.9.17 twice we get

$$r(\tau \cup \alpha \cup i^*\beta) = (-1)^{(p+d)q} r(\beta \cup \tau \cup \alpha)$$

$$(-1)^{(d+p)q}\beta \cup r(\tau \cup \alpha) = r(\tau \cup \alpha) \cup \beta$$

from which the projection formula follows.

<div align="right">Q.E.D.</div>

<u>Definition 2.6.</u> The <u>Euler class</u> $e \in H^d(Z,k)$ is given by
$e = i^*r(\tau)$ where $\tau \in H_Z^d(X,k)$ is the Thom class.

<u>Formula 2.7.</u> $i^*i_*\alpha = e \cup \alpha$; $\alpha \in H^p(Z,k)$. If d is odd,
then $2e = 0$. In particular $e \cup \alpha = \alpha \cup e$ for all $\alpha \in H^{\cdot}(Z,k)$.

<u>Proof.</u> For the first part it suffices to prove that
$\tau \cup i^*i_*\alpha = \tau \cup e \cup \alpha$. Using the formulas II.9.15-16 we get

$$\tau \cup i^*i_*\alpha = \tau \cup i^*r(\tau \cup \alpha) = r(\tau) \cup (\tau \cup \alpha)$$

$$\tau \cup (e \cup \alpha) = \tau \cup i^*r(\tau) \cup \alpha = r(\tau) \cup \tau \cup \alpha$$

Let us finally prove the formula $e = (-1)^p e$ or what amounts to
the same $\tau \cup e = (-1)^p \tau \cup e$. The formula 2 lines above with
$\alpha = 1$ gives $\tau \cup e = r(\tau) \cup \tau$, on the other hand by II.9.17

$$\tau \cup e = \tau \cup i^*r(\tau) = (-1)^{d^2} r(\tau) \cup \tau$$

and the result follows since $d^2 \equiv d \mod 2$.

<div align="right">Q.E.D.</div>

VIII.3   Oriented microbundles

Let X denote a topological space. By a <u>microbundle</u> of fibre dimension d we understand a triple (E,p,i) where E is a topological space, p: E → X a continuous map and i: X → E a continuous section to p such that for each part x of X there exists an open neighbourhood V of i(x) in E and an open neighbourhood U of x in X with p(V) ⊆ U and i(U) ⊆ V and a homeomorphism a: V → U × $\mathbb{R}^d$ which makes the following diagram commutative

$$
\begin{array}{ccc}
 & V & \\
{}^{i}\nearrow & \downarrow a & \searrow^{p} \\
U & & U \\
{}_{\times 0}\searrow & \downarrow & \nearrow_{pr_1} \\
 & U \times \mathbb{R}^d &
\end{array}
$$

Microbundles (E,p,i) and (F,q,j) are said to be isomorphic, if there exists an open neighbourhood V of i(X) in E and an open neighbourhood W of j(X) in F such that the triples (V,p,i) and (W,q,j) are isomorphic. The concept of microbundles was introduced by Milnor (1). It is known that any microbundle over a paracompact base space is isomorphic to an $\mathbb{R}^d$-bundle, Kister (1), Holm (1).

Example 3.1. Let E be a d dimensional vector bundle over the space X and V and open neighbourhood of the zero section in F. The projection of V onto X and the zero section makes V into a microbundle.

Example 3.2. Let M denote a d-dimensional topological manifold. The projection p_1: M × M → M and the diagonal Δ: M → M × M constitutes a microbundle: For x ∈ M choose an open neighbourhood U of x in M and a homeomorphism f: U $\xrightarrow{\sim}$ $\mathbb{R}^d$: The neighbourhood U × U of (x,x) is mapped homeomorphically onto U × $\mathbb{R}^n$ by the map (x,y) ↦ (x,f(y) − f(x)). This microbundle is called the <u>tangent microbundle</u>. − In case of a smooth manifold the tangent microbundle is isomorphic to the tangent bundle, Milnor (1).

Theorem 3.3. Let (E,p,i) be a microbundle of fibre dimension d over the topological space X. The presheaf

$$
U \longmapsto H_{i(U)}^{\ d}(p^{-1}(U),k)
$$

is a locally constant k-sheaf. The stalk of x ∈ X is

$$
H_{\{i(x)\}}^{\ d}(p^{-1}(x),k)
$$

Proof. Let us calculate the stalk at x ∈ X of the sheaf $R^s i^! \underset{\sim}{k}$, s ∈ $\mathbb{Z}$. According to 1.4 we have

$$
R^s i^! \underset{\sim}{k}_{\ x} = \varinjlim H_{i(U)}^{\ s}(p^{-1}(U),k)
$$

where the limit is taken over all open neighbourhoods U of x in X. Consider the long exact commutative ladder

$$
\begin{array}{ccccc}
H^{s-1}(p^{-1}(U))-i(U),k) & \longrightarrow & H_{i(U)}^{\ s}(p^{-1}(U),k) & \longrightarrow & H^s(p^{-1}(U),k) \\
\downarrow & & \downarrow & & \downarrow \\
H^{s-1}(p^{-1}(x)-\{i(x)\},k) & \longrightarrow & H_{\{i(x)\}}^{\ s}(p^{-1}(x),k) & \longrightarrow & H^s(p^{-1}(x),k)
\end{array}
$$

By excission we may assume that $E = X \times \mathbb{R}^d$. Pass to the
direct limit over all open neighbourhoods U of x to obtain
a new exact commutative ladder. Apply theorem IV.1.6 to the
fibrations E and $E-i(X)$ over X and use the five lemma to
conclude that the stalk of $R^s i_! \underset{\sim}{k}$ at the point x is as de-
scribed.

From this we conclude that $R^s i_! \underset{\sim}{k} = 0$ for $s \neq d$. It
follows from 1.4 that the sections of $R^d i^! k$ over the open set
U of X are given by

$$\Gamma(U, R^d i^! k) = H_{i(U)}{}^d(p^{-1}(U), k)$$

which shows that our presheaf is indeed a sheaf.

To prove that our sheaf is locally constant we may assume
that the bundle is the trivial bundle $pr_1 : X \times \mathbb{R}^d \to X$. Let
us pick a generator $\tau \in H_{\{0\}}{}^d(\mathbb{R}^d, k)$ and pull this back along
$pr_2 : X \times \mathbb{R}^d \to \mathbb{R}^d$ to obtain a class

$$pr_2{}^* \tau \in H_{X \times \{0\}}{}^d(X \times \mathbb{R}^d, k)$$

From the first part of the proof follows that $pr_2{}^* \tau$ generates
our k-sheaf.

$$Q.E.D.$$

The microbundle (E,p,i) is said to be __orientable__ relative
to k if the k-sheaf of the theorem is isomorphic to the
constant k-sheaf $\underset{\sim}{k}$. Expressed otherwise

__Definition 3.4.__ By an __orientation__ relative to k of the
microbundle (E,p,i) of fibre dimension d we understand a coho-
mology class

$$\tau \in H_{i(X)}{}^d(E, k)$$

which for each $x \in X$ restricts to a generator of the k-mdoule
$H_{\{i(x)\}}{}^d(p^{-1}\{x\}, k)$.

If (E,p,i,τ) is an oriented microbundle of fiber dimension
d, τ will also be called the __Thom class__ of the bundle. The image
of τ by the composite

$$H_X{}^n(E,k) \xrightarrow{\ r\ } H^n(E,k) \xrightarrow{\ i^*\ } H^n(X,k)$$

is called the __Euler class__, $e(E)$ of the oriented microbundle.

Pull back of microbundles

Let $f : X \to Y$ be a continuous map and (F,j,q) a micro-
bundle on Y of fibre dimension d. Let

$$f^*F = \{(x,v) \in X \times F \mid f(x) = q(v)\}$$

and let $p : f^*F \to X$ denote the restriction of the first projection
and let $i : X \to f^*E$ denote the map given by $i(x) = (x, jf(x))$. In
this way we have constructed a microbundle (f^*F, i, p), the __pull
back of__ (F,j,q) along f.

If $\tau \in H_{j(Y)}{}^d(F, k)$ is an orientation of F then
$f^*\tau \in H_{i(X)}{}^d(f^*F, k)$ is an orientation of f^*E which follows by iden-
tifying the fibers $p^{-1}(x)$ and $q^{-1}(x)$, $x \in X$. If $e(F)$ is

the Euler class of F then the Euler class $e(f^*F)$ is given by

3.5
$$e(f^*F) = f^*e(F)$$

Products of microbundles

Let (E,p,i) be a microbundle on X of fibre dimension m and (F,q,j) a microbundle on Y of fibre dimension n. Then $(E \times F, p \times q, i \times j)$ is a fibre bundle on $X \times Y$ of fibre dimension $m+n$ as one easily verifies.

Given orientations $\sigma \in H_X^m(E,k)$ and $\tau \in H_Y^n(F,k)$ respectively. Then the cross product, II.10.7

$$\sigma \times \tau \in H_{X \times Y}^{m+n}(E \times F,k)$$

is an orientation of $(E \times F, p \times q, i \times j)$ as it follows from Corollary 3.9 below. If $e(E) \in H^m(X,k)$ and $e(F) \in H^n(Y,k)$ denotes the Euler class of E and F respectively, then the Euler class $e(E \times F)$ of the product is given by

3.6
$$e(E \times F) = e(E) \times e(F)$$

Whitney sum of microbundles

Let E and F be microbundles on X and $E \times F$ the product bundle on $X \times X$. Consider the diagonal map $\Delta: X \to X \times X$ and let the Whitney sum $E \oplus F$ be defined by

$$E \oplus F = \Delta^* E \times F$$

In case E and F are oriented microbundles the Euler class $e(E \oplus F)$ of $E \oplus F$ is given by

3.7
$$e(E \oplus F) = e(E) \cup e(F)$$

as it follows from 3.5, 3.6 and II.10.8.

Proposition 3.8. Let D denote a d-dimensional real vector-space and $\tau \in H_0^d(D,k)$ an orientation. For any pair X,Z consisting of a topological space X and a closed subset Z, the cross product with τ will induce an isomorphism

$$\times \tau : H_Z^p(X,k) \longrightarrow H_{Z \times 0}^{p+d}(X \times D,k) \quad ; \quad p \in \mathbb{Z}$$

Proof. Let us first remark that in case Z is empty the result follows from 3.3 and 2.3. In the general case consider the commutative exact ladder II.10.2

$$
\begin{array}{ccccccc}
\longrightarrow & H^{p-1}(X-Z,k) & \longrightarrow & H_Z^p(X,k) & \longrightarrow & H^p(X,k) & \longrightarrow \\
& \downarrow \times \tau & & \downarrow \times \tau & & \downarrow \times \tau & \\
\longrightarrow & H_{(X-Z) \times 0}^{p-1+d}((X-Z) \times D,k) & \to & H_{Z \times 0}^{p+d}(X \times D,k) & \longrightarrow & H_{X \times 0}^{p+d}(X \times D,k) &
\end{array}
$$

Finally apply the 5-lemma.

Q.E.D.

Corollary 3.9. Let E and F be finite dimensional real vector spaces. Cross product induces an isomorphism

$$H_{\{0\}}^{\cdot}(E,k) \otimes H_{\{0\}}^{\cdot}(F,k) \xrightarrow{\sim} H_{\{0\}}^{\cdot}(E \times F,k)$$

<u>Example 3.10</u>. Let E denote an arbitrary microbundle of fibre dimension d. The Whitney sum $E \oplus E$ has a <u>canonical orientation</u> relative to $\mathbb{Z}$: For $x \in X$ let τ_x generate $H_{\{x\}}^{\ d}(E_x, \mathbb{Z})$. Then $\tau_x \times \tau_x$ is a local orientation of $E \oplus E$, which is independent of τ_x since $(-\tau_x) \times (-\tau_x) = \tau_x \times \tau_x$. These local data fit together to give a global orientation of $E \oplus E$, 3.3.

In the following two sections we shall consider the case where the coefficient ring k equals $\mathbb{F}_2$. Any microbundle has a unique orientation relative to $\mathbb{F}_2$ as it follows from the fact that a locally constant $\mathbb{F}_2$-sheaf of rank 1 is uniquely isomorphic to $\underset{\sim}{\mathbb{F}}_2$, compare V.3.8.

<u>3.11. The Möbius band</u> is the classical microbundle M of fibre dimension 1 over S^1. Let us investigate the Euler class of M in cohomology with coefficients in $\mathbb{F}_2$. In this case the Thom class is unique V.3.8. - From the fact that $M-S^1$ is connected follows that restriction from $H_{S^1}^{\ 1}(M, \mathbb{F}_2) \to H^1(M, \mathbb{F}_2)$ is a monomorphism and consequently by dimension reasons, restriction is an isomorphism. It follows that the Euler class of the Möbius band is the non trivial-element of $H^1(S^1, \mathbb{F}_2)$.

VIII.4 Cohomology of real projective space

The set of lines in $\mathbb{R}^{n+1}$ constitute $P^n(\mathbb{R})$, real projective n-space. The multiplicative group $\mathbb{R}^*$ of real numbers $\neq 0$ acts on $\mathbb{R}^{n+1}-(0)$ and we can represent $P^n(\mathbb{R})$ as the orbit space

$$P^n(\mathbb{R}) = \mathbb{R}^{n+1}-0 \ / \ \mathbb{R}^*$$

Projective space carries a <u>canonical line bundle L</u> given as the orbit space

$$L = (\mathbb{R}^{n+1}-0) \times \mathbb{R} \ / \ \mathbb{R}^*$$

where the action of $\mathbb{R}^*$ is given by

$$r(v,x) = (rv, rx) \quad ; \quad r \in \mathbb{R}^*, \ v \in \mathbb{R}^{n+1}-0, \ x \in \mathbb{R}$$

The projection $p: L \to P^n$ is induced by the projection of $(\mathbb{R}^n-0) \times \mathbb{R}$ onto its first factor.

<u>Theorem 4.1</u>. Let $e \in H^1(P^n(\mathbb{R}), \mathbb{F}_2)$ denote the Euler class of the canonical line bundle L on $P^n(\mathbb{R})$. The powers

$$1, e, e^2, \ldots, e^n$$

are a basis for the $\mathbb{F}_2$-vector space $H^{\cdot}(P^n(\mathbb{R}), \mathbb{F}_2)$.

<u>Proof</u>. The Thom class $\tau \in H_{P^n}^{\ 1}(L, \mathbb{F}_2)$ of L induces an isomorphism for each $p \in \mathbb{Z}$

$$H^p(P^n, \mathbb{F}_2) \xrightarrow{\ \sim\ } H_{P^n}^{\ p+1}(L, \mathbb{F}_2) \quad ; \quad \alpha \longmapsto \tau \cup \alpha$$

The complement of the zero section in L may be identified with $\mathbb{R}^{n+1}-0$ by mapping v into the orbit of $(v,1)$ in $(\mathbb{R}^{n+1}-0) \times \mathbb{R}$. Thus the long exact sequence of local cohomology may be written, compare 2.7

$$\begin{array}{ccccccc}
H^p(L-0,\mathbb{F}_2) & \longrightarrow & H_p^{p+1}(L,\mathbb{F}_2) & \to & H^{p+1}(L,\mathbb{F}_2) & \to & H^{p+1}(L-0,\mathbb{F}_2) \\
\downarrow & & \downarrow \tau^{-1} & & \downarrow i^* & & \downarrow \\
H^p(\mathbb{R}^{n+1}-0,\mathbb{F}_2) & \to & H^p(P,\mathbb{F}_2) & \xrightarrow{\cup e} & H^{p+1}(P,\mathbb{F}_2) & \xrightarrow{p^*} & H^{p+1}(\mathbb{R}^{n+1}-0,\mathbb{F}_2)
\end{array}$$

The fact that $\mathbb{R}^{n+1}-0$ is connected and $H^p(\mathbb{R}^{n+1}-0,\mathbb{F}_2) = 0$ for $p = 1,\ldots,n-1$ implies that

$$H^p(P^n,\mathbb{F}_2) \xrightarrow{\cup e} H^{p+1}(P^n,\mathbb{F}_2)$$

is an isomorphism for $p = 0,\ldots,n-2$ and a monomorphism for $p = n-1$. Using that $H^{n+1}(P^n,\mathbb{F}_2) = 0$ we find that

$$H^n(\mathbb{R}^{n+1}-0,\mathbb{F}_2) \longrightarrow H^n(P^n,\mathbb{F}_2)$$

is an epimorphism. From this first fact we find that $H^n(P^n,\mathbb{F}_2) \neq 0$ and from the second fact we conclude that $H^n(P^n,\mathbb{F}_2)$ is one dimensional over $\mathbb{F}_2$.

Q.E.D.

Remark 4.2. The canonical projection $p: S^n \to P^n$ induces the zero map in mod 2 cohomology

$$p^*: H^n(P^n,\mathbb{F}_2) \longrightarrow H^n(S^n,\mathbb{F}_2)$$

as it follows from the previous proof.

As an example of application of the idea of characteristic class let us consider a theorem of K. Borzuk proved in 1933 on the basis of a conjecture of St.Ulam.

Borzuk-Ulam theorem 4.3. Any continuous map $f: S^n \to \mathbb{R}^n$ must identify a pair of antipodal points.

Proof. Suppose to the contrary that

$$f(-x) \neq f(x) \qquad ; \quad x \in S^n$$

Then the map $g: S^n \to S^{n-1}$ given by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

will satisfy $g(-x) = -g(x)$ for all $x \in S^n$. The formula

$$f(x) = |x| g\left(\frac{x}{|x|}\right) \qquad ; \quad x \in \mathbb{R}^{n+1}-0$$

defines a continuous map $f: \mathbb{R}^{n+1}-0 \to \mathbb{R}^n-0$, which satisfies

$$f(rx) = rf(x) \qquad ; \quad x \in \mathbb{R}^{n+1}-0$$

From this formula follows that f induces a continuous map $F: P^n \to P^{n-1}$ with the property $F^*L = L$. The last fact is deduced from the commutative diagram

$$
\begin{array}{ccc}
(\mathbb{R}^{n+1}-0) \times \mathbb{R} & \xrightarrow{\ f \times 1\ } & (\mathbb{R}^n-0) \times \mathbb{R} \\
\downarrow & & \downarrow \\
\mathbb{R}^{n+1}-0 & \xrightarrow{\quad f \quad} & \mathbb{R}^n-0
\end{array}
$$

which is $\mathbb{R}^*$-equivariant. Let us now remark that $F^*e_{n-1} = e_n$, where e_{n-1} and e_n denote the Euler classes of the canonical bundles on P^{n-1} and P^n. This gives

$$e_n{}^n = F^*e_{n-1}{}^n = F^*0 = 0$$

which is a contradiction.

$$\text{Q.E.D.}$$

We shall present two applications of the Borzuk-Ulam theorem; the first result goes back to about 1930.

Lusternik-Snirelmann theorem 4.4. If S^n is covered by $n+1$ closed sets, then one of the sets contains a pair of antipodal points.

Proof. Let the closed sets $Z_0, Z_1, \ldots, Z_n$ cover S^n. For $i = 1, \ldots, n$ let $d(x, Z_i)$ denote the distance from x to Z_i. These n functions define a continuous map $d: S^n \to \mathbb{R}^n$. By 4.4 we can find $x \in S^n$ with $d(x) = d(-x)$. If $d(x, Z_i) > 0$ for $i = 1, \ldots, n$, then x and $-x$ lie in Z_0. On the other hand, if $d(x, Z_i) = 0$ for some $i > 0$, then x and $-x$ lie in Z_i, since Z_i is a closed set.

$$\text{Q.E.D.}$$

Theorem 4.5. Let $K_1, \ldots, K_n$ be compact subsets of $\mathbb{R}^n$. There exists a hyperplane in $\mathbb{R}^n$ that divides each of $K_1, \ldots, K_n$ into two parts of equal volume.

Proof. We shall identify $\mathbb{R}^n$ with the hyperplane $\mathbb{R}^n \times 0$ in $\mathbb{R}^{n+1}$. For $x \in S^n$ let $P(x)$ denote hyperplane in $\mathbb{R}^{n+1}$ through $(0, \ldots, 0, \tfrac{1}{2})$ orthogonal to x. Notice that $P(x) = P(-x)$. For $i = 1, \ldots, n$ and $x \in S^n$ let $\mu_i(x)$ denote the volume of the part of K_i which is on the same side of $P(x)$ as x. Let us now apply the Borsuk-Ulam theorem to the map

$$x \longmapsto (\mu_1(x), \ldots, \mu_n(x)) \qquad ; \quad x \in S^n$$

to find an $x \in S^n$ such that $\mu_i(x) = \mu_i(-x)$ for $i = 1, \ldots, n$. It is left to the reader to prove that $P(x) \cap \mathbb{R}^n$ solves our problem.

$$\text{Q.E.D.}$$

VIII.5   Stiefel-Whitney classes

This section is intended to give a brief introduction to Stiefel-Whitney classes. The general reference is Milnor and Stasheff (1). Recall from Bredon (1) II.21:

Steenrod squares

For a closed subspace Z of the topological space X and $i \in \mathbb{N}$ there is given a linear map "square upper i"

$$Sq^i: H_Z^{\ p}(X, \mathbb{F}_2) \longrightarrow H_Z^{\ p+i}(X, \mathbb{F}_2) \qquad ; \ p \in \mathbb{Z}$$

with the following properties: For $a \in H_Z^{\ n}(X, \mathbb{F}_2)$

5.1 $$Sq^0(a) = a, \quad Sq^n(a) = a \cup a, \quad Sq^i(a) = 0 \quad \text{for} \ i > n$$

As a consequence of this we write

$$Sq(a) = Sq^0(a) + Sq^1(a) + \ldots + Sq^n(a)$$

Given a continuous map $f: X \to Y$ and closed sets Z and W of X and Y with $f(X-Z) \subseteq Y-W$, then

5.2 $$Sq(f*a) = f*Sq(a) \qquad ; \ a \in H_W^{\ ''}(Y, \mathbb{F}_2)$$

For closed subsets Z and W of X we have

5.3 $$Sq(\alpha \cup \beta) = Sq(\alpha) \cup Sq(\beta) \qquad ; \ \beta \in H_Z^{\ ''}(X, \mathbb{F}_2), \ \alpha \in H_W^{\ \cdot}(X, \mathbb{F}_2)$$

Stiefel-Whitney classes

Let E denote a rank n vector bundle on the topological space X and $\tau \in H_X^{\ n}(E, \mathbb{F}_2)$ the Thom class. Recall that the Thom isomorphism is given by

$$\alpha \longmapsto \tau \cup \alpha \ ; \qquad H^p(X, \mathbb{F}_2) \to H^{p+n}(X, \mathbb{F}_2) \qquad ; \ p \in \mathbb{Z}$$

The total $\underline{\text{Stiefel-Whitney}}$ class $w(E) \in H(X, \mathbb{F}_2)$ of E is given as the image of $Sq(\tau)$ by the inverse of the Thom isomorphism, i.e.

Thoms formula 5.4.

$$Sq(\tau) = \tau \cup w(E)$$

From the formula 5.1 follows that we can write

5.5 $$w(E) = 1 + w_1(E) + \ldots + w_n(E) \qquad ; \ w_i(E) \in H^i(X, \mathbb{F}_2)$$

From the proof of 2.7 we see that the Euler class e satisfies $\tau \cup \tau = \tau \cup e$, consequently by 5.1

5.6 $$w_n(E) \quad \text{is the Euler class of the rank n bundle} \ E$$

For a continuous map $f: Z \to X$ we have

5.7 $$w(f*E) = f*w(E)$$

as it follows from Thoms formula 5.4. - For bundles E and F on the space X we have

5.8 $$w(E \oplus F) = w(E) \cup w(F)$$

<u>Proof</u>. Consider the bundle $E \times F$ on $X \times X$. It suffices to prove that

$$w(E \times F) = w(E) \times w(F)$$

To see this let $\sigma \in H^m_X(E, \mathbb{F}_2)$ and $\tau \in H^n_X(F, \mathbb{F}_2)$ denote the Thom classes for E and F respectively. The Thom class of $E \times F$ is $\sigma \times \tau$ as it follows from 3.9. Using Thoms formula 5.4 and the formulas 5.2, 5.3 we get

$$Sq(\sigma \times \tau) = Sq(\sigma) \times Sq(\tau) = (\sigma \cup w(E)) \times (\tau \cup w(F)) = (\sigma \times \tau) \cup (v(E) \times w(F))$$

from which the result follows.

<div align="right">Q.E.D.</div>

The formula 5.8 has the following variant

<u>Proposition 5.9</u>. Let X denote a locally compact space, countable at infinity.[*] Given an exact sequence of vector bundles

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

Then $w(F) = w(E) \cup w(G)$.

<u>Proof</u>. Introduce a Riemannian metric on F to obtain an isomorphism $F \xrightarrow{\sim} E \oplus G$.

<div align="right">Q.E.D.</div>

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[*] It suffices that X is a paracompact topological space.

Normal bundles

Let us consider a smooth manfold X and the inclusion $i \colon Z \to X$ of a closed smooth submanifold of codimension d. We have

$$5.10 \qquad R^p i^! \underset{\sim}{\mathbb{F}}_2 = \begin{cases} \mathbb{F}_2 & \text{for } p = d \\ 0 & \text{for } p \neq d \end{cases}$$

as it follows by noticing that the result is in fact local, which allows us to view i as the zero section of a microbundle. With the notation of VIII.2 we deduce a Gysin map

$$i_* \colon H^p(Z, \mathbb{F}_2) \longrightarrow H^{p+d}(X, \mathbb{F}_2) \qquad ; \ p \in \mathbb{Z}$$

in case X is countable at infinity we can find a diffeomorphism of the <u>normal bundle N</u> of Z in X onto an open neighbourhood of Z in X which transforms the zero section into i, Milnor and Stasheff (1). It follows that $i \colon Z \to X$ and the zero section $Z \to N$ have the same Euler class. In particular we get from 2.7

$$5.11 \qquad \boxed{i^* i_* \alpha = w_d(N) \cup \alpha} \qquad ; \quad \alpha \in H^{\cdot}(Z, \mathbb{F}_2)$$

From this formula we can deduce a useful non imbedding criterion.

<u>Proposition 5.12</u>. Let X be an n-dimensional smooth manifold which admits a closed, smooth imbedding into $\mathbb{R}^{n+d}$, $d > 0$. Then

$$[w(X)^{-1}]_d = 0$$

i.e. the d'th homogeneous component of the inverse of the total

Stiefel-Whitney class of the tangent bundle of X is zero.

Proof. Let N denote the normal bundle for $i: X \to \mathbb{R}^{n+k}$.
From the canonical exact sequence

$$0 \longrightarrow T(X) \longrightarrow i^*T(\mathbb{R}^{n+k}) \longrightarrow N \longrightarrow 0$$

we deduce from 5.9 with $w(X) = w(TX)$ that

$$w(X) \cup w(N) = 1$$

From 5.11 and the fact that $H^d(\mathbb{R}^{n+d}, \mathbb{F}_2) = 0$ we deduce that

$$w_d(N) = i^*i_*1 = 0$$

Combine these two formulas to get $[w(X)^{-1}]_d = 0$.

Q.E.D.

Imbeddings of projective n-space

Recall from VIII.3 that $\mathbb{P}^n$ comes equipped with a canonical line bundle L. The bundle $L \otimes \mathbb{R}^{n+1}$ has a canonical section s. We can represent $L \otimes \mathbb{R}^{n+1}$ as the orbit space

$$(\mathbb{R}^{n+1}-0) \times \mathbb{R}^{n+1}/\mathbb{R}^*$$

under the action $r(v,z) = (rv, rz)$. The section $s: \mathbb{P}^n \to L \otimes \mathbb{R}^{n+1}$ is then induced by the $\mathbb{R}^*$-equivariant map

$$\mathbb{R}^{n+1}-0 \longrightarrow (\mathbb{R}^{n+1}-0) \times \mathbb{R}^{n+1} ; \qquad v \longmapsto (v,v)$$

There results an exact sequence of bundles

$$5.13 \qquad 0 \longrightarrow P \times \mathbb{R} \xrightarrow{\ s\ } L \otimes \mathbb{R}^{n+1} \longrightarrow T \longrightarrow 0$$

where T is the tangent bundle on $\mathbb{P}^n$. We shall calculate $w(P) = w(T)$ from this sequence using 5.9 and the fact that $w(L) = 1+e$ where e is the non trivial element of $H^1(\mathbb{P}^n, \mathbb{F}_2)$. The result is

$$5.14 \qquad w(\mathbb{P}^n) = (1+e)^{n+1} \qquad ; \ e \in H^1(\mathbb{P}^n, \mathbb{F}_2), \quad e \neq 0$$

Example 5.15. For $n = 4$ we find that

$$w(\mathbb{P}^n) = 1 + e + e^4$$

A simple calculation yields

$$w(\mathbb{P}^4)^{-1} = 1 + e + e^2 + e^3$$

We deduce from 5.12 that $\mathbb{P}^n$ cannot be imbedded in $\mathbb{R}^7$. It is known that $\mathbb{P}^5$ can be imbedded in $\mathbb{R}^8$. For additional information see Milnor and Stasheff (1).

VIII.6   Chern classes

Let V be a real d-dimensional vector space. Recall that we have a canonical isomorphism

$$H_{\{0\}}{}^d(V,\mathbb{Z}) = H_c{}^d(V,\mathbb{Z})$$

It follows from III.8.10 that a linear automorphism f of V acts on $H_{\{0\}}{}^d(V,\mathbb{Z})$ as multiplication by $\mathrm{sign}(\det(f))$.

An ordered basis $a_1,\ldots,a_d$ for V will determine a generator for $H_{\{0\}}{}^d(V,E)$ as it follows by decomposing $V = \mathbb{R}a_1 \oplus \ldots \oplus \mathbb{R}a_d$ and using 3.9. A second basis $b_1,\ldots,b_d$ will determine the same generator if and only if the transision matrix from the first basis to the second basis has positive determinant. Accordingly we shall identify an _orientation of V_ with a generator for $H_{\{0\}}{}^d(V,\mathbb{Z})$.

Let us consider a complex vector space E . An ordered complex basis $e_1,\ldots,e_n$ for E give rise to an ordered real basis

$$e_1 , ie_1 , e_2 , ie_2 , \ldots , e_n , ie_n .$$

The resulting orientation of E is independent of the complex basis considered as it follows from the fact that a complex linear automorphism of E , considered as an automorphism of the underlying space, has positive determinant.

Let E denote a complex vector bundle of fibre dimension n. According to the discussion above E has a canonical orientation, i.e. a Thom class

6.1 $\tau(E) \in H_X{}^{2n}(E,\mathbb{Z})$

which for each $x \in X$ restricts to the complex orientation class of E_x, compare 3.3. According to 2.6 this gives rise to a

characteristic class

6.2 $e(E) \in H^{2n}(X,\mathbb{Z})$

the Euler class of the complex vector bundle E.

Proposition 6.3. The Euler class of the tensor product of two complex line bundles K and L on the topological space X is given by

$$e(K \otimes L) = e(K) + e(L)$$

Proof. We shall first prove that the projections of the Whitney sum $K \oplus L$ onto its factors induce an isomorphism

$$H_X{}^2(K,\mathbb{Z}) \oplus H_X{}^2(L,\mathbb{Z}) \to H_{KUL}{}^2(K \oplus L,\mathbb{Z})$$

We leave it to the reader to show that the projection of $K \oplus L$ onto its factors induce isomorphisms

$$H_X{}^{\cdot}(K,\mathbb{Z}) \stackrel{\sim}{\to} H_L{}^{\cdot}(K \oplus L,\mathbb{Z}) \qquad H_X{}^{\cdot}(L,\mathbb{Z}) \stackrel{\sim}{\to} H_K{}^{\cdot}(K \oplus L,\mathbb{Z})$$

The Mayer-Vietoris sequence II.9.11 for the subspaces $K = K \times 0$ and $L = 0 \times L$ of $K \oplus L$ gives us an exact sequence

$$\to H_X{}^2(K \oplus L,\mathbb{Z}) \to H_K{}^2(K \oplus L,\mathbb{Z}) \oplus H_L{}^2(K \oplus L,\mathbb{Z}) \to H_{KUL}{}^2(K \oplus L,\mathbb{Z})$$

and the result follows from the fact that $H_X{}^i(K \oplus L,\mathbb{Z}) = 0$ for $i = 0,1,2,3$.

The map $K \oplus L \to K \otimes L$ given by $(x,y) \mapsto x \otimes y$ induces a map on local cohomology

$$H_X^2(K \otimes L, \mathbb{Z}) \to H_{KUL}^2(K \oplus L, \mathbb{Z})$$

Compose this with the inverse of the isomorphism established above to get a map

$$H_X^2(K \otimes L, \mathbb{Z}) \to H_X^2(K, \mathbb{Z}) \oplus H_X^2(L, \mathbb{Z})$$

We are going to prove that this map transforms the Thom class of $K \otimes L$ into the direct sum of the Thom class of K and the Thom class of L.

According to 3.3 it suffices to treat the case where X is a point. Thus we may assume that $K = L = \mathbb{C}$ in which case we may identify three relevant local cohomology groups with $H^1(\mathbb{C}^*, \mathbb{Z})$. It suffices to prove the following.

Let p and q denote two projections of $\mathbb{C}^* \times \mathbb{C}^*$ onto its factors and $\pi: \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ the product morphism $\pi(z,w) = zw$; $z, w \in \mathbb{C}^*$. Then

$$\pi^*\alpha = p^*\alpha + q^*\alpha \quad ; \quad \alpha \in H^1(\mathbb{C}^*, \mathbb{Z})$$

To prove this formula, notice that the two canonical injections $i_1, i_2 = \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$ give rise to an isomorphism

$$H^1(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z}) \xrightarrow{\sim} H^1(\mathbb{C}^*, \mathbb{Z}) \oplus H^1(\mathbb{C}^*, \mathbb{Z})$$

as it follows from the Künneth formula. This completes the proof of the statement about the Thom class. From this it is easy to deduce the statement about the Euler class.

Let $p: E \to X$ be a complex vector bundle of rank n on the topological space X. Define

$$P(E) = E-X/\mathbb{C}^*$$
$$L(E) = (E-X) \times \mathbb{C}/\mathbb{C}^*$$

i.e. the quotient spaces under the action of the complex multiplicative group $\mathbb{C}^*$. In the latter case the action is given by

$$z(e,x) = (ze, zx) \quad ; \quad z \in \mathbb{C}^*, \, e \in E-X, \, z \in X$$

The projection of $(E-X) \in \mathbb{C}$ onto its first factor induces a map $L(E) \to P(E)$ which in fact makes $L(E)$ a line bundle on $P(E)$, the canonical line bundle on $P(E)$. If we let $\pi: P(E) \to X$ denote the projection and E^V the dual bundle, evaluation defines a $\mathbb{C}^*$-equivariant map

$$(E-X) \times E^V \to (E-X) \times \mathbb{C} \quad ; \quad (e,f) \to (e, f(e))$$

and consequently a map of bundles

6.4 $$\pi^*E^V \longrightarrow L(E) \longrightarrow 0$$

A different way of describing this is to start with the $\mathbb{C}^*$-equivariant map

$$(E-0) \to (E-0) \times E \quad ; \quad e \mapsto (e,e)$$

where the action on the second space is given by $r(e,v) = (re, rv)$. This defines a non vanishing section

6.5 $$s: P(E) \longrightarrow L(E) \otimes \pi^*E$$

<u>Theorem 6.6.</u> Let $p: E \to X$ denote a complex vector bundle of rank n, and $\xi \in H^2(P(E),\mathbb{Z})$ the Euler class of the canonical line bundle $L(E) \to P(E)$. Then $H^{..}(P(E),\mathbb{Z})$, is a free $H^{..}(X,\mathbb{Z})$-module with basis

$$1,\xi,\ldots,\xi^{n-1}$$

<u>Proof.</u> The fibre of $P(E)$ at $x \in X$ is $P(E_x)$ and the restriction of $L(E)$ to $P(E_x)$ is the canonical line bundle $L(E_x)$ on $P(E_x)$. Thus the image $\xi_x \in H^2(P(E_x),\mathbb{Z})$ of ξ is the Euler class of $L(E_x)$. It follows by a proof similar to that of 4.1 that $H^{..}(P(E_x),\mathbb{Z})$ is a free $\mathbb{Z}$-module with basis $1,\xi_x,\ldots,\xi_x^{n-1}$. The theorem is now a consequence of the following Lemma.

Q.E.D.

<u>Lemma 6.7.</u> Let $f: P \to X$ be a fibre bundle whose fibres have the homotopy type of a compact space and let k denote a commutative ring. Given homogeneous elements $\alpha_1,\ldots,\alpha_n$ of $H^{..}(P,\underline{k})$ which for each $x \in X$ restricts to a basis for the k-module $H^{..}(P_x,k)$. Then the $H^{..}(X,k)$-module $H^{..}(P,k)$ is free with basis $\alpha_1,\ldots,\alpha_n$.

<u>Proof.</u> Let $I^{..}$ denote an injective resolution of $\underline{k}$ in $Sh(P,k)$. For $i = 1,\ldots,n$ let d_i denote the degree of α_i and interprete $\alpha_i \in H^{d_i}(P,k)$ as a morphism

$$\alpha_i: \underline{k}[-d_i] \to I^{..} \qquad \text{in } K^+(P,k)$$

Using that $f^*\underline{k} = \underline{k}$, we may by adjunction interprete α_i as a morphism

$$\alpha_i: \underline{k}[-d_i] \to f_*I^{..} \qquad \text{in } K^+(X,k)$$

Let us form the direct sum of these morphisms

$$\oplus_i \underline{k}[-d_i] \longrightarrow f_*I^{..}$$

By Theorem IV.1.6 this is a quasi-isomorphism of complexes of k-sheaves on X. Now apply $R\Gamma(X,-)$ to this quasi-isomorphism to get a quasi-isomorphism

$$\oplus_i R^{..}\Gamma(X,k)[-d_i] \longrightarrow \Gamma(P,I^{..})$$

and the result follows by passing to cohomology.

Q.E.D.

With the notation of the previous theorem there is a unique sequence of cohomology classes

$$c_1(E),\ldots,c_n(E) \qquad ; \quad c_i(E) \in H^{2i}(X,\mathbb{Z})$$

such that the following relation holds in $H^{..}(P(E),\mathbb{Z})$, here and in the following we omit the $\cup$-sign,

6.8
$$\xi^n + c_1(E)\xi^{n-1} + \ldots + c_{n-1}(E)\xi + c_n(E) = 0$$

With the conventions

$$c_0(E) = 1, \quad c_i(E) = 0 \quad \text{for } i > n = rk\ E$$

we have defined the i'th Chern class of E

$$c_i(E) \in H^{2i}(X, \mathbb{Z}) \qquad ; \; i \in \mathbb{N}$$

and the total Chern class $c_.(E) \in H^.(X, \mathbb{Z})$ of E

$$c_.(E) = c_0(E) + c_1(E) + \ldots$$

Given a continuous map f: W → X, then

6.9 $$c_.(f^*E) = f^*c_.(E)$$

as it follows rather immediately from the construction of $c_.(E)$.

Proposition 6.10. Given an exact sequence of bundles

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

on the topological space X. Then

$$c_.(F) = c_.(E) \cup c_.(G)$$

Proof. Let $\xi \in H^2(P(F), \mathbb{Z})$ denote the Euler class of L(F) and consider the cohomology classes α and β in $H^.(P(F), \mathbb{Z})$

$$\alpha = \xi^m + c_1(E)\xi^{m-1} + \ldots + c_m(E) \qquad ; \; m = \mathrm{rk}\, E$$

$$\beta = \xi^n + c_1(F)\xi^{n-1} + \ldots + c_n(F) \qquad ; \; n = \mathrm{rk}\, G$$

According to the relation 6.8 the sought for formula is equivalent to $\alpha \cup \beta = 0$. To see this, remark first that the restriction of α to P(E) is zero, since L(F) restricts to L(E) on P(E).

second let us remark that the restriction of β to P(F) - P(E) is zero: the pull back of L(G) along the projection

$$P(F) - P(E) \to P(G)$$

equals the restriction of L(F) to P(F) - P(E). Conclusion by lemma 6.12 below.

Q.E.D.

Corollary 6.11. Let p: F → X be a rank n bundle on X. If p has a continuous section s: X → F with s(x) ≠ 0 for all x ∈ X, then

$$c_n(F) = 0$$

Proof. This implies the existence of a short exact sequence of bundles as in 6.10 where E is the trivial bundle. Hence $c_.(F) = c_.(G)$.

Q.E.D.

Lemma 6.12. Let Z denote a closed subspace of the topological space X and k a commutative ring. Given $\alpha \in H^p(X, k)$ whose image in $H^p(Z, k)$ is zero and $\beta \in H^q(X, k)$ whose image in $H^q(X-Z, k)$ is zero. Then $\alpha \cup \beta = 0$ in $H^{p+q}(X, k)$.

Proof. If we let i: Z → X denote the inclusion we have $i^*\alpha = 0$. In virtue of the exact sequence

$$H_Z^{\,p}(X, k) \xrightarrow{\; r \;} H^p(X, k) \longrightarrow H^p(X-Z, k)$$

we can write $\beta = r(\gamma)$ where $\gamma \in H_Z^P(X,k)$. Thus

$$\alpha \cup \beta = \alpha \cup r(\gamma) = r(\alpha \cup \gamma) = (-1)^{pq}(\gamma \cup i^*\alpha) = 0$$

where we have used the formula II.9.17.

Q.E.D.

Proposition 6.13. For a rank n vector bundle E on the topological space X, the Euler class of E equals $c_n(E)$, the top Chern class of E.

Proof. Let us first treat the case $n = 1$. From the definition 6.8 and 6.4 we find that $c_1(E) = -e(E^V)$, which according to 6.3 equals $e(E)$, using that $E \otimes E^V$ is the trivial bundle.

In the general case we can refer to the proof of 6.14 which establishes that

$$c_n(E) = \theta_n \, e(E)$$

where θ_n is a constant depending only on n. To determine this constant consider the canonical line bundle L on P^n and apply this formula to $E = L^{\oplus n}$. We get

$$e(E) = e(L)^n, \quad c_n(E) = c_1(L)^n$$

as it follows from 3.7 and 6.10 respectively. Thus we can conclude that $\theta_n = 1$.

Q.E.D.

A Chern class formula for the Thom class

Let E denote a rank n vector bundle on the topological space X. Put $\widetilde{E} = E \oplus \mathbb{C}$ and consider $P(\widetilde{E})$. The canonical map $E \to E \oplus \mathbb{C} - 0$, $e \longmapsto (e,1)$ induces an imbedding $j: E \to P(\widetilde{E})$ of E onto an open subset of $P(\widetilde{E})$. We shall consider X as a closed subset of E and $P(\widetilde{E})$ through the zero section of E. This gives an excision isomorphism

$$j^*: H_X^{\,\cdot\cdot}(P(\widetilde{E}), \mathbb{Z}) \xrightarrow{\ \sim\ } H_X^{\,\cdot\cdot}(E, \mathbb{Z})$$

which allows us to view the Thom class of E as

$$\tau(E) \in H_X^{\,\cdot\cdot}(P(\widetilde{E}), \mathbb{Z})$$

The restriction map

$$r: H_X^{\,\cdot\cdot}(P(\widetilde{E}), \mathbb{Z}) \longrightarrow H^{\cdot}(P(\widetilde{E}), \mathbb{Z})$$

is injective and the following formula holds

6.14 $$r(\tau(E)) = c_n(E \otimes L(E))$$

Proof. Let us first prove that the cohomology class

$$c_n(E \otimes L(E)) \in H^{2n}(P(\widetilde{E}), \mathbb{Z})$$

has image zero in $H^{2n}(P(\widetilde{E}) - X, \mathbb{Z})$. According to 6.11 it

suffices to prove that $E \otimes L(\widetilde{E})$ has a section over $P(\widetilde{E})$ which is non vanishing over $P(\widetilde{E}) - X$. In fact the standard section, 6.7 of $\widetilde{E} \otimes L(\widetilde{E})$ has a projection onto $E \otimes L(\widetilde{E})$ of the required sort. This section is induced by the $\mathbb{C}^*$-equivariant map

$$E \oplus \mathbb{C} - 0 \longrightarrow (E \oplus \mathbb{C} - 0) \times E \quad ; \quad (e,z) \longmapsto (e,z,e)$$

Let us exploit this in case X is a point. We conclude that $c_1(L)^n$ the free generator of $H^{2n}(P^n, \mathbb{Z})$ can be lifted back to $H_0^{2n}(P^n, \mathbb{Z})$. This group we have identified with $H_0^{2n}(\mathbb{C}^n, \mathbb{Z})$ so let $\tau \in H_0^{2n}(P^n, \mathbb{Z})$ denote the Thom class. Thus we can write

$$r(\tau) = \theta_n c_1(L) \quad ; \quad \theta_n = \pm 1$$

The constant θ_n will be determined indirectly.

Returning to the global situation we can choose a cohomology class $\alpha \in H_X^{2n}(P(\widetilde{E}), \mathbb{Z})$ with $r(\alpha) = \theta_n c_n(E \otimes L(\widetilde{E}))$. We shall now prove that $\alpha = \tau(E)$. Using 3.3 we see that it suffies to treat the case where X is a point, which has been accomplished above. Hence

$$r(\tau(E)) = \theta_n c_n(E \otimes L(\widetilde{E}))$$

where $\theta_n = \pm 1$ is a constant which depends only on $n \in \mathbb{N}$.

If we pull the formula above back to X we get $e(E) = \theta_n c_n(E)$.

It is established during the proof of 6.13 that $\theta_n = 1$.

Let us now prove that the restriction map r mentioned in the line before 6.14 is injective. Put $c_1(L(E)) = \xi$. Then $1, \xi, \ldots, \xi^n$ form a basis for the $H^\cdot(X, \mathbb{Z})$-module $H^\cdot(P(\widetilde{E}), \mathbb{Z})$ according to 6.6, while the restriction of $1, \xi, \ldots, \xi^{n-1}$ to $P(E) - X$ form a basis for the $H^\cdot(X, \mathbb{Z})$-module $H^\cdot(P(E) - X, \mathbb{Z})$ as it follows from 6.7 and our investigations above of the case where X is a point. This proves that the map

$$H^\cdot(P(\widetilde{E}), \mathbb{Z}) \longrightarrow H^\cdot(P(\widetilde{E}) - X, \mathbb{Z})$$

is surjective. The long exact sequence of local cohomology proves that r is injective.

Q.E.D.

The splitting principle 6.15. Let E be a complex vector bundle on the topological space X. There exists a continuous map $f: W \to X$ such that

1) $f^*: H^\cdot(X, \mathbb{Z}) \longrightarrow H^\cdot(W, \mathbb{Z})$ is injective.

2) f^*E admits a finite filtration by sub-bundles whose quotients are line bundles.

Proof. Repeated applications of 6.4 and 6.6.

Q.E.D.

Let us give some applications of the splitting principle. For a complex vector bundle E, we have for the dual bundle $E^\vee$

6.16
$$c_i(\overset{\vee}{E}) = (-1)^i c_i(E)$$

<u>Proof</u>. Let us look at the even part of $H^{\cdot}(X,\mathbb{Z})$

$$H^{ev}(X,\mathbb{Z}) = \bigoplus_{i \in \mathbb{N}} H^{2i}(X,\mathbb{Z})$$

Notice that this is a commutative ring. For this reason we shall omit the cup product sign. This ring has an automorphism $\alpha \longmapsto \bar{\alpha}$ given by

$$\bar{\alpha} = (-1)^p \alpha \quad ; \quad \alpha \in H^{2p}(X,\mathbb{Z})$$

In order to prove the formula 6.16 we may assume that E admits a filtration whose quotients are line bundles $L_1,\ldots,L_n$. It follows from 6.3 that $c_\cdot(L_s^V) = \bar{c}_\cdot(L_s)$. Using 6.10 we get

$$c_\cdot(E^V) = c_\cdot(L_1^V)\ldots c_\cdot(L_n^V) = \bar{c}_\cdot(L_1)\ldots\bar{c}_\cdot(L_n) = \tilde{c}_\cdot(E)$$

from which the formula 6.16 follows.

<div align="right">Q.E.D.</div>

For a rank n bundle E and a line bundle L we have

6.17 $\qquad c_n(E \otimes L) = c_n(E) + c_{n-1}(E)c_1(L) + c_{n-2}(E)c_1(L)^2 + \ldots$

<u>Proof</u>. This will be done by induction on $n \in \mathbb{N}$. By the splitting principle we may assume that E has a sub bundle F such that $E/F = K$ is a line bundle. Notice that

$$c_i(E) = c_i(F) + c_{i-1}(F)c_1(K) \quad ; \quad i \in \mathbb{N}$$

By the induction hypothesis we have

$$c_{n-1}(F \otimes L) = c_{n-1}(F) + c_{n-2}(F)c_1(L) + \ldots$$

Consequently we get

$$c_n(E \otimes L) = c_{n-1}(F \otimes L)c_1(K \otimes L) =$$

$$(c_{n-1}(F) + c_{n-2}(F)c_2(L) + \ldots)(c_1(K) + c_1(L))$$

from which the formula follows.

<div align="right">Q.E.D.</div>

For a complex vector bundle E of rank n, let $\bar{E}$ denote the conjugate bundle: The underlying real bundle is the same as that of E but the action of $i = \sqrt{-1}$ on $\bar{E}$ is that of $-i$ on E. We have

6.18 $\qquad c_i(\bar{E}) = (-i)^i c_i(E) \qquad ; \quad z \in \mathbb{N}$

<u>Proof</u>. Let us notice in general that

6.19 $\qquad \tau(\bar{E}) = (-1)^n \tau(E) \qquad ; \quad n = \mathrm{rk}\, E$

as it follows by inspection of the case where X is a point, compare the introductory remarks of this section.

<div align="right">Q.E.D.</div>

VIII.7 Pontrjagin classes

Consider a real vector bundle E of rank n on the topological space X. To this we can assign the complexified bundle $F \otimes_{\mathbb{R}} \mathbb{C}$. Let us remark that the Chern classes of this bundle satisfy

7.1 $$2c_i(E \otimes_{\mathbb{R}} \mathbb{C}) = 0 \quad \text{for i odd}$$

as it follows from 6.19 by noticing that the bundle $E \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the conjugate bundle

$$E \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} E \otimes_{\mathbb{R}} \mathbb{C} \quad ; \quad e \otimes z \mapsto e \otimes \bar{z}$$

In consequence of 7.1 we define the <u>i'th Pontrjagin class</u> of the real bundle E on the space X to be

7.2 $$p_i(E) = (-1)^i c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C})$$

which is considered as a class in rational cohomology

7.3 $$p_i(E) \in H^{4i}(X, \mathbb{Q})$$

The <u>total Pontrjagin class</u> $p.(E) \in H^*(X, \mathbb{Q})$ is given by

$$p(E) = p_0(E) + p_1(E) + \ldots$$

The following properties of Pontrjagin classes follow immediately from the properties of the Chern class

7.4 $$p_0(E) = 1, \quad p_i(E) = 0 \quad \text{for } i > \tfrac{1}{2} + \text{rk } E$$

For a continuous map $f: W \to X$ we have

7.5 $$p.(f^*E) = f^* p.(E)$$

For an exact sequence of bundles on X

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

we have the product formula

7.6 $$p.(F) = p(E) \cup p.(G)$$

We shall draw attention to

<u>Hirzebruch's signature theorem 7.7</u>. Let X denote a compact oriented 4n-dimensional smooth manifold. The pairing

$$\int_M \alpha \cup \beta \quad ; \quad \alpha, \beta \in H^{2n}(M, \mathbb{R})$$

is symmetric and non-degenerated by Poincaré duality. The <u>signature of M, sign(M)</u> is by definition the signature of this quadratic form. The <u>Pontrjagin classes of M</u>, $p_1(M)$, $p_2(M)$, ... are by definition the Pontrjagin classes of the tangent bundle of M. According to Hirzebruch one has

$$n = 1: \quad \text{sign}(M) = \int_M \tfrac{1}{3} p_1(M)$$

$$n = 2: \quad \text{sign}(M) = \int_M \tfrac{1}{45} (7p_2(M) - p_2(M)^2)$$

In general there is a polynomium $L_n \in \mathbb{Q}[T_1, \ldots, T_n]$ such that

$$\text{sign}(M) = \int_M L_n(p_1, \ldots, p_n)$$

X Application to Algebraic Geometry

X.1   Dimension of algebraic varieties

In this chapter we shall give an introduction to the topology of algebraic varieties over the complex numbers. For all unexplained notation we refer to Fulton (1).

The set of geometric points of a <u>complex algebraic scheme</u> X carries the structure of a locally compact space, which we also denote by X. Let us recall three fundamental facts about complex <u>algebraic varieties,</u> i.e. reduced and irreducible schemes.

1.1. A non-singular point x of X admits an open neighbourhood homeomorphic to $\mathbb{C}^n$ where n = dim $0_{X,x}$, the local ring of X at x.

1.2. A non empty Zariski open subset of an algebraic variety X is dense in X.

1.3. An algebraic variety is connected.

<u>Proposition 1.4.</u> For a point $x \in X$ of an algebraic scheme X we have

$$\dim_x X = 2 \dim 0_{X,x}$$

where the first symbol refers to the concept of dimension introduced in III.9.

<u>Proof.</u> Let us introduce the ad hoc notation dlc X for the concept of dimension introduced in III.9 while dim X denotes the dimension as an algebraic scheme. - Let us first prove that

$$\text{dlc } X \leq 2 \dim X$$

This will be done by induction on d = dim X. In case X is irreducible, let X_{ns} denote the non singular part of X. The complement $X_s = X - X_{ns}$ is an algebraic scheme of dimension < d. For any sheaf F on X we have the exact sequence

$$\to H_c^{2d+1}(X_{ns},F) \to H_c^{2d+1}(X,F) \to H_c^{2d+1}(X_s,F) \to$$

and the result follows from 1.1. - In the general case we shall do induction on the number s of irreducible components of X. Let Y be the union of s-1 of the irreducible components of X. For a sheaf F on X we have an exact sequence

$$\to H_c^{2d+1}(X-Y,F) \to H_c^{2d+1}(X,F) \to H_c^{2d+1}(Y,F) \to$$

and the result follows, since X-Y is irreducible of dimension $\leq$ d.

To prove the opposite inequality put d = dim $0_{X,x}$ and choose an irreducible component Z through x of dimension d. Since $\text{dlc}_x Z \leq \text{dlc}_x X$ it suffices to treat the case where X is irreducible. - Let U be an open neighbourhood of x in X. It follows from 1.2 that $U \cap X_{ns}$ is non empty. Thus we conclude from 1.1 that U contains an open subset homeomorphic to $\mathbb{R}^{2d}$ and consequently dlc U $\geq$ 2d.

Q.E.D.

X.2   The cohomology class of a subvariety

A non-singular algebraic variety has a natural orientation relative to $\mathbb{Z}$. This comes from the fact that open subsets of $\mathbb{C}^n$ inherits the natural orientation of $\mathbb{C}^n$ and the fact that complex analytic transition functions have complex Jacobi matrices and finally that complex linear transformations have positive determinants considered as real linear transformations.

Theorem 2.1. Let X be a non-singular algebraic variety and Z a closed subvariety of codimension d. Then

$$H_Z^p(X,\mathbb{Z}) = 0 \quad \text{for } p < 2d$$

Proof. By Poincaré duality with $n = \dim X$

$$H_Z^p(X,\mathbb{Z}) \xrightarrow{\sim} H_{2n-p}(Z,\mathbb{Z})$$

Conclusion by 1.4 and IX.1.6.

Q.E.D.

Remark 2.2. With the notation of 2.1 we have

$$H^i_{Z \cap U}(U,\mathbb{Z}) = 0 \quad ; \quad i < d$$

for any open subset of X as it follows from the proof of 2.1. With the terminology of VIII.1, let $i: Z \to X$ denote the inclusion. We conclude from VIII.1.3 that

$$R^p i^! \mathbb{Z} = 0 \quad ; \quad p < 2d$$

and from VIII.1.3 that the presheaf

$$U \longmapsto H_{Z \cap U}^{2d}(U,\mathbb{Z})$$

is a sheaf on X, indeed equal to $i_* R^{2d} i^! \mathbb{Z}$.

Corollary 2.3. Let X be a non-singular variety and Z a closed subvariety of codimension d. For any Zariski open subset U of X which meets Z, the map

$$H_Z^{2d}(X,\mathbb{Z}) \to H_{Z \cap U}^{2d}(U,\mathbb{Z})$$

is an isomorphism.

Proof. The excision sequence II.9.5

$$H_{Z-U}^{2d}(X,\mathbb{Z}) \to H_Z^{2d}(X,\mathbb{Z}) \to H_{Z \cap U}^{2d}(U,\mathbb{Z}) \to H_{Z-U}^{2d+1}(X,\mathbb{Z})$$

combined with 2.1 yields the result.

Q.E.D.

Definition 2.4. Let X denote a non singular variety and Z a closed subvariety of codimension d. The Thom class of Z in X

$$\tau_Z \in H_Z^{2d}(X,\mathbb{Z}) \quad ; \quad d = \operatorname{codim}(Z,X)$$

is the class whose restriction to $X-Z_s$ is the Thom class for

the oriented submanifold Z_{ns} of the oriented manifold $X-Z_s$.

- The image of τ_Z in absolute cohomology will be denoted

$$cl^X(Z) \in H^{2d}(X,\mathbb{Z}) \qquad ; \quad d = codim(Z,X)$$

or just $cl(Z)$ when no confusion is possible.

Proposition 2.5. Let X be a non-singular variety and Z a Zariski closed subset of codimenson d. If $Z_1,\ldots,Z_s$ denote the irreducible components of X of codimension d. Then the canonical map

$$\oplus H_{Z_i}^{2d}(X,\mathbb{Z}) \to H_Z^{2d}(X,\mathbb{Z})$$

is an isomorphism.

Proof. Use the Mayer-Vietoris sequence II.9.11 in combination with 2.1.

$$Q.E.D.$$

Intersection numbers

Let V and W denote closed subvarieties of the of the non-singular variety X. Recall that for any irreducible component Z of $V \cap W$ we have

2.6 $$codim\ Z \leq codim\ V + codim\ W$$

Let us assume that V and W intersects properly in X, i.e. that the inequality 2.6 is an equality for all irreducible components Z of $V \cap W$. Let V and W have codimension c and d respectively and consider the cohomology class

2.7 $$\tau_V \cup \tau_W \in H_{V \cap W}^{2(c+d)}(X,\mathbb{Z})$$

Taking the isomorphism 2.5 into account we can write

2.8 $$\tau_V \cup \tau_W = \sum_Z i(Z,V \cdot W;X) \tau_Z$$

the sum being over all irreducible components Z of $V \cap W$. The integer $i(Z,V \cdot W;X)$ is called the local intersection symbol. Push the formula 2.8 into $H^\cdot(X,\mathbb{Z})$ to get

2.9 $$cl^X(V) \cap cl^X(W) = \sum_Z i(Z,V \cdot W;X) cl^X(Z)$$

The local intersection number can be calculated by means of Serre's alternating Tor formula

2.10 $$i(Z,V \cdot W;X) = \sum_i (-1)^i \ell(Tor_i^O(O_V,O_W))$$

where O denotes the local ring of X of the generic point of Z. Compare Iversen (1).

X.3   Homology class of a subvariety

In this section we shall generalize some of the constructions from the previous section to singular spaces and Borel-Moore homology. The basic principle is that for a scheme X we have

$$3.1 \qquad H_i(X,\mathbb{Z}) = 0 \qquad ; \ i > 2\dim X$$

as it follows from 1.4 and IX.1.6. In Case X is a variety of dimension n and U a non-empty Zariski open subset we have a long exact sequence

$$H_{2n}(X-U,\mathbb{Z}) \to H_{2n}(X,\mathbb{Z}) \to H_{2n}(U,\mathbb{Z}) \to H_{2n-1}(X-U,\mathbb{Z})$$

and consequently an isomorphism

$$3.2 \qquad H_{2n}(X,\mathbb{Z}) \overset{\sim}{\to} H_{2n}(U,\mathbb{Z})$$

In particular if $U = X_{ns}$ we find a _fundamental homology class_

$$3.3 \qquad \mu_X \in H_{2n}(X,\mathbb{Z}) \qquad ; \qquad n = \dim X$$

characterized by requiring its restriction to X_{ns} to be the orientation class.

Theorem 3.4. Let X be an algebraic shceme of dimension n and let $X_1,\dots,X_s$ denote the irreducible components of dimension n The images of $\mu_{X_1},\dots,\mu_{X_s}$ in $H_{2n}(X,\mathbb{Z})$ form a basis for that group.

Proof. In case X is a non singular variety, this is a consequence of Poincaré duality and the fact that X is connected. In case X is a variety, the result follows from the iso-morphism 3.2. In the general case we have an isomorphism

$$\underset{i}{\oplus} \ H_{2n}(X_i,\mathbb{Z}) \overset{\sim}{\to} H_{2n}(X,\mathbb{Z})$$

by an excision or Mayer-Vietoris argument.

Q.E.D.

For a scheme X let $Z_k(X)$ denote the group of _k-cycles_ on X, i.e. the free abelian group based on the set of subva-rieties of X of dimension k. The cycle corresponds to a subvariety V is denoted [V]. The cycle map

$$3.5 \qquad cl_X: Z_k(X) \to H_{2k}(X,\mathbb{Z})$$

is given by the formula

$$3.6 \qquad cl_X([V]) = i_*(\mu_V)$$

where i: V → X denotes the inclusion of a closed k-dimensional subvariety.

Let f: X → Y be a proper map of algebraic schemes. For a closed subvariety V of X, the image W = f(V) is a closed subvariety of Y. Define $f_*[V] = \deg(V/W)[W]$, where

$$\deg(V/W) = \begin{cases} 0 & \text{if } \dim V < \dim W \\ [R(V): R(W)] & \text{if } \dim V = \dim W \end{cases}$$

Here   R(-)   denotes the function field. This defines

3.7 $$f_*: \ Z_\cdot(X) \longrightarrow Z_\cdot(Y)$$

The following diagram is commutative

3.8
$$
\begin{array}{ccc}
Z_k(X) & \xrightarrow{\ f_*\ } & Z_k(Y) \\
\downarrow & & \downarrow \\
H_{2k}(X,\mathbb{Z}) & \xrightarrow{\ f_*\ } & H_{2k}(Y,\mathbb{Z})
\end{array}
$$

Proof. It suffices to prove that if $f: V \to W$ is a sur-
jective and proper map of algebraic varieties then

$$f_*(\mu_V) \ = \ \deg(W/V)\,\mu_W$$

In case $\dim W < \dim V = k$ then $H_{2k}(W,\mathbb{Z}) = 0$ and the result
is clear. In case $\dim W = k$ we may replace W by an open sub-
variety and assume that $V \to W$ is a covering space of degree
$\deg(V/W)$ and the result follows from IX.11.

Q.E.D.

X.4   Intersection theory

Consider a non-singular variety X of dimension n.
Poincaré duality

$$H^p(X,\mathbb{Z}) \xrightarrow{\ \mu_X\cap\ } H_{2n-p}(X,\mathbb{Z}) \qquad ; \ p \in \mathbb{Z}$$

is a isomorphism. In IX.7 we used Poincaré duality to trans-
port stucture from homology to cohomology. In accordance
with Fulton (1) we shall here transport structure from cohomology
to homology. This has several advantages: better and more
suggestive formulas which in many cases may be generalized to
singular varities.

Let Z denote a closed subvariety of X. Then we have the
fundamental relation, compare 2.4

4.1 $$\mu_Z \ = \ \mu_X \cap \tau_Z$$

which follows from 3.3 and IX.4.9 using the second square in
IX.3.5.

Let $i: Z \to X$ denote the inclusion. From 4.1 follows

$$\mu_Z \cap i^*\xi \ = \ (\mu_X \cap \tau_Z) \cap i^*\xi \qquad ; \ \xi \in H^\cdot(X,\mathbb{Z})$$

Using IX.3.3 and II.9.17 and taking advantage of the fact that
τ_Z has even degree we get

4.2 $$\mu_Z \cap i^*\xi \ = \ (\mu_X \cap \xi) \cap \tau_Z$$

In case Z is non singular we can interpret the formula 4.2
as expressing that the Poincaré dual of $i^*: H^\cdot(X,\mathbb{Z}) \to H^\cdot(Z,\mathbb{Z})$

is given by

4.3 $$\alpha \longmapsto \alpha \cap \tau_Z \quad ; \quad H_\cdot(X,\mathbb{Z}) \to H_\cdot(Z,\mathbb{Z})$$

At the same time this suggests immediately a generalization to the case where Z is singular.

Let $f: X \to Y$ be a morphism between non-singular varieties and $\tau_\Gamma \in H_\Gamma^\cdot(X \times Y, \mathbb{Z})$ the Thom class. The following formula gives an expression for the Poincaré dual of $f^*: H^\cdot(Y,\mathbb{Z}) \to H^\cdot(X,\mathbb{Z})$ or more generally the symbol $\xi \cup f^*\eta$,

4.4 $$\mu_X \cap (\xi \cup f^*\eta) = (-1)^{pq}((\mu_X \cap \xi) \times (\mu_Y \cap \eta)) \cap \tau_\Gamma$$

where $\xi \in H^p(X,\mathbb{Z})$ and $\eta \in H^q(Y,\mathbb{Z})$.

Proof. Let $\gamma: X \to X \times Y$ denote the graph map and $\delta: X \to X \times X$ the diagonal map. Notice the formula

$$\xi \cup f^*\eta = \delta^*(\xi \times f^*\eta) = \delta^*(1 \times f)^*(\xi \times \eta) = \gamma^*(\xi \times \eta)$$

Next, apply the formula 4.2 to $\gamma: X \to X \times Y$ to get

$$\mu_X \cap \gamma^*(\xi \times \eta) = (\mu_X \times \mu_Y) \cap (\xi \times \eta) \cap \tau_\Gamma$$

$$= (-1)^{pq}((\mu_X \cap \xi) \times (\mu_Y \cap \eta)) \cap \tau_\Gamma$$

where we have used IX.5.9. The result follows by combining these two formulas.

Q.E.D.

The formula 4.4 suggest the following definition, compare Fulton (1) Ch.8

4.5 $$\alpha \cdot_f \beta = (-1)^{pq}\alpha \times \beta \cap \tau_\Gamma \quad ; \alpha \in H_p(X,\mathbb{Z}), \beta \in H_q(Y,\mathbb{Z})$$

In particular, for $f = 1_X$ the <u>intersection product</u> $\alpha \cdot \beta$ <u>is defined by</u>

4.6 $$\alpha \cdot \beta = (\beta \times \alpha) \cap \tau_X \quad ; \quad \alpha, \beta \in H_\cdot(X,\mathbb{Z})$$

The definition 4.6 may be <u>refined</u> in the following sense. Let $i: V \to X$ and $h: W \to X$ be inclusions of closed subvarieties. The symbol

$$(\beta \times \alpha) \cap \tau_X \quad ; \quad \beta \in H_\cdot(V,\mathbb{Z}), \alpha \in H_\cdot(W,\mathbb{Z})$$

defines an element of $H_\cdot(V \cap W, \mathbb{Z})$. If we push this forward into $H_\cdot(X,\mathbb{Z})$ we obtain $i_*\beta \cdot h_*\alpha$ as it follows from the projection formula IX.3.7.

Theorem 4.7. Let V and W be closed subvarieties of dimension v and w. Then

$$\mu_V \cdot \mu_W = \mu_V \cap \tau_W = \mu_W \cap \tau_V$$

in $H_{2(w+v-n)}(V \cap W, \mathbb{Z})$.

Proof. Let $\delta: X \to X \times X$ denote the diagonal map. We start from the formula II.10.8 $\tau_V \cup \tau_W = \delta^*(\tau_V \times \tau_W)$

valid in $H_{V \cap W}^{\cdot}(X, \mathbb{Z})$ and the formula

$$\mu_X = (\mu_X \times \mu_X) \cap \tau_X$$

valid in $H_{\cdot}(X, \mathbb{Z})$. Cap these together to get

$$(\mu_X \cap \tau_V) \cap \tau_W = ((\mu_X \times \mu_X) \cap \tau_X) \cap \delta^*(\tau_V \times \tau_W)$$

and using the formula 4.1, IX.3.4 and IX.5.9

$$\mu_V \cap \tau_W = (\mu_X \times \mu_X) \cap (\tau_X \cup (\tau_V \times \tau_W))$$

$$= (\mu_X \times \mu_X) ((\tau_V \times \tau_W) \cup \tau_X)$$

$$= ((\mu_X \cap \tau_V) \times (\mu_X \cap \tau_W)) \cap \tau_X$$

from which the result follows.

Corollary 4.8. If the two subvarieties V and W intersect properly then

$$\mu_V \cdot \mu_W = \sum_Z i(Z, V \cdot W; X) \mu_Z$$

in $H_{2(v+w-n)}(V \cap W, \mathbb{Z})$.

Proof. In $H_{V \cap W}^{\cdot}(X, \mathbb{Z})$ we have the formula

$$\tau_V \cup \tau_W = \sum_Z i(Z, V \cdot W; X) \tau_Z$$

and the result follows by capping this with μ_X.

Q.E.D.

Projection formula 4.9. Let $f: X \to Y$ be a proper map between non-singular varieties. Then

$$f_*(\xi \cdot_f \eta) = f_*(\xi) \cdot \eta \qquad ; \ \xi \in H_{\cdot}(X, \mathbb{Z}), \ \eta \in H_{\cdot}(Y, \mathbb{Z})$$

Proof. With the notation of 4.5 we shall prove that

4.10 $$(f \times 1_Y)^* \tau_Y = \tau_\Gamma$$

To do so consider the commutative diagram

This may be viewed as a pull back diagram of microbundles and 4.10 follows from the remarks preceding VIII.3.5. combining 4.5 and 4.10 we get

$$\xi \cdot_f \eta = \alpha \times \beta \cap (f \times 1)^* \tau_Y (-1)^{pq}$$

From the projection formula IX.3.7

$$f_*(\alpha \cdot_f \beta) = (f \times 1)_*(\alpha \times \beta) \cap \tau_Y (-1)^{pq}$$

$$= (f_* \alpha \times \beta \cap \tau_Y (-1)^{pq} = (f_* \alpha) \cdot \beta$$

where we have used the definition 4.6.

Q.E.D.

X.5 Algebraic families of cycles

Let $f: V \to T$ be a map of algebraic varities where T is non singular. Let the dimension of V and T be n and d respectively. We define the class

5.1
$$\tau_I \in H^{2d}(V \times T, \mathbb{Z})$$

to be the pull back of $\tau_T \in H_\Delta^{2n}(T \times T)$ along $f \times 1$. Let us notice that

5.2
$$\mu_V \times \mu_T \cap \tau_I = \mu_V$$

as it follows from an excision argument, compare 4.10.

Proposition 5.3. For $t \in T$ let $\tau_t \in H_{\{t\}}^{2d}(T, \mathbb{Z})$ denote the Thom class. Then we may identify

$$\mu_V \cap f^*(\tau_t) \in H_{2(n-d)}(f^{-1}(t), (\mathbb{Z}))$$

with the refined intersection class

$$\mu_T \cap p^*(\tau_t) = (\mu_V \times \mu_t) \cap \tau_T$$

where $p: V \times T \to T$ denotes the projection.

Proof. Let us first identify $\mu_V \cap f^*(\tau_t)$ with the class $(\mu_V \times \mu_t) \cap \tau_T$. To do so let $i_t: V \to V \times T$ given by $i_t(v) = (v,t)$. We must prove that

$$i_t^*(\tau_I) = f^*(\tau_t) \quad \text{in} \quad H_{f^{-1}(t)}(V, \mathbb{Z}).$$

This follows from the commutative diagram where $V_t = f^{-1}(t)$

$$
\begin{array}{ccc}
(V, V-V_t) & \xrightarrow{\ i_t\ } & (V \times T, V \times T - I) \\
\downarrow{f} & & \downarrow{f \times 1} \\
(T, T-\{t\}) & \xrightarrow[\ s \mapsto (s,t)\ ]{} & (T \times T, T \times T - \Delta)
\end{array}
$$

and the theory of the diagonal class IX.6.13. The second formula follows from

$$(\mu_V \times \mu_T) \cap \tau_I \cap (1 \times \tau_t) = (\mu_V \times \mu_T) \cap (1 \times \tau_t) \cap \tau_I$$

the formula 5.2 and $\mu_T \cap \tau_t = \mu_t$.

$$Q.E.D.$$

In case $f: V \to T$ is __proper__ we have the

__Projection formula 5.4.__

$$f_*(\alpha \cdot_f \beta) = f_*(\alpha) \cdot \beta \quad ; \alpha \in H_*(V, \mathbb{Z}), \beta \in H_*(T, \mathbb{Z})$$

as it follows from the proof of 4.9.

In case the fibers of $f: V \to T$ all have dimension $n-d$ we can define $V_t \in Z_{n-d}(V)$ by

5.5
$$cl(V_t) = \mu_V \cap f^*(\tau_t)$$

in $H_{2(n-d)}(f^{-1}(t), \mathbb{Z})$. Let us remark that

5.6 $\qquad$ $cl(V_t) \in H_{2(n-d)}(V, \mathbb{Z})$ is independent of $t \in T$.

$\underline{Proof}$. Let $r(\tau_{\{t\}}) \in H^{2d}(T, \mathbb{Z})$ denote the image of $\tau_{\{t\}}$. According to IX.3.5, 1, square, the image of $cl(V_t)$ in $H_{2(n-d)}(V, \mathbb{Z})$ is $\mu_V \cap f^*(r(\tau_t))$. Conclusion by the fact that $r(\tau_t) \in H^{2d}(T, \mathbb{Z})$ is independent of $t \in T$, as it follows from IX.6.14 and a homotopy argument.

$$Q.E.D.$$

Consider a cycle $Y \in Z_{k+d}(X \times T)$ where X is a variety of dimension n and T a non-singular variety of dimension d. For a point $t \in T$ for which

5.7 $\qquad$ $\dim X \times \{t\} \cap \text{Supp } Y = k$

we can define a k-cycle Y_t on X characterized by

5.8 $\qquad$ $cl(Y_t) \times \mu_t = cl(Y) \cap p^*(\tau_t)$

where $p: X \times T \to T$ is the projection and $\tau_t \in H_{\{t\}}^{2d}(T, \mathbb{Z})$ the local Thom class.

A family $(Z_t)_{t \in T}$ of k-cycles on X is called an $\underline{algebraic}$ $\underline{family}$ if there exists Y on $Z(X \times T)$ which satisfies 5.7 for all $t \in T$ and such that $Y_t = Z_t$ for all $t \in T$.

$\underline{Proposition\ 5.9}$. Given an algebraic family $(Z_t)_{t \in T}$ of k-cycles on the algebraic variety X. The homology class $cl(Z_t) \in H_{2k}(X, \mathbb{Z})$ is independent of $t \in T$.

$\underline{Proof}$. But $\beta = cl(Z)$. For $t \in T$ define $\beta_t \in H_.(X, \mathbb{Z})$ such that

$$\beta_t \times \mu_t = \beta \cap p^*(\tau_t)$$

where $\tau_t \in H_{\{t\}}(T, \mathbb{Z})$ is the local Thom class and $\mu_t \in H_.(\{t\}, \mathbb{Z})$ the canonical generator. In this formulation the problem becomes local on T, thus we may assume that $T = \mathbb{R}^n$. Using the Künneth formula IX.5.8 we see that β has the form $\alpha \times \mu_T$ with $\alpha \in H_.(X, \mathbb{Z})$. Thus we get

$$\beta \cap p^*(\tau_t) = (\alpha \times \mu_T) \cap (1 \times \tau_t) = \alpha \times \mu_t$$

i.e. $\beta_t = \alpha$ for all $t \in T$.

$$Q.E.D.$$

In case X is non-singular consider algebraic cycles $Y \in Z_{d+k}(X \times T)$ and $Z \in Z_{d+1}(X \times T)$ and a fixed $t \in T$ such that Y_t, Z_t and $Y_t \cdot Z_t$ are defined. Then

5.10 $\qquad$ $cl(Y_t \cdot Z_t) = (cl(Y) \cdot cl(Z))_t$

in $H_.(\text{Supp } Y_t \cap \text{Supp } Z_t, \mathbb{Z})$.

Proof. Put $\alpha = cl(Y)$ and $\beta = cl(Z)$. This gives $cl(Y_t) = \alpha \cap p^*(\tau_t)$ and $cl(Z_t) = \beta \cap p^*(\tau_t)$ and the formula reads

$$(\alpha \times \beta) \cap (p^*(\tau_t) \times p^*(\tau_t)) \cap \tau_X = (\alpha \times \beta) \cap \tau_{X \times T} \cap p^*(\tau_t)$$

Thus it suffices to prove that $\tau_{X \times T} = \tau_X \times \tau_T$ and that

$$\tau_t \times \tau_t = \tau_T \cup 1 \times \tau_t$$

which is left to the reader.

Q.E.D.

We shall now prove a result which will justify more intuitive ways of defining the local intersection symbol.

Theorem 5.11. Let X denote a non-singular variety of dimension n, $(Z_t)_{t \in T}$ an algebraic family of cycles on X and $(W_t)_{t \in T}$ an algebraic family of cycles of complementary dimension such that $\text{Supp } Z_t \cap \text{Supp } W_t$ is finite for all $t \in T$.

Given a compact subset K of X and a connected open subset D of T such that

$$\text{Supp } Z_t \cap \text{Supp } W_t \subseteq K \quad \text{for all} \quad t \in D$$

Then the sum of the local intersection numbers

$$\sum_x i(\{x\}, Z_t \cdot W_t; X)$$

is independent of $t \in D$.

Proof. It follows from the hypothesis that the cycles Z and W on $X \times T$ intersect properly. Let $W \cdot Z \in Z_0(X \times T)$ be the intersection cycle. Let Y be an irreducible component of the support of $Z \cdot W$. It suffices to prove that $\deg Y_t$ is independent of $t \in D$. Let $f: Y \to T$ denote the restriction of the projection. Notice that $f^{-1}(D) = K \times D$ as a consequence of the second assumption and consequently that $f^{-1}(D) \to D$ is proper. Let $j: f^{-1}(D) \to Y$ denote the inclusion and consider the cohomology class $j^\# \mu_Y$. According to the principle of preservation of numbers IX.10.3

$$\deg Y_t = \deg(j^\# \mu_Y \cap f^* \tau_t) \quad ; \quad t \in D$$

is independent of $t \in D$.

Q.E.D.

Example 5.12. Let W be a plane curve, $(V_t)_{t \in \mathbb{P}^1}$ the family of lines through a fixed non singular point x of W and K a compact neighbourhood of x in the plane. Let V_{t_0} be the tangent to W at x. Variations of t in a suitable neighbourhood D of t_0 will often reveal the number $i(\{x\}, W \cdot V_{t_0})$.

X.6 Algebriac cycles and Chern classes

Let X denote an algebraic variety of dimension n, $p: E \to X$ an algebraic bundle of rank d and $s: X \to E$ an algebraic section. We shall assume that

6.1
$$\dim s^{-1}(0) = n-d$$

The pull back of the Thom class VIII.6.1

$$s^* \tau_E \in H^{2(n-d)}_{s^{-1}(0)}(X, \mathbb{Z})$$

defines a homology class

$$\mu_X \cap s^* \tau_E \in H_{2(n-d)}(s^{-1}(0), \mathbb{Z}).$$

By the assumption 6.1 this determines an algebraic cycle $Z(s) \in Z_{n-d}(X)$ by the formula

6.2
$$cl_X(Z(s)) = \mu_X \cap s^* \tau_E$$

in $H_{2(n-d)}(s^{-1}(0), \mathbb{Z})$. The image of $s^* \tau_E$ in $H^{2(n-d)}(X, \mathbb{Z})$ is $c_d(E)$: Let $r: H_X^{2d}(E, \mathbb{Z}) \to H^{2d}(E, \mathbb{Z})$ denote the restriction. We have $c_d(E) = i^*(r(\tau_E))$ where $i: X \to E$ denote the zero section, VIII.6.2. On the other hand $i^* = s^*$ since p^* is an isomorphism and i^* and s^* both are left inverses to p^*. Applying XI.3.5 to 6.2 we get

6.3
$$\boxed{cl_X(Z(s)) = \mu_X \cap c_d(E)}$$

in $H_*(X, \mathbb{Z})$.

In case X is non-singular we have

6.4
$$\boxed{cl^X(Z(s)) = c_d(E)}$$

Proof. By Poincaré duality we must show that

$$\mu_X \cap cl^X(Z(s)) = \mu_X \cap c_d(E)$$

But for any algebraic cycle Z we have

$$\mu_X \cap cl^X(Z) = cl_X(Z)$$

and the result follows from 6.3.

Q.E.D.

Theorem 6.5. Let E denote a complex vector space and $\mathrm{Grass}_i(E)$ the variety of i-subplanes in E and Q the canonical quotient bundle on $\mathrm{Grass}_i(E)$. For a linear subspace F of E let $\sigma(F) \subseteq \mathrm{Grass}_i(E)$ denote the set of i-planes which meet F in a non zero subspace. If $1 \leq \mathrm{rk}\, F \leq \mathrm{rk}\, Q$ then $\sigma(F)$ is a subvariety of $\mathrm{Grass}_i(E)$ with

$$\mathrm{codim}\, \sigma(E) = \mathrm{rk}\, Q - \mathrm{rk}\, F + 1$$

and for $p = \mathrm{codim}\, F$ we have

$$c_d(Q) = cl(\sigma(F))$$

Proof. For the first part of the statement, see Hodge-Pedoe
(1) XIV.2. - The restriction of the bundle Q to the comple-
ment of $\sigma(F)$ contains F as a subbundle; consequently the
restriction of $c_p(Q)$ to the complement of $\sigma(F)$ is zero.
It follows that $c_p(Q)$ "comes from" $H_{\sigma(F)}^{2p}(\mathrm{Grass}_i(E), \mathbb{Z})$, i.e.
is a multiple of $cl(\sigma(F))$. Write

$$c_p(Q) = m \; cl(\sigma(F)) \qquad ; \; m \in \mathbb{Z}$$

In order to prove that $m = 1$ choose an $(i-1)$-plane D with
$D \cap F = 0$ and consider the canonical projection

$$p: \mathrm{Grass}_1(E/D) \longrightarrow \mathrm{Grass}_i(E)$$

Notice that $p^{-1}(\sigma(F+D/D)) = \sigma(F)$ and that these two subvarieties
have the same codimension. Thus we can write

$$p^*cl(\sigma(F)) = n \; cl(\sigma(F+D/D)) \qquad ; \; n \in \mathbb{Z}$$

Combining the two formulas we get

$$c_p(p^*Q) = mn \; cl(\sigma(F+D/D))$$

Thus we have reduced the problem to $i = 1$.

Let L denote the canonical line bundle on $P(E) = \mathrm{Gras}_1(E)$.
We have an exact sequence of bundles on $P(E)$

$$0 \longrightarrow L^{\vee} \longrightarrow E \longrightarrow Q \longrightarrow 0$$

which gives $C.(L^{\vee})c.(Q) = 1$. Writing $C.(L^{\vee}) = 1-c_1(L)$
we get immediately

$$c.(Q) = 1 + c_1(L) + c_1(L)^2 + \ldots$$

in particular $c_p(Q) = c_1(L)^p$. On the other hand $\sigma(F)$ may be
identified with $P(F)$, a linear subspace of $P(E)$ a codimension
p.

Q.E.D.

XI. Derived Categories

XI.1 Categories of fractions

Let us consider an additive category K and a system S
of morphisms subjected to conditions FR 1, 2, 3 below. To
these data we shall associate an additive category $S^{-1}K$

FR 1. If f and g are composable morphisms belonging
to S then g∘f belongs to S. The identity of every object
of K belongs to S.

FR 2. Any diagram in K with s in S
can be completed to a commutative diagram
in K with t in S.

Ditto, with all arrows reversed.

FR 3. For morphisms f,g: Y → Z in K and t: Z → W
with tf = tg, there exists s: X → Y in S with fs = gs. —
Ditto with all arrows reversed.

We shall now introduce the additive category $S^{-1}K$

Objects in $S^{-1}K$: The same as those of K.

Morphisms in $S^{-1}K$: Given objects X and Y in K:
By a right fraction (b,s) from X to Y we understand a diagram
of the form

$$X \xleftarrow{\ \ s\ \ } \cdot \xrightarrow{\ \ b\ \ } Y$$

where s is in S. Given two right fractions (a,r) and (b,s)
from X to Y. We write (a,r) ∼ (b,s) if there exists a commu-
tative diagram as below with u in S.

1.1

Let us prove that ∼ is an equivalence relation on the fractions
from X to Y. So let there be given three fractions with
(a,r) ∼ (b,s) and (b,s) ∼ (c,t). Let the former equivalence be
realized by the diagram above and the latter by the commutative
diagram below

In order to construct an equivalence between (a,r) and (c,t)
consider the following diagram which is commutative except for
the middle diamond

Explanation: Complete first u, v to a commutative square u, v, x, y with x in S. This implies $s(hx) = s(ky)$. Choose w in S with $(xh)w = (yk)w$. Consequently

$$(aixw, rixw) = (bhxw, tjyw) = (bkyw, tjyw) = (cjyw, tjyw)$$

which provides an equivalence between (a, r) and (c, t).

The morphisms or arrows in $S^{-1}K$ from X to Y are the equivalence classes of right fractions from X to Y. The equivalence class of a right fraction (a, s) is denoted a/s. The basic relation for calculation with right fractions is

1.2 $$ac/sc = a/s \qquad ; \; s \in S, \; sc \in S$$

Composition in $S^{-1}K$. Given objects X, Y, Z in K and a right fraction (a, r) from X to Y and a right fraction (b, s) from Y to Z. Consider the following diagram

1.3

$$b/s \circ a/r = bc/rt$$

where a commutative square has been filled in according to FR 3 with t in S. The arrow bc/rt from X to Z is easily seen to be independent of the way the square has been filled in. We leave it to the reader to verify that bc/rt depends only on a/r and b/s. It is also left to the reader to verify that this composition law gives rise to a category.

Addition in $S^{-1}K$. Given objects X and Y in $S^{-1}K$. It is easy to see that two morphisms from X to Y can be represented with a common denominator a/t, b/t. We put

1.4 $$a/t + b/t = (a + b)/t$$

and leave it to the reader to verify that this defines a structure of abelian group on $\mathrm{Hom}_{S^{-1}K}(X, Y)$ which makes composition in $S^{-1}K$ bilinear.

The functor $K \to S^{-1}K$. A morphism $f : X \to Y$ in K gives rise to a morphism $f/1 : X \to Y$ in $S^{-1}K$. This functor is seen to transform $X \oplus Y$ into a direct sum in $S^{-1}K$. This completes the proof of the fact that $\underline{S^{-1}K \text{ is an additive category}}$. Notice that $K \longrightarrow S^{-1}K$ is an additive functor.

Proposition 1.5. Let $U : K \to L$ be an additive functor which transforms the morphisms in S into isomorphisms in L. Then there exists a unique additive functor $V : S^{-1}K \to L$ whose composite with $K \longrightarrow S^{-1}K$ is U.

Proof. Left to the reader.

Q.E.D.

Definition 1.6. The multiplicative system S is said to be

saturated if any morphism $f: X \to Y$, for which there exists

$h: W \to X$ and $k: Y \to Z$ such that fh and kf is in S,

belongs to S.

Proposition 1.7. Let S be a saturated multiplicative system

in K. Then, a morphism $f: X \to Y$ is an isomorphism in $S^{-1}K$

if and only if f belongs to S.

Proof. If f is an isomorphism in $S^{-1}K$ then f has a

right inverse $Y \stackrel{s}{\Longleftarrow} Z \stackrel{h}{\longrightarrow} X$. A simple consideration shows that

this can be chosen such that $s = fh$ in K. The opposite relation

can be obtained from the discussion below.

Q.E.D.

Left fractions. Let X and Y be objects in K. By a

left fraction from X to Y we understand a pair of morphisms

(s,b) with s in S

$$X \stackrel{b}{\longrightarrow} \bullet \stackrel{s}{\Longleftarrow} Y$$

This gives rise to a morphism

1.8 $\qquad\qquad s\diagdown b = (1/s) \circ (b/1)$

It follows from FR 2 that any morphism in $S^{-1}K$ can be repre-

sented by a left fraction. Two left fractions (r,a) and (s,b)

give $r\diagdown a = s\diagdown b$ if and only if there exists a commutative

diagram

1.9

with t in S. – Let us reproduce the diagram for composing left

fractions

1.10

$$s\diagdown b \circ r\diagdown a = ts\diagdown ca$$

XI.2 The derived category, D(A)

Let A denote an additive category and K(A) the homotopy category of complexes over A. Let us recall the basic properties of the triangles in K(A).

TR1 Any diagram isomorphic to a triangle is a triangle. Any morphism can be completed to a triangle. The diagram (X,X,0,1,0,0) is a triangle.

TR2 A diagram of the form

is a triangle if and only if the diagram

is a triangle.

TR3 Given a diagram

consisting of two triangles and morphisms f and g making the bottom square commutative. Then the dotted arrow may be filled in making the remaining two squares commutative.

Remark 2.1. The dotted arrow in TR3 is not unique in general. Here is a collection of examples

Any dotted arrow will meet the requirements.

Let us use TR 1.2.3 to prove two familiar results about triangles.

Proposition 2.2. From an object X and a triangle

result two exact sequences

$$\longrightarrow [X^{"},E^{\cdot}] \longrightarrow [X^{"},F^{"}] \longrightarrow [X^{"},G^{"}] \longrightarrow [X^{"},E^{"}[1]] \longrightarrow$$

$$\longrightarrow [G^{"},X^{"}] \longrightarrow [F^{"},X^{"}] \longrightarrow [E^{"},X^{"}] \longrightarrow [G^{"}[-1],X^{"}] \longrightarrow$$

Proof. We shall restrict ourselves to prove exactness of the following sequence

$$[X^{"},E^{"}] \rightarrow [X^{"},F^{"}] \rightarrow [X^{"},G^{"}]$$

Let us first notice that $g \circ e = 0$ as it follows from

$$
\begin{array}{ccc}
G" & \xrightarrow{\quad 1 \quad} & G" \\
\end{array}
$$

Given $x \in [X",F"]$ with $ex = 0$ then we can find $y \in [X",E"]$

with $x = gy$ from the following diagram

Proposition 2.3. Two triangles

give rise to a direct sum

which again is a triangle.

Proof. Complete $g \oplus t$ to a triangle $(\ldots,W,g \oplus t)$. Next

choose $i: G" \to W"$ to make the following diagram commutative

Construct $j: T" \to W"$ in a similar way. Finally prove that

$(i,j) : G" \oplus T" \to W"$ is an isomorphism by using the following

commutative diagram

Proposition 2.2 and the 5-lemma.

Q.E.D.

Let us now assume that A is an abelian category. We shall

construct the derived category $D(A)$ of A on the basis of the

following proposition.

Proposition 2.4. The class Q of $K(A)$ consisting of

all quasi-isomorphisms is a multiplicative system in $K(A)$,

i.e. satisfies FR 1,2,3 of the previous section.

Q.E.D.

Proof. The axiom FR1 is clearly satisfied. The proof of FR2 is based on the observation that given a triangle

$$\begin{array}{ccc} & S^{\cdot} & \\ {}^{y}\swarrow & & \nwarrow^{x} \\ X^{\cdot} & \xrightarrow{\quad s \quad} & Y^{\cdot} \end{array}$$

then s is in Q if and only if $H^{\cdot}(S^{\cdot}) = 0$, compare I.5.5. To prove FR2 let there be given morphisms

$$E^{\cdot} \longrightarrow Y^{\cdot} \Longleftarrow X^{\cdot} \qquad ; \quad s \ \text{in} \ Q$$

On this basis we can construct the following diagram

where we have used TR1 to construct the triangle $D^{\cdot}E^{\cdot}S^{\cdot}$ and TR3 to fill in the dotted arrow. The proof of the dual statement is similar.

To prove FR3 let there be given morphisms $f,g: X^{\cdot} \to Y^{\cdot}$ and $t: Y^{\cdot} \to Z^{\cdot}$ in Q with $tf = tg$. Put $h = g-f$ to get $th = 0$. We are looking for $s: W^{\cdot} \to X^{\cdot}$ in Q with $hs = 0$.

Let us first complete $t: Y^{\cdot} \to Z^{\cdot}$ to a triangle $Y^{\cdot}Z^{\cdot}T^{\cdot}$, next construct the diagram

Explanation: Since $th = 0$ we can choose an arrow $k: X^{\cdot} \to T^{\cdot}$ with $zk = h$. Next, fill in the triangle $X^{\cdot}T^{\cdot}K^{\cdot}$ at random. Notice that s is in Q and that $hs = (zk)s = 0$.

$$Q.E.D.$$

The category obtained from $K(A)$ by inverting the class Q of quasi-isomorphisms is the derived category $D(A)$ of A

$$2.5 \qquad D(A) = Q^{-1}K(A)$$

Quite similarly we can invert the class of all quasi-isomorphisms in $K^{+}(A)$, the homotopy category of bounded below complexes over A to get

$$2.6 \qquad D^{+}(A) = Q^{-1}K^{+}(A)$$

Proposition 2.7. Let A be an abelian category and I the full subcategory of injective objects. If every object of A admits a monomorphism into an object of I, then

$$D^{+}(A) \xrightarrow{\ \sim\ } K^{+}(I)$$

the homotopy category of bounded below injectives.

Proof. Let I denote the additive category of injective objects in A and consider the resolution functor

$$\rho: K^{+}(A) \to K^{+}(I)$$

The class Q is transformed into isomorphism which allows us to

factor ρ through $D^+(A)$ to obtain

$$r: D^+(A) \longrightarrow K^+(I)$$

The inclusion $I \rightarrow A$ will induce a functor

$$i: K^+(I) \longrightarrow D^+(A)$$

which is a right adjoint to r, I.6.6

$$[X^{"},iI^{"}] = [rX^{"},I^{"}]$$

Let us notice that the adjunction morphism

$$r\,i\,I^{"} \longrightarrow I^{"}$$

is an isomorphism. From this we conclude that the functor
$i: K^+(I) \rightarrow D^+(A)$ is fully faithful, i.e. gives isomorphisms

$$[I_1,I_2] \xrightarrow{\sim} [i_*I_1,i_*I_2]$$

Thus it follows from I.6.1 that i is an equivalence of categories.
The same is true for r, r being a left adjoint to i.

$$Q.E.D.$$

Let us record the dual notions. The result of inverting
the quasi-isomorphisms in $K^-(A)$ is denoted

$$2.6^O \qquad\qquad D^-(A) = Q^{-1}K^-(A)$$

Let P denote the full subcategory of projective objects in A.
If A has enough projectives, then

$$2.7^O \qquad\qquad D^-(A) \xrightarrow{\sim} K^-(P)$$

Triangles in $D(A)$. Let us first remark that the trans-
lation functor $X^{"} \rightarrow X^{"}[1]$ gives rise to an endofunctor of
$D(A)$, 1.5. By a triangle in $D(A)$ we understand a diagram iso-
morphic to the transform of a triangle from $K(A)$.

Proposition 2.8. The triangles in $D(A)$ has the proper-
ties TR1,2,3.

Proof. Only TR3 needs to be treated seriously. We leave
it to the reader to check that it suffices to take as point of
departure a commutative diagram in $K(A)$ of the form

We can now fill in arrows $F^{"}C^{"}$ and $F^{"}Z^{"}$ according to TR3. The
arrow $F^{"}C^{"}$ is necessarily a quasi-isomorphism as it follows
from the long exact homology ladder resulting from the morphism
$D^{"}E^{"}F^{"} \rightarrow A^{"}B^{"}C^{"}$ and the five lemma.

$$Q.E.D.$$

Homology in $\mathcal{D}(A)$. The homology functor

$$H^0: K(A) \longrightarrow A$$

transforms the multiplicative system Q into isomorphisms. Thus by 1.5 the homology functor may be factored through $D(A)$. The resulting functor will be denoted

$$H^0: D(A) \longrightarrow A$$

Given a triangle in $D(A)$

the resulting sequence in A

2.9 $\qquad H^0(E^{\cdot}) \xrightarrow{H^0(g)} H^0(F^{\cdot}) \xrightarrow{H^0(e)} H^0(G^{\cdot}) \xrightarrow{H^0(f)} H^0(E^{\cdot}[1]) \xrightarrow{H^0(g[1])} H^0(F[1])$

is exact.

Ext groups. For complexes $X^{\cdot}$ and $Y^{\cdot}$ over we define for each integer n

2.10 $\qquad \mathrm{Ext}^n(X^{\cdot},Y^{\cdot}) = \mathrm{Hom}(X^{\cdot},Y^{\cdot}[n])$

where Hom is calculated in $D(A)$. Define

2.11 $\qquad \alpha \cup \beta = \alpha[q] \circ \beta$

for $\quad \alpha \in \mathrm{Ext}^p(Y^{\cdot},Z^{\cdot}) \quad$ and $\quad \beta \in \mathrm{Ext}^q(X^{\cdot},Y^{\cdot})$.

Let us note a case where we can identify $D(A)$

Proposition 2.12. Let A be an abelian category in which every object has an injective resolution of length n, where n is a fixed integer. Then

$$D(A) = K(I)$$

where I denotes the category of injective objects in A.

Proof. According to I.7.7 every object of $K(A)$ admits a quasi-isomorphism into a complex of injective objects. Thus it suffices to prove that any quasi-isomorphism $f: I^{\cdot} \to J^{\cdot}$ in $K(I)$ is a homotopy equivalence. It suffices to treat the case where $J^{\cdot} = 0$ by a mapping cone argument. To prove that $I^{\cdot}$ is homotopic to zero it suffices to prove that $\mathrm{Ker}\, \partial^p$ is injective for all $p \in \mathbb{Z}$. By translation it suffices to treat the case $p = 0$. From the long exact sequence

$$0 \to \mathrm{Ker}\, \partial^{-n} \to I^{-n} \to \dots I^{-n} \to \mathrm{Ker}\, \partial^0 \to 0$$

we derive for each object N in A an isomorphism

$$\mathrm{Ext}^1(N,\mathrm{Ker}\, \partial^0) \xrightarrow{\sim} \mathrm{Ext}^{n+1}(N,\mathrm{Ker}\, \partial^n)$$

Hence $\mathrm{Ext}^1(N,\mathrm{Ker}\, \partial^0) = 0$ for all N in A shows that $\mathrm{Ker}\, \partial^0$ is an injective object. For the sake of completness let us show that

2.13 $[X^{\cdot},I^{\cdot}] = 0$ for $I^{\cdot}$ in $K(I)$ and

$\quad\quad\quad\quad\quad\quad$ $X^{\cdot}$ in $K(A)$ with $H^{\cdot}(X^{\cdot}) = 0$

So let there be given $f: X^{\cdot} \to I^{\cdot}$ in $K(A)$.   According to FR2

applied to f and $X^{\cdot} \to 0$ we can find a quasi-isomorphism

$k: I^{\cdot} \to J^{\cdot}$ such that $kf = 0$.   By the first part of the proof

we may assume that $J^{\cdot}$ is in $K(I)$ and by the second part k

is a homotopy equivalence. Whence $f = 0$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ Q.E.D.

XI.3 Triangle associated to an exact sequence

$\quad\quad$ Let us consider an abelian category A and the correspond-

ing derived category $D(A)$ constructed in the previous section.

To a short exact sequence of complexes

3.1 $\quad\quad\quad\quad\quad\quad\quad$ $0 \to P^{\cdot} \overset{u}{\to} Q^{\cdot} \overset{v}{\to} R^{\cdot} \to 0$

we are going to associate a triangle in $D(A)$

3.2

$$
\begin{array}{ccc}
& R^{\cdot} & \\
{}^{w}\swarrow & & \nwarrow{}^{v} \\
P^{\cdot} & \underset{u}{\longrightarrow} & Q^{\cdot}
\end{array}
$$

The arrow $w: R^{\cdot} \to P^{\cdot}[1]$ will sometimes be referred to as the

characteristic arrow of the short exact sequence 3.1.

$\quad\quad$ Construction: Let us notice that $(0,v): P[1] \oplus Q \to R$ in

fact defines a morphism of complexes

$$(0,v): \mathrm{Con}^{\cdot}(u) \longrightarrow R^{\cdot}$$

We are going to prove that this is a quasi-isomorphism. As in

the proof of I.4.15 consider the exact commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P^{\cdot} & \overset{\binom{0}{1}{0}}{\longrightarrow} & \mathrm{Cyl}^{\cdot}(u) & \overset{\left(\begin{smallmatrix}-1 & 0 & 0 \\ 0 & 0 & 1\end{smallmatrix}\right)}{\longrightarrow} & \mathrm{Con}^{\cdot}(u) & \longrightarrow & 0 \\
& & \downarrow{}^{1} & & \downarrow{}^{(0,u,1)} & & \downarrow{}^{(0,v)} & & \\
0 & \longrightarrow & P^{\cdot} & \underset{u}{\longrightarrow} & Q^{\cdot} & \underset{v}{\longrightarrow} & R^{\cdot} & \longrightarrow & 0
\end{array}
$$

It follows from the discussions preceding I.4.14 that the vertical

arrow in the middle is a homotopy equivalence. Now form the
long homology ladder and apply the 5-lemma to conclude that
$(0,v)$ is a quasi-isomorphism.

We can now define the characteristic arrow $w: R^. \to P^{"}[1]$
in $D(A)$ to be the fraction

3.3
$$w: R^{"} \xleftarrow{(0,v)} Con^{"}(u) \xrightarrow{(1,0)} P^{"}[1]$$

We can conclude that 3.2 is a triangle in $D(A)$ by referring
to the standard mapping cone triangle I.4.14.

Functoriality 3.4. Given an exact commutative diagram of
complexes in A

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P^{"} & \xrightarrow{u} & Q^{"} & \xrightarrow{v} & R^{"} & \longrightarrow & 0 \\
 & & \downarrow p & & \downarrow q & & \downarrow r & & \\
0 & \longrightarrow & E^{"} & \xrightarrow{f} & F^{"} & \xrightarrow{g} & G^{"} & \longrightarrow & 0
\end{array}
$$

Then (p,q,r) is a morphism from the triangle associated the
upper row to the triangle associated the lower row.

Proof. This results immediately from the following diagram
of complexes in A

$$
\begin{array}{ccccc}
R^{"} & \longleftarrow & Con^{"}(u) & \longrightarrow & P^{"}[1] \\
\downarrow r & & \downarrow \begin{pmatrix} p[1] & 0 \\ 0 & q \end{pmatrix} & & \downarrow p[1] \\
G^{"} & \longleftarrow & Con^{"}(f) & \longrightarrow & E^{"}[1]
\end{array}
$$

which is commutative.

<div align="right">Q.E.D.</div>

The formula 3.3 represents w as a right fraction. The
following formula represents w as a <u>left fraction</u>

3.5
$$w: R^{.} \xrightarrow{-\binom{0}{1}} Con^{"}(v) \xleftarrow{\binom{u[1]}{0}} P[1]$$

Proof. Consider the following diagram

$$
\begin{array}{ccccccc}
Q^{"} & \xrightarrow{-\binom{0}{1}} & Con^{"}(u) & \xrightarrow{(1,0)} & P^{.}[1] & \xrightarrow{u[1]} & Q^{"}[1] \\
\downarrow 1 & & \downarrow (0,v) & & \downarrow \binom{u[1]}{0} & & \downarrow 1 \\
Q^{"} & \xrightarrow{-v} & R^{"} & \xrightarrow{-\binom{0}{1}} & Con^{"}(v) & \xrightarrow{(1,0)} & Q^{.}[1]
\end{array}
$$

Notice that the two rows represents triangles. In fact the dia-
gram is a morphism of triangles in $K(A)$ since the middle square
is homotopy commutative

$$
\binom{u[1]}{0}(1,0) - \binom{0}{-1}(0,v) = \binom{u[1] \quad 0}{0 \quad v} = \begin{pmatrix} -\partial & 0 \\ -v & \partial \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\partial & 0 \\ -u & \partial \end{pmatrix}
$$

From the morphism of triangles results a homology ladder. Apply
the 5-lemma to this to conclude that $\binom{u[1]}{0}$ is a quasi-isomorphism.
Thus 3.5 represents indeed a left fraction. To see that this
equals 3.3 we have to appeal once more to the homotopy commutative
square just established.

<div align="right">Q.E.D.</div>

The 3 × 3 diagram. Let us consider a commutative exact diagram in the category of complexes over A of the form

3.6

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1^{\cdot \cdot} & \xrightarrow{x} & A_2^{\cdot} & \xrightarrow{y} & A_3^{\cdot \cdot} & \longrightarrow & 0 \\
& & \downarrow a_1 & & \downarrow a_2 & & \downarrow a_3 & & \\
0 & \longrightarrow & B_1^{\cdot \cdot} & \xrightarrow{z} & B_2^{\cdot \cdot} & \xrightarrow{u} & B_3^{\cdot \cdot} & \longrightarrow & 0 \\
& & \downarrow b_1 & & \downarrow b_2 & & \downarrow b_3 & & \\
0 & \longrightarrow & C_1^{\cdot \cdot} & \xrightarrow{v} & C_2^{\cdot} & \xrightarrow{w} & C_3^{\cdot \cdot} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Let w_1, w_2, w_3 denote the characteristic arrows of the three columns and w_a, w_b, w_c the characteristic arrows of the rows. Then

3.7

$$\boxed{w_a[1] \circ w_3 = -w_1[1] \circ w_c}$$

or otherwise expressed: the diagram below is a morphism of triangles in $D(A)$.

3.8

Proof. Consider the following commutative diagram of complexes over A

3.9

The sequence in the middle is exact: check directly that wb_2 is a kernel for $(a_2, -z)$ and that $\begin{pmatrix} x \\ a_1 \end{pmatrix}$ is a kernel for $(a_2, -z)$.

Let us make a general remark concerning an exact sequence of complexes of the form

3.10
$$0 \longrightarrow D_1^{\cdot} \longrightarrow D_2^{\cdot \cdot} \longrightarrow D_3^{\cdot \cdot} \longrightarrow D_4^{\cdot \cdot} \longrightarrow 0$$

By factoring the middle arrow in an epimorphism followed by a monomorphism we obtain two short exact sequences of complexes

$$0 \to D_1^{\cdot \cdot} \to D_2^{\cdot \cdot} \to E^{\cdot} \to 0, \quad 0 \to E^{\cdot} \to D_3^{\cdot} \to D_4^{\cdot} \to 0$$

Let $u: E^{\cdot} \to D_1^{\cdot \cdot}[1]$ and $v: D_4^{\cdot} \to E^{\cdot}[1]$ denote respective characteristic arrows. The composite

$$u[1] \circ v: D_4 \longrightarrow D_1[2]$$

will be called the characteristic class of the sequence 3.10. Notice that this construction is functorial in a sense similar to 3.4. If we apply this to the diagram 3.9 we see that the top row and the bottom row has the same characteristic arrow. The characteristic arrow of the top row is $w_a[1] \circ w_3$ while the characteristic class of the bottom row is $-w_1[1] \circ w_c$.

Q.E.D.

XI.4 Yoneda extensions

Let us consider two objects M and N of an abelian category A. We shall consider n-fold extensions of M by N, i.e. exact sequences of the form

4.1
$$0 \to N \to R_{n-1} \to R_{n-2} \to \cdots \to R_0 \to M \to 0$$

An n-fold extension gives rise to a complex

4.2
$$R.: 0 \to N \to R_{n-1} \to \cdots \to R_1 \to R_0 \to 0$$

and two morphisms of complexes over A

4.3
$$M \xleftarrow{s} R. \xrightarrow{f} N[n]$$

As indicated, the morphism s is a quasi-isomorphism, so that f/s defines an arrow in $D(A)$. The characteristic class of the extension 4.1 is by definition

4.4
$$(-1)^{n(n+1)/2} \; f/s \in \mathrm{Ext}^n(M,N)$$

The reason for the sign will appear shortly.

Proposition 4.5. In case $n = 1$ the characteristic class of the extension $0 \to N \to R_0 \to M \to 0$ coincides with the third side in the triangle associated to the exact sequence, 3.2.

Proof. Consider an extension of M by N
$$0 \to N \xrightarrow{u} R_0 \xrightarrow{v} M \to 0$$

The rule from 3.3 assigns the fraction

$$M \xLeftarrow{(0,v)} \mathrm{Con}^{\cdot}(u) \xrightarrow{(1,0)} N[1]$$

Notice that $\mathrm{Con}^{\cdot}(u)$ and the complex $R.$ from 4.2 only differs by the sign of the differential. Consider the commutative diagram

From which we conclude that the fraction above equals $-f/s$ with the notation of 4.3. This is consistent with 4.4 since $(-1)^{n(n+1)/2} = -1$ in case $n = 1$.

Q.E.D.

Proposition 4.6. Given extensions in A

$$0 \longrightarrow P \longrightarrow S_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow S_1 \xrightarrow{\partial_1} S_0 \xrightarrow{\lambda} N \longrightarrow 0$$

$$0 \longrightarrow N \xrightarrow{\mu} R_{m-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow R_1 \xrightarrow{\partial_1} R_0 \longrightarrow M \longrightarrow 0$$

with characteristic classes $\theta \in \mathrm{Ext}^m(N,P)$ and $\varphi \in \mathrm{Ext}^n(M,N)$ respectively. Then the spliced extension

$$0 \longrightarrow P \longrightarrow S_{m-1} \longrightarrow \cdots \longrightarrow S_0 \xrightarrow{\mu\lambda} R_{n-1} \longrightarrow \cdots \longrightarrow R_0 \longrightarrow M \longrightarrow 0$$

has characteristic class

$$\theta \cup \varphi \in \mathrm{Ext}^{n+m}(M,P)$$

Proof. Let us define three complexes

$$S. \qquad 0 \longrightarrow P \xrightarrow{\nu} S_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow S_1 \xrightarrow{\partial_1} S_0 \longrightarrow 0$$

$$R. \qquad 0 \longrightarrow N \xrightarrow{\mu} R_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow R_1 \xrightarrow{\partial_1} R_0 \longrightarrow 0$$

$$T. \qquad 0 \longrightarrow P \longrightarrow S_{m-1} \rightarrow \cdots S_0 \xrightarrow{\mu\lambda} R_{n-1} \rightarrow R_0 \longrightarrow 0$$

Let us first exhibit a quasi-isomorphism $u: T. \Rightarrow R.$

$$
\begin{array}{ccccccccccc}
0 & \to & P & \to & S_{m-1} & \cdots & S_1 & \to & S_0 & \xrightarrow{\mu\lambda} & R_{n-1} & \to & & \to & R_0 & \to & 0 \\
 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow \lambda & & \downarrow 1 & & & & \downarrow 1 & & \\
0 & \to & 0 & \to & 0 & \cdots & 0 & \to & N & \xrightarrow{\mu} & R_{n-1} & \to & & \to & R_0 & \to & 0
\end{array}
$$

and a morphism $h: T. \to S.[n]$

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & P & \xrightarrow{\nu} & S_{m-1} & \xrightarrow{\partial_{m-1}} & \cdots & \longrightarrow & S_1 & \xrightarrow{\partial_1} & S_0 & \xrightarrow{\mu\lambda} & R_{n-1} & \longrightarrow & R_{n-2} & \longrightarrow \\
 & & \downarrow {\scriptstyle (-1)^{mn}} & & \downarrow {\scriptstyle (-1)^{(m-1)n}} & & & & \downarrow {\scriptstyle (-1)^n} & & \downarrow 1 & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P & \xrightarrow{(-1)^n \nu} & S_{m-1} & \xrightarrow{(-1)^n \partial_{m-1}} & \cdots & \longrightarrow & S_1 & \xrightarrow{(-1)^n \partial_1} & S_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow
\end{array}
$$

Let the two extensions give rise to diagrams

$$N \xleftarrow{\;t\;} S. \xrightarrow{\;g\;} P[m]$$

$$M \xleftarrow{\;s\;} R. \xrightarrow{\;f\;} N[n]$$

Then we have the following commutative diagram

$$
\begin{array}{ccc}
 & T & \\
{\scriptstyle u}\swarrow\!\!\!\Downarrow & & \searrow{\scriptstyle h} \\
R & & S[n] \\
{\scriptstyle s}\swarrow\!\!\!\Downarrow \quad \searrow{\scriptstyle f} & & \Downarrow{\scriptstyle t[n]} \quad \searrow {\scriptstyle (-1)^n g[n]} \\
M \qquad\qquad N[n] & & P[m+n]
\end{array}
$$

where the data 4.3 relative to the spliced sequence is represented by the composites su and $(-1)^{nm}g[n]h$. Thus the spliced sequence has characteristic class

$$(-1)^{(n+m)(n+m+1)/2}(-1)^{nm}g[n]/t[n] \circ f/s$$

We shall finally involve the elementary identity

$$m(m+1)/2 + n(n+1)/2 = (m+n)(m+n+1)/2 - mn$$

and the formula follows.

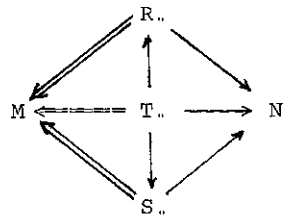Q.E.D.

Proposition 4.7. Two n-fold extensions in A

$$0 \to N \to R_{n-1} \to \cdots \to R_1 \to R_0 \to M \to 0$$

$$0 \to N \to S_{n-1} \to \cdots \to S_1 \to S_0 \to M \to 0$$

has the same characteristic class if and only if there exists a commutative exact diagram of the form

$$0 \to N \to R_{n-1} \to \cdots \to R_1 \to R_0 \to M \to 0$$
$$0 \to N \to T_{n-1} \to \cdots \to T_1 \to T_0 \to M \to 0$$
$$0 \to N \to S_{n-1} \to \cdots \to S_1 \to S_0 \to M \to 0$$

The same statement is true with the vertical arrows reversed.

Proof. Given a diagram as above. Let $R_{.}$, $S_{.}$ and $T_{.}$ denote the corresponding complexes in the sense of 4.3. We deduce a commutative diagram

$$\begin{array}{ccc} & R_{.} & \\ M & T_{.} & N \\ & S_{.} & \end{array}$$

from which we conclude that the two relevant right fractions are identical. - Suppose conversely that the two extensions have the same characteristic class. Then we deduce the existence of a diagram as above where T'' is some complex. In the diagram above we can replace T'' by the truncation $\tau_{\leq 0}\tau_{\geq -n}T''$ without disturbing the diagram. This gives a diagram of the promised sort.

In order to prove the statement relative to the reversed vertical arrows we need a representation of the characteristic class by a left fraction. This is accomplieshed in 4.9 below.

Q.E.D.

Proposition 4.8. Let M and N be objects in the abelian category A, and $n \in \mathbb{N}$. Any element of $\text{Ext}^n(M,N)$ can be realized as the characteristic class of an n-fold extension of M by N.

Proof. An element of $\text{Ext}^n(M,N)$ can be realized as a right fraction f/s

$$M \xleftarrow{\quad s \quad} R_{.} \xrightarrow{\quad f \quad} N[n]$$

Replace $R_{.}$ by the truncation $\tau_{\leq 0}\tau_{\geq -n}R_{.}$ to obtain the desired n-fold extension of M by N.

Q.E.D.

Representation by a left fraction. Let there be given an n-fold extension of M by N as in 4.1. Let us introduce the complex

$$S_{.}: \quad 0 \to R_{n-1} \to R_{n-2} \to \cdots \to R_0 \to M \to 0$$

and the two morphisms

$$M \xrightarrow{\quad g \quad} S_{.} \xleftarrow{\quad t \quad} N[n]$$

With the notation of 4.3 let us prove the formula

4.9 $$f/s = (-1)^n \, t \diagdown g$$

Proof. We shall prove that the following diagram

$$
\begin{array}{ccc}
R. & \xrightarrow{\ 1\ } & R. \\
\| s & & \downarrow f \\
M & & N[n] \qquad (-1)^n tf = gs \\
\downarrow g & & \| t \\
S. & \xrightarrow{(-1)^n} & S.
\end{array}
$$

is homotopy commutative. Put $\theta_0 = gs$ and $\theta_n = tf$. For $i = 1, \ldots, n-1$ let $\theta_i : R. \to S.$ be given by

$$
\begin{array}{c}
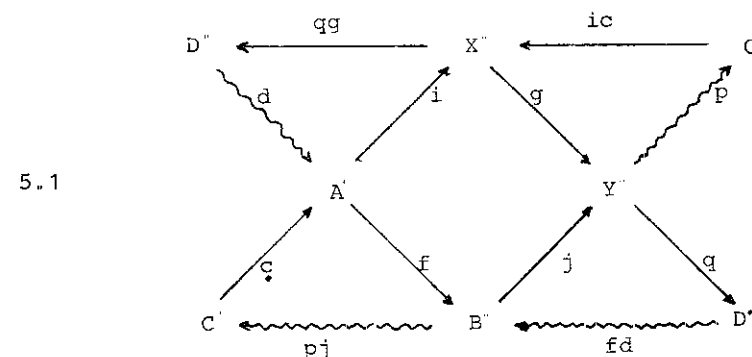N \longrightarrow R_{n-1} \longrightarrow \cdots R_{i+1} \xrightarrow{r_{i+1}} R_i \xrightarrow{r_i} R_{i-1} \longrightarrow \cdots R_1 \xrightarrow{r_1} R_0 \\
\end{array}
$$

with vertical maps 0, the dashed arrow labelled $(-1)^i$, $(-1)^i r_i$, and

$$
R_{n-1} \xrightarrow{r_{n-1}} R_{n-2} \longrightarrow \cdots R_i \xrightarrow{r_i} R_{i-1} \xrightarrow{r_{i-1}} R_{i-2} \longrightarrow \cdots R_0 \xrightarrow{r_0} M
$$

The single dotted arrow above provides a homotopy from θ_i to θ_{i+1}. A simple induction will complete the proof.

Q.E.D.

XI.5. Octahedra

Let A denote an additive category by an octahedra in $K = K(A)$ we understand a diagram of the form
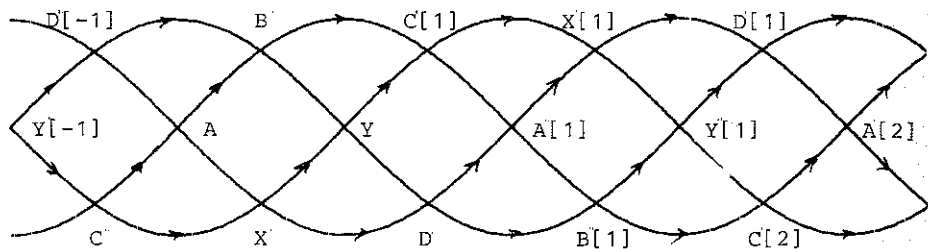
5.1

consisting of four triangles and such that

5.2 $$ gi = jf \qquad cp = dq $$

The first of these relations expresses commutativity of the middle diamond in 5.1. The second relation is nicely realized by reflecting the diagram 5.1 in the line AY and attaching it to the old diagram
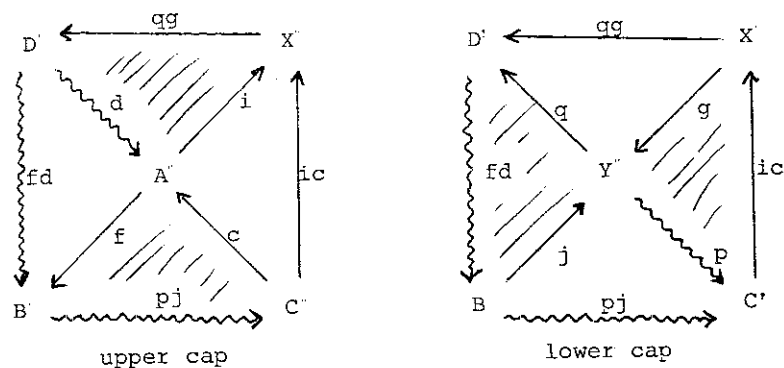

Alternatively the diagram can be represented as a <u>braid</u>

This representation reveals that an octahedron contains four morphisms of triangles.

To explain the name of the diagram 5.1 we shall represent it as an octahedron

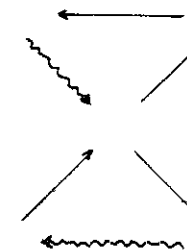upper cap   lower cap

where the shaded areas represent triangles and the unshaded areas represent commutative diagrams. To the general octahedron we can associate two sequences

5.3 $\qquad A^{\cdot} \xrightarrow{\binom{i}{-f}} X^{\cdot} \oplus B^{\cdot} \xrightarrow{(g,j)} Y^{\cdot} \xrightarrow{\quad cp \quad} A^{\cdot}[1]$

5.4 $\qquad A^{\cdot} \xrightarrow{gi} Y^{\cdot} \xrightarrow{\binom{-p}{q}} C^{\cdot}[1] \oplus D^{\cdot} \xrightarrow{(c,d)} A^{\cdot}[1]$

which we call the Mayer-Vietoris sequences of the octahedron.

<u>Theorem 5.5.</u> Any diagram consisting of two triangles with a common vertex can be completed to an octahedron whose Mayer-Vietoris sequences are triangles.

<u>Proof.</u> The triangle I.4.22 can be completet to the diagram

which is easily verified to be an octahedron.

To establish the Mayer-Vietoris sequence 5.3 we notice that the chainwise split exact sequence

$$Y^{\cdot} \xrightarrow{\binom{0}{-1}{v}} \mathrm{Con}^{\cdot}(u) \oplus Z^{\cdot} \xrightarrow{\binom{100}{0v1}} \mathrm{Con}^{\cdot}(vu)$$
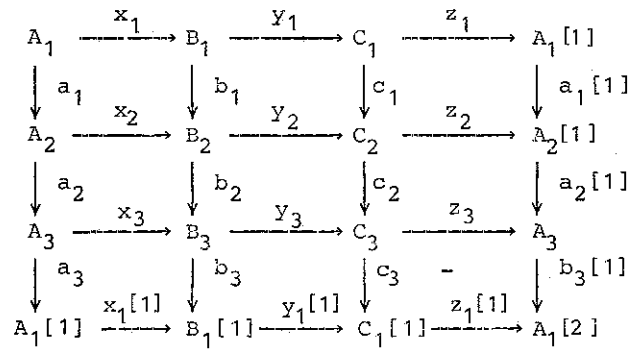
has homotopy invariant $(u,0)$ as it follows from

$$\begin{pmatrix} -\partial 0 0 \\ -u\partial 0 \\ 0 0 \partial \end{pmatrix}\begin{pmatrix} 1 0 \\ 0 0 \\ 0 1 \end{pmatrix} - \begin{pmatrix} 1 0 \\ 0 0 \\ 0 1 \end{pmatrix}\begin{pmatrix} -\partial & 0 \\ -vu & \partial \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -u & 0 \\ vu & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ v \end{pmatrix}(u,0)$$
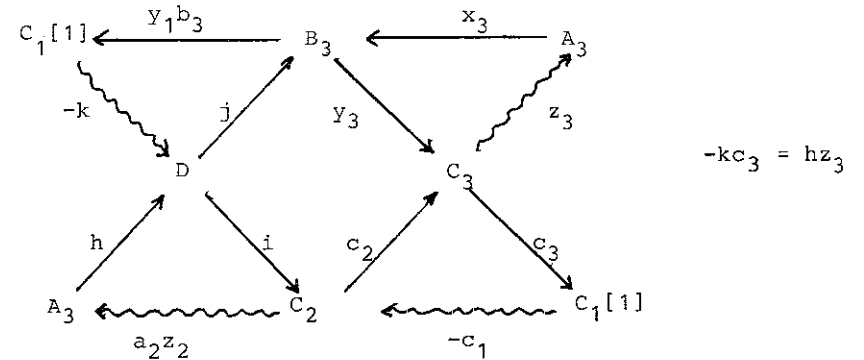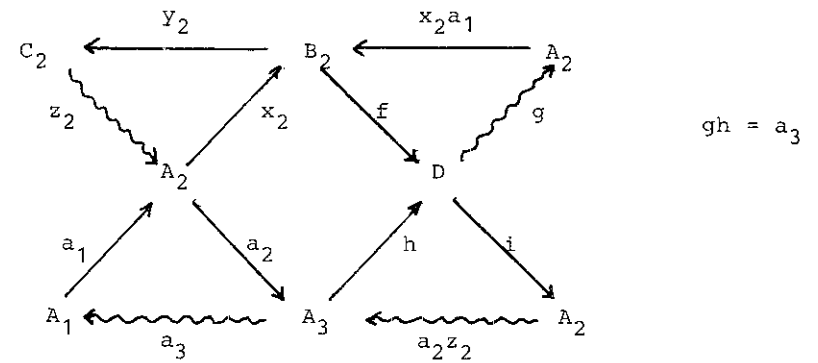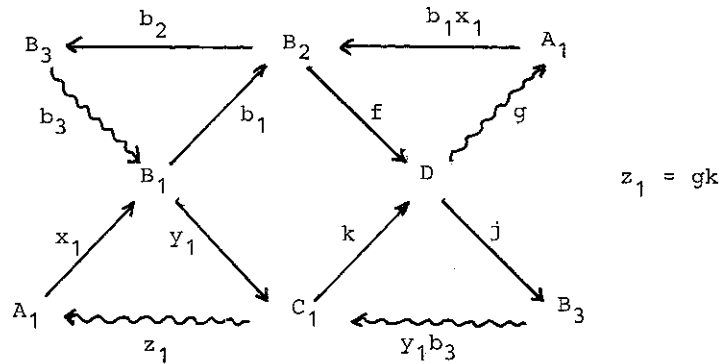
We leave the sequence 5.4 to the reader.

Q.E.D.

Proposition 5.6. Any commutative square $A_1A_2B_1B_2$ can be completed to a 9 diagram

$$
\begin{array}{ccccccc}
A_1 & \xrightarrow{x_1} & B_1 & \xrightarrow{y_1} & C_1 & \xrightarrow{z_1} & A_1[1] \\
\downarrow a_1 & & \downarrow b_1 & & \downarrow c_1 & & \downarrow a_1[1] \\
A_2 & \xrightarrow{x_2} & B_2 & \xrightarrow{y_2} & C_2 & \xrightarrow{z_2} & A_2[1] \\
\downarrow a_2 & & \downarrow b_2 & & \downarrow c_2 & & \downarrow a_2[1] \\
A_3 & \xrightarrow{x_3} & B_3 & \xrightarrow{y_3} & C_3 & \xrightarrow{z_3} & A_3 \\
\downarrow a_3 & & \downarrow b_3 & & \downarrow c_3 & - & \downarrow b_3[1] \\
A_1[1] & \xrightarrow{x_1[1]} & B_1[1] & \xrightarrow{y_1[1]} & C_1[1] & \xrightarrow{z_1[1]} & A_1[2]
\end{array}
$$

in this diagram the three first rows and the three first columns are triangles and all squares are commutative, except the one marked - anticommutative.

Proof. Choose first triangles based on x_1, b_1, a_1, x_2 and complete the two octahedrons below, finally complete the third octahedron. We let the diagrams speak for themselves.

$z_1 = gk$

$gh = a_3$

$-kc_3 = hz_3$

Let us verify the required anti commutativity

$$z_1 c_3 = gkc_3 = g(-hz_3) = -a_3 z_3$$

The commutativity relations are left to the reader.
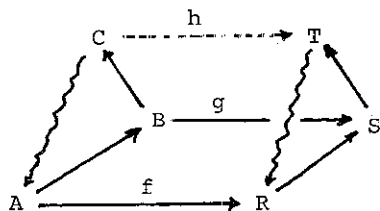
Q.E.D.

XI.6 Localization

Let us consider the homotopy category $K(A)$ of complexes over an additive category and M a subcategory of $K = K(A)$, which satisfies the following two conditions

6.1   The category M is stable under translation.
        If two of the vertices of a triangle belongs to
        M then the third vertex belongs to M.

6.2   If $X \oplus Y$ belongs to M, then X and Y
        belongs to M.

Let S denote the class of morphisms s in K which are the basis for a triangle in K whose third vertex are in M. We are going to prove that S is a multiplicative system in K i.e. that it satisfies FR1,2,3 from XI.1. Let us add
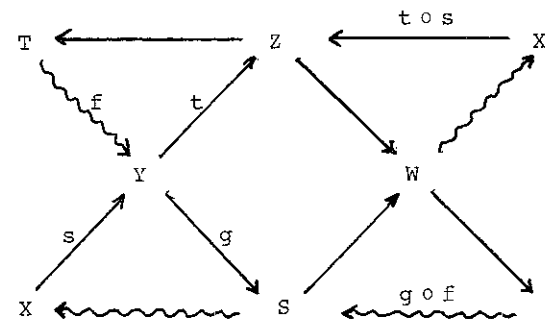
FR4 S is stable under decalage.

FR5 Given a diagram consisting of two triangles and a commutative square

with f and g in S. Then there exists $h: C \to T$ in S making the remaining two squares commutative.

Theorem 6.3. The class S of morphisms in K is a multiplicative system in the sense that it satisfies FR1,2,3 from XI.1 and FR 4,5 above. Moreover, the system S is saturated, 1.6.

**Proof.** Let $s: X \to Y$ and $t: Y \to Z$ be morphisms from S. Consider the octahedron

Notice that S and T belong to M to conclude that W belongs to M i.e. that $t \circ s$ belongs to S. This proves FR1. The proof of FR2 and FR3 can be read off the proof of 2.4. The condition FR4 is a consequence of 6.1 while FR5 follows from 5.6.

We shall now prove that S is saturated, so let there be given morphisms $r: V \to X$, $s: X \to Y$ $t: Y \to Z$ such that sr and ts are in S. With the notation from the octahedron above remark that $g(sr) = (gs)r = 0$. From this follows that g can be factored through the third vertex of a triangle based on sr. In particular we conclude that $g \circ f$ can be factored through an object of M. Apply the lemma below to the morphism $g \circ f$ and conclude that S is in M, i.e. that s belongs to S.

Q.E.D.

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Lemma 6.4. Given a triangle with W in M

$$
\begin{array}{ccc}
 & W & \\
\swarrow & & \nwarrow \\
T & \xrightarrow{\ \ h\ \ } & S
\end{array}
$$
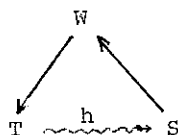
If the morphism h can be factored through an object of M, then T and S belong to M.

Proof. Let $T \xrightarrow{f} Y \xrightarrow{g} S$ be a factorization of h with Y in M. Consider the octahedron above and notice that Y and W belong to M. It follows from the Mayer-Vietoris sequences that $Z \oplus S$ and $T \oplus X$ belong to M. We conclude from 6.2 that S and T belong to M.

Q.E.D.

On the basis of this theorem we can construct the category $S^{-1}K$. Let us say that a diagram in $S^{-1}K$ is a triangle if it is isomorphic to the transform of a triangle from K. The category $S^{-1}K$ is a triangulated category in that the triangles satisfy TR1,2,3 from XI.2.

It is easy to see that the construction $M \to S$ gives a complete description of the saturated multiplicative ystems in K in terms of subcategories of K satisfying 6.1 and 6.2.

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Coherent Analytic Sheaves

Contents: Complex Spaces. - Local **Weierstrass** Theory - Finite Holomorphic Maps - Analytic Sets Coherence of Ideal Sheaves - Dimension Theory. - Analyticity of the Singular Locus. Normalization of the Structure Sheaf. - **Riemann** Extension Theorem and Analytic Coverings. - Normalization of Complex Spaces - Irreducibility and Connectivity. Extension of Analytic Sets. - Direct Image Theorem - Annex. Theory of Sheaves. Notion of Coherence. - Bibliography. - Index of Names. - Index

The theory of coherent analytic sheaves revolutionized complex analysis. Function theory based on such sheaves reached its first high point in 1953 at the Brussels Colloquium, when H Cartan and J-P. Serre presented what has since then been termed Theorems A and B for Stein Manifolds There are four fundamental coherence theorems in complex analysis: Coherence of the structure sheaf $\mathcal{O}_X$ of any complex space X; coherence of the ideal sheaf i(A) of any analytic set A; coherence of the normalization sheaf $\mathcal{O}_X$ of any structure sheaf; and coherence of the direct image sheaves under proper holomorphic maps. Complex analysis is developed in this book so to speak ab ovo; general sheaf theory, as far as it is needed, is given in an appendix Along with the basic theorems on coherent sheaves the following theorems necessary either as tools or as applications are included in the text:
local description lemma, on which the proof of the coherence of the structure sheaf is based; Rückert's Nullstellensatz; a profound discussion of finite holomorphic maps and dimension theory of complex spaces; Riemann's extension theorem; singular locus and construction of normalization spaces; theorem on removable singularities of analytic sets; theorems of Chow, Levi and Hurewitz-Weierstrass.
The book concludes with a proof of the coherence of all direct image sheaves under proper holomorphic maps and its applications to the proper mapping theorem, families of complex spaces and the **Stein** factorization. The approach taken is goemetric; detailed historical notes provide further insight into the theory of coherent analytic sheaves The authors who played a prominent part in the development of modern complex analysis give proofs of today's point of view This aspect and the self-contained exposition make the theory of coherent sheaves, which actually means the theory of complex spaces, easily accessible to a wide readership.