SPHERICAL FIBRATIONS

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Introduction

Suppose that $p : E \to B$ and $p' : E' \to B$ are two fiber spaces with the same fiber S^m, where S^m is the standard Euclidean m-sphere. Recall that a fiber homotopy equivalence

$$\varphi : E \to E'$$

is a fiber-preserving map which covers the identity map $B \to B$ and which is, at the same time, a homotopy equivalence. The object of this article is to answer in part the following problem.

Problem A. Determine when one can replace a given homotopy equivalence

$$f : (E, S^m) \to (E', S^m)$$

by a fiber homotopy equivalence where S^m is identified with the fiber above the basepoint.

This problem is related to the classification part of the second problem, which follows.

Problem B. Given the pair (E, S^m), determine when E is fibered by S^m over a space B, and classify the various possible fibrations.

Problem B is the subject of [1]. There are necessary and sufficient conditions (which generalize the conditions on the existence of Hopf fibrations when E is contractible) on $\Omega(E, S^m)$, the space of based paths in E ending in S^m, which insure that E is fibered by S^m and, therefore, in some sense provide an answer for Problem A. But whereas these conditions are perfectly satisfactory for the purposes of existence of fibrations, it is generally difficult to see how to use them, particularly when the possible fiber homotopy equivalences between E and E' are not necessarily in the homotopy class of f. Our object is to provide a different kind of answer.

It is easy enough to see that some additional conditions are necessary if one hopes for a positive answer. For example, two Hopf fibrations $S^7 \to S^4$ are equivalent if, and only if, the multiplications on S^3 which give rise to them are equivalent (see also [4]).

James and Whitehead study in [3] the problem of the homotopy-type classification of E and E' when B is a sphere. Their study has some bearing on Problem A. It indicates that there are two essentially different cases, depending upon whether or not the fibrations admit sections, and that one cannot hope for a positive answer, even in the easier case when there is a section, unless one stabilizes the problem.

The positive answer we give is a stable result. The fiber homotopy equivalence

$$\varphi : E \to E'$$

is constructed by induction on the skeletons of the base. The given maps f and φ are related in a certain way which can be described roughly as follows: the stable nature of the problem allows us to think of f as effecting a horizontal twist of B by S^m and a vertical twist of S^m by B. (This is precisely the situation if the fibrations are trivial.) In constructing φ one insures that it has the vertical effect of B.

To make the preceding precise is a delicate task and requires use of the RPT-category; the relevant facts and results are summarized in §2. In §3 the problem of the Theorem is given. We conclude with some remarks and examples in §4.

1. Statement of the Main Theorem

It will be assumed tacitly throughout this article (unless otherwise noted) that spaces and maps are in the category of based spaces and basepoint-preserving maps.

Suppose now that

$$(\xi) \qquad p : E \to B$$

is a Hurewicz fibration with fiber F, and assume that B is a simply connected finite CW-complex. By definition, let $E(\Sigma^n \xi)$ be the space obtained from $E \times S^n$, where S^n is the standard Euclidean n-sphere, by collapsing $p^{-1}(x) \vee S^n$ to a point, for all $x \in B$. One can prove easily, either directly or by using the classification theorem of [2], that the map

$$\Sigma^n p : E(\Sigma^n \xi) \to B$$

induced by p is also a Hurewicz fibration. We shall denote it by $\Sigma^n \xi$ and call it the _n_th suspension of ξ. Two Hurewicz fibrations

$$(\xi) \qquad p : E \to B, \text{ and}$$

$$(\xi') \qquad p' : E' \to B,$$

with the same fiber F and over the simply connected finite CW-complex B are said to be stably fiber homotopically equivalent if, and only if, there is a fiber homotopy equivalence

$$\varphi : E(\Sigma^n \xi) \to E(\Sigma^n \xi'),$$

for some n, covering the identity map $B \to B$.

The main theorem is the following.

THEOREM 1.1. Suppose that

$$(\xi) \qquad p : E \to B, \quad \text{and}$$

$$(\xi') \qquad p' : E' \to B$$

are the Hurewicz fibrations over the simply connected finite CW-complex B and with the same fiber S^m, where S^m is the standard Euclidean m-dimensional sphere. Then ξ and ξ' are stably fiber homotopically equivalent if, and only if, there is a homotopy equivalence

$$f : (E(\Sigma^n \xi, S^{n+m}) \to (E(\Sigma^n \xi'), S^{n+m}),$$

where $n + m > \dim B$ and S^{n+m} is identified with the fiber over the basepoint.

When B is a sphere, the theorem follows from the classification, up to homotopy type, of the total spaces of sphere bundles over spheres carried out by James and Whitehead in [3]. In general the fiber homotopy equivalences $E \to E'$ obtainable are not homotopic to f but are, however, related to f in certain ways to be described in §3, where the proof is given.

2. The RPT-category

Before we can give the proof, we need a few concepts from the theory of RPT-complexes [1, 2]. Recall that a special complex is a countable CW-complex with a single vertex which we take as a basepoint. An RPT-complex (i.e., a complex of the reduced product type) is a special complex with an associative multiplication

$$A \times A \to A,$$

for which the vertex is a two-sided identity and such that the product of two cells of A is again a cell of A. (See [1, 2] for details.) The purpose of the RPT-theory is to provide suitable combinatorial models for loopspaces and path spaces. For example, if B is a special complex and $\Omega(B)$ is its space of Moore loops, then there is an RPT-complex M and a homomorphism

(2.1) $$\gamma : M \to \Omega(B)$$

which is also a homotopy equivalence, and the indecomposable cells of M are in one-to-one correspondence with the cells of ΩB. Moreover, there is a universal quasi-fibration

$$q : \mathcal{B}M \to B,$$

for M, whose total space $\mathcal{B}M$ is a contractible special complex on which M acts on the right, and a map

$$\mathcal{B}M \to \mathcal{P}B$$

of an M-space to an $\Omega(B)$-space which covers the identity $B \overset{=}{\to} B$ ($\mathcal{P}(B)$ being the space of Moore paths).

These combinatorial models for $\Omega(B)$ and $\mathcal{P}(B)$ allow us to prove a classification theorem [2] for fibrations.

THEOREM 2.1. *Suppose that*

$$p : E \to B$$

is a Hurewicz fibration with fiber F; *and assume that* E, B, *and* F *are special complexes and that* B *is simply connected. Then there is a fiber-preserving map*

$$\varphi' : \mathcal{B}M \times F \to E$$

which induces a fiber homotopy equivalence

$$\varphi : \mathcal{B}M \times_M F \to E$$

which covers the identity map of B *and such that the maps* $(x, f) \to F$ *defined by the induced action*

$$M \times F \to F$$

give a homomorphism

$$\bar{\gamma} : M \to G(F),$$

where $G(F)$ is the monoid of homotopy equivalences of F with itself. Moreover, if $p : E \to B$ admits a section, then $\bar{\gamma}$ takes M to the submonoid $G_0(F)$ of basepoint-preserving homotopy equivalences.

The following corollary is the starting point of the proof of Theorem 1.1.

COROLLARY 2.2. Suppose that

$$p : E \to B$$

is a Hurewicz fibration as in Theorem 2.1, and let

$$E_r = p^{-1}B_r, \qquad r \geq 0,$$

where B_r stands for the r-skeleton of B. Then there are fiber-preserving maps

$$\alpha_i : \partial D^r \times F \to E_r,$$

where $i = 1, \ldots, k_r$, (k_r being the number of cells of $B_r - B_{r-1}$), and a fiber homotopy equivalence

$$E_r \cong E_{r-1} \cup_{\alpha_1} D^r \times F \cup_{\alpha_2} \ldots \cup_{\alpha_{k_r}} D^r \times F$$

which is the identity on E_{r-1}. Moreover, if $p : E \to B$ admits a section, then

$$B_r = B_{r-1} \cup \alpha_1(D^r \times *) \cup \ldots \cup_{\beta_{k_r}} D^r \times *,$$

where $*$ stands for the basepoint of F.

3. <u>The Proof of Theorem</u> 1.1.

In order to prove the theorem, it is enough to consider the following. Suppose that

$$p : E \to B, \quad \text{and}$$
$$p' : E' \to B$$

are two Hurewicz fibrations with fiber S^m over the simply connected finite CW-complex B; and let

$$f : (E, S^m) \to (E', S^m)$$

be a homotopy equivalence where S^m is identified with the fiber above the base-point in B. To prove Theorem 1.1, it is enough to show that there is a fiber homotopy equivalence

$$\varphi : E \to E'$$

covering the identity map of B, provided that $\dim B \leq m - 1$.

Observe that, since $\dim B < m$, we can find sections

$$s : B \to E \quad \text{and} \quad s' : B \to E'$$

for the two fibrations $p : E \to B$ and $p' : E' \to B$. Identify B with the subsets of E and E' given by the sections, and choose cellular structures on E and E' which make the sum $B \vee S^m$ a subcomplex of E and E'. It is clear that the given homotopy equivalence f can be assumed (after a deformation, if necessary) to induce a homotopy equivalence of pairs

$$f : (E, B \vee S^m) \to (E', B \vee S^m)$$

which takes each summand of $B \vee S^m$ into itself. We can also arrange it, after changing $p' : E' \to B$ by a fiber homotopy equivalence if necessary, so that

$$f \mid B \vee S^m = \text{identity}.$$

As in §2, denote the r-skeleton of B by B_r, and let

$$E_r = p^{-1}B_r \quad \text{and} \quad E'_r = p'^{-1}B_r.$$

According to Corollary 2.2, we can assume that

$$E_r = E_{r-1} \cup_{\alpha_1} D^r \times S^m \cup_{\alpha_2} \cdots \cup_{\alpha_{k_r}} D^r \times S^m$$
$$E' = E'_{r-1} \cup_{\alpha'_1} D^r \times S^m \cup_{\alpha'_2} \cdots \cup_{\alpha'_{k_r}} D^r \times S^m,$$

where

(3.1)
$$\alpha_i : \partial D^r \times S^m \to E_{r-1} \quad \text{and}$$
$$\alpha'_i : \partial D^r \times S^m \to E'_{r-1},$$

for $i = 1, \ldots, k_r$ (k_r being the number of cells in $B_r - B_{r-1}$), are suitable fiber maps. Since $p : E \to B$ and $p' : E' \to B$ admit sections, we can assume that the maps α_i and α'_i have been so chosen that

$$\alpha_i(\partial D^r \times \{x_0\}) \subset B_{r-1} \subset E_{r-1} \quad \text{and}$$
$$\alpha'_i(\partial D^r \times \{x_0\}) \subset B_{r-1} \subset E'_{r-1},$$

where x_0 is the basepoint of S^m. Observe that the (r+m)-cells of E_r and E'_r are obtained from the (r+m)-cells of products $D^r \times S^m$. Let

(3.2)
$$\beta_i : \partial D^{r+m} \to E_{r-1} \cup B_r \quad \text{and}$$
$$\beta'_i : \partial D^{r+m} \to E'_{r-1} \cup B_r,$$

for $i = 1, \ldots, k_r$, be the attaching maps of the (r+m)-cells.

Consider now the filtration

$$S^m \vee B \subset \ldots \subset E_r \cup B \subset \ldots \subset E.$$

Suppose that M is the RPT-complex representing ΩE, and let

$$\gamma : M \to \Omega E$$

be the representing homomorphism. The filtration of E induces an ascending filtration

$$\ldots \subset {}^{(r)}M \subset \ldots \subset M$$

of M by sub-RPT-complexes ${}^{(r)}M$ such that

(1) $\gamma|{}^{(r)}M$ is a homotopy equivalence

$${}^{(r)}M \to \Omega(E_r \cup B), \quad \text{and}$$

(2) ${}^{(r)}M$ is generated by ${}^{(r-1)}M$ and the cells $\{e_i^r\}$ in one-to-one correspondence with the cells of $E_r \cup B - E_{r-1} \cup B$, whose characteristic maps

$$\beta_i : (D^{r+m}, \partial D^{r+m}) \to (E_r \cup B,\ E_{r-1} \cup B)$$

are those of (3.2) above.

We shall construct now a complex useful in measuring the difference between two given homomorphisms of ${}^{(r)}M$ into G, an associative H-space. To start out with, denote by A the RPT-complex of M which corresponds to $\Omega(B)$, and note that ${}^{(r)}M$ is generated by A and the set of cells $\{e^{m+i-1}\}$, with $0 \leq i \leq r$, which are in one-to-one correspondence with the cells of $E_r \cup B - B$. Therefore, according to [1, 2], an element of ${}^{(r)}M$ is an equivalence class of the form

$$(x_1, \ldots, x_n),$$

where each x_j is either an element of A or one of the generating cells $\{e^{m+i-1}\}$. One can easily see that the condition $\dim B < m$ implies that the subset N_r' of ${}^{(r)}M$ consisting of those elements represented by sequences

$$(x_1, \ldots, x_n)$$

such that <u>at most</u> one x_j belongs to a generating cell e^{m+i-1} is actually a subcomplex of ${}^{(r)}M$. Now let $N_r' \cup_A N_r'$ be the complex obtained from two distinct copies of N_r' by joining them along A; and put

$$L_r = \bigcup \partial(D^{i+m-1} \times I)_{j_i},$$

with the disjoint summation ranging over the set of indices (i, j_i), where $0 \leq i \leq r$ and, for each i, $1 \leq j_i \leq k_i$ (k_i being the number of indecomposable cells of dimension i). By definition, let

$$N_r = (N'_r \cup_A N'_r) \cup A \times L_r \times A,$$

where the attaching map

$$A \times \bigcup (D^{i+m-1} \times \partial I)_{j_i} \times A \to N'_r \cup_A N'_r$$

is the one induced by the characteristic map of the cells $D^{i+m-1} \times \{0\}$ and $D^{i+m-1} \times \{1\}$ onto the cells e^{i+m-1} in the left and right copies of N'_r in $N'_r \cup_A N'_r$ respectively. We shall call N_r the <u>difference complex relative to</u> A.

We can now result the proof of the theorem. We shall construct the desired fiber-preserving map

$$\varphi : E \to E'$$

by induction on the skeletons of B. So we put

$$\varphi_0 : E_0 = S^m \to E'_0 = S^m \subset E'$$

equal to the identity map of S^m. Suppose we have been able to define a fiber-preserving map

$$\varphi_r : E_r \to E'_r$$

which covers the identity map $B_r \xrightarrow{=} B_r$ such that the following conditions are satisfied

$(3.3)_r$ $\qquad$ <u>$\varphi_r\beta_k$ and $f\beta_k$ are homotopic as maps</u>

$$S^{r+m-1} \to E'_r \cup B,$$

<u>where β_k are the maps defined in</u> (3.2) <u>above.</u>

In order to state the second condition, we need a new concept. Consider therefore the adjoints

$$\tilde{\varphi}_r, \tilde{f} : {}^{(r)}M \rightarrow \Omega(E'_r \cup B) \subset \Omega E'.$$

Together they define a map

$$\delta(\tilde{\varphi}_r, \tilde{f}) : N_r/A \rightarrow \Omega(E'_r \cup B) \subset \Omega E'$$

by first putting $\delta(\tilde{\varphi}_r, \tilde{f})$ equal to $\tilde{\varphi}_r$ on $D^{m+i-1} \times \{0\}$ and to $\tilde{f}$ on $D^{m+i-1} \times \{1\}$ and then using the homotopy of $(3.3)_r$ to define it on $\partial D^{m+i-1} \times I$. (Cf. [5].) Here N_r/A is the complex obtained from N_r by collapsing A to the vertex.

We shall say that a map

$$h : K \rightarrow \Omega E'$$

of the complex K into the loopspace $\Omega E'$ is <u>horizontal</u> if, and only if, the maps

$$\eta h, h : K \rightarrow \Omega E'$$

are homotopic. Here η is the composite

$$E' \xrightarrow{\Omega p'} B \xrightarrow{\Omega i'} \Omega E'.$$

The second condition is the following

$(3.4)_r$ <u>The map</u>

$$\delta(\tilde{\varphi}_r, \tilde{f}) : N_r/A \rightarrow \Omega E'$$

<u>is horizontal.</u>

We wish to extend the fiber homotopy equivalence φ to the (r+1)-skeleton. For the sake of simplicity, we shall give the proof when $B_{r+1} - B_r$ consists of one cell only, the general case being similar. Then

$$E_{r+1} = E_r \cup_\alpha (D^{r+1} \times S^m) \quad \text{and} \quad E'_{r+1} = E'_r \cup_{\alpha'} D^{r+1} \times S^m,$$

where the attaching maps

$$\alpha : \partial D^{r+1} \times S^m \to E_r \quad \text{and} \quad \alpha' : \partial D^{r+1} \times S^m \to E'_r$$

are those given in $(3.1)_{r+1}$ above. Next choose an RPT-complex $P_{r,m}$ of the form

$$P_{r,m} = e^0 \cup e^{r-1} \cup e^{m-1} \cup e^{r+m-1} \cup \text{higher dimensional cells}$$

to represent the loopspace $\Omega(\partial D^{r+1} \times S^m)$ (see [1, 2]). The maps α and α' define homomorphisms

$$\widetilde{\alpha} : P_{r,m} \to {}^{(r)}M \simeq \Omega(E_r \cup B) \quad \text{and} \quad \widetilde{\alpha}' : P_{r,m} \to \Omega(E'_r \cup B),$$

which correspond to the adjoints of α and α'. Observe now that the two homomorphisms

$$\widetilde{\varphi}_r\widetilde{\alpha},\ \widetilde{f}\widetilde{\alpha} : P_{r,m} \to \Omega(E'_r \cup B)$$

agree on the m-skeleton $e^0 \cup e^{r-1} \cup e^{m-1}$ and hence induce a map

$$d(\widetilde{\varphi}_r\widetilde{\alpha}, \widetilde{f}\widetilde{\alpha}) : S^{r+m-1} \to \Omega(E'_r \cup B)$$

as in [5].

Next we note that

$$d(\widetilde{\varphi}_r\widetilde{\alpha}, \widetilde{f}\widetilde{\alpha}) = \delta(\widetilde{\varphi}_r, \widetilde{f}) \circ (\widetilde{\alpha} \vee \widetilde{\alpha}),$$

where $\widetilde{\alpha} \vee \widetilde{\alpha}$ is the folding map

$$P_{r,m} \vee P_{r,m} \to N_r$$

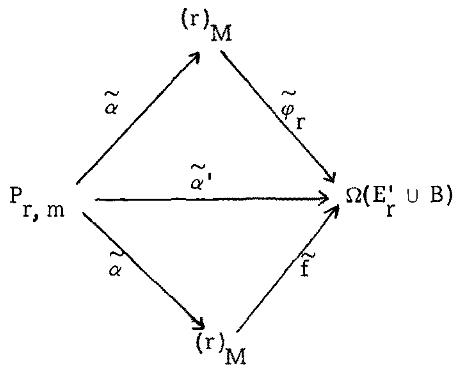
induced by the mapping $P_{r,m}$ into N'_r by $\widetilde{\alpha}$ [1]. Since $\delta(\widetilde{\varphi}_r, \widetilde{f})$ is horizontal by induction, we conclude the following.

(3.5) <u>The mapping</u>

$$d(\widetilde{\varphi}_r\widetilde{\alpha}, \widetilde{f}\widetilde{\alpha}) : S^{r+m-1} \to \Omega(E'_r \cup B)$$

<u>is horizontal</u>.

Consider next the diagram



where the triangles are not necessarily commutative. One can easily see that (cf. [5])

$$d(\tilde{\varphi}_r\tilde{\alpha}, \tilde{f}\tilde{\alpha}) = d(\tilde{\varphi}_r\tilde{\alpha}, \tilde{\alpha}') \cdot d(\tilde{\alpha}', \tilde{f}\tilde{\alpha})$$

as maps of S^{m+r-1} into $\Omega(E_r' \cup B)$, or that

(3.6) $$[d(\tilde{\varphi}_r\tilde{\alpha}, \tilde{f}\tilde{\alpha})] = [d(\tilde{\varphi}_r\tilde{\alpha}, \tilde{\alpha}')] + [d(\tilde{\alpha}', \tilde{f}\tilde{\alpha})]$$

as elements of $\pi_{m+r-1}\Omega(E_r' \cup B)$. To compute $[d(\tilde{\varphi}_r\tilde{\alpha}, \tilde{\alpha}')]$, consider the commutative diagram

$$\begin{array}{ccc} P_{r,m} & \xrightarrow{\ \tilde{\alpha}\ } & \Omega(E_r \cup B) \\ {\scriptstyle\tilde{\sigma}}\downarrow & & \downarrow{\scriptstyle\tilde{\varphi}_r} \\ P_{r,m} & \xrightarrow{\ \tilde{\alpha}'\ } & \Omega(E_r' \cup B) \end{array}$$

where $\tilde{\sigma}$ is the loopspace map induced by the map

$$\alpha'^{-1}\varphi_r\alpha : \partial D^{r+1} \times S^m \to \partial D^{r+1} \times S^m$$

and α'^{-1} is a fiber homotopy inverse to α'. Now let

$$\tau : \partial D^{r+1} \to G(S^m)$$

be the map of ∂D^{r+1} into $G_0(S^m)$, the monoid of basepoint-preserving homotopy equivalences of S^m defined by the equation

$$\alpha'^{-1}\varphi_r\alpha(x, y) = (x, \tau(x)y),$$

for all (x, y) in $\partial D^{r+1} \times S^m$.

LEMMA 3.7. <u>The element</u> $[d(\tilde{\varphi}_r\tilde{\alpha}, \tilde{\alpha}')]$ <u>lies in</u>

$$\text{im}(\tilde{i}_* : \pi_{r+m-1} S^m \to \pi_{r+m-1}\Omega(E'_r \cup B)),$$

<u>where</u> i <u>is the injection of</u> S^m <u>in</u> $E'_r \cup B$ <u>as the fiber above the basepoint.</u>

<u>Proof.</u> Note that

$$(\tilde{\alpha}')_{\#}[d(\tilde{\sigma}, \text{id})] = [d(\tilde{\alpha}'\tilde{\sigma}, \tilde{\alpha}')] = [d(\tilde{\varphi}_r\tilde{\alpha}, \tilde{\alpha}')],$$

where id stands for the identity map of $P_{r,m}$. But $d(\tilde{\sigma}, \text{id})$ is the adjoint of the map

$$d(\sigma, \text{id}) : S^{r+m} \to \partial D^{r+1} \times S^m.$$

Since the two maps σ and id agree when projected onto $S^r = \partial D^{r+1}$, it follows that the component of $d(\sigma, \text{id})$ which lies in ∂D^{r+1} is trivial. This implies that $d(\tilde{\sigma}, \text{id})$ factors through S^m --which proves the lemma.

LEMMA 3.8. $d(\alpha', f\alpha)$ <u>is null-homotopic as a map of</u> S^{r+m} <u>into</u> $E'_r \cup B$.

<u>Proof.</u> Observe that f induces a homotopy equivalence of pairs

$$(E_{r+1} \cup B, E_r \cup B) \to (E'_{r+1} \cup B, E_r \cup B).$$

But

$$E_{r+1} \cup B = (E_r \cup B) \cup_\beta D^{r+m+1} \quad \text{and}$$
$$E'_{r+1} \cup B = (E'_r \cup B) \cup_{\beta'} D^{r+m+1},$$

where β and β' are the attaching maps for the (r+m+1)-cells (see (3.2) above).

Hence the maps

$$\beta f, \beta' : S^{m+r} \to E'_r \cup B$$

are homotopic. After a more or less straightforward computation (see [6]), one sees that

$$d(\beta f, \beta') = d(\alpha f, \alpha')$$

--which proves the lemma.

An immediate consequence of the preceding two lemmas is that

$$[d(\widetilde{\varphi}_r \widetilde{\alpha}, \widetilde{\alpha}')] = 0,$$

since they imply that the element under consideration is both horizontal and lies in the image of ΩS^m. It follows now that we can extend φ_r to a map

$$\varphi'_{r+1} : E_{r+1} \to E'_{r+1}$$

such that $\varphi'_{r+1} | B_{r+1}$ = identity and $\varphi'_{r+1} | E_r$ is the fiber homotopy equivalence $E_r \to E'_r$ obtained by induction. According to [6], or computing directly in the spirit of [3], one can find a fiber homotopy equivalence

$$\varphi''_{r+1} : E_{r+1} \to E'_{r+1}$$

such that

$$\varphi''_{r+1} | E_{r+1} = \varphi_r.$$

The map φ''_{r+1}, however, may not satisfy Condition $(3.4)_r$ of the induction assumption without further modification. So consider the map

$$\delta(\widetilde{\varphi}''_{r+1}, \widetilde{f}) : N_{r+1}/A \to \Omega E',$$

where N_{r+1}/A is the difference complex and $\delta(\widetilde{\varphi}''_{r+1}, \widetilde{f})$ is the map induced by the adjoints $\widetilde{\varphi}''_{r+1}$ and $\widetilde{f}$ of φ''_{r+1} and f respectively. Observe that

$$\delta(\tilde{\varphi}''_{r+1}, \tilde{f}) | (N_r/A) = \delta(\tilde{\varphi}_r, \tilde{f}).$$

The necessary modification is intended to make $\delta(\tilde{\varphi}''_{r+1}, \tilde{f})$ horizontal by choosing φ''_{r+1} appropriately. So let us consider the diagram

$$\begin{array}{ccc} N_{r+1}/A & \xrightarrow{\delta(\tilde{\varphi}_{r+1}, \tilde{f})} & \Omega E' \\ & \searrow^{\delta(\tilde{\varphi}''_{r+1}, \tilde{f})} & \downarrow \eta \\ & & \Omega E' \end{array}$$

where η is the composite

$$\Omega E' \xrightarrow{\Omega p'} \Omega B \xrightarrow{\Omega i'} \Omega E'$$

introduced earlier. We wish to modify φ''_{r+1} so that the preceding diagram becomes homotopy commutative. To begin with, we change $\delta(\tilde{\varphi}''_{r+1}, \tilde{f})$ up to homotopy if necessary, so that the diagram becomes strictly commutative when restricted to N_r/A. Then the difference (in the sense of [4]) of the two maps $\delta(\tilde{\varphi}''_{r+1}, \tilde{f})$ and $\eta \circ \delta(\tilde{\varphi}''_{r\ 1}, \tilde{f})$ is a map

$$h : (N_{r+1}/A)/(N_r/A) \to \Omega E',$$

where $(N_{r+1}/A)/(N_r/A)$ is the complex obtained from N_{r+1}/A by collapsing N_r/A to a point. But

$$(N_{r+1}/A)/(N_r/A) = (A \times (S^{m+r} \vee S^{m+r} \vee S^{m+r}) \times A)/(A \times A),$$

as follows readily from the structure of N_{r+1}. Next we note that h can be factored according to the diagram

$$\begin{array}{ccc} (N_{r+1}/A)/(N_r/A) & \xrightarrow{h} & \Omega E' \\ \downarrow & & \downarrow \| \\ A \times S^{m+r} \times A/(A \times A) & \longrightarrow & \Omega E' , \end{array}$$

where the vertical map on the left is that induced by the folding map

$S^{m+r} \vee S^{m+r} \vee S^{m+r} \to S^{m+r}$ and h is a map of two-sided A-spaces. Therefore, to insure that h is homotopic to 0, it is enough to consider its restriction,

$$h|S^{m+r} : S^{m+r} \to \Omega E'.$$

Now the fact that $p' : E' \to B$ admits a section implies that

$$\pi_{m+r}\Omega E' \cong \pi_{m+r}\Omega S^m \oplus \pi_{m+r}\Omega B.$$

The component of $[h|S^{m+r}]$ in $\pi_{m+r}\Omega S^m$ is represented by a map $h' : S^{m+r} \to \Omega S^m$ which we regard as a map

$$\sigma : S^{r+1} \to G_0(S^m),$$

where $G_0(S^m)$ is the monoid of basepoint-preserving homotopy equivalences of S^m with itself. By definition, let

$$\varphi_{r+1} : E_{r+1} \to E'_{r+1}$$

be the fiber homotopy equivalence satisfying the following:

(i) $\varphi_{r+1}|E_r = \varphi_r$, and

(ii) $\varphi_{r+1}(x, y) = \varphi'_{r+1}(x, \sigma'(x)y)$,

where σ' is the composite

$$D^{r+1} \to S^{r+1} \xrightarrow{\sigma} G_0(S^m) \subset (S^m)^{S^m},$$

the map $D^{r+1} \to S^{r+1}$ being that which sends ∂D^{r+1} to the basepoint. One can now easily check that φ_{r+1} satisfies the two conditions of the induction assumption. This proves the theorem.

4. <u>Remarks</u>.

Suppose that in Problem A the spaces E, E' and B are all smooth closed manifolds and that f is a diffeomorphism. Then, assuming that $\dim B < m$ and $\dim E \geq 5$, one can adjust f up to diffeotopy so that it takes B diffeomorphically onto itself, where B is regarded as a smooth submanifold of E and E' (this is possible since dim $B < m$). Now, by appealing to the Tubular Neighborhood Theorem, one can change f up to diffeotopy so that it induces a map of closed disk bundles of a tubular neighborhood E_0 of B in E onto a tubular neighborhood E'_0 of B in E'. Note that $E_1 = E - \text{int } E_0$ and $E'_1 = E' - E'_0$ are also closed disk subbundles of E and E'. Hence one can find an orthogonal bundle equivalence $\varphi : E \to E'$ which obviously need not be homotopic to f.

The preceding remark raises the question as to whether or not one can find an equivalence $\varphi : E \to E'$ of the same nature as f. For example, if the given fibrations were Top (S^m)-bundles, Top (S^m) being the group of homeomorphisms of S^m, and f a homeomorphism, would it be possible to find a Top (S^m)-bundle equivalence $E \to E'$?

It is not clear what the answer is, but it seems likely that further conditions on f need to be imposed.

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