A new proof of the signature formula for surface bundles

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Abstract

Let \( E \to X \) be an oriented surface bundle over a closed surface. Then the signature \( \text{sign}(E) \) is determined by the first Chern class of the flat vector bundle \( \Gamma \) associated to the monodromy homomorphism \( \chi : \pi_1(X) \to \text{Sp}_{2h}(\mathbb{Z}) \) of \( E \), it is equal to \(-4\langle c_1(\Gamma), [X] \rangle\). The aim of this paper is to give an algebro-topological proof of this formula, i.e., one that does not use the Atiyah–Singer index theorem. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let \( E \to X \) be a smooth oriented surface bundle over a surface, more precisely a smooth fibre bundle with fiber \( S_h \), an oriented closed surface of genus \( h \), and base \( X \) an oriented compact surface with boundary \( \partial X \). Thus the structure group of \( E \) is the group \( G := \text{Diffeo}^+(S_h) \) of orientation preserving diffeomorphisms of \( S_h \) with the usual \( C^\infty \)-topology.

For \( h \geq 2 \) the connected components of \( G \) are contractible, so the bundle \( E \) is determined by its monodromy homomorphism

\[
\rho : \pi_1(X) \to G/G_0 = \mathcal{M}_h
\]

into the mapping class group \( \mathcal{M}_h \) of \( S_h \) (\( G_0 \) the connected component of the identity). Let \( \text{Sp}_{2h}(\mathbb{Z}) \) be the Siegel modular group, i.e., the subgroup of the automorphism group \( \text{Sp}_{2h}(\mathbb{R}) \) of the standard symplectic space \((\mathbb{R}^{2h}, \omega_0)\) consisting of the matrices with integer
coefficients. Because every element of \( \mathcal{M}_h \) leaves the intersection form on \( H_1(Sh; \mathbb{Z}) \) invariant and because \( (H_1(Sh; \mathbb{R}), \cdot) \cong (\mathbb{R}^{2h}, \omega_0) \), there is a natural homomorphism

\[
\sigma : \mathcal{M}_h \to \text{Sp}_{2h}(\mathbb{Z}) \subset \text{Sp}_{2h}(\mathbb{R}).
\]

Composing these two we get a homomorphism \( \chi := \sigma \circ \rho : \pi_1(X) \to \text{Sp}_{2h}(\mathbb{Z}) \). If \( X \) has genus \( g \geq 1 \), then it is an Eilenberg–MacLane space \( K(\pi_1(X), 1) \), thus \( X \) is a classifying space for its fundamental group. So the singular cohomology of \( X \) is canonically isomorphic to the Eilenberg–MacLane cohomology of the group \( \pi_1(X) \) and \( \chi \) induces a homomorphism

\[
\chi^* : H_{EM}^*(\text{Sp}_{2h}(\mathbb{Z}); \mathbb{Z}) \to H^*(X; \mathbb{Z}).
\]

Now Meyer [5] showed, that the signature of the 4-manifold \( E \) can be computed in terms of the Leray spectral sequence of \( E \to X \) and equals \( \text{sign}(E) = \text{sign}(X, \partial X; H^1(Sh; \mathbb{R})) \). By cutting the base \( X \) into spheres with three boundary components each he also gave an explicit method to compute such signatures \( \text{sign}(X, \partial X; \Gamma) \) for flat symplectic vector bundles \( \Gamma \) in terms of a special cocycle \( \tau_h \) of the symplectic group (cf. Section 2). For the case of a surface bundle considered here, this gives the following formula.

**Theorem 1** (Meyer [5,6]). Let \( E \to X \) be a smooth oriented surface bundle over an oriented, closed surface \( X \) with genus \( g \geq 1 \), and with fibre \( Sh \) an oriented closed surface of genus \( h \geq 2 \). Then the signature of the total space is equal to

\[
\text{sign}(E) = -\langle \chi^*[\tau_h], [X] \rangle,
\]

where \( \chi = \sigma \circ \rho \) is the monodromy map of \( E \) followed by the homology representation of \( \mathcal{M}_h \) in \( \text{Sp}_{2h}(\mathbb{Z}) \).

It follows, for example, that every such bundle \( E \) with \( h \leq 2 \) has vanishing signature, because \( \mathcal{M}_2 \) is \( \mathbb{Q} \)-acyclic by a result of Igusa [4] (that torus bundles have vanishing signature can be seen in an elementary way).

By using the Atiyah–Singer index theorem for families Meyer derived another formula expressing the signature of a smooth oriented fibre bundle over a closed manifold \( X \) in terms of characteristic classes of flat vector bundles on \( X \). In the special case of a surface bundle over a closed surface this leads to the following formula.

Let \( \Gamma \) be a flat vector bundle with structure group \( \text{Sp}_{2h}(\mathbb{R}) \). The standard symplectic form \( \omega_0 \) on \( \mathbb{R}^{2h} \) induces a symplectic form on \( \Gamma \) and any compatible complex structure \( J \) on \( \Gamma \) turns \( \Gamma \) into a complex vector bundle. Let \( c_1(\Gamma) \in H^2(\mathbb{R}; \mathbb{Z}) \) denote its first Chern class.

**Theorem 2** (Meyer [5]). Let \( E \to X \) be a smooth oriented surface bundle over an oriented, closed surface \( X \) with fibre \( Sh \). Then the signature of the total space is equal to

\[
\text{sign}(E) = -4\langle c_1(\Gamma), [X] \rangle,
\]

where \( \Gamma = H^1(Sh; \mathbb{R}) \).
Combining these two theorems, respectively the more general forms of them dealing with signatures \( \text{sign}(X, \Gamma) \), identifies the pull back of the signature class \([\tau_h]\) with a characteristic class of a flat vector bundle over \(X\).

**Theorem 3.** For any oriented, closed surface \(X\) with genus \( g \geq 1 \), any homomorphism \( \chi : \pi_1(X) \to \text{Sp}_{2h}(\mathbb{Z}) \) and \( \Gamma \) the associated flat symplectic vector bundle there is the equality

\[
\chi^*[\tau_h] = 4c_1(\Gamma)
\]

in \(H^2(X; \mathbb{Z})\).

The aim of this paper is to give an algebro-topological proof of this last theorem, which does not use the Atiyah–Singer index theorem hidden in Theorem 2. From this and Theorem 1 of course we get a new proof for Meyer’s Theorem 2. In fact a universal version of Theorem 3 will be proved.

2. The signature cocycle

For any topological group \(G\) let \(G^\delta\) be the underlying discrete group and \(D : BG^\delta \to BG\) the continuous map of classifying spaces induced by the identity \(G^\delta \to G\). Let \(X\) be a compact oriented surface with boundary \(\partial X\). Let \(\Gamma\) be a flat vector bundle over \(X\) with structure group \(\text{Sp}_{2h}(\mathbb{R})\). Thus the classifying map of \(\Gamma\) factors over \(D : B\text{Sp}_{2h}(\mathbb{R})^\delta \to B\text{Sp}_{2h}(\mathbb{R})\).

Then \(\Gamma\) is determined by the monodromy homomorphism \(\chi : \pi_1(X) \to \text{Sp}_{2h}(\mathbb{R})\), namely \(\Gamma = \tilde{X} \times_{\pi_1(X)} \mathbb{R}^{2h}\), where \(\pi_1(X)\) acts on the universal cover \(\tilde{X}\) of \(X\) by deck transformation and on \(\mathbb{R}^{2h}\) via \(\chi\). The standard symplectic form \(\omega_0 = (0 \ 1 \ -1 \ 0)\) induces a non-degenerate, antisymmetric form on \(\Gamma\). Combining this form with the cup product gives a symmetric bilinear map

\[H^1(X, \partial X; \Gamma) \times H^1(X, \partial X; \Gamma) \to H^2(X, \partial X; \mathbb{R})\]

and by evaluation on the orientation class of \(X\) a symmetric bilinear form on \(H^1(X, \partial X; \Gamma)\). Let \(\text{sign}(X, \partial X; \Gamma)\) be the signature of this bilinear form.

Now let \(X_0\) be a sphere with three open discs removed. The fundamental group \(\pi_1(X_0)\) is a free group on two generators, so that any two elements \(\alpha, \beta \in \text{Sp}_{2h}(\mathbb{R})\) define a homomorphism

\[\chi_{\alpha, \beta} : \pi_1(X_0) \to \text{Sp}_{2h}(\mathbb{R})\]

and by this a flat symplectic vector bundle \(\Gamma_{\alpha, \beta}\) over \(X_0\). Using a triangulation of \((X_0, \partial X_0)\) Meyer [5] showed that there is an isomorphism of \(H^1(X_0, \partial X_0; \Gamma_{\alpha, \beta})\) onto

\[H_{\alpha, \beta} := \{x = (x_1, x_2) \in \mathbb{R}^{2h} \oplus \mathbb{R}^{2h} | (\alpha^{-1} - 1)x_1 + (\beta - 1)x_2 = 0\}\]

and that under this isomorphism the bilinear form on \(H^1(X_0, \partial X_0; \Gamma_{\alpha, \beta})\) becomes

\[-\langle \cdot, \cdot \rangle_{\alpha, \beta}, \text{ where for } x, y \in H_{\alpha, \beta}\]

\[\langle x, y \rangle_{\alpha, \beta} = \omega_0(x_1 + x_2, (1 - \beta)y_2).\]
Finally let $\tau_h$ be defined by
\[ \tau_h : \text{Sp}^h_{2h}(\mathbb{R}) \times \text{Sp}^h_{2h}(\mathbb{R}) \to \mathbb{Z}, \]
\[ \tau_h(\alpha, \beta) := -\text{sign}(X_0, \partial X_0; \Gamma_{\alpha, \beta}) = \text{sign}(H_{\alpha, \beta}, \langle \cdot, \cdot \rangle_{\alpha, \beta}). \]

Then by cutting a sphere with four open discs removed into two copies of $X_0$ in two different ways and using the Novikov additivity of the signature it is easy to see, that for every $\alpha, \beta, \gamma \in \text{Sp}^h_{2h}(\mathbb{R})$ the equality
\[ \tau_h(\alpha, \beta) + \tau_h(\alpha\beta, \gamma) = \tau_h(\alpha, \beta\gamma) + \tau_h(\beta, \gamma) \]
holds. Thus $\tau_h$ is a cocycle of the symplectic group and defines a class $[\tau_h]$ in the Eilenberg–MacLane cohomology $H^2_{\text{EM}}(\text{Sp}^h_{2h}(\mathbb{R}); \mathbb{Z})$ of this group, which of course is canonically isomorphic to the singular cohomology $H^2(B\text{Sp}^h_{2h}\delta; \mathbb{Z})$ of the classifying space of $\text{Sp}^h_{2h}\delta$.

3. Identification of the signature class

The maximal compact subgroup of $\text{Sp}^h_{2h}(\mathbb{R})$ is $U(h)$. So we can consider the first universal Chern class as an element $c_1 \in H^2(B\text{Sp}^h_{2h}(\mathbb{R}); \mathbb{Z})$. Let $X$ be as in Theorem 3, $\chi : \pi_1(X) \to \text{Sp}^h_{2h}(\mathbb{R})$ a homomorphism and $\Gamma$ the associated flat vector bundle. Then $c_1(\Gamma) = \chi^*D^*c_1$. Thus to prove Theorem 3 it would be sufficient to prove $[\tau_h] = 4D^*c_1$ in $H^2(B\text{Sp}^h_{2h}\delta; \mathbb{Z}) = H^2_{\text{EM}}(\text{Sp}^h_{2h}(\mathbb{R}); \mathbb{Z})$. To do this we use the results of Milnor on the Friedlander conjecture.

**Proposition 1** (Milnor [7]). Let $G$ be a Lie group with finitely many components and $A$ a finite Abelian group. If the connected component $G_0$ of the identity is solvable or $G$ is a Chevalley group, then $D^* : H^2(BG; A) \to H^2(BG^\delta; A)$ and $D_\alpha : H_2(BG^\delta; A) \to H_2(BG; A)$ are isomorphisms.

We will use this result for the Abelian group $G = U(1)$ and the Chevalley group $G = \text{Sp}^h_{2h}(\mathbb{R})$. This proposition will enable us in Section 4 to prove the following.

**Lemma 1.** Let $h \geq 1$ and $n \in \mathbb{N}$. Then there is the equality $[\tau_h] = 4D^*c_1$ in $H^2(B\text{Sp}^h_{2h}\delta; \mathbb{Z}) = H^2_{\text{EM}}(\text{Sp}^h_{2h}(\mathbb{R}); \mathbb{Z})$.

Then in fact the same is true with integer coefficients. The following theorem is our main result.

**Theorem 4.** In $H^2_{\text{EM}}(\text{Sp}^h_{2h}(\mathbb{R}); \mathbb{Z}) = H^2(B\text{Sp}^h_{2h}(\mathbb{R}); \mathbb{Z})$ the equality $[\tau_h] = 4D^*c_1$ holds for $h \geq 1$.

**Proof.** Let $X$ be an oriented, closed surface with genus $g \geq 1$ and $\chi : \pi_1(X) \to \text{Sp}^h_{2h}(\mathbb{R})$ a homomorphism. Look at the long exact cohomology sequence induced by the coefficient sequence $0 \to \mathbb{Z}^n \xrightarrow{n} \mathbb{Z}^n \to \mathbb{Z}^n \to 0$ for $n \in \mathbb{N}$:
Because of Lemma 1 and the exactness of the lower row \( \chi^*([\tau_h] - 4D^*c_1) \) is in \( \bigcap_{n \in \mathbb{N}} nH^2(X; \mathbb{Z}) \). But because \( H^2(X; \mathbb{Z}) \) is finitely generated, this intersection is trivial. So \( \chi^*[\tau_h] = 4c_1(\Gamma) \) in \( H^2(X; \mathbb{Z}) \).

Suppose now that \( [\tau_h] - 4D^*c_1 \neq 0 \) in \( H^2(BSp_{2h}(\mathbb{R})^\delta; \mathbb{Z}) \). \( Sp_{2h}(\mathbb{R}) \) is perfect, i.e.,

\[
H_1(BSp_{2h}(\mathbb{R})^\delta; \mathbb{Z}) = H^1_{EM}(Sp_{2h}(\mathbb{R}); \mathbb{Z}) = 0.
\]

Thus

\[
H^2(BSp_{2h}(\mathbb{R})^\delta; \mathbb{Z}) = \text{Hom}(H_2(BSp_{2h}(\mathbb{R})^\delta; \mathbb{Z}), \mathbb{Z}).
\]

So there is a class \( x \in H_2(BSp_{2h}(\mathbb{R})^\delta; \mathbb{Z}) \) with \( \langle [\tau_h] - 4D^*c_1, x \rangle \neq 0 \). But by Hopf's theorem [2] every class in \( H^2_{EM}(Sp_{2h}(\mathbb{R}); \mathbb{Z}) \) can be represented by a surface \( X \) as above. So \( x = \chi^*[X] \) and \( \langle [\tau_h] - 4D^*c_1, x \rangle = \langle \chi^*[\tau_h] - 4\chi^*D^*c_1, [X] \rangle = 0 \). That is a contradiction. \( \Box \)

Hence to show Theorem 3 it remains to prove Lemma 1.

4. Proof of Lemma 1

**Lemma 2.** If \( [\tau_1] = 4D^*c_1 \) in \( H^2(BU(1)^\delta; \mathbb{Z}_n) \) then Lemma 1 is true for all \( h \geq 1 \) in \( H^2(BSp_{2h}(\mathbb{R})^\delta; \mathbb{Z}_n) \).

**Proof.** The natural inclusion \( \mathbb{C} \subset \mathbb{C}^h : z \mapsto (z, 0, \ldots, 0)^t \) induces an embedding \( \iota : U(1) \hookrightarrow U(h) \). Then \( \iota^* : H^*(BU(h); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_h] \to H^*(BU(1); \mathbb{Z}) = \mathbb{Z}[c_1] \) is the projection \( \iota^*(c_1) = c_1, \iota^*(c_i) = 0 \) for \( i \geq 2 \). So \( \iota \) induces an isomorphism

\[
H^2(BSp_{2h}(\mathbb{R}); \mathbb{Z}) = H^2(BU(h); \mathbb{Z}) \to H^2(BU(1); \mathbb{Z}).
\]

Furthermore \( \iota \) induces an isomorphism \( H^2(BSp_{2h}(\mathbb{R}); \mathbb{Z}_n) \to H^2(BU(1); \mathbb{Z}_n) \). Because of Proposition 1 this descends to an isomorphism \( H^2(BSp_{2h}(\mathbb{R})^\delta; \mathbb{Z}_n) \to H^2(BU(1)^\delta; \mathbb{Z}_n) \).

Now looking at the explicit presentation of the cocycle \( \tau_h \) given by \( \tau_h(\alpha, \beta) = \text{sign}(H_{\alpha, \beta}, \langle \cdot, \cdot \rangle_{\alpha, \beta}) \) shows that \( \tau_h(t(z_1), t(z_2)) = \tau_1(z_1, z_2) \) for \( z_1, z_2 \in U(1) \). Thus \( \iota^*[\tau_h] = [\iota^*\tau_h] = [\tau_1] = 4D^*c_1 = \iota^*(4D^*c_1) \). But \( \iota^* \) is an isomorphism, so Lemma 1 follows. \( \Box \)

Thus it is enough to show that \( [\tau_1] = 4D^*c_1 \) in \( H^2_{EM}(U(1); \mathbb{Z}_n) \).

For \( \alpha, \beta \in U(1) \), say \( \alpha = e^{ia}, \beta = e^{ib} \), the vector space \( H_{\alpha, \beta} \) becomes

\[
H_{\alpha, \beta} = \{ x = (x_1, x_2) \in \mathbb{C} \oplus \mathbb{C} | (e^{-ia} - 1)x_1 + (e^{ib} - 1)x_2 = 0 \}
\]
and $(x, y)_{\alpha, \beta}$ is computed to be

$$ (x, y)_{\alpha, \beta} = \frac{\sin(a + b) - \sin a - \sin b}{1 - \cos b} \cdot \text{re}(\bar{x}_1 y_1). $$

Thus $\tau_1(e^{i\alpha}, e^{i\beta}) = 2 \text{sgn}(\sin(a + b) - \sin a - \sin b)$. So $\tau_1(e^{2\pi i}, e^{2\pi i})$ takes the following values on $[0, 1] \times [0, 1]$:

\begin{align*}
1 & \quad +2 \\
0 & \quad -2 \\
1 & \quad 1
\end{align*}

And $\tau_1 \equiv 0$ on the lines.

The easiest way to compare $[\tau_1]$ and $D^*c_1$ would be to evaluate both classes on a generator of $H_2^{EM}(U(1); \mathbb{Z}_n) = \mathbb{Z}_n$, for $H_1(BU(1)^{\delta}; \mathbb{Z}_n) = 0$ implies that $H_2^{EM}(U(1); \mathbb{Z}_n) = \text{Hom}(H_2^{EM}(U(1); \mathbb{Z}_n), \mathbb{Z}_n)$.

Let $C_\bullet(G)$ be the Eilenberg–MacLane complex of $G := U(1)$ (cf. [1, II.3]). So $C_k(G)$ is the free Abelian group generated by tuples $[g_1|\cdots|g_k]$, $g_i \in G$, and the differential $d : C_k(G) \to C_{k-1}(G)$ is defined as $\sum_{i=0}^k (-1)^i d_i$, where

$$
d_i[g_1|\cdots|g_k] = \begin{cases} 
[g_2|\cdots|g_k] & i = 0, \\
[g_1|\cdots|g_i g_{i+1}|\cdots|g_k] & 0 < i < k, \\
[g_1|\cdots|g_{k-1}] & i = k.
\end{cases}
$$

\textbf{Lemma 3.} Let $\omega_n := e^{2\pi i/n}$. Then the class of the 2-cycle $Q_n := \sum_{j=1}^n [\omega_n|\omega_n^j]$ satisfies $[\tau_1], [Q_n] \equiv 4 \text{ mod } n$.

\textbf{Proof.} It is $d\cdot Q_n = \sum_{j=1}^n ([\omega_n^j] - [\omega_n^{j+1}] + [\omega_n]) = n[\omega_n] \equiv 0 \text{ mod } n$. So $Q_n$ is indeed a $\mathbb{Z}_n$-2-cycle of $U(1)$. Furthermore

$$
[\tau_1], [Q_n] = \sum_{j=1}^n \tau_1(\omega_n, \omega_n^j) = (n - 1) \cdot (-2) + 0 + 2 \equiv 4 \text{ mod } n.
$$

Considering the fact that

$$
H_2^{EM}(U(1); \mathbb{Z}_n) = H_2(BU(1)^{\delta}; \mathbb{Z}_n) \xrightarrow{D^*} H_2(BU(1); \mathbb{Z}_n) \xrightarrow{\bar{\partial}} H_2(\mathbb{C}P^\infty; \mathbb{Z}_n) = H_2(\mathbb{C}P^1; \mathbb{Z}_n)
$$

is an isomorphism and $\langle c_1, [\mathbb{C}P^1] \rangle = 1$, it remains to show that $D_n[Q_n] = [\mathbb{C}P^1]$. This will be done in two steps in Lemmas 4 and 5. Define the singular 2-cycle $P_n$ of $\mathbb{C}P^1$ by $P_n := \sum_{j=1}^n (P_n^j - P_\infty)$, where

$$
P_n^j : \Delta^2 \to \mathbb{C}P^1 : (t_0, t_1, t_2) \mapsto [\sqrt{t_0} + \sqrt{t_1} \omega_n : \sqrt{t_2} \omega_n^{j+1}],
$$

$$
P_\infty : \Delta^2 \to \mathbb{C}P^1 : (t_0, t_1, t_2) \mapsto [1 : 0].$$
Lemma 4. The image under the map $H_2(\mathbb{C}P^1; \mathbb{Z}) \to H_2(\mathbb{C}P^1; \mathbb{Z}_n)$ of the class $[P_n] \in H_2(\mathbb{C}P^1; \mathbb{Z})$ is $D_n[Q_n]$.

Proof. Let $F_* (G)$ be the standard resolution (cf. [1, I.5]) of $\mathbb{Z}$ over $\mathbb{Z}G$, $G := U(1)$. So the tuples $(g_0, \ldots, g_k), g_i \in G$, build a $\mathbb{Z}$-basis of $F_k(G)$ and the $\mathbb{Z}G$-module structure is given by $g \cdot (g_0, \ldots, g_k) = (gg_0, \ldots, gg_k)$. The differential $d: F_k(G) \to F_{k-1}(G)$ is defined to be $\sum_{i=0}^k (-1)^i d_i$, where

$$d_i(g_0, \ldots, g_k) = (g_0, \ldots, g_i g_i^{-1}, g_{i+1}, \ldots, g_k).$$

Then the Eilenberg–MacLane complex is $C_k(G) = F_k(G) \otimes G$. The $\mathbb{Z}$-basis elements of $C_k(G)$ are given by the $\mathbb{Z}G$-basis elements of $F_k(G)$. By the form $[g_1|g_2| \cdots |g_k] = (1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_k)$.

Take the Milnor model of the universal $G^\delta$-bundle $EG^\delta \to BG^\delta$ (cf. [3, 4.11]), $EG^\delta = G^\delta \ast G^\delta \ast \cdots$ and write an element $g \in EG^\delta$ as $g = (g_0, t_0; g_1, t_1; \ldots)$, where $g_i \in G_i$, only finitely many $t_i \neq 0$ and $\sum_{i=1}^\infty t_i = 1$. Then define $\mathbb{Z}G$-homomorphisms $\Phi_i : F_i(G) \to S_i(EG^\delta)$, $i = 0, 1, 2$, into the singular chain complex of $EG^\delta$ by

$$\Phi_0(g_0) := (g_0, t_0; 1, 0; \ldots),$$

$$\Phi_1(g_0, g_1) := (g_0, t_0; g_1, t_1; 1, 0; \ldots) - (g_1, t_0; g_1, t_1; 1, 0; \ldots),$$

$$\Phi_2(g_0, g_1, g_2) := (g_0, t_0; g_1, t_1; g_2, t_2; 1, 0; \ldots) - (g_0, t_0; g_2, t_1; g_2, t_2; 1, 0; \ldots) + (g_1, t_0; g_1, t_1; g_2, t_2; 1, 0; \ldots).$$

Here the terms depending on the $t_i$ represent singular simplices in $EG^\delta$. Then $d\Phi_i = \Phi_{i-1}d$ for $i = 1, 2$. Because $F_*(G^\delta)$ is free and $S_*(EG^\delta)$ is acyclic, there is a chain map $\Phi_* : F_*(G) \to S_*(EG^\delta)$, unique up to homotopy, extending the given $\Phi_i$’s [1, Lemma 1.7.4]. Because this $\Phi_*$ is an augmentation-preserving chain map between free resolutions of $\mathbb{Z}$ over $\mathbb{Z}G$, it is a homotopy equivalence [1, Theorem 1.7.5]. Finally $(-)_G$ is an additive functor, so $\Phi_*$ induces a homotopy equivalence

$$\overline{\Phi}_* : C_*(G) \to S_*(BG^\delta).$$

Computing $\overline{\Phi}_*(Q_n)$ gives

$$\overline{\Phi}_*(Q_n) = \sum_{j=1}^n \left( [1, t_0; \omega_n, t_1; \omega_n^{j+1}, t_2; 1, 0; \ldots] - [1, t_0; \omega_n^{j+1}, t_1; \omega_n^{j+1}, t_2; 1, 0; \ldots] + [\omega_n, t_0; \omega_n^{j+1}, t_1; \omega_n^{j+1}, t_2; 1, 0; \ldots] - [\omega_n, t_0; \omega_n, t_1; \omega_n^{j+1}, t_2; 1, 0; \ldots] \right)$$

$$= \sum_{j=1}^n \left( [1, t_0; \omega_n, t_1; \omega_n^{j+1}, t_2; 1, 0; \ldots] - [1, t_0; 1, t_1; \omega_n^{j}, t_2; 1, 0; \ldots] \right).$$

Pulling it back to $\mathbb{C}P^\infty$ via the homeomorphism

$$\mathbb{C}P^\infty \to BU(1) : [z_0 : z_1 : \cdots] \mapsto \left[ \frac{z_0}{|z_0|^2}, \frac{|z_0|^2}{|z_1|^2}, \frac{|z_1|^2}{|z_1|^2}, \ldots \right]$$

...
we get a cycle $P'_n$ in $\mathbb{C}P^2$, namely
\[
P'_n = \sum_{j=1}^{n} \left( [\sqrt{i0} : \sqrt{i1} \omega_n : \sqrt{i2} \omega_n^{j+1}] - [\sqrt{i0} : \sqrt{i1} : \sqrt{i2} \omega_n^{j+1}] \right).
\]
Because of
\[
d\left[ \sqrt{i0} : \sqrt{i1} + \sqrt{i2} \omega_n : \sqrt{i3} \omega_n^{j+1} \right] = [0 : \sqrt{i0} + \sqrt{i1} \omega_n : \sqrt{i2} \omega_n^{j+1}] - [\sqrt{i0} : \sqrt{i1} \omega_n : \sqrt{i2} \omega_n^{j+1}] + [\sqrt{i0} : \sqrt{i1} : \sqrt{i2} \omega_n^{j+1}] - [\sqrt{i0} : \sqrt{i1} + \sqrt{i2} \omega_n : 0].
\]
$P'_n$ is homologous to
\[
P'_n \sim \sum_{j=1}^{n} \left( [0 : \sqrt{i0} + \sqrt{i1} \omega_n : \sqrt{i2} \omega_n^{j+1}] - [\sqrt{i0} : \sqrt{i1} + \sqrt{i2} \omega_n : 0] \right)
\]
\[
= \sum_{j=1}^{n} [0 : \sqrt{i0} + \sqrt{i1} \omega_n : \sqrt{i2} \omega_n^{j+1}] \mod n.
\]
Using that $\mathbb{C}P^2 \to \mathbb{C}P^2 : [z_1 : z_2 : z_3] \mapsto [z_2 : z_3 : z_1]$ induces the identity in homology and adding $-nP_n$ to make the last cocycle become an integer cocycle completes the proof. \(\square\)

**Lemma 5.** $[P_n] = [\mathbb{C}P^1]$ in $H_2(\mathbb{C}P^1; \mathbb{Z})$.

**Proof.** Look at the relative homeomorphism $f : (\mathbb{D}^2, \partial \mathbb{D}^2) \to (\mathbb{C}P^1, [1 : 0]) : z \mapsto [z : 1 - |z|]$. Let $T_n : \mathbb{D}^2 \to \mathbb{D}^2 : z \mapsto \omega_n z$ be the rotation with angle $\omega_n$ and $\sigma : \Delta^2 \to \mathbb{D}^2$ the following simplex, homeomorphic onto its image:

$\sigma (i_0, i_1, i_2) := \frac{\sqrt{i0} + \sqrt{i1} \omega_n}{|\sqrt{i0} + \sqrt{i1} \omega_n| + \sqrt{i2}}$

Then the cycle $C := \sum_{j=1}^{n} T'_n(\sigma)$ represents the fundamental class of $(\mathbb{D}^2, \partial \mathbb{D}^2)$. But
\[
f_*C = \sum_{j=1}^{n} \left[ \frac{\omega_n}{|\sqrt{i0} + \sqrt{i1} \omega_n| + \sqrt{i2}} : 1 - \frac{|\sqrt{i0} + \sqrt{i1} \omega_n|}{|\sqrt{i0} + \sqrt{i1} \omega_n| + \sqrt{i2}} \right]
\]
\[
= \sum_{j=1}^{n} \left[ \sqrt{i0} + \sqrt{i1} \omega_n : \sqrt{i2} \omega_n^{j+1} \right].
\]
So in $H_2(\mathbb{C}P^1; \mathbb{Z}) = H_2(\mathbb{C}P^1, [1 : 0]; \mathbb{Z})$ we have $[\mathbb{C}P^1] = f_*[C] = [P_n]$. \(\square\)

Combining Lemmas 4 and 5 we get $D_*[Q_n] = [\mathbb{C}P^1]$. So by Lemma 3 we have $[\{\tau_1, [Q_n]\}] = 4 = 4e_1, [\mathbb{C}P^1] = 4e_1, D_*[Q_n], \mathbb{C}P^1 = 4D*e_1, [Q_n])$. Thus $[\tau_1] = 4D*e_1$ because $\{Q_n\}$ generates $H^*_EM(U(1); \mathbb{Z})$. By Lemma 2 this proves Lemma 1. That completes the proof of Theorem 4.
References