THE SIGNATURE THEOREM: REMINISCENCES AND RECREATION

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In the years immediately following 1952, when I came to the United States for the first time, I learned a lot of my mathematics in Princeton, at the Institute and at Fine Hall.

It may be a proper occasion to remember these old days. I shall recall (§1, §2) how the general Riemann-Roch theorem for complex manifolds is related to the signature theorem for differentiable manifolds. The signature theorem could be proved by Thom's cobordism theory ([25], [26]) and was then the basic tool for proving the Riemann-Roch theorem for algebraic manifolds [13]. This Riemann-Roch theorem (for complex manifolds) is nowadays a special case of the Atiyah-Singer index theorem for elliptic operators ([5], [6], [7], [8]): namely, the index theorem is applied to the Cauchy-Riemann equation. Atiyah-Bott-Singer generalized the index theorem to the equivariant case ([2], [3], [5], [7], [8]). This is a fantastic generalization of the Lefschetz fixed point theorem. The equivariant index theorem can be specialized to the signature operator ([8], p. 582). This gives a fixed point theorem involving the signature and generalizing the old signature theorem.

The old signature theorem involves Bernoulli numbers, and has many relations to number theory and applications in topology. Exotic spheres were discovered using it [21]. (We shall mention this in §3). The equivariant signature theorem has many more number theoretical connections. In the second half of this lecture we shall point out some rather elementary connections to number theory obtained by studying the equivariant signature theorem for four-dimensional manifolds. Perhaps these connections still belong to recreational mathematics because no deeper explanation, for example of the occurrence of Dedekind sums both in the theory of modular forms and in the study of 4-dimensional manifolds, is known.

As a theme (familiar to most topologists) under the general title "Prospects of mathematics" we propose "More and more number theory in topology."

Remark

The paper follows the original lecture rather closely, except for §6, §7, and §8, which were added later.

§1. *Conjecturing the Riemann-Roch theorem*

Let X be a compact complex manifold and W a holomorphic vector bundle over X. Let $\Omega(W)$ be the sheaf of germs of holomorphic sections of W. The cohomology groups $H^i(X,\Omega(W))$ vanish for $i > \dim_C X = n$ and are all finite-dimensional complex vector spaces.

We define the "Euler number"

$$\chi(X,W) = \sum_{i=0}^{n} (-1)^i \dim H^i(X,\Omega(W)).$$

Serre conjectured in a letter to Kodaira and Spencer (Sept. 29, 1953) that $\chi(X,W)$ is (for algebraic manifolds X) expressible in terms of the Chern classes of X and those of W.

How does one come from this general conjecture to an explicit one? This will be answered here in a very fast way using modern terminology (including the functor K which was introduced much later ([1], [4], [9])).

For holomorphic vector bundles W_1, W_2 over the compact complex manifold X we have

$$\chi(X,W_1 \oplus W_2) = \chi(X,W_1) + \chi(X,W_2). \tag{1}$$

For a holomorphic vector bundle W_1 over the compact complex manifold X_1 and a holomorphic vector bundle W_2 over the compact complex manifold X_2, we have

$$\chi((X_1 \times X_2, W_1 \otimes W_2) = \chi(X_1,W_1) \cdot \chi(X_2,W_2). \tag{2}$$

Serre's conjecture implies that $\chi(X,W)$ depends only on the topological class of the bundle. Therefore we introduce the ring $K(X)$, constructed from the semi-ring of isomorphism classes of topological complex vector bundles over X, and, because of (1), wish to construct an additive homomorphism

$$\tilde{\chi} : K(X) \to Q$$

such that

$$\chi(X,W) = \tilde{\chi}([W]).$$

where $[W]$ is the element of $K(X)$ represented by W.

Suppose

$$ch : K(X) \to H^{ev}(X,Q)$$

is an additive homomorphism from $K(X)$ to the ring of even-dimensional cohomology classes and that

$$td(X) \in H^{ev}(X,Q)$$

is a fixed cohomology class. Then

$$\xi \to (ch(\xi) \cdot td(X))\,[X], \quad \xi \in K(X), \tag{3}$$

is a candidate for $\tilde{\chi}$. Here $\alpha[X]$ means the evaluation of the cohomology class α on the fundamental cycle of X. Condition (1) is satisfied; condition (2) will be satisfied if ch is a ring homomorphism defined for all X and having the usual naturality properties, and if $td(X)$ is defined for all X and satisfies

$$td(X_1 \times X_2) = td(X_1) \otimes td(X_2). \tag{4}$$

How can one find such a $td(X)$? Assuming it depends only on the tangent bundle of X, we try to define $td(E) \in H^{ev}(B,Q)$ for any complex vector bundle E with base B, satisfying the usual naturality properties and such that

(5) $$td(E_1 \oplus E_2) = td(E_1) \cdot td(E_2)$$

if E_1, E_2 are complex vector bundles over the same base space B. If td(X) is defined as the value of td for the tangent bundle of X, then (4) is a consequence of (5). We normalize by requiring

(6) $td(E) = 1$, if E is the trivial line bundle.

To get a precise conjecture for the Riemann-Roch theorem in the form

(7) $$\chi(X,W) = (\text{ch}\,[W] \cdot td(X))\,[X],$$

we must specify ch and td. We choose for X the complex projective space $P_n(C)$ and put $W = F^k$, where F is the line bundle associated to the hyperplane of X which has characteristic class $x \in H^2(P_n(C),Z)$. Here x is the so-called positive generator of $H^2(P_n(C),Z)$; see [13], p. 138.
We have

(8) $$\chi(P_n(C),F^k) = \binom{n+k}{k}.$$

In view of the desired properties of ch and td considered above, the special example (8) determines ch and td completely. We recall this for td. The tangent bundle of $P_n(C)$ plus the trivial line bundle equals $F \oplus \ldots \oplus F = (n+1)F$. Therefore by (5) and (6),

$$td(P_n(C)) = (td(F))^{n+1}$$

with

$$td(F) = 1 + b_1 x + b_2 x^2 + \ldots = f(x), \quad b_i \in Q,$$

where f is an infinite formal power series determined uniquely (by naturality) for $n \to \infty$. This is the characteristic power series for td. Since the arithmetic genus of $P_n(C)$ equals 1, (this is the case $k = 0$ in (8), where F^0 is the trivial line bundle), we must have, by (7),

(9) $$td(P_n(C))\,[P_n(C)] = 1.$$

For the characteristic power series f, the formula (9) means

(10) *the coefficient of* x^n *in* $(f(x))^{n+1}$ *equals* 1 *for all* n.

The only power series with constant term 1 satisfying (10) is

$$f(x) = \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}$$

Here Bernoulli numbers show up in topology.

The final result for ch (Chern character) and td (Todd class) is the following: Let E be a complex vector bundle (fibre C^n) with base space B, and c(E) its total Chern class,

$$c(E) = 1 + c_1(E) + c_2(E) + \ldots + c_n(E) \in H^{ev}(B,Q).$$

Suppose there is a space B′ and a map $\pi : B' \to B$ such that π^* is injective for rational cohomology and π^*E is a direct sum of line bundles $E_i (1 \leq i \leq n)$ where E_i has the characteristic class $x_i \in H^2(B',Z)$. Such a B′ always exists (splitting principle). We regard x_i as element of $H^2(B',Q)$. Then

(11)
$$\pi^* c(E) = (1+x_1) \ldots (1+x_n),$$
$$\pi^* \text{ch}(E) = e^{x_1} + \ldots + e^{x_n},$$
$$\pi^* td(E) = \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}}$$

After ch and td have thus been found, (7) is the explicit form of the Riemann-Roch conjecture.

§2. *The signature theorem*

Let X be a compact oriented manifold of dimension $4k$ without boundary. Then $H^{2k}(X,R)$ is a finite dimensional real vector space over which we have a bilinear symmetric non-degenerate form B defined by

$$B(x,y) = (x \cup y)[X], \text{ for } x,y \in H^{2k}(X,R). \tag{12}$$

The signature of this form B, i.e., the number of positive entries minus the number of negative entries in a diagonalized version, is called sign (X). It was first introduced by H. Weyl [27]. The signature theorem ([13], p. 86) claims that (for a differentiable manifold X) the signature of X is a universal linear combination of Pontrjagin numbers. For example, we have for a 12-dimensional manifold

$$\operatorname{sign}(X^{12}) = \frac{1}{3^3.5.7}(62\,p_3 - 13\,p_2p_1 + 2p_1{}^3)[X] \tag{13}$$

($p_i \in H^{4i}(X,Z)$ are the Pontrjagin classes).

Formula (13) implies, for example, that for a 12-dimensional compact oriented differentiable manifold X without boundary the signature is divisible by 62 if the fourth Betti number of X is zero.

How can one arrive at a conjecture for the signature theorem? I shall now give an answer to this question. But I do not claim that this is the way that I arrived at the conjecture.

Let X be a projective algebraic manifold with $\dim_C X = 2k$. Let T^* be its dual tangent bundle. Then a theorem of Hodge (compare [13], p. 125) can be written in the form

$$\chi(X,\Lambda^*T^*) = \operatorname{sign} X. \tag{14}$$

Here Λ^*T^* is the exterior algebra bundle of T. We write for the total Chern class of X

$$c(X) = \prod_{i=1}^{2k}(1+x_i), \text{ where } x_i \in H^2(\ ,Q).$$

(We do not specify the space over which the tangent bundle of X splits and forget to write π^*; compare (11).) Using known formulas for $ch(\Lambda^*T^*)$ we get from (7) and (14)

$$\operatorname{sign} X = \left(\prod_{i=1}^{2k}(1+e^{-x_i})\prod_{i=1}^{2k}\left(\frac{x_i}{1-e^{-x_i}}\right)\right)[X].$$

Thus

$$\operatorname{sign} X = \prod_{i=1}^{2k}\frac{x_i}{\tanh\frac{x_i}{2}}[X] = \prod_{i=1}^{2k}\frac{x_i}{\tanh x_i}[X]. \tag{15}$$

Since

$$\frac{x}{\tanh x} = 1 + \sum_{k=1}^{\infty}(-1)^{k+1}B_k\frac{2^{2k}x^{2k}}{(2k)!}$$

is in fact a power series in x^2, and the i^{th} elementary symmetric function of the $x_j{}^2$ is the Pontrjagin class p_i of X, formula (15) makes sense for differentiable manifolds and is the signature theorem to be conjectured.

How to prove it? After conjecturing it I went to the library of the Institute for Advanced Study (June 2, 1953). Thom's *Comptes Rendus* note [25] had just arrived. This finished the proof. The signature theorem (for algebraic manifolds a special case of the Riemann-Roch theorem) was the starting point for the proof of the Riemann-Roch theorem in the case of algebraic manifolds [13].

Remark

The actual process of conjecturing the Riemann-Roch theorem and the signature theorem was not so straightforward as the process described in §1 and §2. In the Princeton days of 1952-54, much of my work arose from discussions with Kodaira and Spencer and was motivated by the old papers of J. A. Todd. Kodaira and Spencer explained their work to me. For example, I learned from them that for a line bundle F over a given X, the number $\chi(X,F)$ depends only on the cohomology class of F.

Studying $\chi(X,F)$, especially the arithmetic genus (Todd genus), led to the power series

$$f(x) = \frac{x}{1-e^{-x}}$$

in a way very close to the story told in §1. When dealing with Pontrjagin classes, we need a power series in x^2. Fortunately, $f(x) - \frac{x}{2}$ is such a power series. We have

$$f(2x) = x + \frac{x}{\tanh x}.$$

Already this simple observation motivated the study of the multiplicative sequence with characteristic power series $\frac{x}{\tanh x}$ and gave rise to the signature theorem and its relation to the Riemann-Roch theorem (see for example [13], §13.6). Serre's letter of September 1953 helped to clarify the situation. The Riemann-Roch theorem was proved shortly afterwards (December 10, 1953 approximately).

Much of the machinery of characteristic classes I learned from discussions with A. Borel. Perhaps I might use this occasion to express my thanks to Princeton and the mathematicians who were there in 1952-54.

§3. *An exotic sphere*

Milnor's original idea [21] for constructing an exotic sphere and the later work by Kervaire and Milnor [19] are recognizable in the following example arising from the work of Brieskorn [10]. This example is a typical application of the signature theorem.

Consider (for small $\varepsilon \neq 0$) the subset of C^7 defined by

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^3 + z_7^5 = \varepsilon,$$

$$\sum_{i=1}^{7} z_i \bar{z}_i \leq 1.$$

This is a 12-dimensional manifold M^{12} with boundary. According to Brieskorn, $\partial M^{12} = \Sigma^{11}$ is a topological sphere and M^{12} / Σ^{11} (i.e., we collapse the boundary to a point) is a 12-dimensional manifold N^{12} without boundary. N^{12} is 5-connected and has signature -8. If N^{12} could be made into a differentiable manifold, then sign $N^{12} = -8$ would be divisible by 62 (see (13)). Therefore N^{12} does not carry any differentiable structure. (A manifold without any differentiable structure was first constructed by Kervaire [18].) It follows that Σ^{11} is not diffeomorphic to the standard sphere, i.e., the differentiable manifold Σ^{11} is an exotic sphere. Compare also Kuiper [20].

§4. *The equivariant signature theorem in the 4-dimensional case*

In the preceding sections we have told the oid story "How to reach the signature theorem," and have given a typical application. As explained in the introduction, the Atiyah-Bott-Singer index and fixed point theorems contain as a special case the equivariant signature theorem. This theorem has many relations to number theory; its full impact can be seen only by studying it for manifolds of arbitrary even dimension. We restrict ourselves to 4-dimensional compact oriented differentiable manifolds without boundary. For such a manifold M^4 the ordinary signature theorem simply says

$$\text{sign } M^4 = \frac{1}{3} p_1 [M^4]. \tag{16}$$

where $p_1 \in H^4(M^4,Z)$ is the Pontrjagin class. Let us formulate the equivariant signature theorem in the 4-dimensional case.

Let M be a compact oriented differentiable manifold without boundary and G a finite group acting on M by orientation preserving diffeomorphisms. As in §2, consider over $H^2(M,R)$ the bilinear symmetric non-degenerate form B. Then $H^2(M,R)$ is a G-module, and the action of G on $H^2(M,R)$ preserves B. We can decompose $H^2(M,R)$ as follows:

$$H^2(M,R) = H_+ \oplus H_-,$$

where H_+ and H_- are B-orthogonal, B is positive-definite on H_+ and negative-definite on H_-, and

$$g(H_+) = H_+, g(H_-) = H_-, \text{ for all } g \in G.$$

For any $g \in G$ we define

$$\operatorname{sign}(g,M) = \operatorname{tr}(g|H_+) - \operatorname{tr}(g|H_-),$$

where tr denotes the trace. It is easy to show that sign (g,M) does not depend on the choice of H_+ and H_- ([8], p. 578). If g acts on M as the identity, then

$$\operatorname{sign}(g,M) = \operatorname{sign} M.$$

The number sign (g,M) depends only on the action of the individual element g on M and not on the action of the whole group G. (The definition for sign (g,M) given here works, of course, for all 4k-dimensional manifolds, not only for 4-dimensional manifolds.)

We assume from now on that M is connected and G acts effectively. Let M^g be the set of fixed point of g, which for $g \neq 1$ consists of isolated points x_j and connected 2-manifolds Y_k. Following [8], Proposition 6.18, we formulate the equivariant G-signature theorem. For each j, let the action of g on the tangent space at x_j be given by the matrix

$$(17) \qquad \begin{pmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{pmatrix} \oplus \begin{pmatrix} \cos\beta_j & -\sin\beta_j \\ \sin\beta_j & \cos\beta_j \end{pmatrix}$$

relative to an oriented basis of the tangent space. For each k, let the diffeomorphism g induce a rotation in the normal bundle of Y_k by an angle $\pm\theta_k$. Let $Y_k \cdot Y_k$ denote the self-intersection number of Y_k (which is also defined when Y_k is not orientable; see for example [14], p. 152).

We have (for $g \neq 1$)

$$(18) \qquad \operatorname{sign}(g,M) = \sum_j \left(-\cot\frac{\alpha_j}{2} \cdot \cot\frac{\beta_j}{2}\right) + \sum_k \frac{1}{\sin^2\frac{\theta_k}{2}} \cdot (Y_k \cdot Y_k).$$

Formula (18) is the equivariant G-signature theorem in the 4-dimensional case. Observe that $\cot\frac{\alpha_j}{2} \cdot \cot\frac{\beta_j}{2}$ does not change if the pair (α_j, β_j) is permuted or replaced by $(-\alpha_j, -\beta_j)$.

Let M and G be as before, with G acting effectively by orientation preserving diffeomorphisms. The orbit space M/G is a rational homology manifold with oriented fundamental cycle and the signature of M/G can be defined as usual though M/G is not a manifold. If $\pi : M \to M/G$ is the canonical map, then π^* maps $H^2(M/G,R)$ bijectively on $H^2(M,R)^G$, the subspace of the vector space $H^2(M,R)$ consisting of the elements invariant under all $g \in G$. An elementary formula of representation theory together with the fact that π^* preserves the cup-product yields

$$(19) \qquad \operatorname{sign} M/G = \frac{1}{|G|} \sum_{g \in G} \operatorname{sign}(g,M)$$

or equivalently,

$$(20) \qquad \operatorname{sign} M = |G| \cdot \operatorname{sign} M/G - \sum_{g \neq 1} \operatorname{sign}(g,M).$$

If g acts freely, then by (18) the number sign (g,M) vanishes for $g \neq 1$, and

$$(21) \qquad \operatorname{sign} M = |G| \cdot \operatorname{sign} M/G,$$

a fact which also follows immediately from the ordinary signature Theorem (16). In view of (20) and (21) we call $\sum_{g \neq 1} \operatorname{sign}(g,M)$ the (total) *signature-defect* of the given effective G-action on M.

For any point $x \in M$, we consider the subset $G(x)$ of G consisting of those elements $g \in G$ which have x as an isolated fixed point. For only finitely many x is the set $G(x)$ non-empty. For any $g \in G(x)$ we have the angles $\alpha_{g,x}$ and $\beta_{g,x}$ (see (17)).

We define the *signature-defect at* x by the formula

$$\text{(22)} \qquad \operatorname{def}_x = \sum_{g \in G(x)} -\cot(\alpha_{g,x}/2) \cdot \cot(\beta_{g,x}/2).$$

For any 2-dimensional connected submanifold Y of M (not necessarily orientable) we consider the subset $G(Y)$ of G consisting of those elements $g \in G$ for which Y is a connectedness component of M^g. There are only finitely many Y for which the set $G(Y)$ is not empty. For any $g \in G(Y)$ we have the angle $\theta_{g,Y}$ (see (18)). We define the *signature-defect at* Y by the formula

$$\text{(23)} \qquad \operatorname{def}_Y = \sum_{g \in G(Y)} \frac{1}{\sin^2(\theta_{g,Y}/2)} (Y \cdot Y).$$

We obtain for the signature defect of the whole G-action the formula

$$\text{(24)} \qquad |G| \cdot \operatorname{sign} M/G - \operatorname{sign} M = \sum_{g \neq 1} \operatorname{sign}(g,M) = \sum_x \operatorname{def}_x + \sum_Y \operatorname{def}_Y.$$

We wish to study the signature defects at Y and x in some particular cases. For a connected 2-dimensional manifold Y the elements $g \in G$ with $Y \subset M^g$ constitute a subgroup G_Y of G and we have $G(Y) = G_Y - \{1\}$. The group G_Y is always cyclic, say of order n. Then

$$\operatorname{def}_Y = \sum_{k=1}^{n-1} \frac{1}{\sin^2 \frac{\pi k}{n}} \cdot (Y \cdot Y).$$

It is easy to verify

$$\text{(25)} \qquad \sum_{k=1}^{n-1} \frac{1}{\sin^2 \frac{\pi k}{n}} = \frac{n^2-1}{3}.$$

Therefore

$$\text{(26)} \qquad \operatorname{def}_Y = \frac{n^2-1}{3} (Y \cdot Y) \text{ where } n = |G_Y|.$$

It is much more complicated to calculate the defect at points x. We shall study def_x in the case where the isotropy group G_x is cyclic, of order p say, and where G_x operates freely on the tangent space at x (with the origin removed). Let ρ be a generator of G_x. The action of ρ on the tangent space at x is given, relative to an oriented basis, by a matrix of the form (17) with (α_j, β_j) replaced by $(2\pi\frac{q}{p}, 2\pi\frac{r}{p})$, where q,r are relatively prime to p. When ρ is chosen, the residue classes of q and r modulo p are determined up to their order and up to replacing q,r by $-q, -r$ modulo p.

For a natural number $p \geq 1$ and integers q,r prime to p we define

$$\text{(27)} \qquad \operatorname{def}(p;q,r) = -\sum_{k=1}^{p-1} \cot\frac{\pi qk}{p} \cdot \cot\frac{\pi rk}{p}.$$

By (22) this is the signature defect at x if the isotropy group G_x acts on the tangent space at x as described above. We have

i) $\operatorname{def}(p;q,r) = \operatorname{def}(p;r,q)$
ii) $\operatorname{def}(p;q,r) = \operatorname{def}(p;q',r')$ if $q \equiv q' \bmod p$ and $r \equiv r' \bmod p$
iii) $\operatorname{def}(p;sq,sr) = \operatorname{def}(p;q,r)$ if s is prime to p.
iv) $\operatorname{def}(p;-q,r) = -\operatorname{def}(p;q,r)$.

The equation iii) corresponds to different choices of the generator ρ of G_x. If, according to ii), we regard q,r as residue classes mod p we may write in view of iii) and i)

v) $\operatorname{def}(p;q,r) = \operatorname{def}(p;\frac{q}{r},1) = \operatorname{def}(p;\frac{r}{q},1)$.

Therefore, we only have to calculate the signature defects $\operatorname{def}(p;q,1)$. For $q = 1$ we have by (25),

(28) $$\operatorname{def}(p;1,1) = -\frac{(p-1)\cdot(p-2)}{3}.$$

Remark

As an invariant for a free G-manifold of odd dimension a complex-valued function α on $G-\{1\}$ can be defined using the equivariant signature theorem. (See [8], p. 590; the function σ introduced in [8] equals $-\alpha$. The function α was used for example in [16]. Compare the lecture of C.T.C. Wall in this conference.) The free action given by q,r of the cyclic group μ_p of order p on the 3-sphere has a lens space as orbit space. The function $\xi \to \alpha(\xi, S^3)$, $\xi \in \mu_p$, $\xi \neq 1$, can be used for the classification of lens spaces [3]. We have, for the free action of μ_p on S^2 given by q,r,

$$\operatorname{def}(p;q,r) = -\sum_{\xi \in \mu_p - \{1\}} \alpha(\xi, S^3).$$

The summands in this sum are exactly the summands in (27).

By summing over ξ we lose information, like passing from a character to its degree. Nevertheless, the "degrees" def(p;q,r) are quite interesting – as we shall see.

§5. *The equivariant signature theorem and some number theory.*

It is amusing that the signature defects,

(29) $$\operatorname{def}(p;q,1) = -\sum_{k=1}^{p-1} \cot\frac{\pi k}{p} \cdot \cot\frac{\pi qk}{p},$$

occur in the classical literature.

PROPOSITION. *Let* $p \geq 1$ *be a natural number and* q *an integer prime to* p. *Then*

(30) $$\operatorname{def}(p;q,1) = -\frac{2}{3}(q,p),$$

where (q,p) *is the Dedekind symbol (also called Dedekind sum) introduced in* [11], *formulas* (11) *and* (12). *The Dedekind symbol is always an integer.*

The proof of (30) is in Rademacher [23], and uses the function $((\ )) : R \to R$ defined by

$$((x)) = x - [x] - \tfrac{1}{2}, \text{ if } x \text{ is not an integer,}$$
$$((x)) = 0 \qquad , \text{ if } x \text{ is an integer.}$$

Rademacher proves ([23], pp. 276-277)

(31) $$\sum_{k=1}^{p-1} \cot\frac{\pi k}{p} \cdot \cot\frac{\pi qk}{p} = 4p \sum_{k=0}^{p-1} ((\tfrac{k}{p})) \cdot ((\tfrac{qk}{p})).$$

[Dedekind also uses a function (( )). It is the above function (( )) used by Rademacher with a shift of $\frac{1}{2}$ in the independent variable. We use the (( )) of Rademacher.] We have by [11], formula (32),

(32) $$(q,p) = 6p \sum_{k=0}^{p-1} ((\tfrac{k}{p})) \cdot ((\tfrac{qk}{p})).$$

Formulas (31) and (32) imply (30). Dedekind proves that (q,p) is an integer.

Let us recall how Dedekind [11] defined his symbol and why he was interested in it. Consider in the upper half plane H the function

$$F(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1-e^{2\pi inz})^{24}$$

It is a modular function of weight 12 (compare e.g. [24], p. 154).

Since H is simply connected we can define a holomorphic function f(z) in H with

$$e^{f(z)} = F(z)$$

and

$$\lim_{\substack{y \to \infty \\ y > 0}} (f(x+iy) - 2\pi i(x+iy)) = 0.$$

We have

$$f(z+b) = f(z) + 2\pi ib, \text{ for } b \in Z.$$

Dedekind investigates the behaviour of $f(z)$ under the modular group. His result is the following.

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z)$ *there is a number* (d,c) *depending only on* d *and* c *such that*

$$(33) \qquad c \cdot f\left(\frac{az+b}{cz+d}\right) = c \cdot f(z) + 6c \cdot \log\{-(cz+d)^2\} + 2\pi i\,(a+d-2(d,c)),$$

where the function $c \log\{-(cz+d)^2\}$ *is defined in the upper half plane without ambiguity by requiring that the absolute value of its imaginary part is* $\leq |\pi c|$.

Of course, for $c \neq 0$ the number $c^{-1} \cdot (a+d-2(d,c))$ is an integer. Dedekind uses (33) as definition of his symbol (d,c) for any pair of coprime integers. First consequences of (33) are

i) $(1,0) = 1$
ii) $(-d,-c) = -(d,c)$.

Replacing in (33) the variable z by $-\bar{z}$ leads to

$$(d,c) = (d,-c),$$

so by ii) we have

iii) $(-d,c) = -(d,c)$,

which implies

$$(0,1) = 0.$$

Replacing in (33) the variable z by $z+1$ or by $-\frac{1}{z}$ respectively, leads to the equations

iv) $(d,c) = (d',c)$ for $d \equiv d' \bmod c$
v) $2d(d,c) + 2c(c,d) = 1+c^2+d^2-3|cd|$.

Formula v) is the reciprocity law for Dedekind symbols.

Clearly, i) – v) allow calculating (d,c) for any coprime pair. *There is exactly one and only one real valued symbol defined for pairs of coprime integers satisfying* i) – v).

The reciprocity law v) and the other properties of the Dedekind symbol imply for $c \geq 0$.

$$(34) \qquad (1,c) = \frac{(c-1)\cdot(c-2)}{2}, \quad (2,c) = \frac{(c-1)\cdot(c-5)}{4};$$

for the first formula compare (28). In the second formula c has to be odd.

Formula (30) shows that our signature defect $\operatorname{def}(p;q,1)$ equals the Dedekind symbol (up to a factor). This followed from formulas (31) and (32) of Rademacher and Dedekind. We might try to prove (30) alternatively by establishing properties i) – v) for the signature defect. We only defined $\operatorname{def}(p;q,1)$ for $p \geq 1$ and q relatively prime to p. Therefore we do not have i) and ii), but we have

$$(35) \qquad \begin{aligned} &\operatorname{def}(1;0,1) = 0 \\ &\operatorname{def}(p;q,1) = \operatorname{def}(p;q',1) \text{ if } q \equiv q' \bmod p. \end{aligned}$$

Therefore, we only have to prove the reciprocity law for $\operatorname{def}(p;q,1)$ corresponding to v) because, clearly, (35) and the reciprocity law establish already an inductive process for the calculation of $\operatorname{def}(p;q,1)$, and this will prove again what we want:

$$\operatorname{def}(p;q,1) = -\frac{2}{3}(q,p).$$

PROPOSITION (Reciprocity law). *Let* p,q,r *be pairwise coprime natural numbers* ≥ 1. *Then*

$$(36) \qquad qr \operatorname{def}(p;q,r) + pr \operatorname{def}(q;p,r) + pq \operatorname{def}(r;p,q) = pqr - \frac{p^2+q^2+r^2}{3}.$$

This uses the notation of (27). For $r=1$, formula (36) is the reciprocity law corresponding to v). Formula (36) is essentially a formula due to Rademacher ([23], p. 272). In the next paragraph we shall prove (36) by using the equivariant signature theorem for suitable 4-dimensional manifolds and group actions. It is pretty clear that (36) must be related to a situation where we have

(37) qr *points with isotropy group cyclic of order* p *and induced representation in the tangent space given by* q,r, *and correspondingly for* p,q,r *cyclic permuted.*

Before constructing an example for (37) we make some number theoretical remarks. The right side of (36) vanishes if and only if

$$p^2+q^2+r^2 = 3\,pqr. \tag{38}$$

A triple (p,q,r) satisfying (38), for example (194,13,5), is called a Markoff triple; compare [23] and [12]. For a Markoff triple the signature defects of the qr + pr + pq points in a situation (37) do not give a contribution to the total signature defect (20), since their sum is zero. Indeed, each one is zero: *If* (p,q,r) *is a Markoff triple, then*

$$\mathrm{def}(p;q,r) = \mathrm{def}(q;p,r) = \mathrm{def}(r;p,q) = 0.$$

Proof. $q^2+r^2 \equiv 0 \bmod p$ implies $\frac{q}{r} \equiv -\frac{r}{q} \bmod p$. The result follows from iv) and v) in §4.

We close this section by pointing out the connection between the Dedekind symbol and the quadratic reciprocity law.

For coprime integers p,q *with* p *odd,* $p \geq 1$,

$$3\,\mathrm{def}(p;q,1) \equiv 2\left(\frac{q}{p}\right)-p-1 \pmod 8, \tag{39}$$

i.e., $\left(\frac{q}{p}\right) + (q,p) \equiv \frac{p+1}{2} \bmod 4$, *where* $\left(\frac{q}{p}\right)$ *is the Jacobi-Legendre symbol.*

From this fact and the Dedekind reciprocity v) the quadratic reciprocity theorem follows easily. Presumably, the congruence (39) exists in the classical literature. Don Zagier told me a very nice proof for (39).

§6. *Proof of the Rademacher reciprocity theorem by the equivariant G-signature theorem.*

Consider the non-singular algebraic surface V_n in $P_3(C)$,

$$V_n : z_0{}^n+z_1{}^n+z_2{}^n+z_3{}^n = 0.$$

Let μ_n be the group of n–th roots of unity. It acts on V_n by

$$(z_0,z_1,z_2,z_3) \to (\zeta^{-1}z_0,z_1,z_2,z_3),\ \zeta \in \mu_n.$$

The orbit space is $P_2(C)$ which has signature 1; therefore by (24),

$$n - \mathrm{sign}\ V_n = \mathrm{def}_Y$$

where Y is the curve in V_n given by $z_0 = 0$.

Since $Y \cdot Y = n$, we obtain by (26),

$$\mathrm{sign}\ V_n = n - \frac{n(n^2-1)}{3} = \frac{n(4-n^2)}{3} \tag{40}$$

Of course, (40) is well-known and could be used conversely to prove the trigonometrical formula (25).

Let p,q,r be pairwise coprime natural numbers ≥ 1. Consider

$$V = V_{pqr} : z_0{}^{pqr} + z_1{}^{pqr} + z_2{}^{pqr} + z_3{}^{pqr} = 0,$$

and on it the action of $H = \mu_{pqr} \times \mu_p \times \mu_q \times \mu_r$ given by

$$(z_0,z_1,z_2,z_3) \to (\zeta^{-1}z_0,\alpha z_1,\ \beta z_2,\ \gamma z_3)$$

for $(\zeta,\alpha,\beta,\gamma) \in H$. We put $\mu_{pqr} = G_1$ and $\mu_p \times \mu_q \times \mu_r = G_2$. We have

$$V/H = (V/G_1)/G_2 = (V/G_2)/G_1.$$

Since V/G_1 is the complex projective plane whose cohomology remains invariant under the action of G_2, we have

$$\mathrm{sign}\ V/H = \mathrm{sign}\ (V/G_2)/G_1 = 1. \tag{41}$$

We now calculate sign V/G_2 by applying (24) to the case M=V and $G=G_2$. There are three connected 2-dimensional manifolds (curves) Y_1,Y_2,Y_3 in V given by $z_i = 0$ (i=1,2,3). The sets $G(Y_i)$ are $\mu_p-\{1\}$, $\mu_q-\{1\}$, $\mu_r-\{1\}$

Therefore by (26)

$$(42) \qquad \sum_{i=1}^{3} \operatorname{def}_{Y_i} = pqr\left(\frac{p^2-1}{3} + \frac{p^2-1}{3} + \frac{r^2-1}{3}\right).$$

There are three sets of pqr points each. The first set is given by $z_2=z_3=0$, the second by $z_1=z_3=0$, the third by $z_1=z_2=0$. The isotropy group $(G_2)_x$ for any point x in the first set is $\mu_q \times \mu_r$.

The action of $(G_2)_x$ in the tangent space at x is the direct sum of the standard one-dimensional complex representation of μ_q with that of μ_r. Therefore

$$\operatorname{def}_x = -\left(\sum_{j=1}^{q-1} \cot\frac{\pi j}{q}\right)\cdot\left(\sum_{k=1}^{r-1} \cot\frac{\pi k}{r}\right) = 0$$

(each factor in the preceding line is 0, because

$$\cot\frac{\pi j}{q} = -\cot\frac{\pi(q-j)}{q}).$$

The same holds for the second and third set. Thus $\Sigma_x \operatorname{def}_x = 0$. If we apply (24) to our case, then by (40), (42), and (24),

$$(43) \qquad \operatorname{sign} V/G_2 = \frac{1}{3}(p^2+q^2+r^2-p^2q^2r^2+1).$$

V/G_2 *is non-singular*. This follows immediately by looking at the action in the neighborhood of the curves Y_i and the points x. We may therefore apply (24) to the action of G_1 on V/G_2. There is one curve Y, namely the curve in V/G_2 given by $z_0=0$. We have $Y \cdot Y = 1$ and

$$(44) \qquad \operatorname{def}_Y = \frac{p^2q^2r^2-1}{3}, \text{ (see (26))}.$$

$V/G_2 - Y$ can be identified under $t_1=z_1{}^p, t_2=z_2{}^q, t_3=z_3{}^r$ with the non-singular affine surface

$$W : t_1{}^{qr} + t_2{}^{pr} + t_3{}^{pq} = -1$$

on which $G_1 = \mu_{pqr}$ acts by

$$(t_1,t_2,t_3) \to (\zeta^p t_1, \zeta^q t_2, \zeta^r t_3), \text{ for } \zeta \in G_1.$$

On W we have exactly the situation (37).

Therefore

$$(45) \qquad \sum_{x \in W} \operatorname{def}_x = qr \operatorname{def}(p;q,r) + pr \operatorname{def}(q;p,r) + pq \operatorname{def}(r;p,q).$$

Applying (24) to the G_1-action on V/G_2 gives, in view of (43), (41), (44), (45),

$$pqr - \frac{1}{3}(p^2+q^2+r^2-p^2q^2r^2+1) =$$

$$\frac{p^2q^2r^2-1}{3} + qr \operatorname{def}(p;q,r) + pr \operatorname{def}(q;p,r) + pq \operatorname{def}(r;p,q)$$

which is exactly Rademacher's reciprocity formula (36).

§7. *An identity due to Mordell*

Mordell [22] has proved a relation between Dedekind sums and the number $N_3(p,q,r)$ of lattice points in the tetrahedron

$$0 \le x < p,\ 0 \le y < q,\ 0 \le z < r,\ 0 < \frac{x}{p} + \frac{y}{q} + \frac{z}{r} < 1,$$

where p,q,r are pairwise coprime. Let $N_3'(p,q,r)$ be the number of lattice points satisfying

$$0 < x < p,\ 0 < y < q,\ 0 < z < r,\ 0 < \frac{x}{p} + \frac{y}{q} + \frac{z}{r} < 1.$$

Then

$$N_3(p,q,r) = N_3'(p,q,r) + \frac{1}{2}(qr+pr+pq-3).$$

The signature t(p,q,r) of the affine surface