Singularities and exotic spheres


Brieskorn has proved [4] that the n-dimensional affine algebraic variety

\[ z_0^a + z_1^a + \ldots + z_n^a = 0 \quad (n \text{ odd}, \quad n \geq 1) \]

is a topological manifold though the variety has an isolated singular point (which is normal for \( n \geq 2 \)). Such a phenomenon cannot occur for normal singularities of 2-dimensional varieties, as was shown by Mumford ([12], [6]). Brieskorn's result stimulated further research on the topology of isolated singularities (Brieskorn [5], Milnor [11] and the speaker [6], [7]). Brieskorn [5] uses the paper of F. Pham [14], whereas the speaker studied certain singularities from the point of view of transformation groups using results of Bredon ([2], [3]), W. C. Hsiang and W. Y. Hsiang [8] and Jänich [9].

§ 1. The integral homology of some affine hypersurfaces

Pham [14] studies the non-singular subvariety \( V_\alpha = V(\alpha_0, \alpha_1, \ldots, \alpha_n) \) of \( \mathbb{C}^{n+1} \) given by

\[ z_0^a + z_1^a + \ldots + z_n^a = 1 \quad (n \geq 0), \]

where \( \alpha = (\alpha_0, \ldots, \alpha_n) \) consists of integers \( \alpha_j \geq 2 \).

Let \( G_{\alpha} \) be the cyclic group of order \( \alpha_j \) multiplicatively written and generated by \( w_j \). Define the group \( G_\alpha = G_{\alpha_0} \times G_{\alpha_1} \times \ldots \times G_{\alpha_n} \) and put \( \alpha_j = \exp (2 \pi i / \alpha_j) \).

Then \( w_0^a w_1^b \ldots w_n^c \) is an element of \( G_\alpha \) whereas \( \delta_0^a \delta_1^b \ldots \delta_n^c \) is a complex number. \( G_\alpha \) operates on \( V_\alpha \) by

\[ w_0^a w_1^b \ldots w_n^c (x_0, \ldots, x_n) = (\alpha_j x_0, \ldots, \alpha_j x_n). \]

Let \( \hat{G}_\alpha \) be the group of \( a_j \)-th roots of unity and \( x \mapsto \hat{x} \) the isomorphism \( G_\alpha \to \hat{G}_\alpha \) given by \( w_j \mapsto \hat{w}_j \). Pham considers the following subspace \( U_\alpha \) of \( V_\alpha \)

\[ U_\alpha = \{ z \mid z \in V_\alpha \text{ and } z_j \text{ real } \geq 0 \text{ for } j = 0, \ldots, n \}. \]

Lemma. The subspace \( U_\alpha \) is a deformation retract of \( V_\alpha \) by a deformation compatible with the operations of \( G_\alpha \).

For the proof see Pham [14], p. 338.

\( U_\alpha \) can also be described by the conditions

\[ z_j = u_j | z_j | \quad \text{with} \quad u_j \in \hat{G}_\alpha \quad (j = 0, \ldots, n). \]

Put \( | z^u | = t_j \). Then \( U_\alpha \) becomes the space of \((n+1)\)-tuples of complex numbers

\[ t_0 u_0 \oplus t_1 u_1 \oplus \ldots \oplus t_n u_n \]

where \( u_j \in \hat{G}_\alpha \), \( t_j \geq 0 \), \( \sum_{j=0}^{n} t_j = 1 \).

Thus \( U_\alpha \) can be identified with the join \( G_{\alpha_0} * G_{\alpha_1} * \ldots * G_{\alpha_n} \) of the finite sets \( G_{\alpha_i} \) (see Milnor [10]).

Lemma 2.1 in [10] states in particular that the reduced integral homology groups of the join \( A * B \) of two spaces \( A, B \) without torsion are given by a canonical isomorphism

\[ \tilde{H}_{r+1} (A * B) \cong \bigoplus_{i+j=r} \tilde{H}_i (A) \otimes \tilde{H}_j (B), \]

whereas Lemma 2.2 in [10] shows that \( A * B \) is simply connected provided \( B \) is arcwise connected and \( A \) is any non-vacuous space. These properties of the join together with its associativity imply

Theorem. The subvariety \( V_\alpha \) of \( \mathbb{C}^{n+1} \) is \((n-1)\)-connected. Moreover

\[ \tilde{H}_r (V_\alpha) \cong \tilde{H}_0 (G_{\alpha_0}) \otimes \tilde{H}_0 (G_{\alpha_1}) \otimes \ldots \otimes \tilde{H}_0 (G_{\alpha_n}) \].

This is a free abelian group of rank \( r = \prod (\alpha_j - 1) \).

The isomorphism (1) is compatible with the operations of \( G_{\alpha} \). All other reduced integral homology groups of \( V_\alpha \) vanish.

It can be shown that \( V_\alpha \) has the homotopy type of a connected union \( S^a \vee \ldots \vee S^a \) of spheres of dimension \( n \).

The identification of \( U_\alpha \) with a join was explained to the speaker by Milnor.

\( U_\alpha = G_{\alpha_0} * G_{\alpha_1} * \ldots * G_{\alpha_n} \) is an \( n \)-dimensional simplicial complex which has an \( n \)-simplex for each element of \( G_{\alpha} \). The n-simplex belonging to the unit of \( G_{\alpha} \) is denoted by \( e \). All other n-simplices are obtained from \( e \) by operations of \( G_{\alpha} \). Thus we have for the \( n \)-dimensional simplicial chain group

\[ C_n (U_\alpha) = J_n e \]

where \( J_n \) is the group ring of \( G_{\alpha} \). The homology group \( \tilde{H}_n (U_\alpha) = \tilde{H}_n (V_\alpha) \) is an additive subgroup of \( J_n e = C_n (U_\alpha) \cong J_n \).

The face operator \( \partial_1 \) commutes with all operations of \( G_{\alpha} \) on \( C_n (U_\alpha) \) and furthermore satisfies \( \partial_1 \cdot w_j = \partial_1 \cdot w_j \). Therefore

\[ h = (1-w_0) (1-w_1) \ldots (1-w_n) e \]

is a cycle. Thus \( h \in \tilde{H}_n (U_\alpha) \). It follows easily that \( \tilde{H}_n (V_\alpha) = J_n h \). This yields the

Theorem. The map \( w \mapsto w h \) (w \in \mathbb{G}_n) induces an isomorphism

\[ J_n / I_n \cong \tilde{H}_n (V_\alpha) = J_n h \]

where \( I_n \subset J_n \) is the annihilator ideal of \( h \) which is generated by the elements

\[ 1 + w_0 + w_1^2 + \ldots + w_0^{n+1} \quad (j = 0, \ldots, n). \]

Therefore \( \omega_j w_1^k \ldots w_n^k h \) (where \( 0 \leq k_j \leq a_j - 2 \), \( j = 0, \ldots, n \)) is a basis of \( \tilde{H}_n (V_\alpha) \).
F. Hirzebruch

We recall that \( \bar{H}_e(V_a) \) is the integral singular homology group (of course with compact support). \( V_a \) is a 2\( n \)-dimensional oriented manifold without boundary (non-compact for \( n \geq 1 \)). Therefore the bilinear intersection form \( S \) is well defined over \( \bar{H}_e(V_a) \). It is symmetric for \( n \) even, skew-symmetric for \( n \) odd. It is compatible with the operations of \( G_a \).

Pham ([14], p. 358) constructs an \( n \)-dimensional cycle \( h \) in \( V_a \) which is homologous to \( h \) and intersects \( U_\gamma \) exactly in two interior points of the simplices \( e \) and \( w_0w_1 \ldots w_{n-1}e \) (sign questions have to be observed). In this way he obtains (using the \( G_a \)-invariance of \( S \)) the following result, reformulated somewhat for our purposes.

Theorem. Put \( \eta = (1 - w_0) \ldots (1 - w_a) \). The bilinear form \( S \) over \( J_a \eta \cong \bar{H}_e(V_a) \) is given by

\[
S(x \eta, y \eta) = f(\tilde{y} x \eta) \quad (x, y \in J_a),
\]

where \( f: J_a \to \mathbb{Z} \) is the additive homomorphism with

\[
f(1) = -f(w_0 \ldots w_a) = (-1)^{\frac{n(n-1)}{2}},
\]

and \( y \mapsto \tilde{y} \) is the ring automorphism of the group ring \( J_a \) induced by \( w \mapsto w^{-1} \) (in \( G_a \)).

\[\text{§ 2. The quadratic form of } V_a\]

Let \( G \) be a finite abelian group, \( J(G) \) its group ring. The ring automorphism of \( J(G) \) induced by \( g \mapsto g^{-1} (g \in G) \) is denoted by \( x \mapsto \bar{x} (x \in J(G)) \). Give an element \( \eta \in J(G) \) and a function \( f: G \to \mathbb{Z} \). The additive homomorphism \( J(G) \to \mathbb{Z} \) induced by \( f \) is also called \( f \). Put \( \tilde{f} = \sum_{x \in G} f(x) \bar{x} \). We assume

a) \( f(x \eta) = f(x \eta) \) for all \( x \in J(G) \) \( \Leftrightarrow \tilde{f} \eta = f \eta \)

or

b) \( f(x \eta) = -f(x \eta) \) for all \( x \in J(G) \) \( \Leftrightarrow \tilde{f} \eta = -f \eta \).

The bilinear form \( S \) over the lattice \( J(G) \eta \) defined by

\[
S(x \eta, y \eta) = f(\tilde{y} x \eta) \quad (x, y \in J(G)),
\]

is symmetric in case a), skew symmetric in case b). Since \( S \) is a form with integral coefficients, its determinant is well-defined. The signature

\[
\tau(S) = \tau^+(S) - \tau^-(S), \quad \text{case a},
\]

is the number \( \tau^+(S) \) of positive minus the number \( \tau^-(S) \) of negative diagonal entries in a diagonalisation of \( S \) over \( \mathbb{R} \). Let \( \chi \) run through the characters of \( G \).

Lemma. With the preceding assumptions

\[
\pm \det S = \left( \prod_{\chi(n) > 0} \chi(\tilde{f}) \right) \cdot \text{order of the torsion subgroup of } J(G)/J(G) \eta
\]

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and in case a)

\[
\tau^+(S) = \text{number of characters } \chi \text{ with } \chi(\tilde{f} \eta) > 0
\]

\[
\tau^-(S) = \text{number of characters } \chi \text{ with } \chi(\tilde{f} \eta) < 0.
\]

The proof is an exercise as in [1], p. 444.

The lemma and the last theorem of § 1 imply for the affine hypersurface \( V_a = V(a_0, \ldots, a_n) \) the

Theorem. Let \( S \) be the intersection form of \( V_a \). Then

\[
\pm \det S = \prod_{1 \leq i < j \leq n-1} (1 - e^{2 \pi i a_i/\lambda_i}),
\]

where \( \lambda_i = \exp(2 \pi i/a_i) \). For \( n \) even, we have

\[
\tau^+(S) = \text{number of } (n+1) \text{-tuples of integers } (x_0, \ldots, x_n), 0 < x_j < a_j,
\]

\[
\text{with } 0 < \sum_{j=0}^{n} x_j < 1 \text{ mod } 2 \mathbb{Z},
\]

\[
\tau^-(S) = \text{number of } (n+1) \text{-tuples of integers } (x_0, \ldots, x_n), 0 < x_j < a_j,
\]

\[
\text{with } -1 < \sum_{j=0}^{n} x_j < 0 \text{ mod } 2 \mathbb{Z}.
\]


Remark. The intersection form of \( V(a_0, \ldots, a_n) \) with \( n = 0 \) mod 2 is even, i.e. \( S(x, x) \equiv 0 \) mod 2 for \( x \in \bar{H}_e(V_a) \). Therefore, by a well-known theorem, \( \det S = \pm 1 \) implies \( \tau^+(S) - \tau^-(S) = \tau(S) \equiv 0 \) mod 8.

Following Milnor we introduce for \( a = (a_0, \ldots, a_n) \) the graph \( \Gamma(a) \): \( \Gamma(a) \) has the \( (n+1) \) vertices \( a_0, \ldots, a_n \). Two of them (say \( a_i, a_j \)) are joined by an edge if and only if the greatest common divisor \( (a_i, a_j) \) is greater than 1. Then we have [5]

Lemma. \( \det S \) as given in the preceding theorem equals \( \pm 1 \) if and only if \( \Gamma(a) \) satisfies

a) \( \Gamma(a) \) has at least two isolated points, or,

b) it has one isolated point and at least one connectedness component \( K \) with an odd number of vertices such that \( (a_i, a_j) = 2 \) for \( a_i, a_j \in K (i \neq j) \).

Now suppose \( n \) even and \( a = (a_0, \ldots, a_n) = (p, q, 2, \ldots, 2) \) with \( p, q \) odd and \( (p, q) = 1 \). Then \( \det S = \pm 1 \) and

\[
\tau(S) = \left( \frac{(p-1)(q-1)}{2} + 2 (N_{p,q} + N_{q,p}),
\]

where \( N_{p,q} \) is the number of \( q \cdot x \) \( \left( 1 \leq x \leq \frac{p-1}{2} \right) \) whose remainder mod \( p \) of smallest absolute value is negative. This follows from the preceding theorem.

Observe that by the above remark \( \tau(S) \) is divisible by 4 (even by 8) and that this is related to one of the proofs of the quadratic reciprocity law ([1], p. 450).
In particular, for \( n \) even and \((a_0, \ldots, a_n) = (3, 6k - 1, 2, \ldots, 2)\) the signature \( (S) \) equals \((-1)^{n/2} \cdot 8k\).

### § 3. Exotic spheres

A \( k \)-dimensional compact oriented differentiable manifold is called a \( k \)-sphere if it is homeomorphic to the \( k \)-dimensional standard sphere. A \( k \)-sphere not diffeomorphic to the standard \( k \)-sphere is said to be exotic. The first exotic sphere was discovered by Milnor in 1956. Two \( k \)-spheres are said to be equivalent if there exists an orientation preserving diffeomorphism between them. The equivalence classes of \( k \)-spheres constitute for \( k \geq 5 \) a finite abelian group \( \Theta_k \) under the connected sum operation. \( \Theta_k \) contains the subgroup \( bP_{k+1} \) of those \( k \)-spheres which bound a parallelizable manifold. \( P_{4m} (m \geq 2) \) is cyclic of order

\[
2^{2m-2} (2^{2m-1}) \text{ numerator} \left( \frac{4B_m}{m} \right),
\]

where \( B_m \) is the \( m \)-th Bernoulli number. Let \( g_m \) be the Milnor generator of \( P_{4m} \); see § 5. If \( (4m-1) \)-sphere \( \Sigma \) bounds a parallelizable manifold \( B \) of dimension \( 4m \), then the signature \( \tau (B) \) of the intersection form of \( B \) is divisible by \( 8 \) and

\[
\tau (B) = \frac{\Sigma}{8} g_m
\]

\( \tau \) should be chosen in such a way that we have always the plus-sign in (1). If \( m = 2 \) and \( 4 \) we have

\[bP_8 = \Theta_1 = \mathbb{Z}_{28}, \quad bP_{12} = \Theta_{11} = \mathbb{Z}_{992}.\]

All these results are due to Milnor-Kervaire. The group \( bP_{2n} \) (\( n \) odd, \( n \geq 3 \)) either 0 or \( \mathbb{Z}_2 \). It contains only the standard sphere and the Kervaire sphere obtained by plumbing two copies of the tangent bundle of \( S^3 \). It is known that \( bP_{2n} \) is \( \mathbb{Z}_2 \) (equivalently that the Kervaire sphere is exotic) if \( \equiv 1 \mod 4 \) and \( n \geq 5 \) (E. Brown-F. Peterson).

Let \( V = V^a (a_0, a_1, \ldots, a_n) \subset \mathbb{C}^{n+1} \) \((a_0 \equiv 2)\) be defined by

\[z_0^2 + z_1^2 + \ldots + z_n^2 = 0,
\]

his affine variety has exactly one singular point, namely the origin of \( \mathbb{C}^{n+1} \).

**S^2n+1** is compact oriented differentiable manifold (without boundary) of dimension \( 2n-1 \).

**Theorem.** Let \( n \geq 3 \). Then \( \Sigma_n \) is \((n-2)\)-connected. It is a \((2n-1)\)-sphere if and only if the graph \( \Gamma (a) \) defined in § 2 satisfies the condition \( a \) or \( b \). If \( \Sigma_n \) is a

### § 4. Manifolds with actions of the orthogonal group

\( O(n) \) denotes the real orthogonal group with \( O(m) \subset O(n), m < n \), by

\[
A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \quad (A \in O(m), \ 1 = \text{unit of } O(n-m)).
\]

Let \( X \) be a compact differentiable manifold of dimension \( 2n-1 \) on which \( O(n) \) acts differentiably \((n \equiv 2)\). Suppose each isotropy group is conjugate to \( O(n-2) \) or \( O(n-1) \). Then the orbits are either Stiefel manifolds \( O(n)/O(n-2) \)
is the standard sphere for \( d \equiv \pm 1 \mod 8 \) and the Kervaire sphere for \( d = \pm 3 \mod 8 \), in agreement with a more general result in [5].

**Remarks.** The \( O(n) \)-manifold \( W^{2n-1}(d) \) coincides with Bredon's manifolds \( M^{2n-1}_{d'k+1} \) for \( d = 2k+1 \), see Bredon [3]. \( \Sigma(3,2,2,2) \) is the standard 5-sphere (since \( \phi_3 = 0 \)). Therefore \( S^3 \) admits a differentiable involution \( \alpha \) with the lens space \( L(3,1) \) as fixed point set and a diffeomorphism \( \beta \) of period 3 with the real projective 3-space as fixed point set. Compare [3]. \( \alpha, \beta \) and \( \epsilon \) are defined on \( \Sigma(3,2,2,2) \) given by (1) as follows

\[
\alpha(z_0, z_1, z_2, z_3) = (z_0, z_1, z_2, -z_3) \\
\beta(z_0, z_1, z_2, z_3) = (\epsilon z_0, z_1, z_2, z_3), \quad \text{where} \quad \epsilon = \exp(2\pi i/3).
\]

Many more such examples of "exotic" involutions etc. which are not differentiably equivalent to orthogonal involutions etc. can be constructed.

**§ 5. Manifolds associated to knots**

Let \( X \) be a compact differentiable manifold of dimension \( 2n-1 \) on which \( O(n-1) \) acts differentiably on \( n \geq 3 \). Suppose each isotropy group is conjugate to \( O(n-3) \) or \( O(n-2) \) or \( O(n-1) \). Then the orbits are either Stiefel manifolds \( O(n-1)/O(n-3) \) (of dimension \( 2n-5 \)) or spheres \( O(n-1)/O(n-2) \) (of dimension \( n-2 \)) or points (fixed points of the whole action). The representations of the isotropy groups \( O(n-3), O(n-2) \) and \( O(n-1) \) respectively normal to the orbit are supposed to be the 4-dimensional trivial representation, the 3-dimensional trivial plus the standard representation of \( O(n-2) \), the 1-dimensional trivial plus the sum of two copies of the standard representation of \( O(n-1) \). The orbit space \( X' \) is then a 4-dimensional manifold with boundary. We suppose that \( X' \) is the 4-dimensional disk \( D^4 \).

Then the points of the interior of \( D^4 \) correspond to Stiefel-manifold-orbits, the points of \( \partial D^4 \) to \( S^3 \) to the other orbits. The set \( F \) of fixed points corresponds to a 1-dimensional submanifold of \( S^3 \), also called \( F \).

We suppose \( F \) non-empty and connected, it is then a knot in \( S^3 \). We shall call an \( O(n-1) \)-manifold of dimension \( 2n-1 \) a "knot manifold" if all the above conditions are satisfied.

Let \( K \) be the set of isomorphism classes of differentiable knots (i.e. isomorphism classes of pairs \( (S^3, F) \) - F a compact connected 1-dimensional submanifold — under diffeomorphisms of \( S^3 \)). For the following theorem see Jänich ([9], § 6), compare also W.C. Hsiang and W.Y. Hsiang [8].

**Theorem.** For any \( n \geq 3 \) there is a one-to-one correspondence

\[
x_n^a : K \rightarrow \Phi_{2n-1}^a,
\]

where \( \Phi_{2n-1}^a \) is the set of isomorphism classes of \( (2n-1) \)-dimensional knot manifolds under equivariant diffeomorphisms. \( x_n^a \) associates to a knot manifold the knot \( F \) considered above.
Remark. The 2-fold branched covering of $S^3$ along a knot $F$ is an $O(1)$-manifold which will be denoted by $x_0(F)$.

If we plumb $8$ copies of the tangent bundles of $S^n$ ($n \equiv 1$) according to the tree $E_8$

we get a $(2n-1)$-dimensional manifold $M^{2n-1}(E_8)$. For $n = 2$ this is $S^3/G$, where $G$ is the binary pentagonal dodecahedron group [6]. For $n$ odd, $M^{2n-1}(E_8)$ is the standard sphere, as the Arf invariant shows. For $n = 2m + 4$, the manifold $M^{4m-1}(E_8)$ is an exotic sphere, it is the famous Milnor sphere which represents the generator $\pm g_2$ of $bP_{4m}$ (see § 3).

$M^{2n-1}(E_8)$ admits an action of $O(n-1)$ as follows: $O(n-1)$ operates as subgroup of $O(n+1)$ on $S^n$ and thus on the unit tangent bundle of $S^n$. The action on $S^n$ leaves a great circle fixed.

When plumbing the eight copies of the tangent bundle, we put the center of the plumbing operation always on this great circle (for one copy, corresponding to the central vertex of the $E_8$-tree, we need three such centers, therefore, we cannot have an action of $O(n)$ which has only 2 fixed points on $S^n$).

Then the action of $O(n-1)$ on each copy of the tangent bundle is compatible with the plumbing and extends to an action of $O(n-1)$ on $M^{2n-1}(E_8)$ which, for $n \equiv 3$, becomes a knot manifold as can be checked. The resulting knot can be seen on a picture attached at the end of this lecture. The speaker had shown himself that this is the torus knot $t(3,5)$, but Zieschang and Vogt showed him a better proof. This implies the

Theorem. Suppose $n \equiv 3$. Then $x_0(t(3,5))$ is equivariantly diffeomorphic to $M^{2n-1}(E_8)$ with the $O(n-1)$-action defined by equivariant plumbing. (This is still true for $n = 2$, see Remark above.)

We now consider the manifold $\Sigma(p,q,2,\ldots,2) \subset C^{n+1}$ given by the equations (see § 3)

$$z_0^p + z_1^q + z_1^2 + \ldots + z_n^2 = 0$$
$$\sum_{i=0}^n z_i z_i = 1 \quad (n \equiv 3).$$

This is an $O(n-1)$-manifold, the action being defined similarly as in § 4. Suppose $(p,q) = 1$. Then it can be shown that $\Sigma(p,q,2,\ldots,2)$ is a knot manifold: It is $x_0(t(p,q))$ where $t(p,q)$ is the torus knot. Therefore, by the preceding theorem we have an equivariant diffeomorphism

$$M^{2n-1}(E_8) \cong \Sigma(3,5,2,\ldots,2) \quad (n \equiv 3).$$

This gives a different proof (based on the classification of knot manifolds) that $\Sigma(3,5,2,\ldots,2)$ represents for $m \equiv 2$ a generator of $bP_{4m}$ (compare § 3).

\section{Singularities and exotic spheres}

\subsection{A theorem on knot manifolds}

Let $F$ be a knot in $S^3$. Then the signature $\tau(F)$ can be defined in the following way which Milnor explained to the speaker in a letter. Milnor also considers higher dimensional cases. We cite from his letter, but restrict to classical knots:

Let $X$ be the complement of an open tubular neighbourhood of $F$ in $S^3$. Then the cohomology

$$H^* = H^*(\tilde{X}, \partial \tilde{X}; \mathbb{R})$$

where $\tilde{X}$ is the infinite cyclic covering of $X$, satisfies Poincaré duality just as if $\tilde{X}$ were a 2-dimensional manifold bounded by $F$.

In particular, the pairing

$$\cup: H^1 \otimes H^1 \to H^2 \cong \mathbb{R}$$

is non-degenerate. Let $t$ denote a generator for the group of covering transformations of $\tilde{X}$. Then for $a, b \in H^1$ the pairing

$$\langle a, b \rangle = a \cup t^* b + b \cup t^* a$$

is symmetric and non-degenerate. Hence, the signature $\tau^*(F) - \tau^*(F) = \tau(F)$ is defined.

There exist earlier definitions of the signature by Murasugi [13] and Trotter [17]. The signature is a cobordism invariant of the knot. A cobordism invariant mod 2, was introduced by Robertello [15] inspired by an earlier paper of Kervaire-Milnor. Let $F$ be a knot and $\Delta$ its Alexander polynomial, then the Robertello invariant $c(F)$ is an integer mod 2, namely

$$c(F) = 0, \quad 1$$

if $\Delta(1) \equiv \pm 1 \mathrm{mod} 8$, \begin{align*}
\quad & c(F) = 0, \quad 1 \quad \text{if} \quad \Delta(1) \equiv \pm 3 \mathrm{mod} 8.
\end{align*}

We recall that the first integral homology group of $x_0(F)$, the 2-fold branched covering of the knot $F$ (see a remark in § 5), is always finite, its order is odd, and equals up to sign the determinant of $F$. We have $\pm \det F = \det(1 - A)$.

Theorem. Let $F$ be a knot, then $x_0(F)$, $n \equiv 2$, is the boundary of a parallelizable manifold. For $n$ odd, $x_0(F)$ is homeomorphic to $S^{2n-1}$ and thus represents an element of $bP_{2n}$, it is the standard sphere if $c(F) = 0$, the Kervaire sphere if $c(F) = 1$. If $n = 2m$, then $x_0(F)$ is $(2 - 2)m$-connected and $H_{2m-1}(x_0(F), \mathbb{Z}) \cong H_1(x_0(F), \mathbb{Z})$. For $m \equiv 2$, it is homeomorphic to $S^{4m-1}$ if and only if $\det F = \mp 1$. Then $x_0(F)$ represents (up to sign) an element of $bP_{4m}$ which is $\pm \tau(F)$ (see § 3).

The proof uses an equivariant handlebody construction starting out from a Seifert surface [16] spanned in the knot $F$. For simplicity, not out of necessity, we have disregarded orientation questions in § 5 and § 6.

Remark. § 2(3) gives up to sign a formula for the signature of the torus knot $t(p,q)$ ($p, q$ odd with $(p, q) = 1$).
40.

Involutionen auf Mannigfaltigkeiten


HEINRICH BEHNKE zum 70. Geburtstag gewidmet


Viele Dinge dieser Arbeit können verallgemeinert werden auf $G$-Mannigfaltigkeiten, wo $G$ eine kompakte Liesche Gruppe ist (vgl. Atiyah and Singer [3]).

1. Beispiele von Involutionen

Wir betrachten die Gleichung

$$z_0^n + z_1^n + \cdots + z_n^n = 0 \quad (n \geq 1).$$

Die Exponenten $a_i$ sollen ganze Zahlen $\geq 2$ sein. Wir setzen $a = (a_0, \ldots, a_n)$. Nach A. Weil [31] ist die Anzahl der Lösungen von (1) über jedem  

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