MAPPINGS BETWEEN ANRs THAT ARE FINE HOMOTOPY EQUIVALENCES

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It is shown in this note that every closed $UV^0$-map between separable ANRs is a fine homotopy equivalence.

We extend Lacher's result [6,7] that a closed $UV^0$-map between locally compact, finite dimensional ANRs is a fine homotopy equivalence to the case of arbitrary separable ANRs. It is hoped that this theorem will be useful in studying manifolds modelled on the Hilbert Cube. (See [1], section PF3. Added in proof. See also [9]).

A set $A \subset X$ has property $UV^0$ if for each open set $U$ containing $A$, there is an open $V$, with $A \subset V \subset U$ such that $V$ is null-homotopic in $U$. A mapping $f: X \rightarrow Y$ of $X$ onto $Y$ is a $UV^0$-map if for each $y \in Y$, $f^{-1}(y)$ is a $UV^0$ subset of $X$. The mapping $f$ is said to be closed if the image of every closed set is closed and proper if the inverse image of every compact set is compact. An absolute neighborhood retract for metric spaces is denoted an ANR. If $\alpha$ is a cover of $Y$ and $g_1$ and $g_2$ are maps of a space $A$ into $Y$, $g_1$ is $\alpha$-near $g_2$ if for each $a \in A$ there is a $U \in \alpha$ containing $g_1(a)$ and $g_2(a)$. The map $g_1$ is $\alpha$-homotopic to $g_2$, $g_1 \cong g_2$, if there is a homotopy $\lambda: A \times I \rightarrow Y$ taking $g_1$ to $g_2$ with the property that for each $a \in A$ there exists $U \in \alpha$ containing $\lambda((a) \times I)$. A map $f: X \rightarrow Y$ is a fine homotopy equivalence if for each open cover, $\alpha$, of $Y$ there exists a map $g: Y \rightarrow X$ such that $fg \cong id_Y$ and $gf \cong id_X$.

Various versions of Lemma 3 have been proven by Smale [8], Armentrout and Price [2], Kozlowski [5] and Lacher [6]. The difference in this lemma is that $K$ is not required to be a finite dimensional complex.

Let $K$ be a locally finite complex and $j$ be a nonnegative integer. When there is no confusion we will not distinguish between the complex $K$ and its underlying point set $|K|$. If $\sigma$ is a simplex of $K$, then $N(\sigma, K) = \{\tau < K \mid \sigma \cap \tau \neq \phi\}$ and $st(\sigma, K) = \{\tau < K \mid \sigma < \tau\}$. Also $K^j$ will denote the $j$-skeleton of $K$ and $iK^j = \{\sigma < K \mid N(\sigma, K) \subset |K^j|\}$. Let $\mathcal{U}$ be a covering of a space $Y$ and $B$ a subset of $Y$. The star of $B$ with respect to $\mathcal{U}$, $st(B, \mathcal{U})$, is the set $\{U \in \mathcal{U} \mid B \cap U \neq \phi\}$. Inductively, $st^*(B, \mathcal{U})$ is defined to be $st(st^{-1}(B, \mathcal{U}))$. A covering $\mathcal{V}$ is called a star$^*$ refinement of $\mathcal{U}$ if the covering $\{st^*(V, \mathcal{V}) \mid V \in \mathcal{V}\}$ refines $\mathcal{U}$. Every open covering of a
metric space has an open star refinement for each positive integer \(n\) (c.f. [3]). We start by stating without proof two easily verified lemmas.

**Lemma 1.** Let \(K\) be a locally finite complex. Suppose \(\phi: K \to Y\) is a map, \(\mathcal{U}\) is an open cover of \(Y\), and \(k\) is a nonnegative integer. Then there is a subdivision \(\tilde{K}\) of \(K\) so that:

(a) if \(\sigma\) is a \(k\)-simplex of \(\tilde{K}\), then \(\phi(N(\sigma, \tilde{K})) \subseteq U\), for some \(U \in \mathcal{U}\),

(b) if \(\sigma < k^{-1}K\), then \(\sigma < \tilde{K}\).

We will call such a subdivision, \(\tilde{K}\), a \((k, \mathcal{U})\)-subdivision of \(K\). We note that for any vertex, \(v\), of \(\tilde{K}\) with \(v \notin k^{-1}K\) it follows that \(\phi(st(v, \tilde{K})) \subseteq U\) for some \(U \in \mathcal{U}\).

**Lemma 2.** Let \(\mathcal{U}\) be an open cover of the paracompact space \(Y\) and \(f: X \to Y\) a closed \(UV^*\)-map. Then there is an open locally finite refinement \(\mathcal{V}\) of \(\mathcal{U}\) such that for each \(V \in \mathcal{V}\), there is a \(U \in \mathcal{U}\) satisfying

(a) \(st(V, \mathcal{V}) \subseteq U\)

(b) if \(m\) is a positive integer and the map \(\gamma: \partial B^m \to f^{-1}(st(V, \mathcal{V}))\) is given, then \(\gamma\) can be extended to \(\tilde{\gamma}: B^m \to f^{-1}(U)\).

We will call such a refinement, \(\mathcal{V}\), a \(UV^*\) star refinement of \(\mathcal{U}\).

**Lemma 3.** Let \(f: X \to Y\) be a closed \(UV^*\)-map of an arbitrary space, \(X\), onto the paracompact space \(Y\). Let \(K\) be a locally finite complex and \(J\) a subcomplex of \(K\). Let \(\phi: K \to Y\) and \(\psi': \tilde{J} \to X\) be mappings such that \(f\psi' = \phi|J\). Then given any open cover, \(\alpha\), of \(Y\) there exists a map \(\psi: K \to X\) extending \(\psi'\) so that \(f\psi\) is \(\alpha\)-near \(\phi\).

**Proof.** Let \(K_0\) be a \((0, \alpha_0)\)-subdivision of \(K\) and let \(\alpha_0 = \alpha\). Define inductively a sequence of covers of \(Y\), \(\{\alpha_i\}_{i=0}^\infty\), and subdivisions of \(K_0\), \(\{K_i\}_{i=0}^\infty\), such that for each \(i > 0, \alpha_i\) is a \(UV^*\) star refinement of \(\alpha_{i-1}\) and \(K_i\) is an \((i, \alpha_i)\)-subdivision of \(K_{i-1}\).

Define \(\psi_0: K_0 \to X\) by letting \(\psi_0(v) = \psi'(v)\) if \(V \in J\) and otherwise an arbitrary element of \(f^{-1}(\phi(v))\). Assume inductively that there exist maps \(\{\psi_i: K_i \to X\}_{i=0}^\infty\) such that for \(0 \leq i \leq n\):

1. \(\psi_i|J \cap K_i = \psi'|J \cap K_i\) and if \(j < i\), \(\psi_i|K_j = \psi_j|K_j\),
2. if \(v\) is a vertex of \(K_n\), \(\psi_i(v) \in f^{-1}(\phi(v))\),
3. if \(\sigma\) is a \(j\)-simplex of \(K_i\) and \(k = \dim(st(\sigma, K_i))\), then \(\phi(st(\sigma, K_i)) \cup f\psi_i(\sigma) \subseteq U\), for some \(U \in \alpha_{k-j}\).

[Note that \(\psi_0: K_0 \to X\) satisfies these conditions since if \(\sigma\) is a 0-simplex of \(K_0\) the dimension of \(st(\sigma, K_0)\) is 0 and the fact that \(K_0\) is a \((0, \alpha_0)\)-subdivision of \(K\) implies that \(\phi(st(\sigma, K_0)) \cup f\psi_0(\sigma) \subseteq U\) for some \(U \in \alpha_0\).

We wish now to define \(\psi_{n+1}: K_{n+1} \to X\) satisfying conditions (1) - (3) for \(i = n + 1\). For each vertex \(v\) of \(K_{n+1}\), let
\[
\psi_{n+1}(v) = \begin{cases} 
\psi_n(v), & \text{if } v \text{ is a vertex of } \ast K_n \\
\psi'(v), & \text{if } v \in J
\end{cases}
\]

an arbitrary element of \(f^{-1}(\phi(v))\), otherwise

Assume (subinductive statement) that \(\psi_{n+1}|K_{n+1}^r\) has been defined so that

\begin{align*}
(1') & \quad \psi_{n+1}|J \cap K_{n+1}^r = \psi'|J \cap K_{n+1}^r \\
(2') & \quad \text{if } v \text{ is a vertex of } K_{n+1}, \psi_{n+1}(v) \in f^{-1}(\phi(v)), \\
(3') & \quad \text{if } \sigma \text{ is a } j\text{-simplex of } K_{n+1}^r \text{ and } k = \dim(st(\sigma, K_{n+1}^r)), \text{ then } \\
& \qquad \phi(st(\sigma, K_{n+1}^r)) \cup f\psi_{n+1}(\sigma) \subset U, \text{ for some } U \in \alpha_{k-j}.
\end{align*}

[Note that \(\psi_{n+1}|K_{n+1}^0\) has been defined in such a manner that properties (1')-(3') are satisfied. Properties (1') and (2') follow immediately from the definition. Let \(v\) be a simplex of \(K_{n+1}^r\). If \(v\) is a vertex of \(K_{n+1}^r\), property (3') follows from the fact that \(\psi_n\) satisfies property (3) of the main inductive statement since in this case \(\dim(st(v, K_{n+1}^r)) = \dim(st(v, K_n^r))\). Suppose \(v\) is not a vertex of \(K_{n+1}^r\). By the remark following Lemma 1, \(\phi(st(v, K_{n+1}^r))\) is contained in some element of \(\alpha_{n+1}\) and hence property (3') is again satisfied.]

Now let \(\sigma\) be an \((r+1)\)-simplex of \(K_{n+1}^r\). If \(\sigma\) is a subset of \(J\), let \(\psi_{n+1}|\sigma = \psi'|\sigma\). If \(\sigma < \ast K_n\), let \(\psi_{n+1}|\sigma = \psi_n|\sigma\). Otherwise, let \(k = \dim(st(\sigma, K_{n+1}^r))\). For each \(r\)-simplex, \(\tau\), in \(\partial \sigma\), there is a \(u, \in \alpha_{k-r}\) containing \(\phi(st(\tau, K_{n+1}^r)) \cup f\psi_{n+1}(\tau)\). Let \(\tau'\) be a fixed \(r\)-simplex in \(\partial \sigma\) and note that \(\psi_{n+1}|\partial \sigma \subset f^{-1}(st(u, \alpha_{k-r}))\). Since \(\alpha_{k-r}\) is a \(UV^*\) star refinement of \(\alpha_{k-r} = \alpha_{k-r+1}\) containing \(st(U, \alpha_{k-r})\) and an extension of \(\psi_{n+1}|\partial \sigma\) which maps \(\sigma\) into \(f^{-1}(U)\). We call this extension \(\psi_{n+1}\) and note that \(\phi(st(\sigma, K_{n+1}^r)) \cup f\psi_{n+1}(\sigma) \subset U\). In this manner, extend \(\psi_{n+1}\) to \(K_{n+1}^r\) and note that conditions (1')-(3') are satisfied. This completes the subinductive argument and hence the main inductive argument.

We now define \(\psi: K \to X\) by \(\psi(x) = \lim_{n \to \infty} \psi_n(x)\). For any \(x \in K\), the local finiteness of \(K\) assures that there exists an integer \(N\) so that \(x \in \chi K_N\). Hence for \(n \geq N\), \(\psi_n(x) = \psi_N(x)\). Therefore \(\psi\) is well-defined and continuous. Let \(x \in K\) and let \(\sigma\) be a simplex of maximal dimension containing \(x\). Then there exists an integer \(N\) such that \(|\sigma| \subset \chi K_N\). Choose a simplex \(B\) in \(\chi K_N\) containing \(x\) and note that \(\psi(x) = \psi_N(x)\). By inductive statement (3), there is an open set \(U \in \alpha_i\), for some \(i \geq 0\), such that \(\phi(st(B, K_N)) \cup f\psi(B) \subset U\). Since \(\alpha_i\) refines \(\alpha_0 = \alpha\), there is a \(V \in \alpha\) such that \(\{\phi(x)\} \cup \{f\psi(x)\} \subset V\). Since \(\psi\) extends \(\psi'\), this completes the proof of Lemma 3.
Remark. By a slightly more cumbersome process, $\psi$ can be chosen so that $f\psi$ is a $\alpha$-homotopic to $\phi$.

Theorem. Let $X$ and $Y$ be separable ANRs and $f: X \to Y$ be a closed $UV^*$-map. Then $f$ is a fine homotopy equivalence.

Proof. Let $\alpha$ be an open cover of $Y$. Let $\alpha_1$ be a star refinement of $\alpha$ and $\alpha_2$ a star refinement of $\alpha_1$. Let $\beta$ be an open refinement of $\alpha_2$ such that any two $\beta$-near maps from any space into $Y$ are $\alpha_2$-homotopic (such refinements exist since $Y$ is an ANR, c.f. [4]).

By Hanner's characterization of separable ANRs (c.f. [4]), there exist a locally finite polyhedron $Q$ and maps $c: Q \to Y$ and $s: Q \to Y$ with property that $sc \cong id_Y$. By Lemma 3, there is a map $v: Q \to X$ such that $fv$ is $\beta$-near $s$. Define $g: Y \to X$ by $g = vc$. Note that $fg$ is $\beta$-near $sc$ and hence $fg \cong sc$. But $sc \cong id_Y$ and hence $fg \cong id_Y$. Denote this $\alpha_1$-homotopy by $h$; then, $h: Y \times I \to Y$ is a $\alpha_1$-homotopy with $h_0 = id_Y$ and $h_1 = fg$.

It remains to be shown that $gf$ is $f^{-1}(\alpha)$ homotopic to $id_X$.

Choose a locally finite polyhedron, $P$, maps $b: P \to P$ and $r: P \to X$ and a homotopy $W: X \times I \to X$ with the following properties:

(a) $W_0 = rb$ and $W_1 = id_X$

(b) $W$ is limited by $f^{-1}(\alpha_1)$ and by $(gf)^{-1}(f^{-1}(\alpha_1))$.

Next, define $H: P \times I \to Y$ by $H(p,t) = h(fr(p))$ and note that $H(p,0) = fr(p)$ and $H(p,1) = fgfr(p)$.

Define $G': P \times [0,1] \to X$ by $G'(p,0) = r(p)$, $G'(p,1) = gf(r(p))$. Then by Lemma 3 there is a map $G: P \times I \to X$ extending $G'$ with the property that $fG$ is $\alpha_1$-near $H$.

Define $\psi: X \times I \to X$ by $\psi(x,t) = G(b(x),t)$.

Note that $\psi_0(x) = G(b(x),0) = G'(b(x),0) = rb(x)$ and $\psi_1(x) = G(b(x),1) = G'(b(x),1) = gf(rb(x))$.

Now, $W$ is a homotopy taking $rb$ to $id_X$ and is limited by $f^{-1}(\alpha_1)$.

Recall that $\alpha_1$ is a star refinement of $\alpha$. Therefore, to show that $id_X \cong (\alpha)gf$, it suffices to show that $f\psi: X \times I \to Y$ is limited by $\alpha_1$. Fix $x \in X$. Since the homotopy $h$ is limited by $\alpha_1$, there exists $U \subset \alpha_1$ with $h(f(x) \times I) \subset U$. We claim that $f(\psi(x \times I)) \subset \text{st}(U)$.

Fix $t \in I$. Recall $f(\psi(x,t)) = f(G(b(x),t))$. There thus exists $U' \in \alpha_1$ such that $f^{-1}(U')$ contains $x$ and $rb(x)$. Hence $f(x)$ and $frb(x)$ are elements of $U'$ and $U \cap U' \neq \phi$. Since $h$ is limited by $\alpha_1$, we can choose $U'' \in \alpha_1$ so that $hfrb(x)$ and $frb(x)$ are elements of $U''$. Note that $U'' \cap U' \neq \phi$. Also, there exists $U''' \in \alpha_1$ containing $H(b(x),t)$ and $f(G(b(x),t))$, since $fg$ is $\alpha_1$-near $H$. But $H(b(x),t) = \psi(x,t) = G(b(x),t)$.
$h_{frb}(x)$. Hence $U'' \cap U'' \neq \emptyset$ and we have completed the proof of the theorem by showing that $f\psi: X \times I \to Y$ is limited by star$^3(\alpha_t)$.

Added in proof. I would like to thank Bob Edwards for some suggestions concerning this paper and for pointing out that George Kozlowski [Images of ANR’s, to appear] has shown that a $UV^*$-map between ANR’s is a homotopy equivalence.

Remark. If in addition it is assumed that $X$ and $Y$ are locally compact and $f$ is a proper map it follows immediately that $f$ is a proper fine homotopy equivalence.

References


Received March 12, 1974 and in revised form July 20, 1974.

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