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ON THE HOMOTOPY GROUPS OF SPHERES AND ROTATION GROUPS[1]

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1. Introduction

One of the outstanding problems in modern topology is that of classifying the mappings of an m-dimensional sphere S^m into a topological space X. In terms of the Hurewicz theory of homotopy groups[2] this problem may be phrased as follows: to determine the structure of the m^{th} homotopy group $\pi_m(X)$. Of particular interest is the case where X itself is an n-sphere S^n. In this case the results of Hopf,[3] Freudenthal,[4] and Pontrjagin[5] have led to the solution of the problem for $m \leqq n + 2$. For $m > n + 2$ almost nothing is known concerning the structure of $\pi_m(S^n)$.

That this problem is closely related to the study of homotopy properties of the rotation group R_n of the n-sphere has been shown by Pontrjagin,[5] who has used the one- and two-dimensional homotopy groups of R_n to compute the groups $\pi_{n+i}(S_n)$ $(i = 1, 2)$.

In the present paper we introduce an operation which associates with each mapping $f(S^m \times S^n) \subset S^n$ a mapping $\phi(S^{m+n+1}) \subset S^{n+1}$. This is a generalization of the procedure of Hopf[6] for the case $m = n$. This operation is shown to induce a homomorphism of $\pi_m(R_n)$ into $\pi_{m+n+1}(S^{n+1})$, which for $m = 1, 2$ turns out to be an isomorphism. The connection of this homomorphism with one introduced by Freudenthal[4] is studied.

In a recent paper Freudenthal[7] has announced without proof a very general theorem on extension of mappings, and used this theorem to construct maps of S^{2n-1} on S^n of Hopf invariant 1[6] for all even n. We shall use the above results to construct a counter-example to Freudenthal's theorem. It is further shown that Freudenthal's construction definitely fails if $n > 2$ and $n \equiv 2 \pmod 4$.

2. Preliminary concepts

In Euclidean $(r + 1)$-space $\mathfrak{E}^{r+1}$ let S^r denote the unit sphere, i.e., the set of points $x = (x_1, \cdots, x_{r+1}) \in \mathfrak{E}^{r+1}$ with

$$|x|^2 = \sum_{i=1}^{r+1} x_i^2 = 1. \tag{1}$$

[1] Presented to the American Mathematical Society, December 30, 1941.

[2] W. Hurewicz, Proc. Akad. Amsterdam 38 (1935), pp. 112–119

[3] H Hopf, Math. Ann. 104 (1931), pp. 637–665. We shall refer to this paper as H I.

[4] H. Freudenthal, Comp. Math. 5 (1937), pp. 299–314. We shall refer to this paper as F I.

[5] L. Pontrjagin, C. R. Acad. Sci. URSS 19 (1938), pp. 147–149, 361–363.

[6] H. Hopf, Fund. Math. 25 (1935), pp. 427–440. We shall refer to this paper as H II.

[7] H. Freudenthal, Proc. Akad. Amsterdam 42 (1939), pp. 139–140. We shall refer to this paper as F II.

Let E_i^r $(i = 1, 2)$ be the hemispheres defined by the conditions $x_{r+1} \geqq 0$, $x_{r+1} \leqq 0$, respectively. E^{r+1} denotes the closed $(r + 1)$-cell $| x | \leqq 1$ bounded by S^r. We shall refer to the points $x^1 = (0, 0, \cdots, 1)$ and $x^2 = (0, 0, \cdots, -1)$ as the *north* and *south poles*, respectively.

Let Y be a metric space with distance function $\rho(y_1, y_2)$, y^0 a fixed point of Y. By Y^{S^r} we shall mean the space of all mappings[8] $f(S^r) \subset Y$ metrized by

$$\rho(f, g) = \underset{x \in S^r}{\text{L.U.B.}}\ \rho[f(x), g(x)] \qquad (f, g \in Y^{S^r}). \tag{2}$$

Let x^0 be the point of S^r with co-ordinates $(1, 0, \cdots, 0)$. Then $Y^{S^r}(x^0, y^0)$ denotes the subspace of Y^{S^r} consisting of those mappings $f(S^r) \subset Y$ such that $f(x^0) = y^0$. Two mappings $f, g \in Y^{S^r}(x^0, y^0)$ are said to be homotopic if they can be joined by an arc in $Y^{S^r}(x^0, y^0)$. The relation of homotopy is reflexive, symmetric, and transitive and divides the space $Y^{S^r}(x^0, y^0)$ into equivalence classes, called *homotopy classes*. The set of all these homotopy classes we denote by $\pi_r(Y)$. We shall denote the homotopy class of any $f \in Y^{S^r}(x^0, y^0)$ by $\mathbf{f}$.

We define an operation of addition between homotopy classes as follows: let f_i $(i = 1, 2) \in Y^{S^r}(x^0, y^0)$. Let ϕ_i $(i = 1, 2)$ be a mapping of E_i^r on S^r such that (1) $\phi_i(S^{r-1}) = x^0$; (2) $\phi_i(E_i^r - S^{r-1}) \subset S^r$ is a topological map of degree 1. Then we define a mapping $f(S^r) \subset Y$ as follows:

$$f(x) = \begin{cases} f_1[\phi_1(x)] & (x \in E_1^r), \\ f_2[\phi_2(x)] & (x \in E_2^r). \end{cases} \tag{3}$$

It is easily verified that the homotopy class of f depends only on the homotopy classes of f_1 and f_2. Let

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2. \tag{4}$$

Hurewicz[2] has proved that under the operation of addition so defined the set $\pi_r(Y)$ becomes a group, called the r^{th} *homotopy group* of Y. This group is abelian if $r > 1$; in all the cases we consider here it is also abelian if $r = 1$.

3. The homomorphism H

Let Euclidean $(m + n + 2)$-space be represented as the product space $\mathcal{E}^{m+1} \times \mathcal{E}^{n+1}$, points $x \in \mathcal{E}^{m+n+2}$ being represented by co-ordinates (p, q) $(p \in \mathcal{E}^{m+1}$, $q \in \mathcal{E}^{n+1})$. Then S^{m+n+1} is defined by

$$| p |^2 + | q |^2 = 1. \tag{5}$$

Let H_1 and H_2 be the subsets of S^{m+n+1} defined by

$$| p | \leqq | q |, \tag{6$_1$}$$

$$| p | \geqq | q |, \tag{6$_2$}$$

[8] All mappings are supposed continuous.

respectively. Let

$$(7_1) \qquad \psi_1(p, q) = (p/|q|, q/|q|) \qquad ((p, q) \in H_1),$$

$$(7_2) \qquad \psi_2(p, q) = (p/|p|, q/|p|) \qquad ((p, q) \in H_2).$$

Evidently $\psi_1 \mid H_1H_2 = \psi_2 \mid H_1H_2$[9] and maps H_1H_2 into $S^m \times S^n$. Denote this mapping by ψ. Then

LEMMA 1. *The mappings ψ_1, ψ_2, and ψ defined above are homeomorphic mappings of H_1 on $E^{m+1} \times S^n$, H_2 on $S^m \times E^{n+1}$, and H_1H_2 on $S^m \times S^n$ respectively.*

Let f be a mapping of $S^m \times S^n$ into S^n. We associate with f the mapping $H(f) = \phi(S^{m+n+1}) \subset S^{n+1}$ as follows: ϕ maps the great circle joining the point $(0, q)$ to the point (p, q) on the great circle joining the north pole z^1 of S^{n+1} to the point $f[\psi^{-1}(p, q)]$, and maps the great circle joining $(p, 0)$ to (p, q) on the great circle joining z^2 to $f[\psi^{-1}(p, q)]$. Evidently $\phi(H_1) \subset E_1^{n+1}$, $\phi(H_2) \subset E_2^{n+1}$, while $\phi = f\psi^{-1}$ on H_1H_2. The functions defining the mapping ϕ are given by

$$(8) \qquad \begin{aligned} \phi_i(p, q) &= 2\,|p|\cdot|q|\cdot f_i(p/|p|, q/|q|) && (|p|\cdot|q| \neq 0),\\ \phi_i(0, q) &= \phi_i(p, 0) = 0 && (i = 1, \cdots, n+1);\\ \phi_{n+2}(p, q) &= |q|^2 - |p|^2. \end{aligned}$$

We use this operation to construct a mapping $\mathbf{H} = \mathbf{H}_{m,n}$ of $\pi_m(R_n)$ into $\pi_{m+n+1}(S^{n+1})$ as follows: let $e \in R_n$ denote the identity mapping of S^n on itself, and let $f \in R_n^{S^m}(p^0, e)$. If $p \in S^m$, $q \in S^n$, let $f^*(p, q)$ denote the point of S^n into which q is carried by the rotation $f(p)$. Let $\phi = H(f^*)$. Then it is easy to verify that $\phi \in S^{n+1 S^{m+n+1}}(x^0, z^2)$, where $x^0 = (p^0, 0)$ and z^2 is the south pole of S^{n+1}. Let $\mathbf{H}(\mathbf{f}) = \boldsymbol{\phi}$. Evidently $\mathbf{f} = \mathbf{g}$ implies $\mathbf{H}(\mathbf{f}) = \mathbf{H}(\mathbf{g})$, so that $\mathbf{H}$ is a well-defined mapping of $\pi_m(R_n)$ into $\pi_{m+n+1}(S^{n+1})$. We have further

THEOREM 1. $\mathbf{H}$ *is a homomorphic mapping of $\pi_m(R_n)$ into $\pi_{m+n+1}(S^{n+1})$.*

For let $\mathbf{f}, \mathbf{g} \in \pi_m(R_n)$, and let h be the constant mapping $h(p) = e$ $(p \in S^m)$. Then $\mathbf{h} = 0$. Hence $\mathbf{f} + \mathbf{h} = \mathbf{f}$, $\mathbf{h} + \mathbf{g} = \mathbf{g}$, so that $\mathbf{H}(\mathbf{f} + \mathbf{h}) = \mathbf{H}(\mathbf{f})$, $\mathbf{H}(\mathbf{h} + \mathbf{g}) = \mathbf{H}(\mathbf{g})$. It is therefore sufficient to prove that

$$(9) \qquad \mathbf{H}(\mathbf{f} + \mathbf{h}) + \mathbf{H}(\mathbf{h} + \mathbf{g}) = \mathbf{H}(\mathbf{f} + \mathbf{g}).$$

Let f', g' be mappings of S^m into R_n defined by

$$(10_1) \qquad f'(p) = \begin{cases} f[\phi_1(p)] & (p \in E_1^m),\\ h[\phi_2(p)] & (p \in E_2^m); \end{cases}$$

$$(10_2) \qquad g'(p) = \begin{cases} h[\phi_1(p)] & (p \in E_1^m),\\ g[\phi_2(p)] & (p \in E_2^m). \end{cases}$$

Then $\mathbf{f}' = \mathbf{f} + \mathbf{h}$, $\mathbf{g}' = \mathbf{h} + \mathbf{g}$. Let $F = H(f'^*)$, $G = H(g'^*)$.

[9] If $f(x) \subset Y$ and A is a closed subset of X, $f \mid A$ denotes the mapping of A into Y obtained by restricting the range of definition of f to the set A.

Let π_i denote the vertical projection of E_i^{m+n+1} on E^{m+n+1} $(i = 1, 2)$. Then $\pi_i(x) = x$ for $x \in S^{m+n}$. Let $F_0 = F \mid E_1^{m+n+1}$, $H'' = F \mid E_2^{m+n+1}$, $H' = G \mid E_1^{m+n+1}$, $G_0 = G \mid E_2^{m+n+1}$. Then it is easily verified that $H'\pi_1^{-1} = H''\pi_2^{-1}$. Call this mapping H_0. Evidently $F_0(x) = G_0(x) = H_0(x)$ $(x \in S^{m+n})$.

Let H_t $(0 \leqq t \leqq 1)$ be a homotopy of H_0 to x^0 keeping x^0 fixed. Then[10] there exist homotopies F_t, G_t $(0 \leqq t \leqq 1)$ of F_0, G_0 respectively, such that $F_t(x) = G_t(x) = H_t(x)$ $(x \in S^{m+n})$. Let

$$(11_1) \qquad F'_t(x) = \begin{cases} F_t(x) & (x \in E_1^{m+n+1}), \\ H_t[\pi_2(x)] & (x \in E_2^{m+n+1}); \end{cases}$$

$$(11_2) \qquad G'(x) = \begin{cases} H_t[\pi_1(x)] & (x \in E_1^{m+n+1}), \\ G_t(x) & (x \in E_2^{m+n+1}). \end{cases}$$

Evidently $\mathbf{F}'_1 = \mathbf{F}$, $\mathbf{G}'_1 = \mathbf{G}$.

Let

$$(12) \qquad H'_t(x) = \begin{cases} F_t(x) & (x \in E_1^{m+n+1}), \\ G_t(x) & (x \in E_2^{m+n+1}). \end{cases}$$

Then $\mathbf{H}'_0 = \mathbf{H}(\mathbf{f} + \mathbf{g})$, while $\mathbf{H}'_1 = \mathbf{F}'_1 + \mathbf{G}'_1 = \mathbf{F} + \mathbf{G} = \mathbf{H}(\mathbf{f} + \mathbf{h}) + \mathbf{H}(\mathbf{f} + \mathbf{g})$.[11] But $\mathbf{H}'_0 = \mathbf{H}'_1$, which proves the theorem.

4. Relations between the homomorphisms F, G, and H

Let S^{m+n} be the equator of S^{m+n+1}, S^n the equator of S^{n+1}, and let f be a mapping of S^{m+n} into S^n. We associate with the mapping f a mapping $F(f) = \psi(S^{m+n+1}) \subset S^{n+1}$ as follows: ψ maps the great circle joining the north pole x^1 of S^{m+n+1} to the point $x \in S^{m+n}$ on the great circle joining z^1 to $f(x)$, and maps the great circle joining x^2 to x on the great circle joining z^2 to $f(x)$. Evidently $\psi(E_1^{m+n+1}) \subset E_1^{n+1}$, $\psi(E_2^{m+n+1}) \subset E_2^{n+1}$, while $\psi = f$ on S^{m+n}. If $f \in S^{nS^{m+n}}(x^0, y^0)$, then $F(f) \in S^{n+1S^{m+n+1}}(x^0, y^0)$; moreover, f homotopic to g implies $F(f)$ homotopic to $F(g)$. Thus F induces a mapping $\mathbf{F}$ of $\pi_{m+n}(S^n)$ into $\pi_{m+n+1}(S^{n+1})$, which was shown by Freudenthal[4] to be a homomorphism.

Let R_{n-1} be the closed subgroup of R_n consisting of those rotations which leave the north pole fixed. Evidently R_{n-1} is isomorphic with the group of rotations of S^{n-1}. Since $R_{n-1} \subset R_n$, there is a natural homomorphism $\mathbf{G}$ of $\pi_m(R_{n-1})$ into $\pi_m(R_n)$.

THEOREM 2. *The homomorphisms* **F**, **G**, *and* **H** *are related by*

$$(13) \qquad \mathbf{F}\mathbf{H}_{m,n-1} = \mathbf{H}_{m,n}\mathbf{G}.$$

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[10] K. Borsuk, Fund. Math. 28 (1937), p. 101.

[11] This follows from the definition of addition in $\pi_{m+n+1}(S^{n+1})$ given by S. Eilenberg (Ann. of Math. 41 (1940), p. 235), which is easily shown to be equivalent to the one given here.

For let $\mathbf{f} \in \pi_m(R_{m-1})$, $g = F[H_{m,n-1}(f)]$, $g' = H_{m,n}[G(f)]$. It is then easily verified that $g = g'$ on S^{m+n}. Moreover $g'(E_1^{m+n+1}) \subset E_1^{n+1}$, $g'(E_2^{m+n+1}) \subset E_2^{n+1}$. Hence for no x is $g'(x) = -g(x)$. It follows that g and g' are homotopic, so that $\mathbf{g} = \mathbf{g}'$.

Let ϕ be a mapping of S^{n-1} into R_{n-1} defined as follows: if $x \in S^{n-1}$, x' is the point in the great circle joining x^1 to x whose angular distance from x^1 is twice that from x^1 to x. Then $\phi(x)$ is that rotation which carries x^1 into x' and leaves each point in the $(n-2)$-sphere orthogonal to x^1 and x fixed. Let $h = H_{n-1,n-1}(\phi)$. Then it can easily be shown[12] that if n is even h has Hopf invariant 2. We have further:

THEOREM 3. *The kernel of the homomorphism* $\mathbf{F}[\pi_{2n-1}(S^n)] \subset \pi_{2n}(S^{n+1})$ *(n even) is the subgroup of* $\pi_{2n-1}(S^n)$ *generated by* $\mathbf{h}$.

The author has recently shown[13] that $\mathbf{G}(\boldsymbol{\phi}) = 0$; in fact, the kernel of the homomorphism $\mathbf{G}$ is the subgroup of $\pi_{n-1}(R_{n-1})$ generated by $\boldsymbol{\phi}$. It follows from Theorem 2 that $\mathbf{F}[\mathbf{H}_{n-1,n-1}(\boldsymbol{\phi})] = \mathbf{F}(\mathbf{h}) = 0$. Let $\mathbf{g} \in \pi_{2n-1}(S^n)$, and suppose that $\mathbf{F}(\mathbf{g}) = 0$. Then the Hopf invariant of g is even,[14] say $2k$. Let $\mathbf{f} = k\mathbf{h}$. Then $\mathbf{F}(\mathbf{f} - \mathbf{g}) = 0$, and $\mathbf{f} - \mathbf{g}$ has Hopf invariant zero. Hence[15] $\mathbf{f} - \mathbf{g} = 0$, i.e., $\mathbf{g} = \mathbf{f} = k\mathbf{h}$.

THEOREM 4. $\mathbf{H}_{m,n}$ *maps* $\pi_m(R_n)$ *isomorphically for* $m = 1, 2$. $\mathbf{H}_{m,n}$ *maps* $\pi_m(R_n)$ *on* $\pi_{m+n+1}(S^{n+1})$ *for* $m = 1$ *and for* $m = 2$, $n > 1$.

Let $h(S^1) \subset R_1$ be defined by

$$h(x) = \begin{Vmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{Vmatrix}.$$

Then h maps S^1 homeomorphically on R_1, and $\mathbf{h}$ is a generator of the free cyclic group $\pi_1(R_1)$. But $H_{1,1}(h)$ maps S^3 on S^2 with Hopf invariant 1[16] and generates the group $\pi_3(S^2)$. It follows from Theorems 2 and 3 that $\mathbf{H}_{1,n}$ maps $\pi_1(R_n)$ isomorphically on $\pi_{n+2}(S^{n+1})$ for $n > 1$.

Since $\pi_2(R_n) = 0$, it follows that $\mathbf{H}_{2,n}$ is an isomorphism. But $\pi_{n+3}(S^{n+1}) = 0$ for $n > 1$[5], and hence $\mathbf{H}_{2,n}$ maps $\pi_2(R_n)$ on $\pi_{n+3}(S^{n+1})$. This completes the proof of the theorem.

5. Freudenthal's theorem

Freudenthal has recently announced[7] without proof a very general theorem on extension of mappings, and used this theorem to construct maps of S^{2n-1} on S^n with Hopf invariant 1 for all even n.[17] In this section the foregoing results are used to construct a counter-example to Freudenthal's theorem, and to show that the above-mentioned construction fails if $n > 2$ and $n \equiv 2 \pmod 4$.

[12] Cf. H II, p. 431.
[13] Ann. of Math. 43 (1942), Theorem 5.
[14] F I, Satz III.
[15] F I, Satz II, 2.
[16] H I, p. 654.
[17] F II, p. 140.

Let points z of Euclidean $2n$-space be represented by complex co-ordinates $(z_1, \cdots, z_n)$. Then S^{2n-1} is represented by the equation $\sum_{i=1}^{n} z_i\bar{z}_i = 1$.

Let P_{n-1} denote complex projective $(n - 1)$-space. Then there is a natural mapping $\phi(S^{2n-1}) \subset P_{n-1}$ defined by mapping each point $z \in S^{2n-1}$ into the point of P_{n-1} with the same coordinates. This is evidently a fibre map in the sense of Hurewicz and Steenrod,[18] the fibres being great circles. This mapping $\phi(S^{2n-1}) \subset P_{n-1}$ can be extended to a mapping $\psi(E^{2n}) \subset P_n$, where $\psi(z_1, \cdots, z_n) = (z_1, \cdots, z_n, (1 - \sum z_i\bar{z}_i)^{\frac{1}{2}})$. It is easily verified that ψ is a homeomorphism on $E^{2n} - S^{2n-1}$ and $\psi = \phi$ on S^{2n-1}.

Let X be a topological space, f a mapping of P_{n-1} into X. Then

THEOREM 5. *The mapping $f(P_{n-1}) \subset X$ can be extended to a mapping $f^*(P_n) \subset X$ if and only if the mapping $f\phi(S^{2n-1}) \subset X$ is inessential.*

For if $f\phi$ is inessential, there is a mapping $F(E^{2n}) \subset X$ such that $F = f\phi$ on S^{2n-1}. Let $f^* = F\psi^{-1}$. Then f^* is the required extension. Conversely, if f^* is an extension of f, let $F = f^*\psi$. Then F maps E^{2n} into X and $F = f\phi$ on S^{2n-1}. Hence $f\phi$ is inessential.

Let $g(S^1) \subset R_{2n-1}$ be defined by

$$g(x) = \begin{Vmatrix} x_1 & x_2 & 0 & 0 & \cdots & 0 & 0 \\ -x_2 & x_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_1 & x_2 & \cdots & 0 & 0 \\ 0 & 0 & -x_2 & x_1 & \cdots & 0 & 0 \\ & & \cdots & \cdots & \cdots & & \\ 0 & 0 & 0 & 0 & \cdots & x_1 & x_2 \\ 0 & 0 & 0 & 0 & \cdots & -x_2 & x_1 \end{Vmatrix}.$$

Then g is essential or inessential according as n is odd or even. For if $n = 1$, **g** is a generator of $\pi_1(R_1)$, so that g is essential. If $n = 2$, we have $g(S^1) \subset Q^3$, where Q^3 is the quaternion subgroup of R_3. But $\pi_1(Q^3) = \pi_1(S^3) = 0$. Hence $\mathbf{g} = 0$ in $Q^3 \subset R_3$, and g is inessential. The proof is completed by induction.

Let $h = H(g)$. Then it follows from Theorem 4 that $h(S^{2n+1}) \subset S^{2n}$ is essential if n is odd and inessential if n is even. Moreover, it can be directly verfied that there is a mapping $h'(P_n) \subset S^{2n}$ such that $h = h'\phi$, and that h' has degree 1. An application of Theorem 5 gives

THEOREM 6. *If n is even, the mapping $h'(P_n) \subset S^{2n}$ can be extended over P_{n+1}. If n is odd, it cannot be so extended.*

The theorem of Freudenthal's referred to above can be phrased as follows:[19] *Let K be a complex, f a normal mapping*[20] *of K^q into S^q. Suppose that f can be extended over K^{q+1}. Then f can be extended over K^{2q-1}.*

Let K be a triangulation of P_{n+1}, so that P_n becomes a closed subcomplex L of K. Then $L \subset K^{2n}$. Let h' be the mapping of L into S^{2n} of degree one

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[18] W. Hurewicz and N. E. Steenrod, Proc. Nat. Acad. 27 (1941), pp. 60–64.

[19] F II, p. 140.

[20] I.e., $f(K^{q-1}) = x^0$.

described above. Then[21] h' can be deformed into a normal map h''; moreover, h'' can be extended over K if and only if the same is true of h'. Let $H^r(K - L)$ denote the r^{th} cohomology group of $K - L$ with integral coefficients. Then $H^r(K - L) = 0$ for $r < 2n + 2$, while $H^{2n+2}(K - L)$ is a free cyclic group. In particular, $H^{2n+1}(K - L) = 0$. It follows from a theorem of Whitney[22] that h'' can be extended over K^{2n+1}. But h'' cannot be extended over K^{2n+2} for n odd.

Freudenthal's construction of maps of S^{4n-1} on S^{2n} is based on an application of his theorem to the case $K = P_{2n}$, $f(K^{2n}) \subset S^{2n}$, where $f(P_n) \subset S^{2n}$ is of degree one. The argument above shows that this construction breaks down if n is odd and >1; for f cannot even be extended over the subspace P_{n+1} of P_{2n}.

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[21] H. Whitney, Duke Journal 3 (1937), p. 53.

[22] Loc. cit., Theorem 2.