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Gauge theory and knot homologies

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Topological gauge theories in four dimensions which admit surface operators provide a natural framework for realizing homological knot invariants. Every such theory leads to an action of the braid group on branes on the corresponding moduli space. This action plays a key role in the construction of homological knot invariants. We illustrate the general construction with a simple example based on surface operators in $\mathcal{N} = 4$ twisted gauge theory which lead to a categorification of a variant of the Casson invariant.

1 Introduction

Topological field theory is a natural framework for "categorification", an informal procedure that turns integers into vector spaces (abelian groups), vector spaces into abelian or triangulated categories, operators into functors between these categories [1]. The number becomes the dimension of the vector space, while the vector space becomes the Grothendieck group of the category (tensored with a field). This procedure can be illustrated by the following diagram [2]:

$$\text{Number} \underset{\text{dimension}}{\overset{\text{Categorification}}{\rightleftarrows}} \text{Vector Space} \underset{\text{Grothendieck Group}}{\overset{\text{Categorification}}{\rightleftarrows}} \text{Category} \tag{1.1}$$

Recently, this idea led to a number of remarkable developments in various branches of mathematics, notably in low-dimensional topology, where many polynomial knot invariants were lifted to homological invariants.

Although the list of homological knot invariants is constantly growing, most of the existing knot homologies fit into the "the A-series" of homological knot invariants associated with the fundamental representation of $sl(N)$ (or $gl(N)$). Each such theory is a doubly graded knot homology whose graded Euler characteristic with respect to one of the gradings gives the corresponding knot invariant,

$$P(q) = \sum_{i,j} (-1)^i q^j \dim \mathcal{H}_{i,j} \tag{1.2}$$

For example, the Jones polynmial can be obtained in this way as the graded Euler characteristic of the Khovanov homology [3]. Similarly, the so-called knot Floer homology [4,5] provides a categorification of the Alexander polynomial $\Delta(q)$. At first, these as well as other homological knot invariants listed in the table below appear to have very different character. Thus, as the name suggests, knot Floer homology is defined as a symplectic Floer homology of two Lagrangian submanifolds in a certain configuration space, while the other theories are defined combinatorially. In addition, the definition of the knot Floer homology admits a generalization to knots in arbitrary 3-manifolds, whereas at present the definition of the other knot homologies (with $N > 0$) is known only for knots in $\mathbf{R}^3$.

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Table 1 A general picture of knot polynomials and knot homologies.

$\mathfrak{g}$	Knot polynomial	Categorification
$gl(1\|1)$	$\Delta(q)$	knot Floer homology $HFK(K)$
"$sl(1)$"	–	Lee's deformed theory $H'(K)$
$sl(2)$	Jones	Khovanov homology $H^{\mathrm{Kh}}(K)$
$sl(N)$	$P_N(q)$	$sl(N)$ homology $HKR^N(K)$

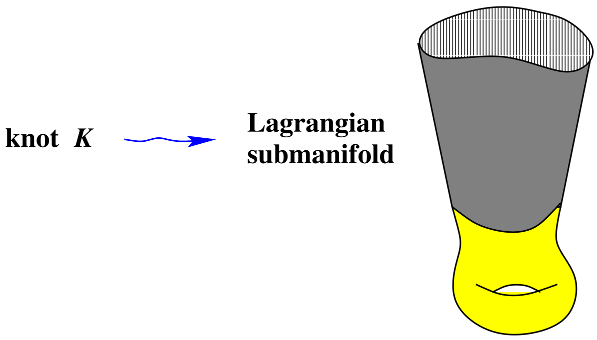


Fig. 1 (online colour at: www.fp-journal.org) A membrane ending on a Lagrangian five-brane.

The $sl(N)$ knot homology [3,6,7] – whose Euler characteristic is the quantum $sl(N)$ invariant $P_N(q)$ – has a physical interpretation as the space of BPS states, $\mathcal{H}_{BPS}$, in string theory [8]. In order to remind the physical setup of [8], let us recall that polynomial knot invariants, such as $P_N(q)$, can be related to open topological srting amplitudes ("open Gromov-Witten invariants") by first embedding Chern-Simons gauge theory in topological string theory [9], and then using the so-called large N duality [10–12], a close cousin of the celebrated AdS/CFT duality [13]. Moreover, open topological string amplitudes and, hence, the corresponding knot invariants can be reformulated in terms of new integer invariants which capture the spectrum of BPS states in the string Hilbert space, $\mathcal{H}_{BPS}$. The BPS states in question are membranes ending on Lagrangian five-branes in M-theory on a non-compact Calabi-Yau space $\mathcal{X} = \mathcal{O}_{\mathbf{P^1}}(-1) \oplus \mathcal{O}_{\mathbf{P^1}}(-1)$. Specifically, the five-branes have world-volume $\mathbf{R}^{2,1} \times L_K$ where $L_K \subset \mathcal{X}$ is a Lagrangian submanifold (which depends on knot K) and $\mathbf{R}^{2,1} \subset \mathbf{R}^{4,1}$.

Surprisingly, the physical interpretation of the $sl(N)$ knot homology naturally leads to a triply-graded (rather than doubly-graded) knot homology [8]. Indeed, the Hilbert space of BPS states, $\mathcal{H}_{BPS}$, is graded by three quantum numbers, which are easy to see in the physical setup described in the previous paragraph. The world-volume of the five-brane breaks the $SO(4) \cong SU(2) \times SU(2)$ rotation symmetry in five dimensions down to a subgroup $U(1)_L \times U(1)_R$, where $U(1)_L$ (resp. $U(1)_R$) is a rotation symmetry in the dimensions parallel (resp. transverse) to the five-brane. Therefore, BPS states in the effective $\mathcal{N} = 2$ theory on the five-brane are labelled by three quantum numbers j_L, j_R, and Q, where $Q \in H_2(\mathcal{X}, L_K) \cong \mathbb{Z}$ is the relative homology class represented by the membrane world-volume. In other words, apart from the $\mathbb{Z}_2$-grading by the fermion number, the Hilbert space of BPS states $\mathcal{H}_{BPS}$ is triply-graded. The properties of this triply-graded theory were studied in [14]; it turns out that this theory unifies all the doubly-graded knot homologies listed in Table 1, including the knot Floer homology. A mathematical definition of the triply-graded knot homology which appears to have many of the expected properties was constructed in [15].

Apart from realization in (topological) string theory, the homological knot invariants are expected to have a physical realization also in topological gauge theory, roughly as polynomial knot invariants have a physical realization in three-dimensional gauge theory (namely, in the Chern-Simons theory [16]) as well as in the topological string theory [9–11]. Although these two realizations are not unrelated, different properties of knot polynomials are easier to see in one description or the other. For example, the dependence on the rank N is manifest in the string theory description, while the skein operations and transformations under surgeries is easier to see in the Chern-Simons gauge theory.

Similarly, as we explained above, string theory realization is very useful for understanding relation between knot homologies of different rank. On the other hand, the formal properties of knot homologies which are hard to see in string theory (which, however, would be very natural in topological field theory) have to do with the fact that, in most cases, knot homologies can be extended to a functor $\mathcal{F}$ from the category of 3-manifolds with links and cobordisms to the category of graded vector spaces and homomorphisms

$$\mathcal{F}(Y;K) = \mathcal{H}_{Y;K} \tag{1.3}$$

$$\mathcal{F}(X;\Sigma) \;:\; \mathcal{H}_{Y;K} \to \mathcal{H}_{Y';K'} \tag{1.4}$$

Moreover, on manifolds with corners, it is expected that $\mathcal{F}$ extends to a 2-functor from the 2-category of oriented and decorated 4-manifolds with corners to the 2-category of triangulated categories [17–19]. In particular, it should associate:

- a triangulated category $\mathcal{F}(\Sigma)$ to a closed oriented 2-manifold Σ;
- an exact functor $\mathcal{F}(Y)$ to a 3-dimensional oriented cobordism Y;
- a natural transformation $\mathcal{F}(X)$ to a 4-dimensional oriented cobordism X.

As we explain below, these are precisely the formal properties of a four-dimensional topological field theory with boundaries and corners. Moreover, links and link cobordisms can be incorporated by introducing "surface operators" in the topological gauge theory.

In Sect. 2, we discuss the general aspects of topological gauge theories which admit surface operators. Of particular importance is the fact that every topological gauge theory which admits surface operators gives rise to an action of the braid group on D-branes. Then, in Sect. 3, we illustrate how these general structures are realized in a simple example of $\mathcal{N}=4$ twisted gauge theory – studied recently in connection with the geometric Langlands program [20,21] – with a simple type of surface operators, which provides a physical framework for categorification of the $G_{\mathbf{C}}$ Casson invariant. Gauge theory realization of other knot homologies (in particular, the Khovanov homology) categorifying Chern-Simons invariants will appear elsewhere [22].

2 Gauge theory and categorification

Let us start by describing the general properties of the topological quantum field theory (TQFT) with boundaries, corners, and surface operators. To a closed 4-manifold X, a four-dimensional TQFT associates a number, $Z(X)$, the partition function of the topological theory on X. Similarly, to a closed 3-manifold Y, it associates a vector space, $\mathcal{H}_Y$, the Hilbert space obtained by quantization of the theory on $X = \mathbf{R} \times Y$. Finally, to a closed surface Σ it associates a triangulated category, $\mathcal{F}(\Sigma)$, which can be understood as the category of D-branes in the topological sigma-model obtained via the dimensional reduction of gauge theory on Σ. The objects of the category $\mathcal{F}(\Sigma)$ describe BRST-invariant boundary conditions in the four-dimensional TQFT on 4-manifolds with corners (locally, such manifolds look like $X = \mathbf{R} \times \mathbf{R}_+ \times \Sigma$). Summarizing,

$$\begin{array}{lcl} \text{gauge theory on } X & \rightsquigarrow & \text{number } Z(X) \\ \text{gauge theory on } \mathbf{R}\times Y & \rightsquigarrow & \text{vector space } \mathcal{H}_Y \\ \text{gauge theory on } \mathbf{R}^2\times \Sigma & \rightsquigarrow & \text{category } \mathcal{F}(\Sigma) \end{array}$$

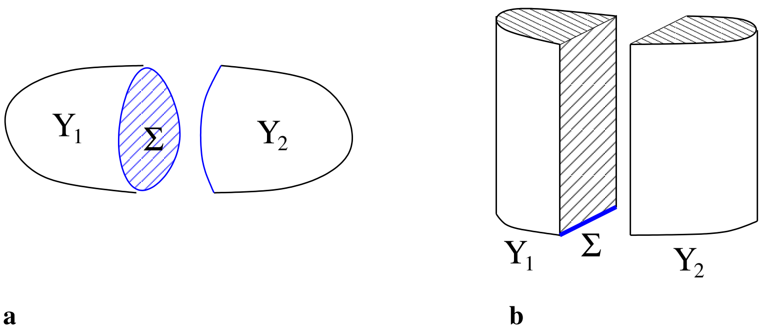


Fig. 2 (online colour at: www.fp-journal.org) **a** A 3-manifold Y can be obtained as a connected sum of 3-manifolds Y_1 and Y_2, joined along their common boundary Σ. **b** In four-dimensional gauge theory, the space $\mathbf{R} \times Y$ is obtained by gluing two 4-manifolds with corners.

where we assume that X, Y, and Σ are closed. Depending on whether the topological reduction of the four-dimensional gauge theory on $\mathbf{R}^2 \times \Sigma$ gives a topological A-model or B-model, the category $\mathcal{F}(\Sigma)$ is either the derived Fukaya category[1], $\mathbf{Fuk}(\mathcal{M})$, or the derived category of coherent sheaves, $D^b(\mathcal{M}) := D^b\mathbf{Coh}(\mathcal{M})$,

$$\text{topological A-model:} \quad \mathcal{F}(\Sigma) = \mathbf{Fuk}(\mathcal{M})$$

$$\text{topological B-model:} \quad \mathcal{F}(\Sigma) = D^b(\mathcal{M})$$

where $\mathcal{M}$ is the moduli space of classical solutions in gauge theory on $\mathbf{R}^2 \times \Sigma$, invariant under translations along $\mathbf{R}^2$. Different topological gauge theories lead to different functors $\mathcal{F}$. For example, in the context of Donaldson-Witten theory [25], Fukaya suggested [26] that the category associated to a closed surface Σ should be A_∞-category of Lagrangian submanifolds in the moduli space of flat G-connections on Σ. This is precisely what one finds from the topological reduction [27] of the twisted $\mathcal{N} = 2$ gauge theory on $\mathbf{R}^2 \times \Sigma$, in agreement with the general principle discussed here.

The Atiyah-Floer conjecture and its variants

It is easy to see that $\mathcal{F}$ defined by the topological gauge theory has all the expected properties of a 2-functor. In particular, to a 3-manifold Y with boundary $\partial Y = \Sigma$ it associates a "D-brane", that is an object in the category $\mathcal{F}(\Sigma)$.

The interpretation of 3-manifolds with boundary as D-branes can be used to reproduce the Atiyah-Floer conjecture, which states [28]:

$$HF_*^{\text{inst}}(Y) \cong HF_*^{\text{symp}}(\mathcal{M}; \mathcal{L}_1, \mathcal{L}_2) \tag{2.1}$$

Here $\mathcal{M} = \mathcal{M}_{flat}^G$ is the moduli space of flat connections on Σ, while $\mathcal{L}_1$ and $\mathcal{L}_2$ are Lagrangian submanifolds in $\mathcal{M}$ associated with the Heegard splitting of Y,

$$Y = Y_1 \cup_\Sigma Y_2$$

such that the points of $\mathcal{L}_i \subset \mathcal{M}$, $i = 1, 2$, correspond to flat connections on Σ which can be extended to Y_i.

Similarly, in the B-model, Y_1 and Y_2 define the corresponding B-branes, which are objects in the derived category of coherent sheaves on $\mathcal{M}$. In both cases, the vector space $\mathcal{H}_Y$ associated with the compact

[1] Notice, according to the Homological Mirror Symmetry conjecture, this category is equivalent to the derived category of the mirror B-model [23]. In particular, the category $\mathbf{Fuk}(\mathcal{M})$, suitably defined, must be a triangulated category [24].

3-manifold Y is the space of "$1-2$ strings":

$$\mathcal{H}_Y = \begin{cases} HF_*^{\mathrm{symp}}(\mathcal{M}; \mathcal{L}_1, \mathcal{L}_2) & \text{A-model} \\ \mathrm{Ext}^*(\mathcal{F}_{Y_1}, \mathcal{F}_{Y_2}) & \text{B-model} \end{cases} \tag{2.2}$$

In the Donaldson-Witten theory, this leads to the Atiyah-Floer conjecture (2.1).

"Decategorification"

The operation represented by the arrow going to the left in (1.1) – "decategorification" – also has a natural interpretation in gauge theory. It corresponds to the dimensional reduction, or compactification on a circle. Indeed, the partition function in gauge theory on $X = \mathbf{S}^1 \times Y$ is the trace (the index) over the Hilbert space $\mathcal{H}_Y$:

$$Z_{\mathbf{S}^1 \times Y} = \chi(\mathcal{H}_Y) \tag{2.3}$$

Similarly, the vector space associated with $Y = \mathbf{S}^1 \times \Sigma$ is the Grothendieck group of the category $\mathcal{F}(\Sigma)$

$$\mathcal{H}_{\mathbf{S}^1 \times \Sigma} = K(\mathcal{F}(\Sigma)) \tag{2.4}$$

In the case of A-model and B-model, respectively, we find

$$K(\mathcal{F}(\Sigma)) = \begin{cases} H^d(\mathcal{M}) & \text{for } \mathcal{F}(\Sigma) = \mathbf{Fuk}(\mathcal{M}) \\ H^*(\mathcal{M}) & \text{for } \mathcal{F}(\Sigma) = D^b(\mathcal{M}) \end{cases} \tag{2.5}$$

where $d = \frac{1}{2} \dim(\mathcal{M})$.

2.1 Incorporating surface operators

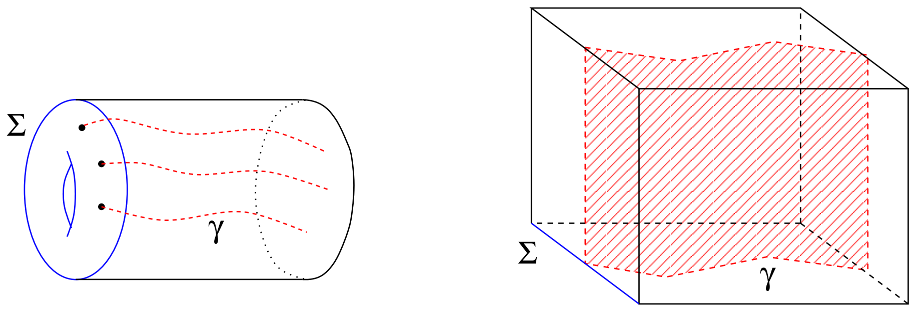
In a three-dimensional TQFT, knots and links can be incorporated by inrtoducing topological loop observables. The familiar example is the Wilson loop observable in Chern-Simons theory,

$$W_R(K) = \mathrm{Tr}_R \Big(P \exp \oint_K A \Big) \tag{2.6}$$

Recall, that canonical quantization of the Chern-Simons theory on $\Sigma \times \mathbf{R}$ associates a vector space $\mathcal{H}_\Sigma$ – the "physical Hilbert space" – to a Riemann surface Σ [16]. In presence of Wilson lines, quantization gives a Hilbert space $\mathcal{H}_{\Sigma; p_i, R_i}$ canonically associated to a Riemann surface Σ together with marked points p_i (points where Wilson lines meet Σ) decorated by representations R_i. For example, to n marked points on the plane colored by the fundamental representation it associates $\mathcal{V}^{\otimes n}$, where $\mathcal{V}$ is a N-dimensional irreducible representation of the quantum group $U_q(sl(N))$.

We wish to lift this to a four-dimensional gauge theory by including the "time" direction, so that the space-time becomes $X = Y \times \mathbf{R}$, where the knot K is represented by a topological defect (which was called a "surface operator" in [21]) localized on the surface $D = K \times \mathbf{R}$. In the part integral, a surface operator is defined by requiring the gauge field A (and perhaps other fields as well) to have a prescribed singularity. For example, the simplest type of singularity studied in [21] creates a holonomy of the gauge field on a small loop around D,

$$V = \mathrm{Hol}(A) \tag{2.7}$$



a **b**

Fig. 3 (online colour at: www.fp-journal.org) **a** Line operators in a three-dimensional TQFT on $\Sigma \times I$ and **b** topological "surface operators" in four-dimensional gauge theory on $\Sigma \times I \times \mathbf{R}$.

Quantization of the four-dimensional topological theory on a 4-manifold $X = Y \times \mathbf{R}$ with a surface operator on $D = K \times \mathbf{R}$ gives rise to a functor that associates to this data (namely, a 3-manifold Y, a knot K, and parameters of the surface operator) a vector space, the space of quantum ground states,

$$\mathcal{F}(Y; K) = \mathcal{H}_{Y;K} \tag{2.8}$$

Moreover, we will be interested in surface operators which preserve topological invariance for more general 4-manifolds X and embedded surfaces $D \subset X$. For example, if the four-dimensional topological gauge theory is obtained by a topological twist of a supersymmetric gauge theory, it is natural to consider a special class of surface operators which preserve supersymmetry, in particular, supercharges which become BRST charges in the twisted theory. Such surface operators can be defined on a more general embedded surface D, which might be either closed or end on the boundary of X. An example of this situation is a four-dimensional TQFT with corners, which arisies when we consider a lift of a 3-manifold with boundary Σ and line operators with end-points on Σ.

To summarize, including topological surface operators in the four-dimensional gauge theory, we obtain a functor from the category of 3-manifolds with links and their cobordisms to the category of graded vector spaces and homomorphisms:

$$\mathcal{F}(X; \Sigma) \ : \ \mathcal{H}_{Y;K} \to \mathcal{H}_{Y';K'} \tag{2.9}$$

Here, the knot homology $\mathcal{H}_{Y;K}$ is the space of quantum ground states in the four-dimensional gauge theory with surface operators and boundaries. Similarly, the functor $\mathcal{F}$ associates a number (the partition function) to a closed 4-manifold with embedded surfaces, and a category $\mathcal{F}(\Sigma; p_i)$ to a surface Σ with marked points, p_i, which correspond to the end-points of the topological surface operators.

As in the theory without surface operators, the category $\mathcal{F}(\Sigma; p_i)$ is either the category of A-branes or the category of B-branes on $\mathcal{M}$, depending on whether the topological reduction of the four-dimensional gauge theory is A-model or B-model. Here, $\mathcal{M}$ is the moduli space of $\mathbf{R}^2$-invariant solutions in gauge theory on $X = \mathbf{R}^2 \times \Sigma$ with surface operators supported at $\mathbf{R}^2 \times p_i$.

2.2 Braid group actions

As we just explained, surface operators are the key ingredients for realizing knot homologies in four-dimensional gauge theory. Our next goal is to explain that every topological gauge theory which admits surface operators is, in a sense, a factory that produces examples of braid group actions on branes, including some of the known examples as well as the new ones[2].

[2] It is worth pointing out that, compared to [21], where the braid group action is associated with *local* singularities in the moduli space $\mathcal{M}$, in the present context the origin of the braid group action is associated with *global* singularities.

In general, the mapping class group of the surface Σ acts on branes on $\mathcal{M}$. In particular, when Σ is a plane with n punctures, the moduli space $\mathcal{M}$ is fibered over the configuration space $\mathrm{Conf}^n(\mathbb{C})$ of n unordered points on $\mathbb{C}$,

$$\begin{array}{c} \mathcal{M} \\ \downarrow \\ \mathrm{Conf}^n(\mathbb{C}) \end{array} \tag{2.10}$$

and the braid group $Br_n = \pi_1(\mathrm{Conf}^n(\mathbb{C}))$ (= the mapping class group of the n-punctured disk) acts on the category $\mathcal{F}(\Sigma)$. Recall, that the braid group on n strands, Br_n, has $n-1$ generators, σ_i, $i = 1, \ldots, n-1$ which satisfy the following relations

$$\begin{aligned} &\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ &\sigma_i \sigma_j = \sigma_j \sigma_i \;, \quad |i-j| > 1 \end{aligned} \tag{2.11}$$

where σ_i can be reprsented by a braid with only one crossing between the strands i and $i+1$, as shown on the figure below.

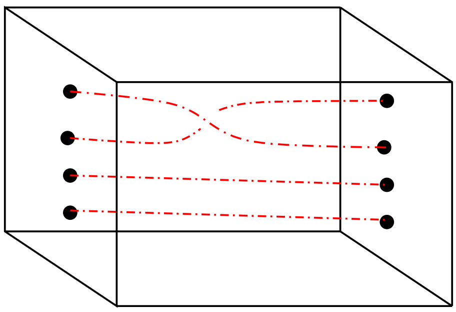


Fig. 4 (online colour at: www.fp-journal.org) A braid on four strands.

In gauge theory, the action of the braid group on branes is induced by braiding of the surface operators. Namely, a braid, such as the one on Fig. 4, corresponds to a non-contactible loop in the configuration space, $\mathrm{Conf}^n(\mathbb{C})$. As we go around the loop, the fibration (2.10) has a monodromy, which acts on the category of branes $\mathcal{F}(\Sigma)$ as an autoequivalence,

$$\begin{aligned} Br_n \quad &\to \quad \mathrm{Auteq}(\mathcal{F}(\Sigma)) \\ \beta \quad &\mapsto \quad \phi_\beta \end{aligned} \tag{2.12}$$

The simplest situation where one finds the action of the braid group Br_n on A-branes (resp. B-branes) on $\mathcal{M}$ is when $\mathcal{M}$ contans A_{n-1} chain of Lagrangian spheres (resp. spherical objects).

We remind that an A_{n-1} chain of Lagrangian spheres is a collection of Lagrangian spheres $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1} \subset \mathcal{M}$, such that

$$|\mathcal{L}_i \cap \mathcal{L}_j| = \begin{cases} 1 & |i-j| = 1 \\ 0 & |i-j| > 1 \end{cases} \tag{2.13}$$

These configurations occur when $\mathcal{M}$ can be degenerated into a manifold with singularity of type A_{n-1}. Indeed, to any Lagrangian sphere $\mathcal{L} \subset \mathcal{M}$, one can associate a symplectic automorphism of $\mathcal{M}$, the so-called generalized Dehn twist $T_{\mathcal{L}}$ along $\mathcal{L}$, which acts on $H_*(\mathcal{M})$ as the Picard-Lefschetz monodromy transformation

$$(T_{\mathcal{L}})_*(x) = \begin{cases} x - ([\mathcal{L}] \cdot x)[\mathcal{L}] & \text{if } \dim(x) = \dim(\mathcal{L}) \\ x & \text{otherwise} \end{cases} \tag{2.14}$$

As shown in [29], Dehn twists $T_{\mathcal{L}_i}$ along A_{n-1} chains of Lagrangian spheres satisfy the braid relations (2.11), and this induces an action of the braid group with n strands on the category of A-branes, $\mathbf{Fuk}(\mathcal{M})$.

The mirror of this construction gives an example of the braid group action on B-branes [30]. In this case, the braid group is generated by the twist functors along spherical objects ("spherical B-branes") which are mirror to the Lagrangian spheres. As the name suggests, an object $\mathcal{E} \in D^b(\mathcal{M})$ is called d-spherical if $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$ is isomorphic to $H^*(S^d, \mathbb{C})$ for some $d > 0$,

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } d \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

A spherical B-brane defines a twist functor $T_{\mathcal{E}} \in \mathrm{Auteq}(D^b(\mathcal{M}))$ which, for any $\mathcal{F} \in D^b(\mathcal{M})$, fits into exact triangle

$$\mathrm{Hom}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow T_{\mathcal{E}}(\mathcal{F}) \tag{2.16}$$

where the first map is evaluation. At the level of D-brane charges, the twist functor $T_{\mathcal{E}}$ acts as, cf. (2.14),

$$x \mapsto x + (v(\mathcal{E}) \cdot x)\, v(\mathcal{E})$$

where $v(\mathcal{E}) = ch(\mathcal{E})\sqrt{Td(\mathcal{M})} \in H^*(\mathcal{M})$ is the D-brane charge (the Mukai vector) of $\mathcal{E}$.

The mirror of an A_{n-1} chain of Lagrangian spheres is an A_{n-1} chain of spherical objects, that is a collection of spherical objects $\mathcal{E}_1, \ldots, \mathcal{E}_{n-1}$ which satisfy the condition analogous to (2.13),

$$\sum_k \dim \mathrm{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = \begin{cases} 1 & |i-j| = 1 \\ 0 & |i-j| > 1 \end{cases} \tag{2.17}$$

With some minor technical assumptions [30], the corresponding twist functors $T_{\mathcal{E}_i}$ generate an action of the braid group Br_n on $D^b(\mathcal{M})$. As we illustrate below, many examples of braid group actions on branes can be found by studying gauge theory with surface operators.

In A-model as well as in B-model, the braid group action on branes can be used to write a convenient expression for knot homology, $\mathcal{H}_K$, of a knot K represented as a braid closure. Let K be a knot obtained by closing a braid β on both ends as shown on Fig. 5. Then, the space of quantum ground states, $\mathcal{H}_K$, in the four-dimensional gauge theory with a surface operator on $D = \mathbf{R} \times K$ can be represented as the space of open string states between branes $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}' = \phi_\beta(\widetilde{\mathcal{B}})$. Here, $\widetilde{\mathcal{B}}$ is the basic brane which corresponds to the configuration on Fig. 5, while $\widetilde{\mathcal{B}}'$ is the brane obtained from it by applying the functor ϕ_β; it corresponds to the braid β closed on one side. These are A-branes (resp. B-branes) in the case of A-model (resp. B-model), and the space open strings is, cf. (2.2),

$$\mathcal{H}_K = \begin{cases} HF_*^{\mathrm{symp}}(\mathcal{M}; \widetilde{\mathcal{B}}, \phi_\beta(\widetilde{\mathcal{B}})) & \text{A-model} \\ \mathrm{Ext}^*(\widetilde{\mathcal{B}}, \phi_\beta(\widetilde{\mathcal{B}})) & \text{B-model} \end{cases} \tag{2.18}$$

In particular, when topological reduction of the gauge theory gives A-model, the branes $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}'$ are represented by Lagrangian submanifolds in $\mathcal{M}$. This leads to a construction of link homologies via symplectic geometry, as in [31–33].

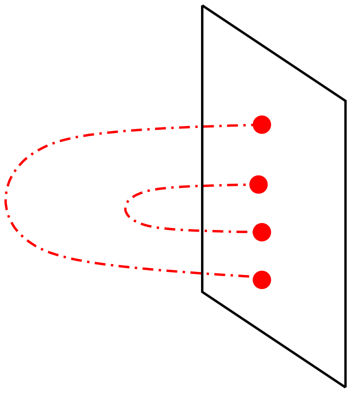


Fig. 5 (online colour at: www.fp-journal.org) A particular brane $\widetilde{\mathcal{B}}$ which corresponds to closing a braid on four strands.

3 Categorification of the $G_{\mathbb{C}}$ Casson invariant

Now, let us illustrate the general structures discussed in the previous section in the context of $\mathcal{N} = 4$ topological super-Yang-Mills theory in four dimensions. Specifically, we shall consider the GL twist of the theory [20], with surface operators labeled by regular semi-simple conjugacy classes [21]. As we shall explain below, this theory provides a natural framework for categorification of the $G_{\mathbb{C}}$ Casson invariant, which counts flat connections of the complexified group $G_{\mathbb{C}}$.

The topological reduction of this theory leads to a $\mathcal{N} = 4$ sigma-model [20,27], whose target space is a hyper-Kahler manifold $\mathcal{M}_H(\Sigma, G)$, the moduli space of solutions to the Hitchin equations on Σ [34]:

$$\begin{aligned} &F - \phi \wedge \phi = 0 \\ &d_A \phi = 0\,, \quad d_A \star \phi = 0 \end{aligned} \tag{3.1}$$

This twist of the $\mathcal{N} = 4$ super-Yang-Mills theory has a rich spectrum of supersymmetric surface operators. In particular, here we will be interested in the most basic type of surface operators, which correspond to the singular behavior of the gauge field A and the Higgs field ϕ of the form [21]:

$$\begin{aligned} &A = \alpha d\theta + \ldots, \\ &\phi = \beta \tfrac{dr}{r} - \gamma d\theta + \ldots \end{aligned} \tag{3.2}$$

where $\alpha, \beta, \gamma \in \mathfrak{t}$, and the dots stand for terms regular at $r = 0$. For generic values of the parameters α, β, γ, (3.2) defines a surface operator associated with the regular semi-simple conjugacy class $\mathfrak{C} \in G_{\mathbb{C}}$.

According to the general rules explained in Sect. 2, this topological field theory associates a homological invariant $\mathcal{H}_Y$ to a closed 3-manifold Y and, more generally, a knot homology $\mathcal{H}_{Y;K}$ to a 3-manifold with a knot (link) K. These homologies can be computed as in (2.2) and (2.18) using the Heegard decomposition of Y as well as the braid group action on branes. The branes in questions[3] are branes of type (A, B, A) with respect to the three complex structures (I, J, K) of the hyper-Kahler space $\mathcal{M}_H$. We can use this fact and analyze the branes in different complex structures in order to gain a better understanding of the homological invariant $\mathcal{H}_{Y,K}$ as well as the $G_{\mathbb{C}}$ Casson invariant itself.

Complex structure J: counting flat connections

The B-model in complex structure J is obtained, e.g. by setting the theta angle to zero, $\mathrm{Re}(\tau) = 0$, and choosing $t = i$ (where t is a complex parameter that labels a family of GL twists of the $\mathcal{N} = 4$ super-Yang-Mills [20]). In complex structure J, the moduli space $\mathcal{M}_H$ is the space of complexified flat connections

[3] e.g. branes $\mathcal{B}_1$ and $\mathcal{B}_2$ associated with the Heegard decomposition $Y = Y_1 \cup_\Sigma Y_2$

$\mathcal{A} = A + i\phi$, and the surface operator (3.2) creates a holonomy,

$$V = \mathrm{Hol}(\mathcal{A}) \,,$$

which is conjugate to $\exp(-2\pi(\alpha - i\gamma))$. Furthermore, at $t = i$ the supersymmetry equations of the four-dimensional gauge theory are equivalent to the flatness equations, $d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$, which explains why (from the viewpoint of complex structure J) the partition function of this theory on $X = \mathbf{S}^1 \times Y$ with a surface operator on $D = \mathbf{S}^1 \times K$ computes the $G_{\mathbf{C}}$ Casson invariant,

$$Z = \lambda_{G_{\mathbf{C}}}(Y, K)$$

The space of ground states, $\mathcal{H}_{Y;K}$, is a categorification of $\lambda_{G_{\mathbf{C}}}(Y;K)$. In general, both $\lambda_{G_{\mathbf{C}}}(Y;K)$ and $\mathcal{H}_{Y;K}$ depend on the holonomy V, which characterizes surface operators. However, if V is regular semi-simple, as we consider here, then $\lambda_{G_{\mathbf{C}}}(Y;K)$ and $\mathcal{H}_{Y;K}$ do not depend on a particular choice of V.

Complex structure K

Since the four-dimensional topological gauge theory (even with surface operators) does not depend on the parameter t that labels different twists, we can take $t = 1$, which leads to the A-model on $\mathcal{M}_H(\Sigma)$ with symplectic structure ω_K. This theory computes the same $G_{\mathbf{C}}$ Casson invariant and its categorification, $\mathcal{H}_{Y;K}$, but via counting solutions to the following equations on Y [20]:

$$\begin{aligned} & F - \phi \wedge \phi = \star\big(D\phi_0 - [A_0, \phi]\big) \\ & \star D\phi = [\phi_0, \phi] + DA_0 \\ & D^* \phi + [A_0, \phi_0] = 0 \end{aligned} \tag{3.3}$$

rather than flat $G_{\mathbf{C}}$ connections. In particular, given a Heegard decomposition $Y = Y_1 \cup_\Sigma Y_2$, the space of solutions to Eqs. (3.3) on Y_1 (resp. Y_2) defines a Lagrangian A-brane in $\mathcal{M}_H(\Sigma)$ with respect to ω_K. This allows to express $\mathcal{H}_{Y;K}$ as the space of open string states between the corresponding A-branes $\mathcal{B}_1$ and $\mathcal{B}_2$, cf. (2.2),

$$\mathcal{H}_{Y;K} = HF_*^{\mathrm{symp}}(\mathcal{M}_H; \mathcal{B}_1, \mathcal{B}_2)$$

This alternative definition of the $G_{\mathbf{C}}$ Casson invariant and its categorification that follows from the twisted $\mathcal{N} = 4$ gauge theory can be especially useful in situations when the (A, B, A) branes $\mathcal{B}_1$ and $\mathcal{B}_2$ intersect at singular points in $\mathcal{M}_H$ or over higher-dimensional subvarieties.

Categorification of the $SL(2, \mathbb{C})$ Casson invariant

Now, let us return to the complex structure J and, for simplicity, take the gauge group to be $G = SU(2)$. Furthermore, we shall consider an important example of the sphere with four punctures:

$$\Sigma = \mathbf{CP}^1 \setminus \{p_1, p_2, p_3, p_4\}$$

which in gauge theory corresponds to inserting four surface operators. In complex structure J, $\mathcal{M}_H$ is the moduli space of flat $G_{\mathbf{C}} = SL(2, \mathbb{C})$ connections with fixed conjugacy class of the monodromy around each puncture. It can be identified with the space of conjugacy classes of monodromy representations

$$\mathcal{M}_H \cong \{\rho : \ \pi_1(\Sigma) \to G_{\mathbf{C}} \mid \rho(\gamma_i) \in \mathfrak{C}_i\} / \sim$$

where the representations are restricted to take the simple loop γ_i around the i-th puncture into the conjugacy class $\mathfrak{C}_i \subset G_{\mathbf{C}}$.

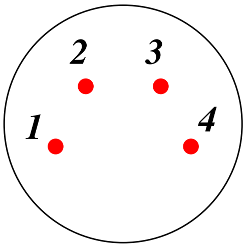


Fig. 6 (online colour at: www.fp-journal.org) Sphere with four punctures.

Using the fact that $\pi_1(\Sigma)$ is free on three generators, we can explicitly describe the moduli space $\mathcal{M}_H$ by introducing holonomies of the flat $SL(2,\mathbb{C})$ connection around each puncture,

$$V_i = \mathrm{Hol}_{p_i}(\mathcal{A}) \quad , \quad i = 1, \ldots, 4 \tag{3.4}$$

where $V_1V_2V_3V_4 = 1$ and each V_i is in a fixed conjugacy class. Following [35–39], we introduce the local monodromy data

$$a_i = \begin{cases} \mathrm{tr}V_i & i = 1,2,3 \\ \mathrm{tr}(V_3V_2V_1) & i = 4 \end{cases} \tag{3.5}$$

and

$$\begin{aligned} \theta_1 &= a_1a_4 + a_2a_3 \\ \theta_2 &= a_2a_4 + a_1a_3 \\ \theta_3 &= a_3a_4 + a_2a_1 \\ \theta_4 &= a_1a_2a_3a_4 + \sum_{i=1}^{4} a_i^2 - 4 \end{aligned} \tag{3.6}$$

which determines the conjugacy classes of V_i. We also introduce the variables

$$\begin{aligned} x_1 &= \mathrm{tr}(V_3V_2) \\ x_2 &= \mathrm{tr}(V_1V_3) \\ x_3 &= \mathrm{tr}(V_2V_1) \end{aligned} \tag{3.7}$$

which will be the coordinates on the moduli space $\mathcal{M}_H$. Namely, the moduli space we are interested in is

$$\mathcal{M}_H = \{(V_1, \ldots, V_4) \mid V_i \in \mathfrak{C}_i,\ V_1V_2V_3V_4 = 1\}/G_{\mathbf{C}}$$

In terms of the variables (3.7), it can be explicitly described as the affine cubic

$$\mathcal{M}_H = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid f(x_i, \theta_m) = 0\} \tag{3.8}$$

where

$$f(x_i, \theta_m) = x_1x_2x_3 + \sum_{i=1}^{3}(x_i^2 - \theta_i x_i) + \theta_4 \tag{3.9}$$

Singularities in $\mathcal{M}_H$

For certain values of the monodromy data, the moduli space $\mathcal{M}_H$ becomes singular. It is important to understand the nature of the singularities and when they develop. In fact, as we shall see below, inetersting examples of branes pass through such singularities.

The descriminant $\Delta(f)$ of the cubic (3.9) is a polynomial in a_i of total degree 16 [40]:

$$\Delta(a) = \left(\prod_{\epsilon_1\epsilon_2\epsilon_3=1} (a_4 + \sum \epsilon_i a_i) - \prod_{i=1}^{3} (a_i a_4 - a_j a_k) \right)^2 \prod_{i=1}^{4} (a_i^2 - 4) \tag{3.10}$$

where $\epsilon_i = \pm 1$. A special subfamily of cubics (3.9), which will play an important role in applications to knot invariants discussed below, corresponds to the case where all monodromy parameters a_i are equal, $a_i = a$, $i = 1, 2, 3, 4$. In this case,

$$\begin{aligned} \theta_i &= 2a^2\ , \quad i = 1,2,3 \\ \theta_4 &= a^4 + 4a^2 - 4 \end{aligned} \tag{3.11}$$

and it is easy to verify that $\Delta(a) = 0$. Specifically, for generic values of the parameter a, the moduli space $\mathcal{M}_H$ has three simple singularities of type A_1 (double points) at

$$(x_i, x_j, x_k) = (a^2 - 2, 2, 2) \tag{3.12}$$

These singularities correspond to reducible flat connections. For special values of a, the singularities can become worse and/or additional singularities can appear. For example, for $a^2 = 0$ a new singularity of type A_1 develops at the point $(x_1, x_2, x_3) = (-2, -2, -2)$. On the other hand, for $a^2 = 4$ the moduli space has a simple singularity of type D_4 at $(x_1, x_2, x_3) = (2, 2, 2)$.

Braid group action

The mapping class group of Σ, which in the present case is the braid group Br_3, acts on the family of cubic surfaces (3.8) by polynomial automorphisms. In particular, one can verify that the generators σ_i, $i = 1, 2, 3$, represented as [35]:

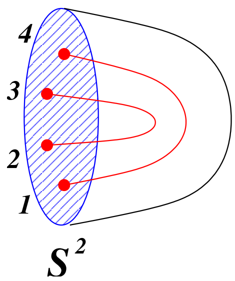
$$\sigma_i \ : \ (x_i, x_j, x_k, \theta_i, \theta_j, \theta_k, \theta_4) \to (\theta_j - x_j - x_k x_i, x_i, x_k, \theta_j, \theta_i, \theta_k, \theta_4) \tag{3.13}$$

satisfy the relations $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ and $\sigma_k = \sigma_i \sigma_j \sigma_i^{-1}$. Here and below we denote by (i, j, k) any cyclic permutation of $(1, 2, 3)$.

Examples

Let us consider examples of (A, B, A) branes that arise from knotted surface operators in $\mathbf{R} \times \mathbf{B}^3$, where $\mathbf{B}^3$ denotes a 3-dimensional ball. We consider surface operators which are extended along the $\mathbf{R}$ direction and which meet the boundary $\mathbf{S}^2 = \partial \mathbf{B}^3$ at four points. The simplest example of such brane is

brane $\widetilde{\mathcal{B}}$ =



(3.14)

We shall denote this brane $\widetilde{\mathcal{B}}$ (or $\widetilde{\mathcal{B}}_{(14)(23)}$ if we wish to specify which pairs of points on $\mathbf{S}^2$ it connects). Since the brane (3.14) identifies the monodromies around the points 1 and 4 (resp. 2 and 3),

$$V_1 = V_4^{-1}\ , \quad V_2 = V_3^{-1} \tag{3.15}$$

it can be explicitly described as a subvariety of $\mathcal{M}_H$ defined by

$$x_1 = \mathrm{tr}(V_3 V_2) = 2 \tag{3.16}$$

Of course, we also need to set $a_1 = a_4$ and $a_2 = a_3$, so that

$$\begin{aligned} \theta_1 &= a_1^2 + a_2^2 \\ \theta_2 &= \theta_3 = 2a_1 a_2 \\ \theta_4 &= a_1^2 a_2^2 + 2a_1^2 + 2a_2^2 - 4 \end{aligned} \tag{3.17}$$

Substituting (3.16) and (3.17) into the cubic equation $f(x_i, \theta_m) = 0$, we find that the brane (3.14) can be described as a degenerate quadric,

$$(x_2 + x_3 - a_1 a_2)^2 = 0 \tag{3.18}$$

One can also think of it as a set of two coincident branes on $x_2 + x_3 = a_1 a_2$. By acting on this brane with the elements of the braid group (3.13), we can construct other examples of (A, B, A) branes in $\mathcal{M}_H$. Furthermore, by closing the braid one can obtain homological invariants of knots (links) in $\mathbf{S}^3$ as spaces of open strings between two such branes. In the rest of this section, we consider a few explicit examples.

Unknot: One way to construct the unknot is to take surface operators which correspond to two branes of type (3.14), as shown on the figure below:

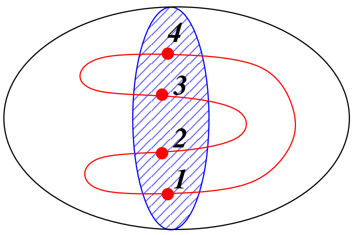


Fig. 7 (online colour at: www.fp-journal.org) Unknot in $\mathbf{S}^3$ can be represented as a union of two branes $\widetilde{\mathcal{B}}$.

The two branes on this figure are branes $\widetilde{\mathcal{B}}_{(14)(23)}$ and $\widetilde{\mathcal{B}}_{(12)(34)}$. We already discussed the first brane: it is described by the conditions (3.16)–(3.18). Similarly, the brane $\widetilde{\mathcal{B}}_{(12)(34)}$ is given by $V_1 = V_2^{-1}$, which implies $V_3 = V_4^{-1}$,

$$x_3 = \mathrm{tr}(V_2 V_1) = 2 \tag{3.19}$$

and the corresponding conditions for θ_i, cf. (3.17). Altogether, the conditions describing these two branes imply that the local monodromy data should be identified,

$$a_1 = a_2 = a_3 = a_4 = a \tag{3.20}$$

This condition is very natural, of course, and will be relevant in all the examples where the resulting link has only one connected component, *i.e.* is actually a knot. Furthermore, for the unknot in Fig. 7 we have:

$$V_1 = V_2^{-1} = V_3 = V_4^{-1} \tag{3.21}$$

These equations describe the intersection points of branes $\widetilde{\mathcal{B}}_{(14)(23)}$ and $\widetilde{\mathcal{B}}_{(12)(34)}$. Using (3.16) and (3.19), it is easy to see that there is only one such point (of multiplicity 2):

$$(x_1, x_2, x_3) = (2, a^2 - 2, 2) \tag{3.22}$$

This is precisely one of the singular points (3.12) where the moduli space $\mathcal{M}_H$ has A_1 singularity (for generic values of a) due to reducible representations. Therefore, we conclude that the cohomology of the unknot, $\mathcal{H}^{sl(2)}_{\mathrm{unknot}}$, is given by the space of open string states for two different branes intersecting at the A_1 singularity in $\mathcal{M}_H$. We point out that the values of x_i in (3.22) can be read off directly from Fig. 7. Indeed, $x_1 = 2$ simply follows from the fact that the combined monodromy around the points 2 and 3 is equal to the identity (similarly for $x_3 = 2$). In order to explain $x_2 = a^2 - 2$, it is convenient to introduce the eigenvalues $m^{\pm 1}$ of the monodromy matrix V_1. Of course, m is related to the local monodromy parameter a, namely $a = m + m^{-1}$. Moreover, since $V_1 = V_3$, we have

$$x_2 = \mathrm{tr}(V_1 V_3) = m^2 + m^{-2} = a^2 - 2 \tag{3.23}$$

One can also construct the unknot using identical branes $\widetilde{\mathcal{B}}$ and the braid group action on one of them:

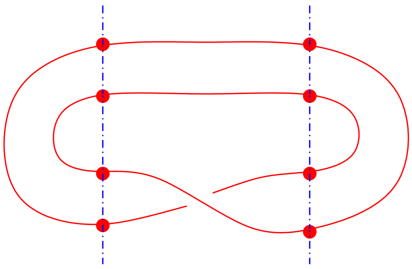


Fig. 8 (online colour at: www.fp-journal.org) Unknot as a union of two branes $\widetilde{\mathcal{B}}$ with a half-twist. Each vertical line represents a surface (topologically a 2-sphere) which divides $\mathbf{S}^3$ into two balls and meets the surface operator at four points.

Here, the two parts of the unknot correspond to the branes $\widetilde{\mathcal{B}}$ and $\phi_{\sigma_1}(\widetilde{\mathcal{B}})$, where $\widetilde{\mathcal{B}}$ is the brane described in (3.14)–(3.18), and σ_1 denotes the generator of the braid group Br_3. Using the explicit form (3.13) of σ_1, we find that the brane $\phi_{\sigma_1}(\widetilde{\mathcal{B}})$ is supported on the line:

$$\phi_{\sigma_1}(\widetilde{\mathcal{B}}) \ : \quad x_2 = 2 \tag{3.24}$$

Together with (3.16), this condition implies that the branes $\widetilde{\mathcal{B}}$ and $\phi_{\sigma_1}(\widetilde{\mathcal{B}})$ meet only at one point (of multiplicity 2):

$$(x_1, x_2, x_3) = (2, 2, a^2 - 2) \tag{3.25}$$

which is precisely one of the A_1 singularities (3.12) in the moduli space $\mathcal{M}_H$. This is in complete agreement with our previous analysis, where the same configuration of D-branes in $\mathcal{M}$ was found starting from the presentation of the unknot shown on Fig. 7. This agreement was expected, of course, since both presentations of the unknot on Figs. 7 and 8 are homotopy equivalent in $\mathbf{S}^3$. The second presentation (on Fig. 8) can be easily generalized to the trefoil knot and more general torus knots (links) of type $(2, k)$.

Trefoil knot: The trefoil can be constructed by joining together the brane (3.14) and the brane obtained by action of three half-twists on $\widetilde{\mathcal{B}}$ (see Fig. 9).

Starting with Eq. (3.16) descrining the brane $\widetilde{\mathcal{B}}$ and applying σ_1 three times, we find that the brane $\phi_{(\sigma_1)^3}(\widetilde{\mathcal{B}})$ is supported on the set of points

$$(x_1, x_2, x_3) = (4z - 2a^2 z + 2a^2 z^2 - 2z^3 + y(1 - z^2), -2 + 2a^2 - 2a^2 z + yz + 2z^2, z) \tag{3.26}$$

where we assumed (3.20). Together with the equation $f(x_i) = 0$, this condition describes a subvariety in $\mathcal{M}_H$ of complex dimension 1. Using (3.16) and (3.26), it is easy to see that the branes $\widetilde{\mathcal{B}}$ and $\phi_{(\sigma_1)^3}(\widetilde{\mathcal{B}})$

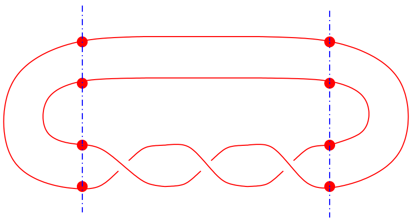


Fig. 9 (online colour at: www.fp-journal.org) The trefoil knot in $\mathbf{S}^3$ can be represented as a union of two branes $\widetilde{\mathcal{B}}$ with three half-twists.

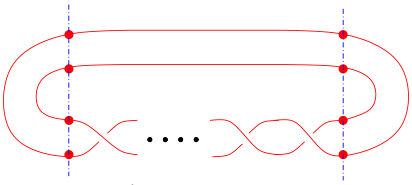


Fig. 10 (online colour at: www.fp-journal.org) The $(2,k)$ torus knot (link) in $\mathbf{S}^3$ can be represented as a union of two branes $\widetilde{\mathcal{B}}$ with k half-twists.

intersect at two points. The first intersection point (of multiplicity 2) is precisely the singular point (3.25), as in the case of the unknot. The second intersection point (of multiplicity 4) is located at the regular point in $\mathcal{M}_H$,

$$(x_1, x_2, x_3) = (2, a^2 - 1, 1) \tag{3.27}$$

Combining the contributions from the two intersection points, we find that the cohomology for the trefoil knot has the following structure

$$\mathcal{H}^{sl(2)}_{\text{trefoil}} = \mathcal{H}^{sl(2)}_{\text{unknot}} \oplus \mathcal{H}^{sl(2)}_{\times} \tag{3.28}$$

where $\mathcal{H}^{sl(2)}_{\text{unknot}}$ is the contribution from the first intersection point, and $\mathcal{H}^{sl(2)}_{\times}$ denotes the contribution from the new intersection point (3.27). Discarding the contribution of reducible connections, we find the *reduced* cohomology of the trefoil knot, which consists only of the term $\mathcal{H}^{sl(2)}_{\times}$,

$$\mathcal{H}^{sl(2)}_{\times} = \mathbb{C}^4 \tag{3.29}$$

Indeed, since $\mathcal{M}_H$ is smooth near the intersection point, the configuration of branes $\widetilde{\mathcal{B}}$ and $\phi_{(\sigma_1)^3}(\widetilde{\mathcal{B}})$ can be locally described (in complex structure J) as an intersection of two sets of B-branes in $\mathbf{C}^2$, such that each set supported on a line in $\mathbf{C}^2$. Let us consider a slightly more general problem where two sets of B-branes in $\mathbf{C}^2$ contain n_1 and n_2 branes, respectively. We denote by $\mathcal{E}_1$ and $\mathcal{E}_2$ the corresponding sheaves, where $\mathcal{E}_1$ (and similarly $\mathcal{E}_2$) is defined by a module of the form $\mathbb{C}[x_1, x_2]/(x^{n_1})$. The space of open string between two such B-branes is given by

$$\mathrm{Ext}^*_{\mathbf{C}^2}(\mathcal{E}_1, \mathcal{E}_2) = \mathbb{C}^{n_1 n_2} \tag{3.30}$$

which, of course, is the expected result since in the present case open strings form a hypermultiplet transforming in (n_1, n_2) under $U(n_1) \times U(n_2)$. Setting $n_1 = n_2 = 2$ gives (3.29).

$(2,k)$ *Torus knots*: A more general torus knot (link) $T_{2,k}$ can be represented as a union of two branes $\widetilde{\mathcal{B}}$ with k half-twists.

In order to describe the action of $(\sigma_1)^k$ on the brane (3.14), again we use (3.13). If the original brane $\widetilde{\mathcal{B}}$ is represented by a set of two coincident branes on the line (cf. (3.18)),

$$(x_1, x_2, x_3) = (2, y, a^2 - y) \tag{3.31}$$

the result of $(\sigma_1)^k$ action is a set of branes supported on a higher-degree curve

$$\phi_{(\sigma_1)^k}(\widetilde{\mathcal{B}}) \ : \quad (x_1, x_2, x_3) = (P_k(y), P_{k-1}(y), a^2 - y) \tag{3.32}$$

where $\{P_i(y)\}_{i \geq -1}$ is a sequence of polynomials in y, such that $P_0(y) = 2$, $P_{-1}(y) = y$, and $P_i(y)$, $i > 1$ are determined by the recursion relation

$$P_i(y) = 2a^2 - (a^2 - y)P_{i-1}(y) - P_{i-2}(y)$$

For example, the first few polynomials $P_i(y)$ look like

$$\begin{aligned}
P_1(y) &= y \\
P_2(y) &= -2 + 2a^2 - a^2 y + y^2 \\
P_3(y) &= 4a^2 - 2a^4 - 3y + 2a^2 y + a^4 y - 2a^2 y^2 + y^3 \\
P_4(y) &= 2 - 4a^4 + 2a^6 + 8a^2 y - 4a^4 y - a^6 y - 4y^2 + 2a^2 y^2 + 3a^4 y^2 - 3a^2 y^3 + y^4 \\
&\ \vdots
\end{aligned}$$

For simplicity, let us focus on torus knots, which correspond to odd values of k (the case of k even, which corresponds to torus links, can be treated similarly). Then, it is easy to see that the brane $\phi_{(\sigma_1)^k}(\widetilde{\mathcal{B}})$ meets the brane (3.16) at $(k+1)/2$ points in $\mathcal{M}_H$. As in the case of the trefoil knot, one intersection point (of multiplicity 2) is the point (3.25) where $\mathcal{M}_H$ has A_1 singularity due to reducible connections. The other $(k-1)/2$ points (each of multiplicity 4) are generically located at regular points in $\mathcal{M}_H$; their precise location is determined by the explicit form of $P_i(y)$. Therefore, extending the earlier result (3.28), we find that cohomology $\mathcal{H}^{sl(2)}_{T_{2,k}}$ of the torus knot $T^{2,k}$ is isomorphic to a direct sum of $\mathcal{H}^{sl(2)}_{\mathrm{unknot}}$ and $(k-1)/2$ copies of $\mathcal{H}^{sl(2)}_{\times} = \mathbb{C}^4$. As usual, it is convenient to remove the contribution of reducible solutions. If we denote by $\widetilde{\mathcal{H}}^{sl(2)}_K$ the "reduced" cohomology of K for the theory considered here, we can state our conclusion as

$$\dim \widetilde{\mathcal{H}}^{sl(2)}_{T_{2,k}} = 2(k-1) \tag{3.33}$$

In general, the cohomology $\widetilde{\mathcal{H}}^{sl(2)}_K$ categorifies a variant of the Casson invariant obtained by counting flat $SL(2,\mathbb{C})$ connections on the knot complement $\mathbf{S}^3 \setminus K$ with fixed conjugacy class of the holonomy around the meridian,

$$\chi(\widetilde{\mathcal{H}}^{sl(2)}_K) = 2\sigma(K) \tag{3.34}$$

We expect that, at least for a certain class of knots, $\sigma(K)$ is equal to the knot signature. Notice, that for $(2,k)$ torus knots, we have $\sigma(T_{2,k}) = (k-1)$.

Notice, one could obtain a different knot invariant (and, presumably, a different knot homologies) by considering the image of the representation variety of the knot complement in the representation variety of the boundary torus, see e.g. [41]. Indeed, the boundary of the knot complement $Y \setminus K$ can be identified with T^2 in the usual way, and the inclusion $T^2 \hookrightarrow Y \setminus K$ induces the restriction map

$$r: \ \mathcal{M}(Y \setminus K) \to \mathcal{M}(T^2) \tag{3.35}$$

which maps a representation $\rho : \pi_1(Y \setminus K) \to G_{\mathbf{C}}$ to its restriction $\rho_{|T^2} : \pi_1(T^2) \to G_{\mathbf{C}}$. In general, $\mathcal{M}(Y \setminus K)$ is a branched cover of its image in $\mathcal{M}(T^2)$ under the restriction map (3.35). For example, if $G_{\mathbf{C}} = SL(2,\mathbb{C})$ and $Y = \mathbf{S}^3$ then the image of the representation variety $\mathcal{M}(\mathbf{S}^3 \setminus K)$ under the restriction map can be described as the zero locus of the A-polynomial [42],

$$A(l,m) = 0 \tag{3.36}$$

where the complex variables l and m parameterize, respectively, the conjugacy classes of the holonomy of the flat $SL(2,\mathbb{C})$ connection along the longitude and the meridian of the knot. The A-polynomial of every knot has a factor $(l-1)$ due to reducible representations. For example, the A-polynomial of a $(2,k)$ torus knot looks like

$$A(T_{2,k}) = (l-1)(lm^{2k}+1) \tag{3.37}$$

Notice, in this example, the part containing irreducible representations consists of a single curve, $lm^{2k}+1 = 0$, of degree one in l. On the other hand, the $SL(2,\mathbb{C})$ representation variety of $T_{2,k}$ is a cover this curve by $\frac{k-1}{2}$ distinct irreducible components which correspond to irreducible representations counted by $\mathcal{N}=4$ topological gauge theory. Restricting the complex variables l and m to be on a unit circle, we obtain the image of the $SU(2)$ representation variety. For $(2,k)$ torus knots, the $SU(2)$ representation variety (again, ignoring reducible representations) is a disjoint union of $\frac{k-1}{2}$ nested open acrs [43,44].

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References

[1] L. Crane and I. Frenkel, J. Math. Phys. **35**, 5136 (1994).
[2] R. Huerfano and M. Khovanov, math.QA/0002060.
[3] M. Khovanov, math.QA/9908171.
[4] P. Ozsvath and Z. Szabo, math.SG/0101206.
[5] J. Rasmussen, math.GT/0306378.
[6] M. Khovanov, math.QA/0304375.
[7] M. Khovanov and L. Rozansky, math.QA/0401268.
[8] S. Gukov, A. Schwarz, and C. Vafa, hep-th/0412243.
[9] E. Witten, Prog. Math. **133**, 637 (1995), hep-th/9207094.
[10] R. Gopakumar and C. Vafa, hep-th/9809187; hep-th/9812127.
[11] H. Ooguri and C. Vafa, Nucl. Phys. B **577**, 419 (2000).
[12] J. M. F. Labastida, M. Marino, and C. Vafa, J. High Energy Phys. **0011**, 007 (2000), hep-th/0010102.
[13] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998), hep-th/9711200.
[14] N. Dunfield, S. Gukov, and J. Rasmussen, math.GT/0505662.
[15] M. Khovanov and L. Rozansky, math.QA/0505056.
[16] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[17] M. Khovanov, Algebr. Geom. Topol. **2**, 665 (2002), math.QA/0103190.
[18] M. Jacobsson, math.GT/0206303.
[19] D. Bar-Natan, math.QA/0201043.
[20] A. Kapustin and E. Witten, hep-th/0604151.
[21] S. Gukov, and E. Witten hep-th/0612073.
[22] Work in progress.
[23] M. Kontsevich, alg-geom/9411018.
[24] A. I. Bondal and M. M. Kapranov, Math. USSR-Sb. **70**, 93 (1991).
[25] E. Witten, Commun. Math. Phys. **117**, 353 (1988).
[26] K. Fukaya, Floer homology for 3-manifolds with boundary, in: Topology, Geometry, and Field Theory, edited by K. Fukaya, M. Furuta, T. Kohno, and D. Kotschirk (World Sci. Publishing, River Edge, NJ, 1994).

[27] M. Bershadsky, A. Johansen, V. Sadov, and C. Vafa, Nucl. Phys. B **448**, 166 (1995), hep-th/9501096.
[28] M. F. Atiyah, Progr. Math. Birkhauser **133**, 105 (1995).
[29] P. Seidel, J. Diff. Geom. **52**, 145 (1999).
[30] P. Seidel and R. Thomas, Duke Math. J. **108**, 37 (2001).
[31] P. Seidel and I. Smith, math.SG/0405089.
[32] C. Manolescu, math.SG/0411015.
[33] C. Manolescu, math.SG/0601629.
[34] N. Hitchin, Proc. Lond. Math. Soc. (3) **55**, 59 (1987).
[35] K. Iwasaki, Commun. Math. Phys. **242**, 185 (2003).
[36] M. Inaba, K. Iwasaki, and M.-H. Saito, Proceedings of Conference Internationale Theories Asymptotiques et Equations de Painleve, Seminaires et Congres, Soc. Math. France, math.AG/0501007.
[37] K. Iwasaki and T. Uehara, math.AG/0512583.
[38] H. Sakai, Commun. Math. Phys. **220**, 165 (2001).
[39] N. Hitchin, J. Geom. Phys. **23**, 287 (1997).
[40] K. Iwasaki Proc. Japan Acad. Ser. A Math. Sci. **78**, 131 (2002).
[41] W. Li, Commun. Contemp. Math. **3**, 501 (2001).
[42] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen, Invent. Math. **118**, 47 (1994).
[43] E. Klassen, Trans. Amer. Math. Soc. **326**, 795 (1991).
[44] H. Boden, C. Herald, P. Kirk, and E. Klassen, Geom. Topol. **5**, 143 (2001), math.GT/9908020.

