An exotic sphere with nonnegative sectional curvature

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In this note we construct an exotic 7-sphere Σ^7 with a metric of nonnegative sectional curvature K. It is obtained as the quotient of a certain isometric action of Sp(1) on Sp(2), and hence as a riemannian submersion from Sp(2). By a formula of O'Neill, Σ^7 automatically inherits nonnegative sectional curvature. It turns out that K is even (strictly) positive on an open dense set of points. It is not known yet whether or not this metric can be deformed into one with positive curvature everywhere. However, there is a conjecture that on any manifold, a metric with nonnegative sectional curvature which is positive at some point, can be deformed into one with positive curvature everywhere. By Aubin's Theorem, a similar result holds for Ricci curvature [1]. We shall see that Σ^7 has naturally many symmetries. In particular, $O(2) \times SO(3)$ acts as an isometry group on Σ^7.

1. The construction of Σ^7

Let Sp(n) denote the group of symplectic $n \times n$ quaternion matrices; i.e., $Q \in \mathrm{Sp}(n)$ if and only if $QQ^* = Q^*Q = \mathrm{Id}$, where Q^* is the transposed conjugate matrix of Q. S^m will denote the standard m-sphere. The field of quaternions will be identified with $\mathbf{R}^4$.

We consider an action of $S^3 \times S^3 \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on Sp(2) given by

$$(q_1 \times q_2, Q) \longmapsto \begin{pmatrix} q_1 & 0 \\ 0 & q_1 \end{pmatrix} Q \begin{pmatrix} \bar{q}_2 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\bar{q}_2$ denotes the conjugate of q_2. This action is clearly free, and the quotient manifold $\mathrm{Sp}(2)/S^3 \times S^3$ is diffeomorphic to S^4. A diffeomorphism $\mathrm{Sp}(2)/S^3 \times S^3 \to S^4$ is given by

$$\mathrm{orbit}_{S^3 \times S^3}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (2\bar{b}d,\ \|b\|^2 - \|d\|^2),$$

as one can check easily. In particular, the diagonal Δ in $S^3 \times S^3$ acts freely on Sp(2). The quotient manifold $\Sigma^7 = \mathrm{Sp}(2)/\Delta$ is an S^3-bundle over S^4:

$$\begin{array}{ll} S^3 \times S^3/\Delta & \cong S^3 \\ \downarrow & \\ \mathrm{Sp}(2)/\Delta & = \Sigma^7 \\ \downarrow & \\ \mathrm{Sp}(2)/S^3 \times S^3 & \cong S^4 \, . \end{array}$$

S^3-bundles over S^4 with structure group $SO(4)$ are classified by $\pi_3(SO(4)) \cong \mathbf{Z} \oplus \mathbf{Z}$. For $(m, n) \in \mathbf{Z} \oplus \mathbf{Z}$ one can construct the corresponding bundle of type (m, n) by glueing two trivial bundles $\mathbf{R}^4 \times S^3$ together on $(\mathbf{R}^4 - 0) \times S^3$ by identifying (u, q) in the first copy with $\left(u \,||\, u \,||^{-2}, (u/|| \, u \, ||)^m q (u/|| \, u \, ||)^n\right)$ in the second copy. According to Milnor [2], whenever $m + n = 1$, the total space is homeomorphic to S^7, and the differentiable structure is exotic if $(m - n)^2 \not\equiv 1 \bmod 7$. We will identify Σ^7 with the total space of the bundle corresponding to $(2, -1)$ and hence with an exotic 7-sphere. It actually was Milnor's description of this sphere which suggested consideration of the above action.

THEOREM 1. *Σ^7 is the exotic 7-sphere of type* $(2, -1)$.

Proof. Consider the maps $h_1, h_2 \colon \mathbf{R}^4 \times S^3 \to \Sigma^7$:

$$h_1(u, q) = \mathrm{orbit}_\Delta \, \varphi(u) \begin{pmatrix} q & \bar{u} \\ -uq & 1 \end{pmatrix} ,$$

$$h_2(v, r) = \mathrm{orbit}_\Delta \, \varphi(v) \begin{pmatrix} \bar{v}r & 1 \\ -r & v \end{pmatrix} ,$$

where $\varphi(u) = (1 + ||\, u \,||^2)^{-1/2}$. Letting $Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2)$, we have

$$h_1(\mathbf{R}^4 \times S^3) = \{\mathrm{orbit}_\Delta \, Q \,|\, d \neq 0\} ,$$

$$h_2(\mathbf{R}^4 \times S^3) = \{\mathrm{orbit}_\Delta \, Q \,|\, b \neq 0\} .$$

Hence, $h_1(\mathbf{R}^4 \times S^3) \cup h_2(\mathbf{R}^4 \times S^3) = \Sigma^7$. Furthermore, the maps h_1, h_2 are differentiable imbeddings; the inverses are given by

$$h_1^{-1}(\mathrm{orbit}_\Delta \, Q) = ||\, d \,||^{-2}(\bar{b}d, \, \bar{d}ad \, ||\, a \,||^{-1}) ,$$

$$h_2^{-1}(\mathrm{orbit}_\Delta \, Q) = ||\, b \,||^{-2}(\bar{b}d, \, -\bar{b}cb \, ||\, c \,||^{-1}) .$$

Finally, $h_2^{-1}h_1(u, q) = \left(u \,||\, u \,||^{-2}, (u/||\, u \,||)^2 q (u/||\, u \,||)^{-1}\right)$, which completes the argument.

2. The action of $O(2) \times SO(3)$ on Σ^7

On Sp(2) we consider the standard bi-invariant metric given by the Killing form. The action of $S^3 \times S^3$ described above is isometric. Therefore,

Σ^7 inherits a natural metric from Sp(2) such that the projection $\mathrm{Sp}(2) \to \Sigma^7$ becomes a riemannian submersion. Now observe that the action of Δ commutes with the action of the group $O(2) \times S^3$ on Sp(2) given by $(A \times q)Q = AQ\begin{pmatrix} 1 & 0 \\ 0 & \bar{q} \end{pmatrix}$, for $A \in O(2)$, $q \in \mathrm{Sp}(1) = S^3$. Thus, $O(2) \times S^3$ acts on Σ^7 by isometries via

$$(A \times q)\mathrm{orbit}_\Delta\, Q = \mathrm{orbit}_\Delta\, AQ\begin{pmatrix} 1 & 0 \\ 0 & \bar{q} \end{pmatrix} .$$

This action on Σ^7 has a kernel $\mathbf{Z}_2 \cong \mathrm{Id} \times \mathbf{Z}_2$, since

$$\mathrm{orbit}_\Delta\, Q\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathrm{orbit}_\Delta\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} Q\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{orbit}_\Delta\, Q .$$

One can see that $O(2) \times SO(3) = O(2) \times S^3/\mathrm{Id} \times \mathbf{Z}_2$ acts effectively on Σ^7.

It is known that $4 = \dim O(2) \times SO(3)$ is the maximal dimension of compact groups that can act effectively on any exotic 7-sphere.* $O(2) \times SO(3)$ has been realized as an isometry group in a different description of Σ^7 given by Brieskorn, where the natural metric, however, has sectional curvatures of either sign.

3. The curvature of Σ^7

To fix notations we briefly review some facts about riemannian submersions. Consider riemannian manifolds $\widetilde{M}$, M with $\dim \widetilde{M} \geqq \dim M$, and a submersion $\pi: \widetilde{M} \to M$; i.e., π is surjective and of maximal rank. For each $p \in M$, we have a submanifold $\pi^{-1}(p)$ of $\widetilde{M}$, the fiber of the submersion over p. The tangent space $\widetilde{M}_q$ of $\widetilde{M}$ at q splits into an orthogonal sum $\widetilde{M}_q = \Delta_q^{\top} \oplus \Delta_q^{\perp}$, where $\Delta_q^{\top}$ is the tangent space of the fiber $\pi^{-1}(\pi(q))$ and $\Delta_q^{\perp}$ is the orthogonal complement. π is called a riemannian submersion if $\pi_*: \Delta_q^{\perp} \to M_{\pi(q)}$ is isometric for all $q \in \widetilde{M}$. $\Delta^{\top}$ and $\Delta^{\perp}$ are called the vertical and horizontal distributions of the riemannian submersion. For a vector field Z on $\widetilde{M}$, let $Z^{\top}$ denote its vertical component. Any vector field X on M has a unique horizontal lift $\widetilde{X}$ on $\widetilde{M}$; i.e., $\widetilde{X}^{\top} = 0$ and $\pi_* \widetilde{X} = X \circ \pi$. The sectional curvatures K of M and $\widetilde{K}$ of $\widetilde{M}$ are related by O'Neill's formula (cf. [3]): If X, Y are orthonormal vector fields on an open subset of M, then

$$(1) \qquad K(X, Y)\circ\pi = \widetilde{K}(\widetilde{X}, \widetilde{Y}) + \frac{3}{4}\|[\widetilde{X}, \widetilde{Y}]^{\top}\|^2 .$$

We shall use this formula to compute the curvature of Σ^7, but first we need another elementary fact.

* Communicated to us by W. C. Hsiang, unpublished.

Let G be a Lie group with Lie algebra $\mathfrak{y}$. L and R denote left and right translations, $\mathrm{Ad} = R_*^{-1}L_*$ the adjoint representation. For $u \in \mathfrak{y}$ consider the left invariant vector field L_*u and the right invariant vector field R_*u with $L_*u|_g = L_{g*}u$, $R_*u|_g = R_{g*}u$. For a bi-invariant metric $\langle\ ,\ \rangle$ on G and $a, b \in \mathfrak{y}$, define a function $f: G \to \mathbf{R}$ by $f(g) = \langle a, \mathrm{Ad}_g\, b\rangle$. Then

$$(2) \qquad (L_{g*}u)f = \langle a, \mathrm{Ad}_g[u, b]\rangle\ .$$

Now we can establish a formula for the curvature of Σ^7 which involves only data of the Lie algebras in question and the adjoint representation.

THEOREM 2. *Let $Q \in \mathrm{Sp}(2)$. For a quaternion a, set $a^+ = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $a^- = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$.*

(a) *The tangent space of the fiber $\pi^{-1}(\pi(Q))$ of the submersion $\pi: \mathrm{Sp}(2) \to \Sigma^7$ is given by*

$$\Delta_Q^\top = \{R_{Q*}(a^+ + a^-) - L_{Q*}a^+ \mid \mathrm{Re}\, a = 0\}\ .$$

(b) *Let u, v be orthonormal vectors in the Lie algebra of $\mathrm{Sp}(2)$ such that $\tilde{u} = L_{Q*}u \in \Delta_Q^\perp$ and $\tilde{v} = L_{Q*}v \in \Delta_Q^\perp$. Then*

$$(3) \qquad K(\pi_*\tilde{u}, \pi_*\tilde{v}) = \frac{1}{4}||[u, v]||^2 + \frac{3}{4} \max_{\substack{a\neq 0 \\ \mathrm{Re}\, a=0}} \frac{\langle \mathrm{Ad}_Q^{-1}(a^+ + a^-) + a^+, [u, v]\rangle^2}{||\mathrm{Ad}_Q^{-1}(a^+ + a^-) - a^+||^2}\ .$$

In particular, $K(\pi_*\tilde{u}, \pi_*\tilde{v}) = 0$ if and only if $[u, v] = 0$.

Proof. (a) The Lie algebra of $\left\{\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \middle| q \in S^3\right\}$ is $\{a^+ + a^- \mid \mathrm{Re}\, a = 0\}$, and the Lie algebra of $\left\{\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \middle| q \in S^3\right\}$ is $\{a^+ \mid \mathrm{Re}\, a = 0\}$. Let φ_a be the curve with $\varphi_a(t) = \exp t(a^+ + a^-)Q \exp t\bar{a}^+$. Then $\varphi_a'(0) = R_{Q*}(a^+ + a^-) - L_{Q*}a^+$, and clearly $\Delta_Q^\top = \{\varphi_a'(0) \mid \mathrm{Re}\, a = 0\}$.

(b) Let $a \in \mathbf{R}^4$, $\mathrm{Re}\, a = 0$. Define ω_a to be the 1-form on $\mathrm{Sp}(2)$ with

$$\omega_a(X)_{|Q} = \langle R_{Q*}(a^+ + a^-) - L_{Q*}a^+, X_Q\rangle\ .$$

Then

$$\Delta_Q^\perp = \{w \mid \omega_a(w) = 0 \ \text{ for all } a,\ \mathrm{Re}\, a = 0\}\ .$$

Since in (1) $\tilde{K}(\tilde{X}, \tilde{Y})|_Q = \tilde{K}(\tilde{u}, \tilde{v}) = \tilde{K}(u, v) = (1/4)||[u, v]||^2$, we have to compute the Lie bracket term in O'Neill's formula. Observe that for any tangent vector w of $\mathrm{Sp}(2)$ at Q we have

$$||w^\top||^2 = \max_{\substack{a\neq 0 \\ \mathrm{Re}\, a=0}} \frac{\omega_a(w)^2}{||R_{Q*}(a^+ + a^-) - L_{Q*}a^+||^2} = \max \frac{\omega_a(w)^2}{||\mathrm{Ad}_Q^{-1}(a^+ + a^-) - a^+||^2}\ ,$$

and therefore,

$$||[\tilde{X}, \tilde{Y}]_Q^\top||^2 = \max_{\substack{a\neq 0 \\ \mathrm{Re}\, a=0}} \frac{\omega_a([\tilde{X}, \tilde{Y}]_Q)^2}{||\mathrm{Ad}_Q^{-1}(a^+ + a^-) - a^+||^2}\ .$$

Computation of $\omega_a([\tilde{X}, \tilde{Y}]_Q)$:

We have

$$(4) \qquad 2d\omega_a(\tilde{X}, \tilde{Y}) = \tilde{X}\omega_a(\tilde{Y}) - \tilde{Y}\omega_a(\tilde{X}) - \omega_a([\tilde{X}, \tilde{Y}]) = -\omega_a([\tilde{X}, \tilde{Y}]) ,$$

since $\tilde{X}$, $\tilde{Y}$ are horizontal, and

$$d\omega_a(\tilde{X}, \tilde{Y})|_Q = d\omega_a(\tilde{u}, \tilde{v}) .$$

On the other hand,

$$(5) \qquad \begin{aligned} 2d\omega_a(\tilde{u}, \tilde{v}) &= 2d\omega_a(L_{Q*}u, L_{Q*}v) \\ &= (L_{Q*}u)\omega_a(L_*v) - (L_{Q*}v)\omega_a(L_*u) - \omega_a(L_{Q*}[u, v]) . \end{aligned}$$

Using the definition of ω_a and (2) we get

$$(L_{Q*}u)\omega_a(L_*v) = (L_{Q*}u)\langle a^+ + a^-, \mathrm{Ad}\, v\rangle = \langle a^+ + a^-, \mathrm{Ad}_Q[u, v]\rangle ,$$

and

$$(L_{Q*}v)\omega_a(L_*u) = \langle a^+ + a^-, \mathrm{Ad}_Q[v, u]\rangle .$$

Finally,

$$\omega_a(L_{Q*}[u, v]) = \langle a^+ + a^- - \mathrm{Ad}_Q a^+, \mathrm{Ad}_Q[u, v]\rangle .$$

Combining this with (4) and (5) yields

$$\omega_a([\tilde{X}, \tilde{Y}])|_Q = -\langle \mathrm{Ad}_Q^{-1}(a^+ + a^-) + a^+, [u, v]\rangle ,$$

which completes the proof of (3).

4. Remarks

It follows from (3) that the curvature of Σ^7 at $\pi(\mathrm{Id})$ is positive for all planes. For this observe that $\Delta_{\mathrm{Id}}^{\top} = \{a^- \mid \mathrm{Re}\, a = 0\}$. Hence, $\Delta_{\mathrm{Id}}^{\perp} = \left\{\begin{pmatrix} a & b \\ -\bar{b} & 0\end{pmatrix} \middle| \mathrm{Re}\, a = 0\right\}$. By Theorem 2, one has only to check: If $u, v \in \Delta_{\mathrm{Id}}^{\perp}$ and $[u, v] = 0$, then u and v are linearly dependent over the reals, which is straightforward.

On the other hand, we obtain curvatue 0 at $\pi(Q)$, $Q = (1/\sqrt{2})\begin{pmatrix} 1 & i \\ i & 1\end{pmatrix}$, exactly for all planes spanned by $\pi_* L_{Q*}u$, $\pi_* L_{Q*}v$ with

$$u = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 0 \\ 0 & \alpha j + \beta k \end{pmatrix}, \qquad \alpha^2 + \beta^2 = 1 .$$

The exact set of points in Σ^7 for which zero plane sections occur can be described in terms of certain quaternion inequalities; it is lower dimensional. Moreover, at any such point, the distribution of zero sections is fairly thin. We might discuss details elsewhere.

Looking only at the first term of (3), one observes immediately that the Ricci curvature of Σ^7 is strictly positive.

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REFERENCES

[1] T. AUBIN, Métriques riemanniennes et courbure, J. Diff. Geom. **4** (1970), 383–424.
[2] J. W. MILNOR, On manifolds homeomorphic to the 7-sphere, Ann. of Math. **64** (1956), 399–405.
[3] B. O'NEILL, The fundamental equations of a submersion, Mich. Math. J. **13** (1966), 459–469.

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