A la Recherche de la Topologie Perdue
Progress in Mathematics, Volume 62

ON THE G-SIGNATURE THEOREM IN DIMENSION FOUR

by

C. McA. Gordon *

0. Introduction

The G-signature theorem of Atiyah-Singer [2] has been successfully applied to several problems in the theory of 4-dimensional manifolds. These applications include Massey's proof [10] of Whitney's conjecture about embeddings of non-orientable surfaces in R^4, and the theorems of Rohlin [11] and Hsiang-Szczarba [8] on the representability of 2-dimensional homology classes in a 4-manifold by embedded surfaces of given genus. It also provides what is perhaps the most appropriate setting for the definition and study of signatures of knots and links, especially when these are applied to concordance questions. Finally, it was used to study knot concordance, from another point of view, in [3] and [4].

The purpose of the present paper is to give an elementary proof of the 4-dimensional G-signature theorem, for finite G, which uses no analysis and only a little bordism. In particular, we remark that we do not require any knowledge about the 4-dimensional cobordism group.

Recently, Gilmer [7] has given a purely topological proof of the G-signature theorem for finite G in all dimensions. Gilmer's proof is in the same spirit as ours, but of course considerably more difficult. Since we do not have to deal with the complexities of the general case, we hope that the present paper may still be of some interest to the low-dimensional topologist.

The plan of the paper is as follows. In §1 we give the basic definitions and state the 2- and 4-dimensional versions of the G-signature theorem (Theorems 1 and 2). In §2, the rudimentary facts from bordism theory which we shall need are

*Partially supported by NSF Grant MCS 78-02995.

collected in Lemma 2, and in Lemma 3 we show that g-signatures vanish for free actions in dimensions 2 and 4. In §3 we prove the 2-dimensional G-signature theorem. Since in this dimension there can only be isolated fixed-points, the proof reduces fairly easily to an explicit computation (Lemma 6). This computation is done in §5. In §4 we prove the 4-dimensional G-signature theorem. We first show that g-signatures are zero for fixed-point free actions (Lemma 7). The case of isolated fixed-points then follows exactly as in the 2-dimensional case. The general case, where there may be some 2-dimensional components in the fixed-point set, is reduced, by means of a standard action on $\mathbb{C}P^2$, to the case where each 2-dimensional component has zero normal euler number, and thence to the isolated fixed-point case. §6 contains some comments on the semi-free case, and on the multiplicativity of signature with respect to finite coverings.

An earlier version of this work (dealing with the semi-free case) was done while the author was visiting the Institute for Advanced Study, Princeton, in 1976-77. I should like to thank the Institute for its support during that period.

I should also like to thank G. Hamrick, R. Litherland, A. Marin, and L. Siebenmann for helpful conversations and comments.

1. Definitions, etc.

We shall work in the smooth category. Let M be a compact, oriented 2n-manifold, possibly with non-empty boundary. The $(-1)^n$-symmetric intersection form

$$\cdot : H_n(M) \times H_n(M) \to \mathbb{Z}$$

induces a hermitian form

$$\varphi : H \times H \to \mathbb{C},$$

where $H = H_n(M;\mathbb{C}) \cong H_n(M) \otimes \mathbb{C}$, by setting

$$\varphi(x \otimes \alpha, y \otimes \beta) = \begin{cases} \alpha\bar{\beta}(x \cdot y), & n \text{ even} \\ i\alpha\bar{\beta}(x \cdot y), & n \text{ odd.} \end{cases}$$

Suppose that G is a finite group which acts on M as a group of orientation-preserving diffeomorphisms. The form φ is then invariant under the induced action of G on H, and we may choose a G-invariant orthogonal (with respect to φ) direct sum decomposition $H = H^+ \oplus H^- \oplus H^0$, where φ is $\pm$-definite on $H^{\pm}$ and zero on H^0. The restrictions of G to H^+ and H^- define representations $\rho^{\pm}$ of G, and the G-<u>signature</u> of (G,M) is defined to be the element

$$\operatorname{sign}(G,M) = \rho^+ - \rho^-$$

of the complex representation ring $R(G)$. This is well-defined, and has the following properties (which hold in all dimensions if we define $\operatorname{sign}(G,M) = 0$ for odd-dimensional M).

(1) $\operatorname{sign}(G,-M) = -\operatorname{sign}(G,M)$

(2) (Multiplicativity.) $\operatorname{sign}(G,M \times N) = \operatorname{sign}(G,M)\operatorname{sign}(G,N)$

(3) (Novikov additivity.) If $(G,M) = (G,M_1) \cup (G,M_2)$, (possibly) identified along some components of the boundaries of M_1 and M_2, then

$$\operatorname{sign}(G,M) = \operatorname{sign}(G,M_1) + \operatorname{sign}(G,M_2)$$

(4) If $(G,M) = \partial(G,W)$ for some W, then $\operatorname{sign}(G,M) = 0$.

In particular, the G-signature defines a ring homomorphism

$$\Omega_*(G) \to R(G).$$

where $\Omega_*(G)$ is the ring of G-bordism classes of closed G-manifolds.

For $g \in G$, the g-<u>signature</u> is defined by

$$\begin{aligned} \mathrm{sign}(g,M) &= \chi(\mathrm{sign}(G,M))(g) \\ &= \mathrm{trace}(g_*|H^+) - \mathrm{trace}(g_*|H^-). \end{aligned}$$

The G-signature theorem of Atiyah-Singer [2] expresses the global invariant $\mathrm{sign}(g,M)$, when M is a closed manifold, in terms of local data, namely the action of g on the normal bundle of the fixed-point set of g.

Before discussing the 2-and-4-dimensional versions of this theorem, we need to establish some notation and conventions.

Since we are assuming that G is finite, it suffices, in disucssing $\mathrm{sign}(g,M)$, to consider the cyclic group generated by g. We shall therefore from now on take G to be the finite cyclic group $\mathbb{Z}_m$, $m > 1$.

It is to be understood throughout that our ambient manifold M is oriented, and that g is orientation-preserving. Also, although M need not be connected, we shall always assume, without loss of generality, that the action is effective on the orbit of each component.

$\mathrm{Fix}(g)$ will denote the set of fixed-points of g. If we need to emphasize the manifold M, we shall write $\mathrm{Fix}(g,M)$.

If $\theta = 2\pi r/m$, with $(r,m) = 1$, we say that θ is <u>of order</u> m. Then (θ,D^2) will denote the 2-disc D^2 with $\mathbb{Z}_m$-action generated by rotation through θ, and we shall write (θ,S^1) for $\partial(\theta,D^2)$.

If E is the total space of the D^2-bundle of some $SO(2)$-bundle, we shall denote simply by θ the automorphism of E induced by rotation of each fibre through θ. If θ is of order m, this defines a $\mathbb{Z}_m$-action (θ,E) on E. If $m = 2$, this also works for $O(2)$-bundles.

We first discuss the 2-dimensional case of the G-signature theorem. Let g generate a $\mathbb{Z}_m$-action on a closed 2-manifold M. Fix(g) will consist of a finite number of points P. Using a $\mathbb{Z}_m$-invariant Riemannian metric on M, we can find a tubular neighborhood of each such P which is equivariantly diffeomorphic to (θ, D^2) for some $\theta = \theta(P)$ (see [6], for example). The G-signature theorem in dimension 2 then states

<u>Theorem 1</u>. <u>With the above notation,</u>

$$\operatorname{sign}(g,M) = -i \sum_P \cot \frac{\theta}{2}.$$

Now consider the 4-dimensional case. Here Fix(g) will consist of a finite set of points P and a finite set of disjoint, closed, connected 2-manifolds F. Again using a $\mathbb{Z}_m$-invariant metric, each P has a tubular neighborhood equivariantly diffeomorphic to $(\theta_1, D^2) \times (\theta_2, D^2)$, say, and each F a tubular neighborhood equivariantly diffeomorphic to (ψ, E), where E is the total space of a D^2-bundle over F. (Of course θ_1, θ_2 depend on P and ψ on F). Note that since the ambient manifold M is oriented, and g preserves its orientation, F must be orientable unless $m = 2$.

$e(F)$ will denote the euler number of the normal bundle of F in M (using local coefficients if F is non-orientable; see [10]). It is equal to the sum of the signed intersections of F with a nearby general position copy.

The G-signature theorem in dimension 4 can now be stated, as follows.

<u>Theorem 2</u>. <u>With the above notation,</u>

$$\operatorname{sign}(g,M) = -\sum_P \cot \frac{\theta_1}{2} \cot \frac{\theta_2}{2} + \sum_F e(F) \operatorname{cosec}^2 \frac{\psi}{2}$$

We conclude with some more conventions and notation.

The generator of any $\mathbb{Z}_m$-action under discussion will usually be denoted by g; we hope that this ambiguous use of g will not cause any confusion.

$\mathcal{F} = \mathcal{F}(M)$ denotes the set of all components of the manifolds $\mathrm{Fix}(g^k)$, $1 \leq k < m$, $\amalg$ denotes disjoint union. Also, as usual, if n is a positive integer, nM denotes the disjoint union of n copies of M, and $(-n)M = n(-M) = -(nM)$.

We shall frequently encounter situations in which we use the fact that, for example, some G-manifold (G,M) represents the zero element of $\Omega_*(G) \otimes \mathbb{Q}$, i.e. $r(G,M) = \partial(G,W)$ for some G-manifold W and non-zero integer r. Since the integer r is irrelevant to our purposes, and would merely complicate the notation, it will be convenient to introduce the notion of a G-object, by which we mean a G-manifold with a non-zero rational coefficient. A statement about G-objects may be interpreted simply as shorthand for the corresponding statement about G-manifolds obtained by multiplying by an appropriate integer. Similarly, we can talk about the fixed-point set of a G-object, a G-bordism between G-objects, the G-signature of a G-object (multiply by the appropriate rational), and so on. It is, of course, not hard to formalize all this, but we shall not do so.

2. Preliminary lemmas

Lemma 1. If g leaves no component of M invariant, then $\mathrm{sign}(g,M) = 0$.

Proof. Let M_0 be a component of M, and let k be the least positive integer such that $g^k M_0 = M_0$. By hypothesis, $k > 1$. We may clearly assume, without loss of generality, that $M = \coprod_{i=0}^{k-1} g^i M_0$. Suppose $\dim M = 2n$, and let

$$H_n(M_0;\mathbb{C}) = H_0^+ \oplus H_0^- \oplus H_0^0 ,$$

where the hermitianized intersection form φ on $H_n(M_0;\mathbb{C})$ is $\pm$-definite on $H_0^{\pm}$ and zero on H_0^0, and the decomposition is orthogonal with respect to φ. Setting

$$H^+ = \sum_{i=0}^{k-1} g_*^i H_0^+ ,$$

and similarly for H^-, H^0, we get a corresponding decomposition

$$H_n(M;\mathbb{C}) = H^+ \oplus H^- \oplus H^0 ,$$

which is g_*-invariant. But clearly $\mathrm{trace}(g_*|H^+) = \mathrm{trace}(g_*|H^-) = 0$, showing that $\mathrm{sign}(g,M) = 0$. □

Let $\Omega_*(X)$ and $N_*(X)$ respectively denote the oriented and unoriented bordism of the space X. There are natural maps $\Omega_*(X) \to H_*(X)$, $N_*(X) \to H_*(X;\mathbb{Z}_2)$.

The only bordism facts we shall need are contained in the following lemma, for which we give the outline of a well-known geometric proof.

Lemma 2. (1) $\Omega_n(X) \to H_n(X)$ *is an isomorphism for* $n \le 3$;

(2) *the sequence* $\Omega_4 \to \Omega_4(X) \to H_4(X) \to 0$ *is exact*;

(3) *the sequence* $N_2 \to N_2(X) \to H_2(X;\mathbb{Z}_2) \to 0$ *is exact*.

Proof. Consider a class in $H_n(X)$, represented by a map $f: K \to X$ for some oriented n-cycle K, and assume, by induction, that K is a manifold away from its (n-k)-skeleton. Then the link L of an (n-k)-simplex σ of K, although it may not be a sphere, will at least be an oriented (k-1)-manifold. Suppose $L = \partial M$ for some oriented k-manifold M. Since the joins $\sigma * L$ and $\sigma * M$ are contractible, $f|\sigma * L$ extends to a map $F: \sigma * M \to X$, and $(f \times id) \cup F$ then defines a map from the oriented (n+1)-chain $K \times I \bigcup_{(\sigma * L)\times\{1\}} \sigma * M$ into X. This provides a homology between f and $f': K' \to X$, where $K' = (K-\sigma * L) \bigcup_{\partial\sigma * L} \partial\sigma * M$, and $f' = f|(K-\sigma * L) \cup F|\partial\sigma * M$. Doing this, if possible, for all (n-k)-simplexes σ, one obtains a representative of the original class in $H_n(X)$ by an n-cycle which is a manifold away from its (n-k-1)-skeleton.

Since $\Omega_1 = \Omega_2 = \Omega_3 = 0$, this shows that $\Omega_n(X) \to H_n(X)$ is surjective for $n \leq 4$.

The same resolution procedure applied to singular $(n+1)$-chains, again using $\Omega_1 = \Omega_2 = \Omega_3 = 0$, shows that $\Omega_n(X) \to H_n(X)$ is injective for $n \leq 3$.

We can also resolve 5-chains to be manifolds away from vertices, the link of a vertex being some oriented 4-manifold. This gives the remaining part of (2).

(3) follows by analogous considerations applied to unoriented cycles, using $N_1 = 0$. $\square$

The following lemma is true in all dimensions (see Remark (2) below), but is rather more elementary in the dimensions that concern us here.

Lemma 3. *Let* g *generate a free* $\mathbb{Z}_m$*-action on a closed manifold* M *of dimension* 2 *or* 4. *Then* $\operatorname{sign}(g,M) = 0$.

Proof. By Lemma 2(1), $\Omega_2(B\mathbb{Z}_m) = 0$. In other words, every free $\mathbb{Z}_m$-action on a closed 2-manifold is freely $\mathbb{Z}_m$-null-bordant. The result in dimension 2 then follows by the G-bordism invariance of the G-signature.

In dimension 4, Lemma 2(2) implies that $\Omega_4 \to \Omega_4(B\mathbb{Z}_m)$ is onto. In other words, every free $\mathbb{Z}_m$-action on a closed 4-manifold M is freely $\mathbb{Z}_m$-bordant to the multiplication action on $\mathbb{Z}_m \times (M/\mathbb{Z}_m)$. But $\operatorname{sign}(g, \mathbb{Z}_m \times (M/\mathbb{Z}_m)) = 0$ by Lemma 1, hence again $\operatorname{sign}(g,M) = 0$ by bordism invariance. $\square$

Remarks. (1) As a generalization of the proof of Lemma 3, recall that for any finite group G, $H_*(BG)$ is annihilated by multiplication by $|G|$. It follows from Lemma 2 that $\Omega_n(BG) \otimes \mathbb{Q} = 0$, $n = 1,2,3$, $\Omega_4 \otimes \mathbb{Q} \to \Omega_4(BG) \otimes \mathbb{Q}$ is onto, and $N_2(BG) \otimes \mathbb{Q} =$

(2) It can be shown more generally, using the bordism spectral sequence (see [6], for example), that for any finite group G, $\Omega_* \otimes \mathbb{Q} \to \Omega_*(BG) \otimes \mathbb{Q}$ is onto. The vanishing of the g-signature for a free action in any dimension then follows much as in the proof of Lemma 3.

(3) If θ has order m, then, as a special case of (1) above, there exists some free $\mathbb{Z}_m$-object $Q(\theta)$ such that

$$\partial Q(\theta) \cong (\theta, S^1) .$$

Define

$$\alpha(\theta) = -\mathrm{sign}(g, Q(\theta)) .$$

(We shall explicitly calculate $\alpha(\theta)$ in Lemma 6.)

We also need the following 4-dimensional analogue of $Q(\theta)$. Let $\overline{Q}(\theta)$ be the closed $\mathbb{Z}_m$-object $Q(\theta) \cup_\partial -(\theta, D^2)$. If θ_i has order m_i, $i = 1,2$, then $\overline{Q}(\theta_1) \times \overline{Q}(\theta_2)$ is a $\mathbb{Z}_m$-object, where $m = \ell cm(m_1, m_2)$, and $\mathrm{Fix}(g)$ has a tubular neighborhood N such that $(g, N) \cong (\theta_1, D^2) \times (\theta_2, D^2)$. Define

$$Q(\theta_1, \theta_2) = -(\overline{Q}(\theta_1) \times \overline{Q}(\theta_2) - \mathring{N}) .$$

Then

$$\partial Q(\theta_1, \theta_2) \cong \partial((\theta_1, D^2) \times (\theta_2, D^2)) ,$$

and

$$\mathrm{sign}(g, Q(\theta_1, \theta_2)) = -\alpha(\theta_1)\alpha(\theta_2) ,$$

by multiplicativity (and additivity).

(4) Although we shall not use this, an explicit choice for $Q(\theta)$ is given by taking $\overline{Q}(\theta)$ to be $\frac{1}{m} F(m)$, where $F(m)$ is the m-fold cyclic cover of S^2 branched

over m points (and the generator g of $\mathbb{Z}_m$ is chosen so that $\mathrm{Fix}(g,F(m))$ has a tubular neighborhood equivariantly diffeomorphic to $-m(\theta,D^2)$. This surface $Q(\theta)$ can be described as m discs $D_1,\dots,D_m$, with D_i joined to D_{i+1} by m twisted bands, $1 \leq i < m$, (compare §5), and a calculation similar to the one done in §5 could be used to determine $\alpha(\theta)$ from this description. However, we shall use a model which, although less natural, is geometrically simpler and seems to make the calculation a little easier.

3. The 2-dimensional case

We first establish

Lemma 4. *Let* g *generate a fixed-point free* $\mathbb{Z}_m$*-action on a closed 2-manifold* M. *Then* $\mathrm{sign}(g,M) = 0$.

Proof. The elements of $\mathcal{F}(M)$ are 0-dimensional. Let P be such; so $P \in \mathrm{Fix}(g^k)$, say, where $1 < k < m$. We may assume without loss of generality that k is minimal (for P). Choose a small 2-disc neighborhood $N(P)$ of P which is invariant under g^k, so that $(g^k,N(P)) \cong (\theta,D^2)$ for some θ of order $\frac{m}{k}$. Then

$$\bigcup_{i=0}^{k-1} g^i\partial N(P) \cong kS^1 ,$$

where the action takes the i-th copy of S^1 to the $(i+1)$-st. (modulo k), by the identity for $0 \leq i < k-1$, and by rotation through θ for $i = k-1$. Consider the corresponding action on $kQ(\theta)$, and let M' be the closed $\mathbb{Z}_m$-object

$$(M - \bigcup_{i=0}^{k-1} g^i\mathring{N}(P)) \cup kQ(\theta) ,$$

equivariantly glued along their boundaries. By Lemma 1, $\mathrm{sign}(g,kQ(\theta)) = 0$, and clearly $\mathrm{sign}(g, \bigcup_{i=0}^{k-1} g^iN(P)) = 0$. Hence, by Novikov additivity,

$$\operatorname{sign}(g,M') = \operatorname{sign}(g,M) .$$

Since the $\mathbb{Z}_m$-action on $kQ(\theta)$ is free, the result of carrying out the above process for each orbit in $\mathcal{F}(M)$ is a closed free $\mathbb{Z}_m$-object M'', with $\operatorname{sign}(g,M'') = \operatorname{sign}(g,M)$. The result now follows from Lemma 1. $\square$

<u>Remark</u>. An entirely analogous argument shows that, in any dimension, $\operatorname{sign}(g,M) = 0$ for any fixed-point free action in which the elements of $\mathcal{F}(M)$ are 0-dimensional.

Now consider the general case. Let g generate a $\mathbb{Z}_m$-action on a closed 2-manifold M. Then $\operatorname{Fix}(g)$ consists of a finite number of points P. Choose disjoint tubular neighborhoods $N(P)$ of these points, with $(g,N(P)) \cong (\theta,D^2)$ for some $\theta = \theta(P)$ of order m. Recall the definition of $\alpha(\theta)$ (§2, Remark (3)).

<u>Lemma 5</u>. <u>With the above notation</u>, $\operatorname{sign}(g,M) = \sum_P \alpha(\theta)$.

<u>Proof</u>. Let

$$M' = (M - \bigcup_P \mathring{N}(P)) \cup \coprod_P Q(\theta) ,$$

glued equivariantly along $\bigcup_P \partial N(P) \cong \coprod_P \partial Q(\theta)$. Clearly $\operatorname{Fix}(g,M') = \emptyset$. Hence, by Lemma 4,

$$\begin{aligned} 0 &= \operatorname{sign}(g,M') \\ &= \operatorname{sign}(g,M) + \sum_P \operatorname{sign}(g,Q(\theta)) \\ &= \operatorname{sign}(g,M) - \sum_P \alpha(\theta). \quad \square \end{aligned}$$

Theorem 1 follows immediately from Lemma 5 and

<u>Lemma 6</u>. $$\alpha(\theta) = -i \cot \frac{\theta}{2} .$$

Lemma 6 is proved by explicit computation with a specific 2-dimensional model; we postpone the proof until §5.

Remark. The above proof applies without essential change to actions in any (even) dimension with the property that the elements of $\mathcal{F}$ are all 0-dimensional, giving a special case of the so-called Atiyah-Bott fixed-point formula [1].

4. The 4-dimensional case

We shall need the following standard action. Let ψ be of order m, and define a $\mathbb{Z}_m$-action on $\mathbb{C}P^2$ by

$$(z_0: z_1: z_2) \longmapsto (e^{i\psi} z_0: e^{i\psi} z_1: z_2) .$$

This action is semi-free, with fixed-point set the union of $\mathbb{C}P^1$ and the point $P = (0: 0: z_2)$. In a neighborhood of P the action is equivalent to $(\psi, D^2) \times (\psi, D^2)$ and on the normal bundle E of $\mathbb{C}P^1$, to $(-\psi, E)$. We shall denote this action by $(\psi, \mathbb{C}P^2)$.

Lemma 7. Let g generate a fixed-point free $\mathbb{Z}_m$-action on a closed 4-manifold M. Then $\operatorname{sign}(g,M) = 0$.

Proof. Step 1. We replace M by a $\mathbb{Z}_m$-object such that the elements of $\mathcal{F}$ are all 2-dimensional, as follows.

Let P be a 0-dimensional component of $\operatorname{Fix}(g^k)$ for some $1 < k < m$. We may assume that k is minimal (for P). Choose a small 4-ball neighborhood $N(P)$ of P such that $(g^k, N(P)) \cong (\theta_1, D^2) \times (\theta_2, D^2)$, say. Let

$$M' = (M - \bigcup_{i=0}^{k-1} g^i \overset{\circ}{N}(P)) \cup kQ(\theta_1, \theta_2) ,$$

equivariantly glued along their boundaries in the obvious way. By Lemma 1, $\operatorname{sign}(g, kQ(\theta_1, \theta_2)) = 0$, and additivity then shows that $\operatorname{sign}(g, M') = \operatorname{sign}(g, M)$. Also, $\mathcal{F}(Q(\theta_1, \theta_2))$ has no 0-dimensional components. Hence in this way we may obtain a $\mathbb{Z}_m$-object M'', say, such that $\mathcal{F}(M'')$ has no 0-dimensional components and $\operatorname{sign}(g, M'') = \operatorname{sign}(g, M)$.

Step 2. By Step 1, we may assume that all the components in $\mathcal{F}(M)$ are 2-dimensional this implies that distinct elements of $\mathcal{F}(M)$ are disjoint.

Let $F \in \mathcal{F}(M)$; so $F \subset \operatorname{Fix}(g^k)$, say, where $1 < k < m$. Let p be the least integer ≥ 1 such that $g^p F = F$. Let $N(F)$ be a tubular neighborhood of F which is equivariant with respect to g^p, and such that $N(F), gN(F), \ldots, g^{p-1}N(F)$ are disjoint.

Suppose for the moment that $p > 1$. g^p generates a free $\mathbb{Z}_{m/p}$-action on $\partial N(F)$, and by Remark (1) in §2, $(g^p, \partial N(F)) \cong \partial W$ for some free $\mathbb{Z}_{m/p}$-object W. Let

$$M' = (M - \bigcup_{i=0}^{p-1} g^i \overset{\circ}{N}(F)) \cup pW ,$$

equivariantly glued along their boundaries in the obvious way. By Lemma 1, $\operatorname{sign}(g, pW) = 0$, and similarly, $\operatorname{sign}(g, \bigcup_{i=0}^{p-1} g^i N(F)) = 0$. Hence, by additivity, $\operatorname{sign}(g, M') = \operatorname{sign}(g, M)$. Note also that the $\mathbb{Z}_m$-action on pW is free.

Step 3. By repeating the process described in Step 2, we may assume that for each $F \in \mathcal{F}(M)$, $gF = F$.

If $F \subset \operatorname{Fix}(g^k)$, and k is minimal for F, then g generates a free $\mathbb{Z}_k$-action on F, and $(g^k, N(F)) \cong (\psi, N(F))$, where $\psi = \frac{2\pi kr}{m}$ for some r coprime to $\frac{m}{k}$. We claim that $e(F) \equiv 0 \pmod{k}$. For, consider the free $\mathbb{Z}_k$-action on $N(F)$ generated by $h = (-\frac{2\pi r}{m})g$. The quotient map $N(F) \to N(F)/h$ is then a k-fold cover of D^2-bundles induced by the k-fold cover $F \to F/g$, and hence $e(F) = ke(F/g)$.

Now consider equivariantly attaching to (M,F), by connected sums of pairs, k copies of $(\mathbb{C}P^2, \mathbb{C}P^1)$, where the action takes the i-th copy to the $(i+1)$-st (modulo k), by the identity for $0 \leq i < k-1$, and by ψ (see the beginning of §4) for $i = k-1$. This gives (M', F'), say, where $e(F') = e(F) + k$, and, for the usual reasons, $\text{sign}(g,M') = \text{sign}(g,M)$. This process introduces points $P, gP, \ldots, g^{k-1}P$ into $\text{Fix}(g^k, M')$, where P has a neighborhood $N(P)$ with $(g^k, N(P)) \cong (\psi, D^2) \times (\psi, D^2)$. However, these may then be eliminated by defining

$$M'' = (M' - \bigcup_{i=0}^{k-1} g^i \mathring{N}(P)) \cup kQ(\psi,\psi) .$$

Since $\text{sign}(g,M'') = \text{sign}(g,M')$, and $kQ(\psi,\psi)$ is a free $\mathbb{Z}_m$-object, this shows that we are essentially able to alter $e(F)$ by multiples of k, without introducing any new components into $\mathcal{F}$. Since $e(F) \equiv 0 \pmod k$, we may therefore assume $e(F) = 0$.

Step 4. We have now shown that we may assume that all the elements of $\mathcal{F}(M)$ are 2-dimensional, have $e(F) = 0$, and satisfy $gF = F$. On such an F, the restriction of g generates a free $\mathbb{Z}_k$-action, say.

Case (a). Suppose F is orientable. Then $(g, N(F)) \cong (g,F) \times (\psi, D^2)$, for some ψ. Since $\Omega_2(B\mathbb{Z}_k) \otimes \mathbb{Q} = 0$, (see §2, proof of Lemma 3 or Remark (1)), $(g,F) \cong \partial V$ for some free $\mathbb{Z}_k$-object V. Regard $V \times D^2$ as a $\mathbb{Z}_m$-bordism (rel ∂) between $F \times D^2$ and $V \times S^1$. Then

$$M \times I \cup V \times D^2 ,$$

where $N(F) \times \{1\}$ is equivariantly identified with $F \times D^2$, is a $\mathbb{Z}_m$-bordism from

$M = M \times \{0\}$ to

$$M' = (M - \overset{\circ}{N}(F)) \cup V \times S^1 .$$

Since the action of $\mathbb{Z}_m$ on $V \times S^1$ is free, passing from M to M' has the effect of removing F from $\mathscr{F}$. Also, $\operatorname{sign}(g,M') = \operatorname{sign}(g,M)$, by bordism invariance.

Case (b). Suppose F is non-orientable (in which case $m = 2$). Then $N(F) \cong F \tilde{\times} D^2$, say. Since $N_2(B\mathbb{Z}_k) \otimes \mathbb{Q} = 0$ (see §2, Remark (1)), $(g,F) \cong \partial V$ for some unoriented free $\mathbb{Z}_k$-object V. Also, our D^2-bundle over F extends to one over V (for example, $F \tilde{\times} D^2$ is just $(F \tilde{\times} I) \times I$, where $F \tilde{\times} I$ is the I-bundle over F corresponding to the 1-st Stiefel-Whitney class $w_1(F)$; now consider $V \tilde{\times} D^2 = (V \tilde{\times} I) \times I$, where $V \tilde{\times} I$ corresponds to $w_1(V)$). Then, as in Case (a), $V \tilde{\times} D^2$ is a $\mathbb{Z}_m$-bordism between $F \tilde{\times} D^2$ and $V \tilde{\times} S^1$, and $M \times I \cup V \tilde{\times} D^2$ is a $\mathbb{Z}_m$-bordism from $M = M \times \{0\}$ to

$$M' = (M - \overset{\circ}{N}(F)) \cup V \tilde{\times} S^1 .$$

Applying (a) and (b), as appropriate, to all components of $\mathscr{F}(M)$, we ultimately obtain a free $\mathbb{Z}_m$-object M" with $\operatorname{sign}(g,M'') = \operatorname{sign}(g,M)$. The fact that $\operatorname{sign}(g,M) = 0$ now follows from Lemma 1. □

Proof of Theorem 2. Case 1. Suppose Fix(g) has no 2-dimensional components.

Choose $P \in \operatorname{Fix}(g)$, and a tubular neighborhood N(P) such that $(g,N(P)) \cong (\theta_1, D^2) \times (\theta_2, D^2)$, say. Let

$$M' = (M - \overset{\circ}{N}(P)) \cup Q(\theta_1, \theta_2) .$$

Then

$$\operatorname{sign}(g,M) = \alpha(\theta_1)\alpha(\theta_2) + \operatorname{sign}(g,M') .$$

Doing this for all $P \in Fix(g)$, and using Lemma 7, we obtain

$$sign(g,M) = \sum_P \alpha(\theta_1)\alpha(\theta_2) \ ,$$

as required.

Case 2. Suppose each 2-dimensional component F of $Fix(g)$ has $e(F) = 0$.

For such an F, $(g, N(F))$ is equivariantly diffeomorphic to $(id,F) \times (\psi, D^2)$ or $(id,F) \tilde{\times} (\psi, D^2)$, for some ψ, according as F is orientable or non-orientable. The procedure described in Step 4 of the proof of Lemma 7 above, only now with $k = 1$, produces a $\mathbb{Z}_m$-bordism from M to a $\mathbb{Z}_m$-object M' such that $Fix(g,M')$ has no 2-dimensional components and the same 0-dimensional components as $Fix(g,M)$. The result now follows from Case 1.

General Case. Consider a 2-dimensional component F of $Fix(g)$, with equivariant neighborhood $(\psi, N(F))$, say. Define $\varepsilon = \pm 1$ by $e(F) = -\varepsilon|e(F)|$, and equivariantly attach to (M,F), by connected sum of pairs, $\#_{i=1}^{|e(F)|} \varepsilon(\psi, (\mathbb{C}P^2, \mathbb{C}P^1))$. This gives (M_1, F'), say, where $e(F') = 0$. Let M' be the result of doing this for all components F. Since $sign(\psi, \mathbb{C}P^2) = 1$, additivity implies

$$sign(g,M') = sign(g,M) - \sum_F e(F) \ .$$

The 0-dimensional components of $Fix(g,M')$ consist of those of $Fix(g,M)$ together with some with equivariant neighborhood $-\sum_F e(F)(\psi, D^2) \times (\psi, D^2)$. Since each 2-dimensional component F' of $Fix(g,M')$ has $e(F') = 0$, Case 2 yields

$$sign(g,M') = \sum_{P'} \alpha(\theta_1)\alpha(\theta_2) = \sum_P \alpha(\theta_1)\alpha(\theta_2) + \sum_F e(F) \cot^2 \frac{\psi}{2} \ .$$

Hence

$$\operatorname{sign}(g,M) = \sum_P \alpha(\theta_1)\alpha(\theta_2) + \sum_F e(F)(\cot^2 \frac{\psi}{2} + 1) ,$$

giving the stated result. □

5. The 2-dimensional computation

Proof of Lemma 6. Suppose θ is of order m. Let M_0 be the bounded 2-manifold shown in Figure 1, consisting of two discs joined by m twisted bands. M_0 has 1 or 2 boundary components according as m is odd or even; let M be the closed 2-manifold obtained by capping off these boundary components with discs.

Let h be the automorphism of M induced by the rotation through $\frac{2\pi}{m}$ indicated in Figure 1. If $(s,m)=1$, h^s generates a $\mathbb{Z}_m$-action on M, whose fixed-point set has a tubular neighborhood equivariantly diffeomorphic to

$$2(\tfrac{2\pi s}{m}, D^2) - (\tfrac{\pi(m+1)s}{m}, D^2), \quad m \text{ odd; or } \quad 2(\tfrac{2\pi s}{m}, D^2), \quad m \text{ even.}$$

Recall that intersections define a hermitian pairing $\varphi\colon H\times H \to \mathbb{C}$, where $H = H_1(M;\mathbb{C})$. Let $\omega = e^{2\pi i/m}$. Then H decomposes as an orthogonal (with respect to φ) direct sum $E_0 \oplus E_1 \oplus \ldots \oplus E_{m-1}$, where E_r is the ω^r-eigenspace of h_*. Let ε_r be the signature of the restriction of φ to E_r. Then

$$\operatorname{sign}(h_*^s, M) = \sum_{r=0}^{m-1} \omega^{rs}\varepsilon_r .$$

Let $x \in H$ be the class indicated in Figure 1. Then $\{h_*^s x\colon 0 \le s \le m-1\}$ generates H, and

$$\varphi(h_*^s x, h_*^t x) = \begin{cases} -1, & s = t+1 \\ 1, & s = t-1 \\ 0, & \text{otherwise} . \end{cases}$$

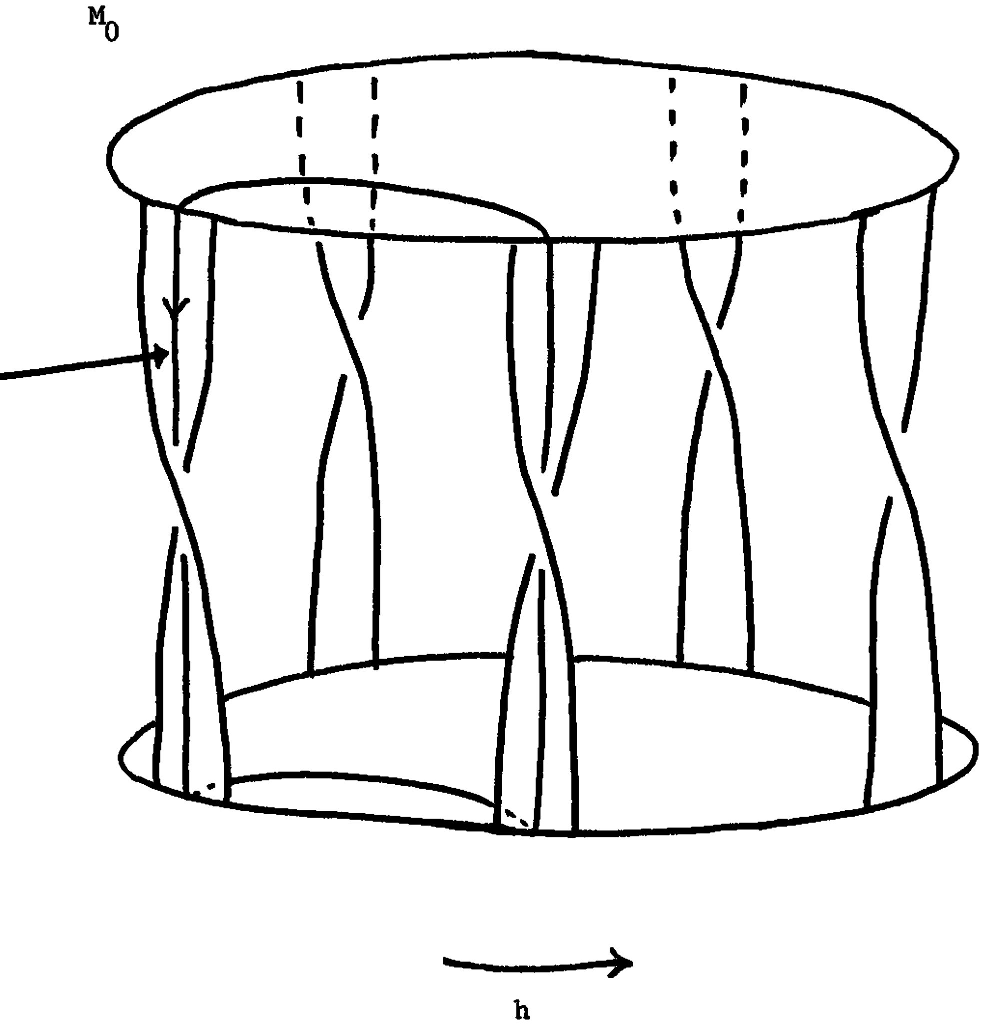

Figure 1

Clearly $e_r = \sum_{s=0}^{m-1} \omega^{-rs} h_*^s x \in E_r$. Also

$$\varphi(e_r, e_r) = i \sum_{s=0}^{m-1} \sum_{t=0}^{m-1} \omega^{-r(s-t)} \varphi(h_*^s x, h_*^t x)$$

$$= i \sum_{s=0}^{m-1} (\omega^r - \omega^{-r})$$

$$= im(\omega^r - \omega^{-r}) = -2m \sin \frac{2\pi r}{m} .$$

We now discuss separately the two cases: m odd, and m even.

m odd. Here, $\dim H = m-1$. Hence, since $e_r \neq 0$, $0 < r < m$ (look at $\varphi(e_r, e_r)$), E_r is 1-dimensional with basis $\{e_r\}$, $0 < r < m$, and $E_0 = 0$. Hence

$$\varepsilon_r = \begin{cases} -1, & 1 \leq r \leq \frac{m-1}{2} \\ 1, & \frac{m+1}{2} \leq r \leq m-1 . \end{cases}$$

Assume $(s,m) = 1$. Since m is odd, we may write $s = 2t$. Let $\theta = \frac{\pi(m+1)s}{m} = \frac{2\pi t}{m}$. Then, by Lemma 5,

$$2\alpha(2\theta) - \alpha(\theta) = \operatorname{sign}(h_*^s, M) = \sum_{r=\frac{m+1}{2}}^{m-1} \omega^{2rt} - \sum_{r=1}^{\frac{m-1}{2}} \omega^{2rt} ,$$

which one easily calculates to be

$$\frac{\omega^t - 1}{\omega^t + 1} = 2\left(\frac{\omega^{2t}+1}{\omega^{2t}-1}\right) - \left(\frac{\omega^t+1}{\omega^t-1}\right) = -2i \cot\theta + i \cot\frac{\theta}{2} .$$

Setting $\beta(\theta) = \alpha(\theta) + i\cot\frac{\theta}{2}$, it follows that $2\beta(2\theta) = \beta(\theta)$, for all θ of order m. Since m is odd, this clearly implies $\beta(\theta) = 0$. Hence $\alpha(\theta) = -i\cot\frac{\theta}{2}$ as stated.

m even. Here, $\dim H = m-2$. Hence, since $e_r \neq 0$ unless $r=0$ or $\frac{m}{2}$, E_r is 1-dimensional with basis $\{e_r\}$, for $r \neq 0, \frac{m}{2}$, and $E_0 = E_{\frac{m}{2}} = 0$. Hence

$$\varepsilon_r = \begin{cases} -1, & 1 \leq r \leq \frac{m}{2} - 1 \\ 1, & \frac{m}{2} + 1 \leq r \leq m-1 . \end{cases}$$

Therefore, assuming $(s,m) = 1$ and writing $\theta = \frac{2\pi s}{m}$,

$$2\alpha(\theta) = \operatorname{sign}(h_*^s, M) = \sum_{r=\frac{m}{2}+1}^{m-1} \omega^{rs} - \sum_{r=1}^{\frac{m}{2}-1} \omega^{rs}$$

$$= -2 \sum_{r=1}^{\frac{m}{2}-1} \omega^{rs} = -2\left(\frac{1+\omega^s}{1-\omega^s}\right) = -2\, i \cot \frac{\theta}{2} . \quad \square$$

6. The semi-free case and multiplicativity of signature

The above proof of the G-signature theorem in dimension 4 simplifies considerably if the action is semi-free. For in this case the process described in the proof of Theorem 2 reduces the problem to proving that $\operatorname{sign}(g,M)$ vanishes for a free action, (as opposed to one that is merely fixed-point free), and one can use Lemma 3 directly, instead of Lemma 7.

The semi-free case is sufficient for all the applications mentioned in §0, (except the calculations in [3]), and in fact for these applications one can specialize further to the case where the fixed-point set is 2-dimensional. A slightly different elementary proof of the 4-dimensional G-signature theorem in the latter case has been given by Litherland [9], in terms of signatures of links.

Finally, we make a few comments about the multiplicativity of signature with respect to coverings.

If the finite group G acts freely on N, then a standard transfer argument shows that

$$|G|\,\mathrm{sign}(N/G) - \mathrm{sign}\, N = \sum_{g\in G-\{1\}} \mathrm{sign}(g,N) \ .$$

(Actually, this holds even if the action is not free.)

In particular, if N is closed, the vanishing of $\mathrm{sign}(g,N)$ for $g \in G-\{1\}$ implies that

$$\mathrm{sign}\, N = |G|\,\mathrm{sign}(N/G) \ .$$

However, this multiplicativity can be established directly. Also, it is not necessary to restrict to regular coverings. Thus we have the following well-known theorem, which is often stated as a consequence of the Hirzebruch index theorem (see [5], for example).

<u>Theorem 3</u>. <u>Let</u> $\widetilde{M} \to M$ <u>be a (finite) n-sheeted covering of closed, oriented manifolds</u>. <u>Then</u> $\mathrm{sign}\,\widetilde{M} = n\,\mathrm{sign}\,M$.

<u>Proof</u>. First consider a regular covering $\widetilde{M} \to M$, with group of covering transformations G, so that we have a free action of G on $\widetilde{M}$. Since $\Omega_* \otimes \mathbb{Q} \to \Omega_*(BG) \otimes \mathbb{Q}$ is onto (see §2, Remark (2), and recall that this is elementary in dimension 4), there exists a positive integer r such that $r(G,\widetilde{M})$ is freely G-bordant to $r(G, G\times M)$, where G acts on $G\times M$ by left multiplication. Hence $\mathrm{sign}\,\widetilde{M} = \mathrm{sign}(G\times M) = |G|\,\mathrm{sign}\, M$.

In general, let $\widetilde{M} \to M$ be an n-sheeted covering, corresponding to the subgroup H, say, of index n in $\pi = \pi_1(M)$. Then $H_0 = \bigcap_{x\in\pi} x^{-1}Hx$ is a normal subgroup of π (and H) of finite index; let $\widetilde{M}_0 \to M$ be the corresponding covering. The fact that sign $\widetilde{M} = n \operatorname{sign} M$ now follows from the regular case applied to the regular coverings $\widetilde{M}_0 \to M$, $\widetilde{M}_0 \to \widetilde{M}$. □

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Department of Mathematics
The University of Texas at Austin
Austin, TX 78712