WHAT IS GLOBAL ANALYSIS?

S. SMALE, University of California, Berkeley

There has recently been a lot of activity in that branch of mathematics now referred to as "global analysis." For example, the subject of the 1968 Summer Institute of the American Mathematical Society was global analysis.

My definition of global analysis is simply the study of differential equations, both ordinary and partial, on manifolds and vector space bundles. Thus one might consider global analysis as differential equations from a global, or topological point of view.

Even the earliest studies of differential equations contained an element of global analysis; this element had become quite important for example in the work of Poincaré on ordinary differential equations. G. D. Birkhoff's development of dynamical systems and especially M. Morse's theory of geodesics are both excellent examples of global analysis. After the rapid recent progress in topology, the subject of our exposition has been moving especially fast. After mentioning a couple of references in partial differential equations, I shall devote the rest of my article to an account of a theorem in dynamical systems to illustrate the global analysis point of view.

Recently there have been nice results in the topology of linear elliptic differential operators, especially in the work of Atiyah, Singer, and Bott (see for example [2] and [4]).

One cannot expect to have a satisfactory framework for nonlinear partial differential equations with linear function spaces. Thus it is important that nonlinear partial differential equations are beginning to be attacked by a systematic use of infinite dimensional manifolds of maps. A good survey of this is Eells [3].

The work of Andronov, Pontryagin [1] and Peixoto [5] in dynamical systems (or ordinary differential equations), on one hand can be explained in relatively simple terms and on the other hand gives a real insight into this modern way of looking at differential equations. I shall try to give a brief account of their theory now.

Consider an ordinary differential equation (1st order, autonomous) defined on a domain D in the x, y-plane:

$$\frac{dx}{dt} = P(x, y) \qquad \frac{dy}{dt} = Q(x, y).$$

Stephen Smale received his Ph.D. at the University of Michigan in 1956. He has occupied various positions at the University of Chicago, the Institute for Advanced Study, Columbia University, and his present location, the University of California at Berkeley. For his outstanding research in differential topology and in global analysis, Professor Smale was awarded the Fields Medal of the International Mathematical Union in 1966 and the Veblen Prize of the American Mathematical Society in 1964. *Editor*

We shall assume that these functions P, Q defined on D are continuously differentiable (or of class C^1). Now the fundamental existence theorem of ordinary differential equations yields for each (x_0, y_0) in D and real t sufficiently small in absolute value, $|t| < \epsilon$, functions $f(x_0, y_0, t)$, $g(x_0, y_0, t)$ which satisfy the initial conditions $f(x_0, y_0, 0) = x_0$, $g(x_0, y_0, 0) = y_0$ and the differential equation

$$\left(\frac{df}{dt}\right)(x_0, y_0, t) = P(f(x_0, y_0, t), g(x_0, y_0, t))$$

$$\left(\frac{dg}{dt}\right)(x_0, y_0, t) = Q(f(x_0, y_0, t), g(x_0, y_0, t)).$$

Let us look at this phenomenon from a more geometric point of view and in fact get away from the particular choice of x, y-coordinates.

To each (x, y) in D associate the vector $(P(x, y), Q(x, y))$ of the x, y-plane with the initial point at (x, y). This gives us what is called a C^1 vector field on D. For each point p of D, we will call the associated vector for short $X(p)$. Then the existence theorem we just stated may be interpreted to yield a system of plane curves $\phi_t(p)$ with $\phi_0(p) = p$, and with the property that the tangent of the curve at a point q of D will be the vector $X(q)$ (see Figure 1).

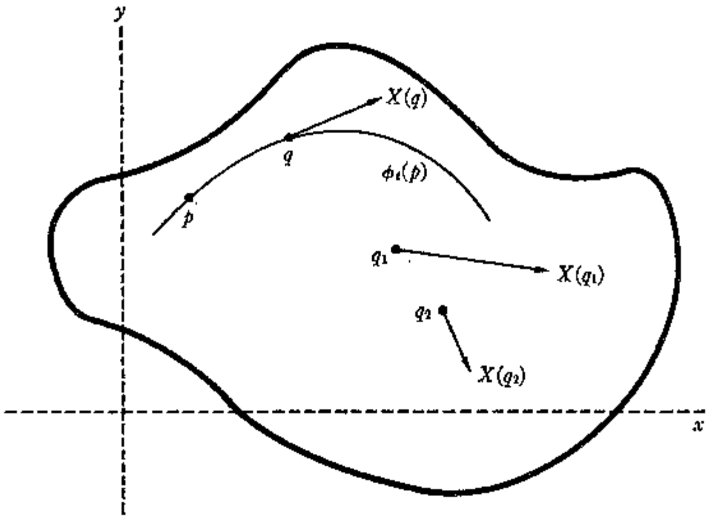

FIG. 1.

The right context for the study of this differential equation becomes clearer now. More generally than a domain of the Euclidean plane, consider a 2-dimensional smooth manifold M. Roughly speaking, one can think of this as a surface in 3-dimensional Euclidean space E^3 or better abstractly as a space on which differentiation makes sense and a neighborhood of each point is a domain in the plane. To each point p of M there is associated a 2-dimensional vector space

$T_p(M)$, the tangent space of M at p. If M is a surface in E^3 then $T_p(M)$ is the plane tangent to M at p.

A vector field X on M is an assignment, continuously differentiable, $p \to X(p)$ for p in M to $X(p)$, a "vector" in $T_p(M)$. The vector field on D defined previously from the differential equation given by the functions P, Q on D is now a vector field on the 2-manifold D in this sense.

To define the basic idea of this article, structural stability of a differential equation, we need to develop two things: one, the space of differential equations on M, $\chi(M)$, and two, an equivalence relation on $\chi(M)$, the phase portrait.

We have seen that the kind of differential equations on M we are studying (which are really pretty general except for the low dimension) correspond to vector fields on M. We call the set of all vector fields (C^1 as usual) on M, $\chi(M)$.

Now $\chi(M)$ has the structure of a vector space, using the fact that for each $p \in M$, the values of all vector fields lie in the same linear space $T_p(M)$. That is if X, Y belong to $\chi(M)$, $(X+Y)(p) = X(p) + Y(p)$. This space $\chi(M)$ will be basic in what follows.

The solution curves $\phi_t(p)$ of a vector field X on M, defined earlier, may be "pieced together " so that for each p, $\phi_t(p)$ will be defined for all $a < t < b$ where the interval (a, b) is maximal. If M is compact, for each p, this interval will be $(-\infty, \infty)$ so that we have a 1-parameter group ϕ_t of transformations on M. Thus for each real t, ϕ_t is a C^1 transformation of M, ϕ_t: $M \to M$, with a C^1 inverse, ϕ_0 is the identity and $\phi_t(\phi_s) = \phi_{t+s}$. In short ϕ_t is a dynamical system.

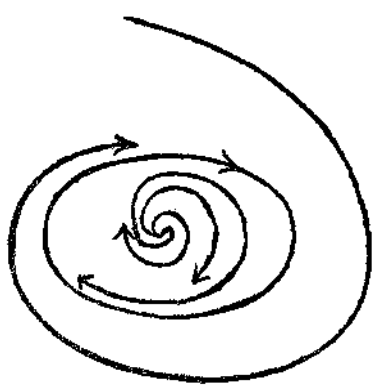

FIG. 2a.

To abstract the qualitative features of a differential equation on M, the concept of a phase portrait becomes important. Usually the phase portrait means the picture of the solution curves of the differential equation. For example, Figure 2a is the phase portrait of a differential equation in the plane.

To give a precise mathematical content to "phase portrait," we proceed as follows. Say X, Y in $\chi(M)$ are ***topologically equivalent*** when there is a homeomorphism $h: M \to M$ taking solution curves of X into those of Y. Thus the differential equation in Figure 2a is topologically equivalent to that described in Figure 2b.

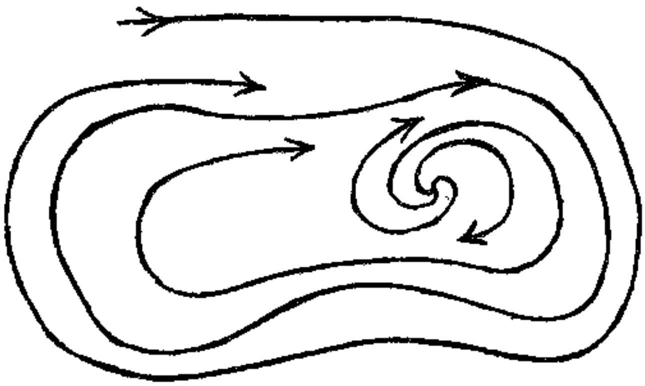

FIG. 2b.

Then two differential equations on M have the same phase portrait if they are topologically equivalent. A definition of ***phase portrait*** is thus a topological equivalence class of differential equations on M. A main goal of the qualitative study of ordinary differential equations is to obtain information on the phase portrait of differential equations.

To make progress in this direction, one soon sees the need to avoid "degenerate" cases. For example a differential equation that is zero on all of M, or even on some nonempty open set of M should be considered degenerate and excluded from most considerations. I think that engineers and physicists will agree with this statement.

To aid in discussing the question of degeneracy, a topology or metric on $\chi(M)$ is useful. To simplify matters in defining this metric, in the rest of our article, we will assume M compact. This excludes many or even most interesting examples, but on the other hand the main features are not lost.

Assuming M compact define a norm $\| \ \|$ on $\chi(M)$ as follows. Let $U_1, \cdots, U_k$ be a covering of M, $\overline{U}_i \subset V_i$, with each V_i a plane domain. Then on each V_i, X in $\chi(M)$ is represented by $P_i(x, y)$, $Q_i(x, y)$ as at the beginning. Then $\|X\|$ is defined as the maximum of the following finite set of numbers:

$$\sup_{(x,y)\in U_i} |P(x, y)| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i} |Q(x, y)| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i}\left|\frac{\partial P}{\partial x}(x, y)\right| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i}\left|\frac{\partial P}{\partial y}(x, y)\right| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i}\left|\frac{\partial Q}{\partial x}(x, y)\right| \qquad i = 1, \cdots, k$$

$$\sup_{(x,y)\in U_i}\left|\frac{\partial Q}{\partial y}(x, y)\right| \qquad i = 1, \cdots, k.$$

This gives $\chi(M)$ the structure of a complete normed space or a Banach space. A metric on $\chi(M)$ is then defined by $d(X, Y) = \|X - Y\|$.

With this metric on $\chi(M)$ it is possible to say when differential equations are "close." In terms of local coordinate representations, two differential equations are close when the P and Q are uniformly close, with their first derivatives uniformly close as well.

With this background, we say that X in $\chi(M)$ is *structurally stable* when there is a neighborhood $N(X)$ in $\chi(M)$ with the property that every Y in $N(X)$ is topologically equivalent to X. Thus X is structurally stable when nearby differential equations have the same phase portrait. A little thought will indicate that this excludes degeneracy; a structurally stable X cannot be degenerate (in some senses at least). It is an important concept for the engineer who studies qualitative differential equations, since in engineering the differential equations one works with are only approximations of the real equations. The engineer wants the qualitative conclusion he makes to be valid for the actual differential equation which describes his world. In fact the original idea of structural stability was the joint work of an engineer, A. Andronov, and a mathematician, L. Pontryagin.

Thus it becomes important to know if most differential equations are structurally stable.

THEOREM. (M. Peixoto) *If M is a compact 2-dimensional manifold, then the structurally stable differential equations in $\chi(M)$ form an open and dense set.*

This theorem is an excellent theorem in global analysis. One sees in two ways how it is global. First the differential equation is defined over a whole manifold, and structural stability depends on its behavior everywhere. Second, the theorem makes a conclusion about the space of all differential equations on M.

The proof gives much information on the structure of differential equations on 2-manifolds.

We state the main lemma which indicates how this is so.

The nonwandering set $\Omega(X)$ of X is defined as the set of x in M such that for every neighborhood U of x and t_0, there is a $t > t_0$ with $\phi_t(U) \cap U \neq \emptyset$.

MAIN LEMMA. *If M is a compact 2-manifold and X is in $\chi(M)$, then X is structurally stable if and only if the following conditions are met:*

(a) *Each closed orbit and each singular point of X is "nondegenerate." This nondegeneracy is defined in terms of derivatives associated to the closed orbits and singular points.*

(b) *The separatrices of saddle points don't meet.*

(c) *$\Omega(X)$ consists of the finite union of closed orbits and singular points.*

[*Separatrices* are the trajectories which come to and leave from the saddle points.]

If α is a singular point or closed orbit, let $W^s(\alpha)$ be the set of x in M with $\phi_t(x)\to\alpha$ as $t\to\infty$. Then if X is structurally stable, it provides for a decomposition of M as the finite union of $W^s(\alpha)$ as α ranges over the closed orbits and singular points. This decomposition gives a good practical understanding of the differential equation X.

A survey of this subject with many references is [6].

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