Hasse–Witt Invariants for \((\alpha, \mu)\)-Reflexive Forms and Automorphisms. I: Algebraic \(K_2\)-Valued Hasse–Witt Invariants*

CHARLES H. GIFFEN

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903

Communicated by I. N. Herstein

Received November 15, 1975

IN MEMORIAM: RALPH HARTZLER FOX (1913–1973)

"We knew not the knots he knew."

0. INTRODUCTION

The classical invariants of (nonsingular) symmetric bilinear forms over a field are rank, discriminant, signatures, and Hasse–Witt invariants. In the theory of \((\alpha, \mu)\)-reflexive forms \[9\], an algebraic \(K_0\)-valued rank and an algebraic \(K_1\)-valued discriminant come into play, leading to various algebraic \(L\)-theories \(L^\alpha, L^\mu, L^\gamma\), as well as to certain periodicity phenomena relating them \[10\].

In this paper, we generalize for \((\alpha, \mu)\)-reflexive forms the Hasse–Witt invariants to an algebraic \(K_2\)-valued invariant. A corresponding invariant for automorphisms (isometrics) of such forms is also defined. This is all accomplished in Section 3, and the relation of the invariants defined with the Hasse–Witt invariant in the case of a field is given in Section 4.

The first section motivates and defines involutions \(T_{\alpha,\mu}\) on the even order general linear groups \(GL(2n, R)\) and Steinberg groups \(St(2n; R)\), compatible with the natural homomorphisms \(St(2n; R) \to GL(2n; R)\), where \((R, \alpha, \mu)\) is a ring with antistructure. As \(n \to \infty\), \(T_{\alpha,\mu}\) induces the right involution on \(K_2R\), \(s = 1, 2\). Then \((\alpha, \mu)\)-reflexive forms and automorphisms of even rank and their discriminants are described in terms of \(T_{\alpha,\mu}\), in Section 2. In Sections 2 and 3, there arise "differentials" \(H^*(Z_2; K_sR) \to H^*(Z_2; K_{s+1}R)\), \(s = 0, 1\), which are carefully described; also, some general computational structure of these differentials is given.

For algebraic \(L\)-theory (and for topological applications), \(K_1R\) and \(K_2R\) are not the right value groups for discriminants and Hasse–Witt invariants of reflexive forms. Hence, in Section 5, we describe Steinberg-type groups

* Research supported by a grant from the National Science Foundation.

Copyright © 1977 by Academic Press, Inc.
All rights of reproduction in any form reserved.

ISSN 0021–8693
St(V; R) for suitable subgroups \( V \subseteq R \), the group of units of \( R \), and an exact sequence
\[
0 \to K_2^V R \to St^V(R) \to GL(R) \to K_1^V R \to 0,
\]
so that \( K_2^V R \)-valued discriminants and \( K_2^V \)-valued Hasse–Witt invariants may be defined with all the right properties. The special choice \( R = \mathbb{Z}_\pi, V = \pm \pi \) gives \( K_1^V R = Wh(\pi), K_2^V R = Wh_2(\pi) \) and the "correct" theory for surgery of manifolds.

In the second paper of this series, we shall give an algebraic \( L \)-theory application and interpretation of the ideas set forth here. In particular, an \( L \)-theory \( L_{st}^s \) will be described, which fits into an exact triangle
\[
\begin{align*}
L_{st}^s & \longrightarrow L_{sy}^s \\
\downarrow & \\
\text{H}^*(\mathbb{Z}_2; K_2) & \leftarrow
\end{align*}
\]
From the standpoint of periodicity, the periodicity sequences of Wall [10] may be extended somewhat farther to the left. Also, the direct connection with the unitary \( K_2 \) of Sharpe [7] may be established.

Subsequently, computational results and topological applications will be given, including the relation of \( L_{st}^s \) to pseudo-isotopy.

A final word on our restriction of all discussion of forms in this paper to reflexive forms (as opposed to Hermitian or quadratic forms) is in order. This is because the theory of reflexive forms is, in a precise sense, the "fixed point theory" of an involution on a theory of finitely generated projective modules (cf. [3, 8]); moreover, the algebraic \( K \)-theory valued invariants we treat arise from this equivariant algebraic \( K \)-theoretic situation.

1. Algebraic Preliminaries

Let \( R \) be an associative ring with unity. An antistructure \( (\alpha, u) \) on \( R \) consists of an antiautomorphism \( \alpha \) of \( R \) together with a unit \( u \in R \) such that \( \alpha(u) u = 1 \) and \( \alpha^2(r) = uru^{-1} \) for every \( r \in R \). There is the contravariant duality functor \( D_\alpha : \mathcal{M}_R \to \mathcal{M}_R \) given by \( D_\alpha M = \text{Hom}_R(M, R) \) for \( M \in \text{Obj} \mathcal{M}_R \), with the conjugate right \( R \)-module structure determined by \( \alpha \),
\[
(fr)(x) = \alpha(r)f(x), \quad r \in R, \quad x \in M, \quad f \in \text{Hom}_R(M, R),
\]
and \( D_\alpha h = \text{Hom}_R(h, R) \) for \( h \in \text{Mor} \mathcal{M}_R \).

**Lemma 1.1** The formula \((\eta_{\alpha, u} M)(x)(f) = \alpha(f(x)) u\) defines a natural transformation \( \eta_{\alpha, u} : 1_{\mathcal{M}_R} \to D_\alpha^2 \).
Proof. First, we check that \((\eta_{a,u}M)(x) \in D_a^2M:\)
\[
(\eta_{a,u}M)(x)(fr) - \alpha((fr)(x)) u - \alpha(\omega(r)f(x)) u = \alpha(f(x)) u \alpha(r) u
\]
\[
= \alpha(f(x)) ur = (\eta_{a,u}M)(x)(f) r.
\]
Next, we check that \(\eta_{a,u}M \in \text{Hom}_\mathcal{R}(M, D_a^2M):\)
\[
(\eta_{a,u}M)(xr)(f) = \alpha(f(xr)) u = \alpha(f(x) r) u = \alpha(r) \alpha(f(x)) u
\]
\[
= \alpha(r)(\eta_{a,u}M)(x)(f) = ((\eta_{a,u}M)(x) r)(f).
\]

Hence, \(\eta_{a,u}M\) is well defined for every \(M \in \text{Obj } \mathcal{M}_R\). That it is a natural transformation follows easily. 

\textbf{PROPOSITION 1.2.} The natural transformation \(\eta_{a,u}\) defines a self-adjunction of \(D_a\); that is, \((D_a\eta_{a,u})(\eta_{a,u}D_a) = 1_{D_a}\), and the associated homomorphisms
\[
t_{a,u} = t_{a,u}(M, N): \text{Hom}_\mathcal{R}(N, D_aM) \to \text{Hom}_\mathcal{R}(M, D_aN)
\]
defined by \(t_{a,u}(M, N)(f) = (D_a f)(\eta_{a,u}M)\) are natural isomorphisms and satisfy
\[
t_{a,u} = 1, t e ,
\]
\[
t_{a,u}(N, M) t_{a,u}(M, N) = 1_{\text{Hom}_\mathcal{R}(N, D_aM)}.
\]

\textit{Proof.} It suffices to show that \((D_a\eta_{a,u})(\eta_{a,u}D_a) = 1_{D_a}\), the other parts then being routine (cf. Eilenberg and Moore [2], for example). Let \(f \in \text{Hom}_\mathcal{R}(M, R)\), \(g \in \text{Hom}_\mathcal{R}(D_aM, R)\), and \(h \in \text{Hom}_\mathcal{R}(D_a^2M, R)\); then, \((\eta_{a,u}D_aM)(f)(g) = \alpha(g(f)) u\) and
\[
(D_a\eta_{a,u}M)(h)(x) = h((\eta_{a,u}M)(x)).
\]

Hence, we have the composition
\[
(D_a\eta_{a,u}M)((\eta_{a,u}D_aM)(f))(x) = (\eta_{a,u}D_aM)(f)((\eta_{a,u}M)(x)) = \alpha((\eta_{a,u}M)(x)(f)) u
\]
\[
= \alpha(\alpha(f(x)) u) u = u^{-1} \omega f(x) u = f(x) \]

\textit{Note} Adjacent and self-adjoint contravariant functors and the reflexive structures determined by the latter are discussed in greater detail in [3].

It is helpful to have an alternate description of \(D_a^2M\) in the cases where \(\eta_{a,u}M\) is an isomorphism. If \(\phi: R \to S\) is a ring homomorphism then there is the base change functor \(J_\phi: \mathcal{M}_S \to \mathcal{M}_R\) given by \(J_\phi M = M, J_\phi f = f\) for \(M \in \text{Obj } \mathcal{M}_S\), \(f \in \text{Mor } \mathcal{M}_S\), where \(J_\phi M\) has the induced \(R\)-module structure \(x \circ r = x\phi(r)\) for \(x \in M, r \in R\). The following is trivial.

\textbf{LEMMA 13.} The formula \((j_{a,u}M)(x) = xu^{-1}\) defines a natural equivalence of functors \(j_{a,u}: 1_{\mathcal{M}_R} \to J_{a^2}\).

It follows that the natural transformation \(\theta_{a,u} = \eta_{a,u}j_{a,u}^{-1}: J_{a^2} \to D_a^2\) has the especially simple form
\[
(\theta_{a,u}M)(x)(f) = \alpha(f(xu)) u = \alpha^{-1}(f(x)).
\]
Thus, for example, \((\theta_{a,u}M)(x)(f) = 0\) or 1 according as \(f(x) = 0\) or 1. In particular, let \(F\) be the free right \(R\)-module with finite basis \(e_1, \ldots, e_n\); then the dual basis of \(D_\alpha F\) to \(e_1, \ldots, e_n\) is the basis \(e_1^*, \ldots, e_n^*\) of \(D_\alpha F\) determined by \(e_i^*(e_j) = \delta_{ij}\) (Kronecker delta). Similarly, \(D_\alpha^2 F\) has the basis \(e_1^{**}, \ldots, e_n^{**}\) dual to \(e_1^*, \ldots, e_n^*\); then, if \(J_\alpha F\) has the basis \(e_1, \ldots, e_n\), we have shown the following.

**Lemma 1.4.** For \(i = 1, \ldots, n\), \((\theta_{a,u}F)(e_i) = e_i^{**}\). Moreover, \(\theta_{a,u}M: J_\alpha^2 M \to D_\alpha^2 M\) is an isomorphism whenever \(\eta_{a,u} M\) is; in particular, \(\theta_{a,u} P\) is an isomorphism for every finitely generated projective \(R\)-module \(P\).

Let \(\mathcal{P}_R \subset \mathcal{M}_R\) be the subcategory of finitely generated projective \(R\)-modules and isomorphisms of such. The restriction of \(D_\alpha\) defines a contravariant functor on \(\mathcal{P}_R\) by (1.4), however, since the morphisms of \(\mathcal{P}_R\) are isomorphisms, it is more convenient to consider the covariant functor \(T_\alpha: \mathcal{P}_R \to \mathcal{P}_R\) given by \(T_\alpha P = D_\alpha P\), \(T_\alpha f = D_\alpha f^{-1}\) for \(P \in \text{Obj} \mathcal{P}_R\), \(f \in \text{Mor} \mathcal{P}_R\). We have \(T_\alpha^2 = D_\alpha^2\) on \(\mathcal{P}_R\), and \(J_\alpha\) restricts to a functor on \(\mathcal{P}_R\). Hence, by (1.4) we have the natural equivalences

\[
\eta_{a,u}: 1_{\mathcal{P}_R} \simeq T_\alpha^2, \quad \theta_{a,u}: J_\alpha \simeq T_\alpha^2
\]

of functors on \(\mathcal{P}_R\). As before, we also have the natural involutions

\[
t_{a,u}(P, Q): \mathcal{P}_R(Q, T_\alpha P) \simeq \mathcal{P}_R(P, T_\alpha Q).
\]

Since \(T_\alpha\) is a product preserving functor and \(\eta_{a,u}\) is a natural equivalence of functors on \(\mathcal{P}_R\), it follows that \(T_\alpha\) induces an involution, also denoted \(T_\alpha\), on \(K_\mathcal{P}_R\), the Quillen–Segal algebraic \(K\)-theory of \(R\). Although \(T_\alpha^2\) is the identity on \(K_\mathcal{P}_R\), it is not the case that, for \(K_{\mathcal{P}_R}\) or \(K_{\mathcal{M}_R}\), \(T_\alpha\) is induced from an involution on \(\text{GL}(P, R)\) or on \(\text{St}(n, R)\). However, for isomorphisms between projectives of the form \(P \oplus T_\alpha P\), we can do considerably better.

Let \(H_\alpha: \mathcal{P}_R \to \mathcal{P}_R\) be the hyperbolic module functor, given by \(H_\alpha P = P \oplus T_\alpha P\), \(H_\alpha f = f \oplus T_\alpha f\) for \(P \in \text{Obj} \mathcal{P}_R\), \(f \in \text{Mor} \mathcal{P}_R\). Then \(H_\mathcal{P}_R\) denotes the category with objects \(H_\alpha P\) for \(P \in \text{Obj} \mathcal{P}_R\) and with morphisms from \(H_\alpha P\) to \(H_\alpha Q\) all the morphisms in \(\mathcal{P}_R\) from \(P\) to \(Q\). Since \(T_\alpha H_\alpha P \simeq H_\alpha T_\alpha P\) naturally in \(P \in \text{Obj} \mathcal{P}_R\), the functor \(T_\alpha: H_\mathcal{P}_R \to H_\mathcal{P}_R\) is defined, and \(\eta_{a,u}: 1_{H_\mathcal{P}_R} \simeq T_\alpha^2\).

For \(P \in \text{Obj} \mathcal{P}_R\), there is the hyperbolic form

\[
\psi_{a,u} P = \begin{pmatrix} 0 & \psi_{a,u} P \\ \psi_{a,u} P & 0 \end{pmatrix}
\]

in \(H_\mathcal{P}_R\). Let \(T_{a,u}: H_\mathcal{P}_R \to H_\mathcal{P}_R\) be the functor given by \(T_{a,u} H_\alpha P = H_\alpha P, \quad T_{a,u} f = (\psi_{a,u} Q)^{-1}(T_\alpha f)(\psi_{a,u} P)\) for \(P \in \text{Obj} \mathcal{P}_R\), \(f \in \mathcal{P}_R(H_\alpha P, H_\alpha Q)\). Then the following is immediate.

**Lemma 1.5.** \(\psi_{a,u}\) is a natural equivalence of functors on \(H_\mathcal{P}_R\). \(\psi_{a,u}: T_{a,u} \simeq T_\alpha\).
Noting that $\psi_{a,u} T_a = T_a \psi_{a,u}$ and that $(T_a \psi_{a,u})(\psi_{a,u}) = \eta_{a,u}, 1_{B_a \mathcal{P}_R} \cong T_a^2$, we have, not only that $T_{a,u}$ fixes the objects of $H_a \mathcal{P}_R$, but also that $T_{a,u}^2 = 1_{B_a \mathcal{P}_R}$.

Since the canonical functor $H_a \mathcal{P}_R \rightarrow \mathcal{P}_R$ is cofinal, we have

$$K_i H_a \mathcal{P}_R \cong (1 + T_a) K_0 R \quad \text{if } i = 0,$$

$$\cong K_i R \quad \text{if } i > 0.$$

Furthermore, $T_{a,u} = T_a$ on $K_i R$, $i > 0$, and $T_{a,u}^2 = 1$ on $GL(H_a F; R)$ for $P \in \text{Obj} \mathcal{P}_R$. We shall see that the same holds for the Steinberg group $St(H_a F; R)$ for $F \in \text{Obj} \mathcal{P}_R$ a free module with specified basis $e_1, \ldots, e_n$. For then $H_a F$ has the basis $e_1, \ldots, e_n, e_1^*, \ldots, e_n^*$, and in matrix notation,

$$T_{a,u} M = \begin{pmatrix} 0 & I_n u^{-1} \\ I_n & 0 \end{pmatrix} M^{-\alpha} \begin{pmatrix} 0 & I_n \\ I_n u & 0 \end{pmatrix}$$

in $GL(2n; R) \cong GL(H_a F; R)$, where $M^\alpha$ denotes conjugate transpose of $M$ by $\alpha$, $M^{-\alpha}$ the inverse of $M^\alpha$, and $I_n$ is the identity of $GL(n; R)$.

**Proposition 16** If $e_{ij}$ is an elementary matrix in $E(2n; R) \subset GL(2n; R) \cong GL(H_a F; R)$, then $T_{a,u} e_{ij} = e_{pq}$ for suitable unique $p, q, s$ (depending upon notational convention only).

**Proof** We simply give below the result of applying $T_{a,u}$ to the elementary matrix $e_{ij}$. The proposition follows from the same kind of arguments given in Milnor [5, 9.2, 9.4], except that we are working in $E(2n; R)$. Let the basis element $e_t$ correspond to the integer $i$, and let the basis element $e_i^*$ correspond to the integer $-i$, where $i = 1, \ldots, n$. Then

$$T_{a,u} e_{ij} = \begin{cases} e^{-\alpha(t)}_{-j,-i} & \text{if } i, j > 0, \\ e^{-\alpha(t)}_{-j,-i} & \text{if } i > 0, j < 0, \\ e^{-\alpha(t)u}_{-j,-i} & \text{if } i < 0, j > 0, \\ e^{-\alpha(t)-1}_{-i,-i} & \text{if } i, j < 0. \end{cases}$$

For convenience, we record how $T_{a,u}$ acts on $E(2n; R)$ in blockwise notation (subscripts are to be ordered $1, \ldots, n, -1, \ldots, -n$).

$$T_{a,u} \begin{pmatrix} I_n & E_i \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} I_n & -u^{-1}E_i \\ 0 & I_n \end{pmatrix},$$

$$T_{a,u} \begin{pmatrix} I_n & 0 \\ E_i & I_n \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ -E_i u & I_n \end{pmatrix},$$

$$T_{a,u} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} E_{1^{-\alpha}} & 0 \\ 0 & E_{1^{-\alpha}} \end{pmatrix},$$
Also, for \( (\frac{A}{C} \frac{B}{D}) \in \text{GL}(2n; R) \), we have
\[
T_{a,u} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)^{-1} = \left( \begin{array}{cc} D^{-1} & u^{-1}B^* \\ C^*u & A^* \end{array} \right)
\]

Using the Steinberg relations for \( E(2n; R) \), the following is easily checked (cf. Milnor [5, Chap 5, 10.4]).

**Corollary 1.7.** The involution \( T_{a,u} \) on \( E(2n; R) \) lifts to an involution \( T_{a,u} \) on \( \text{St}(2n; R) \) defined by the permutation of generators

\[
\begin{align*}
T_{a,u}^* & = x_{i,j}^{(\alpha)} \\
& = x_{i,j}^{-1} \quad i, j \text{ both } > 0, \\
& = x_{i,j}^{(\beta)} \quad i > 0, j < 0, \\
& = x_{i,j}^{-1} \quad i < 0, j > 0, \\
& = x_{i,j}^{(\gamma)} \quad i, j \text{ both } < 0.
\end{align*}
\]

Letting \( n \to \infty \) gives the corresponding result for \( \text{St}(R) = \text{St}(2\infty; R) \). Thus we have a commutative diagram with exact rows and vertical morphisms all involutions (here, \( \text{GL}(R) = \text{GL}(2\infty; R) \), also)

\[
\begin{array}{ccc}
0 & \to & \mathbb{K}_2 R \\
T_{a,u} \downarrow & & \downarrow T_{a,u} \\
& \to & \text{St}(R) \\
\end{array}
\]

2. **Reflexive Forms on Hyperbolic Modules**

Recall that a (nonsingular) \((\alpha, u)\)-reflexive form on \( P \in \text{Obj} \mathcal{P}_R \) is an element \( g \in \mathcal{P}_R(P, T_{\alpha}P) \) such that \( \eta_{\alpha,u}P = (T_{\alpha}g)g \) (in other words, \( t_{\alpha,u}(P, T_{\alpha}P)(g) = (D_{\alpha}g)(\eta_{\alpha,u}P) = g \)) A basic observation in our approach to reflexive forms is the following.

**Proposition 2.1.** If \( P \in \text{Obj} \mathcal{P}_R \), then the (nonsingular) \((\alpha, u)\)-reflexive forms \( g \) on \( H_{\alpha}P \) are in one-to-one correspondence with the elements \( g' \in \text{GL}(H_{\alpha}P; R) \) such that \( 1_{H_{\alpha}P} = (T_{\alpha}g')g' \), according to the rule \( g = (\psi_{\alpha,u}P)g' \).

**Proof.** Since \( T_{\alpha}\psi_{\alpha,u}P = \psi_{\alpha,u}T_{\alpha}P \) and \( (\psi_{\alpha,u}T_{\alpha}P)(\psi_{\alpha,u}P) = \eta_{\alpha,u}H_{\alpha}P \), we have
\[
T_{a,u}(\psi_{\alpha,u}P)^{-1} g = (\psi_{\alpha,u}P)^{-1}(T_{\alpha}\psi_{\alpha,u}P)^{-1}(T_{\alpha}g)(\psi_{\alpha,u}P)
\]
\[
= (\eta_{\alpha,u}P)^{-1}(T_{\alpha}g)(\psi_{\alpha,u}P)
\]
Hence, if \( g = (\psi_{\alpha,u}P)g' \), then \( (T_{a,u}g')g' = 1_{H_{\alpha}P} \) if and only if \( 1_{H_{\alpha}P} = (\eta_{\alpha,u}P)^{-1}(T_{\alpha}g)g' \), i.e., if and only if \( g \) is \((\alpha, u)\)-reflexive. \( \square \)
Of course, the hyperbolic form $\psi_{a,u}P$ itself is an $(\alpha, u)$-reflexive form on $H_aP$, and it corresponds in Proposition 2.1 to $1_{H_aP}$. The following is clear.

**Lemma 2.2** Let $P, Q \in \text{Obj} P_R$, and let $g = (\psi_{a,u}P)g'$, $h = (\psi_{a,u}Q)h'$ be $(\alpha, u)$-reflexive forms on $P, Q$, respectively; if $\tau: H_aP \oplus H_aQ \cong H_a(P \oplus Q)$ is the canonical isomorphism which interchanges the middle two summands, then

$$g \oplus h = (\tau \tau)(g \oplus h) \tau^{-1} = (\psi_{a,u}(P \oplus Q))(g' \oplus h')$$

is an $(\alpha, u)$-reflexive form on $H_a(P \oplus Q)$, and $g' \oplus h' = \tau(g' \oplus h') \tau^{-1}$.

Thus, stabilizing the $(\alpha, u)$-reflexive form $g = (\psi_{a,u}P)g'$ on $H_aP$ by orthogonal direct sum with the hyperbolic form $\psi_{a,u}Q$ on $H_aQ$ corresponds in Proposition 2.1 to stabilizing $g' \in \text{GL}(H_aP, R)$ by direct sum with $1_{H_aQ}$. The process of stabilizing $g$ or $g'$ this way will always be followed by $\tau^*$, viz.,

$$\text{GL}(H_aP; R) \subseteq \text{GL}(H_aP \oplus H_aQ; R) \cong \tau_a \text{GL}(H_a(P \oplus Q); R),$$

as with the direct sum operations $\oplus$. Thus is basically in agreement with notation adopted in the previous section for free modules $F$ and for $\text{GL}(H_aF, R)$.

Having identified $g' \in \text{GL}(H_aP; R)$ such that $1_{H_aP} = (T_{a,u}g')g'$ with the $(\alpha, u)$-reflexive form $(\psi_{a,u}P)g'$, we should point out that an element $f \in \text{GL}(H_aP; R)$ satisfies $T_{a,u}f = f$ if and only if $f$ is an automorphism of the hyperbolic form $\psi_{a,u}P$. More generally, we have the following.

**Lemma 2.3.** If $g = (\psi_{a,u}P)g'$, $h = (\psi_{a,u}Q)h'$ are $(\alpha, u)$-reflexive forms on $P, Q \in \text{Obj} P_R$, respectively, then $f \in \text{GL}(H_aP, H_aQ)$ is an isomorphism of $(\alpha, u)$-reflexive forms from $g$ to $h$ if and only if $T_{a,u}f = h'fg'^{-1}$.

**Proof.** Expanding $T_{a,u}f = (\psi_{a,u}Q)^{-1}(T_{a,u})(\psi_{a,u}P)$, we have $T_{a,u}f = h'fg'^{-1}$ if and only if $(D_{a,u})hf = g$, which is just the condition for $f$ to be an isomorphism from $g$ to $h$.

The discriminant with respect to $P \in \text{Obj} P_R$ of an $(\alpha, u)$-reflexive form $g$ on $H_aP$ is defined to be the class of $(\psi_{a,u}P)^{-1}g \in \text{GL}(H_aP; R)$ in $K_1 R$,

$$\text{disc}_g = [(\psi_{a,u}P)^{-1}g] \in K_1 R.$$ 

Similarly, if $e \in P_R(P, Q)$ and $f \in P_R(H_aP, H_aQ)$ is an isomorphism of $(\alpha, u)$-reflexive forms from $g$ to $h$, the discriminant with respect to $e$ of $f$ is defined to be the class of $(H_a)e^{-1}f \in \text{GL}(H_aP; R)$ in $K_1 R$,

$$\text{disc}_e f = [(H_a)e^{-1}f] \in K_1 R.$$ 

If $P = Q$, $e = 1_P$, and $g = h$, then $\text{disc}_1 f$ is simply the determinant of the automorphism $f$ of $g$. From Proposition 2.1, it is clear that

$$(1 + T_a) \text{disc}_P g = 0;$$
on the other hand, since $T_{\alpha,\gamma}H_{\alpha} = H_{\alpha}$, we have from Lemma 2.3 that

$$(1 - T_{\alpha}) \text{disc}_e f = \text{disc}_P g - \text{disc}_Q h,$$

which vanishes if $g$ and $h$ have the same relative discriminants, e.g., when $f$ is an automorphism. From a somewhat different viewpoint, Lemma 2.3 tells us that, up to stable isomorphism of $(g, P)$, $\text{disc}_P g$ is well-defined modulo $(1 - T_{\alpha}) K_1 R$. Similarly, up to stable choice of $e$, $\text{disc}_e f$ is well-defined modulo $(1 + T_{\alpha}) K_1 R$.

Now, if $f \in \mathcal{P}_R(H_{\alpha} P, H_{\alpha} Q)$ is an isomorphism from $g$ to $h$ of $(\alpha, \gamma)$-reflexive forms, then, in the absence of any $e \in \mathcal{P}_R(P, Q)$, we have from Lemma 2.3 that

$$(\psi_{\alpha,\gamma} P)^{-1} g = (T_{\alpha,\gamma} f)^{-1} f f^{-1} (\psi_{\alpha,\gamma} Q)^{-1} hf,$$

and since conjugation does not alter a class modulo commutators,

$$\text{disc}_P g = \text{disc}_Q h + [(T_{\alpha,\gamma} f)^{-1} f] \in K_1 R.$$

The element $[(T_{\alpha,\gamma} f)^{-1} f] \in K_1 R$ is of particular interest, as explained by the following, whose proof is omitted.

**Lemma 2.4.** Let $x \in Z^-(Z_2; K_0 R)$, then there are $P, Q \in \text{Obj } \mathcal{P}_R$ such that $H_{\alpha} P \cong H_{\alpha} Q$ and $x = [P] - [Q]$. If $f \in \mathcal{P}_R(H_{\alpha} P, H_{\alpha} Q)$, then, modulo $(1 - T_{\alpha}) K_1 R$, the element $d_{\alpha,\gamma}^x = [(T_{\alpha,\gamma} f)^{-1} f]$ is well defined. The resulting function

$$d_{\alpha,\gamma}^x: Z^-(Z_2; K_0 R) \rightarrow K_1 R/(1 - T_{\alpha}) K_1 R$$

is a homomorphism which vanishes on $(1 - T_{\alpha}) K_0 R$ and satisfies $(1 + T_{\alpha}) d_{\alpha,\gamma}^x = 0$, and so induces a homomorphism

$$d_{\alpha,\gamma}^x: H^-(Z_2; K_0 R) \rightarrow H^-(Z_2; K_1 R).$$

**Note.** $Z^*(Z_2; ) = \text{Ker}(1 \mp T_{\alpha})$; $H^*(Z_2; ) = Z^*(Z_2; )/\text{Im}(1 \mp T_{\alpha})$.

**Corollary 2.5.** If $g$ is a nonsingular $(\alpha, \gamma)$-reflexive form on $H_{\alpha} P, P \in \text{Obj } \mathcal{P}_R$, then the (global) discriminant of $g$

$$\text{disc } g = [\text{disc}_P g] \in \text{Coker}[d_{\alpha,\gamma}^x: H^-(Z_2; K_0 R) \rightarrow H^-(Z_2; K_1 R)]$$

is well defined on the stable isomorphism class of $g$ (as opposed to the stable isomorphism class of the pair $(g, P)$).

Suppose now that $P \in \text{Obj } \mathcal{P}_R$ satisfies $P \cong T_{\alpha} P$; then, for $g \in \mathcal{P}_R(P, T_{\alpha} P)$, there is defined the hyperbolic automorphism of $\psi_{\alpha,\gamma} P$

$$\psi_{\alpha,\gamma} g = \begin{pmatrix} 0 & t_{\alpha,\gamma} g^{-1} \\ g & 0 \end{pmatrix} \in \text{GL}(H_{\alpha} P; R)$$

As complement to Lemma 2.4, we have the following, which is easily verified...
LEMMA 2.6 Let $x \in \mathcal{Z}^+(\mathbb{Z}_a; K_0 \mathcal{R})$; then there are $P, Q \in \text{Obj} \mathcal{P}_\mathcal{R}$ such that $P \simeq T_a P$, $Q \simeq T_a Q$, and $x = [P] - [Q]$. If $g \in \mathcal{P}_\mathcal{R}(P, T_a P)$, $h \in \mathcal{P}_\mathcal{R}(Q, T_a Q)$, then, modulo $(1 + T_a) K_1 \mathcal{R}$, the element

$$d_{\alpha, u}^+ x = [\psi_{\alpha, u} g] - [\psi_{\alpha, u} h] \in K_1 \mathcal{R}/(1 + T_a) K_1 \mathcal{R}$$

is well defined. The resulting function

$$d_{\alpha, u}^+: \mathcal{Z}^+(\mathbb{Z}_a; K_0 \mathcal{R}) \rightarrow K_1 \mathcal{R}/(1 + T_a) K_1 \mathcal{R}$$

is a homomorphism which vanishes on $(1 + T_a) K_0 \mathcal{R}$ and satisfies $(1 - T_a) d_{\alpha, u}^+ x = 0$, and so induces a homomorphism

$$d_{\alpha, u}^+: H^+(\mathbb{Z}_a; K_0 \mathcal{R}) \rightarrow H^+(\mathbb{Z}_a; K_1 \mathcal{R}).$$

Hence, modulo both the stable choice of $e$ and stable hyperbolic automorphisms (or modulo hyperbolic stabilization), the discriminant of an automorphism of an $(\alpha, u)$-hyperbolic form is well defined as an element of

$$\text{Coker}(d_{\alpha, u}^+: H^+(\mathbb{Z}_a; K_0 \mathcal{R}) \rightarrow H^+(\mathbb{Z}_a; K_1 \mathcal{R})).$$

If stabilization is via $H_\alpha F$, $F$ free, then topological applications require that the determinant or relative discriminant only be considered modulo $[\psi_{\alpha, u}(F \rightarrow T_a F)]$, where $F \rightarrow T_a F$ is the dual basis map which sends $e_i$ to $e_i^*$. At any rate, we have defined $(\ldots)$ discriminant homomorphisms $d_{\alpha, u}^+: H^+(\mathbb{Z}_a; K_0 \mathcal{R}) \rightarrow H^+(\mathbb{Z}_a; K_1 \mathcal{R})$, which are of some interest in algebraic $K$-theory. Presumably, they may be identified with a row of differentials in the $E_\infty$-term of the equivariant algebraic $K$-theory spectral sequence due to Vance [8].

Recall that if $(\alpha, \psi)$ is an antistructure on $R$, then $(\alpha, \psi)$ is an antistructure on $R$ if and only if $v = \psi e u$, $e \in Z = Z(R)$, and $\alpha(e) e = 1$. For example, there are always the cases $e = \pm 1$.

PROPOSITION 2.7. Let $e \in Z = Z(R)$ be a unit satisfying $\alpha(e) e = 1$. If $P \in \text{Obj} \mathcal{P}_\mathcal{R}$ supports an $(\alpha, e u)$-reflexive form, then, in the pairing $K_1 Z \otimes K_0 \mathcal{R} \rightarrow K_1 \mathcal{R}$,

$$d_{\alpha, u}^+ [P] = ([\psi_{\alpha, u} g]) \in H^+(\mathbb{Z}_a; K_1 \mathcal{R}).$$

In particular, if $P$ supports an $(\alpha, u)$-reflexive form, then $d_{\alpha, u}^+ [P] = 0$.

Proof. Let $g \in \mathcal{P}_\mathcal{R}(P, T_a P)$ satisfy $t_{\alpha, e u} g = g$; since $t_{\alpha, e u} = e_{\alpha, u}$, we have

$$\psi_{\alpha, u} g = \begin{pmatrix} 0 & t_{\alpha, u} g^{-1} \\ g & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-1} g^{-1} \\ g & 0 \end{pmatrix} \in \text{GL}(H_\alpha P, R),$$

Using $1_p \oplus g: P \oplus P \rightarrow H_\alpha P$, we have

$$(1 \oplus g)^{-1} (\psi_{\alpha, u} g)(1 \oplus g) = \begin{pmatrix} 0 & \epsilon_1 p \\ 1_p & 0 \end{pmatrix} \in \text{GL}(P \oplus P; R),$$
which gives the desired result under the identifications $R = \mathbb{Z} \otimes Z R$, $P = \mathbb{Z} \otimes Z P$. ☐

In the same vein, we have the following (proof omitted).

**Proposition 2.8.** Let $Z = Z(R)$ and $P \in \text{Obj } \mathcal{P}_Z$ support an $(\alpha | Z, \varepsilon)$-reflexive form; then, in the pairing $K_i R \otimes K_0 Z \to K_i R$,

$$d^+_{\alpha, u}[R \otimes Z P] - [[-\varepsilon u][P^*]] \in H^*(Z_2, K_1 R).$$

In particular, $d^+_{\alpha, u}[R] = [-u]$. ☐

As for $d^+_{\alpha, u}$, we have the following, using Lemma 2.3.

**Proposition 2.9.** If $P, Q \in \text{Obj } \mathcal{P}_R$ and the hyperbolic forms $\psi_{\alpha, u} P, \psi_{\alpha, u} Q$ are isomorphic, then $d^+_{\alpha, u}([P] - [Q]) = 0$. ☐

3. Reflexive Forms with Vanishing Discriminant

Let $g$, $h$ be $(\alpha, u)$-reflexive forms on $H_2 P$, $H_2 Q$, respectively, where $P, Q \in \text{Obj } \mathcal{P}_R$; then, by Lemma 2.2, the relative discriminant is additive:

$$\text{disc}_{P \oplus Q} g \boxplus h - \text{disc}_P g + \text{disc}_Q h.$$

Thus, for example, we may fix $g$, $P$ and then choose $Q$ such that $P \oplus Q = F$ is free, say with basis $e_1, \ldots, e_n$; of course, $\text{disc}_F g + \psi_{\alpha, u} Q = \text{disc}_P g$. If this vanishes, and if $n$ is sufficiently large, then, with respect to the canonical basis $e_1, \ldots, e_n, e_1^*, \ldots, e_n^*$ of $H_2 F$, the matrix of

$$(\psi_{\alpha, u} F)^{-1} (g \boxplus \psi_{\alpha, u} Q) \in \text{GL}(H_2 F; R)$$

is an element $\tilde{g} \in E(2n; R) \subseteq \text{GL}(2n; R)$ satisfying $(T_{\alpha, u} \tilde{g}) \tilde{g} = 1$ by Proposition 2.1.

Now let $\tilde{g} \in \text{St}(2n; R)$ be an element such that $\psi \tilde{g} = \tilde{g}$, where $\psi : \text{St}(m; R) \to E(m; R)$ denotes the canonical epimorphism. Since $\psi T_{\alpha, u} = T_{\alpha, u} \psi$ by Corollary 1.7, we have

$$(T_{\alpha, u} \tilde{g}) \tilde{g} \in C_{2n} R = \text{Ker}[\phi : \text{St}(2n; R) \to E(2n; R)];$$

hence, there is defined the element $\tilde{G}_{\alpha, u} \tilde{g} = [(T_{\alpha, u} \tilde{g}) \tilde{g}] \in K_2 R$.

**Proposition 3.1.** If $\tilde{g} \in E(2n; R)$ satisfies $(T_{\alpha, u} \tilde{g}) \tilde{g} = 1$, and if $\tilde{g} \in \text{St}(2n; R)$ satisfies $\psi \tilde{g} = \tilde{g}$, then

$$\tilde{G}_{\alpha, u} \tilde{g} \in Z^+(Z_2; K_2 R).$$
If $g' \in \text{St}(2n; R)$ also satisfies $\phi g' = \tilde{g}$, then

$$\mathcal{G}_{a,u} g' - \mathcal{G}_{a,u} \tilde{g} - (1 + T) [\tilde{g}^{-1}] \in (1 + T) K_2 R;$$

hence, $\mathcal{G}_{a,u} \tilde{g} = [\mathcal{G}_{a,u} \tilde{g}] \in H^+(Z_2; K_2 R)$ is well defined.

**Proof.** That $\mathcal{G}_{a,u} \tilde{g} = T_a \mathcal{G}_{a,u} \tilde{g}$ follows from the fact that $T^2_a = 1$ on St$(2n; R)$ by Corollary 1.7. The second assertion follows from the computation

$$[(T_a \mathcal{G}_{a,u} \tilde{g})^j] - [(T_a \mathcal{G}_{a,u} \tilde{g})^j] = [(T_a \mathcal{G}_{a,u} \tilde{g})^j \tilde{g}^{-1}]$$

by Lemma 2.3 and $T_a \mathcal{G}_{a,u} \tilde{g} = 1$, $\mathcal{G}_{a,u} \tilde{g} = \tilde{g}$, $\mathcal{G}_{a,u} \tilde{g} = \tilde{g}$, hence there is defined the element

$$\chi(a, \tilde{g}, g) = A - A.$$
For the second assertion, we have

\[
G_{a,u}(f'; g, h) - G_{a,u}(f, g, h) = [(T_{a,u}f')g^{-1}h^{-1}f^{-1}g^{-1}(T_{a,u}f)^{-1}]
\]
\[
= [f'^{-1}f] + [T_{a,u}(f'f)^{-1}] = (1 - Ta)[f'^{-1}f].
\]

Thus, if \(G_{a,u}g = G_{a,u}h\), then \(G_{a,u}(f; g, h) \in Z^{-}(\mathbb{Z}_a; K_a)\) and \(G_{a,u}(f; g, h) \in H^{-}(\mathbb{Z}_a; K_a)\). This is the case, for example, when \(g = h\) (and hence \(g = h\)); moreover, in this situation, we have the following, whose proof is trivial.

**Proposition 3.3.** Let \(f, g \in E(2n; R)\) satisfy \((T_{a,u}g)f = 1\) and \((T_{a,u}f)g = 1\), if \(f, g \in St(2n; R)\) are elements such that \(f^{-1}g f = f, g^{-1}g = g\), then the elements

\[
G_{a,u}(f; g) = G_{a,u}(f; g, g) \in Z^{-}(\mathbb{Z}_a; K_a),
\]

\[
G_{a,u}(f; g) = G_{a,u}(f; g, g) \in H^{-}(\mathbb{Z}_a; K_a)
\]

are well defined, independent of the choice of \(g \in St(2n; R)\) such that \(f^{-1}g f = g\).  

Suppose that, in the situation preceding (Proposition 3.2), we are given \(f, g \in St(2n; R)\) such that \(f^{-1}g f = f, g^{-1}g = g\), \(i = 1, 2\). Since inner automorphism of \(GL(R)\) by an element \(x \in GL(R)\), say \(y \mapsto x^{-1}y x\), induces an automorphism of \(E(R)\), then covering this automorphism of \(E(R)\) is a unique automorphism, \(y \mapsto y\), of \(St(R)\). In our case, we have \((T_{a,u}f)g_1 = g_2^{-H_a f^{-1}} f\), and so we have the elements

\[
G_{a,u}(f; g_1, g_2) = G_{a,u}(f; g_1, g_2) \in K_a R,
\]

\[
G_{a,u}(f; g_1, g_2) = G_{a,u}(f; g_1, g_2) \in K_a R/(1 - Ta) K_a R.
\]

Thus, for all practical purposes, we may take \(e = 1\)

Suppose now that \(f, g, h \in GL(2n, R)\) satisfy \((T_{a,u}g)f = 1\), \((T_{a,u}h)g = 1\), and \((T_{a,u}f)g = h f\). If, as elements of \(GL(R)\), we have \(g, h \in E(R)\), then \(h f \in E(R)\) and hence also \(f^{-1}(T_{a,u}f) = h f^{-1} - 1 \in E(R)\). Let \(g, h, \tilde{y} \in St(R)\) satisfy \(g g = g, g h = h, g \tilde{y} = f^{-1}(T_{a,u}f)\), and \(g \tilde{y} = h f\). Then, since \(K_a R\) acts trivially on \(K_a R\), we have

\[
G_{a,u}h = [(T_{a,u}h)h] = [(T_{a,u}h)f f] = [(T_{a,u}h)g g] = [(T_{a,u}h)\tilde{y} g]
\]
\[
= [\tilde{y} (T_{a,u}h)g] - [\tilde{y} (T_{a,u}h)g] = [(T_{a,u}h)(T_{a,u}h)] - [\tilde{y} (T_{a,u}h)g]
\]

As for the element \(\tilde{y} (T_{a,u}h)g\), we have the following analog of Lemma 2.4.

**Proposition 3.4.** Let \(x \in Z^{+}(\mathbb{Z}_a; K_a)\); then, if \(f \in GL(R)\) represents \(x\), \((T_{a,u}f)^{-1} f \in E(R)\). If \(\tilde{y} \in St(R)\) satisfies \(f^{-1}g f = (T_{a,u}f)^{-1} f\), then, modulo
(1 + T_a) K_2 R, the element \( d_{a,u}^+ x = [(T_{a,u} \tilde{y}) \tilde{y}] \) is well defined. The resulting function

\[
d_{a,u}^+: Z^+(Z_2; K_1 R) \rightarrow K_2 R/(1 + T_a) K_2 R
\]

is a homomorphism which vanishes on \((1 + T_a) K_1 R\) and satisfies \((1 - T_a) d_{a,u}^+ = 0\), and so induces a homomorphism

\[
d_{a,u}^+: H^+(Z_2; K_1 R) \rightarrow H^+(Z_2; K_2 R)
\]

**Proof.** Clearly, modulo \((1 + T_a) K_2 R\), \([(T_{a,u} \tilde{y}) \tilde{y}]\) depends on \( \tilde{f} \) and not on the choice of \( \tilde{y} \in \text{St}(R) \) such that \( \tilde{y} = (T_{a,u} f)^{-1} \tilde{f} \). Suppose \( \tilde{f}' = \tilde{f} \tilde{e} \), where \( \tilde{e} \in E(R) \), and \( \tilde{y}' \in \text{St}(R) \) satisfies \( \phi \tilde{y}' = (T_{a,u} \tilde{f}')^{-1} \tilde{f}' \), if \( \tilde{e} \in \text{St}(R) \) satisfies \( \phi \tilde{e} = \tilde{e} \), then \( \tilde{y}' = (\tilde{y} \tilde{e}^{-1}(T_{a,u} \tilde{e}))^j \) satisfies

\[
\phi \tilde{y}' = (T_{a,u} \tilde{f}')^{-1} \tilde{f}' \tilde{e}^{-1}(T_{a,u} \tilde{e}) \tilde{f} = (T_{a,u} \tilde{f})^{-1} \tilde{f}
\]

Hence, we have

\[
[(T_{a,u} \tilde{y}) \tilde{y}] = [(T_{a,u} \tilde{y}' (\tilde{e}^{-1}(T_{a,u} \tilde{e}))^j)] (T_{a,u} \tilde{e})^j
\]

since \((T_{a,u} \tilde{f})(\phi \tilde{y}') = \tilde{f}'\), and thus \( d_{a,u}^+ x \) is well defined. To see that \( d_{a,u}^+ \) vanishes on \((1 + T_a) K_1 R\), let \( \tilde{e} \in \text{GL}(n, R) \) represent \( x \in K_1 R \); then \( \tilde{f} = H_\tilde{e} \in \text{GL}(2n; R) \) represent \((1 + T_a)x\), and \((T_{a,u} \tilde{f})^{-1} \tilde{f} = 1\), so that \( d_{a,u}^+(1 + T_a)x = 0\). The rest follows easily.

The exposition above has been arranged so that the following is now a routine verification.

**Corollary 3.5.** If \( g \) is an \((\alpha, u)\)-reflexive form on \( H_a P, P \in \text{Obj} \mathcal{P}_R \), and if \( \text{disc} \, g = 0 \), then, for \( g \in E(2n; R) \) the matrix of \((\psi_{a,u} \tilde{F})^{-1} (g \oplus \psi_{a,u} Q)\) as before, the element

\[
G_{a,u} g = [G_{a,u} \tilde{y}] \in \text{Coker}[d_{a,u}^+: H^+(Z_2; K_1 R) \rightarrow H^+(Z_2; K_2 R)]
\]

is well defined on the stable isomorphism class of \( g \) (as opposed to the class of \((g, P)\) up to stable isomorphisms with vanishing discriminant).

Continuing the analogy with Section 2, suppose that \( \tilde{e} \in \text{GL}(n, R) \) satisfies \( H_\tilde{e} \in E(2n, R) \); since \( T_{a,u} H_\tilde{e} = H_\tilde{e} \), if \( \tilde{y} \in \text{St}(2n; R) \) satisfies \( \phi \tilde{y} = H_\tilde{e} \), then \((T_{a,u} \tilde{y}) \tilde{y}^{-1} \in C_{2n} R\).

**Proposition 3.6.** Let \( x \in Z^-(Z_2, K_1 R) \), then \( x \) is represented by \( \tilde{e} \in \text{GL}(n; R) \) such that \( H_\tilde{e} \in E(2n; R) \) for some \( n \). If \( \tilde{y} \in \text{St}(2n; R) \) satisfies \( \phi \tilde{y} = H_\tilde{e} \), then
modulo \(1 - T_a)K_2R\), the element \(d_{a-u}^{-1}x = [(T_{a,u}\tilde{y})\tilde{y}^{-1}]\) is well defined. The resulting function

\[
d_{a-u}^{-1} : Z^-(Z_2; K_1R) \to K_2R/(1 - T_a)K_2R
\]

is a homomorphism which vanishes on \((1 - T_a)K_1R\) and satisfies \((1 + T_a) d_{a-u}^{-1} = 0\), and so induces a homomorphism

\[
d_{a-u}^{-1} : H^-(Z_2; K_1R) \to H^-(Z_2; K_2R).
\]

**Proof.** Exercise. 1

There are various ways of relating Proposition 3.6 with \(G_{a,u}(\tilde{T}; \tilde{g})\) and the like. Perhaps the most obvious comes from the fact that, for \(\tilde{e} \in GL(n; R)\), \(H_{a,\tilde{e}}\) is the matrix of an automorphism of \(\psi_{a,u}F\) with respect to a basis of the form \(e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\). Thus, if \([\tilde{e}] \in Z^-(Z_2; K_1R)\), then \(d_{a-u}[\tilde{e}] = \hat{G}_{a,u}(H_{a,\tilde{e}}; 1) \in H^-(Z_2; K_2R)\). The following result for \(d_{a-u}^{-1}\) is quite basic.

**Theorem 3.7** Let \(\hat{e} \in Z = Z(R)\) be a unit satisfying \(\alpha(\hat{e}) = 1\) If \(\tilde{g} \in GL(2n; R)\) satisfies \((T_{a,u}\tilde{g})\tilde{g} = 1\), then, in the pairing \(K_2Z \otimes K_1R \to K_2R\),

\[
d_{a-u}^{-1}[\tilde{g}] = [[-\tilde{e}][\tilde{g}]] \in H^-(Z_2; K_2R)
\]

In particular, if \(g\) is an \((\alpha, -u)\)-reflexive form on \(H_0P\), \(P \in \text{Obj} \mathcal{P}_R\), then \(d_{a-u}[\text{disc}_P g] = 0\) (\(\text{disc}_P g\) is taken in the \((\alpha, -u)\) antistructure).

**Proof.** From the fact that \((T_{a,u}\tilde{g})\tilde{g} = 1\), we have \(T_{a,\tilde{g}} = \tilde{g}^{-1}g^{-1}\), where

\[
\hat{g} = \begin{pmatrix} 0 & 1 \\ e_{u} & 0 \end{pmatrix} \in GL(2n; R).
\]

Now, following the notational scheme of Proposition 1.6, let block elementary matrices be denoted

\[
e_{a+s}^\pm = \begin{pmatrix} 1 & \hat{a} \\ 0 & 1 \end{pmatrix}, \quad e_{a+s}^\pm = \begin{pmatrix} 1 & 0 \\ \hat{a} & 1 \end{pmatrix} \in E(2m; R) \quad \text{for} \quad \hat{a} \in M_nR,
\]

and let \(x_{a+s}^\pm \in \text{St}(2m; R)\) denote the block Steinberg generator such that \(\hat{g}x_{a+s}^\pm = e_{a+s}^\pm\). In other words, if \(e_{a+s}^\pm\) is expressed as the canonical product of \(a^2\) ordinary elementary matrices in \(E(2m; R)\), then \(x_{a+s}^\pm\) is expressed as the corresponding product of \(a^2\) ordinary Steinberg generators in \(\text{St}(2m, R)\). For \(\hat{h} \in GL(m; R)\), let

\[
w_{a+s}(\hat{h}) = x_{a+s}^\pm x_{a+s}^{-1} x_{a+s}^- x_{a+s}^+, \quad h_{a+s}(\hat{h}) = w_{a+s}(\hat{h}) w_{a+s}(-1)
\]
in \text{St}(2m; R) be the block analogs of the elements \(w_{ij}(u), h_{ij}(u)\) in Milnor [5, Sect. 9]; then

\[
\phi w_{\pm\pm}(h) = \begin{pmatrix}
0 & \pm h^\pm 1 \\
\mp h^\mp 1 & 0
\end{pmatrix}, \quad \phi h_{\pm\mp}(h) = \begin{pmatrix}
h^\pm 1 & 0 \\
0 & h^\mp 1
\end{pmatrix}.
\]

Hence, with \(m = 2n\), we have

\[
H_{\alpha \beta} = \begin{pmatrix}
1 & 0 \\
0 & \phi
\end{pmatrix} \begin{pmatrix}
\hat{g} & 0 \\
0 & \phi^{-1}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & \phi
\end{pmatrix} - \begin{pmatrix}
1 & 0 \\
0 & \phi
\end{pmatrix} \phi h_{\pm\mp}(\hat{g}) \begin{pmatrix}
1 & 0 \\
0 & \phi^{-1}
\end{pmatrix}
\]

\[
= \phi w_{\pm\mp}(\hat{g}^{-1}) \phi w_{\pm\mp}(-\phi^{-1}) = \phi h_{\pm\mp}(\hat{g}^{-1}) \phi h_{\pm\mp}(-\phi^{-1})^{-1},
\]

much as in [5, Corollary 9.4]. Now, using Corollary 1.7, we have

\[
T_{a,u} w_{\pm\mp}(h) = w_{\pm\mp}(-u^{-1} h^\alpha)
\]

\[
T_{a,u} w_{\pm\mp}(h) = w_{\pm\mp}(-h^u)
\]

for \(h \in \text{GL}(m, R)\). Hence, with \(\tilde{y} = w_{\pm\mp}(-\phi^{-1}) w_{\pm\mp}(-\phi^{-1})\), we have

\[
T_{a,u} \tilde{y} = w_{\pm\mp}(-u^{-1} \phi^{-1} \hat{g}) w_{\pm\mp}(u^{-1} \phi^{-1}) = w_{\pm\mp}(-\phi^{-1} \hat{g}) w_{\pm\mp}(\phi^{-1})
\]

\[
- w_{\pm\mp}(-\phi^{-1} \hat{g}) w_{\pm\mp}(\phi^{-1}),
\]

since \(\phi^{-1} \hat{g} = \hat{g}^{-1}\). Thus we have

\[
(T_{a,u} \tilde{y}) \tilde{y}^{-1} = w_{\pm\mp}(-\phi^{-1} \hat{g}) w_{\pm\mp}(\phi^{-1}) w_{\pm\mp}(-\phi^{-1}) w_{\pm\mp}(\phi^{-1})
\]

\[
= h_{\pm\mp}(-\phi^{-1} \hat{g}) \ h_{\pm\mp}(-\phi^{-1}) \ h_{\pm\mp}(\phi^{-1}) \ h_{\pm\mp}(-\phi^{-1})^{-1},
\]

since \(w_{\pm\mp}(h) w_{\pm\mp}(-h) = 1\) and \(w_{\pm\mp}(h_1) w_{\pm\mp}(h_2) = h_{\pm\mp}(h_1) h_{\pm\mp}(-h_2)^{-1}\), as in [5, Sect. 9]. Letting \(n \to \infty\) and continuing our calculation, we have, upon inserting \(h_{\pm\mp}(-\epsilon) h_{\pm\mp}(-\epsilon)^{-1}\) between \(h_{\pm\mp}(\phi^{-1})\) and \(h_{\pm\mp}(\phi^{-1})^{-1}\),

\[
[(T_{a,u} \tilde{y}) \tilde{y}^{-1}] = -[-\epsilon][\phi^{-1}] + [h_{\pm\mp}(-\phi^{-1} \hat{g}) h_{\pm\mp}(-\phi^{-1}) h_{\pm\mp}(\phi^{-1}) h_{\pm\mp}(-\phi^{-1})^{-1}]
\]

\[
= [-\epsilon][\phi] + [-\epsilon][\phi^{-1}] = [-\epsilon][\hat{g}],
\]

by mild extension of [5, Lemmas 8.2, 9.4]. The theorem follows.

For the appropriate analog of Proposition 2.8, we offer the following without proof.

**Proposition 3.8.** Let \(\epsilon \in Z = Z(R)\) be a unit satisfying \(\alpha(\epsilon) \epsilon = 1\). If \(\hat{g} \in \text{GL}(n; Z)\) satisfies \((T_{a,u} \hat{g}) \hat{g} = 1\), then, in the pairing \(K_1 R \otimes K_2 Z \to K_2 R\),

\[
d_{a,u}[R \otimes Z \hat{g}] = [-\epsilon u][\hat{g}] \in H^*(Z \otimes K_2 R).
\]

In particular, \(d_{a,u}[-1] = [[-u][-1]]\).

A vanishing result for \(d_{a,u}^+\) is the following.
THEOREM 3.9. If \( \tilde{f} \in \text{GL}(2n; R) \) is the matrix with respect to \( e_1, \ldots, e_n \), \( e_1^*, \ldots, e_n^* \) of an automorphism of an \((\alpha, \mu)\)-reflexive form \( g \) with vanishing relative discriminant on \( H, F \), then \( d_{a,u}^+[\tilde{f}] = 0 \).

Proof. Let \( \tilde{g} \in \text{GL}(2n; R) \) be the matrix of \((\psi_{a,u} F)^{-1} g \); then \( \tilde{g} \in E(R) \), and \((T_{a,u}, \tilde{f}) \tilde{g}^i = \tilde{g} \tilde{f} \), by Lemma 2.3. Thus \((T_{a,u}, \tilde{f})^{-1} \tilde{g}^i = \tilde{g} \tilde{f} \). Let \( g \in \text{St}(R) \) satisfy \( \phi g = \tilde{g} \) and set \( \tilde{y} = \tilde{g} \tilde{g}^{-1} \). Then, noting that \( \tilde{g} = (T_{a,u}, \tilde{g})^{-1} \) near the end, we have

\[
d_{a,u}^+ \tilde{f} = [(T_{a,u}, \tilde{f}) \tilde{y}] = [(T_{a,u}, \tilde{g})(T_{a,u}, \tilde{g})^{-1} g^{-1} g^i - f] = \tilde{G}_{a,u} \tilde{g} + [(T_{a,u}, \tilde{g})^{-1} - \tilde{g}^{-1}] = 0.
\]

Now we can show how the \( d_{a,u}^\pm \) of this and the preceding section are differentials.

THEOREM 3.10. \( d_{a,u}^\pm d_{a,\pm u}^\mp = 0 \).

Proof. If \( g \in \mathcal{P}_R(P, T_{a} P) \), then \( T_{a,u} \psi_{a,u} g = \psi_{a,u} g \), so that \( d_{a,u}^\pm g = 0 \) by Theorem 3.9. To show that \( d_{a,u}^\pm d_{a,\pm u}^\mp = 0 \), let \( x \in \mathcal{Z}(Z_2; K \otimes R) \); clearly, in the representation \( x = [P] - [Q] \) of Lemma 2.4, we may take \( P = F \) to be free, say with basis \( e_1, \ldots, e_n \). Now let \( f \in \mathcal{P}_R(H, F, H \otimes Q) \) and let \( g \in \text{GL}(2n; R) \) be the matrix of \((T_{a,u} g)^{-1} f \in \text{GL}(H, F; R) \) with respect to \( e_1, \ldots, e_n, e_1^*, \ldots, e_n^* \). Then \((T_{a,u} \tilde{g}) \tilde{g} = 1 \), so \( d_{a,u}^\pm [\tilde{g}] = 0 \) by Theorem 3.7. Since \( [\tilde{g}] = d_{a,\pm u}^\mp \), it follows that \( d_{a,u}^\pm d_{a,\pm u}^\mp = 0 \).

Let \( d_{a,u}^{p,q} : H^p(Z_2; K_{-q} R) \to H^{p+2}(Z_2; K_{-q+1} R) \) for \( p \geq 0, q = 0, -1 \), be defined by

\[
d_{a,u}^{p,0} = d_{a,u}^+, \quad \text{for} \quad p = 0
\]

\[
= d_{a,u}^-, \quad \text{for} \quad p = 0 \quad (\text{mod} \ 4),
\]

\[
= d_{a,\pm u}^+ \quad (\text{mod} \ 4),
\]

\[
= d_{a,\pm u}^- \quad (\text{mod} \ 4).
\]

\[
d_{a,u}^{p,-1} = d_{a,u}^- \quad \text{for} \quad p = 0
\]

\[
= d_{a,u}^+, \quad \text{for} \quad p = 0 \quad (\text{mod} \ 4),
\]

\[
= d_{a,u}^- \quad (\text{mod} \ 4),
\]

\[
= d_{a,\pm u}^+ \quad (\text{mod} \ 4).
\]

\[
= d_{a,\pm u}^- \quad (\text{mod} \ 4).
\]
Then these should be differentials in the $E_2$-term of the spectral sequence of Vance [8] for equivariant algebraic $K$-theory:

$$E_2^{p,q} = H^p(Z_2 ; K, -q R) \Rightarrow KR_2^{p,q}(S^\infty, A),$$

$$(E_r^{p,q} = E_r^{p,q}(R, \alpha, u), KR_2^w = K(R, \alpha, u)_2^w).$$

### 4. The Case of a Commutative Ring

In this section, $R$ will be a commutative ring; hence, $\alpha$ is an automorphism of period 1 or 2, and $u$ is a unit such that $\alpha(u) u = 1$.

**Proposition 4.1.** Let $v \in R$ be a unit such that $T_v v = v \pm 1$ (i.e., $\alpha(v) v = 1$); then $[v] \in Z^*(Z_2 ; K_1 R)$ and

$$d^\pm_{a,u}[v] = [[\pm u][v]] \in H^+(Z_2 ; K_2 R).$$

**Proof.** If $T_v v = v$, then $(T_v, a(v \oplus 1))^{-1} (v \oplus 1) = v \oplus v^{-1} = \phi h_{-1} v$; on the other hand, if $T_v, v = v^{-1}$, then $H_v v = v \oplus v^{-1} = \phi h_{-1} v$. In either case, let $\tilde{y} = h_{-1}(v) \in St(2, R)$; then, using Corollary 1.7, we have

$$[(T_v, a(v \oplus 1))^{-1} \phi h_{-1} v] = [[h_{-1}(-u^{-1} v^{-1}) h_{-1}(-u^{-1})^{-1}] \pm h_{-1}(v)]$$

in $K_2 R$. Hence, $d^\pm_{a,u}[v] = [[\pm u][v]]$.

**Corollary 4.2.** If $H^*(Z_2 ; SK_1 R) = 0$, then $d^\pm_{a, \pm 1} : H^+(Z_2 ; K_1 R) \rightarrow H^+(Z_2 ; K_2 R)$ vanishes

This is the case, for example, if $R$ is a Dedekind domain or the group ring of a finite abelian group with either cyclic or elementary abelian Sylow 2-subgroup. We shall be particularly interested in the case of a field.

The following calculation leads directly to the main result of this section.

**Lemma 4.3.** Let $a, b \in R$ be units, and let

$$\hat{g} = \begin{pmatrix}
0 & a^{-1} b^{-1} & 0 \\
0 & 1 & 0 \\
a & 0 & 0
\end{pmatrix} \in GL(4; R).$$

Then $\hat{g} \in E(4; R)$, $(T_{u \hat{v}, \hat{y}}(\hat{g})) \hat{g} = 1$, and $\hat{G}_{u \hat{v}, \hat{y}} \hat{g} = [[-a][-b]] \in H^+(Z_2 ; K_2 R)$. 
Proof. We have the factorization

$$\tilde{g} = \begin{pmatrix}
\circ & \circ & -a^{-1} & 0 \\
\circ & \circ & 0 & -b^{-1} \\
a & 0 & 0 & \circ \\
b & 0 & 0 & \circ \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & \circ \\
0 & 1 & \circ \\
\circ & \circ & -b^{-1} & 0 \\
\circ & \circ & 0 & -b \\
\end{pmatrix}$$

$$= \phi(w_{-1,1}(a) w_{-2,2}(b) h_{-2,-1}(-b)) \in E(4; R).$$

Setting $\tilde{g} = w_{-1,1}(a) w_{-2,2}(b) h_{-2,-1}(-b) \in St(4; R)$ and using Corollary 1.7, we have $T_{1d,1} \tilde{g} = w_{-1,1}(-a) w_{-2,2}(-b) h_{2,1}(-b^{-1})$ Now using [5, Lemmas 8.2, 9.6, Corollary 9.4], we have

$$G_{zd,1} \tilde{g} = b^{-1,1}(-a) b^{-2,2}(-b) b^{-1,1}(-b)$$

$$= \left[ h_{-1,1}(-b^{-1,1}) h_{-2,2}(-b^{-2,2}) \right] = \left[ h_{-1,1}(-b^{-1}) h_{-2,2}(-b^{-2}) \right]$$

in $Z^+(Z_2; K_2 R)$. Thus $G_{zd,1} \tilde{g} = [-a][-b]$.  

Now we can identify our invariant $G_{zd,1}$ in the case $R = E$ is a field. Note that, by Corollary 4.2, $G_{zd,1}$ takes values in $H^-(Z_2; K_2 E) \cong K_2 E / 2K_2 E$, the isomorphism following directly from Matsumoto’s theorem [5, Theorem 11.1].

Theorem 4.4. Let $E$ be a field, $g$ an $(zd, 1)$-reflexive (i.e., symmetric bilinear) form of even rank with vanishing discriminant $\text{disc}_g$; then, for every symbol $\varphi: E^* \times E \to \{\pm 1\}$, the Hasse–Witt invariant $h_g \varphi$ equals $\tilde{\varphi} G_{zd,1} g$, where $\tilde{\varphi}$ is the unique homomorphism such that the following diagram is commutative.

$$
\begin{array}{ccc}
K_1 E \times K_1 E & \xrightarrow{\det \times \det} & K_2 E \\
\cong & & K_2 E / 2K_2 E \\
\downarrow & & \downarrow \varphi \\
E^* \times E^* & \xrightarrow{\varphi} & \{\pm 1\}
\end{array}
$$

($h_g$ is defined in Milnor and Husemoller [6, p. 80]).

Proof. Clearly, $\text{disc}_g$ lives in $H^-(Z_2; K_1 E) = E / E^2$ and coincides with the usual discriminant for forms of even rank. Thus, in the Witt ring $W(E)$, $g$ represents an element of $I^2$, $I \subseteq W(E)$ being the fundamental ideal generated by forms of even rank. Since both $h_g$ and $G_{zd,1}$ are homomorphisms of $I^2$, it suffices to show that $h_g = \tilde{\varphi} G_{zd,1} g$ for a set of generators for $I^2$. But such a set is given by the forms

$$g = \langle a \rangle + \langle b \rangle + \langle a^{-1} b^{-1} \rangle + \langle 1 \rangle, \quad a, b \in E^*.$$
Now, Lemma 4.3 computes $G_{ rd,1} = \hat{G}_{ rd,1} = \mathcal{I}[-a][-b]$; hence, we must have $h_{ rd,g} = \hat{g}[-a][-b] = g(-a, -b)$. This is indeed the case, as is easily checked from the definition of $h$.

It follows from Theorem 4.4 that $G_{ rd,1} : I^2/I^3 \cong K_{a', E}/2K_{a', E}$ is the universal Hasse–Witt invariant in the case of a field $E$. For this reason, the invariants $\hat{G}_{ rd,u}, \hat{G}_{ rd,u}, \hat{G}_{ rd,u}$ of the preceding section are all called generalized Hasse–Witt invariants.

5. Extensions of the Hasse–Witt Invariants

We begin by defining a Steinberg-like group $St_v(n; R)$, where $V \subseteq R^r$ is a subgroup of the group $R^r$ of units of the ring $R$, $n \geq 1$. In terms of generators and relations, $St_v(n; R)$ is presented as follows

**Generators.** (1) $x_{ij}^r, r \in R, 1 \leq i, j \leq n, i \neq j$,
(2) $y(v_1, \ldots, v_n), v_1, \ldots, v_n \in V$.

**Relations.** (1) $x_{ij}^r x_{jk}^s = x_{ki}^{r+s}$,
(2) $y(u_1, \ldots, u_n) y(v_1, \ldots, v_n) = y(u_1 v_1, \ldots, u_n v_n)$,
(3) $y(v_1, \ldots, v_n) x_{ij}^r y(v_1, \ldots, v_n)^{-1} = x_{ij}^{v_i^r v_j^{-1}}$,
(4) $[x_{ij}^r, x_{kl}^s] = 1$ if $i \neq j, j \neq k$,
$= x_{ki}^{r+s}$ if $i \neq j, j = k$,
$= x_{kj}^{r+s}$ if $i = 1, j \neq k$,
(5) $w_{ij}(u) x_{ij}^r w_{ij}(-u) = x_{ik}^{w_{ij} r u}$, where $u \in R$ and $w_{ij}(u) = x_{ij}^{w_{ij} u} x_{ij}^{-1} x_{ij}^{-1}$,
(6) $h_{ij}(v) = y(1, \ldots, v_i, \ldots, v_j, \ldots, 1)$, $v \in V$, where $h_{ij}(u) = w_{ij}(u) w_{ij}(-1)$.

Clearly, for $V = \{1\}$, $St_v(n; R) = St(n; R)$.

There are canonical homomorphisms $\psi_v : St(n; R) \rightarrow St_v(n; R)$ given by $\psi_v x_{ij} = x_{ij}^r$ and $\phi_v : St_v(n; R) \rightarrow GL(n; R)$ given by $\phi_v x_{ij} = e_{ij}^r, \phi_v y(v_1, \ldots, v_n) = \text{diag}(v_1, \ldots, v_n)$. If $E_v(n; R) = \phi_v St_v(n; R)$, then $E(n; R)$ is a normal subgroup of $E_v(n; R)$. Let $W_1 v(n; R) = E_v(n; R)/E(n; R)$ and $W_2 v(n; R) = \text{Ker}[\psi_v : St(n; R) \rightarrow St_v(n; R)]$. From relation (3), it follows that $\psi_v St(n; R)$ is a normal subgroup of $St_v(n; R)$, and so $\phi_v$ induces a surjection $\bar{\phi}_v : \text{Coker} \psi_v \rightarrow W_1 v(n; R)$.

If $\bar{\phi}_v : \text{Coker} \psi_v \rightarrow W_1 v(n; R)$ is an isomorphism, then $V$ is called an $St(n)$-subgroup of $R$; $V$ is called an $St(\infty)$-subgroup of $R$ if it is an $St(n)$-subgroup for almost all $n$, and $V$ is called an $St$-subgroup of $R$ if it is an $St(n)$-subgroup for all $n$. 
The significance of an $\text{St}(n)$-subgroup $V \subseteq R$ is that it leads immediately to the following commutative diagram with exact rows and columns, where $K_2^V(n; R) = \text{Ker}[\phi_V : \text{St}^V(n; R) \to E^V(n; R)]$.

\[
\begin{array}{c}
1 \\
\downarrow \\
W_2^V(n; R) \longrightarrow K_2(n; R) \longrightarrow K_2^V(n; R) \longrightarrow 1 \\
\downarrow \\
1 \longrightarrow \text{St}(n; R) \longrightarrow \text{St}^V(n; R) \longrightarrow W_1^V(n; R) \longrightarrow 1 \\
\downarrow \phi \\
1 \longrightarrow E(n; R) \longrightarrow E^V(n; R) \longrightarrow W_1^V(n; R) \longrightarrow 1 \\
1 1
\end{array}
\]

Letting $n \to \infty$, where $\text{St}^V(n; R) \subseteq \text{St}^V(n + 1; R)$, etc., in the obvious way, we have, for an $\text{St}(\infty)$-subgroup $V \subseteq R$, the following commutative diagram with exact rows and columns, where $K_1^V(R) = K_1(R)/W_1^V(R)$.

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
W_2^V(R) \longrightarrow K_2(R) \longrightarrow K_2^V(R) \longrightarrow 0 \\
\downarrow \\
W_2^V(R) \longrightarrow \text{St}(R) \longrightarrow \text{St}^V(R) \longrightarrow W_1^V(R) \longrightarrow 0 \\
\downarrow \phi \\
1 \longrightarrow E(R) \longrightarrow \text{GL}(R) \longrightarrow K_2(R) \longrightarrow 0 \\
\downarrow \\
1 \longrightarrow K_1^V(R) \longrightarrow K_1^V(R) \\
0 0
\end{array}
\]

It could perhaps be that all subgroups of $R$ are, say, $\text{St}(\infty)$-subgroups. The best we can do is the following, which provides a source of $\text{St}$-subgroups.

**Proposition 5.1.** If the kernel of the canonical homomorphism $V \to K_1R$ is the commutator subgroup of $V$, then $V \subseteq R$ is an $\text{St}$-subgroup.
Proof. For $n = 1$, $\bar{\phi}_V = id: V \cong V$ is clearly an isomorphism. Suppose $n > 1$, and let $w \in St^r(n; R)$ be an element such that $\bar{\phi}_V[w] = 1$ in $W_1^r(n; R)$, i.e., such that $\psi_V w \in E(n; R)$. Using relations (1), (2), (3), we may suppose that $w = xy(v_1, \ldots, v_n)$, where $x$ is a word in the $x_i^r$. Now, using relation (6), we have $y(v_1, \ldots, v_n) = h_{1,2}(v_2) y(1, v_1, v_2, \ldots, v_n)$. Continuing inductively, we have

$$y(v_1, \ldots, v_n) = h_{1,2}(v_1) h_{n-1,n}(v_1 \cdots v_{n-1}) y(1, \ldots, 1, v_1 \cdots v_n),$$

and so we may in fact suppose that $w = xy(1, \ldots, 1, v_0)$, where $x$ is a word in the $x_i^r$. From the assumption that the kernel of $V \to K_4R$ is the commutator subgroup of $V$, it follows that $v_0$ is an element of the commutator subgroup of $V$. Now, for $u, v \in V$, we have

$$y(1, \ldots, 1, [u, v]) = y(1, \ldots, 1, u^{-1}v^{-1}, uv) y(1, \ldots, 1, u, v, u^{-1}) y(1, \ldots, 1, v, v^{-1})$$

$$= h_{n,n-1}(uv) h_{n,n-1}(u)^{-1} h_{n,n-1}(v)^{-1},$$

again using relation (6). Hence, $w$ may be expressed as a word in the $x_i^r$, and so $w$ is in the image of $\psi_V$. That is, $[w] \in \text{Coker } \psi_V$, and $V$ is an $St(n)$-subgroup of $R$: for every $n$.

Two important examples of rings and $St$-subgroups, in virtue of Proposition 5.1, are the following.

Example 1. Any subgroup $V$ of the group of units of a commutative ring $R$ is an $St$-subgroup. The particular choice $V = R^*$ gives $W_1^R(n; R) = W_1^R(R) = R$, and since $SK_4(R) = K_4(R)/R = K_2^R(R)$, we define $SK_2(R) = K_2^R(R)$.

By the work of Dennis and Stein, the ring of integers in $\mathbb{Q}(-17)^{1/2}$ is a ring $R$ with $SK_2R \neq 0 [1, \text{Sect } 3]$.

Example 2. Any subgroup $V \subseteq \pm \pi$ such that $V/V'$ injects monomorphically into $\pm \pi/\pi'$ is an $St$-subgroup of the group of units of the group ring $\mathbb{Z}\pi$. In particular, we may choose $V = \pm \pi$, and then we have $K_1^\pm(\mathbb{Z}\pi) = Wh(\pi)$ and, in the notation of Hatcher [4], $W_2^+(\mathbb{Z}\pi) = W(\pi) \cap K_2\mathbb{Z}\pi$, and so we have $Wh_2(\pi) = K_2^+(\mathbb{Z}\pi)$.

It would appear that basic objects of topological algebraic $K_2$-theory should somehow be the groups in the exact sequence

$$0 \to Wh_2(\pi) \to \text{St}(\pi) \to \text{GL}(\mathbb{Z}\pi) \to Wh(\pi) \to 0,$$

as well as the sequence itself, where $\text{St}(\pi) = St^\pm(\mathbb{Z}\pi)$. 
Now suppose that $R$ is a ring with antistructure $(\alpha, u)$, and that $V \subseteq R$ is an $\mathrm{St}(\infty)$-subgroup such that $\alpha V = V$. Then $T_{\alpha} W_{i}^{\prime}(R) = W_{i}^{\prime}(R)$, $i = 1, 2$, and so $T_{\alpha}$ induces an involution $T_{\alpha}$ on $K_{i}^{\prime}(R)$, $i = 1, 2$.

If $V \subseteq R$ is an $\mathrm{St}(2n)$-subgroup such that $\alpha V = V$, then, with the notational convention of Proposition 1.6–Corollary 1.7, there is an involution $T_{\alpha, u}$ on $\mathrm{St}^{\prime}(2n; R)$ given by

\[ T_{\alpha, u} x_{i, j} = x_{-a_{i}, -b_{j}} \quad t, j \text{ both } 0, \]

\[ = x_{-a_{i}, -b_{j}}^{-1} \quad i > 0, \quad j < 0, \]

\[ = x_{a_{i}, -b_{j}} \quad i < 0, \quad j > 0, \]

\[ = x_{a_{i}, -b_{j}}^{-1} \quad i, j \text{ both } 0, \]

\[ T_{\alpha, u}(\tau_{1}, \ldots, \tau_{n}) = y(\alpha_{-1}^{-1} a_{1}, \ldots, \alpha_{n}^{-1} a_{n}). \]

For an $\mathrm{St}(\infty)$-subgroup $V$ with $\alpha V = V$, these involutions are compatible with stabilization and commute with $\phi_{V}$, $\psi_{V}$. Hence, in this case the commutative diagram preceding Proposition 5.1 is a commutative diagram of groups-with-involution.

Now, if $V \subseteq R$ is an $\mathrm{St}$-subgroup (or, with care being taken to stabilize appropriately, an $\mathrm{St}(\infty)$-subgroup) such that $\alpha V = V$, then the program of Sections 2 and 3 may be repeated with $K_{i}^{R}$ replacing $K_{i}^{R}$, $i = 1, 2$, $E^{V}(n; R)$ replacing $E(n; R)$, $\mathrm{St}^{V}(n; R)$ replacing $\mathrm{St}(n; R)$, etc. For example, we obtain relative discriminants $\mathrm{disc}^{V}$ with values in $K_{1}^{V} R$ and homomorphisms

\[ d_{\alpha, u}^{\pm}: H_{i}(Z_{2}; K_{1} R) \to H_{i}(Z_{2}; K_{1}^{V} R). \]

Similarly, Hasse–Witt invariants $G_{\alpha, u}^{\prime}$ in $Z_{i}(Z_{2}; K_{1}^{V} R)$, $\hat{G}_{\alpha, u}^{\prime}$ in $H_{i}(Z_{2}; K_{1}^{V} R)$, and homomorphisms

\[ d_{\alpha, u}^{\pm}: H_{i}(Z_{2}; K_{1}^{V} R) \to H_{i}(Z_{2}; K_{1}^{V} R) \]

are defined. With the obvious rewording, the results of both sections hold true in this situation.

Algebraic $L$-theoretic ramifications will be taken up in the next paper of this series, along with topological applications.

Note added in proof. S. C. Geller has shown the sufficient condition provided by Proposition 5.1 is also necessary.

References


