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A SHORT PROOF OF THE LOCAL ATIYAH–SINGER INDEX THEOREM*

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IN THIS paper, we will give a simple proof of the local Atiyah–Singer index theorem, first proved by Patodi [9]; in fact, his earlier proof of the Gauss–Bonnet–Chern theorem (Patodi [8]) is quite close to ours. (Perhaps he did not find the proof for Dirac operators given in this paper because he was unaware of the symbol calculus for Clifford algebras.) A paper of Kotake [5] contains a proof of the Riemann–Roch theorem for Riemann surfaces along similar lines, and a recent paper of Bismut [3] is also very closely related.

It might be helpful to give a short history of this theorem. As explained in Atiyah *et al.* [1], all of the common geometric complexes, namely, the twisted Dirac operators, $\bar{\partial}$-operators, signature operators and the De Rham complex, are, locally, Dirac operators. We shall refer to all of these operators as Dirac operators, although this is not globally correct on non-spin manifolds. The index theorem for Dirac operators was first proven, at least for Kahler manifolds, by Hirzebruch using cobordism theory. A few years later, McKean and Singer gave their famous formula for the index of the Dirac operator:

$$\operatorname{Index}(D\!\!\!/) = \operatorname{Str} e^{tD\!\!\!/^2} (= \operatorname{Tr} e^{tD\!\!\!/^- D\!\!\!/^+} - \operatorname{Tr} e^{tD\!\!\!/^+ D\!\!\!/^-})$$
$$= \int_M \operatorname{str} \langle x | e^{tD\!\!\!/^2} | x \rangle \, dx,$$

and asked if the integrand converges as $t \to 0$, by some "fantastic cancellations". Notice that this is a completely local question, only depending on the metric and connection in a small neighbourhood of x, since we are sending t to 0. Thus, we may as well assume that our operator is a Dirac operator on $\mathbf{R}^n$, with twisting bundle $\mathbf{C}^m$. Patodi established this convergence and identified the limit; for a review of this phase of the history of the theorem, see Atiyah *et al.* [1].

In Getzler [0], motivated by the ideas of the physicists Witten and Alvarez-Gaume, it was shown that these cancellations are not fantastic at all, but quite natural, and for the first time, the local index theorem was proven in a completely analytic fashion, without appealing to any topological calculations as in earlier proofs. However, the machinery developed in that paper is rather general, so the goal of this paper is to show how simple the basic idea is.

The proof of the theorem splits into two steps:

(a) the asymptotic expansion of the heat kernel of $D\!\!\!/^2$;
(b) an algebraic calculation of the top order in the asymptotic expansion.

The second part is the same as in Getzler [0]; the only question is what is the clearest method of establishing the asymptotic expansion. In this paper, we use a new technique that requires the estimation of the heat kernel of uniformly elliptic operators on $\mathbf{R}^n$ whose first and zeroth order coefficients increase linearly and quadratically fast at infinity. This estimate is accomplished in the first appendix, by using the Feynman–Kac representation of the heat kernel.

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An alternative method is to use Patodi's original asymptotic expansion, which is technically more difficult to establish than anything else that we use, but is better known. Granted this, the proof becomes extraordinarily simple. This is dealt with briefly in the second appendix. (I wish to thank M. Vergne for suggesting this strategy after reading an earlier version of the paper.)

To state the theorem, we need some results from the theory of Clifford algebras. If n is an even integer, so that the space of spinors $\Delta = \Delta^+ \oplus \Delta^-$ has dimension $2^{n/2}$, then the endomorphism algebra of the spinors End(Δ) is the complex Clifford algebra on $\mathbf{R}^n$, which we shall write as Cliff(n). This algebra may be thought of as the exterior algebra $\Lambda(n) = \Lambda^*_{\mathbf{C}}\mathbf{R}^n$ with a twisted multiplication defined as follows. The vector space $\mathbf{R}^n$ acts on the exterior algebra $\Lambda(n) = \Lambda^*_{\mathbf{C}}\mathbf{R}^n$ by the formula

$$v \circ a = v \wedge a + v \lrcorner a, \quad \text{where } v \in \mathbf{R}^n \quad \text{and} \quad a \in \Lambda(n),$$

and this gives an isomorphism from Cliff(n) to $\Lambda(n)$, called the symbol map, with which we shall identify these spaces. If $a \in \Lambda(n)$ and $b \in \Lambda(n)$, then we have the formula,

$$a \circ b - a \wedge b \in \Lambda^{i+j-2}(n). \tag{1}$$

We will denote the projection of $a \in \Lambda(n)$ onto $\Lambda^m(n)$ by a_m. Recall that there is a isomorphism from the Lie algebra $o(n)$ to Cliff(n), given by sending the antisymmetric matrix a_{ij} to the element of Cliff(n) with symbol $1/2 \sum_{i<j} a_{ij} e^i \wedge e^j$. This is the isomorphism that is used to obtain the connection on the spinor bundle from the Reimannian connection.

Let g be a Riemannian metric on $\mathbf{R}^n$ satisfying the following three conditions:

(i) g is asymptotically Euclidean, so that $g_{ij}(x) = \delta_{ij}$ for $|x|$ large;
(ii) g is a small C^∞ perturbation of the Euclidean metric;
(iii) the coordinates are normal around $0 \in \mathbf{R}^n$, that is, the exponential map at the origin is an isomorphism.

Using parallel translation along the geodesics to the origin, the tangent and spinor bundles on $\mathbf{R}^n$ may be trivialized. Indices for tensors in this frame will be written with the indices $a, b \ldots$. For example, the spinor connection is

$$\Gamma_i \circ = \tfrac{1}{2}\Gamma_{iab}(\mathrm{e}^a \wedge \mathrm{e}^b) \circ .$$

In Atiyah *et al.* [1], it is shown that

$$\begin{aligned} g_{ij}(x) &= \delta_{ij} + O(|x|^2), \\ \Gamma_{iab} &= -\tfrac{1}{2}R_{ijab}(0)x^j(\mathrm{e}^a \wedge \mathrm{e}^b) \circ + O(|x|^2), \end{aligned} \tag{2}$$

where R_{ijab} is the Riemannian curvature of the metric g.

Let A_i be a connection on the bundle with fibre $\mathbf{C}^m$ on $\mathbf{R}^n$, satisfying conditions corresponding to (i)–(iii) above. Namely, $A_i(x)$ vanishes for $|x|$ large, is a small perturbation of the zero connection, and parallel translation along the geodesics out of the origin is the identity. (This is called the radial gauge.) We have

$$A_i = -\tfrac{1}{2}F_{ij}(0)x^j + O(|x|^2), \tag{3}$$

where F_{ij} is the curvature of the connection A_i.

If $\not{D}$ is the twisted Dirac operator for the metric g and connection A, and ∇_i is the covariant derivative

$$\nabla_i = \partial_i + \tfrac{1}{2}\Gamma_i \circ + A_i$$

then Lichnerowicz's formula (Lichnerowicz [6]) states that

$$\begin{aligned} D^2 &= \nabla^*\nabla + F \circ - \frac{R}{4} \\ &= g^{ij}(\nabla_i\nabla_j + \Gamma_{ij}^k\nabla_k) + F_{ab}(e^a \wedge e^b) \circ - \frac{R}{4}, \end{aligned}$$

where R is the scalar curvature for the metric g. This explicit formula is basic to our proof.

It follows from the Kato–Rellich theorem (Reed and Simon [10], p. 162) that D^2 is a small enough perturbation of the flat Laplacian Δ that it is self-adjoint on the domain of Δ. The Schwartz kernel $k_t(x, y)$ of the operator e^{tD^2}, which lies in

$$C^\infty((0,\infty) \times \mathbf{R}^n \times \mathbf{R}^n) \otimes \mathrm{End}(\Delta \otimes \mathbf{C}^m)$$

by parabolic regularity theory is called the heat kernel of D^2.

We shall denote by $\mathbf{k}_t(x) \in C^\infty((0,\infty)\times\mathbf{R}^n) \otimes \Lambda(n) \otimes \mathrm{End}(\mathbf{C}^m)$ the symbol of the heat kernel $k_t(x, 0) \in C^\infty((0,\infty)\times\mathbf{R}^n) \otimes \mathrm{Cliff}(n) \otimes \mathrm{End}(\mathbf{C}^m)$. Thinking of D^2 as an operator $\mathbf{D}$ on $C^\infty(\mathbf{R}^n) \otimes \Lambda(n) \otimes \mathrm{End}(\mathbf{C}^m)$, $\mathbf{k}_t(x)$ satisfies the heat equation

$$(\partial/\partial_t - \mathbf{D})\mathbf{k}_t(x) = 0, \qquad (4)$$

$$\lim_{t\to 0} \mathbf{k}_t(x) = \delta(x) \otimes 1.$$

Let T_ε be a rescaling operator defined by the following assignment of degrees (this was the main idea of the proof in Getzler [0]):

$$\deg(x_i) = 1, \qquad \deg(t) = 2, \qquad \deg(e_i) = -1.$$

That is, $T_\varepsilon(t^p x^q e^r) = \varepsilon^{2p+q-r} t^p x^q e^r$. Clearly, $\mathbf{k}_t^\varepsilon(x) = \varepsilon^n T_\varepsilon \mathbf{k}_t(x)$ is the solution of the heat equation (4) for the operator $\mathbf{D}^\varepsilon = \varepsilon^2 T_\varepsilon \mathbf{D}(T_\varepsilon)^{-1}$. This turns out to be enough to calculate $\lim_{\varepsilon\to 0} \mathbf{k}_\varepsilon(x)$.

THEOREM. *Let $\mathbf{k}_t^0(x) \in C^\infty((0,\infty)\times\mathbf{R}^n) \otimes \Lambda(n) \otimes \mathrm{End}(\mathbf{C}^m)$ be defined by the formula*

$$\mathbf{k}_t^0(x) = (4\pi t)^{-n/2}\hat{A}(t\Omega)\exp\left[tF - \frac{1}{4t}\left(\frac{t\Omega/2}{\tanh t\Omega/2}\right)_{ij} x^i x^j\right],$$

where the symbol Ω denotes the Riemannian curvature of the metric g at the point 0 thought of as an antisymmetric $n \times n$ matrix of 2-forms, F is the curvature of the connection B at 0 thought of as a hermitian $m \times m$ matrix of 2-forms, and the characteristic class $\hat{A}(\theta)$ is the $\hat{A}$-genus

$$\det\left[\frac{\theta/2}{\sinh\theta/2}\right]^{1/2}.$$

Then for all t small enough and $\delta > 0$,

$$|\mathbf{k}_t^\varepsilon(x) - \mathbf{k}_t^0(x)| \le c\varepsilon t^{-n/2+1-\delta} e^{-a|x|^2/t}.$$

Proof. The basic idea of the proof is the observation that as $\varepsilon \to 0$, $\mathbf{D}^\varepsilon$ converges to the operator

$$\mathbf{D}^0 = \sum_{i=1}^{2n} (\partial_i - \tfrac{1}{4}\Omega_{ij}x^j \wedge)^2 + F \wedge.$$

for which the solution of (4) is precisely $\mathbf{k}_t^0(x)$. Using an *a priori* bound on $\mathbf{k}_t^\varepsilon(x)$, it is then easy to prove the uniform convergence. At $\varepsilon > 0$, the noncommutative Clifford multiplication $a \circ b$

is replaced by the product $a \circ_\varepsilon b$, defined by

$$v \circ_\varepsilon a = v \wedge a + \varepsilon^2 v \lrcorner a, \quad \text{where } v \in \mathbf{R}^n \text{ and } a \in \Lambda(n).$$

The limit $\varepsilon \to 0$ is like the classical limit in quantum theory; as $\varepsilon \to 0$, the noncommutative product $a \circ_\varepsilon b$ converges to $a \wedge b$.

Using Lichnerowicz's formula, it is easy to show that $\mathbf{D}^\varepsilon$ has the following form:

$$\mathbf{D}^\varepsilon = g^{ij}(\varepsilon x)((\partial_i + \varepsilon^{-1}\Gamma_i(\varepsilon x)\circ_\varepsilon + \varepsilon A_i(\varepsilon x))(\partial_j + \varepsilon^{-1}\Gamma_j(\varepsilon x)\circ_\varepsilon + \varepsilon A_j(\varepsilon x))$$
$$+ \Gamma_{ij}^k(\varepsilon x)(\partial_k + \varepsilon^{-1}\Gamma_k(\varepsilon x)\circ_\varepsilon + \varepsilon^{-1}A_k(\varepsilon x))) + F_{ab}(\varepsilon x)(\mathrm{e}^a \wedge \mathrm{e}^b)\circ_\varepsilon - \frac{\varepsilon^2 R}{4}.$$

This shows that the coefficients b_i^ε and c^ε of the first and zeroth derivatives in $\mathbf{D}^\varepsilon$ are bounded uniformly in ε for t small enough:

$$|b_i^\varepsilon(x)| \leq c(1+|x|), \tag{5}$$

$$|c^\varepsilon(x)| \leq c(1+|x|)^2.$$

Using the Taylor expansions for g and A, it follows that for small t:

$$|(\mathbf{D}^\varepsilon - \mathbf{D}^0)k_t^0(x)| \leq c\varepsilon t^{-n/2}\mathrm{e}^{-a|x|^2/t}. \tag{6}$$

We will only prove this for the terms involving ∂^2, which should be enough to give an idea of how the inequality is proved. The terms in question may be bounded for small t by $|g(\varepsilon x) - g(0)|.O(t^{-1} + |x|^2/t^2).t^{-n/2}\mathrm{e}^{-|x|^2/4t}$. For $|x| > \varepsilon^{-1}$, the inequality holds because the exponential factor decreases exponentially in t^{-1}, while for $|x| < \varepsilon^{-1}$, we can bound $|g(\varepsilon x) - g(0)|$ by $\varepsilon^2|x|^2$, obtaining

$$t^{-n/2}.O(|x|^2/t + |x|^4/t^2).\mathrm{e}^{-|x|^2/4t} \leq t^{-n/2}\mathrm{e}^{-|x|^2/8}.$$

We now show that the solution to the heat equation (4) for $\mathbf{D}^0$ is $\mathbf{k}_t^0(x)$. By Mehler's formula (Glimm and Jaffe [4], p. 19), the heat kernel $\langle x|\mathrm{e}^{-tH}|y\rangle$ of the one dimensional harmonic oscillator $H = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + a^2x^2$ equals

$$(4\pi t)^{-1/2}\left(\frac{2at}{\sinh 2at}\right)^{1/2}\exp -\frac{1}{4t}\left[\frac{2at}{\sinh 2at}(\cosh 2at\,(x^2+y^2) - 2xy)\right].$$

If θ is an $m \times m$ positive matrix, then the heat kernel of the m-dimensional harmonic oscillator $-\Delta + \theta_{ij}x^ix^j$ is

$$(4\pi t)^{-m/2}\det\left[\frac{2t\sqrt{\theta}}{\sinh 2t\sqrt{\theta}}\right]^{1/2}\exp -\frac{1}{4t}\left[\left(\frac{2t\sqrt{\theta}}{\tanh 2t\sqrt{\theta}}\right)_{ij}(x^ix^j + y^iy^j) - 2\left(\frac{2t\sqrt{\theta}}{\sinh 2t\sqrt{\theta}}\right)_{ij}x^iy^j\right].$$

In the expansion of this formula as a power series in $\sqrt{\theta}$ near $\theta = 0$, only the even powers contribute, so the heat kernel is actually analytic in θ. We will use this formula in a setting in which θ is a matrix of 2-forms; this uses the same analytic continuation from complex variables to even differential forms that is used to define characteristic classes in differential geometry.

Observe that the operator $\mathbf{D}^0$ is equal to the sum of the two commuting operators $K = \Delta - \frac{1}{16}\Omega_{ij}\Omega_{jk}x^ix^k$ and $L = F - \frac{1}{4}\Omega_{ij}\partial^ix^j$. We have calculated the heat kernel of K above (let $\theta = \frac{1}{16}\Omega_{ij}\Omega_{jk}x^ix^k$), so the result follows from the formula $\mathbf{k}_t^0 = (\mathrm{e}^{tK}\mathrm{e}^{tL})(\delta(x)\otimes 1)$, and the fact that $\mathrm{e}^{tL}(\delta(x)\otimes 1) = \mathrm{e}^{tF}(\delta(x)\otimes 1)$.

To prove that $\mathbf{k}_t^\varepsilon(x)$ converges to $\mathbf{k}_t^0(x)$ as $\varepsilon \to 0$, we use Duhamel's formula, which states that

$$\mathbf{k}_t^\varepsilon(x) - \mathbf{k}_t^0(x) = \int_0^t ds\, e^{-(t-s)\mathbf{D}^\varepsilon}(\mathbf{D}^\varepsilon - \mathbf{D}^0)\mathbf{k}_s^0(x).$$

This follows from the operator identity

$$e^{tA}e^{-tB} = 1 + \int_0^t ds\, e^{sA}(A-B)e^{-sB}.$$

The theorem now follows from Theorem A.1 in Appendix A, which applies uniformly to the operators $\mathbf{D}^\varepsilon$, $0 < \varepsilon < 1$, and the bound in formula (6). □

If a is an element of Cliff($\mathbf{R}^n$), then the supertrace of a, denoted Str a, is defined to be $\mathrm{Tr}|_{\Delta^+} a - Tr|_{\Delta^-} a$, where a is identified with an endomorphism of the spinors $\Delta = \Delta^+ \oplus \Delta^-$. In terms of the gradation of Cliff(n), we have the following suggestive formula for Str a:

$$\mathrm{Str}\, a = (2/i)^{n/2} \textstyle\int a,$$

where the linear map $\int : \Lambda(n) \to \mathbf{C}$ (Berezin's integral) is given by taking the inner product of a_n with the volume element in $\Lambda^n\mathbf{R}^n$. A proof may be found in [0].

From this formula for the supertrace and the theorem that we have proved, it follows that

$$\lim_{t\to 0} \mathrm{Str}\, k_t(0,0) = \lim_{\varepsilon\to 0} (2/i)^{n/2} \textstyle\int \mathbf{k}_t^\varepsilon(0) = (2\pi i)^{-n/2}(\hat{A}(\Omega)\,\mathrm{ch}\,(F))_n.$$

This is the local Atiyah–Singer index theorem.

APPENDIX A

In this appendix, we prove the theorem on the heat kernel for elliptic operators that was used in the proof of the index theorem.

THEOREM A.1. *Let D be a second order elliptic operator on $\mathbf{R}^n$ given by the formula*

$$D = \frac{1}{2}\Delta + \sum_{i=1}^m b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

where Δ is the Laplacian for a Riemannian metric g on $\mathbf{R}^n$ such that $g(x)$ and $g(x)^{-1}$ are both uniformly bounded, and b_i and c are bounded continuous functions from $\mathbf{R}^n$ to $m \times m$ complex matrices satisfying the bounds

$$|b_i(x)| \le c(1+|x|),$$

$$|c(x)| \le c(1+|x|)^2.$$

Then for t small enough, the heat kernel $k_t(x,y)$ of D satisfies the following bound for any $\delta > 0$:

$$\int_0^t ds \int |k_{s-t}(x,y)s^{-n/2}e^{-|y|^2/s}|dy \le ct^{-n/2+1-\delta}e^{-a|x|^2/t}.$$

Proof. We use Stroock's version of the Feynman–Kac formula [11]. Let Ω_x be the set of all paths $\omega: [0,\infty) \to \mathbf{R}^n$ such that $\omega(0) = x$, let E_x be the Brownian expectation on Ω_x corresponding to the metric g. If B is a ball in $\mathbf{R}^n$, let b_B and c_B be the functions b and c multiplied by the characteristic function of B. If $\Phi: \Omega_x x[0,\infty) \to$ End ($\mathbf{C}^m$) is the solution of the stochastic integral equation

$$\Phi(\omega,t) = 1 + \int_0^t (\Phi(\omega,\tau)b(\omega(\tau)), d\omega(\tau)) + \int_0^t \Phi(\omega,\tau)c(\omega(\tau))d\tau,$$

then the heat kernel of D satisfies

$$\int k_t(x, y) f(y)\,dy = E_x[\Phi(\omega, t) f(\omega(t))].$$

In particular, $\int p_t(x, y) f(y)\,dy = E_x[f(\omega(t))]$, where $p_t(x, y)$ is the heat kernel of the Laplacian Δ for the metric g.

Furthermore, the following estimates hold:

(a) $E_x[|\Phi(\omega,t)|^p] \le c\,e^{O(\|b\|^2+\|c\|)t}$ for $p \ge 2$, where $\|\cdot\|$ denotes the uniform norm,

(b) (McKean [7], p. 93) $P_x\left(\max_{0\le\tau\le t} |\omega(\tau) - x| \ge R\right) \le c\,e^{-aR^2/t}$.

(c) (Baldi [2]) $p_t(x, y) \le c\,t^{-n/2} e^{-a|x|^2/t}$ for some positive constants a and c.

In proving the theorem, we divide the set of paths starting at x into the countable partition $U_m = \left\{\omega \mid \omega(0) = x,\ \sqrt{m} \le \max_{0\le\tau\le t-s} |\omega(\tau) - x| \le \sqrt{m+1}\right\}$, $m \ge 0$, and bound the expectation of $\Phi(\omega, t-s)$ on each of these sets. On the set U_m, we estimate the expectation of Φ using (a) and (b) above, by making use of the bounds on $\|b\|$ and $\|c\|$ on the ball around x of radius $\sqrt{m+1}$. Using the Feynman–Kac formula, we have all for $p > 1$

$$\begin{aligned}\int |k_{t-s}(x, y) s^{-n/2} e^{-|y|^2/s}|\,dy &\le \sum_{m=0}^{\infty} E_x[s^{-n/2} e^{-|\omega(t-s)|^2/s} \chi(U_m) |\Phi(w, t-s)|]\\ &\le E_x[(s^{-n/2} e^{-|\omega(t-s)|^2/s})^p]^{1/p} E_x[\chi(U_m)]^{1/2p'} E_x[|\Phi(\omega, t-s)|^{2p'}]^{1/2p'}\\ &\le c\left[\int p_{t-s}(x, y) s^{-np/2} e^{-p|y|^2/s}\,dy\right]^{1/p} \sum_{m=0}^{\infty} e^{-am/(t-s)} e^{O(|x|^2+m)(t-s)}\\ &\le c' s^{-n/2p'} e^{-a|x|^2/t}, \quad \text{for } t \text{ small enough.}\end{aligned}$$

Integrating over s gives the stated result, since we can choose p' arbitrarily large. □

APPENDIX B

In this appendix, we give another method of proving the local index theorem, based on Patodi [8]. Using Patodi's asymptotic expansion for the solution of the heat equation (4) around $t = 0$, we may write

$$\mathbf{k}_t^\varepsilon(x) \sim (4\pi t)^{-n/2} e^{-|x|^2/4t} \sum_{\alpha, \beta, m \ge 0} c_{m\alpha\beta}^\varepsilon t^m x^\alpha e^\beta$$

where α is a multi-index for the symmetric algebra $S^*(\mathbf{R}^n)$ and β is a multi-index for the antisymmetric algebra $\Lambda^*(\mathbf{R}^n)$.

It is immediate from Patodi's proof of the asymptotic expansion that the numbers $c_{m\alpha\beta}^\varepsilon$ are universal polynomials in the derivatives of the coefficients of $\mathbf{D}^\varepsilon$ at $x = 0$. It follows from the explicit formula for $\mathbf{D}^\varepsilon$ that

$$c_{m\alpha\beta}^\varepsilon = c_{m\alpha\beta}^0 + O(\varepsilon).$$

The proof of the local index theorem follows from the calculation of $\mathbf{k}_t^0(x)$ in the body of the paper.

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