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GENERALIZED POINCARÉ’S CONJECTURE
IN DIMENSIONS GREATER THAN FOUR

BY STEPHEN SMALE*

(Received October 11, 1960)
(Revised March 27, 1961)

Poincaré has posed the problem as to whether every simply connected closed 3-manifold (triangulated) is homeomorphic to the 3-sphere, see [18] for example. This problem, still open, is usually called Poincaré’s conjecture. The generalized Poincaré conjecture (see [11] or [28] for example) says that every closed \( n \)-manifold which has the homotopy type of the \( n \)-sphere \( S^n \) is homeomorphic to the \( n \)-sphere. One object of this paper is to prove that this is indeed the case if \( n \geq 5 \) (for differentiable manifolds in the following theorem and combinatorial manifolds in Theorem B).

**THEOREM A.** Let \( M^n \) be a closed \( C^\infty \) manifold which has the homotopy type of \( S^n \), \( n \geq 5 \). Then \( M^n \) is homeomorphic to \( S^n \).

Theorem A and many of the other theorems of this paper were announced in [20]. This work is written from the point of view of differential topology, but we are also able to obtain the combinatorial version of Theorem A.

**THEOREM B.** Let \( M^n \) be a combinatorial manifold which has the homotopy of \( S^n \), \( n \geq 5 \). Then \( M^n \) is homeomorphic to \( S^n \).

J. Stallings has obtained a proof of Theorem B (and hence Theorem A) for \( n \geq 7 \) using different methods (Polyhedral homotopy-spheres, Bull. Amer. Math. Soc., 66 (1960), 485–488).

The basic theorems of this paper, Theorems C and I below, are much stronger than Theorem A.

A *nice* function \( f \) on a closed \( C^\infty \) manifold is a \( C^\infty \) function with non-degenerate critical points and, at each critical point \( \beta \), \( f(\beta) \) equals the index of \( \beta \). These functions were studied in [21].

**THEOREM C.** Let \( M^n \) be a closed \( C^\infty \) manifold which is \((m - 1)\)-connected, and \( n \geq 2m \), \((n, m) \neq (4, 2)\). Then there is a nice function \( f \) on \( M \) with type numbers satisfying \( M_0 = M_n = 1 \) and \( M_i = 0 \) for \( 0 < i < m \), \( n - m < i < n \).

Theorem C can be interpreted as stating that a cellular structure can be imposed on \( M^n \) with one 0-cell, one \( n \)-cell and no cells in the range \( 0 < i < m \), \( n - m < i < n \). We will give some implications of Theorem C.

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First, by letting \( m = 1 \) in Theorem C, we obtain a recent theorem of M. Morse [13].

**Theorem D.** Let \( M^n \) be a closed connected \( C^\infty \) manifold. There exists a (nice) non-degenerate function on \( M \) with just one local maximum and one local minimum.

In § 1, the handlebodies, elements of \( \mathcal{H}(n, k, s) \) are defined. Roughly speaking if \( H \in \mathcal{H}(n, k, s) \), then \( H \) is defined by attaching \( s \)-disks, \( k \) in number, to the \( n \)-disk and "thickening" them. By taking \( n = 2m + 1 \) in Theorem C, we will prove the following theorem, which in the case of 3-dimensional manifolds gives the well known Heegaard decomposition.

**Theorem F.** Let \( M \) be a closed \( C^\infty (2m + 1) \)-manifold which is \((m - 1)\)-connected. Then \( M = H \cup H' \), \( H \cap H' = \partial H = \partial H' \) where \( H, H' \in \mathcal{H}(2m + 1, k, m) \) are handlebodies (\( \partial V \) means the boundary of the manifold \( V \)).

By taking \( n = 2m \) in Theorem C, we will get the following.

**Theorem G.** Let \( M^{2m} \) be a closed \((m - 1)\)-connected \( C^\infty \) manifold, \( m \neq 2 \). Then there is a nice function on \( M \) whose type numbers equal the corresponding Betti numbers of \( M \). Furthermore \( M \), with the interior of a \( 2m \)-disk deleted, is a handlebody, an element of \( \mathcal{H}(2m, k, m) \) where \( k \) is the \( m \)th Betti number of \( M \).

Note that the first part of Theorem G is an immediate consequence of the Morse relation that the Euler characteristic is the alternating sum of the type numbers [12], and Theorem C.

The following is a special case of Theorem G.

**Theorem H.** Let \( M^{2m} \) be a closed \( C^\infty \) manifold \( m \neq 2 \) of the homotopy type of \( S^{2m} \). Then there exists on \( M \) a non-degenerate function with one maximum, one minimum, and no other critical point. Thus \( M \) is the union of two \( 2m \)-disks whose intersection is a submanifold of \( M \), diffeomorphic to \( S^{2m-1} \).

Theorem H implies the part of Theorem A for even dimensional homotopy spheres.

Two closed \( C^\infty \) oriented \( n \)-dimensional manifolds \( M \) and \( M' \) are \( J \)-equivalent (according to Thom, see [25] or [10]) if there exists an oriented manifold \( V \) with \( \partial V \) diffeomorphic to the disjoint union of \( M \) and \( -M' \), and each \( M' \) is a deformation retract of \( V \).

**Theorem I.** Let \( M_i \) and \( M \) be \((m - 1)\)-connected oriented closed \( C\infty (2m + 1) \)-dimensional manifolds which are \( J \)-equivalent, \( m \neq 1 \). Then \( M_i \) and \( M \) are diffeomorphic.
We obtain an orientation preserving diffeomorphism. If one takes $M_i$ and $M_j$ $J$-equivalent disregarding orientation, one finds that $M_i$ and $M_j$ are diffeomorphic.

In studying manifolds under the relation of $J$-equivalence, one can use the methods of cobordism and homotopy theory, both of which are fairly well developed. The importance of Theorem I is that it reduces diffeomorphism problems to $J$-equivalence problems for a certain class of manifolds. It is an open question as to whether arbitrary $J$-equivalent manifolds are diffeomorphic (see [10, Problem 5]). (Since this was written, Milnor has found a counter-example).

A short argument of Milnor [10, p. 33] using Mazur’s theorem [7] applied to Theorem I yields the odd dimensional part of Theorem A. In fact it implies that, if $M^{2m+1}$ is a homotopy sphere, $m \neq 1$, then $M^{2m+1}$ minus a point is diffeomorphic to euclidean $(2m+1)$-space (see also [9, p. 440]).

Milnor [10] has defined a group $\mathcal{H}^n$ of $C^\infty$ homotopy $n$-spheres under the relation of $J$-equivalence. From Theorems A and I, and the work of Milnor [10] and Kervaire [5], the following is an immediate consequence.

**Theorem J.** If $n$ is odd, $n \neq 3$, $\mathcal{H}^n$ is the group of classes of all differentiable structures on $S^n$ under the equivalence of diffeomorphism. For $n$ odd there are a finite number of differentiable structures on $S^n$.

**For example:**

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Differentiable Structures on $S^n$</td>
<td>0</td>
<td>0</td>
<td>28</td>
<td>8</td>
<td>992</td>
<td>3</td>
<td>16256</td>
</tr>
</tbody>
</table>

Previously it was known that there are a countable number of differentiable structures on $S^n$ for all $n$ (Thom), see also [9, p. 442]; and unique structures on $S^n$ for $n \leq 3$ (e.g., Munkres [14]). Milnor [8] has also established lower bounds for the number of differentiable structures on $S^n$ for several values of $n$.

A group $\Gamma^n$ has been defined by Thom [24] (see also Munkres [14] and Milnor [9]). This is the group of all diffeomorphisms of $S^{n-1}$ modulo those which can be extended to the $n$-disk. A group $A^n$ has been studied by Milnor as those structures on the $n$-sphere which, minus a point, are diffeomorphic to euclidean space [9]. The group $\Gamma^n$ can be interpreted (by Thom [22] or Munkres [14]) as the group of differentiable structures on $S^n$ which admit a $C^\infty$ function with the non-degenerate critical points, and hence one has the inclusion map $i: \Gamma^n \rightarrow A^n$ defined. Also, by taking $J$-equivalence classes, one gets a map $p: A^n \rightarrow \mathcal{H}^n$. 
THEOREM K. With notation as in the preceding paragraph, the following sequences are exact:

(a) \[ A^n \xrightarrow{p} \mathcal{A}^n \rightarrow 0, \quad n \neq 3, 4 \]
(b) \[ \Gamma^n \xrightarrow{i} A^n \rightarrow 0, \quad n \text{ even} \neq 4 \]
(c) \[ 0 \rightarrow A^n \xrightarrow{p} \mathcal{H}^n, \quad n \text{ odd} \neq 3. \]

Hence, if \( n \) is even, \( n \neq 4 \), \( \Gamma^n = A^n \) and, if \( n \) is odd \( \neq 3 \), \( A^n = \mathcal{H}^n \).

Here (a) follows from Theorem A, (b) from Theorem H, and (c) from Theorem I.

Kervaire [4] has also obtained the following result.

THEOREM L. There exists a manifold with no differentiable structure at all.

Take the manifold \( W_0 \) of Theorem 4.1 of Milnor [10] for \( k = 3 \). Milnor shows \( \partial W_0 \) is a homotopy sphere. By Theorem A, \( \partial W_0 \) is homeomorphic to \( S^{11} \). We can attach a 12-disk to \( W_0 \) by a homeomorphism of the boundary onto \( \partial W_0 \) to obtain a closed 12 dimensional manifold \( M \). Starting with a triangulation of \( W_0 \), one can easily obtain a triangulation of \( M \). If \( M \) possessed a differentiable structure it would be almost parallelizable, since the obstruction to almost parallelizability lies in \( H^q(M, \pi_3(SO(12))) = 0 \). But the index of \( M \) is 8 and hence by Lemma 3.7 of [10] \( M \) cannot possess any differentiable structure. Using Bott’s results on the homotopy groups of Lie groups [1], one can similarly obtain manifolds of arbitrarily high dimension without a differentiable structure.

THEOREM M. Let \( C^{2m} \) be a contractible manifold, \( m \neq 2 \), whose boundary is simply connected. Then \( C^{2m} \) is diffeomorphic to the \( 2m \)-disk. This implies that differentiable structures on disks of dimension \( 2m \), \( m \neq 2 \), are unique. Also the closure of the bounded component \( C \) of a \( C^\infty \) imbedded \((2m - 1)\)-sphere in euclidean \( 2m \)-space, \( m \neq 2 \), is diffeomorphic to a disk.

For these dimensions, the last statement of Theorem M is a strong version of the Schoenflies problem for the differentiable case. Mazur’s theorem [7] had already implied \( C \) was homeomorphic to the \( 2m \)-disk.

Theorem M is proved as follows from Theorems C and I. By Poincaré duality and the homology sequence of the pair \((C, \partial C)\), it follows that \( \partial C \) is a homotopy sphere and \( J \)-equivalent to zero since it bounds \( C \). By Theorem I, then, \( \partial C \) is diffeomorphic to \( S^n \). Now attach to \( C^{2m} \) a \( 2m \)-disk by a diffeomorphism of the boundary to obtain a differentiable manifold \( V \). One shows easily that \( V \) is a homotopy sphere and, hence by Theorem H, \( V \) is the union of two \( 2m \)-disks. Since any two \( 2m \) sub-disks of \( V \) are
equivalent under a diffeomorphism of $V$ (for example see Palais [17]), the original $C^{2m} \subset V$ must already have been diffeomorphic to the standard $2m$-disk.

To prove Theorem B, note that $V = (M$ with the interior of a simplex deleted) is a contractible manifold, and hence possesses a differentiable structure [Munkres 15]. The double $W$ of $V$ is a differentiable manifold which has the homotopy type of a sphere. Hence by Theorem A, $W$ is a topological sphere. Then according to Mazur [7], $\partial V$, being a differentiable submanifold and a topological sphere, divides $W$ into two topological cells. Thus $V$ is topologically a cell and $M$ a topological sphere.

**Theorem N.** Let $C^{2m}$, $m \neq 2$, be a contractible combinatorial manifold whose boundary is simply connected. Then $C^{2m}$ is combinatorially equivalent to a simplex. Hence the Hauptvermutung (see [11]) holds for combinatorial manifolds which are closed cells in these dimensions.

To prove Theorem N, one first applies a recent result of M. W. Hirsch [3] to obtain a compatible differentiable structure on $C^{2m}$. By Theorem M, this differentiable structure is diffeomorphic to the $2m$-disk $D^{2m}$. Since the standard $2m$-simplex $\sigma^{2m}$ is a $C^1$ triangulation of $D^{2m}$, Whitehead's theorem [27] applies to yield that $C^{2m}$ must be combinatorially equivalent to $\sigma^{2m}$.

Milnor first pointed out that the following theorem was a consequence of this theory.

**Theorem O.** Let $M^{2m}$, $m \neq 2$, be a combinatorial manifold which has the same homotopy type as $S^{2m}$. Then $M^{2m}$ is combinatorially equivalent to $S^{2m}$. Hence, in these dimensions, the Hauptvermutung holds for spheres.

For even dimensions greater than four, Theorems N and O improve recent results of Gluck [2].

Theorem O is proved by applying Theorem N to the complement of the interior of a simplex of $M^{2m}$.

Our program is the following. We introduce handlebodies, and then prove "the handlebody theorem" and a variant. These are used together with a theorem on the existence of "nice functions" from [21] to prove Theorems C and I, the basic theorems of the paper. After that, it remains only to finish the proof of Theorems F and G of the Introduction.

The proofs of Theorems C and I are similar. Although they use a fair amount of the technique of differential topology, they are, in a certain sense, elementary. It is in their application that we use many recent results.
A slightly different version of this work was mimeographed in May 1960. In this paper J. Stallings pointed out a gap in the proof of the handlebody theorem (for the case \( s = 1 \)). This gap happened not to affect our main theorems.

Everything will be considered from the \( C^\infty \) point of view. All imbeddings will be \( C^\infty \). A differentiable isotopy is a homotopy of imbeddings with continuous differential.

\[
E^n = \{ x = (x_1, \cdots, x_n) \}, \quad ||x|| = (\sum_{i=1}^n x_i^2)^{1/2},
\]
\[
D^n = \{ x \in E^n | ||x|| \leq 1 \}, \quad \partial D^n = S^{n-1} = \{ x \in E^n | ||x|| = 1 \};
\]
\[
D^n_i \text{ etc. are copies of } D^n.
\]

A. Wallace’s recent article [26] is related to some of this paper.

1. Let \( M^n \) be a compact manifold, \( Q \) a component of \( \partial M \) and

\[
f_i: \partial D^n_i \times D_i^{n-s} \rightarrow Q, \quad i = 1, \cdots, k
\]

imbeddings with disjoint images, \( s \geq 0, n \geq s \). We define a new compact \( C^\infty \) manifold \( V = \chi(M, Q; f_i, \cdots, f_k; s) \) as follows. The underlying topological space of \( V \) is obtained from \( M \), and the \( D^n_i \times D_i^{n-s} \) by identifying points which correspond under some \( f_i \). The manifold thus defined has a natural differentiable structure except along corners \( \partial D^n_i \times \partial D_i^{n-s} \) for each \( i \). The differentiable structure we put on \( V \) is obtained by the process of “straightening the angle” along these corners. This is carried out in Milnor [10] for the case of the product of manifolds \( W_i \) and \( W_2 \) with a corner along \( \partial W_i \times \partial W_2 \). Since the local situation for the two cases is essentially the same, his construction applies to give a differentiable structure on \( V \). He shows that this structure is well-defined up to diffeomorphism.

If \( Q = \partial M \) we omit it from the notation \( \chi(M, Q; f_i, \cdots, f_k; s) \), and we sometimes also omit the \( s \). We can consider the “handle” \( D^n_i \times D_i^{n-s} \subset V \) as differentially imbedded.

The next lemma is a consequence of the definition.

(1.1) Lemma. Let \( f_i: \partial D^n_i \times D_i^{n-s} \rightarrow Q \) and \( f_i': \partial D^n_i \times D_i^{n-s} \rightarrow Q, \quad i = 1, \cdots, k \) be two sets of imbeddings each with disjoint images, \( Q, M \) as above. Then \( \chi(M, Q; f_1, \cdots, f_k; s) \) and \( \chi(M, Q; f_1', \cdots, f_k'; s) \) are diffeomorphic if

(a) there is a diffeomorphism \( h: M \rightarrow M \) such that \( f_i' = hf_i, \quad i = 1, \cdots, k \); or

(b) there exist diffeomorphisms \( h_i: D^n \times D^{n-s} \rightarrow D^n \times D^{n-s} \) such that \( f_i' = f_i h_i, \quad i = 1, \cdots, k \); or

(c) the \( f_i' \) are permutations of the \( f_i \).

If \( V \) is the manifold \( \chi(M, Q; f_1, \cdots, f_k; s) \), we say \( \sigma = (M, Q; f_1, \cdots, f_k; s) \)
is a presentation of $V$.

A handlebody is a manifold which has a presentation of the form $(D^n; f_1, \ldots, f_k; s)$. Fixing $n$, $k$, $s$ the set of all handlebodies is denoted by $\mathcal{H}(n, k, s)$. For example, $\mathcal{H}(n, k, 0)$ consists of one element, the disjoint union of $(k+1)$ $n$-disks; and one can show $\mathcal{H}(2, 1, 1)$ consists of $S^1 \times I$ and the Möbius strip, and $\mathcal{H}(3, k, 1)$ consists of the classical handlebodies [19; Henkelkörper], orientable and non-orientable, or at least differentiable analogues of them. The following is one of the main theorems used in the proof of Theorem C. An analogue in §5 is used for Theorem I.

(1.2) **Handlebody Theorem.** Let $n \geq 2s + 2$ and, if $s = 1$, $n \geq 5$; let $H \in \mathcal{H}(n, k, s)$, $V = \chi(H; f_1, \ldots, f_r; s + 1)$, and $\pi_s(V) = 0$. Also, if $s = 1$, assume $\pi_s(\chi(H; f_1, \ldots, f_r; s + 1)) = 1$. Then $V \in \mathcal{H}(n, r - k, s + 1)$. (We do not know if the special assumption for $s = 1$ is necessary.)

The next three sections are devoted to a proof of (1.2).

2. Let $G_r = G_r(s)$ be the free group on $r$ generators $D_1, \ldots, D_r$ if $s = 1$, and the free abelian group on $r$ generators $D_1, \ldots, D_r$ if $s > 1$. If $\sigma = (M, Q; f_1, \ldots, f_r; s + 1)$ is a presentation of a manifold $V$, define a homomorphism $f_\sigma: G_r \to \pi_s(Q)$ by $f_\sigma(D_i) = \varphi_i$, where $\varphi_i \in \pi_s(Q)$ is the homotopy class of $f_i: \partial D^{s+1} \times 0 \to Q$, the restriction of $f_i$. To take care of base points in case $\pi_s(Q) \neq 1$, we will fix $x_0 \in \partial D^{s+1} \times 0$, $y_0 \in Q$, Let $U$ be some cell neighborhood of $y_0$ in $Q$, and assume $f_i(x_0) \in U$. We say that the homomorphism $f_\sigma$ is induced by the presentation $\sigma$.

Suppose now that $F: G_r \to \pi_s(Q)$ is a homomorphism where $Q$ is a component of the boundary of a compact $n$-manifold $M$. Then we say that a manifold $V$ realizes $F$ if some presentation of $V$ induces $F$. Manifolds realizing a given homomorphism are not necessarily unique.

The following theorem is the goal of this section.

(2.1) **Theorem.** Let $n \geq 2s + 2$, and if $s = 1$, $n \geq 5$; let $\sigma = (M, Q; f_1, \ldots, f_r; s + 1)$ be a presentation of a manifold $V$, and assume $\pi_s(Q) = 1$ if $n = 2s + 2$. Then for any automorphism $\alpha: G_r \to G_r$, $V$ realizes $f_\sigma \alpha$.

Our proof of (2.1) is valid for $s = 1$, but we have application for the theorem only for $s > 1$. For the proof we will need some lemmas.

(2.2) **Lemma.** Let $Q$ be a component of the boundary of a compact manifold $M^n$ and $f_i: \partial D^i \times D^{n-i} \to Q$ an imbedding. Let $\tilde{f}_2: \partial D^i \times 0 \to Q$ be an imbedding, differentially isotopic in $Q$ to the restriction $\tilde{f}_i$ of $f_i$ to $\partial D^i \times 0$. Then there exists an imbedding $f_2: \partial D^i \times D^{n-i} \to Q$ extending $\tilde{f}_2$ and a diffeomorphism $h: M \to M$ such that $hf_i = f_i$. 
PROOF. Let \( \bar{f}_t : \partial D^s \times 0 \to Q, 1 \leq t \leq 2, \) be a differentiable isotopy between \( \bar{f}_1 \) and \( \bar{f}_2. \) Then by the covering homotopy property for spaces of differentiable imbeddings (see Thom [23] and R. Palais, Comment. Math. Helv. 34 (1960)), there is a differentiable isotopy \( F_t : \partial D^s \times D^{n-s} \to Q, 1 \leq t \leq 2, \) with \( F_t = f_t \) and \( F_t \) restricted to \( \partial D^s \times 0 = \{ \bar{f}_t \}. \) Now by applying this theorem again, we obtain a differentiable isotopy \( G_t : M \to M, 1 \leq t \leq 2, \) with \( G_t \) equal the identity, and \( G_t \) restricted to image \( F_t = F_t^{-1}F_{t+1}. \) Then taking \( h = G_2^{-1}, F_t \) satisfies the requirements of \( f_t \) of (2.2); i.e., \( hF_2 = G_2^{-1}F_2 = F_1F_2^{-1}F_2 = f_1. \)

(2.3) THEOREM (H. Whitney, W.T. Wu). Let \( n \geq \max \{ 2k+1, 4 \} \) and \( f, g : M^k \to X^n \) be two imbeddings, \( M \) closed, \( M \) connected and \( X \) simply connected if \( n = 2k+1. \) Then, if \( f \) and \( g \) are homotopic, they are differentiably isotopic.

Whitney [29] proved (2.3) for the case \( n \geq 2k+2. \) W. T. Wu [30] (using methods of Whitney) proved it where \( X^n \) was euclidean space, \( n = 2k+1. \) His proof also yields (2.3) as stated.

(2.4) LEMMA. Let \( Q \) be a component of the boundary of a compact manifold \( M^n, n \geq 2s+2 \) and if \( s = 1, n \geq 5, \) and \( \pi_1(Q) = 1 \) if \( n = 2s+2. \) Let \( f_1 : \partial D^{s+1} \times D^{n-s-1} \to Q \) be an imbedding, and \( f_2 : \partial D^{s+1} \times 0 \to Q \) an imbedding homotopic in \( Q \) to \( f_1, \) the restriction of \( f_2 \) to \( \partial D^{s+1} \times 0. \) Then there exists an imbedding \( f_2 : \partial D^{s+1} \times D^{n-s-1} \to Q \) extending \( f_2 \) such that \( \chi(M, Q; f_i) \) is diffeomorphic to \( \chi(M, Q; f_i). \)

PROOF. By (2.3), there exists a differentiable isotopy between \( f_1 \) and \( f_2. \) Apply (2.2) to get \( f_2 : \partial D^{s+1} \times D^{n-s-1} \to Q \) extending \( f_2, \) and a diffeomorphism \( h : M \to M \) with \( h f_2 = f_1. \) Application of (1.1) yields the desired conclusion.

See [16] for the following.

(2.5) LEMMA (Nielson). Let \( G \) be a free group on \( r \) generators \( \{ D_1, \ldots, D_r \}, \) and \( \mathcal{A} \) the group of automorphisms of \( G. \) Then \( \mathcal{A} \) is generated by the following automorphisms:

\[
R : D_i \to D_i^{-1}, \quad D_i \to D_i \quad i > 1
\]

\[
T_j : D_i \to D_i, \quad D_i \to D_j, \quad D_j \to D_j \quad j \neq 1, j \neq i, i = 2, \ldots, r
\]

\[
S : D_i \to D_iD_i, \quad D_i \to D_i \quad i > 1.
\]

The same is true for the free abelian case (well-known).

It is sufficient to prove (2.1) with \( \alpha \) replaced by the generators of \( \mathcal{A} \) of (2.5).

First take \( \alpha = R. \) Let \( h : D^{s+1} \times D^{n-s-1} \to D^{s+1} \times D^{n-s-1} \) be defined by
$h(x, y) = (r, x, y)$ where $r: D^{s+1} \rightarrow D^{s+1}$ is a reflection through an equatorial $s$-plane. Then let $f'_i = f_i h$. If $\sigma' = (M, Q; f'_1, f'_2, \ldots, f'_r; s + 1), \chi(\sigma')$ is diffeomorphic to $V$ by (1.1). On the other hand $\chi(\sigma')$ realizes $f'_{\sigma'} \alpha$.

The case $\alpha = T_1$ follows immediately from (1.1). So now we proceed with the proof of (2.1) with $\alpha = S$.

Define $V_1$ to be the manifold $\chi(M, Q; f_1, \ldots, f_r; s + 1)$ and let $\varphi_i \subset \partial V_1$ be $\varphi_i = \partial V_i - (\partial M - Q)$. Let $\varphi_i \in \pi_s(Q), i = 1, \ldots, r$ denote the homotopy class of $f'_i; \partial D^{s+1}_{\sigma'} \times 0 \rightarrow Q$, the restriction of $f'_i$. Let $\gamma: \pi_s(Q \cap Q_1) \rightarrow \pi_s(Q)$ and $\beta: \pi_s(Q \cap Q_1) \rightarrow \pi_s(Q_1)$ be the homomorphisms induced by the respective inclusions.

(2.6) Lemma. With notations and conditions as above, $\varphi_2 \in \gamma \ker \beta$.

Proof. Let $q \in \partial D^{n-s-1}_2$ and $\psi: \partial D^{s+1}_2 \times q \rightarrow Q \cap Q_1$ be the restriction of $f'_2$. Denote by $\varphi_2 \in \pi_s(Q \cap Q_1)$ the homotopy class of $\psi$. Since $\varphi_2$ and $\bar{f}_2$ are homotopic in $Q, \gamma \bar{f}_2 = \varphi_2$. On the other hand $\beta \bar{f}_2 = 0$, thus proving (2.6).

By (2.6), let $\varphi_2 \in \pi_s(Q \cap Q_1)$ with $\gamma \bar{f}_2 = \varphi_2$ and $\beta \bar{f}_2 = 0$. Let $g = y + \bar{y}$ (or $y \bar{f}_2$ in case $s = 1$; our terminology assumes $s > 1$) where $y \in \pi_s(Q \cap Q_1)$ is the homotopy class of $\bar{f}_2; \partial D^{s+1}_2 \times 0 \rightarrow Q \cap Q_1$. Let $\bar{g}: \partial D^{s+1}_2 \times 0 \rightarrow Q \cap Q_1$ be an imbedding realizing $g$ (see [29]).

If $n = 2s + 2$, then from the fact that $\pi_s(Q) = 1$, it follows that also $\pi_s(Q_1) = 1$. Then since $\bar{g}$ and $\bar{f}_i$ are homotopic in $Q$, i.e., $\beta g = \beta y$, (2.4) applies to yield an imbedding $e: \partial D^{s+1} \times D^{n-s-1} \rightarrow Q_1$ extending $\bar{g}$ such that $\chi(V_1, Q; e)$ and $\chi(V, Q; f_i)$ are diffeomorphic.

On one hand $V = \chi(V, Q; f_1, \ldots, f_r) = \chi(V_1, Q; f_1)$ and, on the other hand, $\chi(V, Q; e, f_1, \ldots, f_r) = \chi(V_1, Q; e)$, so by the preceding statement, $V$ and $\chi(V, Q; e, f_1, \ldots, f_r)$ are diffeomorphic. Since $\gamma g = g_1 + g_2, f_\sigma \alpha(D) = f_\sigma(D_1 + D_2) = g_1 + g_2, f_\sigma \alpha = f_\sigma'$, where $\sigma' = (V, Q; e, f_2, \ldots, f_r)$. This proves (2.1).

3. The goal of this section is to prove the following theorem.

(3.1) Theorem. Let $n \geq 2s + 2$ and, if $s = 1, n \geq 5$. Suppose $H \in \mathcal{H}(n, k, s)$. Then given $r \geq k$, there exists an epimorphism $g: G_r \rightarrow \pi_s(H)$ such that every realization of $g$ is in $\mathcal{H}(n, r - k, s + 1)$.

For the proof of 3.1, we need some lemmas.

(3.2) Lemma. If $\mathcal{H}(n, k, s)$ then $\pi_s(H)$ is

(a) a set of $k + 1$ elements if $s = 0$,
(b) a free group on $k$ generators if $s = 1$,
(c) a free abelian group on $k$ generators if $s > 1$.

Furthermore if $n \geq 2s + 2$, then $\pi_s(\partial H) \rightarrow \pi_s(H)$ is an isomorphism for $i \leq s$.

Proof. We can assume $s > 0$ since, if $s = 0, H$ is a set of $n$-disks $k + 1$
in number. Then $H$ has as a deformation retract in an obvious way the wedge of $k$ $s$-spheres. Thus (b) and (c) are true. For the last statement of (3.2), from the exact homotopy sequence of the pair $(H, \partial H)$, it is sufficient to show that $\pi_i(H, \partial H) = 0$, $i \leq s + 1$.

Thus let $f: (D^i, \partial D^i) \to (H, \partial H)$ be a given continuous map with $i \leq s + 1$. We want to construct a homotopy $f_i: (D^i, \partial D^i) \to (H, \partial H)$ with $f_0 = f$ and $f_i(D^i) \subset \partial H$.

Let $f_i: (D^i, \partial D^i) \to (H, \partial H)$ be a differentiable approximation to $f$. Then by a radial projection from a point in $D^s$ not in the image of $f_i$, $f_i$ is homotopic to a differentiable map $f_i: (D^i, \partial D^i) \to (H, \partial H)$ with the image of $f_i$ not intersecting the interior of $D^s \subset H$. Now for dimensional reasons $f_i$ can be approximated by a differentiable map $f_i: (D^i, \partial D^i) \to (H, \partial H)$ with the image of $f_i$ not intersecting any $D^i \times 0 \subset H$. Then by other projections, one for each $i$, $f_i$ is homotopic to a map $f_i: (D^i, \partial D^i) \to (H, \partial H)$ which sends all of $D^i$ into $\partial H$. This shows $\pi_i(H, \partial H) = 0$, $i \leq s + 1$, and proves (3.2).

If $\beta \in \pi_{s-1}(O(n - s))$, let $H_\beta$ be the $(n - s)$-cell bundle over $S^s$ determined by $\beta$.

(3.3) Lemma. Suppose $V = \chi(H_\beta; f; s + 1)$ where $\beta \in \pi_{s-1}(O(n - s))$, $n \geq 2s + 2$, or if $s = 1$, $n \geq 5$. Let also $\pi_s(V) = 0$. Then $V$ is diffeomorphic to $D^n$.

Proof. The zero-cross-section $\sigma: S^s \to H_\beta$ is homotopic to zero, since $\pi_i(V) = 0$, and so is regularly homotopic in $V$ to a standard $s$-sphere $S_0^s$ contained in a cell neighborhood by dimensional reasons [29]. Since a regular homotopy preserves the normal bundle structure, $\sigma(S^s)$ has a trivial normal bundle and thus $\beta = 0$. Hence $H_\beta$ is diffeomorphic to the product of $S^s$ and $D^{n-s}$.

Let $\sigma^*: S^s \to \partial H_\beta$ be a differentiable cross section and $\bar{f}: \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$ the restriction of $f: \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$. Then $\sigma$ and $\bar{f}$ are homotopic in $\partial H_\beta$ (perhaps after changing $f$ by a diffeomorphism of $D^{s+1} \times D^{n-s-1}$ which reverses orientation of $\partial D^{s+1} \times D^{n-s-1}$) since $\pi_s(V) = 0$, and hence differentiably isotopic. Thus we can assume $\bar{f}$ and $\sigma^*$ are the same.

Let $f_{\varepsilon}$ be the restriction of $f$ to $\partial D^{s+1} \times D^{n-s-1}$ where $D^{n-s-1}$ denotes the disk $\{x \in D^{n-s-1} \mid ||x|| \leq \varepsilon\}$, and $\varepsilon > 0$. Then the imbedding $g: \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$ is differentiably isotopic to $f$ where $g_t(x, y) = f_t r_t(x, y)$ and $r_t(x, y) = (x, \varepsilon y)$. Define $k: \partial D^{s+1} \times D^{n-s-1} \to \partial H_\beta$ by $p_2 g_t(x, y)$ where $p_2: g_t(x \times D^{n-s-1}) \to F_x$ is projection into the fibre $F_x$ of $\partial H_\beta$ over $\sigma^{-1} g_t(x, 0)$. If $\varepsilon$ is small enough, $k_t$ is well-defined and an imbedding. In fact if $\varepsilon$ is small enough, we can even suppose that for each $x$, $k_t$ maps $x \times D^{n-s-1}$ linearly onto image $k_{t x} \cap F_x$ where image $k_{tx} \cap F_x$ has a linear structure.
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induced from $F_s$.

It can be proved $k_s$ and $g_s$ are differentiably isotopic. (The referee has remarked that there is a theorem, Milnor's "tubular neighborhood theorem", which is useful in this connection and can indeed be used to make this proof clearer in general.)

We finish the proof of (3.3) as follows. Suppose $V$ is as in (3.3) and $V' = \chi(H_\beta; f'; s + 1), \pi_s(V') = 0$. It is sufficient to prove $V$ and $V'$ are diffeomorphic since it is clear that one can obtain $D^n$ by choosing $f'$ properly and using the fact that $H_\beta$ is a product of $S^s$ and $D^{n-s}$. From the previous paragraph, we can replace $f$ and $f'$ by $k_s$ and $k_s'$ with those properties listed. We can also suppose without loss of generality that the images of $k_s$ and $k_s'$ coincide. It is now sufficient to find a diffeomorphism $h$ of $H_\beta$ with $hf = f'$. For each $x$, define $h$ on image $f \cap F_s$ to be the linear map which has this property. One can now easily extend $h$ to all of $H_\beta$ and thus we have finished the proof of (3.3).

Suppose now $M^s$ and $M^s_2$ are compact manifolds and $f_i: D^{n-1} \times i \to \partial M_i$ are imbeddings for $i = 1$ and 2. Then $\chi(M_1 \cup M_2; f_1 \cup f_2; 1)$ is a well defined manifold, where $f_1 \cup f_2: \partial D^1 \times D^{n-1} \to \partial M_1 \cup \partial M_2$ is defined by $f_1$ and $f_2$, the set of which, as the $f_i$ vary, we denote by $M_1 + M_2$. (If we pay attention to orientation, we can restrict $M_1 + M_2$ to have but one element.) The following lemma is easily proved.

(3.4) Lemma. The set $M^a + D^n$ consists of one element, namely $\bigcup M^a$.

(3.5) Lemma. Suppose an imbedding $f: \partial D^s \times D^{n-s} \to \partial M^a$ is null-homotopic where $M$ is a compact manifold, $n \geq 2s + 2$ and, if $s = 1$, $n \geq 5$. Then $\chi(M; f) \in M + H_\beta$ for some $\beta \in \pi_s(O(n - s))$.

Proof of (3.5). Let $\tilde{f}: \partial D^s \times q \to \partial M$ be the restriction of $f$ where $q$ is a fixed point in $\partial D^{n-s}$. Then by dimensional reasons [29], $\tilde{f}$ can be extended to an imbedding $\varphi: D^s \to \partial M$ where the image of $\varphi$ intersects the image of $f$ only on $\tilde{f}$. Next let $T$ be a tubular neighborhood of $\varphi(D^s)$ in $M$. This can be done so that $T$ is a cell, $T \cup (D^s \times D^{n-s})$ is of the form $H_\beta$ and $V \subset M + H_\beta$. We leave the details to the reader.

To prove (3.1), let $H = \chi(D^s; f_1, \ldots, f_k; s)$. Then $f_i$ defines a class $\gamma_i \in \pi_s(H, D^n)$. Let $\gamma_i \in \pi_s(\partial H)$ be the image of $\gamma_i$ under the inverse of the composition of the isomorphisms $\pi_s(\partial H) \to \pi_s(H) \to \pi_s(H, D^n)$ (using (3.2)). Define $g$ of (3.1) by $gD_i = \gamma_i$, $i \leq k$, and $gD_i = 0, i > k$. That $g$ satisfies (3.1) follows by induction from the following lemma.

(3.6) Lemma. $\chi(H; g, s + 1) \in \mathcal{A}(n, k - 1, s)$ if the restriction of $g$, to $\partial D^{k+1} \times 0$ has homotopy class $\gamma_i \in \pi_s(\partial H)$.

Now (3.6) follows from (3.3), (3.4) and (3.5), and the fact that $g$, is dif-
ferentially isotopic to \( g' \) whose image is in \( \partial H_\beta \cap \partial H \), where \( H_\beta \) is defined by (3.5) and \( f_i \).

4. We prove here (1.2). First suppose \( s = 0 \). Then \( H \in \mathcal{A}(n, k, 0) \) is the disjoint union of \( n \)-disks, \( k+1 \) in number, and \( V = \chi(H; f_1, \ldots, f_r; 1) \). Since \( \pi_\alpha(V) = 1 \), there exists a permutation of \( 1, \ldots, r, i_1, \ldots, i_r \) such that \( Y = \chi(H; f_{i_1}, \ldots, f_{i_r}, 1) \) is connected. By (3.4), \( Y \) is diffeomorphic to \( D^n \). Hence \( V = \chi(Y; f_{i_{k+1}}, \ldots, f_{i_r}, 1) \) is in \( \mathcal{A}(n, r-k, 1) \).

Now consider the case \( s = 1 \). Choose, by (3.1), \( g: G_k \twoheadrightarrow \pi_\alpha(\partial H) \) such that every manifold derived from \( g \) is diffeomorphic to \( D^n \). Let \( Y = \chi(H; f_1, \ldots, f_{r-k}) \). Then \( \pi_\alpha(Y) = 1 \) and by the argument of (3.2), \( \pi_\alpha(\partial Y) = 1 \). Let \( \tilde{g}_i: \partial D^3 \times 0 \to \partial H \) be disjoint imbeddings realizing the classes \( g(D_i) \in \pi_\alpha(\partial H) \) which are disjoint from the images of all \( f_i \), \( i = 1, \ldots, k \). Then by (2.4) there exist imbeddings \( g_1, \ldots, g_k: \partial D^2 \times D^{n-2} \to \partial H \) extending the \( \tilde{g}_i \) such that \( V = \chi(Y; f_{r-k+1}, \ldots, f_r) \) and \( \chi(Y; g_1, \ldots, g_k) \) are diffeomorphic. But

\[
\chi(Y; g_1, \ldots, g_k) = \chi(H; g_1, \ldots, g_k, f_1, \ldots, f_{r-k}) = \chi(D^n, f_1, \ldots, f_{r-k}) \in \mathcal{A}(n, r-k, 2).
\]

Hence so does \( V \).

For the case \( s > 1 \), we use an algebraic lemma.

(4.1) **Lemma.** If \( f, g: G \to G' \) are epimorphisms where \( G \) and \( G' \) are finitely generated free abelian groups, then there exists an automorphism \( \alpha: G \to G \) such that \( f \alpha = g \).

**Proof.** Let \( G'' \) be a free abelian group of rank equal to \( \text{rank} G - \text{rank} G' \), and let \( p: G' + G'' \to G' \) be the projection. Then, identifying elements of \( G \) and \( G' + G'' \) under some isomorphism, it is sufficient to prove the existence of \( \alpha \) for \( g = p \). Since the groups are free, the following exact sequence splits

\[
0 \to f^{-1}(0) \to G \xrightarrow{f} G' \to 0.
\]

Let \( h: G \to f^{-1}(0) \) be the corresponding projection and let \( k: f^{-1}(0) \to G'' \) be some isomorphism. Then \( \alpha: G \to G' + G'' \) defined by \( f + kh \) satisfies the requirements of (4.1).

**Remark.** Using Grusko's Theorem [6], one can also prove (4.1) when \( G \) and \( G' \) are free groups.

Now take \( \sigma = (H; f_1, \ldots, f_r; s + 1) \) of (1.2) and \( g: G_r \to \pi_\alpha(\partial H) \) of (3.1). Since \( \pi_\alpha(V) = 0 \), and \( s > 1 \), \( f_\sigma: G_r \to \pi_\alpha(\partial H) \) is an epimorphism. By (3.2) and (4.1) there is an automorphism \( \alpha: G_r \to G_r \) such that \( f_\sigma \alpha = g \). Then (2.1) implies that \( V \) is in \( \mathcal{A}(n, r-k, s+1) \) using the main property of \( g \).

5. The goal of this section is to prove the following analogue of (1.2).
(5.1) **Theorem.** Let \( n \geq 2s + 2 \), or if \( s = 1, n \geq 5 \), \( M^{n-1} \) be a simply connected, \((s - 1)\)-connected closed manifold and \( \mathcal{H}_n(n, k, s) \) the set of all manifolds having presentations of the form \( (M \times [0, 1], M \times 1; f_1, \ldots, f_s; s, k) \). Now let \( H \in \mathcal{H}_n(n, k, s), Q = \partial H - M \times 0, V = \chi(H, Q); g_1, \ldots, g_s; s + 1 \) and suppose \( \pi_s(M \times 0) \to \pi_s(V) \) is an isomorphism. Also suppose if \( s = 1, \chi(H, Q); g_1, \ldots, g_{r-k}, 2) \). Then \( V \in \mathcal{H}_n(n, r - k, s + 1) \).

One can easily obtain (1.2) from (5.1) by taking for \( M \), the \((n - 1)\)-sphere. The following lemma is easy, following (3.2).

(5.2) **Lemma.** With definitions and conditions as in (5.1), \( \pi_s(Q) = G_k \) if \( s = 1 \), and if \( s > 1 \), \( \pi_s(Q) = \pi_s(M) + G_k \).

Let \( p_1: \pi_s(Q) \to \pi_s(M), p_2: \pi_s(Q) \to G_k \) be the respective projections.

(5.3) **Lemma.** With definitions and conditions as in (5.1), there exists a homomorphism \( g: G_r \to \pi_s(Q) \) such that \( p_2g \) is trivial, \( p_2g \) is an epimorphism, and every realization of \( g \) is in \( \mathcal{H}_n(n, r - k, s + 1) \), each \( r \geq k \).

The proof follows (3.1) closely.

We now prove (5.1). The cases \( s = 0 \) and \( s = 1 \) are proved similarly to these cases in the proof of (1.2). Suppose \( s > 1 \). From the fact that \( \pi_s(M \times 0) - \pi_s(V) \) is an isomorphism, it follows that \( p_1f_{\sigma} \) is trivial and \( p_2f_{\sigma} \) is an epimorphism where \( \sigma = (H, Q); g_1, \ldots, g_r, s + 1 \). Then apply (4.1) to obtain an automorphism \( \alpha: G_r \to G_r \) such that \( p_2f_{\sigma} \alpha = p_2g \) where \( g \) is as in (5.3). Then \( f_{\sigma} \alpha = g \), hence using (2.1), we obtain (5.1).

6. The goal of this section is to prove the following two theorems.

(6.1) **Theorem.** Suppose \( f \) is a \( C^\infty \) function on a compact manifold \( W \) with no critical points on \( f^{-1}[-\varepsilon, \varepsilon] = N \) except \( k \) non-degenerate ones on \( f^{-1}(0) \), all of index \( \lambda \), and \( N \cap \partial W = \emptyset \). Then \( f^{-1}[-\infty, \epsilon] \) has a presentation of the form \( f^{-1}(-\infty, -\varepsilon], f^{-1}(-\varepsilon], f_1, \ldots, f_s; \lambda) \).

(6.2) **Theorem.** Let \( (M, Q; f_1, \ldots, f_s; s) \) be a presentation of a manifold \( V \), and \( g \) be a \( C^\infty \) function on \( M \), regular, in a neighborhood of \( Q \), and constant with its maximum value on \( Q \). Then there exists a \( C^\infty \) function \( G \) on \( V \) which agrees with \( g \) outside a neighborhood of \( Q \), is constant and regular on \( \partial V - (\partial M - Q) \), and has exactly \( k \) new critical points, all non-degenerate, with the same value and with index \( s \).

**Sketch of proof of (6.1).** Let \( \beta_i \) denote the critical points of \( f \) at level zero, \( i = 1, \ldots, k \) with disjoint neighborhoods \( V_i \). By a theorem of Morse [13] we can assume \( V_i \) has a coordinate system \( x = (x_1, \ldots, x_n) \) such that for \( \| x \| \leq \delta \), some \( \delta > 0, f(x) = -\sum_{i=1}^{l} x_i^2 + \sum_{i=\lambda+1}^{n} x_i^2 \). Let \( E_0 \) be the \((x_1, \ldots, x_n)\) plane of \( V_i \) and \( E_1 \) be the \((x_1, \ldots, x_{\lambda+1})\) plane. Then for \( \varepsilon_1 > 0 \) sufficiently small \( E_1 \cap f^{-1}[-\varepsilon, \varepsilon] \) is diffeomorphic to \( D^\lambda \). A sufficiently
small tubular neighborhood $T$ of $E_i$ will have the property that $T' = T \cap f^{-1}([-\varepsilon_i, \varepsilon_i])$ is diffeomorphic to $D^\lambda \times D^{n-\lambda}$ with $T \cap f^{-1}(-\varepsilon_i)$ corresponding to $\partial D^\lambda \times D^{n-\lambda}$.

As we pass from $f^{-1}(-\infty, -\varepsilon_i]$ to $f^{-1}(-\infty, \varepsilon_i]$, it happens that one such $T'$ is added for each $i$, together with a tubular neighborhood of $f^{-1}(-\varepsilon_i)$ so that $f^{-1}(-\infty, \varepsilon_i]$ is diffeomorphic to a manifold of the form $\chi(f^{-1}(-\infty, -\varepsilon_i); f_i, \cdots, f_{2i}; \lambda)$. Since there are no critical points between $-\varepsilon_i$ and $-\varepsilon_i, \varepsilon_i$ and $\varepsilon_i$, $\varepsilon_i$ can be replaced by $\varepsilon$ in the preceding statement thus proving (6.1).

Theorem (6.2) is roughly a converse of (6.1) and a sketch of the proof can be constructed similarly.

7. In this section we prove Theorems C and I of the Introduction.

The following theorem was proved in [21].

(7.1) **Theorem.** Let $V^n$ be a $C^\infty$ compact manifold with $\partial V$ the disjoint union of $V_i$ and $V_2$, each $V_i$ closed in $\partial V$. Then there exists a $C^\infty$ function $f$ on $V$ with non-degenerate critical points, regular on $\partial V$, $f(V_1) = -(1/2), f(V_2) = n + (1/2)$ and at a critical point $\beta$ of $f, f(\beta) = \text{index } \beta$.

Functions described in (7.1) are called *nice* functions.

Suppose now $M^n$ is a closed $C^\infty$ manifold and $f$ is the function of (7.1). Let $X_s = f^{-1}(0, s + (1/2)), s = 0, \cdots, n$.

(7.2) **Lemma.** For each $s$, the manifold $X_s$ has a presentation of the form $(X_{s-1}; f_i, \cdots, f_{s}; s)$.

This follows from (6.1).

(7.3) **Lemma.** If $H \in \mathcal{H}(n, k, s)$, then there exists—a $C^\infty$ non-degenerate function $f$ on $H, f(\partial H) = s+(1/2), f$ has one critical point of index 0, value 0, $k$ critical points of index $s$, value $s$ and no other critical points.

This follows from (6.2).

The proof of Theorem C then goes as follows. Take a nice function $f$ on $M$ by (7.1), with $X_s$ defined as above. Note that $X_s \in \mathcal{H}(n, q, 0)$ and $\pi_q(X_s) = 0$, hence by (7.2) and (1.2), $X_s \in \mathcal{H}(n, k, 1)$. Suppose now that $\pi_q(M) = 1$ and $n \geq 6$. The following argument suggested by H. Samelson simplifies and replaces a complicated one of the author. Let $X'_s$ be the sum of $X_s$ and $k$ copies $H_s, \cdots, H_s$ of $D^{n-2} \times S^2$. Then since $\pi_q(X_s) = 0$, (1.2) implies that $X'_s \in H(n, r, 2)$. Now let $f_i: \partial D^3 \times D^{n-3} \to \partial H_i \cap \partial X'_s$ for $i = 1, \cdots, k$ be differentiable imbeddings such that the composition

$$
\pi_q(\partial D^3 \times D^{n-3}) \to \pi_q(\partial H_i \times \partial X'_s) \to \pi_q(\partial H_i)
$$
is an isomorphism. Then by (3.3) and (3.4), $\chi(X_2', f_1, \ldots, f_k; 3)$ is diffeomorphic to $X_2$. Since $X_3 = \chi(X_2; g_1, \ldots, g_i; 3)$ we have

$$X_3 = \chi(X_2', f_1, \ldots, f_k, g_1, \ldots, g_i; 3),$$

and another application of (1.2) yields that $X_3 \in H(n, k + l - r, 3)$.

Iteration of the argument yields that $X_3^* \in \mathcal{H}(n, r, m)$. By applying (7.3), we can replace $g$ by a new nice function $h$ with type numbers satisfying $M_0 = 1$, $M_i = 0$, $0 < i < m$. Now apply the preceding arguments to $-h$ to yield that $h^{-1}[n - m - (1/2), n] = X_m^* \in \mathcal{H}(n, k, m)$. Now we modify $h$ by (7.3) on $X^*_m$ to get a new nice function on $M$ agreeing with $h$ on $M - X^*_m$ and satisfying the conditions of Theorem C.

The proof of Theorem I goes as follows. Let $V^*$ be a manifold with $\partial V = V_1 - V_2$, $n = 2m + 2$. Take a nice function $f$ on $V$ by (7.1) with $f(V_1) = -(1/2)$ and $f(V_2) = n + (1/2)$.

Following the proof of Theorem C, replacing the use of (1.2) with (5.1), we obtain a new nice function $g$ on $V$ with $g(V_1) = -(1/2)$, $g(V_2) = n + (1/2)$ and no critical points except possibly of index $m + 1$. The following lemma can be proved by the standard methods of Morse theory [12].

(7.4) Lemma. Let $V$ be as in (7.1) and $f$ be a $C^\infty$ non-degenerate function on $V$ with the same boundary conditions as in (7.1). Then

$$\chi_V = \sum (-1)^q M_q + \chi_{V_1},$$

where $\chi_V, \chi_{V_1}$ are the respective Euler characteristics, and $M_q$ denote the $q^{th}$ type number of $f$.

This lemma implies that our function $g$ has no critical points, and hence $V_1$ and $V_2$ are diffeomorphic.

8. We have yet to prove Theorems F and G. For Theorem F, observe by Theorem C, there is a nice function $f$ on $M$ with vanishing type numbers except in dimensions $M_0$, $M_m$, $M_{m+1}$, $M_n$, and $M_0 = M_n = 1$. Also, by the Morse relation, observe that the Euler characteristic is the alternating sum of the type numbers, $M_m = M_{m+1}$. Then by (7.2), $f^{-1}[0, m + (1/2)], f^{-1}[m + (1/2), 2m + 1] \in \mathcal{H}(2m + 1, M_m, m)$ proving Theorem F.

All but the last statement of Theorem G has been proved. For this just note that $M - D^{2m}$ is diffeomorphic to $f^{-1}[0, m + (1/2)]$ which by (7.2) is in $\mathcal{H}(2m, k, m)$.

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