THE THEORY OF MATRICES

BY

F. R. GANTMACHER

VOLUME ONE

CHELSEA PUBLISHING COMPANY
NEW YORK, N. Y.

Library of Congress Catalog Card Number 59-11779

The present work, published in two volumes, is an English translation by K. A. Hirsch, of the Russian-language book TEORIYA MATRITS by F. R. Gantmacher (Гантмахер)

SBN 8284-0131-4

Printed in the United States of America

PREFACE

The matrix calculus is widely applied nowadays in various branches of mathematics, mechanics, theoretical physics, theoretical electrical engineering, etc. However, neither in the Soviet nor the foreign literature is there a book that gives a sufficiently complete account of the problems of matrix theory and of its diverse applications. The present book is an attempt to fill this gap in the mathematical literature.

The book is based on lecture courses on the theory of matrices and its applications that the author has given several times in the course of the last seventeen years at the Universities of Moscow and Tiflis and at the Moscow Institute of Physical Technology.

The book is meant not only for mathematicians (undergraduates and research students) but also for specialists in allied fields (physics, engineering) who are interested in mathematics and its applications. Therefore the author has endeavoured to make his account of the material as accessible as possible, assuming only that the reader is acquainted with the theory of determinants and with the usual course of higher mathematics within the programme of higher technical education. Only a few isolated sections in the last chapters of the book require additional mathematical knowledge on the part of the reader. Moreover, the author has tried to keep the individual chapters as far as possible independent of each other. For example, Chapter V, *Functions of Matrices*, does not depend on the material contained in Chapters II and III. At those places of Chapter V where fundamental concepts introduced in Chapter IV are being used for the first time, the corresponding references are given. Thus, a reader who is acquainted with the rudiments of the theory of matrices can immediately begin with reading the chapters that interest him.

The book consists of two parts, containing fifteen chapters.

In Chapters I and III, information about matrices and linear operators is developed *ab initio* and the connection between operators and matrices is introduced.

Chapter II expounds the theoretical basis of Gauss's elimination method and certain associated effective methods of solving a system of n linear equations, for large n. In this chapter the reader also becomes acquainted with the technique of operating with matrices that are divided into rectangular 'blocks.'

In Chapter IV we introduce the extremely important 'characteristic' and 'minimal' polynomials of a square matrix, and the 'adjoint' and 'reduced adjoint' matrices.

In Chapter V, which is devoted to functions of matrices, we give the general definition of $f(A)$ as well as concrete methods of computing it—where $f(\lambda)$ is a function of a scalar argument λ and A is a square matrix. The concept of a function of a matrix is used in §§ 5 and 6 of this chapter for a complete investigation of the solutions of a system of linear differential equations of the first order with constant coefficients. Both the concept of a function of a matrix and this latter investigation of differential equations are based entirely on the concept of the minimal polynomial of a matrix and—in contrast to the usual exposition—do not use the so-called theory of elementary divisors, which is treated in Chapters VI and VII.

These five chapters constitute a first course on matrices and their applications. Very important problems in the theory of matrices arise in connection with the reduction of matrices to a normal form. This reduction is carried out on the basis of Weierstrass' theory of elementary divisors. In view of the importance of this theory we give two expositions in this book: an analytic one in Chapter VI and a geometric one in Chapter VII. We draw the reader's attention to §§ 7 and 8 of Chapter VI, where we study effective methods of finding a matrix that transforms a given matrix to normal form. In § 8 of Chapter VII we investigate in detail the method of A. N. Krylov for the practical computation of the coefficients of the characteristic polynomial.

In Chapter VIII certain types of matrix equations are solved. We also consider here the problem of determining all the matrices that are permutable with a given matrix and we study in detail the many-valued functions of matrices $\sqrt[m]{A}$ and $\ln A$.

Chapters IX and X deal with the theory of linear operators in a unitary space and the theory of quadratic and hermitian forms. These chapters do not depend on Weierstrass' theory of elementary divisors and use, of the preceding material, only the basic information on matrices and linear operators contained in the first three chapters of the book. In § 9 of Chapter X we apply the theory of forms to the study of the principal oscillations of a system with n degrees of freedom. In § 11 of this chapter we give an account of Frobenius' deep results on the theory of Hankel forms. These results are used later, in Chapter XV, to study special cases of the Routh-Hurwitz problem.

The last five chapters form the second part of the book [the second volume, in the present English translation]. In Chapter XI we determine normal forms for complex symmetric, skew-symmetric, and orthogonal matrices and establish interesting connections of these matrices with real matrices of the same classes and with unitary matrices.

In Chapter XII we expound the general theory of pencils of matrices of the form $A + \lambda B$, where A and B are arbitrary rectangular matrices of the same dimensions. Just as the study of regular pencils of matrices $A + \lambda B$ is based on Weierstrass' theory of elementary divisors, so the study of singular pencils is built upon Kronecker's theory of minimal indices, which is, as it were, a further development of Weierstrass's theory. By means of Kronecker's theory—the author believes that he has succeeded in simplifying the exposition of this theory—we establish in Chapter XII canonical forms of the pencil of matrices $A + \lambda B$ in the most general case. The results obtained there are applied to the study of systems of linear differential equations with constant coefficients.

In Chapter XIII we explain the remarkable spectral properties of matrices with non-negative elements and consider two important applications of matrices of this class: 1) homogeneous Markov chains in the theory of probability and 2) oscillatory properties of elastic vibrations in mechanics. The matrix method of studying homogeneous Markov chains was developed in the book [25] by V. I. Romanovskiĭ and is based on the fact that the matrix of transition probabilities in a homogeneous Markov chain with a finite number of states is a matrix with non-negative elements of a special type (a 'stochastic' matrix).

The oscillatory properties of elastic vibrations are connected with another important class of non-negative matrices—the 'oscillation matrices.' These matrices and their applications were studied by M. G. Kreĭn jointly with the author of this book. In Chapter XIII, only certain basic results in this domain are presented. The reader can find a detailed account of the whole material in the monograph [7].

In Chapter XIV we compile the applications of the theory of matrices to systems of differential equations with variable coefficients. The central place (§§ 5-9) in this chapter belongs to the theory of the multiplicative integral (Produktintegral) and its connection with Volterra's infinitesimal calculus. These problems are almost entirely unknown in Soviet mathematical literature. In the first sections and in § 11, we study reducible systems (in the sense of Lyapunov) in connection with the problem of stability of motion; we also give certain results of N. P. Erugin. Sections 9-11 refer to the analytic theory of systems of differential equations. Here we clarify an inaccuracy in Birkhoff's fundamental theorem, which is usually applied to the investigation of the solution of a system of differential equations in the neighborhood of a singular point, and we establish a canonical form of the solution in the case of a regular singular point.

In § 12 of Chapter XIV we give a brief survey of some results of the fundamental investigations of I. A. Lappo-Danilevskiĭ on analytic functions of several matrices and their applications to differential systems.

The last chapter, Chapter XV, deals with the applications of the theory of quadratic forms (in particular, of Hankel forms) to the Routh-Hurwitz problem of determining the number of roots of a polynomial in the right half-plane ($\operatorname{Re} z > 0$). The first sections of the chapter contain the classical treatment of the problem. In § 5 we give the theorem of A. M. Lyapunov in which a stability criterion is set up which is equivalent to the Routh-Hurwitz criterion. Together with the stability criterion of Routh-Hurwitz we give, in § 11 of this chapter, the comparatively little known criterion of Liénard and Chipart in which the number of determinant inequalities is only about half of that in the Routh-Hurwitz criterion.

At the end of Chapter XV we exhibit the close connection between stability problems and two remarkable theorems of A. A. Markov and P. L. Chebyshev, which were obtained by these celebrated authors on the basis of the expansion of certain continued fractions of special types in series of decreasing powers of the argument. Here we give a matrix proof of these theorems.

This, then, is a brief summary of the contents of this book.

F. R. Gantmacher

PUBLISHERS' PREFACE

THE PUBLISHERS WISH TO thank Professor Gantmacher for his kindness in communicating to the translator new versions of several paragraphs of the original Russian-language book.

The Publishers also take pleasure in thanking the VEB Deutscher Verlag der Wissenschaften, whose many published translations of Russian scientific books into the German language include a counterpart of the present work, for their kind spirit of cooperation in agreeing to the use of their formulas in the preparation of the present work.

No material changes have been made in the text in translating the present work from the Russian except for the replacement of several paragraphs by the new versions supplied by Professor Gantmacher. Some changes in the references and in the Bibliography have been made for the benefit of the English-language reader.

CONTENTS

CHAPTER I

MATRICES AND OPERATIONS ON MATRICES

§ 1. Matrices. Basic Notation

1. Let F be a given number field.[1]

DEFINITION 1: *A rectangular array of numbers of the field* F

$$\left\| \begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{matrix} \right\| \tag{1}$$

is called a matrix*. When $m = n$, the matrix is called square and the number* ***m, equal to n,*** *is called its order. In the general case the matrix is called* ***rectangular*** *(of dimension $m \times n$). The numbers that constitute the matrix* ***are called its elements****.*

NOTATION: In the double-subscript notation for the elements, the first subscript always denotes the row and the second subscript the column containing the given element.

As an alternative to the notation (1) for a matrix we shall also use the abbreviation

$$\| a_{ik} \| \quad (i = 1, 2, \dots, m;\ k = 1, 2, \dots, n). \tag{2}$$

Often the matrix (1) will also be denoted by a single letter, for example A. If A is a square matrix of order n, then we shall write $A = \| a_{ik} \|_1^n$. The determinant of a square matrix $A = \| a_{ik} \|_1^n$ will be denoted by $| a_{ik} |_1^n$ or by $| A |$.

---

[1] A *number field* is defined as an arbitrary collection of numbers within which the four operations of addition, subtraction, multiplication, and division by a non-zero number can always be carried out.

Examples of number fields are: the set of all rational numbers, the set of all real numbers, and the set of all complex numbers.

All the numbers that will occur in the sequel are assumed to belong to the number field given initially.

We introduce a concise notation for determinants formed from elements of the given matrix:

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} = \begin{vmatrix} a_{i_1k_1} & a_{i_1k_2} & \dots & a_{i_1k_p} \\ a_{i_2k_1} & a_{i_2k_2} & \dots & a_{i_2k_p} \\ \dots & \dots & \dots & \dots \\ a_{i_pk_1} & a_{i_pk_2} & \dots & a_{i_pk_p} \end{vmatrix}. \tag{3}$$

The determinant (3) is called a *minor* of A of order p, provided $1 \leqq i_1 < i_2 < \dots < i_p \leqq m$ and $1 \leqq k_1 < k_2 < \dots < k_p \leqq n$. A rectangular matrix $A = \| a_{ik} \|$ $(i = 1, 2, \dots, m;\ k = 1, 2, \dots, n)$ has $\binom{m}{p}\cdot\binom{n}{p}$ minors of order p

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} \qquad \left(\begin{matrix} 1 \leqq i_1 < i_2 < \dots < i_p \leqq m \\ 1 \leqq k_1 < k_2 < \dots < k_p \leqq n \end{matrix};\ p \leqq m, n\right). \tag{3'}$$

The minors (3') in which $i_1 = k_1,\ i_2 = k_2,\ \dots,\ i_p = k_p$, are called *principal* minors.

In the notation (3) the determinant of a square matrix $A = \| a_{ik} \|_1^n$ can be written as follows:

$$|A| = A\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}.$$

The largest among the orders of the non-zero minors generated by a matrix is called the *rank* of the matrix. If r is the rank of a rectangular matrix A of dimension $m \times n$, then obviously $r \leqq \min(m, n)$.

A rectangular matrix consisting of a single column

$$\begin{Vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Vmatrix}$$

is called a *column matrix* and will be denoted by $(x_1, x_2, \dots, x_n)$.

A rectangular matrix consisting of a single row

$$\| z_1, z_2, \dots, z_n \|$$

is called a *row matrix* and will be denoted by $[z_1, z_2, \dots, z_n]$.

A square matrix in which all the elements outside the main diagonal are zero

$$\begin{Vmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_n \end{Vmatrix}$$

is called a *diagonal matrix* and is denoted by[2] $\| d_i \delta_{ik} \|_1^n$ or by

$$\{d_1, d_2, \dots, d_n\}.$$

Suppose that m quantities $y_1, y_2, \dots, y_m$ have linear and homogeneous expressions in terms of n other quantities $x_1, x_2, \dots, x_n$:

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n, \end{aligned}\right\} \tag{4}$$

or more concisely,

$$y_i = \sum_{k=1}^{n} a_{ik}x_k \qquad (i = 1, 2, \dots, m). \tag{4'}$$

The transformation of the quantities $x_1, x_2, \dots, x_n$ into the quantities $y_1, y_2, \dots, y_m$ by means of the formulas (4) is called a *linear transformation*.

The coefficients of this transformation form a rectangular matrix (1) of dimension $m \times n$.

The linear transformation (4) determines the matrix (1) uniquely, and vice versa.

In the next section we shall define the basic operations on rectangular matrices using the properties of the linear transformations (4) as our starting point.

§ 2. Addition and Multiplication of Rectangular Matrices

We shall define the basic operations on matrices: addition of matrices, multiplication of a matrix by a number, and multiplication of matrices.

1. Suppose that the quantities $y_1, y_2, \dots, y_m$ are expressed in terms of the quantities $x_1, x_2, \dots, x_n$ by means of the linear transformation

$$y_i = \sum_{k=1}^{n} a_{ik}x_k \qquad (i = 1, 2, \dots, m) \tag{5}$$

---

[2] Here δ_{ik} is the Kronecker symbol: $\delta_{ik} = \begin{cases} 1 & (i = k), \\ 0 & (i \neq k). \end{cases}$

and the quantities $z_1, z_2, \ldots, z_m$ in terms of the same quantities $x_1, x_2, \ldots, x_n$ by means of the transformation

$$z_i = \sum_{k=1}^{n} b_{ik} x_k \qquad (i=1, 2, \ldots, m). \tag{6}$$

Then

$$y_i + z_i = \sum_{k=1}^{n} (a_{ik} + b_{ik})\, x_k \qquad (i=1, 2, \ldots, m). \tag{7}$$

In accordance with this,we formulate the following definition.

Definition 2: *The sum of two rectangular matrices* $A = \| a_{ik} \|$ *and* $B = \| b_{ik} \|$, *both of dimension* $m \times n$, *is the matrix* $C = \| c_{ik} \|$, *of the same dimension, whose elements are the sums of the corresponding elements of the given matrices:*

$$C = A + B,$$

where

$$c_{ik} = a_{ik} + b_{ik} \quad (i=1, 2, \ldots, m;\ k=1, 2, \ldots, n).$$

The operation of forming the sum of given matrices is called addition.

Example.

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix} + \begin{Vmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{Vmatrix} = \begin{Vmatrix} a_1 + c_1 & a_2 + c_2 & a_3 + c_3 \\ b_1 + d_1 & b_2 + d_2 & b_3 + d_3 \end{Vmatrix}.$$

According to Definition 2, only rectangular matrices of equal dimension can be added.

By virtue of the same definition, the coefficient matrix of the transformation (7) is the sum of the coefficient matrices of the transformations (5) and (6).

From the definition of matrix addition it follows immediately that this operation has the properties of commutativity and associativity:

1. $A + B = B + A$;
2. $(A + B) + C = A + (B + C)$.

Here A, B, and C are arbitrary rectangular matrices all of equal dimension.

The operation of addition of matrices extends in a natural way to the case of an arbitrary finite number of summands.

2. Let us multiply the quantities $y_1, y_2, \ldots, y_m$ in the transformation (5) by some number α of F. Then

$$\alpha y_i = \sum_{k=1}^{n} (\alpha a_{ik})\, x_k \qquad (i=1, 2, \ldots, m).$$

In accordance with this, we formulate the following definition.

Definition 3. *The product of a matrix* $A = \| a_{ik} \|$ $(i=1, 2, \ldots, m;$ $k=1, 2, \ldots, n)$ *by a number* α *of* F *is the matrix* $C = \| c_{ik} \|$ $(i=1, 2, \ldots, m;$ $k=1, 2, \ldots, n)$ *whose elements are obtained from the corresponding elements of* A *by multiplication by* α:

$$C = \alpha A,$$

where

$$c_{ik} = \alpha a_{ik} \qquad (i=1, 2, \ldots, m;\ k=1, 2, \ldots, n).$$

The** operation of forming the product of a matrix by a number is called* ***multiplication *of the matrix by the number.*

Example.

$$\alpha \begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix} = \begin{Vmatrix} \alpha a_1 & \alpha a_2 & \alpha a_3 \\ \alpha b_1 & \alpha b_2 & \alpha b_3 \end{Vmatrix}.$$

It is easy to see that

1. $\alpha(A + B) = \alpha A + \alpha B$,
2. $(\alpha + \beta) A = \alpha A + \beta A$,
3. $(\alpha\beta) A = \alpha(\beta A)$.

Here A and B are rectangular matrices of equal dimension and α and β are numbers of F.

The *difference* $A - B$ of two rectangular matrices of equal dimension is defined by

$$A - B = A + (-1) B.$$

If A is a square matrix of order n and α a number of F, then[3]

$$|\alpha A| = \alpha^n |A|.$$

3. Suppose that the quantities $z_1, z_2, \ldots, z_m$ are expressed in terms of the quantities $y_1, y_2, \ldots, y_n$ by the transformation

$$z_i = \sum_{k=1}^{n} a_{ik} y_k \qquad (i=1, 2, \ldots, m) \tag{8}$$

and that the quantities $y_1, y_2, \ldots, y_n$ are expressed in terms of the quantities $x_1, x_2, \ldots, x_q$ by the formulas

$$y_k = \sum_{j=1}^{q} b_{kj} x_j \qquad (k=1, 2, \ldots, n) \tag{9}$$

Then on substituting these expressions for the y_k $(k=1, 2, \ldots, n)$ in (8) we can express $z_1, z_2, \ldots, z_m$ in terms of $x_1, x_2, \ldots, x_q$ by means of the composite transformation:

---

[3] Here the symbols $|A|$ and $|\alpha A|$ denote the determinants of the matrices A and αA (see p. 1).

$$z_i = \sum_{k=1}^{n} a_{ik} \sum_{j=1}^{q} b_{kj} x_j = \sum_{j=1}^{q} \Bigl(\sum_{k=1}^{n} a_{ik} b_{kj}\Bigr) x_j \qquad (i = 1, 2, \ldots, m). \tag{10}$$

In accordance with this we formulate the following definition.

DEFINITION 4. *The product of two rectangular matrices*

$$A = \begin{Vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \multicolumn{2}{c}{\dotfill} \\ a_{m1} & a_{m2} \ldots a_{mn} \end{Vmatrix}, \qquad B = \begin{Vmatrix} b_{11} & b_{12} \ldots b_{1q} \\ b_{21} & b_{22} \ldots b_{2q} \\ \multicolumn{2}{c}{\dotfill} \\ b_{n1} & b_{n2} \ldots b_{nq} \end{Vmatrix}$$

is the matrix

$$C = \begin{Vmatrix} c_{11} & c_{12} \ldots c_{1q} \\ c_{21} & c_{22} \ldots c_{2q} \\ \multicolumn{2}{c}{\dotfill} \\ c_{m1} & c_{m2} \ldots c_{mq} \end{Vmatrix}$$

in which the element c_{ij} at the intersection of the i-th row and the j-th column is the 'product'[4] *of the i-th row of the first matrix A into the j-th column of the second matrix B:*

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, q). \tag{11}$$

The operation of forming the product of given matrices is called matrix multiplication.

Example.

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{Vmatrix} \begin{Vmatrix} c_1 & d_1 & e_1 & f_1 \\ c_2 & d_2 & e_2 & f_2 \\ c_3 & d_3 & e_3 & f_3 \end{Vmatrix} =$$

$$= \begin{Vmatrix} a_1c_1 + a_2c_2 + a_3c_3 & a_1d_1 + a_2d_2 + a_3d_3 & a_1e_1 + a_2e_2 + a_3e_3 & a_1f_1 + a_2f_2 + a_3f_3 \\ b_1c_1 + b_2c_2 + b_3c_3 & b_1d_1 + b_2d_2 + b_3d_3 & b_1e_1 + b_2e_2 + b_3e_3 & b_1f_1 + b_2f_2 + b_3f_3 \end{Vmatrix}.$$

By Definition 4 the coefficient matrix of the transformation (10) is the product of the coefficient matrices of (8) and (9).

Note that *the operation of multiplication of two rectangular matrices can only be carried out when the number of columns of the first factor is equal to the number of rows of the second.* In particular, multiplication is always possible when both factors are square matrices of one and the same order.

---

[4] The product of two sequences of numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ is defined as the sum of the products of the corresponding numbers: $\sum\limits_{i=1}^{n} a_i b_i$.

The reader should observe that even in this special case the multiplication of matrices does not have the property of commutativity. For example,

$$\begin{Vmatrix} 1 & 2 \\ 3 & 4 \end{Vmatrix} \begin{Vmatrix} 2 & 0 \\ 3 & -1 \end{Vmatrix} = \begin{Vmatrix} 8 & -2 \\ 18 & -4 \end{Vmatrix}, \quad \text{but} \quad \begin{Vmatrix} 2 & 0 \\ 3 & -1 \end{Vmatrix} \begin{Vmatrix} 1 & 2 \\ 3 & 4 \end{Vmatrix} = \begin{Vmatrix} 2 & 4 \\ 0 & 2 \end{Vmatrix}.$$

If $AB = BA$, then the matrices A and B are called *permutable* or *commuting.*

Example. The matrices

$$A = \begin{Vmatrix} 1 & 2 \\ -2 & 0 \end{Vmatrix} \quad \text{and} \quad B = \begin{Vmatrix} -3 & 2 \\ -2 & -4 \end{Vmatrix}$$

are permutable, because

$$AB = \begin{Vmatrix} -7 & -6 \\ 6 & -4 \end{Vmatrix} \quad \text{and} \quad BA = \begin{Vmatrix} -7 & -6 \\ 6 & -4 \end{Vmatrix}.$$

It is very easy to verify the *associative* property of matrix multiplication and also the *distributive* property of multiplication with respect to addition:

1. $(AB)\,C = A\,(BC)$,
2. $(A + B)\,C = AC + BC$,
3. $A\,(B + C) = AB + AC$.

The definition of matrix multiplication extends in a natural way to the case of several factors.

When we make use of the multiplication of rectangular matrices, we can write the linear transformation

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

as a single matrix equation

$$\begin{Vmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{Vmatrix} = \begin{Vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \multicolumn{2}{c}{\dotfill} \\ a_{m1} & a_{m2} \ldots a_{mn} \end{Vmatrix} \begin{Vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Vmatrix},$$

or in abbreviated form,

$$y = Ax.$$

Here $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ are column matrices and $A = \| a_{ik} \|$ is a rectangular matrix of dimension $m \times n$.

Let us treat the special case when in the product $C = AB$ the second factor is a square diagonal matrix $B = \{d_1, d_2, \ldots, d_n\}$. Then it follows from (11) that

$$c_{ij} = a_{ij} d_j \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n),$$

i.e.,

$$\begin{Vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{Vmatrix} \begin{Vmatrix} d_1 & 0 \ldots 0 \\ 0 & d_2 \ldots 0 \\ \ldots & \ldots \\ 0 & 0 \ldots d_n \end{Vmatrix} = \begin{Vmatrix} a_{11}d_1 & a_{12}d_2 \ldots a_{1n}d_n \\ a_{21}d_1 & a_{22}d_2 \ldots a_{2n}d_n \\ \ldots\ldots & \ldots\ldots \\ a_{m1}d_1 & a_{m2}d_2 \ldots a_{mn}d_n \end{Vmatrix}.$$

Similarly,

$$\begin{Vmatrix} d_1 & 0 \ldots 0 \\ 0 & d_2 \ldots 0 \\ \ldots & \ldots \\ 0 & 0 \ldots d_m \end{Vmatrix} \begin{Vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{Vmatrix} = \begin{Vmatrix} d_1a_{11} & d_1a_{12} \ldots d_1a_{1n} \\ d_2a_{21} & d_2a_{22} \ldots d_2a_{2n} \\ \ldots\ldots & \ldots\ldots \\ d_ma_{m1} & d_ma_{m2} \ldots d_ma_{mn} \end{Vmatrix}.$$

Hence: *When a rectangular matrix A is multiplied on the right (left) by a diagonal matrix $\{d_1, d_2, \ldots\}$, then the columns (rows) of A are multiplied by $d_1, d_2, \ldots$, respectively.*

4. Suppose that a square matrix $C = \| c_{ij} \|_1^m$ is the product of two rectangular matrices $A = \| a_{ik} \|$ and $B = \| b_{kj} \|$ of dimension $m \times n$ and $n \times m$, respectively:

$$\begin{Vmatrix} c_{11} & \ldots & c_{1m} \\ \ldots & \ldots & \ldots \\ c_{m1} & \ldots & c_{mm} \end{Vmatrix} = \begin{Vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{Vmatrix} \begin{Vmatrix} b_{11} & \ldots & b_{1m} \\ b_{21} & \ldots & b_{2m} \\ \ldots & \ldots & \ldots \\ b_{n1} & \ldots & b_{nm} \end{Vmatrix}, \tag{12}$$

i.e.,

$$c_{ij} = \sum_{\alpha=1}^{n} a_{i\alpha} b_{\alpha j} \qquad (i, j = 1, 2, \ldots, m). \tag{13}$$

We shall establish the important *Binet-Cauchy formula*, which expresses the determinant $| C |$ in terms of the minors of A and B:

$$\begin{vmatrix} c_{11} & \ldots & c_{1m} \\ \ldots & \ldots & \ldots \\ c_{m1} & \ldots & c_{mm} \end{vmatrix} = \sum_{1 \leqq k_1 < k_2 < \cdots < k_m \leqq n} \begin{vmatrix} a_{1k_1} & \ldots & a_{1k_m} \\ \ldots & \ldots & \ldots \\ a_{mk_1} & \ldots & a_{mk_m} \end{vmatrix} \begin{vmatrix} b_{k_11} & \ldots & b_{k_1m} \\ \ldots & \ldots & \ldots \\ b_{k_m1} & \ldots & b_{k_mm} \end{vmatrix} \tag{14}$$

or, in the notation[5] of page 2,

$$C\begin{pmatrix} 1 & 2 & \ldots & m \\ 1 & 2 & \ldots & m \end{pmatrix} = \sum_{1 \leqq k_1 < k_2 < \cdots < k_m \leqq n} A\begin{pmatrix} 1 & 2 & \ldots & m \\ k_1 & k_2 & \ldots & k_m \end{pmatrix} B\begin{pmatrix} k_1 & k_2 & \ldots & k_m \\ 1 & 2 & \ldots & m \end{pmatrix}. \tag{14'}$$

According to this formula *the determinant of C is the sum of the products of all possible minors of the maximal (m-th) order*[5] *of A into the corresponding minors of the same order of B.*

Derivation of the Binet-Cauchy formula. By (13) the determinant of C can be represented in the form

$$\begin{aligned} \begin{vmatrix} c_{11} & \ldots & c_{1m} \\ \ldots & \ldots & \ldots \\ c_{m1} & \ldots & c_{mm} \end{vmatrix} &= \begin{vmatrix} \sum\limits_{\alpha_1=1}^{n} a_{1\alpha_1} b_{\alpha_1 1} & \ldots & \sum\limits_{\alpha_m=1}^{n} a_{1\alpha_m} b_{\alpha_m m} \\ \ldots & \ldots & \ldots \\ \sum\limits_{\alpha_1=1}^{n} a_{m\alpha_1} b_{\alpha_1 1} & \ldots & \sum\limits_{\alpha_m=1}^{n} a_{m\alpha_m} b_{\alpha_m m} \end{vmatrix} \\ &= \sum_{\alpha_1, \ldots, \alpha_m = 1}^{n} \begin{vmatrix} a_{1\alpha_1} b_{\alpha_1 1} & \ldots & a_{1\alpha_m} b_{\alpha_m m} \\ \ldots & \ldots & \ldots \\ a_{m\alpha_1} b_{\alpha_1 1} & \ldots & a_{m\alpha_m} b_{\alpha_m m} \end{vmatrix} \\ &= \sum_{\alpha_1, \ldots, \alpha_m = 1}^{n} A\begin{pmatrix} 1 & 2 & \ldots & m \\ \alpha_1 & \alpha_2 & \ldots & \alpha_m \end{pmatrix} b_{\alpha_1 1} b_{\alpha_2 2} \ldots b_{\alpha_m m}. \end{aligned} \tag{15}$$

If $m > n$, then among the numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ there are always at least two that are equal, so that every summand on the right-hand side of (15) is zero. Hence in this case $| C | = 0$.

Now let $m \leqq n$. Then in the sum on the right-hand side of (15) all those summands will be zero in which at least two of the subscripts $\alpha_1, \alpha_2, \ldots, \alpha_m$ are equal. All the remaining summands of (15) can be split into groups of $m!$ terms each by combining into one group those summands that differ from each other only in the order of the subscripts $\alpha_1, \alpha_2, \ldots, \alpha_m$ (so that

---

[5] When $m > n$, the matrices A and B do not have minors of order m. In that case the right-hand sides of (14) and (14′) are to be replaced by zero.

within each such group the subscripts $\alpha_1, \alpha_2, \ldots, \alpha_m$ have one and the same set of values). Now within one such group the sum of the corresponding terms is[6]

$$\sum \varepsilon(\alpha_1, \alpha_2, \ldots, \alpha_m) A\begin{pmatrix} 1 & 2 & \ldots & m \\ k_1 & k_2 & \ldots & k_m \end{pmatrix} b_{\alpha_1 1} b_{\alpha_2 2} \cdots b_{\alpha_m m} =$$
$$= A\begin{pmatrix} 1 & 2 & \ldots & m \\ k_1 & k_2 & \ldots & k_m \end{pmatrix} \sum \varepsilon(\alpha_1, \alpha_2, \ldots, \alpha_m) b_{\alpha_1 1} b_{\alpha_2 2} \cdots b_{\alpha_m m}$$
$$= A\begin{pmatrix} 1 & 2 & \ldots & m \\ k_1 & k_2 & \ldots & k_m \end{pmatrix} B\begin{pmatrix} k_1 & k_2 & \ldots & k_m \\ 1 & 2 & \ldots & m \end{pmatrix}.$$

Hence from (15) we obtain (14′).

Example 1.

$$\left\| \begin{matrix} a_1c_1 + a_2c_2 + \cdots + a_nc_n & a_1d_1 + a_2d_2 + \cdots + a_nd_n \\ b_1c_1 + b_2c_2 + \cdots + b_nc_n & b_1d_1 + b_2d_2 + \cdots + b_nd_n \end{matrix} \right\| = \left\| \begin{matrix} a_1 & a_2 \ldots a_n \\ b_1 & b_2 \ldots b_n \end{matrix} \right\| \left\| \begin{matrix} c_1 & d_1 \\ c_2 & d_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ c_n & d_n \end{matrix} \right\|.$$

Therefore formula (14) yields the so-called *Cauchy identity*

$$\begin{vmatrix} a_1c_1 + a_2c_2 + \cdots + a_nc_n & a_1d_1 + a_2d_2 + \cdots + a_nd_n \\ b_1c_1 + b_2c_2 + \cdots + b_nc_n & b_1d_1 + b_2d_2 + \cdots + b_nd_n \end{vmatrix} = \sum_{1 \leqq i < k \leqq n} \begin{vmatrix} a_i & a_k \\ b_i & b_k \end{vmatrix} \begin{vmatrix} c_i & d_i \\ c_k & d_k \end{vmatrix}. \quad (16)$$

Setting $a_i = c_i$, $b_i = d_i$ $(i = 1, 2, \ldots, n)$ in this identity, we obtain:

$$\begin{vmatrix} a_1^2 + a_2^2 + \cdots + a_n^2 & a_1b_1 + a_2b_2 + \cdots + a_nb_n \\ a_1b_1 + a_2b_2 + \cdots + a_nb_n & b_1^2 + b_2^2 + \cdots + b_n^2 \end{vmatrix} = \sum_{1 \leqq i < k \leqq n} \begin{vmatrix} a_i & a_k \\ b_i & b_k \end{vmatrix}^2.$$

If a_i and b_i $(i = 1, 2, \ldots, n)$ are real numbers, we deduce the well-known inequality

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)\,(b_1^2 + b_2^2 + \cdots + b_n^2). \quad (17)$$

Here the equality sign holds if and only if all the numbers a_i are proportional to the corresponding numbers b_i $(i = 1, 2, \ldots, n)$.

Example 2.

$$\left\| \begin{matrix} a_1c_1 + b_1d_1 & \ldots & a_1c_n + b_1d_n \\ \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ a_nc_1 + b_nd_1 & \ldots & a_nc_n + b_nd_n \end{matrix} \right\| = \left\| \begin{matrix} a_1 & b_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ a_n & b_n \end{matrix} \right\| \left\| \begin{matrix} c_1 \ldots c_n \\ d_1 \ldots d_n \end{matrix} \right\|.$$

---

[6] Here $k_1 < k_2 < \ldots < k_m$ is the normal order of the subscripts $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\varepsilon(\alpha_1, \alpha_2, \ldots, \alpha_m) = (-1)^N$, where N is the number of transpositions of the indices needed to put the permutation $\alpha_1, \alpha_2, \ldots, \alpha_m$ into normal order.

Therefore for $n > 2$

$$\begin{vmatrix} a_1c_1 + b_1d_1 & \ldots & a_1c_n + b_1d_n \\ \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ a_nc_1 + b_nd_1 & \ldots & a_nc_n + b_nd_n \end{vmatrix} = 0.$$

Let us consider the special case where A and B are square matrices of one and the same order n. When we set $m = n$ in (14′), we arrive at the well-known multiplication theorem for determinants:

$$C\begin{pmatrix} 1 & 2 \ldots n \\ 1 & 2 \ldots n \end{pmatrix} = A\begin{pmatrix} 1 & 2 \ldots n \\ 1 & 2 \ldots n \end{pmatrix} B\begin{pmatrix} 1 & 2 \ldots n \\ 1 & 2 \ldots n \end{pmatrix}$$

or, in another notation,

$$|C| = |AB| = |A| \cdot |B|. \quad (18)$$

Thus, *the determinant of the product of two square matrices is equal to the product of the determinants of the factors.*

5. The Binet-Cauchy formula enables us, in the general case also, to express the minors of the product of two rectangular matrices in terms of the minors of the factors. Let

$$A = \| a_{ik} \|, \quad B = \| b_{kj} \|, \quad C = \| c_{ij} \|$$
$$(i = 1, 2, \ldots, m;\; k = 1, 2, \ldots, n;\; j = 1, 2, \ldots, q)$$

and

$$C = AB.$$

We consider an arbitrary minor of C:

$$C\begin{pmatrix} i_1 & i_2 \ldots i_p \\ j_1 & j_2 \ldots j_p \end{pmatrix} \quad \left(\begin{matrix} 1 \leqq i_1 < i_2 < \cdots < i_p \leqq m \\ \leqq j_1 < j_2 < \cdots < j_p \leqq q \end{matrix} ;\; p \leqq m \text{ and } p \leqq q \right).$$

The matrix formed from the elements of this minor is the product of two rectangular matrices

$$\left\| \begin{matrix} a_{i_1 1} & a_{i_1 2} \ldots a_{i_1 n} \\ \cdot\cdot\cdot\cdot & \cdot\cdot\cdot\cdot \\ a_{i_p 1} & a_{i_p 2} \ldots a_{i_p n} \end{matrix} \right\|, \quad \left\| \begin{matrix} b_{1j_1} \ldots b_{1j_p} \\ b_{2j_1} \ldots b_{2j_p} \\ \cdot\cdot\cdot\cdot\cdot \\ b_{nj_1} \ldots b_{nj_p} \end{matrix} \right\|.$$

Therefore, by applying the Binet-Cauchy formula, we obtain:[7]

$$C\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ j_1 & j_2 & \dots & j_p \end{pmatrix} = \sum_{1 \leqq k_1 < k_2 < \dots < k_p \leqq n} A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} B\begin{pmatrix} k_1 & k_2 & \dots & k_p \\ j_1 & j_2 & \dots & j_p \end{pmatrix}. \qquad (19)$$

For $p = 1$ formula (19) goes over into (11). For $p > 1$ formula (19) is a natural generalization of (11).

We mention another consequence of (19).

The rank of the product of two rectangular matrices does not exceed the rank of either factor.

If $C = AB$ and r_A, r_B, r_C are the ranks of A, B, C, then

$$r_C \leqq \min (r_A, r_B).$$

§ 3. Square Matrices

1. The square matrix of order n in which the main diagonal consists entirely of units and all the other elements are zero is called the *unit matrix* and is denoted by $E^{(n)}$ or simply by E. The name 'unit matrix' is connected with the following property of E: For every rectangular matrix

$$A = \| a_{ik} \| \qquad (i = 1, 2, \dots, m;\ k = 1, 2, \dots, n)$$

we have

$$E^{(m)} A = A E^{(n)} = A .$$

Clearly

$$E^{(n)} = \| \delta_{ik} \|_1^n .$$

Let $A = \| a_{ik} \|_1^n$ be a square matrix. Then the *power of the matrix* is defined in the usual way:

$$A^p = \underbrace{AA \cdots A}_{p \text{ times}} \qquad (p = 1, 2, \dots); \qquad A^0 = E .$$

From the associative property of matrix multiplication it follows that

$$A^p A^q = A^{p+q} .$$

Here p and q are arbitrary non-negative integers.

---

[7] It follows from the Binet-Cauchy formula that the minors of order p in C for $p > n$ (if minors of such orders exist) are all zero. In that case the right-hand side of (19) is to be replaced by zero. See footnote 5, p. 9.

We consider a polynomial (integral rational function) with coefficients in the field F:

$$f(t) = \alpha_0 t^m + \alpha_1 t^{m-1} + \dots + \alpha_m .$$

Then by $f(A)$ we shall mean the matrix

$$f(A) = \alpha_0 A^m + \alpha_1 A^{m-1} + \dots + \alpha_m E .$$

We define in this way a *polynomial in a matrix.*

Suppose that $f(t)$ is the product of two polynomials $g(t)$ and $h(t)$:

$$f(t) = g(t) h(t). \qquad (21)$$

The polynomial $f(t)$ is obtained from $g(t)$ and $h(t)$ by multiplication term by term and collection of similar terms. In this we make use of the multiplication rule for powers: $t^p \cdot t^q = t^{p+q}$. Since all these operations remain valid when the scalar t is replaced by the matrix A, it follows from (21) that

$$f(A) = g(A) h(A).$$

Hence, in particular,[8]

$$g(A) h(A) = h(A) g(A) ; \qquad (22)$$

i.e., *two polynomials in one and the same matrix are always permutable.*

Examples.

Let the sequence of elements a_{ik} for which $k - i = p$ $(i - k = p)$ in a rectangular matrix $A = \| a_{ik} \|$ be called the p-th *superdiagonal* (*subdiagonal*) of the matrix. We denote by $H^{(n)}$ the square matrix order n in which all the elements of the first superdiagonal are units and all the other elements are zero. The matrix $H^{(n)}$ will also be denoted simply by H. Then

$$H = H^{(n)} = \left\| \begin{matrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \cdot \\ \cdot & \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ 0 & 0 & 0 & \dots & 0 \end{matrix} \right\|, \qquad H^2 = \left\| \begin{matrix} 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & 1 \\ \cdot & & & \cdot & 0 \\ 0 & 0 & 0 & \dots & 0 \end{matrix} \right\|, \dots ,$$

$$H^p = 0 \qquad (p \geqq n) .$$

---

[8] Since each of these products is equal to one and the same $f(A)$, by virtue of the fact that $h(t) g(t) = f(t)$. It is worth mentioning that the substitution of matrices in an algebraic identity in *several* variables is not valid. The substitution of matrices that commute with one another, however, is allowable in this case.

By these equations, if

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1} + \cdots$$

is a polynomial in t, then

$$f(H) = a_0 E + a_1 H + a_2 H^2 + \cdots = \left\| \begin{matrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_2 \\ \vdots & & & \ddots & a_1 \\ 0 & 0 & 0 & \cdots & a_0 \end{matrix} \right\| .$$

Similarly, if F is the square matrix of order n in which all the elements of the first subdiagonal are units and all others are zero, then

$$f(F) = a_0 E + a_1 F + a_2 F^2 + \cdots = \left\| \begin{matrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n-1} & \cdots & a_1 & a_0 \end{matrix} \right\| .$$

We leave it to the reader to verify the following properties of the matrices H and F:

1. *When an arbitrary rectangular matrix A of dimension $m \times n$ is multiplied on the left by the matrix H (or F) of order m, then all the rows of A are shifted upward (or downward) by one place, the first (last) row of A disappears, and the last (first) row of the product is filled by zeros.* For example,

$$\left\| \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \right\| \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\| = \left\| \begin{matrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ 0 & 0 & 0 & 0 \end{matrix} \right\| ,$$

$$\left\| \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \right\| \left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\| = \left\| \begin{matrix} 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix} \right\| .$$

2. *When an arbitrary rectangular matrix A of dimension $m \times n$ is multiplied on the right by the matrix H (or F) of order n, then all the columns of A are shifted to the right (left) by one place, the last (first) column of A disappears, and the first (last) column of the product is filled by zeros.* For example,

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\| \left\| \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \right\| = \left\| \begin{matrix} 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \\ 0 & c_1 & c_2 & c_3 \end{matrix} \right\| ,$$

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\| \left\| \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} \right\| = \left\| \begin{matrix} a_2 & a_3 & a_4 & 0 \\ b_2 & b_3 & b_4 & 0 \\ c_2 & c_3 & c_4 & 0 \end{matrix} \right\| .$$

2. A square matrix A is called *singular* if $|A| = 0$. Otherwise A is called *non-singular*.

Let $A = \| a_{ik} \|_1^n$ be a non-singular matrix ($|A| \neq 0$). Let us consider the linear transformation with coefficient matrix A

$$y_i = \sum_{k=1}^{n} a_{ik} x_k \quad (i = 1, 2, \ldots, n). \tag{23}$$

When we regard (23) as equations for $x_1, x_2, \ldots, x_n$ and observe that the determinant of the system of equations (23) is, by assumption, different from zero, then we can express $x_1, x_2, \ldots, x_n$ in terms of $y_1, y_2, \ldots, y_n$ by means of the well-known formulas:

$$x_i = \frac{1}{|A|} \left| \begin{matrix} a_{11} & \cdots & a_{1,i-1} & y_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & y_2 & a_{2,i+1} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{n,i-1} & y_n & a_{n,i+1} & \cdots & a_{nn} \end{matrix} \right| \equiv \sum_{k=1}^{n} a_{ik}^{(-1)} y_k \quad (i = 1, 2, \ldots, n). \tag{24}$$

We have thus obtained the 'inverse' transformation of the transformation (23). The coefficient matrix of this transformation

$$A^{-1} = \| a_{ik}^{(-1)} \|_1^n$$

will be called the *inverse matrix* of A. From (24) it is easy to see that

$$a_{ik}^{(-1)} = \frac{A_{ki}}{|A|} \quad (i, k = 1, 2, \ldots, n), \tag{25}$$

where A_{ki} is the algebraic complement (the cofactor) of the element a_{ki} in the determinant $|A|$ $(i, k = 1, 2, \ldots, n)$.

For example, if

$$A = \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\| \text{ and } |A| \neq 0,$$

then

$$A^{-1} = \frac{1}{|A|} \left\| \begin{matrix} b_2c_3-b_3c_2 & a_3c_2-a_2c_3 & a_2b_3-a_3b_2 \\ b_3c_1-b_1c_3 & a_1c_3-a_3c_1 & a_3b_1-a_1b_3 \\ b_1c_2-b_2c_1 & a_2c_1-a_1c_2 & a_1b_2-a_2b_1 \end{matrix} \right\| .$$

By forming the composite transformation of the given transformation (23) and the inverse (24), in either order, we obtain in both cases the identity transformation (with the unit matrix as coefficient matrix); therefore

$$AA^{-1} = A^{-1}A = E. \qquad (26)$$

The validity of equation (26) can also be established by direct multiplication of the matrices A and A^{-1}. In fact, by (25) we have[9]

$$[AA^{-1}]_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}^{(-1)} = \frac{1}{|A|} \sum_{k=1}^{n} a_{ik}A_{jk} = \delta_{ij} \qquad (i, j = 1, 2, \ldots, n).$$

Similarly,

$$[A^{-1}A]_{ij} = \sum_{k=1}^{n} a_{ij}^{(-1)}a_{kj} = \frac{1}{|A|} \sum_{k=1}^{n} A_{ki}a_{kj} = \delta_{ij} \qquad (i, j = 1, 2, \ldots, n).$$

It is easy to see that the matrix equations

$$AX = E \text{ and } XA = E \qquad (|A| \neq 0) \qquad (27)$$

have no solutions other than $X = A^{-1}$. For by multiplying both sides of the first (second) equation on the left (right) by A^{-1} and using the associative property of matrix multiplication we obtain from (26) in both cases:[10]

$$X = A^{-1}.$$

---

[9] Here we make use of the well-known property of determinants that the sum of the products of the elements of an arbitrary column into the cofactors of the elements of that column is equal to the value of the determinant and the sum of the products of the elements of a column into the cofactors of the corresponding element of another column is zero.

[10] If A is a singular matrix, then the equations (27) have no solution. For if one of these equations had a solution $X = \| x_{ik} \|_1^n$, then we would have by the multiplication theorem of determinants (see formula (18)) that $|A| \cdot |X| = |E| = 1$, and this is impossible when $|A| = 0$.

In the same way it can be shown that each of the matrix equations

$$AX = B, XA = B \qquad (|A| \neq 0), \qquad (28)$$

where X and B are rectangular matrices of equal dimensions and A is a square matrix of appropriate order, have one and only one solution,

$$X = A^{-1}B \quad \text{and} \quad X = BA^{-1}, \qquad (29)$$

respectively. The matrices (29) are the 'left' and the 'right' quotients on 'dividing' B by A. From (28) and (29) we deduce (see p. 12) that $r_B \leqq r_X$ and $r_X \leqq r_B$, so that $r_X = r_B$. On comparing this with (28), we have:

When a rectangular matrix is multiplied on the left or on the right by a non-singular matrix, the rank of the original matrix remains unchanged.

Note that (26) implies $|A| \cdot |A^{-1}| = 1$, i.e.

$$|A^{-1}| = \frac{1}{|A|}.$$

For any two non-singular matrices we have

$$(AB)^{-1} = B^{-1}A^{-1}. \qquad (30)$$

3. All the matrices of order n form a ring[11] with unit element $E^{(n)}$.

Since in this ring the operation of multiplication by a number of F is defined, and since there exists a basis of n^2 linearly independent matrices in terms of which all the matrices of order n can be expressed linearly,[12] the ring of matrices of order n is an algebra.[13]

---

[11] A *ring* is a collection of elements in which two operations are defined and can always be carried out uniquely: the 'addition' of two elements (with the commutative and associative properties) and the 'multiplication' of two elements (with the associative and distributive properties with respect to addition); moreover, the addition is reversible. See, for example, van der Waerden, ***Modern Algebra***, § 14.

[12] For, an arbitrary matrix $A = \| a_{ik} \|_1^n$ with elements in F can be represented in the form $A = \sum_{i,k=1}^{n} a_{ik}E_{ik}$, where E_{ik} is the matrix of order n in which there is a 1 at the intersection of the i-th row and the k-th column and all the other elements are zeros.

[13] See, for example, van der Waerden, ***Modern Algebra***, § 17.

All the square matrices of order n form a commutative group with respect to the operation of addition.[14] All the non-singular matrices of order n form a (non-commutative) group with respect to the operation of multiplication.

A square matrix $A = \| a_{ik} \|_1^n$ is called *upper triangular* (*lower triangular*) if all the elements below (above) the main diagonal are zero:

$$A = \left\| \begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & 0 & \dots & a_{nn} \end{matrix} \right\|, \quad A = \left\| \begin{matrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{matrix} \right\|.$$

A diagonal matrix is a special case both of an upper triangular matrix and a lower triangular matrix.

Since the determinant of a triangular matrix is equal to the product of its diagonal elements, a triangular (and, in particular, a diagonal) matrix is non-singular if and only if all its diagonal elements are different from zero.

It is easy to verify that the sum and the product of two diagonal (upper triangular, lower triangular) matrices is a diagonal (upper triangular, lower triangular) matrix and that the inverse of a non-singular diagonal (upper triangular, lower triangular) matrix is a matrix of the same type. Therefore:

1. All the diagonal matrices of order n form a commutative group under the operation of addition, as do all the upper triangular matrices or all the lower triangular matrices.

2. All the non-singular diagonal matrices form a commutative group under multiplication.

3. All the non-singular upper (lower) triangular matrices form a (non-commutative) group under multiplication.

4. We conclude this section with a further important operation on matrices —*transposition.*

---

[14] A *group* is a set of objects in which an operation is defined which associates with any two elements a and b of the set a well-defined third element $a * b$ of the same set provided that

1) the operation has the associative property $((a * b) * c = a * (b * c))$,

2) there exists a unit element e in the set $(a * e = e * a = a)$, and

3) for every element e of the set there exists an inverse element a^{-1} $(a * a^{-1} = a^{-1} * a = e)$.

A group is called *commutative,* or abelian, if the group operation has the commutative property. Concerning the group concept see, for example, [53], pp. 245ff.

If $A = \| a_{ik} \|$ $(i = 1, 2, \dots, m;\ k = 1, 2, \dots, n)$, then the *transpose* A^{T} is defined as $A^{\mathsf{T}} = \| a_{ik}^{\mathsf{T}} \|$, where $a_{ki}^{\mathsf{T}} = a_{ik}$ $(i = 1, 2, \dots, m;\ k = 1, 2, \dots, n)$. If A is of dimension $m \times n$, then A^{T} is of dimension $n \times m$.

It is easy to verify the following properties:[15]

1. $(A + B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}}$,
2. $(\alpha A)^{\mathsf{T}} = \alpha A^{\mathsf{T}}$,
3. $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$,
4. $(A^{-1})^{\mathsf{T}} = (A^{\mathsf{T}})^{-1}$.

If a square matrix $S = \| s_{ij} \|_1^n$ coincides with its transpose $(S^{\mathsf{T}} = S)$, then it is called *symmetric*. In a symmetric matrix elements that are symmetrically placed with respect to the main diagonal are equal. Note that the product of two symmetric matrices is not, in general, symmetric. By 3., this holds if and only if the two given symmetric matrices are permutable.

If a square matrix $K = \| k_{ij} \|_1^n$ differs from its transpose by a factor -1 $(K^{\mathsf{T}} = -K)$, then it is called *skew-symmetric*. In a skew-symmetric matrix any two elements that are symmetrical to the main diagonal differ from each other by a factor -1 and the diagonal elements are zero. From 3. it follows that the product of two permutable skew-symmetric matrices is a symmetric matrix.[16]

§ 4. Compound Matrices. Minors of the Inverse Matrix

1. Let $A = \| a_{ik} \|_1^n$ be a given matrix. We consider all possible minors of A of order p $(1 \leqq p \leqq n)$:

$$A \begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} \qquad \left(1 \leqq \begin{matrix} i_1 < i_2 < \dots < i_p \\ k_1 < k_2 < \dots < k_p \end{matrix} \leqq n \right). \tag{31}$$

The number of these minors is N^2, where $N = \binom{n}{p}$ is the number of combinations of n objects taken p at a time. In order to arrange the minors (31) in a square array, we enumerate in some definite order—lexicographic order, for example—all the N combinations of p indices selected from among the indices $1, 2, \dots, n$.

---

[15] In formulas 1., 2., 3., A and B are arbitrary rectangular matrices for which the corresponding operations are feasible. In 4., A is an arbitrary square non-singular matrix.

[16] As regards the representation of a square matrix A in the form of a product of two symmetric matrices $(A = S_1S_2)$ or two skew-symmetric matrices $(A = K_1K_2)$, see [357].

If the combinations of indices $i_1 < i_2 < \dots < i_p$ and $k_1 < k_2 < \dots < k_p$ have the numbers α and β, then the minors (31) will also be denoted as follows:

$$\mathfrak{a}_{\alpha\beta} = A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix}.$$

By giving to α and β independently all the values from 1 to N, we obtain all the minors of $A = \| a_{ik} \|_1^n$ of order p.

The square matrix of order N

$$\mathfrak{A}_p = \| \mathfrak{a}_{\alpha\beta} \|_1^N$$

is called the p-th *compound matrix* of $A = \| a_{ik} \|_1^n$; p can take the values $1, 2, \dots, n$. Here $\mathfrak{A}_1 = A$, and $\mathfrak{A}_n$ consists of the single element $| A |$.

Note. The order of enumeration of the combination of indices is fixed once and for all and does not depend on the choice of A.

Example. Let

$$A = \begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{Vmatrix}.$$

We enumerate all combinations of the indices 1, 2, 3, 4 taken two at a time by arranging them in the following order:

$$(12) \quad (13) \quad (14) \quad (23) \quad (24) \quad (34).$$

Then

$$\mathfrak{A}_2 = \begin{Vmatrix} A\begin{pmatrix}1&2\\1&2\end{pmatrix} & A\begin{pmatrix}1&2\\1&3\end{pmatrix} & A\begin{pmatrix}1&2\\1&4\end{pmatrix} & A\begin{pmatrix}1&2\\2&3\end{pmatrix} & A\begin{pmatrix}1&2\\2&4\end{pmatrix} & A\begin{pmatrix}1&2\\3&4\end{pmatrix} \\ A\begin{pmatrix}1&3\\1&2\end{pmatrix} & A\begin{pmatrix}1&3\\1&3\end{pmatrix} & A\begin{pmatrix}1&3\\1&4\end{pmatrix} & A\begin{pmatrix}1&3\\2&3\end{pmatrix} & A\begin{pmatrix}1&3\\2&4\end{pmatrix} & A\begin{pmatrix}1&3\\3&4\end{pmatrix} \\ A\begin{pmatrix}1&4\\1&2\end{pmatrix} & A\begin{pmatrix}1&4\\1&3\end{pmatrix} & A\begin{pmatrix}1&4\\1&4\end{pmatrix} & A\begin{pmatrix}1&4\\2&3\end{pmatrix} & A\begin{pmatrix}1&4\\2&4\end{pmatrix} & A\begin{pmatrix}1&4\\3&4\end{pmatrix} \\ A\begin{pmatrix}2&3\\1&2\end{pmatrix} & A\begin{pmatrix}2&3\\1&3\end{pmatrix} & A\begin{pmatrix}2&3\\1&4\end{pmatrix} & A\begin{pmatrix}2&3\\2&3\end{pmatrix} & A\begin{pmatrix}2&3\\2&4\end{pmatrix} & A\begin{pmatrix}2&3\\3&4\end{pmatrix} \\ A\begin{pmatrix}2&4\\1&2\end{pmatrix} & A\begin{pmatrix}2&4\\1&3\end{pmatrix} & A\begin{pmatrix}2&4\\1&4\end{pmatrix} & A\begin{pmatrix}2&4\\2&3\end{pmatrix} & A\begin{pmatrix}2&4\\2&4\end{pmatrix} & A\begin{pmatrix}2&4\\3&4\end{pmatrix} \\ A\begin{pmatrix}3&4\\1&2\end{pmatrix} & A\begin{pmatrix}3&4\\1&3\end{pmatrix} & A\begin{pmatrix}3&4\\1&4\end{pmatrix} & A\begin{pmatrix}3&4\\2&3\end{pmatrix} & A\begin{pmatrix}3&4\\2&4\end{pmatrix} & A\begin{pmatrix}3&4\\3&4\end{pmatrix} \end{Vmatrix}$$

We mention some properties of compound matrices:

1. *From* $C = AB$ *it follows that* $\mathfrak{C}_p = \mathfrak{A}_p \cdot \mathfrak{B}_p (p = 1, 2, \dots, n)$.

For when we express the minors of order p $(1 \leqq p \leqq n)$ of the matrix product C, by formula (19), in terms of the minors of the same order of the factors, then we have:

$$C\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} = \sum_{1 \leqq l_1 < l_2 < \dots < l_p \leqq n} A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ l_1 & l_2 & \dots & l_p \end{pmatrix} B\begin{pmatrix} l_1 & l_2 & \dots & l_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix}$$
$$\left(1 \leqq \begin{matrix} i_1 < i_2 < \dots < i_p \\ k_1 < k_2 < \dots < k_p \end{matrix} \leqq n\right). \tag{32}$$

Obviously, in the notation of this section, equation (32) can be written as follows:

$$\mathfrak{c}_{\alpha\beta} = \sum_{\lambda=1}^{N} \mathfrak{a}_{\alpha\lambda} \mathfrak{b}_{\lambda\beta} \qquad (\alpha, \beta = 1, 2, \dots, N)$$

(here α, β, and λ are the numbers of the combinations of indices $i_1 < i_2 < \dots < i_p$; $k_1 < k_2 < \dots < k_p$; $l_1 < l_2 < \dots < l_p$). Hence

$$\mathfrak{C}_p = \mathfrak{A}_p \mathfrak{B}_p \qquad (p = 1, 2, \dots, n).$$

2. *From* $B = A^{-1}$ *it follows that* $\mathfrak{B}_p = \mathfrak{A}_p^{-1}$ $(p = 1, 2, \dots, n)$.

This result follows immediately from the preceding one when we set $C = E$ and bear in mind that $\mathfrak{C}_p$ is the unit matrix of order $N = \binom{n}{p}$

From 2. there follows an important formula that expresses the minors of the inverse matrix in terms of the minors of the given matrix:

If $B = A^{-1}$, then for arbitrary $\left(1 \leqq \begin{matrix} i_1 < i_2 < \dots < i_p \\ k_1 < k_2 < \dots < k_p \end{matrix} \leqq n\right)$

$$B\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} = \frac{(-1)^{\sum_{\nu=1}^{p} i_\nu + \sum_{\nu=1}^{p} k_\nu} A\begin{pmatrix} k'_1 & k'_2 & \dots & k'_{n-p} \\ i'_1 & i'_2 & \dots & i'_{n-p} \end{pmatrix}}{A\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}}, \tag{33}$$

here $i_1 < i_2 < \dots < i_p$ and $i'_1 < i'_2 < \dots < i'_{n-p}$ form a complete system of dices $1, 2, \dots, n$, as do $k_1 < k_2 < \dots < k_p$ and $k'_1 < k'_2 < \dots < k'_{n-p}$.

For it follows from $AB = E$ that

$$\mathfrak{A}_p \mathfrak{B}_p = \mathfrak{C}_p$$

or in more explicit form:

$$\sum_{\alpha=1}^{N} \mathfrak{a}_{\gamma\alpha}\mathfrak{b}_{\alpha\beta} = \delta_{\gamma\beta} = \begin{cases} 1 & (\gamma=\beta), \\ 0 & (\gamma\neq\beta). \end{cases} \tag{34}$$

Equations (34) can also be written as follows:

$$\sum_{1\leqq i_1<i_2<\cdots<i_p\leqq n} A\begin{pmatrix} j_1\, j_2 \cdots j_p \\ i_1\, i_2 \ldots i_p \end{pmatrix} B\begin{pmatrix} i_1\, i_2 \ldots i_p \\ k_1\, k_2 \ldots k_p \end{pmatrix} = \begin{cases} 1, & \text{if} \quad \sum\limits_{\nu=1}^{p} (j_\nu - k_\nu)^2 = 0, \\ 0, & \text{if} \quad \sum\limits_{\nu=1}^{p} (j_\nu - k_\nu)^2 > 0 \end{cases} \tag{34'}$$

$$\left(1 \leqq \begin{matrix} j_1 < j_2 < \ldots < j_p \\ k_1 < k_2 < \ldots < k_p \end{matrix} \leqq n\right).$$

On the other hand, when we apply the well-known Laplace expansion to the determinant $|A|$, we obtain

$$\sum_{1\leqq i_1<i_2<\cdots<i_p\leqq n} A\begin{pmatrix} j_1\; j_2 \cdots j_p \\ i_1\; i_2 \ldots i_p \end{pmatrix} \cdot (-1)^{\sum\limits_{\nu=1}^{p} i_\nu + \sum\limits_{\nu=1}^{p} k_\nu} A\begin{pmatrix} k'_1\; k'_2 \ldots k'_{n-p} \\ i'_1\; i'_2 \ldots i'_{n-p} \end{pmatrix} =$$

$$= \begin{cases} |A , & \text{if} \quad \sum\limits_{\nu=1}^{p} (j_\nu - k_\nu)^2 = 0, \\ 0, & \text{if} \quad \sum\limits_{\nu=1}^{p} (j_\nu - k_\nu)^2 > 0, \end{cases} \tag{35}$$

where $i_1 < i_2 < \cdots < i_p$ and $i'_1 < i'_2 < \cdots < i'_{n-p}$ form a complete system of indices $1, 2, \ldots, n$, as do $k_1 < k_2 < \cdots < k_p$ and $k'_1 < k'_2 < \cdots < k'_{n-p}$. Comparison of (35) with (34′) and (34) shows that the equations (34) are satisfied if we take together with $\mathfrak{b}_{\alpha\beta}$ not $B\begin{pmatrix} i_1\, i_2 \ldots i_p \\ k_1\, k_2 \ldots k_p \end{pmatrix}$ but rather

$$\frac{(-1)^{\sum\limits_{\nu=1}^{p} i_\nu + \sum\limits_{\nu=1}^{p} k_\nu} A\begin{pmatrix} k'_1\, k'_2 \ldots k'_{n-p} \\ i'_1\, i'_2 \ldots i'_{n-p} \end{pmatrix}}{A\begin{pmatrix} 1\; 2 \ldots n \\ 1\; 2 \ldots n \end{pmatrix}}.$$

Since the elements $\mathfrak{b}_{\alpha\beta}$ of the inverse matrix of $\mathfrak{A}_p$ are uniquely determined by (34), equation (33) must hold.

CHAPTER II

THE ALGORITHM OF GAUSS AND SOME OF ITS APPLICATIONS

§ 1. Gauss's Elimination Method

1. Let

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n \end{aligned}\right\} \tag{1}$$

be a system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$ with right-hand sides $y_1, y_2, \ldots, y_n$.

In matrix form this system may be written as

$$Ax = y. \tag{1'}$$

Here $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are columns and $A = \| a_{ik} \|_1^n$ is the square coefficient matrix.

If A is non-singular, then we can rewrite this as

$$x = A^{-1}y, \tag{2}$$

or in explicit form:

$$x_i = \sum_{k=1}^{n} a_{ik}^{(-1)} y_k \qquad (i = 1, 2, \ldots, n). \tag{2'}$$

Thus, the task of computing the elements of the inverse matrix $A^{-1} = \| a_{ik}^{(-1)} \|_1^n$ is equivalent to the task of solving the system of equations (1) for *arbitrary* right-hand sides $y_1, y_2, \ldots, y_n$. The elements of the inverse matrix are determined by the formulas (25) of Chapter I. However, the actual computation of the elements of A^{-1} by these formulas is very tedious for large n. Therefore, effective methods of computing the elements of an inverse matrix—and hence of solving a system of linear equations—are of great practical value.[1]

---

[1] For a detailed account of these methods, we refer the reader to the book by Faddeev [15] and the group of papers that appeared in *Uspehi Mat. Nauk*, Vol. 5, 3 (1950).

In the present chapter we expound the theoretical basis of some of these methods; they are variants of Gauss's elimination method, whose acquaintance the reader first made in his algebra course at school.

2. Suppose that in the system of equations (1) we have $a_{11} \neq 0$. We eliminate x_1 from all the equations beginning with the second by adding to the second equation the first multipled by $-\frac{a_{21}}{a_{11}}$, to the third the first multiplied by $-\frac{a_{31}}{a_{11}}$, and so on. The system (1) has now been replaced by the equivalent system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a^{(1)}_{22}x_2 + \cdots + a^{(1)}_{2n}x_n &= y^{(1)}_2 \\ \dots\dots\dots\dots\dots & \\ a^{(1)}_{n2}x_2 + \cdots + a^{(1)}_{nn}x_n &= y^{(1)}_n \end{aligned}\right\} \tag{3}$$

The coefficients of the unknowns and the constant terms of the last $n - 1$ equations are given by the formulas

$$a^{(1)}_{ij} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}, \qquad y^{(1)}_i = y_i - \frac{a_{i1}}{a_{11}} y_1 \qquad (i, j = 2, \dots, n). \tag{3'}$$

Suppose that $a^{(1)}_{22} \neq 0$. Then we eliminate x_2 in the same way from the last $n - 2$ equations of the system (3) and obtain the system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= y_1 \\ a^{(1)}_{22}x_2 + a^{(1)}_{23}x_3 + \cdots + a^{(1)}_{2n}x_n &= y^{(1)}_2 \\ a^{(2)}_{33}x_3 + \cdots + a^{(2)}_{3n}x_n &= y^{(2)}_3 \\ \dots\dots\dots\dots\dots & \\ a^{(2)}_{n3}x_3 + \cdots + a^{(2)}_{nn}x_n &= y^{(2)}_n. \end{aligned}\right\} \tag{4}$$

The new coefficients and the new right-hand sides are connected with the preceding ones by the formulas:

$$a^{(2)}_{ij} = a^{(1)}_{ij} - \frac{a^{(1)}_{i2}}{a^{(1)}_{22}} a^{(1)}_{2j}, \quad y^{(2)}_i = y^{(1)}_i - \frac{a^{(1)}_{i2}}{a^{(1)}_{22}} y^{(1)}_2 \qquad (i, j = 3, \dots, n). \tag{5}$$

Continuing the algorithm, we go in $n - 1$ steps from the original system (1) to the triangular recurrent system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= y_1 \\ a^{(1)}_{22}x_2 + a^{(1)}_{23}x_3 + \cdots + a^{(1)}_{2n}x_n &= y^{(1)}_2 \\ a^{(2)}_{33}a_3 + \cdots + a^{(2)}_{3n}x_n &= y^{(2)}_3 \\ \dots\dots\dots\dots\dots & \\ a^{(n-1)}_{nn}x_n &= y^{(n-1)}_n. \end{aligned}\right\} \tag{6}$$

This reduction can be carried out if and only if in the process all the numbers $a_{11}, a^{(1)}_{22}, a^{(2)}_{33}, \dots, a^{(n-2)}_{n-1,n-1}$ turn out to be different from zero.

This *algorithm of Gauss* consists of operations of a simple type such as can easily be carried out by present-day computing machines.

3. Let us express the coefficients and the right-hand sides of the reduced system in terms of the coefficients and the right-hand sides of the original system (1). We shall not assume here that in the reduction process all the numbers $a_{11}, a^{(1)}_{22}, a^{(2)}_{33}, \dots, a^{(n-2)}_{n-1,n-1}$ turn out to be different from zero; we consider the general case, in which the first p of these numbers are different from zero:

$$a_{11} \neq 0, \; a^{(1)}_{22} \neq 0, \; \dots, \; a^{(p-1)}_{pp} \neq 0 \qquad (p \leqq n - 1). \tag{7}$$

This enables us (at the p-th step of the reduction) to put the original system of equations into the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots\cdots + a_{1n}x_n &= y_1 \\ a^{(1)}_{22}x_2 + \cdots\cdots\cdots\cdots + a^{(1)}_{2n}x_n &= y^{(1)}_2 \\ \dots\dots\dots\dots\dots\dots\dots & \\ a^{(p-1)}_{pp}x_p + \cdots\cdots\cdots + a^{(p-1)}_{pn}x_n &= y^{(p-1)}_p \\ a^{(p)}_{p+1,p+1}x_{p+1} + \cdots + a^{(p)}_{p+1,n}x_n &= y^{(p)}_{p+1} \\ \dots\dots\dots\dots\dots\dots & \\ a^{(p)}_{n,p+1}x_{p+1} + \cdots + a^{(p)}_{nn}x_n &= y^{(p)}_n. \end{aligned}\right\} \tag{8}$$

We denote the coefficient matrix of this system of equations by G_p:

$$G_p = \left\| \begin{matrix} a_{11} & a_{12} & \dots & a_{1p} & a_{1,p+1} & \dots & a_{1n} \\ 0 & a^{(1)}_{22} & \dots & a^{(1)}_{2p} & a^{(1)}_{2,p+1} & \dots & a^{(1)}_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a^{(p-1)}_{pp} & a^{(p-1)}_{p,p+1} & \dots & a^{(p-1)}_{pn} \\ 0 & 0 & \dots & 0 & a^{(p)}_{p+1,p+1} & \dots & a^{(p)}_{p+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & a^{(p)}_{n,p+1} & \dots & a^{(p)}_{nn} \end{matrix} \right\|. \tag{9}$$

The transition from A to G_p is effected as follows: To every row of A in succession from the second to the n-th there are added some preceding rows (from the first p) multiplied by certain factors. Therefore all the minors of order h contained in the first h rows of A and G_p are equal:

$$A\begin{pmatrix} 1 & 2 & \dots & h \\ k_1 & k_2 & \dots & k_h \end{pmatrix} = G_p\begin{pmatrix} 1 & 2 & \dots & h \\ k_1 & k_2 & \dots & k_h \end{pmatrix} \quad \begin{pmatrix} 1 \leqq k_1 < k_2 < \cdots < k_h \leqq n \\ h = 1, 2, \dots, n) \end{pmatrix}. \tag{10}$$

From these formulas we find, by taking into account the structure (9) of G_p,

$$A\begin{pmatrix}1 & 2 & . & . & . & p\\ 1 & 2 & . & . & . & p\end{pmatrix} = a_{11}a_{22}^{(1)} \cdot \cdot \cdot a_{pp}^{(p-1)}, \tag{11}$$

$$A\begin{pmatrix}1 & 2 & . & . & . & p & i\\ 1 & 2 & . & . & . & p & k\end{pmatrix} = a_{11}a_{22}^{(1)} \cdots a_{pp}^{(p-1)}a_{ik}^{(p)} \qquad (i, k = p+1, \ldots, n), \tag{12}$$

When we divide the second of these equations by the first, we obtain the fundamental formulas[2]

$$a_{ik}^{(p)} = \frac{A\begin{pmatrix}1 & 2 & \ldots & p & i\\ 1 & 2 & \ldots & p & k\end{pmatrix}}{A\begin{pmatrix}1 & 2 & \ldots & p\\ 1 & 2 & \ldots & p\end{pmatrix}} \qquad (i, k = p+1, \ldots, n). \tag{13}$$

If the conditions (7) hold for a given value of p, then they also hold for every smaller value of p. Therefore the formulas (13) are valid not only for the given value of p but also for all smaller values of p. The same holds true of (11). Hence instead of this formula we can write the equations

$$A\begin{pmatrix}1\\ 1\end{pmatrix} = a_{11}, \quad A\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} = a_{11}a_{22}^{(1)}, \quad A\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix} = a_{11}a_{22}^{(1)}a_{33}^{(2)}, \quad \ldots . \tag{14}$$

Thus, the conditions (7), i.e., the necessary and sufficient conditions for the feasibility of the first p steps in Gauss's algorithm, can be written in the form of the following inequalities:

$$A\begin{pmatrix}1\\ 1\end{pmatrix} \neq 0, \quad A\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} \neq 0, \ \ldots, \quad A\begin{pmatrix}1 & 2 & \ldots & p\\ 1 & 2 & \ldots & p\end{pmatrix} \neq 0 \tag{15}$$

From (14) we then find:

$$a_{11} = A\begin{pmatrix}1\\ 1\end{pmatrix}, \ a_{22}^{(1)} = \frac{A\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}}{A\begin{pmatrix}1\\ 1\end{pmatrix}}, \ a_{33}^{(2)} = \frac{A\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}}{A\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}}, \ldots, \ a_{pp}^{(p-1)} = \frac{A\begin{pmatrix}1 & 2 & \ldots & p\\ 1 & 2 & \ldots & p\end{pmatrix}}{A\begin{pmatrix}1 & 2 & \ldots & p-1\\ 1 & 2 & \ldots & p-1\end{pmatrix}}. \tag{16}$$

In order to eliminate $x_1, x_2, \ldots, x_p$ *consecutively* by Gauss's algorithm it is necessary that all the values (16) should be different from zero, i.e., that the inequalities (15) should hold. However, the formulas for $a_{ik}^{(p)}$ make sense if only the last of the conditions (15) holds.

---

[2] See [181], p. 89.

4. Suppose the coefficient matrix of the system of equations (1) to be of rank r. Then, by a suitable permutation of the equations and a renumbering of the unknowns, we can arrange that the following inequalities hold:

$$A\begin{pmatrix}1 & 2 & \ldots & j\\ 1 & 2 & \ldots & j\end{pmatrix} \neq 0 \qquad (j = 1, 2, \ldots, r). \tag{17}$$

This enables us to eliminate $x_1, x_2, \ldots, x_r$ consecutively and to obtain the system of equations

$$\left.\begin{array}{rl}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots\cdots\cdots + a_{1n}x_n & = y_1\\
a_{22}^{(1)}x_2 + \cdots\cdots\cdots\cdots + a_{2n}^{(1)}x_n & = y_2^{(1)}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{rr}^{(r-1)}x_r + \cdots\cdots\cdots + a_{rn}^{(r-1)}x_n & = y_r^{(r-1)}\\
a_{r+1,r+1}^{(r)}x_{r+1} + \cdots + a_{r+1,n}^{(r)}x_n & = y_{r+1}^{(r)}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{n,r+1}^{(r)}x_{r+1} + \cdots + a_{nn}^{(r)}x_n & = y_n^{(r)}.
\end{array}\right\} \tag{18}$$

Here the coefficients are determined by the formulas (13). From these formulas it follows, because the rank of the matrix $A = \| a_{ik} \|_1^n$ is equal to r, that

$$a_{ik}^{(r)} = 0 \qquad (i, k = r+1, \ldots, n). \tag{19}$$

Therefore the last $n - r$ equations (18) reduce to the consistency conditions

$$y_i^{(r)} = 0 \qquad (i = r+1, \ldots, n). \tag{20}$$

Note that in the elimination algorithm the column of constant terms is subjected to the same transformations as the other columns, of coefficients. Therefore, by supplementing the matrix $A = \| a_{ik} \|_1^n$ with an $(n+1)$-th column of the constant terms we obtain:

$$y_i^{(p)} = \frac{A\begin{pmatrix}1 & \ldots & p & i\\ 1 & \ldots & p & n+1\end{pmatrix}}{A\begin{pmatrix}1 & \ldots & p\\ 1 & \ldots & p\end{pmatrix}} \qquad (i = 1, 2, \ldots, n;\ p = 1, 2, \ldots, r). \ . \tag{21}$$

[illegible] particular, the consistency conditions (20) reduce to the well-known [illegible]uations

$$A\begin{pmatrix}1 & \ldots & r & r+j\\ 1 & \ldots & r & n+1\end{pmatrix} = 0 \qquad (j = 1, 2, \ldots, n-r). \tag{22}$$

If $n = r$, i.e. if the matrix $A = \| a_{ik} \|_1^n$ is non-singular, and

$$A\begin{pmatrix}1 & 2 & \dots & j\\ 1 & 2 & \dots & j\end{pmatrix} \neq 0 \qquad (j = 1, 2, \dots, n),$$

then we can eliminate $x_1, x_2, \dots, x_{n-1}$ in succession by means of Gauss's algorithm and reduce the system of equations to the form (6).

§ 2. Mechanical Interpretation of Gauss's Algorithm

1. We consider an arbitrary elastic statical system S supported on edges (for example, a string, a rod, a multispan rod, a membrane, a lamina, or a discrete system) and choose n points (1), (2), . . . , (n) on it. We shall consider the displacements (sags) $y_1, y_2, \dots, y_n$ of the points (1), (2), . . . , (n) of S under the action of forces $F_1, F_2, \dots, F_n$ applied at these points.

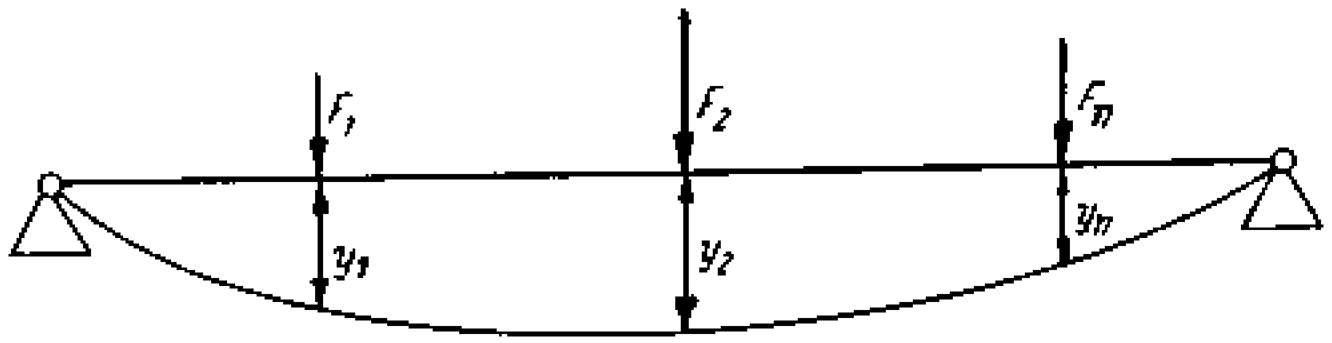

Fig. 1

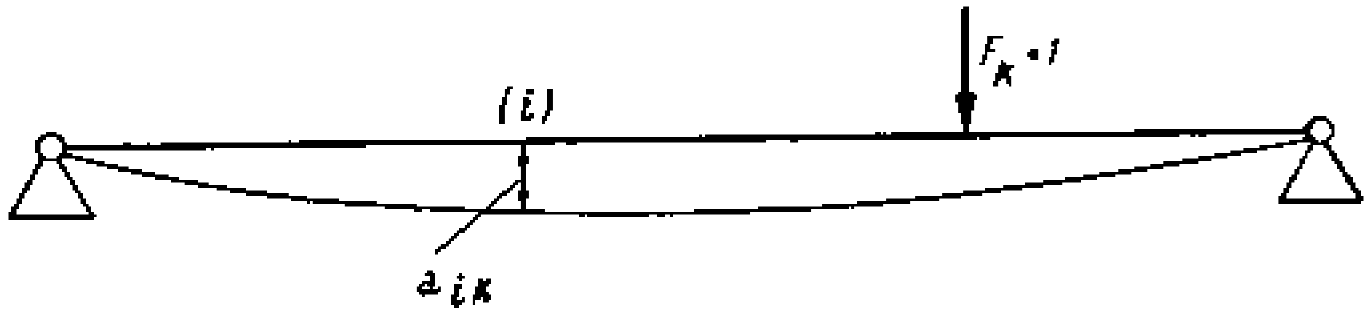

Fig. 2

We assume that the forces and the displacements are parallel to one and the same direction and are determined, therefore, by their algebraic magnitudes (Fig. 1). Moreover, we assume the *principle of linear superposition of forces*:

1. *Under the combined action of two systems of forces the corresponding displacements are added together.*

2. *When the magnitudes of all the forces are multiplied by one and the same real number, then all the displacements are multiplied by the same number.*

We denote by a_{ik} the coefficient of influence of the point (k) on the point (i), i.e., the displacement of (i) under the action of a unit force applied at (k) ($i, k = 1, 2, \dots, n$) (Fig. 2). Then under the combined action of the forces $F_1, F_2, \dots, F_n$ the displacements $y_1, y_2, \dots, y_n$ are determined by the formulas

$$\sum_{k=1}^{n} a_{ik} F_k = y_i \qquad (i = 1, 2, \dots, n). \tag{23}$$

Comparing (23) with the original system (1), we can interpret the task of solving the system of equations (1) as follows:

The displacements $y_1, y_2, \dots, y_n$ being given, we are required to find the corresponding forces $F_1, F_2, \dots, F_n$.

We denote by S_p the statical system that is obtained from S by introducing p fixed hinged supports at the points (1), (2), . . . , (p) ($p \leqq n$). We denote the coefficients of influence for the remaining movable points ($p + 1$), . . . , (n) of the system S_p by

$$a_{ik}^{(p)} \qquad (i, k = p + 1, \dots, n)$$

(see Fig. 3 for $p = 1$).

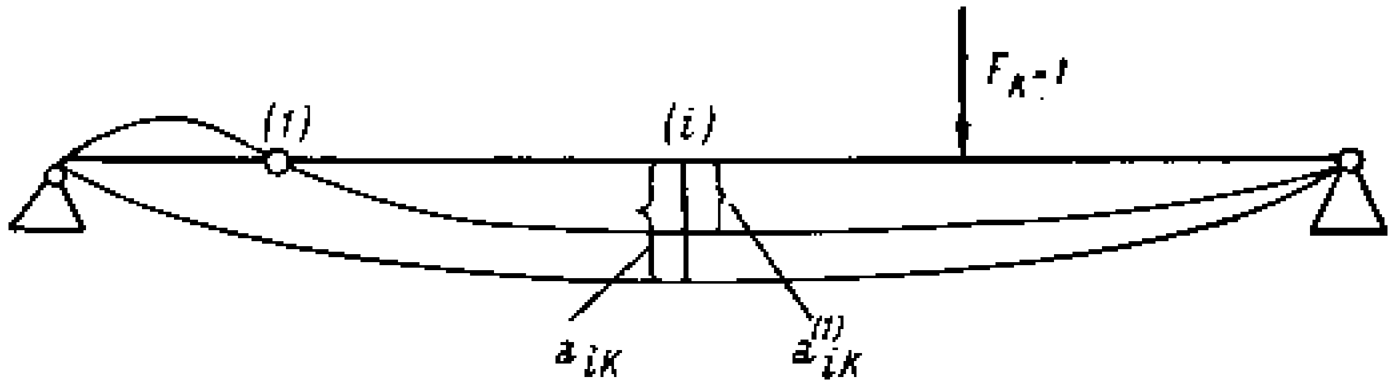

Fig. 3

The coefficient $a_{ik}^{(p)}$ can be regarded as the displacement at the point (i) of S under the action of a unit force at (k) and of the reactions $R_1, R_2, \dots, R_p$ at the fixed points (1), (2), . . . , (p). Therefore

$$a_{ik}^{(p)} = R_1 a_{i1} + \cdots + R_p a_{ip} + a_{ik}. \tag{24}$$

On the other hand, under the same forces the displacements of the system S at the points (1), (2), . . . , (p) are zero:

$$\left.\begin{array}{l} R_1 a_{11} + \cdots + R_p a_{1p} + a_{1k} = 0 \\ \dots\dots\dots\dots\dots\dots \\ R_1 a_{p1} + \cdots + R_p a_{pp} + a_{pk} = 0. \end{array}\right\} \tag{25}$$

If

$$A\begin{pmatrix}1 & 2 & \cdot & \cdot & \cdot & p\\ 1 & 2 & \cdot & \cdot & \cdot & p\end{pmatrix}\neq 0,$$

then we can determine $R_1, R_2, \ldots, R_p$ from (25) and substitute the expressions so obtained in (24). This elimination of $R_1, R_2, \ldots, R_p$ can be carried out as follows. To the system of equations (25) we adjoin (24) written in the form

$$R_1a_{i1}+\cdots+R_pa_{ip}+a_{ik}-a_{ik}^{(p)}=0. \qquad (24')$$

Regarding (25) and (24′) as a system of $p+1$ homogeneous equations with non-zero solutions $R_1, R_2, \ldots, R_p, R_{p+1}=1$, we see that the determinant of the system must be zero:

$$\begin{vmatrix} a_{11} & \cdots & a_{1p} & a_{1k}\\ \cdot & \cdot & \cdot & \cdot\\ a_{p1} & \cdots & a_{pp} & a_{pk}\\ a_{i1} & \cdots & a_{ip} & a_{ik}-a_{ik}^{(p)}\end{vmatrix}=0.$$

Hence

$$a_{ik}^{(p)}=\frac{A\begin{pmatrix}1 & 2 & \ldots & p & i\\ 1 & 2 & \ldots & p & k\end{pmatrix}}{A\begin{pmatrix}1 & 2 & \ldots & p\\ 1 & 2 & \ldots & p\end{pmatrix}} \qquad (i, k=p+1, \ldots, n). \qquad (26)$$

These formulas express the coefficients of influence of the 'support' system S_p in terms of those of the original system S.

But formulas (26) coincide with formulas (13) of the preceding section. Therefore *for every* p ($\leqq n-1$) *the coefficients* $a_{ik}^{(p)}$ $(i, k=p+1, \ldots, n)$ *in the algorithm of Gauss are the coefficients of influence of the support system* S_p.

The truth of this fundamental proposition can also be ascertained by purely mechanical considerations without recourse to the algebraic derivation of formulas (13). For this purpose we consider, to begin with, the special case of a single support: $p=1$ (Fig. 3). In this case, the coefficients of influence of the system S_1 are given by the formulas (we put $p=1$ in (26)):

$$a_{ik}^{(1)}=\frac{A\begin{pmatrix}1 & i\\ 1 & k\end{pmatrix}}{A\begin{pmatrix}1\\ 1\end{pmatrix}}=a_{ik}-\frac{a_{i1}}{a_{11}}a_{1k} \qquad (i, k=1, 2, \ldots, n)$$

These formulas coincide with the formulas (3′).

Thus, if the coefficients a_{ik} $(i, k=1, 2, \ldots, n)$ in the system of equations (1) are the coefficients of influence of the statical system S, then the coefficients $a_{ik}^{(1)}$ $(i, k=2, \ldots, n)$ in Gauss's algorithm are the coefficients of influence of the system S_1. Applying the same reasoning to the system S_1 and introducing a second support at the point (2) in this system, we see that the coefficients $a_{ik}^{(2)}$ $(i, k=3, \ldots, n)$ in the system of equations (4) are the coefficients of influence of the support system S_2 and, in general, for every p ($\leqq n-1$) the coefficients $a_{ik}^{(p)}$ $(i, k=p+1, \ldots, n)$ in Gauss's algorithm are the coefficients of influence of the support system S_p.

From mechanical considerations it is clear that *the successive introduction of* p *supports is equivalent to the simultaneous introduction of these supports.*

Note. We wish to point out that in the mechanical interpretation of the elimination algorithm it was not necessary to assume that the points at which the displacements are investigated coincide with the points at which the forces $F_1, F_2, \ldots, F_n$ are applied. We can assume that $y_1, y_2, \ldots, y_n$ are the displacements of the points (1), (2), . . . , (n) and that the forces $F_1, F_2, \ldots, F_n$ are applied at the points (1′), (2′), . . . , (n'). Then a_{ik} is the coefficient of influence of the point (k') on the point (k). In that case we must consider instead of the support at the point (j) a generalized support at the points (j), (j') under which the displacement at the point (j) is maintained all the time equal to zero at the expense of a suitably chosen auxiliary force R_j at the point (j'). The conditions that allow us to introduce p generalized supports at the points (1), (1′); (2), (2′), . . . ; (p), (p'), i.e., that allow us to satisfy the conditions $y_1=0$, $y_2=0, \ldots, y_p=0$ for arbitrary $F_{p+1}, \ldots, F_n$ at the expense of suitable $R_1=F_1, \ldots, R_p=F_p$, can be expressed by the inequality

$$A\begin{pmatrix}1 & 2 & \ldots & p\\ 1 & 2 & \ldots & p\end{pmatrix}\neq 0.$$

§ 3. Sylvester's Determinant Identity

1. In § 1, a comparison of the matrices A and G_p led to equations (10) and (11).

These equations enable us to give an easy proof of the important *determinant identity of Sylvester*. For from (10) and (11) we find:

$$|A|=A\begin{pmatrix}1 & 2 & \ldots & n\\ 1 & 2 & \ldots & n\end{pmatrix}=A\begin{pmatrix}1 & 2 & \ldots & p\\ 1 & 2 & \ldots & p\end{pmatrix}\begin{vmatrix}a_{p+1,p+1}^{(p)} & \ldots & a_{p+1,n}^{(p)}\\ \cdot & \cdot & \cdot\\ a_{n,p+1}^{(p)} & \ldots & a_{nn}^{(p)}\end{vmatrix}. \qquad (27)$$

We introduce borderings of the minor $A\begin{pmatrix}1&2&\dots&p\\1&2&\dots&p\end{pmatrix}$ by the determinants

$$b_{ik}=A\begin{pmatrix}1&2&\dots&p&i\\1&2&\dots&p&k\end{pmatrix}\qquad (i,\ k=p+1,\ \dots,\ n).$$

The matrix formed from these determinants will be denoted by

$$B=\|b_{ik}\|_{p+1}^{n}.$$

Then by formulas (13)

$$\begin{vmatrix}a_{p+1,p+1}^{(p)}&\dots&a_{p+1,n}^{(p)}\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ a_{n,p+1}^{(p)}&\dots&a_{nn}^{(p)}\end{vmatrix}=\frac{\begin{vmatrix}b_{p+1,p+1}&\dots&b_{p+1,n}\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ b_{n,p+1}&\dots&b_{nn}\end{vmatrix}}{\left[A\begin{pmatrix}1&2&\dots&p\\1&2&\dots&p\end{pmatrix}\right]^{n-p}}=\frac{|B|}{\left[A\begin{pmatrix}1&2&\dots&p\\1&2&\dots&p\end{pmatrix}\right]^{n-p}}.$$

Therefore equation (27) can be rewritten as follows:

$$|B|=\left[A\begin{pmatrix}1&2&\dots&p\\1&2&\dots&p\end{pmatrix}\right]^{n-p-1}|A|.\tag{28}$$

This is Sylvester's determinant identity. It expresses the determinant $|B|$ formed from the bordered determinants in terms of the original determinant and the bordered minor.

We have established equation (28) for a matrix $A\begin{pmatrix}1&2&\dots&p\\1&2&\dots&p\end{pmatrix}$ whose elements satisfy the inequalities

$$A\begin{pmatrix}1&2&\dots&j\\1&2&\dots&j\end{pmatrix}\neq 0\tag{29}$$

$$(j=1,\ 2,\ \dots,\ p).$$

However, we can show by a 'continuity argument' that this restriction may be removed and that Sylvester's identity holds for an arbitrary matrix $A=\|a_{ik}\|_1^n$. For suppose that the inequalities (29) do not hold. We introduce the matrix

$$A_\varepsilon=A+\varepsilon E.$$

Obviously $\lim\limits_{\varepsilon\to 0}A_\varepsilon=A$. On the other hand, the minors

$$A_\varepsilon\begin{pmatrix}1&2&\dots&j\\1&2&\dots&j\end{pmatrix}=\varepsilon^j+\cdots$$

$$(j=1,\ 2,\ \dots,\ p)$$

are p polynomials in ε that do not vanish identically. Therefore we can choose a sequence $\varepsilon_m\to 0$ such that

$$A_{\varepsilon_m}\begin{pmatrix}1&2&\dots&j\\1&2&\dots&j\end{pmatrix}\neq 0\qquad (j=1,\ 2,\ \dots,\ p;\ m=1,\ 2,\ \dots).$$

We can write down the identity (28) for the matrices A_{ε_m}. Taking the limit $m\to\infty$ on both sides of this identity, we obtain Sylvester's identity for the limit matrix[3] $A=\lim\limits_{m\to\infty}A_{\varepsilon_m}$.

If we apply the identity (28) to the determinant

$$A\begin{pmatrix}1&2&\dots&p&i_1&i_2&\dots&i_q\\1&2&\dots&p&k_1&k_2&\dots&k_q\end{pmatrix}\qquad \left(p<\begin{matrix}i_1<i_2<\cdots<i_q\\k_1<k_2<\cdots<k_q\end{matrix}\leq n\right)$$

then we obtain a form of Sylvester's identity particularly convenient for applications

$$B\begin{pmatrix}i_1&i_2&\dots&i_q\\k_1&k_2&\dots&k_q\end{pmatrix}=\left[A\begin{pmatrix}1&2&\dots&p\\1&2&\dots&p\end{pmatrix}\right]^{q-1}A\begin{pmatrix}1&2&\dots&p&i_1&i_2&\dots&i_q\\1&2&\dots&p&k_1&k_2&\dots&k_q\end{pmatrix}.\tag{30}$$

§ 4. The Decomposition of a Square Matrix into Triangular Factors

1. Let $A=\|a_{ik}\|_1^n$ be a given matrix of rank r. We introduce the following notation for the successive principal minors of the matrix

$$D_k=A\begin{pmatrix}1&2&\dots&k\\1&2&\dots&k\end{pmatrix}\qquad (k=1,\ 2,\ \dots,\ n).$$

Let us assume that the conditions for the feasibility of Gauss's algorithm are satisfied:

$$D_k\neq 0\qquad (k=1,\ 2,\ \dots,\ r).$$

We denote by G the coefficient matrix of the system of equations (18) to which the system

$$\sum_{k=1}^{n}a_{ik}x_k=y_i\qquad (i=1,\ 2,\ \dots,\ n)$$

---

[3] By the *limit* (for $p\to\infty$) *of a sequence of matrices* $X_p=\|x_{ik}^{(p)}\|_1^n$ we mean the matrix $X=\|x_{ik}\|_1^n$, where $x_{ik}=\lim\limits_{p\to\infty}x_{ik}^{(p)}$ $(i,k=1,2,\dots,n)$.

has been reduced by the elimination method of Gauss. The matrix G is of upper triangular form and the elements of its first r rows are determined by the formulas (13), while the elements of the last $n - r$ rows are all equal to zero:[4]

$$G = \left\| \begin{matrix} a_{11} & a_{12} \dots a_{1r} & a_{1,r+1} \dots a_{1n} \\ 0 & a_{22}^{(1)} \dots a_{2r}^{(1)} & a_{2,r+1}^{(1)} \dots a_{2n}^{(1)} \\ \dots & \dots\dots\dots & \dots\dots\dots \\ 0 & 0 \quad \dots a_{rr}^{(r-1)} & a_{r,r+1}^{(r-1)} \dots a_{rn}^{(r-1)} \\ 0 & 0 \quad \dots 0 & 0 \quad \dots 0 \\ \dots & \dots\dots\dots & \dots\dots\dots \\ 0 & 0 \quad \dots 0 & 0 \quad \dots 0 \end{matrix} \right\| .$$

The transition from A to G is effected by a certain number N of operations of the following type: to the i-th row of the matrix we add the j-th row ($j < i$), after a preliminary multiplication by some number α. Such an operation is equivalent to the multiplication on the left of the matrix to be transformed by the matrix

$$\begin{array}{c} \begin{matrix} & & (j) & & (i) & & \end{matrix} \\ \left\| \begin{matrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ \vdots & & 1 & & \vdots & & \vdots \\ \vdots & & & \ddots & \vdots & & \vdots \\ 0 & \dots & \alpha & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{matrix} \right\| . \end{array} \tag{31}$$

In this matrix the main diagonal consists entirely of units, and all the remaining elements, except α, are zero.

Thus,

$$G = W_N \cdots W_2 W_1 A ,$$

where each matrix $W_1, W_2, \dots, W_N$ is of the form (31) and is therefore a lower triangular matrix with diagonal elements equal to 1.

---

[4] See formulas (19). G coincides with the matrix G_p (p. 25) for $p = r$.

Let

$$W = W_N \cdots W_2 W_1 . \tag{32}$$

Then

$$G = WA . \tag{33}$$

We shall call W the *transforming* matrix for A in Gauss's elimination method. Both matrices G and W are uniquely determined by A. From (32) it follows that W is lower triangular with diagonal elements equal to 1.

Since W is non-singular, we obtain from (33):

$$A = W^{-1} G . \tag{33'}$$

We have thus represented A in the form of a product of a lower triangular matrix W^{-1} and an upper triangular matrix G. The problem of decomposing a matrix A into factors of this type is completely answered by the following theorem:

THEOREM 1: *Every matrix* $A = \| a_{ik} \|_1^n$ *of rank* r *in which the first* r *successive principal minors are different from zero*

$$D_k = A \begin{pmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{pmatrix} \neq 0 \quad \textit{for} \quad k = 1, 2 \dots, r \tag{34}$$

can be represented in the form of a product of a lower triangular matrix B *and an upper triangular matrix* C

$$A = BC = \left\| \begin{matrix} b_{11} & 0 & \dots & 0 \\ b_{21} & b_{22} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{matrix} \right\| \left\| \begin{matrix} c_{11} & c_{12} & \dots & c_{1n} \\ 0 & c_{22} & \dots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & c_{nn} \end{matrix} \right\| . \tag{35}$$

Here

$$b_{11} c_{11} = D_1, \quad b_{22} c_{22} = \frac{D_2}{D_1}, \quad \dots, \quad b_{rr} c_{rr} = \frac{D_r}{D_{r-1}} . \tag{36}$$

The values of the first r *diagonal elements of* B *and* C *can be chosen arbitrarily subject to the conditions* (36).

When the first r *diagonal elements of* B *and* C *are given, then the elements of the first* r *rows of* B *and of the first* r *columns of* C *are uniquely determined, and are given by the following formulas:*

$$b_{gk} = b_{kk} \frac{A \begin{pmatrix} 1 & 2 & \dots & k-1 & g \\ 1 & 2 & \dots & k-1 & k \end{pmatrix}}{A \begin{pmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{pmatrix}} , \quad c_{kg} = c_{kk} \frac{A \begin{pmatrix} 1 & 2 & \dots & k-1 & k \\ 1 & 2 & \dots & k-1 & g \end{pmatrix}}{A \begin{pmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{pmatrix}} \tag{37}$$

$$(g = k, k+1, \dots, n; \; k = 1, 2, \dots, r) .$$

If $r < n$ ($|A| = 0$), then all the elements in the last $n - r$ rows of B can be put equal to zero and all the elements of the last $n - r$ columns of C can be chosen arbitrarily; or, conversely, the last $n - r$ rows of C can be filled with zeros and the last $n - r$ rows of B can be chosen arbitrarily.

Proof. That a representation of a matrix satisfying conditions (34) can be given in the form of a product (35) has been proved above (see (33′)).

Now let B and C be arbitrary lower and upper triangular matrices whose product is A. Making use of the formulas for the minors of the product of two matrices we find:

$$A\begin{pmatrix}1 & 2 & \dots & k-1 & g\\ 1 & 2 & \dots & k-1 & k\end{pmatrix} = \sum_{\alpha_1<\alpha_2<\dots<\alpha_k} B\begin{pmatrix}1 & 2 & \dots & k-1 & g\\ \alpha_1 & \alpha_2 & \dots & \alpha_{k-1} & \alpha_k\end{pmatrix} C\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_k\\ 1 & 2 & \dots & k\end{pmatrix} \quad (38)$$

$$(g = k, k+1, \dots, n;\ k = 1, 2, \dots, r).$$

Since C is an upper triangular matrix, the first k columns of C contain only one non-vanishing minor of order k, namely $C\begin{pmatrix}1 & 2 & \dots & k\\ 1 & 2 & \dots & k\end{pmatrix}$. Therefore, equation (38) can be written as follows:

$$A\begin{pmatrix}1 & 2 & \dots & k-1 & g\\ 1 & 2 & \dots & k-1 & k\end{pmatrix} = B\begin{pmatrix}1 & 2 & \dots & k-1 & g\\ 1 & 2 & \dots & k-1 & k\end{pmatrix} C\begin{pmatrix}1 & 2 & \dots & k\\ 1 & 2 & \dots & k\end{pmatrix}$$

$$= b_{11}b_{22}\cdots b_{k-1,k-1}b_{gk}c_{11}c_{22}\cdots c_{kk} \quad (39)$$

$$(g = k, k+1, \dots, n;\ k = 1, 2, \dots, r).$$

We put $g = k$ in this equation, obtaining

$$b_{11}b_{22}\cdots b_{kk}c_{11}c_{22}\cdots c_{kk} = D_k \qquad (k = 1, 2, \dots, r), \quad (40)$$

and relations (36) follow.

Without violating equation (35) we may multiply the matrix B in that equation on the right by an arbitrary non-singular diagonal matrix $M = \|\mu_i\delta_{ik}\|_1^n$, while multiplying C at the same time on the left by $M^{-1} = \|\mu_i^{-1}\delta_{ik}\|_1^n$. But this is equivalent to multiplying the columns of B by $\mu_1, \mu_2, \dots, \mu_n$, respectively, and the rows of C by $\mu_1^{-1}, \mu_2{}^1, \dots, \mu_n^{-1}$. We may therefore give arbitrary values to the diagonal elements $b_{11}, b_{22}, \dots, b_{rr}$ and $c_{11}, c_{22}, \dots, c_{rr}$, provided they satisfy (36).

Further, from (39) and (40) we find:

$$b_{gk} = b_{kk}\frac{A\begin{pmatrix}1 & 2 & \dots & k-1 & g\\ 1 & 2 & \dots & k-1 & k\end{pmatrix}}{A\begin{pmatrix}1 & 2 & \dots & k\\ 1 & 2 & \dots & k\end{pmatrix}} \qquad (g = k, k+1, \dots, n;\ k = 1, 2, \dots, r),$$

i.e., the first formulas in (37). The second formulas in (37), for the elements of C, are established similarly.

We observe that in the multiplication of B and C the elements b_{kg} of the last $n - r$ columns of B and the elements c_{gk} of the last $n - r$ rows of C are multiplied only among each other. We have seen that all the elements of the last $n - r$ rows of C may be chosen to be zero.[5] But as a consequence, the elements of the last $n - r$ columns of B may be chosen arbitrarily. Clearly the product of B and C does not change if we choose the last $n - r$ columns of B to be zeros and choose the elements of the last $n - r$ rows of C arbitrarily.

This completes the proof of the theorem.

From this theorem there follow a number of interesting corollaries.

COROLLARY 1: *The elements of the first r columns of B and the first r rows of C are connected with the elements of A by the recurrence relations*

$$\left.\begin{aligned} b_{ik} &= \frac{a_{ik} - \sum_{j=1}^{k-1} b_{ij}c_{jk}}{c_{kk}} \qquad (i \geqq k;\ i = 1, 2, \dots, n;\ k = 1, 2, \dots, r),\\ c_{ik} &= \frac{a_{ik} - \sum_{j=1}^{i-1} b_{ij}c_{jk}}{b_{ii}} \qquad (i \leqq k;\ i = 1, 2, \dots, r;\ k = 1, 2, \dots, n).\end{aligned}\right\} \quad (41)$$

The relations (41) follow immediately from the matrix equation (35); they can be used to advantage in the actual computation of the elements of B and C.

COROLLARY 2: *If $A = \|a_{ik}\|_1^n$ is a non-singular matrix ($r = n$) satisfying (34), then the matrices B and C in the representation (35) are uniquely determined as soon as the diagonal elements of these matrices are chosen in accordance with (36).*

COROLLARY 3: *If $S = \|s_{ik}\|_1^n$ is a symmetric matrix of rank r and*

$$D_k = S\begin{pmatrix}1 & 2 & \dots & k\\ 1 & 2 & \dots & k\end{pmatrix} \neq 0 \qquad (k = 1, 2, \dots, r),$$

then

$$S = B\,B',$$

where $B = \|b_{ik}\|_1^n$ is a lower triangular matrix in which

---

[5] This follows from the representation (33′). Here, as we have shown already, arbitrary values may be given to the diagonal elements $b_{11}, \dots, b_{rr}, c_{11}, \dots, c_{rr}$ provided (36) is satisfied by the introduction of suitable factors $\mu_1, \mu_2, \dots, \mu_r$.

$$b_{gk}=\begin{cases}\dfrac{1}{\sqrt{D_kD_{k-1}}}A\begin{pmatrix}1&2&\dots&k-1&g\\1&2&\dots&k-1&k\end{pmatrix} & (g=k,k+1,\dots,n;\ k=1,2,\dots,r),\\ 0 & (g=k,k+1,\dots,n;\ k=r+1,\dots,n).\end{cases} \tag{42}$$

2. In the representation (35) let the elements of the last $n-r$ columns of C be zero. Then we may set

$$B=F\begin{Vmatrix}b_{11}&&&&0\\&\ddots&&&\\&&b_{rr}&&\\&&&0&\\&&&&\ddots\\0&&&&0\end{Vmatrix},\qquad C=\begin{Vmatrix}c_{11}&&&&0\\&\ddots&&&\\&&c_{rr}&&\\&&&0&\\&&&&\ddots\\0&&&&0\end{Vmatrix}L, \tag{43}$$

where F and L are upper and lower triangular matrices respectively; the first r diagonal elements of F and L are 1 and the elements of the last $n-r$ columns of F and the last $n-r$ rows of L can be chosen completely arbitrarily. Substituting (43) for B and C in (35) and using (36), we obtain the following theorem:

Theorem 2: *Every matrix* $A=\|a_{ik}\|_1^n$ *of rank* r *in which*

$$D_k=A\begin{pmatrix}1&2&\dots&k\\1&2&\dots&k\end{pmatrix}\neq 0\qquad \textit{for } k=1,2,\dots,r$$

can be represented in the form of a product of a lower triangular matrix F, *a diagonal matrix* D, *and an upper triangular matrix* L:

$$A=FDL=\begin{Vmatrix}1&0&\dots&0\\f_{21}&1&\dots&0\\\cdot&\cdot&\cdot&\cdot\\f_{n1}&f_{n2}&\dots&1\end{Vmatrix}\begin{Vmatrix}D_1&&&&&\\&\dfrac{D_2}{D_1}&&&&\\&&\ddots&&&\\&&&\dfrac{D_r}{D_{r-1}}&&\\&&&&0&\\&&&&&\ddots\\&&&&&&0\end{Vmatrix}\begin{Vmatrix}1&l_{12}&\dots&l_{1n}\\0&1&\dots&l_{2n}\\\cdot&\cdot&\cdot&\cdot\\0&0&\dots&1\end{Vmatrix}, \tag{44}$$

where

$$f_{gk}=\frac{A\begin{pmatrix}1&2&\dots&k-1&g\\1&2&\dots&k-1&k\end{pmatrix}}{A\begin{pmatrix}1&2&\dots&k\\1&2&\dots&k\end{pmatrix}},\qquad l_{kg}=\frac{A\begin{pmatrix}1&2&\dots&k-1&k\\1&2&\dots&k-1&g\end{pmatrix}}{A\begin{pmatrix}1&2&\dots&k\\1&2&\dots&k\end{pmatrix}}, \tag{45}$$

$$(g=k+1,\dots,n;\ k=1,2,\dots,r),$$

and f_{gk} and l_{kg} *are arbitrary for* $g=k+1,\dots,n;\ k=r+1,\dots,n$.

3. The elimination method of Gauss, when applied to a matrix $A=\|a_{ik}\|_1^n$ of rank r for which $D_k\neq 0$ $(k=1,2,\dots,r)$, yields two matrices: a lower triangular matrix W with diagonal elements 1 and an upper triangular matrix G in which the first r diagonal elements are $D_1,\dfrac{D_2}{D_1},\dots,\dfrac{D_r}{D_{r-1}}$ and the last $n-r$ rows consist entirely of zeros. G is the *Gaussian form* of the matrix A; W is the transforming matrix.

For actual computation of the elements of W we recommend the following device.

We obtain the matrix W when we apply to the unit matrix E all the transformations (given by $W_1,\dots,W_N$) that we have performed on A in the algorithm of Gauss (in this case we shall have instead of the product WA, equal to G, the product WE, equal to W). Let us, therefore, write the unit matrix E on the right of A:

$$\begin{Vmatrix}a_{11}&\dots&a_{1n}&1&\dots&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\a_{n1}&\dots&a_{nn}&0&\dots&1\end{Vmatrix}. \tag{46}$$

By applying all the transformations of the algorithm of Gauss to this rectangular matrix we obtain a rectangular matrix consisting of the two square matrices G and W:

$$(G,W).$$

Thus, *the application of Gauss's algorithm to the matrix* (46) *gives the matrices* G *and* W *simultaneously.*

If A is non-singular, so that $|A|\neq 0$, then $|G|\neq 0$ as well. In this case, (33) implies that $A^{-1}=G^{-1}W$. Since G and W are determined by means of the algorithm of Gauss, the task of finding the inverse matrix A^{-1} reduces to determining G^{-1} and multiplying G^{-1} by W.

Although there is no difficulty in finding the inverse matrix G^{-1} once the matrix G has been determined, because G is triangular, the operations involved can nevertheless be avoided. For this purpose we introduce, together with the matrices G and W, similar matrices G_1 and W_1 for the transposed matrix A^{T}. Then $A^{\mathsf{T}}=W_1^{-1}G_1$, i.e.,

$$A=G_1^{\mathsf{T}}W_1^{\mathsf{T}-1}. \tag{47}$$

Let us compare (33′) with (44):

$$A=W^{-1}G,\ A=FDL.$$

These equations may be regarded as two distinct decompositions of the form (35); here we take the product DL as the second factor C. Since the first r diagonal elements of the first factors are the same (they are equal to 1), their first r columns coincide. But then, since the last $n-r$ columns of F may be chosen arbitrarily, we chose them such that

$$F = W^{-1}. \tag{48}$$

On the other hand, a comparison of (47) with (44),

$$A = G_1^{\mathsf{T}} W_1^{\mathsf{T}-1}, \quad A = FDL,$$

shows that we may also select the arbitrary elements of L in such a way that

$$L = W_1^{\mathsf{T}-1}. \tag{49}$$

Replacing F and L in (44) by their expressions (48) and (49), we obtain

$$A = W^{-1} D W_1^{\mathsf{T}-1}. \tag{50}$$

Comparing this equation with (33′) and (47) we find:

$$G = D W_1^{\mathsf{T}-1}, \quad G_1^{\mathsf{T}} = W^{-1} D. \tag{51}$$

We now introduce the diagonal matrix

$$\widehat{D} = \left\{\frac{1}{D_1}, \frac{D_1}{D_2}, \ldots, \frac{D_{r-1}}{D_r}, 0, \ldots, 0\right\}. \tag{52}$$

Since

$$D = D\widehat{D}D,$$

it follows from (50) and (51) that

$$A = G_1^{\mathsf{T}} \widehat{D} G. \tag{53}$$

Formula (53) shows that the decomposition of A into triangular factors can be obtained by applying the algorithm of Gauss to the matrices A and A^{T}.

Now let A be non-singular ($r = n$). Then $|D| \neq 0$, $\widehat{D} = D^{-1}$. Therefore it follows from (50) that

$$A^{-1} = W_1^{\mathsf{T}} \widehat{D} W. \tag{54}$$

This formula yields an effective computation of the inverse matrix A^{-1} by the application of Gauss's algorithm to the rectangular matrices

$$(A, E) \qquad (A^{\mathsf{T}}, E).$$

If, in particular, we take as our A a symmetrical matrix S, then G_1 coincides with G and W_1 with W, and therefore formulas (53) and (54) assume the form

$$S = G^{\mathsf{T}} \widehat{D} G, \tag{55}$$

$$S^{-1} = W^{\mathsf{T}} \widehat{D} W. \tag{56}$$

§ 5. The Partition of a Matrix into Blocks. The Technique of Operating with Partitioned Matrices. The Generalized Algorithm of Gauss

It often becomes necessary to use matrices that are partitioned into rectangular parts—'cells' or 'blocks.' In the present section we deal with such partitioned matrices.

1. Let a rectangular matrix

$$A = \| a_{ik} \| \qquad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n) \tag{57}$$

be given.

By means of horizontal and vertical lines we dissect A into rectangular blocks:

$$A = \left(\begin{array}{cccc} \overbrace{A_{11}}^{n_1} & \overbrace{A_{12}}^{n_2} & \cdots & \overbrace{A_{1t}}^{n_t} \\ A_{21} & A_{22} & \cdots & A_{2t} \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots} \\ A_{s1} & A_{s2} & \cdots & A_{st} \end{array}\right) \begin{array}{l} \}\, m_1 \\ \}\, m_2 \\ \vdots \\ \}\, m_s \end{array}. \tag{58}$$

We shall say of matrix (58) that it is *partitioned* into *st blocks*, or *cells* $A_{\alpha\beta}$ of dimensions $m_\alpha \times n_\beta$ ($\alpha = 1, 2, \ldots, s$; $\beta = 1, 2, \ldots, t$), or that it is represented in the form of a *partitioned*, or blocked, matrix. Instead of (58) we shall simply write

$$A = (A_{\alpha\beta}) \qquad (\alpha = 1, 2, \ldots, s;\ \beta = 1, 2, \ldots, t). \tag{59}$$

In the case $s = t$ we shall use the following notation:

$$A = (A_{\alpha\beta})_1^s. \tag{60}$$

Operations on partitioned matrices are performed according to the same formal rules as in the case in which we have numerical elements instead of blocks. For example, let A and B be two rectangular matrices of equal dimensions partitioned into blocks in exactly the same way:

$$A=(A_{\alpha\beta}),\ \ B=(B_{\alpha\beta}) \qquad (\alpha=1,2,\dots,s;\ \beta=1,2,\dots,t). \tag{61}$$

It is easy to verify that

$$A+B=(A_{\alpha\beta}+B_{\alpha\beta}) \qquad (\alpha=1,2,\dots,s;\ \beta=1,2,\dots,t). \tag{62}$$

We have to consider multiplication of partitioned matrices in more detail. We know (see Chapter I, p. 6) that for the multiplication of two rectangular matrices A and B the length of the rows of the first factor A must be the same as the height of the columns of the second factor B. For 'block' multiplication of these matrices we require, in addition, that the partitioning into blocks be such that the horizontal dimensions in the first factor are the same as the corresponding vertical dimensions in the second:

$$A=\begin{pmatrix} \overbrace{A_{11}}^{n_1} & \overbrace{A_{12}}^{n_2} & \dots & \overbrace{A_{1t}}^{n_t} \\ A_{21} & A_{22} & \dots & A_{2t} \\ \dots & \dots & \dots & \dots \\ A_{s1} & A_{s2} & \dots & A_{st} \end{pmatrix} \begin{matrix} \}\,m_1 \\ \}\,m_2 \\ \vdots \\ \}\,m_s \end{matrix}, \qquad B=\begin{pmatrix} \overbrace{B_{11}}^{p_1} & \overbrace{B_{12}}^{p_2} & \dots & \overbrace{B_{1u}}^{p_u} \\ B_{21} & B_{22} & \dots & B_{2u} \\ \dots & \dots & \dots & \dots \\ B_{t1} & B_{t2} & \dots & B_{tu} \end{pmatrix} \begin{matrix} \}\,n_1 \\ \}\,n_2 \\ \vdots \\ \}\,n_t \end{matrix}. \tag{63}$$

Then it is easy to verify that

$$AB=C=(C_{\alpha\beta}),\ \text{ where }\ \ C_{\alpha\beta}=\sum_{\delta=1}^{t}A_{\alpha\delta}B_{\delta\beta} \qquad \begin{pmatrix}\alpha=1,2,\dots,s\\ \beta=1,2,\dots,u\end{pmatrix}. \tag{64}$$

We mention separately the special case in which one of the factors is a *quasi-diagonal* matrix. Let A be quasi-diagonal, i.e., let $s=t$ and $A_{\alpha\beta}=O$ for $\alpha\neq\beta$. In this case formula (64) gives

$$C_{\alpha\beta}=A_{\alpha\alpha}B_{\alpha\beta} \qquad (\alpha=1,2,\dots,s;\ \beta=1,2,\dots,u). \tag{65}$$

When a partitioned matrix is multiplied on the left by a quasi-diagonal matrix, then the rows of the matrix are multiplied on the left by the corresponding diagonal blocks of the quasi-diagonal matrix.

Now let B be a quasi-diagonal matrix, i.e., let $t=u$ and $B_{\alpha\beta}=O$ for $\alpha\neq\beta$. Then we obtain from (64):

$$C_{\alpha\beta}=A_{\alpha\beta}B_{\beta\beta} \qquad (\alpha=1,2,\dots,s;\ \beta=1,2,\dots,u). \tag{66}$$

When a partitioned matrix is multiplied on the right by a quasi-diagonal matrix, then all the columns of the partitioned matrix are multiplied on the right by the corresponding diagonal cells of the quasi-diagonal matrix.

Note that the multiplication of *square* partitioned matrices of one and the same order is always feasible if the factors are split into equal quadratic schemes of blocks and there are square matrices on the diagonal places in each factor.

The partitioned matrix (58) is called *upper* (*lower*) *quasi-triangular* if $s=t$ and all $A_{\alpha\beta}=O$ for $\alpha>\beta$ ($\alpha<\beta$). A quasi-diagonal matrix is a special case of a quasi-triangular matrix.

From the formulas (64) it is easy to see that:

The product of two upper (*lower*) *quasi-triangular matrices is itself an upper* (*lower*) *quasi-triangular matrix*;[6] *the diagonal cells of the product are obtained by multiplying the corresponding diagonal cells of the factors.*

For when we set $s=t$ in (64) and

$$A_{\alpha\beta}=O,\quad B_{\alpha\beta}=O\quad \text{for}\quad \alpha<\beta,$$

we find

and

$$\left.\begin{matrix} C_{\alpha\beta}=O & \text{for} & \alpha<\beta \\ C_{\alpha\alpha}=A_{\alpha\alpha}B_{\alpha\alpha} & & \end{matrix}\right\} \quad (\alpha,\beta=1,2,\ \dots,\ s).$$

The case of lower quasi-triangular matrices is treated similarly.

We mention a rule for the calculation of the determinant of a quasi-triangular matrix. This rule can be obtained from the Laplace expansion.

If A is a quasi-triangular matrix (in particular, a quasi-diagonal matrix), then the determinant of the matrix is equal to the product of the determinant of the diagonal cells:

$$|A|=|A_{11}|\,|A_{22}|\ \cdots\ |A_{ss}|. \tag{67}$$

2. Let a partitioned matrix

$$A=\begin{pmatrix} \overbrace{A_{11}}^{n_1} & \overbrace{A_{12}}^{n_2} & \dots & \overbrace{A_{1t}}^{n_t} \\ A_{21} & A_{22} & \dots & A_{2t} \\ \dots & \dots & \dots & \dots \\ A_{s1} & A_{s2} & \dots & A_{st} \end{pmatrix} \begin{matrix} \}\,m_1 \\ \}\,m_2 \\ \vdots \\ \}\,m_s \end{matrix}. \tag{68}$$

---

[6] It is assumed here that the block multiplication is feasible.

be given. To the α-th row of submatrices we add the β-th row, multiplied on the left by a rectangular matrix X of dimension $m_\alpha \times n_\beta$. We obtain a partitioned matrix

$$B=\begin{pmatrix} A_{11} & \dots & A_{1t} \\ \dots & \dots & \dots \\ A_{\alpha 1}+XA_{\beta 1} & \dots & A_{\alpha t}+XA_{\beta t} \\ \dots & \dots & \dots \\ A_{\beta 1} & \dots & A_{\beta t} \\ \dots & \dots & \dots \\ A_{s1} & \dots & A_{st} \end{pmatrix}. \tag{69}$$

We introduce an auxiliary square matrix V, which we give in the form of a square scheme of blocks:

$$V=\left(\begin{array}{ccccccc} \overbrace{E}^{m_1} & \dots & \overbrace{O}^{m_\alpha} & \dots & \overbrace{O}^{m_\beta} & \dots & \overbrace{O}^{m_s} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ O & \dots & E & \dots & X & \dots & O \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ O & \dots & O & \dots & E & \dots & O \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ O & \dots & O & \dots & O & \dots & E \end{array}\right)\begin{array}{l} \}m_1 \\ \vdots \\ \}m_\alpha \\ \vdots \\ \}m_\beta \\ \vdots \\ \}m_s \end{array} \tag{70}$$

In the diagonal blocks of V there are unit matrices of order $m_1, m_2, \dots, m_s$, respectively; all the non-diagonal blocks of V are equal to zero except the block X that lies at the intersection of the α-th row and β-th column.

It is easy to see that

$$VA=B. \tag{71}$$

As V is non-singular, we have[7] for the ranks of A and B:

$$r_A=r_B. \tag{72}$$

In the special case where A is a square matrix, we have from (70):

$$|V||A|=|B|. \tag{73}$$

But the determinant of the quasi-triangular matrix V is 1:

$$|V|=1. \tag{74}$$

Hence

$$|A|=|B|. \tag{75}$$

---

[7] See p. 12.

The same conclusion holds when we add to an arbitrary column of (68) another column multiplied on the right by a rectangular matrix x of suitable dimensions.

The results obtained can be formulated as the following theorem.

THEOREM 3: *If to the α-th row (column) of the blocks of the partitioned matrix A we add the β-th row (column) multiplied on the left (right) by a rectangular matrix X of the corresponding dimensions, then the rank of A remains unchanged under this transformation and, if A is a square matrix, the determinant of A is also unchanged.*

3. We now consider the special case in which the diagonal block A_{11} in A is square and non-singular ($|A_{11}| \neq 0$).

To the α-th row of A we add the first row multiplied on the left by $-A_{\alpha 1}A_{11}^{-1}$ ($\alpha = 2, \dots, s$). We thus obtain the matrix

$$B_1=\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1t} \\ O & A_{22}^{(1)} & \dots & A_{2t}^{(1)} \\ \dots & \dots & \dots & \dots \\ O & A_{s2}^{(1)} & \dots & A_{st}^{(1)} \end{pmatrix}, \tag{76}$$

where

$$A_{\alpha\beta}^{(1)}=-A_{\alpha 1}A_{11}^{-1}A_{1\beta}+A_{\alpha\beta} \qquad (\alpha=2, \dots, s;\ \beta=2, \dots, t). \tag{77}$$

If the matrix $A_{22}^{(1)}$ is square and non-singular, then the process can be continued. In this way we arrive at the *generalized algorithm of Gauss.*

Let A be a square matrix. Then

$$|A|=|B_1|=|A_{11}|\begin{vmatrix} A_{22}^{(1)} & \dots & A_{2t}^{(1)} \\ \cdot & \cdot \quad \cdot & \cdot \\ A_{s2}^{(1)} & \dots & A_{st}^{(1)} \end{vmatrix}. \tag{78}$$

Formula (78) reduces the computation of the determinant $|A|$, consisting of st blocks to the computation of a determinant of lower order consisting of $(s-1)\cdot(t-1)$ blocks.[8]

Let us consider a determinant Δ partitioned into four blocks:

$$\Delta=\begin{vmatrix} A & B \\ C & D \end{vmatrix}, \tag{79}$$

where A and D are square matrices.

Suppose $|A| \neq 0$. Then from the second row we subtract the first multiplied on the left by CA^{-1}. We obtain

---

[8] If $A_{22}^{(1)}$ is a square matrix and $|A_{22}^{(1)}| \neq 0$, then this determinant of $(s-1)\ (t-1)$ blocks can again be subjected to such a transformation, etc.

$$\Delta = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A|\ |D - CA^{-1}B|. \tag{I}$$

Similarly, if $|D| \neq 0$, we subtract from the first row in Δ the second multiplied on the left by BD^{-1}, obtaining

$$\Delta = \begin{vmatrix} A - BD^{-1}C & O \\ C & D \end{vmatrix} = |A - BD^{-1}C|\ |D|. \tag{II}$$

In the special case in which all four matrices A, B, C, D are square (of one and the same order n), we deduce from (I) and (II) the *formulas of Schur*, which reduce the computation of a determinant of order $2n$ to the computation of a determinant of order n:

$$\Delta = |AD - ACA^{-1}B| \qquad (A \neq 0), \tag{Ia}$$
$$\Delta = |AD - BD^{-1}CD| \qquad (D \neq 0). \tag{IIa}$$

If the matrices A and C are permutable, then it follows from (Ia) that

$$\Delta = |AD - CB| \qquad (\text{provided } AC = CA). \tag{Ib}$$

Similarly, if C and D are permutable, then

$$\Delta = |AD - BC| \qquad (\text{provided } CD = DC). \tag{IIb}$$

Formula (Ib) was obtained under the assumption $|A| \neq 0$, and (IIb) under the assumption $|D| \neq 0$. However, these restrictions can be removed by continuity arguments.

From formulas (I)-(IIb) we can obtain another six formulas by replacing A and D on the right-hand sides simultaneously by B and C.

Example.

$$\Delta = \begin{vmatrix} 1 & 0 & b_1 & b_2 \\ 0 & 1 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{vmatrix}.$$

By formula (Ib),

$$\Delta = \begin{vmatrix} d_1 - c_1b_1 - c_2b_3 & d_2 - c_1b_2 - c_2b_4 \\ d_3 - c_3b_1 - c_4b_3 & d_4 - c_3b_2 - c_4b_4 \end{vmatrix}.$$

4. From Theorem 3 there follows also

THEOREM 4: *If a rectangular matrix R is represented in partitioned form*

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{80}$$

where A is a square non-singular matrix of order n ($|A| \neq 0$), then the rank of R is equal to n if and only if

$$D = CA^{-1}B. \tag{81}$$

Proof. We subtract from the second row of blocks of R the first, multiplied on the left by CA^{-1}. Then we obtain the matrix

$$T = \begin{pmatrix} A & B \\ O & D - CA^{-1}B \end{pmatrix}. \tag{82}$$

By Theorem 3, the matrices R and T have the same rank. But the rank of T coincides with the rank of A (namely, n) if and only if $D - CA^{-1}B = O$, i.e., when (80) holds. This proves the theorem.

From Theorem 4 there follows an algorithm[9] for the construction of the inverse matrix A^{-1} and, more generally, the product $CA^{-1}B$, where B and C are rectangular matrices of dimensions $n \times p$ and $q \times n$.

By means of Gauss's algorithm,[10] we reduce the matrix

$$\begin{pmatrix} A & B \\ -C & O \end{pmatrix} \qquad (|A| \neq 0) \tag{83}$$

to the form

$$\begin{pmatrix} G & B_1 \\ O & X \end{pmatrix}. \tag{84}$$

We will show that

$$X = CA^{-1}B. \tag{85}$$

For, the same transformation that was applied to the matrix (83) reduces the matrix

---

[9] See [181].

[10] We do not apply here the entire algorithm of Gauss to the matrix (83) but only the first n steps of the algorithm, where n is the order of the matrix. This can be done if the conditions (15) hold for $p = n$. But if these conditions do not hold, then, since $|A| \neq 0$, we may renumber the first n rows (or the first n columns) of the matrix (83) so that the n steps of Gauss's algorithm turn out to be feasible. Such a modified Gaussian algorithm is sometimes applied even when the conditions (15), with $p = n$, are satisfied.

$$\begin{pmatrix} A & B \\ -C & -CA^{-1}B \end{pmatrix} \tag{86}$$

to the form

$$\begin{pmatrix} G & B_1 \\ O & X-CA^{-1}B \end{pmatrix}. \tag{87}$$

By Theorem 4, the matrix (86) is of rank n (n is the order of A). But then (87) must also be of rank n. Hence $X - CA^{-1}B = O$, i.e., (85) holds.

In particular, if $B = y$, where y is a column matrix, and $C = E$, then

$$X = A^{-1}y.$$

Therefore, when we apply Gauss's algorithm to the matrix

$$\begin{pmatrix} A & y \\ -E & O \end{pmatrix},$$

we obtain the solution of the system of equations

$$Ax = y.$$

Further, if in (83) we set $B = C = E$, then by applying the algorithm of Gauss to the matrix

$$\begin{pmatrix} A & E \\ -E & O \end{pmatrix},$$

we obtain

$$\begin{pmatrix} G & W \\ O & X \end{pmatrix},$$

where

$$X = A^{-1}.$$

Let us illustrate this method by finding A^{-1} in the following example.

Example. Let

$$A = \begin{Vmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 2 \end{Vmatrix}.$$

It is required to compute A^{-1}.

We apply a somewhat modified elimination method[11] to the matrix

---

[11] See the preceding footnote.

$$\begin{Vmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{Vmatrix}.$$

To all the rows we add certain multiples of the second row and we arrange that all the elements of the first column, except the second, become zero. Then we add to all the rows, except the second, the third row multiplied by certain factors and see to it that in the second column all the elements, except the second and third, become zero. Then we add to the last three rows the first row with suitable factors and obtain a matrix of the form

$$\begin{Vmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & -2 & -1 & 2 \\ 0 & 0 & 0 & 4 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{Vmatrix}.$$

Therefore

$$A^{-1} = \begin{Vmatrix} -2 & -1 & 2 \\ 4 & 1 & -3 \\ 1 & 1 & -1 \end{Vmatrix}.$$

CHAPTER III

LINEAR OPERATORS IN AN n-DIMENSIONAL VECTOR SPACE

Matrices constitute the fundamental analytic apparatus for the study of linear operators in an n-dimensional space. The study of these operators, in turn, enables us to divide all matrices into classes and to exhibit the significant properties that all matrices of one and the same class have in common.

In the present chapter we shall expound the simpler properties of linear operators in an n-dimensional space. The investigation will be continued in Chapters VII and IX.

§ 1. Vector Spaces

1. Let $\boldsymbol{R}$ be a set of arbitrary elements $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \dots$ in which two operations are defined:[1] the operation of 'addition' and the operation of 'multiplication by a number of the field F.' We postulate that these operations can always be performed uniquely in $\boldsymbol{R}$ and that the following rules hold for arbitrary elements $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ of $\boldsymbol{R}$ and numbers α, β of F:

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$.
2. $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$.
3. There exists an element $\boldsymbol{o}$ in $\boldsymbol{R}$ such that the product of the number 0 with any element $\boldsymbol{x}$ of $\boldsymbol{R}$ is equal to $\boldsymbol{o}$:

   $0 \cdot \boldsymbol{x} = \boldsymbol{o}$.
4. $1 \cdot \boldsymbol{x} = \boldsymbol{x}$.
5. $\alpha(\beta \boldsymbol{x}) = (\alpha\beta)\boldsymbol{x}$.
6. $(\alpha + \beta)\boldsymbol{x} = \alpha\boldsymbol{x} + \beta\boldsymbol{x}$.
7. $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$.

---

[1] These operations will be denoted by the usual signs '+' and '.'; the latter sign will sometimes be omitted.

DEFINITION 1: *A set $\boldsymbol{R}$ of elements in which two operations—'addition' of elements and 'multiplication of elements of $\boldsymbol{R}$ by a number of F'—can always be performed uniquely and for which postulates 1.-7. hold is called a vector space (over the field F) and the elements are called vectors.*[2]

DEFINITION 2. *The vectors $\boldsymbol{x}, \boldsymbol{y}, \dots, \boldsymbol{u}$ of $\boldsymbol{R}$, are called linearly dependent if there exist numbers $\alpha, \beta, \dots, \delta$ in F, not all zero, such that*

$$\alpha\boldsymbol{x} + \beta\boldsymbol{y} + \cdots + \delta\boldsymbol{u} = \boldsymbol{o}. \tag{1}$$

If such a linear dependence does not hold, then the vectors $\boldsymbol{x}, \boldsymbol{y}, \dots, \boldsymbol{u}$ are called linearly independent.

If the vectors $\boldsymbol{x}, \boldsymbol{y}, \dots, \boldsymbol{u}$ are linearly dependent, then one of the vectors can be repesented as a linear combination, with coefficients in R, of the remaining ones. For example, if $\alpha \neq 0$ in (1), then

$$\boldsymbol{x} = -\frac{\beta}{\alpha}\boldsymbol{y} - \cdots - \frac{\delta}{\alpha}\boldsymbol{u}.$$

DEFINITION 3. *The space $\boldsymbol{R}$ is called finite-dimensional and the number n is called the dimension of the space if there exist n linearly independent vectors in $\boldsymbol{R}$, while any $n + 1$ vectors in $\boldsymbol{R}$ are linearly dependent. If the space contains linearly independent systems of an arbitrary number of vectors, then it is called infinite-dimensional.*

In this book we shall study mainly finite-dimensional spaces.

DEFINITION 4. *A system of n linearly independent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ of an n-dimensional space, given in a definite order, is called a basis of the space.*

2. *Example* 1. The set of all ordinary vectors (directed geometrical segments) is a three-dimensional vector space. The part of this space that consists of the vectors parallel to some plane is a two-dimensional space, and all the vectors parallel to a given line form a one-dimensional vector space.

Example 2. Let us call a column $x = (x_1, x_2, \dots, x_n)$ of n numbers of F a vector (where n is a fixed number). We define the basic operations as operations on column matrices:

---

[2] It is easy to see that all the usual properties of the operations of addition and of multiplication by a number follow from properties 1.-7. For example, for arbitrary $\boldsymbol{x}$ of $\boldsymbol{R}$ we have:

$$\boldsymbol{x} + \boldsymbol{o} = \boldsymbol{x}\ [\boldsymbol{x} + \boldsymbol{o} = 1 \cdot \boldsymbol{x} + 0 \cdot \boldsymbol{x} = (1 + 0) \cdot \boldsymbol{x} = 1 \cdot \boldsymbol{x} = \boldsymbol{x}];$$
$$\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{o}, \text{ where } -\boldsymbol{x} = (-1) \cdot \boldsymbol{x};$$

etc.

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),$$
$$\alpha (x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

The null vector is the column $(0, 0, \ldots, 0)$. It is easy to verify that all the postulates 1.-7. are satisfied. The vectors form an n-dimensional space. As a basis of the space we can take, for example, the column of unit matrices of order n:

$$(1, 0, \ldots, 0),\ (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1).$$

The space thus defined is often called the *n-dimensional number space.*

Example 3. The set of all infinite sequences $(x_1, x_2, \ldots, x_n, \ldots)$ in which the operations are defined in a natural way, i.e.,

$$(x_1, x_2, \ldots, x_n, \ldots) + (y_1, y_2, \ldots, y_n, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n, \ldots),$$
$$\alpha (x_1, x_2, \ldots, x_n, \ldots) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n, \ldots),$$

is an infinite-dimensional space.

Example 4. The set of polynomials $a_0 + a_1 t + \ldots + a_{n-1} t^{n-1}$ of degree $< n$ with coefficients in F is an n-dimensional vector space.[3] As a basis of this space we can take, say, the system of powers $t^0, t^1, \ldots, t^{n-1}$.

The set of all such polynomials (without a bound on the degree) form an infinite-dimensional space.

Example 5. The set of all functions defined on a closed interval $[a, b]$ form an infinite-dimensional space.

3. Let the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ forms a basis of an n-dimensional vector space $\boldsymbol{R}$ and let $\mathbf{x}$ be an arbitrary vector of the space. Then the vectors $\mathbf{x}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly dependent (because there are $n + 1$ of them):

$$\alpha_0 \mathbf{x} + \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_n \mathbf{e}_n = \mathbf{o},$$

where at least one of the numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ is different from zero. But in this case we must have $\alpha_0 \neq 0$, since the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ cannot be linearly dependent. Therefore

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \cdots + x_n \mathbf{e}_n \tag{2}$$

where $x_i = -\alpha_i/\alpha_0$ $(i = 1, 2, \ldots, n)$.

Note that the numbers $x_1, x_2, \ldots, x_n$ are uniquely determined when the vector $\mathbf{x}$ and the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are given. For if there is another decomposition of $\mathbf{x}$ besides (2),

$$\mathbf{x} = x_1' \mathbf{e}_1 + x_2' \mathbf{e}_2 + \cdots + x_n' \mathbf{e}_n, \tag{3}$$

---

[3] The basic operations are taken to be ordinary addition of polynomials and multiplication of a polynomial by a number.

---

then, by subtracting (2) from (3), we obtain

$$(x_1' - x_1)\mathbf{e}_1 + (x_2' - x_2)\mathbf{e}_2 + \cdots + (x_n' - x_n)\mathbf{e}_n = \mathbf{o},$$

and since the vectors of a basis are linearly dependent, it follows that

$$x_1' - x_1 = x_2' - x_2 = \cdots = x_n' - x_n = 0,$$

i.e.,

$$x_1' = x_1,\ x_2' = x_2,\ \ldots,\ x_n' = x_n. \tag{4}$$

The numbers $x_1, x_2, \ldots, x_n$ are called the *coordinates* of $\mathbf{x}$ in the basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

If

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i \quad \text{and} \quad \mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{e}_i,$$

then

$$\mathbf{x} + \mathbf{y} = \sum_{i=1}^{n} (x_i + y_i)\mathbf{e}_i \quad \text{and} \quad \alpha \mathbf{x} = \sum_{i=1}^{n} \alpha x_i \mathbf{e}_i.$$

i.e., *the coordinates of a sum of vectors are obtained by addition of the corresponding coordinates of the summands and the product of a vector by a number α is obtained by multiplying all the coordinates of the vector by α.*

4. Let the vectors

$$\mathbf{x}_k = \sum_{i=1}^{n} x_{ik} \mathbf{e}_i$$

be linearly dependent, i.e.,

$$\sum_{k=1}^{m} c_k \mathbf{x}_k = \mathbf{o}, \tag{5}$$

where at least one of the numbers $c_1, c_2, \ldots, c_m$ is not equal to zero.

If a vector is the null vector, then all its components are zero. Hence the vector equation (5) is equivalent to the following system of scalar equations:

$$\left.\begin{array}{l} c_1 x_{11} + c_2 x_{12} + \cdots + c_m x_{1m} = 0 \\ c_1 x_{21} + c_2 x_{22} + \cdots + c_m x_{2m} = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_1 x_{n1} + c_2 x_{n2} + \cdots + c_m x_{nm} = 0. \end{array}\right\} \tag{6}$$

As is well known, this system of homogeneous linear equations for $c_1, c_2, \ldots, c_m$ has a non-zero solution if and only if the rank of the coefficient matrix is less than the number of unknowns, i.e., less than m. A necessary and sufficient condition for the independence of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ is, therefore, that this rank should be m.

Thus, the following theorem holds:

THEOREM 1: *In order that the vectors* $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ *be linearly independent it is necessary and sufficient that the rank r of the matrix formed from the coordinates of these vectors in an arbitrary basis*

$$\left\| \begin{matrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{matrix} \right\| \tag{7}$$

be equal to m, i.e., to the number of vectors.

Note. The linear independence of the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ means that the columns of the matrix (7) are linearly independent, since the k-th column consists of the coordinates of x_k $(k = 1, 2, \ldots, m)$. By the theorem, therefore, if the columns of a matrix are linearly independent, then the rank of the matrix is equal to the number of columns. Hence it follows that in an arbitrary rectangular matrix the maximal number of linearly independent columns is equal to the rank of the matrix. Moreover, if we transpose the matrix, i.e., change the rows into columns and the columns into rows, then the rank obviously remains unchanged. Hence *in a rectangular matrix the number of linearly independent columns is always equal to the number of linearly independent rows and equal to the rank of the matrix.*[4]

5. If in an n-dimensional space a basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ has been chosen, then to every vector $\boldsymbol{x}$ there corresponds uniquely the column $x = (x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n$ are the coordinates of $\boldsymbol{x}$ in the given basis. Thus, the choosing of a basis establishes a one-to-one correspondence between the vectors of an arbitrary n-dimensional vector space $\boldsymbol{R}$ and the vectors of the n-dimensional number space $\boldsymbol{R}'$ considered in Example 2. Here the sum of vectors in $\boldsymbol{R}$ corresponds to the sum of the corresponding vectors of $\boldsymbol{R}'$. The analogous correspondence holds for the product of a vector by a number α of F. In other words, an arbitrary n-dimensional vector space is *isomorphic* to the n-dimensional number space, and therefore *all vector spaces of the same number n of dimensions over the same number field* F *are isomorphic.* This means that to within isomorphism there exists only one n-dimensional vector space for a given number field.

---

[4] This proposition follows from Theorem 1, in the proof of which we have started from the well-known property of a system of linear homogeneous equations: a non-zero solution exists only when the rank of the coefficient matrix is less than the number of unknowns. For a proof of Theorem 1 independent of this property, see § 5.

The reader may ask why we have introduced an 'abstract' n-dimensional space if it coincides to within isomorphism with the n-dimensional number space. Indeed, we could have defined a vector as a system of n numbers given in a definite order and could have introduced the operations on these vectors in the very way it was done in Example 2. But we would then have mixed up properties of vectors that do not depend on the choice of a basis with properties of a particular basis. For example, the fact that all the coordinates of a vector are zero is a property of the vector itself; it does not depend on the choice of basis. But the equality of all its coordinates is not a property of the vector itself, because it disappears under a change of basis. *The axiomatic definition of a vector space immediately singles out the properties of vectors that do not depend on the choice of a basis.*

§ 2. A Linear Operator Mapping an n-Dimensional Space into an m-Dimensional Space

1. We consider a linear transformation

$$\left.\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n , \end{aligned}\right\} \tag{8}$$

whose coefficients belong to the number field F as well as two vector spaces over F: an n-dimensional space $\boldsymbol{R}$ and an m-dimensional space $\boldsymbol{S}$. We choose a basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ in $\boldsymbol{R}$ and a basis $\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_m$ in $\boldsymbol{S}$. Then the transformation (8) associates with every vector $\boldsymbol{x} = \sum_{i=1}^{n} x_i \boldsymbol{e}_i$ of $\boldsymbol{R}$ a certain vector $\boldsymbol{y} = \sum_{k=1}^{m} y_k \boldsymbol{g}_k$ of $\boldsymbol{S}$, i.e., the transformation (8) determines a certain operator $\boldsymbol{A}$ that sets up a correspondence between the vector $\boldsymbol{x}$ and the vector $\boldsymbol{y} : \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. It is easy to see that this operator $\boldsymbol{A}$ has the property of linearity, which we formulate as follows:

DEFINITION 5: *An operator* $\boldsymbol{A}$ *mapping* $\boldsymbol{R}$ *into* $\boldsymbol{S}$, *i.e., associating with every vector* $\boldsymbol{x}$ *of* $\boldsymbol{R}$ *a certain vector* $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ *of* $\boldsymbol{S}$ *is called linear if for arbitrary* $\boldsymbol{x}_1, \boldsymbol{x}_2$ *of* $\boldsymbol{R}$ *and* α *of* F

$$\boldsymbol{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \boldsymbol{A}\boldsymbol{x}_1 + \boldsymbol{A}\boldsymbol{x}_2, \quad \boldsymbol{A}(\alpha\boldsymbol{x}_1) = \alpha\boldsymbol{A}\boldsymbol{x}_1. \tag{9}$$

Thus, the transformation (8), for a given basis in $\boldsymbol{R}$ and a given basis in $\boldsymbol{S}$, determines a linear operator mapping $\boldsymbol{R}$ into $\boldsymbol{S}$.

We shall now show the converse, i.e., that for an arbitrary linear operator $\boldsymbol{A}$ mapping $\boldsymbol{R}$ into $\boldsymbol{S}$ and arbitrary bases $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ in $\boldsymbol{R}$ and $\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_m$ in $\boldsymbol{S}$, there exists a rectangular matrix with elements in F

$$\begin{Vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots & \ldots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{Vmatrix} \tag{10}$$

such that the linear transformation (8) formed by means of this matrix expresses the coordinates of the transformed vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ in terms of the coordinates of the original vector $\boldsymbol{x}$.

Let us, in fact, apply the operator $\boldsymbol{A}$ to the basis vector $\boldsymbol{e}_k$ and let the coordinates in the basis $\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_m$ of the vector $\boldsymbol{A}\boldsymbol{e}_k$ thus obtained be denoted by $a_{1k}, a_{2k}, \ldots, a_{mk}$ $(k = 1, 2, \ldots, n)$:

$$\boldsymbol{A}\boldsymbol{e}_k = \sum_{i=1}^{m} a_{ik}\boldsymbol{g}_i \qquad (k = 1, 2, \ldots, n). \tag{11}$$

Multiplying both sides of (11) by x_k and summing from 1 to n, we obtain

$$\sum_{k=1}^{n} x_k \boldsymbol{A}\boldsymbol{e}_k = \sum_{i=1}^{m} \Big(\sum_{k=1}^{n} a_{ik}x_k\Big)\boldsymbol{g}_i;$$

hence

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\Big(\sum_{k=1}^{n} x_k\boldsymbol{e}_k\Big) = \sum_{k=1}^{n} x_k\boldsymbol{A}\boldsymbol{e}_k = \sum_{i=1}^{m} y_i\boldsymbol{g}_i,$$

where

$$y_i = \sum_{k=1}^{n} a_{ik}x_k \qquad (i = 1, 2, \ldots, m),$$

and this is what we had to show.

Thus, for given bases of $\boldsymbol{R}$ and $\boldsymbol{S}$: *to every linear operator $\boldsymbol{A}$ mapping $\boldsymbol{R}$ into $\boldsymbol{S}$ there corresponds a rectangular matrix of dimension $m \times n$ and, conversely, to every such matrix there corresponds a linear operator mapping $\boldsymbol{R}$ into $\boldsymbol{S}$.*

Here, in the matrix A corresponding to the operator $\boldsymbol{A}$, *the k-th column consists of the coordinates of the vector $\boldsymbol{A}\boldsymbol{e}_k$* $(k = 1, 2, \ldots, n)$.

We denote by $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ the coordinate columns of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Then the vector equation

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

corresponds to the matrix equation

$$y = Ax,$$

which is the matrix form of the transformation (8).

Example. We consider the set of all polynomials in t of degree $\leqq n - 1$ with coefficients in F. This set forms an n-dimensional vector space $\boldsymbol{R}_n$ (see Example 4., p. 52). Similarly, the polynomials in t of degree $\leqq n - 2$ with coefficients in F form a space $\boldsymbol{R}_{n-1}$. The differentiation operator $\frac{d}{dt}$ associates with every polynomial of $\boldsymbol{R}_n$ a certain polynomial in $\boldsymbol{R}_{n-1}$. Thus, this operator maps $\boldsymbol{R}_n$ into $\boldsymbol{R}_{n-1}$. The differentiation operator is linear, since

$$\frac{d}{dt}[\varphi(t) + \psi(t)] = \frac{d\varphi(t)}{dt} + \frac{d\psi(t)}{dt}, \quad \frac{d}{dt}[\alpha\varphi(t)] = \alpha\,\frac{d\varphi(t)}{dt}.$$

In $\boldsymbol{R}_n$ and $\boldsymbol{R}_{n-1}$ we choose bases consisting of powers of t:

$$t^0 = 1, t, \ldots, t^{n-1} \qquad \text{and} \qquad t^0 = 1, t, \ldots, t^{n-2}.$$

Using formulas (11), we construct the rectangular matrix of dimension $(n - 1 \times n)$ corresponding to the differentiation operator $\frac{d}{dt}$ in these bases:

$$\begin{Vmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 2 \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & n-1 \end{Vmatrix}.$$

§ 3. Addition and Multiplication of Linear Operators

1. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two linear operators mapping $\boldsymbol{R}$ into $\boldsymbol{S}$ and let the corresponding matrices be

$$A = \| a_{ik} \|, \quad B = \| b_{ik} \| \qquad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n).$$

DEFINITION 6: *The sum of the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ is the operator $\boldsymbol{C}$ defined by the equation*[5]

$$\boldsymbol{C}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{x} \quad (\boldsymbol{x} \in \boldsymbol{R}). \tag{12}$$

On the basis of this definition it is easy to verify that the sum $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ of the linear operators $\boldsymbol{A}$ and $\boldsymbol{B}$ is itself a linear operator. Furthermore,

$$\boldsymbol{C}\boldsymbol{e}_k = \boldsymbol{A}\boldsymbol{e}_k + \boldsymbol{B}\boldsymbol{e}_k = \sum_{k=1}^{n} (a_{ik} + b_{ik})\,\boldsymbol{e}_k.$$

---

[5] $\boldsymbol{x} \in \boldsymbol{R}$ means that the element $\boldsymbol{x}$ belongs to the set $\boldsymbol{R}$. It is assumed that (12) holds for arbitrary $\boldsymbol{x}$ in $\boldsymbol{R}$.

Hence it follows that the operator $\boldsymbol{C}$ corresponds to the matrix $C = \| c_{ik} \|_1^{}$, where $c_{ik} = a_{ik} + b_{ik}$ $(i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$, i.e., the operator $\boldsymbol{C}$ corresponds to the matrix

$$C = A + B. \tag{13}$$

We would come to the same conclusion starting from the matrix equation

$$Cx = Ax + Bx \tag{14}$$

(x is the coordinate column of the vector $\boldsymbol{x}$) corresponding to the vector equation (12). Since x is an arbitrary column, (13) follows from (14).

2. Let $\boldsymbol{R}$, $\boldsymbol{S}$, and $\boldsymbol{T}$ be three vector spaces of dimension q, n, and m, and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two linear operators, of which $\boldsymbol{B}$ maps $\boldsymbol{R}$ into $\boldsymbol{S}$ and $\boldsymbol{A}$ maps $\boldsymbol{S}$ into $\boldsymbol{T}$; in symbols:

$$\boldsymbol{R} \xrightarrow{\boldsymbol{B}} \boldsymbol{S} \xrightarrow{\boldsymbol{A}} \boldsymbol{T}.$$

DEFINITION 7. *The product of the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ is the operator $\boldsymbol{C}$ for which*

$$\boldsymbol{C}\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{x}) \quad (\boldsymbol{x} \in \boldsymbol{R}). \tag{15}$$

holds for every $\boldsymbol{x}$ of $\boldsymbol{R}$.

The operator $\boldsymbol{C}$ maps $\boldsymbol{R}$ into $\boldsymbol{T}$:

$$\boldsymbol{R} \xrightarrow{\boldsymbol{C}=\boldsymbol{AB}} \boldsymbol{T}.$$

From the linearity of the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ follows the linearity of $\boldsymbol{C}$. We choose arbitrary bases in $\boldsymbol{R}$, $\boldsymbol{S}$, and $\boldsymbol{T}$ and denote by A, B, and C the matrices corresponding, in this choice of basis, to the operators $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$. Then the vector equations

$$\boldsymbol{z} = \boldsymbol{A}\boldsymbol{y},\ \boldsymbol{y} = \boldsymbol{B}\boldsymbol{x},\ \boldsymbol{z} = \boldsymbol{C}\boldsymbol{x} \tag{16}$$

correspond to the matrix equations:

$$z = Ay,\ \ y = Bx,\ \ z = Cx,$$

where x, y, z are the coordinate columns of the vectors $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$. Hence

$$Cx = A(Bx) = (AB)x$$

and as the column x is arbitrary

$$C = AB. \tag{17}$$

Thus, the product $\boldsymbol{C} = \boldsymbol{AB}$ of the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ corresponds to the matrix $C = \| c_{ij} \|$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, q)$, which is the product of the matrices A and B.

We leave it to the reader to show that the operator[6]

$$\boldsymbol{C} = \alpha\boldsymbol{A} \qquad (\alpha \in \text{F})$$

corresponds to the matrix

$$C = \alpha A.$$

Thus we see that in Chapter I the operations on matrices were so defined that the sum $\boldsymbol{A} + \boldsymbol{B}$, the product $\boldsymbol{AB}$, and the product $\alpha\boldsymbol{A}$ correspond to the matrices $A + B$, AB, and αA, respectively, where A and B are the matrices corresponding to the operators $\boldsymbol{A}$ and $\boldsymbol{B}$, and α is a number of F.

§ 4. Transformation of Coordinates

1. In an n-dimensional vector space we consider two bases: $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ (the 'old' basis) and $\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \ldots, \boldsymbol{e}_n^*$ the 'new' basis).

The mutual disposition of the basis vectors is determined if the coordinates of the vectors of the basis are given relative to the other basis.

We set

$$\left.\begin{array}{l} \boldsymbol{e}_1^* = t_{11}\boldsymbol{e}_1 + t_{21}\boldsymbol{e}_2 + \cdots + t_{n1}\boldsymbol{e}_n \\ \boldsymbol{e}_2^* = t_{12}\boldsymbol{e}_1 + t_{22}\boldsymbol{e}_2 + \cdots + t_{n2}\boldsymbol{e}_n \\ \dots\dots\dots\dots\dots\dots\dots \\ \boldsymbol{e}_n^* = t_{1n}\boldsymbol{e}_1 + t_{2n}\boldsymbol{e}_2 + \cdots + t_{nn}\boldsymbol{e}_n \end{array}\right\} \tag{18}$$

or in abbreviated form,

$$\boldsymbol{e}_k^* = \sum_{i=1}^{n} t_{ik}\boldsymbol{e}_i \quad (k = 1, 2, \ldots, n). \tag{18'}$$

We shall now establish the connection between the coordinates of one and the same vector in the two different bases.

Let $x_1, x_2, \ldots, x_n$ and $x_1^*, x_2^*, \ldots, x_n^*$ be the coordinates of the vector $\boldsymbol{x}$ relative to the 'old' and the 'new' bases, respectively:

$$\boldsymbol{x} = \sum_{i=1}^{n} x_i\boldsymbol{e}_i = \sum_{k=1}^{n} x_k^*\boldsymbol{e}_k^*. \tag{19}$$

In (19) we substitute for the vectors $\boldsymbol{e}_k^*$ the expressions given for them in (18). We obtain:

---

[6] I.e., the operator for which $\boldsymbol{C}\boldsymbol{x} = \alpha\boldsymbol{A}\boldsymbol{x}$ $(\boldsymbol{x} \in \boldsymbol{R})$.

$$\mathbf{x} = \sum_{k=1}^{n} x_k^* \sum_{i=1}^{n} t_{ik} \mathbf{e}_i = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} t_{ik} x_k^* \right) \mathbf{e}_i .$$

Comparing this with (19) and bearing in mind that the coordinates of a vector are uniquely determined when the vector and the basis are given, we find:

$$x_i = \sum_{k=1}^{n} t_{ik} x_k^* \quad (i = 1, 2, \ldots, n) . \tag{20}$$

or in explicit form:

$$\left. \begin{array}{l} x_1 = t_{11} x_1^* + t_{12} x_2^* + \cdots + t_{1n} x_n^* \\ x_2 = t_{21} x_1^* + t_{22} x_2^* + \cdots + t_{2n} x_n^* \\ \dots\dots\dots\dots\dots\dots\dots \\ x_n = t_{n1} x_1^* + t_{n2} x_2^* + \cdots + t_{nn} x_n^* . \end{array} \right\} \tag{21}$$

Formulas (21) determine the transformation of the coordinates of a vector on transition from one basis to another. They express the 'old' coordinates in terms of the 'new' ones. The matrix

$$T = \| t_{ik} \|_1^n \tag{22}$$

is called the *matrix of the coordinate transformation* or the *transforming matrix*. Its k-th column consists of the 'old' coordinates of the k-th 'new' basis vector. This follows from formulas (18) or immediately from (21) if we set in the latter $x_k^* = 1$, $x_i^* = 0$ for $i \neq k$.

Note that the matrix T is non-singular, i.e.,

$$| T | \neq 0. \tag{23}$$

For when we set in (21) $x_1 = x_2 = \ldots = x_n = 0$, we obtain a system of n linear homogeneous equations in the n unknowns $x_1^*, x_2^*, \ldots, x_n^*$ with determinant $| T |$. This system can only have the zero solution $x_1^* = 0$, $x_2^* = 0$, $\ldots$, $x_n^* = 0$, since otherwise (19) would imply a linear dependence among the vectors $\mathbf{e}_1^*, \mathbf{e}_2^*, \ldots, \mathbf{e}_n^*$. Therefore $| T | \neq 0$.[7]

We now introduce the column matrices $x = (x_1, x_2, \ldots, x_n)$ and $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$. Then the formulas (21) for the coordinate transformation can be written in the form of the following matrix equation:

$$x = T x^* . \tag{24}$$

Multiplying both sides of this equation by T^{-1}, we obtain the expression for the inverse transformation

$$x^* = T^{-1} x . \tag{25}$$

---

[7] The inequality (23) also follows from Theorem 1 (p.54), because the elements of T are the 'old' coordinates of the linearly independent vectors $\mathbf{e}_1^*, \mathbf{e}_2^*, \ldots, \mathbf{e}_n^*$.

§ 5. Equivalent Matrices. The Rank of an Operator. Sylvester's Inequality

1. Let $\mathbf{R}$ and $\mathbf{S}$ be two vector spaces of dimension n and m, respectively, over the number field F and let $\mathbf{A}$ be a linear operator mapping $\mathbf{R}$ into $\mathbf{S}$. In the present section we shall make clear how the matrix A corresponding to the given linear operator $\mathbf{A}$ changes when the bases in $\mathbf{R}$ and $\mathbf{S}$ are changed.

We choose arbitrary bases $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in $\mathbf{R}$ and $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m$ in $\mathbf{S}$. In these bases the operator $\mathbf{A}$ corresponds to a matrix $A = \| a_{ik} \|$ $(i = 1, 2, 3, \ldots, m; k = 1, 2, \ldots, n)$. To the vector equation

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{26}$$

there corresponds the matrix equation

$$y = Ax , \tag{27}$$

where x and y are the coordinate columns for the vectors $\mathbf{x}$ and $\mathbf{y}$ in the bases $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ and $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m$.

We now choose other bases $\mathbf{e}_1^*, \mathbf{e}_2^*, \ldots, \mathbf{e}_n^*$ and $\mathbf{g}_1^*, \mathbf{g}_2^*, \ldots, \mathbf{g}_m^*$ in $\mathbf{R}$ and $\mathbf{S}$. In the new bases we shall have x^*, y^*, A^* instead of x, y, A. Here

$$y^* = A^* x^* . \tag{28}$$

Let us denote by Q and N the non-singular square matrices of order n and m, respectively, that realize the coordinate tranformations in the spaces $\mathbf{R}$ and $\mathbf{S}$ on transition from the old bases to the new ones (see § 4):

$$x = Q x^* , \quad y = N y^* . \tag{29}$$

Then we obtain from (27) and (29):

$$y^* = N^{-1} y = N^{-1} A x = N^{-1} A Q x^* . \tag{30}$$

Setting $P = N^{-1}$, we find from (28) and (30):

$$A^* = PAQ . \tag{31}$$

DEFINITION 8: *Two rectangular matrices A and B of the same dimension are called equivalent if there exist two non-singular matrices P and Q such that*[8]

$$B = PAQ . \tag{32}$$

---

[8] If the matrices A and B are of dimension $m \times n$, then in (32) the square matrix P is of order m, and Q of order n. If the elements of the equivalent matrices A and B belong to some number field, then P and Q may be chosen such that their elements belong to the same number field.

From (31) it follows that two matrices corresponding to one and the same linear operator $\boldsymbol{A}$ for different choices of bases in $\boldsymbol{R}$ and $\boldsymbol{S}$ are always equivalent. It is easy to see that, conversely, if a matrix A corresponds to the operator $\boldsymbol{A}$ for certain bases in $\boldsymbol{R}$ and $\boldsymbol{S}$, and if a matrix B is equivalent to A, then it corresponds to the same linear operator for certain other bases in $\boldsymbol{R}$ and $\boldsymbol{S}$.

Thus, to every linear operator mapping $\boldsymbol{R}$ into $\boldsymbol{S}$ there corresponds a class of equivalent matrices with elements in F.

2. The following theorem establishes a criterion for the equivalence of two matrices:

THEOREM 2: *Two rectangular matrices of the same dimension are equivalent if and only if they have the same rank.*

Proof. The condition is *necessary*. When a rectangular matrix is multiplied by an arbitrary non-singular square matrix (on the right or left), then its rank does not change (see Chapter I, p. 17). Therefore it follows from (32) that

$$r_A = r_B .$$

The condition is *sufficient*. Let A be a rectangular matrix of dimension $m \times n$. It determines a linear operator $\boldsymbol{A}$ mapping the space $\boldsymbol{R}$ with the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ into the space $\boldsymbol{S}$ with the basis $\boldsymbol{g}_1, \boldsymbol{g}_2, \dots, \boldsymbol{g}_m$. Let r denote the number of linearly independent vectors among the vectors $\boldsymbol{Ae}_1, \boldsymbol{Ae}_2, \dots, \boldsymbol{Ae}_n$. Without loss of generality we may assume that the vectors $\boldsymbol{Ae}_1, \boldsymbol{Ae}_2, \dots, \boldsymbol{Ae}_r$ are linearly independent[9] and that the remaining $\boldsymbol{Ae}_{r+1}, \boldsymbol{Ae}_{r+2}, \dots, \boldsymbol{Ae}_n$ are expressed linearly in terms of them:

$$\boldsymbol{Ae}_k = \sum_{j=1}^{r} c_{kj}\boldsymbol{Ae}_j \qquad (k = r+1, \dots, n). \tag{33}$$

We define a new basis in $\boldsymbol{R}$ as follows:

$$\boldsymbol{e}_i^* = \begin{cases} \boldsymbol{e}_i & (i = 1, 2, \dots, r), \\ \boldsymbol{e}_i - \sum_{j=1}^{r} c_{ij}\boldsymbol{e}_j & (i = r+1, \dots, n). \end{cases} \tag{34}$$

Then by (33),

$$\boldsymbol{Ae}_k^* = \boldsymbol{o} \qquad (k = r+1, \dots, n). \tag{35}$$

Next, we set

$$\boldsymbol{Ae}_j^* = \boldsymbol{g}_j^* \qquad (j = 1, 2, \dots, r). \tag{36}$$

---

[9] This can be achieved by a suitable numbering of the basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$.

The vectors $\boldsymbol{g}_1^*, \boldsymbol{g}_2^*, \dots, \boldsymbol{g}_r^*$ are linearly independent. We supplement them with suitable vectors $\boldsymbol{g}_{r+1}^*, \boldsymbol{g}_{r+2}^*, \dots, \boldsymbol{g}_m^*$ to obtain a basis $\boldsymbol{g}_1^*, \boldsymbol{g}_2^*, \dots, \boldsymbol{g}_m^*$ of $\boldsymbol{S}$.

The matrix corresponding to the same operator $\boldsymbol{A}$ in the new bases $\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \dots, \boldsymbol{e}_n^*$, $\boldsymbol{g}_1^*, \boldsymbol{g}_2^*, \dots, \boldsymbol{g}_m^*$ has now, by (35) and (36), the form

$$I_r = \begin{array}{c} \overbrace{\hphantom{1 \quad 0 \quad \cdots \quad 0}}^{r}\hphantom{\quad 0 \quad \cdots \quad 0 \quad} \\ \left\| \begin{array}{ccccccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} \right\| \end{array} . \tag{37}$$

Along the main diagonal of I_r, starting at the top, there are r units; all the remaining elements of I_r are zeros. Since the matrices A and I_r correspond to one and the same operator $\boldsymbol{A}$, they are equivalent. As we have proved, equivalent matrices have the same rank. Hence the rank of the original matrix A is r.

We have shown that an arbitrary rectangular matrix of rank r is equivalent to the 'canonical' matrix I_r. But I_r is completely determined by specifying its dimensions $m \times n$ and the number r. Therefore all rectangular matrices of given dimension $m \times n$ and of given rank r are equivalent to one and the same matrix I_r and consequently to each other. This completes the proof of the theorem.

3. Let $\boldsymbol{A}$ be a linear operator mapping an n-dimensional space $\boldsymbol{R}$ into an n-dimensional space $\boldsymbol{S}$. The set of all vectors of the form $\boldsymbol{Ax}$, where $\boldsymbol{x} \in \boldsymbol{R}$, forms a vector space.[10] This space will be denoted by $\boldsymbol{AR}$; it is part of the space $\boldsymbol{S}$ or, as we shall say, is a *subspace* of $\boldsymbol{S}$.

Together with the subspace $\boldsymbol{AR}$ of $\boldsymbol{S}$ we consider the set of all vectors $\boldsymbol{x} \in \boldsymbol{R}$ that satisfy the equation

$$\boldsymbol{Ax} = \boldsymbol{o} \tag{38}$$

These vectors also form a subspace of $\boldsymbol{R}$, which we shall denote by $\boldsymbol{N}_{\boldsymbol{A}}$.

---

[10] The set of vectors of the form $\boldsymbol{Ax}$ $(\boldsymbol{x} \in \boldsymbol{R})$ satisfies the postulates 1.-7. of § 1, because the sum of two such vectors and the product of such a vector by a number are also vectors of this form.

DEFINITION 9: *If a linear operator $\boldsymbol{A}$ maps $\boldsymbol{R}$ into $\boldsymbol{S}$, then the dimension r of the space $\boldsymbol{AR}$ is called the rank of $\boldsymbol{A}$,*[11] *and the dimension d of the space $\boldsymbol{N_A}$ consisting of all vectors $\boldsymbol{x} \in \mathbf{R}$ that satisfy the condition* (38) *is called the defect, or nullity, of $\boldsymbol{A}$.*

Among all the equivalent rectangular matrices that describe a given operator $\boldsymbol{A}$ in distinct bases there occurs the canonical matrix I_r (see (37)). We denote the corresponding bases of $\boldsymbol{R}$ and $\boldsymbol{S}$ by $\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \ldots, \boldsymbol{e}_n^*$ and $\boldsymbol{g}_1^*, \boldsymbol{g}_2^*, \ldots, \boldsymbol{g}_m^*$. Then

$$\boldsymbol{Ae}_1^* = \boldsymbol{g}_1^*, \ \ldots, \ \boldsymbol{Ae}_r^* = \boldsymbol{g}_r^*, \quad \boldsymbol{Ae}_{r+1}^* = \ldots = \boldsymbol{Ae}_n^* = \boldsymbol{o}.$$

From the definition of $\boldsymbol{AR}$ and $\boldsymbol{N_A}$ it follows that the vectors $\boldsymbol{g}_1^*, \boldsymbol{g}_2^*, \ldots, \boldsymbol{g}_r^*$ form a basis of $\boldsymbol{AR}$ and that the vectors $\boldsymbol{e}_{r+1}^*, \boldsymbol{e}_{r+2}^*, \ldots, \boldsymbol{e}_n^*$ form a basis of $\boldsymbol{N_A}$. Hence it follows that r is the rank of the operator $\boldsymbol{A}$ and that

$$d = n - r. \tag{39}$$

If A is an arbitrary matrix corresponding to $\boldsymbol{A}$, then it is equivalent to I_r and therefore has the same rank r. Thus, *the rank of an operator $\boldsymbol{A}$ coincides with the rank of the rectangular matrix A*

$$A = \left\| \begin{matrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{matrix} \right\|$$

determined by $\boldsymbol{A}$ in arbitrary bases $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n \in \boldsymbol{R}$ and $\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_m \in \boldsymbol{S}$.

The columns of A are formed by the coordinate vectors $\boldsymbol{A}_1\boldsymbol{e}_1, \ldots, \boldsymbol{A}_n\boldsymbol{e}_n$. Since it follows from $\boldsymbol{x} = \sum_{i=1}^{n} x_i \boldsymbol{e}_i$ that $\boldsymbol{Ax} = \sum_{i=1}^{n} x_i \boldsymbol{Ae}_i$, the rank of $\boldsymbol{A}$, i.e., the dimension of $\boldsymbol{RA}$, is equal to the maximal number of linearly independent vectors among $\boldsymbol{Ae}_1, \boldsymbol{Ae}_2, \ldots, \boldsymbol{Ae}_n$. Thus:

The rank of a matrix coincides with the number of linearly independent columns of the matrix.

Since under transposition the rows of a matrix become its columns and the rank remains unchanged:

---

[11] The dimension of the space $\boldsymbol{AR}$ never exceeds the dimension of $\boldsymbol{R}$, so that $r \leqq n$. This follows from the fact that the equation $\boldsymbol{x} = \sum_{i=1}^{n} x_i \boldsymbol{e}_i$ (where $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ is a basis of $\boldsymbol{R}$) implies the equation $\boldsymbol{Ax} = \sum_{i=1}^{n} x_i \boldsymbol{Ae}_i$.

The number of linearly independent rows of a matrix is also equal to the rank of the matrix.[12]

4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two linear operators and let $\boldsymbol{C} = \boldsymbol{AB}$ be their product. Suppose that the operator $\boldsymbol{B}$ maps $\boldsymbol{R}$ into $\boldsymbol{S}$ and that the operator $\boldsymbol{A}$ maps $\boldsymbol{S}$ into $\boldsymbol{T}$. Then the operator $\boldsymbol{C}$ maps $\boldsymbol{R}$ into $\boldsymbol{T}$:

$$\boldsymbol{R} \xrightarrow{\boldsymbol{B}} \boldsymbol{S} \xrightarrow{\boldsymbol{A}} \boldsymbol{T}, \quad \boldsymbol{R} \xrightarrow{\boldsymbol{C}} \boldsymbol{T}.$$

We introduce the matrices A, B, C corresponding to $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ in some choice of bases in $\boldsymbol{R}$, $\boldsymbol{S}$, and $\boldsymbol{T}$. Then the matrix equation $C = AB$ will correspond to the operator equation $\boldsymbol{C} = \boldsymbol{AB}$.

We denote by r_A, r_B, r_C the ranks of the operators $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ or, what is the same, of the matrices A, B, C. These numbers determine the dimensions of the subspaces $\boldsymbol{AS}$, $\boldsymbol{BR}$, $\boldsymbol{A}(\boldsymbol{BR})$. Since $\boldsymbol{BR} \subset \boldsymbol{S}$, we have $\boldsymbol{A}(\boldsymbol{BR}) \subset \boldsymbol{AS}$.[13] Moreover, the dimension of $\boldsymbol{A}(\boldsymbol{BR})$ cannot exceed the dimension of $\boldsymbol{BR}$.[14] Therefore

$$r_C \leqq r_A, \quad r_C \leqq r_B.$$

These inequalities were obtained in Chapter I, § 2 from the formula for the minors of a product of two matrices.

Let us regard $\boldsymbol{A}$ as an operator mapping $\boldsymbol{BR}$ into $\boldsymbol{T}$. Then the rank of this operator is equal to the dimension of the space $\boldsymbol{A}(\boldsymbol{BR})$, i.e., to r_C. Therefore, by applying (39) we obtain

$$r_C = r_B - d_1, \tag{40}$$

where d_1 is the maximal number of linearly independent vectors of $\boldsymbol{BR}$ that satisfy the equation

$$\boldsymbol{Ax} = \boldsymbol{o}. \tag{41}$$

But all the solutions of this equation that belong to $\boldsymbol{S}$ form a subspace of dimension d, where

$$d = n - r_A \tag{42}$$

is the defect of the operator $\boldsymbol{A}$ mapping $\boldsymbol{S}$ into $\boldsymbol{T}$. Since $\boldsymbol{BR} \subset \boldsymbol{S}$,

$$d_1 \leqq d. \tag{43}$$

From (40), (42), and (43) we find:

$$r_A + r_B - n \leqq r_C.$$

---

[12] In § 1 we reached these conclusions on the basis of other arguments (see p. 54).

[13] $\boldsymbol{R} \subset \boldsymbol{S}$ means that the set $\boldsymbol{R}$ forms part of the set $\boldsymbol{S}$.

[14] See Footnote 11.

Thus we have obtained *Sylvester's inequality* for the rank of the product of two rectangular matrices A and B of dimensions $m \times n$ and $n \times q$:

$$r_A + r_B - n \leqq r_{AB} \leqq \min(r_A, r_B). \tag{44}$$

§ 6. Linear Operators Mapping an n-Dimensional Space into Itself

1. A linear operator mapping the n-dimensional vector space $\boldsymbol{R}$ into itself (here $\boldsymbol{R} \equiv \boldsymbol{S}$ and $n = m$) will be referred to simply as a *linear operator in* $\boldsymbol{R}$.

The sum of two linear operators in $\boldsymbol{R}$ and the product of such an operator by a number are also linear operators in $\boldsymbol{R}$. Multiplication of two such operators is always feasible, and this product is also a linear operator in $\boldsymbol{R}$. Hence the linear operators in $\boldsymbol{R}$ form a ring.[15] This ring has an identity operator, namely the operator $\boldsymbol{E}$ for which

$$\boldsymbol{E}\boldsymbol{x} = \boldsymbol{x} \qquad (\boldsymbol{x} \in \boldsymbol{R}). \tag{45}$$

For every operator $\boldsymbol{A}$ in $\boldsymbol{R}$ we have

$$\boldsymbol{E}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{E} = \boldsymbol{A}.$$

If $\boldsymbol{A}$ is a linear operator in $\boldsymbol{R}$, then the powers $\boldsymbol{A}^2 = \boldsymbol{A}\boldsymbol{A}$, $\boldsymbol{A}^3 = \boldsymbol{A}\boldsymbol{A}\boldsymbol{A}$, and in general $\boldsymbol{A}^m = \underbrace{\boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}}_{m \text{ times}}$ have a meaning. We set $\boldsymbol{A}^0 = \boldsymbol{E}$. Then it is easy to see that for all non-negative integers p and q we have

$$\boldsymbol{A}^p\boldsymbol{A}^q = \boldsymbol{A}^{p+q}.$$

Let $f(t) = \alpha_0 t^m + \alpha_1 t^{m-1} + \cdots + \alpha_{m-1}t + \alpha_m$ be a polynomial in a scalar argument t with coefficients in the field F. Then we set:

$$f(\boldsymbol{A}) = \alpha_0\boldsymbol{A}^m + \alpha_1\boldsymbol{A}^{m-1} + \cdots + \alpha_{m-1}\boldsymbol{A} + \alpha_m\boldsymbol{E}. \tag{46}$$

Here $f(\boldsymbol{A})g(\boldsymbol{A}) = g(\boldsymbol{A})f(\boldsymbol{A})$ for any two polynomials $f(t)$ and $g(t)$.

Let

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} \qquad (\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}).$$

We denote by $x_1, x_2, \ldots, x_n$ the coordinates of the vector $\boldsymbol{x}$ in an arbitrary basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ and by $y_1, y_2, \ldots, y_n$ the coordinates of $\boldsymbol{y}$ in the same basis. Then

$$y_i = \sum_{k=1}^{n} a_{ik}x_k \qquad (i = 1, 2, \ldots, n). \tag{47}$$

---

[15] This ring is in fact an algebra. See Chapter I, p. 17.

In the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ the linear operator $\boldsymbol{A}$ corresponds to a square matrix $A = \| a_{ik} \|_1^n$.[16] We remind the reader (see § 2.) that in the k-th column of this matrix are to be found the coordinates of the vector $\boldsymbol{A}\boldsymbol{e}_k$ $(k = 1, 2, \ldots, n)$. Introducing the coordinate columns $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, we can write the transformation (47) in matrix form

$$y = Ax. \tag{48}$$

The sum and product of two operators $\boldsymbol{A}$ and $\boldsymbol{B}$ correspond to the sum and product of the corresponding square matrices $A = \| a_{ik} \|_1^n$ and $B = \| b_{ik} \|_1^n$. The product $\alpha\boldsymbol{A}$ corresponds to the matrix αA. The identity operator $\boldsymbol{E}$ corresponds to the square unit matrix $E = \| \delta_{ik} \|_1^n$. Thus, the choice of a basis establishes *an isomorphism between the ring of linear operators in* $\boldsymbol{R}$ *and the ring of square matrices of order* n *with elements in* F. In this isomorphism the polynomial $f(\boldsymbol{A})$ corresponds to the matrix $f(A)$.

Let us consider, apart from the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$, another basis $\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \ldots, \boldsymbol{e}_n^*$ of $\boldsymbol{R}$. Then, in analogy with (48), we have

$$y^* = A^*x^*, \tag{49}$$

where x^*, y^* are the column matrices formed from the coordinates of the vectors $\boldsymbol{x}, \boldsymbol{y}$ in the basis $\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \ldots, \boldsymbol{e}_n^*$ and $A^* = \| a_{ik}^* \|_1^n$ is the square matrix corresponding to the operator $\boldsymbol{A}$ in this basis. We rewrite in matrix form the formulas for the transformation of coordinates

$$x = Tx^*, \quad y = Ty^*. \tag{50}$$

Then from (48) and (50) we find:

$$y^* = T^{-1}ATx^*;$$

and a comparison with (49) gives:

$$A^* = T^{-1}AT. \tag{51}$$

Formula (51) is a special case of (31) on p. 61 (namely, $P = T^{-1}$ and $Q = T$).

DEFINITION 10: *Two matrices A and B connected by the relation*

$$B = T^{-1}AT, \tag{51'}$$

where T is a non-singular matrix, are called similar.[17]

---

[16] See § 2 of this chapter. In this case the spaces $\boldsymbol{R}$ and $\boldsymbol{S}$ coincide; in the same way, the bases $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ and $\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_m$ of these spaces are identified.

Thus, we have shown that *two matrices corresponding to one and the same linear operator in* $\boldsymbol{R}$ *for distinct bases are similar* and the matrix T linking these matrices coincides with the matrix of the coordinate transformation in the transition from the first basis to the second (see (50)).

In other words, to a linear operator in $\boldsymbol{R}$ there corresponds a whole class of similar matrices; they represent the given operator in various bases.

In studying properties of a linear operator in $\boldsymbol{R}$, we are at the same time studying the matrix properties that are common to the whole class of similar matrices, that is, that remain unchanged, or invariant, under transition from a given matrix to a similar one.

We note at once that two similar matrices always have the same determinant. For it follows from (51′) that

$$|B| = |T|^{-1}|A||T| = |A|. \tag{52}$$

The equation $|B| = |A|$ is a necessary, but not a sufficient condition for the similarity of the matrices A and B.

In Chapter VI we shall establish a criterion for the similarity of two matrices, i.e., we shall give necessary and sufficient conditions for two square matrices of order n to be similar.

In accordance with (52) we may define the determinant $|A|$ of a linear operator $\boldsymbol{A}$ in $\boldsymbol{R}$ as the determinant of an arbitrary matrix corresponding to the given operator.

If $|\boldsymbol{A}| = 0$ $(\neq 0)$, then the operator $\boldsymbol{A}$ is called *singular* (*non-singular*). In accordance with this definition a singular (non-singular) operator corresponds to a singular (non-singular) matrix in any basis. For a singular operator:

1) There always exists a vector $\boldsymbol{x} \neq \boldsymbol{o}$ such that $\boldsymbol{Ax} = \boldsymbol{o}$;
2) $\boldsymbol{AR}$ is a proper part of $\boldsymbol{R}$.

For a non-singular operator:

1) $\boldsymbol{Ax} = \boldsymbol{o}$ implies that $\boldsymbol{x} = \boldsymbol{o}$;
2) $\boldsymbol{AR} \equiv \boldsymbol{R}$, i.e., the vectors of the form $\boldsymbol{Ax}$ $(\boldsymbol{x} \in \boldsymbol{R})$ fill out the whole space $\boldsymbol{R}$.

In other words, a linear operator in $\boldsymbol{R}$ is singular or non-singular depending on whether its defect is positive or zero.

---

[17] The matrix T can always be chosen such that its elements belong to the same basic number field F as those of A and B. It is easy to verify the three properties of similar matrices:

Reflexivity (a matrix A is always similar to itself);
Symmetry (if A is similar to B, then B is similar to A); and
Transitivity (if A is similar to B, and B to C, then A is similar to C).

§ 7. Characteristic Values and Characteristic Vectors of a Linear Operator

1. An important role in the study of the structure of a linear operator $\boldsymbol{A}$ in $\boldsymbol{R}$ is played by the vectors $\boldsymbol{x}$ for which

$$\boldsymbol{Ax} = \lambda\boldsymbol{x} \qquad (\lambda \in \mathrm{F}, \quad \boldsymbol{x} \neq \boldsymbol{o}) \tag{53}$$

Such vectors are called *characteristic vectors* and the numbers λ corresponding to them are called *characteristic values* or *characteristic roots* of the operator $\boldsymbol{A}$ (or of the matrix A).†

In order to find the characteristic values and characteristic vectors of an operator $\boldsymbol{A}$ we choose an arbitrary basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ in $\boldsymbol{R}$. Let $\boldsymbol{x} = \sum_{i=1}^{n} x_i \boldsymbol{e}_i$ and let $A = \| a_{ik} \|_1^n$ be the matrix corresponding to $\boldsymbol{A}$ in the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Then if we equate the corresponding coordinates of the vectors on the left-hand and right-hand sides of (53), we obtain a system of scalar equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \lambda x_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \lambda x_n, \end{array}\right\} \tag{54}$$

which can also be written as

$$\left.\begin{array}{l} (a_{11} - \lambda)\, x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22} - \lambda)\, x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)\, x_n = 0 \end{array}\right\} \tag{55}$$

Since the required vector must not be the null vector, at least one of its coordinates $x_1, x_2, \ldots, x_n$ must be different from zero.

In order that the system of linear homogeneous equations (55) should have a non-zero solution it is necessary and sufficient that the determinant of the system be zero:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix} = 0. \tag{56}$$

---

† Other terms in use for the former are: *proper vector, latent vector, eigenvector.* Other terms for the latter are: *proper value, latent value, latent root, latent number, characteristic number, eigenvalue,* etc.

The equation (56) is an algebraic equation of degree n in λ. Its coefficients belong to the same number field F as the elements of the matrix $A = \| a_{ik} \|_1^n$.

Equation (56) occurs in various problems of geometry, mechanics, astronomy, and physics and is known as the *characteristic equation* or the *secular equation*[18] of the matrix $A = \| a_{ik} \|_1^n$ (the left-hand side is called the *characteristic polynomial*).

Thus, every characteristic value λ of a linear operator $\boldsymbol{A}$ is a root of the characteristic equation (56). And conversely, if a number λ is a root of (56), then for this value λ the system (55) and hence (54) has a non-zero solution $x_1, x_2, \ldots, x_n$, i.e., to this number λ there corresponds a characteristic vector $\boldsymbol{x} = \sum x_i \boldsymbol{e}_i$ of the operator $\boldsymbol{A}$.

From what we have shown, it follows that every linear operator $\boldsymbol{A}$ in $\boldsymbol{R}$ has not more than n distinct characteristic values.

If F is the field of complex numbers, then every linear operator in $\boldsymbol{R}$ always has at least one characteristic vector in $\boldsymbol{R}$ corresponding to a characteristic value λ.[19] This follows from the fundamental theorem of algebra, according to which an algebraic equation (56) in the field of complex numbers always has at least one root.

Let us write (56) in explicit form

$$| A - \lambda E | \equiv (-\lambda)^n + S_1 (-\lambda)^{n-1} + S_2 (-\lambda)^{n-2} + \cdots + S_{n-1} (-\lambda) + S_n = 0. \quad (57)$$

It is easy to see that here

$$S_1 = \sum_{i=1}^{n} a_{ii}, \quad S_2 = \sum_{1 \leq i < k \leq n} A \begin{pmatrix} i & k \\ i & k \end{pmatrix}, \quad \ldots \quad (58)$$

and, in general, S_p is the sum of the principal minors of order p of the matrix $A = \| a_{ik} \|_1^n$ $(p = 1, 2, \ldots, n)$.[20] In particular, $S_n = | A |$.

We denote by $\tilde{A}$ the matrix corresponding to the same operator $\boldsymbol{A}$ in another basis. $\tilde{A}$ is similar to A:

---

[18] The name is due to the fact that this equation occurs in the study of secular perturbations of the planets.

[19] This proposition is valid even in the more general case in which F is an arbitrary algebraically closed field, i.e., a field that contains the roots of all algebraic equations with coefficients in the field.

[20] The power $(-\lambda)^{n-p}$ occurs only in those terms of the characteristic determinant (56) that contain precisely $n - p$ of the diagonal elements, say,

$$a_{j_1 j_1} - \lambda, \; a_{j_2 j_2} - \lambda, \; \ldots, \; a_{j_{n-p} j_{n-p}} - \lambda.$$

The product of these diagonal elements occurs in the expansion of the determinant (56)

$$\tilde{A} = T^{-1} A T.$$

Hence

$$\tilde{A} - \lambda E = T^{-1} (A - \lambda E)\, T$$

and therefore

$$| \tilde{A} - \lambda E | = | A - \lambda E |. \quad (59)$$

Thus, similar matrices A and $\tilde{A}$ have the same characteristic polynomial. This polynomial is sometimes called the characteristic polynomial of the operator $\boldsymbol{A}$ and is denoted by $| \boldsymbol{A} - \lambda \boldsymbol{E} |$

If $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots$ are linearly independent characteristic vectors of an operator $\boldsymbol{A}$ corresponding to one and the same characteristic λ, and $\alpha, \beta, \gamma, \ldots$ are arbitrary numbers of F*, then the vector $\alpha \boldsymbol{x} + \beta \boldsymbol{y} + \gamma \boldsymbol{z} + \cdots$ is either equal to zero or is also a characteristic vector of $\boldsymbol{A}$ corresponding to the same λ.*

For from

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}, \; \boldsymbol{A}\boldsymbol{y} = \lambda \boldsymbol{y}, \; \boldsymbol{A}\boldsymbol{z} = \lambda \boldsymbol{z}, \; \ldots$$

it follows that

$$\boldsymbol{A} (\alpha \boldsymbol{x} + \beta \boldsymbol{y} + \gamma \boldsymbol{z} + \cdots) = \lambda (\alpha \boldsymbol{x} + \beta \boldsymbol{y} + \gamma \boldsymbol{z} + \cdots).$$

In other words, linearly independent characteristic vectors corresponding to one and the same characteristic value λ form a basis of a 'characteristic' subspace each vector of which is a characteristic vector for the same λ. In particular, each characteristic vector generates a one-dimensional subspace, a 'characteristic' direction.

However, if characteristic vectors of a linear operator $\boldsymbol{A}$ correspond to distinct characteristic values, then a linear combination of these characteristic vectors is not, in general, a characteristic vector of $\boldsymbol{A}$.

The significance of the characteristic vectors and characteristic numbers for the study of linear operators will be illustrated in the next section by the example of operators of simple structure.

---

with a factor in which the term free of λ is the principal minor

$$A \begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ i_1 & i_2 & \ldots & i_p \end{pmatrix},$$

where $i_1, i_2, \ldots, i_p$ together with $j_1, j_2, \ldots, j_{n-p}$ forms a complete set of indices $1, 2, \ldots, n$; hence in the development of (56) we have

$$| A - \lambda E | = (a_{j_1 j_1} - \lambda)\, (a_{j_2 j_2} - \lambda) \cdots (a_{j_{n-p} j_{n-p}} - \lambda)\, A \begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ i_1 & i_2 & \ldots & i_p \end{pmatrix} + (^*)$$

When we take all possible combinations $j_1, j_2, \ldots, j_{n-p}$ of $n - p$ of the indices $1, 2, \ldots, n$, we obtain for the coefficient S_p of $(-\lambda)^{n-p}$ the sum of all principal minors of order p in A.

§ 8. Linear Operators of Simple Structure

1. We begin with the following lemma.

LEMMA: *Characteristic vectors belonging to pairwise distinct characteristic values are always linearly independent.*

Proof. Let

$$\boldsymbol{A}\boldsymbol{x}_i=\lambda_i\boldsymbol{x}_i \quad (\boldsymbol{x}_i\neq\boldsymbol{o};\ \lambda_i\neq\lambda_k \text{ for } i\neq k;\ i,k=1,2,\ldots,m) \tag{60}$$

and

$$\sum_{i=1}^{m}c_i\boldsymbol{x}_i=\boldsymbol{o}. \tag{61}$$

Applying the operator $\boldsymbol{A}$ to both sides we obtain:

$$\sum_{i=1}^{m}c_i\lambda_i\boldsymbol{x}_i=\boldsymbol{o}. \tag{62}$$

We multiply both sides of (61) by λ_1 and subtract (61) from (62) term by term. Then we obtain

$$\sum_{i=2}^{m}c_i(\lambda_i-\lambda_1)\,\boldsymbol{x}_i=\boldsymbol{o}. \tag{63}$$

We can say that (63) is obtained from (61) by termwise application of the operator $\boldsymbol{A}-\lambda_1\boldsymbol{E}$. If we apply the operators $\boldsymbol{A}-\lambda_2\boldsymbol{E},\ldots,\boldsymbol{A}-\lambda_{m-1}\boldsymbol{E}$ to (63) term by term, we are led to the following equation:

$$c_m(\lambda_m-\lambda_{m-1})\,(\lambda_m-\lambda_{m-2})\cdots(\lambda_m-\lambda_1)\,\boldsymbol{x}_m=\boldsymbol{o},$$

so that $c_m=0$. Since any of the summands in (61) can be put last, we have in (61)

$$c_1=c_2=\ldots=c_m=0,$$

i.e., there is no linear dependence among the vectors $\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_m$. This proves the lemma.

If the characteristic equation of an operator has n distinct roots and these roots belong to F, then by the lemma the characteristic vectors belonging to these roots are linearly independent.

DEFINITION 11: *A linear operator $\boldsymbol{A}$ in $\boldsymbol{R}$ is said to be an operator of simple structure if $\boldsymbol{A}$ has n linearly independent characteristic vectors in $\boldsymbol{R}$, where n is the dimension of $\boldsymbol{R}$.*

Thus, a linear operator in $\boldsymbol{R}$ has simple structure if all the roots of the characteristic equation are distinct and belong to F. However, these conditions are not necessary. There exist linear operators of simple structure whose characteristic polynomial has multiple roots.

Let us consider an arbitrary linear operator $\boldsymbol{A}$ of simple structure. We denote by $\boldsymbol{g}_1,\boldsymbol{g}_2,\ldots,\boldsymbol{g}_n$ a basis of $\boldsymbol{R}$ consisting of characteristic vectors of the operator, i.e.,

$$\boldsymbol{A}\boldsymbol{g}_k=\lambda_k\boldsymbol{g}_k \quad (k=1,2,\ldots,n).$$

If

$$\boldsymbol{x}=\sum_{k=1}^{n}x_k\boldsymbol{g}_k,$$

then

$$\boldsymbol{A}\boldsymbol{x}=\sum_{k=1}^{n}x_k\boldsymbol{A}\boldsymbol{g}_k=\sum_{k=1}^{n}\lambda_kx_k\boldsymbol{g}_k.$$

The effect of the operator $\boldsymbol{A}$ of simple structure on the vector $\boldsymbol{x}=\sum_{k=1}^{n}x_k\boldsymbol{g}_k$ may be put into words as follows:

In the n-dimensional space $\boldsymbol{R}$ there exist n linearly independent 'directions' along which the operator $\boldsymbol{A}$ of simple structure realizes a 'dilatation' with coefficients $\lambda_1,\lambda_2,\ldots,\lambda_n$. An arbitrary vector $\boldsymbol{x}$ may be decomposed into components along these characteristic directions. These components are subject to the corresponding 'dilatations' and their sum then gives the vector $\boldsymbol{A}\boldsymbol{x}$.

It is easy to see that to the operator $\boldsymbol{A}$ in a 'characteristic' basis $\boldsymbol{g}_1,\boldsymbol{g}_2,\ldots,\boldsymbol{g}_n$ there corresponds the diagonal matrix

$$\tilde{A}=\|\lambda_i\delta_{ik}\|_1^n.$$

If we denote by A the matrix corresponding to $\boldsymbol{A}$ in an arbitrary basis $\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_n$, then

$$A=T\,\|\lambda_i\delta_{ik}\|_1^n\,T^{-1}. \tag{64}$$

A matrix that is similar (p. 68) to a diagonal matrix is called a *matrix of simple structure.* Thus, to an operator of simple structure there corresponds in any basis a matrix of simple structure, and vice versa.

2. The matrix T in (64) realizes the transition from the basis $\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_n$ to the basis $\boldsymbol{g}_1,\boldsymbol{g}_2,\ldots,\boldsymbol{g}_n$. The k-th column of T contains the coordinates of a characteristic vector $\boldsymbol{g}_k$ (with respect to $\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_n$) that corresponds to the characteristic value λ_k of A $(k=1,2,\ldots,n)$. The matrix T is called the *fundamental* matrix for A.

We rewrite (64) as follows:

$$A = TLT^{-1} \quad (L = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}). \tag{64'}$$

On going over to the p-th compound matrices $(1 \leqq p \leqq n)$, we obtain (see Chapter I, § 4):

$$\mathfrak{A}_p = \mathfrak{T}_p \mathfrak{L}_p \mathfrak{T}_p^{-1}. \tag{65}$$

$\mathfrak{L}_p$ is a diagonal matrix of order N $(N = \binom{n}{p})$ along whose main diagonal are all the possible products of $\lambda_1, \lambda_2, \ldots, \lambda_n$ taken p at a time. A comparison of (65) with (64′) yields the following theorem:

THEOREM 3: *If a matrix $A = \| a_{ik} \|_1^n$ has simple structure, then for every $p \leqq n$ the compound matrix $\mathfrak{A}_p$ also has simple structure; moreover, the characteristic values of $\mathfrak{A}_p$ are all the possible products $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_p}$ $(1 \leqq i_1 < i_2 < \ldots < i_p \leqq n)$ of p of the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, and the fundamental matrix of $\mathfrak{A}_p$ is the compound $\mathfrak{T}_p$ of the fundamental matrix T of A.*

COROLLARY: *If a characteristic value λ_k of a matrix of simple structure $A = \| a_{ik} \|_1^n$ corresponds to a characteristic vector with the coordinates $t_{1k}, t_{2k}, \ldots, t_{nk}$ $(k = 1, 2, \ldots, n)$ and if $T = \| t_{ik} \|_1^n$, then the characteristic value $\lambda_{k_1} \lambda_{k_2} \cdots \lambda_{k_p}$ $(1 \leqq k_1 < k_2 < \ldots < k_p \leqq n)$ of $\mathfrak{A}_p$ corresponds to the characteristic vector with coordinates*

$$T\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ k_1 & k_2 & \ldots & k_p \end{pmatrix} \quad (1 \leqq i_1 < i_2 < \cdots < i_p \leqq n). \tag{66}$$

An arbitrary matrix $A = \| a_{ik} \|_1^n$ may be represented in the form of a sequence of matrices A_m $(m \to \infty)$ each of which does not have multiple characteristic values and, therefore, has simple structure. The characteristic values $\lambda_1^{(m)}, \lambda_2^{(m)}, \ldots, \lambda_n^{(m)}$ of the matrix A_m converge for $m \to \infty$ to the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A,

$$\lim_{m \to \infty} \lambda_k^{(m)} = \lambda_k \qquad (k = 1, 2, \ldots, n).$$

Hence

$$\lim_{m \to \infty} \lambda_{k_1}^{(m)} \lambda_{k_2}^{(m)} \cdots \lambda_{k_p}^{(m)} = \lambda_{k_1} \lambda_{k_2} \cdots \lambda_{k_p} \quad (1 \leqq k_1 < k_2 < \cdots < k_p \leqq n).$$

Moreover, since $\lim_{m \to \infty} \mathfrak{A}_{(m)p} = \mathfrak{A}_p$, we deduce from Theorem 3:

THEOREM 4 (Kronecker): *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ is a complete system of characteristic values of an arbitrary matrix A, then a complete system of characteristic values of the compound matrix $\mathfrak{A}_p$ consists of all possible products of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ taken p at a time $(p = 1, 2, \ldots, n)$.*

In the present section we have investigated operators and matrices of simple structure. The study of the structure of operators and matrices of general type will be resumed in Chapters VI and VII.

CHAPTER IV

THE CHARACTERISTIC POLYNOMIAL AND THE MINIMAL POLYNOMIAL OF A MATRIX

Two polynomials are associated with every square matrix: the characteristic polynomial and the minimal polynomial. These polynomials play an important role in various problems of the theory of matrices. For example, the concept of a function of a matrix, which we shall introduce in the next chapter, will be based entirely on the concept of the minimal polynomial. In the present chapter, the properties of the characteristic polynomial and the minimal polynomial are studied. A prerequisite to this investigation is some basic information about polynomials with matrix coefficients and operations on them.

§ 1. Addition and Multiplication of Matrix Polynomials

1. We consider a square *polynomial* matrix $A(\lambda)$, i.e., a square matrix whose elements are polynomials in λ (with coefficients in the given number field F):

$$A(\lambda) = \| a_{ik}(\lambda) \|_1^n = \| a_{ik}^{(0)} \lambda^m + a_{ik}^{(1)} \lambda^{m-1} + \cdots + a_{ik}^{(m)} \|_1^n . \tag{1}$$

The matrix $A(\lambda)$ can be represented in the form of a polynomial with matrix coefficients arranged with respect to the powers of λ:

$$A(\lambda) = A_0 \lambda^m + A_1 \lambda^{m-1} + \cdots + A_m , \tag{2}$$

where

$$A_j = \| a_{ik}^{(j)} \|_1^n \qquad (j = 0,\ 1,\ \ldots,\ m). \tag{3}$$

The number m is called the *degree* of the polynomial, provided $A_0 \neq O$. The number n is called the *order* of the polynomial. The polynomial (1) is called *regular* if $| A_0 | \neq 0$.

A polynomial with matrix coefficients will sometimes be called a *matrix polynomial*. In contrast to a matrix polynomial an ordinary polynomial with scalar coefficients will be called a *scalar polynomial*.

We shall now consider the fundamental operations on matrix polynomials. Let two matrix polynomials $A(\lambda)$ and $B(\lambda)$ of the same order be given. We denote by m the larger of their degrees. These polynomials can be written in the form

$$\begin{aligned} A(\lambda) &= A_0 \lambda^m + A_1 \lambda^{m-1} + \cdots + A_m , \\ B(\lambda) &= B_0 \lambda^m + B_1 \lambda^{m-1} + \cdots + B_m . \end{aligned}$$

Then

$$A(\lambda) \pm B(\lambda) = (A_0 \pm B_0)\lambda^m + (A_1 \pm B_1)\lambda^{m-1} + \cdots + (A_m \pm B_m),$$

i.e.: *The sum (difference) of two matrix polynomials of the same order can be represented in the form of a polynomial whose degree does not exceed the larger of the degrees of the given polynomials.*

Let $A(\lambda)$ and $B(\lambda)$ be two matrix polynomials of the same order n and of respective degrees m and p:

$$\begin{aligned} A(\lambda) &= A_0 \lambda^m + A_1 \lambda^{m-1} + \cdots + A_m \qquad (A_0 \neq O), \\ B(\lambda) &= B_0 \lambda^p + B_1 \lambda^{p-1} + \cdots + B_p \qquad (B_0 \neq O). \end{aligned}$$

Then

$$A(\lambda)\, B(\lambda) = A_0 B_0 \lambda^{m+p} + (A_0 B_1 + A_1 B_0)\lambda^{m+p-1} + \cdots + A_m B_p . \tag{4}$$

If we multiply $B(\lambda)$ by $A(\lambda)$ (i.e., interchange the order of the factors), then we obtain, in general, a different polynomial.

2. The multiplication of matrix polynomials has a specific property. In contrast to the product of scalar polynomials, the product (4) of matrix polynomials may have a degree less than $m + p$, i.e., less than the sum of the degrees of the factors. For, in (4) the product $A_0 B_0$ may be the null matrix even though $A_0 \neq O$, $B_0 \neq O$. However, if at least one of the matrices A_0 and B_0 is non-singular, then it follows from $A_0 \neq O$ and $B_0 \neq O$ that $A_0 B_0 \neq O$. Thus: *The product of two matrix polynomials is a polynomial whose degree is less than or equal to the sum of the degrees of the factors. If at least one of the two factors is regular, then the degree of the product is always equal to the sum of the degrees of the factors.*

§ 2. Right and Left Division of Matrix Polynomials

1. Let $A(\lambda)$ and $B(\lambda)$ be two matrix polynomials of the same order n, and let $B(\lambda)$ be regular:

$$\begin{aligned} A(\lambda) &= A_0 \lambda^m + A_1 \lambda^{m-1} + \cdots + A_m \qquad (A_0 \neq O), \\ B(\lambda) &= B_0 \lambda^p + B_1 \lambda^{p-1} + \cdots + B_p \qquad (| B_0 | \neq 0). \end{aligned}$$

We shall say that the matrix polynomials $Q(\lambda)$ and $R(\lambda)$ are the *right quotient* and the *right remainder*, respectively, of $A(\lambda)$ on division by $B(\lambda)$ if

$$A(\lambda) = Q(\lambda)\,B(\lambda) + R(\lambda) \tag{5}$$

and if the degree of $R(\lambda)$ is less than that of $B(\lambda)$.

Similarly, we shall call the polynomials $\widehat{Q}(\lambda)$ and $\widehat{R}(\lambda)$ the *left quotient* and the *left remainder* of $A(\lambda)$ on division by $B(\lambda)$ if

$$A(\lambda) = B(\lambda)\,\widehat{Q}(\lambda) + \widehat{R}(\lambda) \tag{6}$$

and if the degree of $\widehat{R}(\lambda)$ is less than that of $B(\lambda)$.

The reader should note that in the *'right'* division (i.e., when the right quotient and the right remainder are to be found) in (5) the quotient $Q(\lambda)$ is multiplied by the 'divisor' $B(\lambda)$ on the *right*, and in the *'left'* division in (6) the quotient $\widehat{Q}(\lambda)$ is multiplied by the divisor $B(\lambda)$ on the *left*. The polynomials $\widehat{Q}(\lambda)$ and $\widehat{R}(\lambda)$ do not, in general, coincide with $Q(\lambda)$ and $R(\lambda)$.

2. We shall now show that *both right and left division of matrix polynomials of the same order are always possible and unique, provided the divisor is a regular polynomial.*

Let us consider the right division of $A(\lambda)$ by $B(\lambda)$. If $m < p$, we can set $Q(\lambda) = O$, $R(\lambda) = A(\lambda)$. If $m \geqq p$, we apply the usual scheme for the division of a polynomial by a polynomial in order to find the quotient $Q(\lambda)$ and the remainder $R(\lambda)$. We 'divide' the highest term of the dividend $A_0\lambda^m$ by the highest term of the divisor $B_0\lambda^p$. We obtain the highest term $A_0B_0^{-1}\lambda^{m-p}$ of the required quotient. We multiply this term on the right by the divisor $B(\lambda)$ and subtract the product so obtained from $A(\lambda)$. Thus we find the 'first remainder' $A^{(1)}(\lambda)$:

$$A(\lambda) = A_0B_0^{-1}\lambda^{m-p}\,B(\lambda) + A^{(1)}(\lambda). \tag{7}$$

The degree $m^{(1)}$ of $A^{(1)}(\lambda)$ is less than m:

$$A^{(1)}(\lambda) = A_0^{(1)}\lambda^{m^{(1)}} + \cdots \qquad (A_0^{(1)} \neq O,\ m^{(1)} < m). \tag{8}$$

If $m^{(1)} \geqq p$, then we repeat the process and obtain:

$$\left.\begin{aligned} A^{(1)}(\lambda) &= A_0^{(1)}B_0^{-1}\lambda^{m^{(1)}-p}B(\lambda) + A^{(2)}(\lambda), \\ A^{(2)}(\lambda) &= A_0^{(2)}\lambda^{m^{(2)}} + \cdots \qquad (m^{(2)} < m^{(1)}), \end{aligned}\right\} \tag{9}$$

etc.

Since the degrees of $A(\lambda)$, $A^{(1)}(\lambda)$, $A^{(2)}(\lambda)$, ... decrease, at some stage we arrive at a remainder $R(\lambda)$ whose degree is less than p. Then it follows from (7) and (9) that

$$A(\lambda) = Q(\lambda)B(\lambda) + R(\lambda),$$

where

$$Q(\lambda) = A_0B_0^{-1}\lambda^{m-p} + A_0^{(1)}B_0^{-1}\lambda^{m^{(1)}-p} + \cdots. \tag{10}$$

We shall now prove the *uniqueness* of the right division. Suppose we have simultaneously

$$A(\lambda) = Q(\lambda)\,B(\lambda) + R(\lambda) \tag{11}$$

and

$$A(\lambda) = Q^*(\lambda)\,B(\lambda) + R^*(\lambda), \tag{12}$$

where the degrees of $R(\lambda)$ and $R^*(\lambda)$ are less than that of $B(\lambda)$, i.e., less than p. Subtracting (11) from (12) term by term we obtain

$$[Q(\lambda) - Q^*(\lambda)]\,B(\lambda) = R^*(\lambda) - R(\lambda). \tag{13}$$

If we had $Q(\lambda) - Q^*(\lambda) \not\equiv O$, then the degree on the left-hand side of (13) would be the sum of the degrees of $B(\lambda)$ and $Q(\lambda) - Q^*(\lambda)$, because $|B_0| \neq 0$, and would therefore be at least equal to p. This is impossible, since the degree of the polynomial on the right-hand side of (13) is less than p. Thus, $Q(\lambda) - Q^*(\lambda) \equiv O$, and then it follows from (13) that $R(\lambda) - R^*(\lambda) \equiv O$, i.e.,

$$Q(\lambda) = Q^*(\lambda), \qquad R(\lambda) = R^*(\lambda).$$

The existence and uniqueness of the left quotient and left remainder is established similarly.[1]

---

[1] Note that the possibility and uniqueness of the left division of $A(\lambda)$ by $B(\lambda)$ follows from that of the right division of the transposed matrices $A^{\mathsf{T}}(\lambda)$ and $B^{\mathsf{T}}(\lambda)$. (The regularity of $B(\lambda)$ implies that of $B^{\mathsf{T}}(\lambda)$.) For from

$$A^{\mathsf{T}}(\lambda) = Q_1(\lambda)\,B^{\mathsf{T}}(\lambda) + R_1(\lambda)$$

it follows (see Chapter I, p. 19) that

$$A(\lambda) = B(\lambda)\,Q_1^{\mathsf{T}}(\lambda) + R_1^{\mathsf{T}}(\lambda). \tag{6'}$$

By the same reasoning, the left division of $A(\lambda)$ by $B(\lambda)$ is unique; for if it were not, then the right division of $A^{\mathsf{T}}(\lambda)$ by $B^{\mathsf{T}}(\lambda)$ would not be unique.

Comparison of (6) and (6′) gives

$$\widehat{Q}(\lambda) = Q_1^{\mathsf{T}}(\lambda), \quad \widehat{R}(\lambda) = R_1^{\mathsf{T}}(\lambda).$$

Example.

$$A(\lambda)=\left\|\begin{matrix}\lambda^3+\lambda & 2\lambda^3+\lambda^2\\ -\lambda^3-2\lambda^2+1 & 3\lambda^2+\lambda\end{matrix}\right\|$$

$$=\overbrace{\left\|\begin{matrix}1 & 2\\ -1 & 3\end{matrix}\right\|}^{A_0}\lambda^3+\left\|\begin{matrix}0 & 1\\ -2 & 0\end{matrix}\right\|\lambda^2+\left\|\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\|\lambda+\left\|\begin{matrix}0 & 0\\ 1 & 0\end{matrix}\right\|,$$

$$B(\lambda)=\left\|\begin{matrix}2\lambda^2+3 & -\lambda^2+1\\ -\lambda^2-1 & \lambda^2+2\end{matrix}\right\|=\overbrace{\left\|\begin{matrix}2 & -1\\ -1 & 1\end{matrix}\right\|}^{B_0}\lambda^2+\left\|\begin{matrix}3 & 1\\ -1 & 2\end{matrix}\right\|,$$

$$|B_0|=1,\ B_0^{-1}=\left\|\begin{matrix}1 & 1\\ 1 & 2\end{matrix}\right\|,\ A_0B_0^{-1}=\left\|\begin{matrix}3 & 5\\ 2 & 5\end{matrix}\right\|,\ A_0B_0^{-1}B(\lambda)=\left\|\begin{matrix}\lambda^2+4 & 2\lambda^2+13\\ -\lambda^2+1 & 3\lambda^2+12\end{matrix}\right\|,$$

$$A^{(1)}(\lambda)=\left\|\begin{matrix}\lambda^3+\lambda & 2\lambda^3+\lambda^2\\ -\lambda^3-2\lambda^2+1 & 3\lambda^3+\lambda\end{matrix}\right\|-\left\|\begin{matrix}\lambda^3+4\lambda & 2\lambda^3+13\lambda\\ -\lambda^3+\lambda & 3\lambda^3+12\lambda\end{matrix}\right\|$$

$$=\left\|\begin{matrix}-3\lambda & \lambda^2-13\lambda\\ -2\lambda^2-\lambda+1 & -11\lambda\end{matrix}\right\|,$$

$$A^{(1)}(\lambda)=\left\|\begin{matrix}0 & 1\\ -2 & 0\end{matrix}\right\|\lambda^2+\left\|\begin{matrix}-3 & -13\\ -1 & -11\end{matrix}\right\|\lambda+\left\|\begin{matrix}0 & 0\\ 1 & 0\end{matrix}\right\|,$$

$$A_0^{(1)}B_0^{-1}=\left\|\begin{matrix}0 & 1\\ -2 & 0\end{matrix}\right\|\cdot\left\|\begin{matrix}1 & 1\\ 1 & 2\end{matrix}\right\|=\left\|\begin{matrix}1 & 2\\ -2 & -2\end{matrix}\right\|,$$

$$A_0^{(1)}B_0^{-1}B(\lambda)=\left\|\begin{matrix}1 & 2\\ -2 & -2\end{matrix}\right\|\cdot\left\|\begin{matrix}2\lambda^2+3 & -\lambda^2+1\\ -\lambda^2-1 & \lambda^2+2\end{matrix}\right\|=\left\|\begin{matrix}1 & \lambda^2+5\\ -2\lambda^2-4 & -6\end{matrix}\right\|,$$

$$R(\lambda)=A^{(1)}(\lambda)-A_0^{(1)}B_0^{-1}B(\lambda)$$

$$=\left\|\begin{matrix}-3\lambda & \lambda^2-13\lambda\\ -2\lambda^2-\lambda+1 & -11\lambda\end{matrix}\right\|-\left\|\begin{matrix}1 & \lambda^2+5\\ -2\lambda^2-4 & -6\end{matrix}\right\|=\left\|\begin{matrix}-3\lambda-1 & -13\lambda-5\\ -\lambda+5 & -11\lambda+6\end{matrix}\right\|,$$

$$Q(\lambda)=A_0B_0^{-1}\lambda+A_0^{(1)}B_0^{-1}=\left\|\begin{matrix}3 & 5\\ 2 & 5\end{matrix}\right\|\lambda+\left\|\begin{matrix}1 & 2\\ -2 & -2\end{matrix}\right\|=\left\|\begin{matrix}3\lambda+1 & 5\lambda+2\\ 2\lambda-2 & 5\lambda-2\end{matrix}\right\|.$$

As an exercise, the reader should verify that

$$A(\lambda)=Q(\lambda)B(\lambda)+R(\lambda).$$

§ 3. The Generalized Bézout Theorem

1. We consider an arbitrary matrix polynomial of order n

$$F(\lambda)=F_0\lambda^m+F_1\lambda^{m-1}+\cdots+F_m \qquad (F_0\neq O). \tag{14}$$

This polynomial can also be written as follows:

$$F(\lambda)=\lambda^mF_0+\lambda^{m-1}F_1+\cdots+F_m. \tag{15}$$

For a scalar λ, both ways of writing give the same result. However, if we substitute for the scalar argument λ a square matrix A of order n, then the results of the substitution in (14) and (15) will, in general, be distinct, since the powers of A need not be permutable with the matrix coefficients $F_0, F_1, \ldots, F_m$.

We set

$$F(A)=F_0A^m+F_1A^{m-1}+\cdots+F_m \tag{16}$$

and

$$\widehat{F}(A)=A^mF_0+A^{m-1}F_1+\cdots+F_m, \tag{17}$$

and call $F(A)$ the *right value* and $\widehat{F}(A)$ the *left value* of $F(\lambda)$ on substitution of A for λ.[2]

We divide $F(\lambda)$ by the binomial $\lambda E-A$. In this case the right remainder $R(\lambda)$ and left remainder $\widehat{R}(\lambda)$ will not depend on λ. To determine the right remainder we use the usual division scheme:

$$\begin{aligned}F(\lambda)&=F_0\lambda^m+F_1\lambda^{m-1}+\cdots+F_m\\ &=F_0\lambda^{m-1}(\lambda E-A)+(F_0A+F_1)\lambda^{m-1}+F_2\lambda^{m-2}+\cdots\\ &=[F_0\lambda^{m-1}+(F_0A+F_1)\lambda^{m-2}](\lambda E-A)+(F_0A^2+F_1A+F_2)\lambda^{m-2}+F_3\lambda^{m-3}+\cdots\\ &=[F_0\lambda^{m-1}+(F_0A+F_1)\lambda^{m-2}+\cdots\\ &\qquad+F_0A^{m-1}+F_1A^{m-2}+\cdots+F_{m-1}](\lambda E-A)\\ &\qquad+F_0A^m+F_1A^{m-1}+\cdots+F_m.\end{aligned}$$

Thus we have found that

$$R=F_0A^m+F_1A^{m-1}+\cdots+F_m=F(A). \tag{18}$$

Similarly

$$\widehat{R}=\widehat{F}(A). \tag{19}$$

This proves

THEOREM 1 (The Generalized Bézout Theorem): *When the matrix polynomial $F(\lambda)$ is divided on the right by the binomial $\lambda E-A$, the remainder is $F(A)$; when it is divided on the left, the remainder is $\widehat{F}(A)$.*

---

[2] In the 'right' value $F(A)$ the powers of A are at the right of the coefficients; in the 'left' value $\widehat{F}(A)$, at the left.

2. From this theorem it follows that:

A polynomial $F(\lambda)$ is divisible by the binomial $\lambda E - A$ on the right (left) without remainder if and only if $F(A) = O$ $(\widehat{F}(A) = O)$.

Example. Let $A = \| a_{ik} \|_1^n$ and let $f(\lambda)$ be a polynomial in λ. Then

$$F(\lambda) = f(\lambda)E - f(A)$$

is divisible by $\lambda E - A$ (both on the right and on the left) without remainder. This follows immediately from the generalized Bézout Theorem, because in this case $F(A) = \widehat{F}(A) = O$.

§ 4. The Characteristic Polynomial of a Matrix. The Adjoint Matrix

1. We consider a matrix $A = \| a_{ik} \|_1^n$. The *characteristic matrix* of A is $\lambda E - A$. The determinant of the characteristic matrix

$$\Delta(\lambda) = | \lambda E - A | = | \lambda \delta_{ik} - a_{ik} |_1^n,$$

is a scalar polynomial in λ and is called the *characteristic polynomial* of A (see Chapter III, § 7).[3]

The matrix $B(\lambda) = \| b_{ik}(\lambda) \|_1^n$, where $b_{ik}(\lambda)$ is the algebraic complement of the element $\lambda\delta_{ik} - a_{ik}$ in the determinant $\Delta(\lambda)$ is called the *adjoint matrix* of A.

By way of example, for the matrix

$$A = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}$$

we have:

$$\lambda E - A = \begin{Vmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{Vmatrix},$$

$$\Delta(\lambda) = | \lambda E - A | = \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \ldots,$$

$$B(\lambda) = \begin{Vmatrix} \lambda^2 - (a_{22} + a_{33})\lambda + a_{22}a_{33} - a_{23}a_{32} & * & * \\ a_{21}\lambda + a_{23}a_{31} - a_{21}a_{33} & * & * \\ a_{31}\lambda + a_{21}a_{32} - a_{22}a_{31} & * & * \end{Vmatrix}.$$

---

[3] This polynomial differs by the factor $(-1)^n$ from the polynomial $\Delta(\lambda)$ introduced in Chapter III, § 7.

These definitions imply the following identities in λ:

$$(\lambda E - A)\, B(\lambda) = \Delta(\lambda)\, E, \tag{20}$$

$$B(\lambda)\, (\lambda E - A) = \Delta(\lambda)\, E. \tag{20'}$$

The right-hand sides of these equations can be regarded as polynomials with matrix coefficients (each of these coefficients is the product of a scalar and the unit matrix E). The polynomial matrix $B(\lambda)$ can also be represented in the form of a polynomial arranged with respect to the powers of λ. Equations (20) and (20′) show that $\Delta(\lambda)E$ is divisible on the right and on the left by $\lambda E - A$ without remainder. By the Generalized Bézout Theorem, this is only possible when the remainder $\Delta(A)E = \Delta(A)$ is the null matrix. Thus we have proved:

THEOREM 2 (Hamilton-Cayley): *Every square matrix A satisfies its characteristic equation,* i.e.

$$\Delta(A) = O. \tag{21}$$

Example.

$$A = \begin{Vmatrix} 2 & 1 \\ -1 & 3 \end{Vmatrix},$$

$$\Delta(\lambda) = \begin{vmatrix} \lambda - 2 & -1 \\ 1 & \lambda - 3 \end{vmatrix} = \lambda^2 - 5\lambda + 7,$$

$$\Delta(A) = A^2 - 5A + 7E = \begin{Vmatrix} 3 & 5 \\ -5 & 8 \end{Vmatrix} - 5 \begin{Vmatrix} 2 & 1 \\ -1 & 3 \end{Vmatrix} + 7 \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix} = O.$$

2. We denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ all the characteristic values of A, i.e., all the roots of the characteristic polynomial $\Delta(\lambda)$ (each λ_i is repeated as often as its multiplicity as a root of $\Delta(\lambda)$ requires). Then

$$\Delta(\lambda) = | \lambda E - A | = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n). \tag{22}$$

Let $g(\mu)$ be an arbitrary scalar polynomial. We wish to find the characteristic values of $g(A)$. For this purpose we split $g(\mu)$ into linear factors

$$g(\mu) = a_0(\mu - \mu_1)(\mu - \mu_2) \cdots (\mu - \mu_l). \tag{23}$$

On both sides of this identity we substitute the matrix A for μ:

$$g(A) = a_0(A - \mu_1 E)(A - \mu_2 E) \cdots (A - \mu_l E). \tag{24}$$

Passing to determinants on both sides of (24) and using (22) and (23) we find

$$\begin{aligned} |g(A)| &= a_0^n |A-\mu_1 E|\,|A-\mu_2 E| \cdots |A-\mu_l E| \\ &= (-1)^{nl} a_0^n \Delta(\mu_1)\,\Delta(\mu_2) \cdots \Delta(\mu_l) \\ &= (-1)^{nl} a_0^n \prod_{i=1}^{l} \prod_{k=1}^{n} (\mu_i-\lambda_k) = g(\lambda_1)\,g(\lambda_2) \cdots g(\lambda_n)\,. \end{aligned}$$

If in the equation

$$|g(A)| = g(\lambda_1)\,g(\lambda_2) \cdots g(\lambda_n) \tag{25}$$

we replace the polynomial $g(\mu)$ by $\lambda - g(\mu)$, where λ is some parameter, we find:

$$|\lambda E - g(A)| = [\lambda - g(\lambda_1)]\,[\lambda - g(\lambda_2)] \cdots [\lambda - g(\lambda_n)]\,. \tag{26}$$

This leads to the following theorem.

THEOREM 3: *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all the characteristic values (with the proper multiplicities) of a matrix A and if $g(\mu)$ is a scalar polynomial, then $g(\lambda_1), g(\lambda_2), \ldots, g(\lambda_n)$ are the characteristic values of $g(A)$.*

In particular, if A has the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$, then A^k has the characteristic values $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ $(k = 0, 1, 2, \ldots)$.

3. We shall now derive an effective formula expressing the adjoint matrix $B(\lambda)$ in terms of the characteristic polynomial $\Delta(\lambda)$.

Let

$$\Delta(\lambda) = \lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \cdots - p_n\,. \tag{27}$$

The difference $\Delta(\lambda) - \Delta(\mu)$ is divisible by $\lambda - \mu$ without remainder Therefore

$$\delta(\lambda,\mu) = \frac{\Delta(\lambda)-\Delta(\mu)}{\lambda-\mu} = \lambda^{n-1} + (\mu - p_1)\,\lambda^{n-2} + (\mu^2 - p_1\mu - p_2)\,\lambda^{n-3} + \cdots \tag{28}$$

is a polynomial in λ and μ.

The identity

$$\Delta(\lambda) - \Delta(\mu) = \delta(\lambda,\mu)\,(\lambda-\mu) \tag{29}$$

will still hold if we replace λ and μ by the permutable matrices λE and A. Since by the Hamilton-Cayley Theorem $\Delta(A) = O$,

$$\Delta(\lambda)\,E = \delta(\lambda E, A)\,(\lambda E - A)\,. \tag{30}$$

Comparing (20′) with (30), we obtain by virtue of the uniqueness of the quotient the required formula

$$B(\lambda) = \delta(\lambda E, A)\,. \tag{31}$$

Hence by (28)

$$B(\lambda) = E\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \cdots + B_{n-1}\,, \tag{32}$$

where

$$B_1 = A - p_1E, \quad B_2 = A^2 - p_1A - p_2E, \quad \ldots$$

and, in general,

$$B_k = A^k - p_1A^{k-1} - p_2A^{k-2} - \cdots - p_kE \qquad (k = 1, 2, \ldots, n-1)\,. \tag{33}$$

The matrices $B_1, B_2, \ldots, B_{n-1}$ can be computed in succession, starting from the recurrence relation

$$B_k = AB_{k-1} - p_kE \qquad (k = 1, 2, \ldots, n-1;\ B_0 = E)\,. \tag{34}$$

Moreover,[4]

$$AB_{n-1} - p_nE = O\,. \tag{35}$$

The relations (34) and (35) follow immediately from (20) if we equate the coefficients of equal powers of λ on both sides.

If A is non-singular, then

$$p_n = (-1)^{n-1}|A| \neq 0\,,$$

and it follows from (35) that

$$A^{-1} = \frac{1}{p_n}B_{n-1}\,. \tag{36}$$

Let λ_0 be a characteristic value of A, so that $\Delta(\lambda_0) = 0$. Substituting the value λ_0 in (20), we find:

$$(\lambda_0E - A)\,B(\lambda_0) = O\,. \tag{37}$$

Let us assume that $B(\lambda_0) \neq O$ and denote by b an arbitrary non-zero column of this matrix. Then from (37) we have $(\lambda_0E - A)\,b = O$ or

$$Ab = \lambda_0 b. \tag{38}$$

Therefore every non-zero column of $B(\lambda_0)$ determines a characteristic vector corresponding to the characteristic value λ_0.[5]

Thus:

---

[4] From (34) follows (33). If we substitute in (35) the expression for B_{n-1} given in (33), we obtain $\Delta(A) = 0$. This approach to the Hamilton-Cayley Theorem does not require the Generalized Bézout Theorem explicitly, but contains this theorem implicitly.

[5] See Chapter III, § 7. If to the characteristic value λ_0 there correspond d_0 linearly independent characteristic vectors ($n - d_0$ is the rank of $\lambda_0E - A$), then the rank of $B(\lambda_0)$ does not exceed d_0. In particular, if only one characteristic direction corresponds to λ_0, then in $B(\lambda_0)$ the elements of any two columns are proportional.

If the coefficients of the characteristic polynomial are known, then the adjoint matrix can be found by formula (31). *If the given matrix A is non-singular, then the inverse matrix A^{-1} can be found by formula* (36). *If λ_0 is a characteristic value of A, then the non-zero columns of $B(\lambda_0)$ are characteristic vectors of A for $\lambda = \lambda_0$.*

Example.

$$A = \begin{Vmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{Vmatrix},$$

$$\Delta(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 2 & 1 & -1 \\ 0 & \lambda - 1 & -1 \\ 1 & -1 & \lambda - 1 \end{vmatrix} = \lambda^3 - 4\lambda^2 + 5\lambda - 2,$$

$$\delta(\lambda, \mu) = \frac{\Delta(\lambda) - \Delta(\mu)}{\lambda - \mu} = \lambda^2 + \lambda(\mu - 4) + \mu^2 - 4\mu + 5,$$

$$B(\lambda) = \delta(\lambda E, A) = \lambda^2 E + \lambda\underbrace{(A - 4E)}_{B_1} + \underbrace{A^2 - 4A + 5E}_{B_2}.$$

But

$$B_1 = A - 4E = \begin{Vmatrix} -2 & -1 & 1 \\ 0 & -3 & 1 \\ -1 & 1 & -3 \end{Vmatrix}, \quad B_2 = AB_1 + 5E = \begin{Vmatrix} 0 & 2 & -2 \\ -1 & 3 & -2 \\ 1 & -1 & 2 \end{Vmatrix},$$

$$B(\lambda) = \begin{Vmatrix} \lambda^2 - 2\lambda & -\lambda + 2 & \lambda - 2 \\ -1 & \lambda^2 - 3\lambda + 3 & \lambda - 2 \\ -\lambda + 1 & \lambda - 1 & \lambda^2 - 3\lambda + 2 \end{Vmatrix},$$

$$|A| = 2, \quad A^{-1} = \frac{1}{2} B_2 = \begin{Vmatrix} 0 & 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{Vmatrix}.$$

Furthermore,

$$\Delta(\lambda) = (\lambda - 1)^2 (\lambda - 2).$$

The first column of the matrix $B(+1)$ gives the characteristic vector $(+1, +1, 0)$ for the characteristic value $\lambda = 1$.

The first column of the matrix $B(+2)$ gives the characteristic vector $(0, +1, +1)$ corresponding to the characteristic value $\lambda = 2$.

§ 5. The Method of Faddeev for the Simultaneous Computation of the Coefficients of the Characteristic Polynomial and of the Adjoint Matrix

1. D. K. Faddeev[6] has suggested a method for the simultaneous determination of the scalar coefficients $p_1, p_2, \ldots, p_n$ of the characteristic polynomial

$$\Delta(\lambda) = \lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \cdots - p_n \tag{39}$$

and of the matrix coefficients $B_1, B_2, \ldots, B_{n-1}$ of the adjoint matrix $B(\lambda)$.

In order to explain the method of Faddeev[7] we introduce the concept of the trace (or spur) of a matrix.

By the *trace* tr A of a matrix $A = \| a_{ik} \|_1^n$ we mean the sum of the diagonal elements of the matrix:

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}. \tag{40}$$

It is easy to see that

$$\operatorname{tr} A = p_1 = \sum_{i=1}^{n} \lambda_i, \tag{41}$$

if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic values of A, i.e., if

$$\Delta(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n). \tag{42}$$

Since by Theorem 3 A^k has the characteristic values $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ $(k = 0, 1, 2, \ldots,)$, we have

$$\operatorname{tr} A^k = s_k = \sum_{i=1}^{n} \lambda_i^k \qquad (k = 0, 1, 2, \ldots). \tag{43}$$

The sums s_k $(k = 1, 2, \ldots, n)$ of powers of the roots of the polynomial (39) are connected with the coefficients by Newton's formulas[8]

$$kp_k = s_k - p_1 s_{k-1} - \cdots - p_{k-1} s_1 \qquad (k = 1, 2, \ldots, n). \tag{44}$$

If the traces $s_1, s_2, \ldots, s_n$ of the matrices $A, A^2, \ldots, A^n$ are computed, then the coefficients $p_1, p_2, \ldots, p_n$ can be determined from (44). This is the method of Leverrier for the determination of the coefficients of the characteristic polynomial from the traces of the powers of the matrix.

2. Faddeev has proposed to compute successively, instead of the traces of the powers $A, A^2, \ldots, A^n$, the traces of certain other matrices $A_1, A_2, \ldots, A_n$

---

[6] See [14], p. 160.

[7] In Chapter VII, § 8, we shall discuss another effective method, due to A. N. Krylov, of computing the coefficients of the characteristic polynomial.

[8] See, for example, G. Chrystal, *Textbook of Algebra*, Vol. I, pp. 436ff.

and so to determine $p_1, p_2, \ldots, p_n$ and $B_1, B_2, \ldots, B_n$ by the following formulas:

$$\left.\begin{array}{lll} A_1 = A, & p_1 = \operatorname{tr} A_1, & B_1 = A_1 - p_1 E, \\ A_2 = AB_1, & p_2 = \frac{1}{2} \operatorname{tr} A_2, & B_2 = A_2 - p_2 E, \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ A_{n-1} = AB_{n-2}, & p_{n-1} = \frac{1}{n-1} \operatorname{tr} A_{n-1}, & B_{n-1} = A_{n-1} - p_{n-1} E, \\ A_n = AB_{n-1}, & p_n = \frac{1}{n} \operatorname{tr} A_n, & B_n = A_n - p_n E = O. \end{array}\right\} \quad (45)$$

The last equation $B_n = A_n - p_n E = O$ may be used to check the computation.

In order to convince ourselves that the numbers $p_1, p_2, \ldots, p_n$ and the matrices $B_1, B_2, \ldots, B_{n-1}$ that are determined successively by (45) are, in fact, the coefficients of $\Delta(\lambda)$ and $B(\lambda)$, we note that the following formulas for A_k and B_k $(k = 1, 2, \ldots, n)$ follow from (45):

$$A_k = A^k - p_1 A^{k-1} - \cdots - p_{k-1} A, \quad B_k = A^k - p_1 A^{k-1} - \cdots - p_{k-1} A - p_k E. \quad (46)$$

Equating the traces on the left-hand and right-hand sides of the first of these formulas, we obtain

$$kp_k = s_k - p_1 s_{k-1} - \cdots - p_{k-1} s_1.$$

But these formulas coincide with Newton's formulas (44) by which the coefficients of the characteristic polynomial $\Delta(\lambda)$ are determined successively. Therefore the numbers $p_1, p_2, \ldots, p_n$ determined by (45) are also the coefficients of $\Delta(\lambda)$. But then the second of formulas (46) coincide with formulas (33) by which the matrix coefficients $B_1, B_2, \ldots, B_{n-1}$ of the adjoint matrix $B(\lambda)$ are determined. Therefore, formulas (45) also determine the coefficients $B_1, B_2, \ldots, B_{n-1}$ of the matrix polynomial $B(\lambda)$.

Example.[9]

$$A = \underset{\textstyle 2 \quad\;\; 2 \quad\;\; 4 \quad\;\; 3}{\begin{Vmatrix} 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{Vmatrix}}, \quad p_1 = \operatorname{tr} A = 4, \quad B_1 = A - 4E = \begin{Vmatrix} -2 & -1 & 1 & 2 \\ 0 & -3 & 1 & 0 \\ -1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -4 \end{Vmatrix};$$

---

[9] As a check on the computation, we write under each matrix A_1, A_2, A_3 a row whose elements are the sums of the elements above it. The product of this row of 'column-sums' of the first factor into the columns of the second factor must give the elements of the column-sum of the product.

$$A_2 = AB_1 = \underset{\textstyle -5 \;\; -1 \;\; -5 \;\; -4}{\begin{Vmatrix} -3 & 4 & 0 & -3 \\ -1 & -2 & -2 & 1 \\ 2 & 0 & -2 & -5 \\ -3 & -3 & -1 & 3 \end{Vmatrix}}, \; p_2 = \frac{1}{2} \operatorname{tr} A_2 = -2, \; B_2 = A_2 + 2E = \begin{Vmatrix} -1 & 4 & 0 & -3 \\ -1 & 0 & -2 & 1 \\ 2 & 0 & 0 & -5 \\ -3 & -3 & -1 & 3 \end{Vmatrix};$$

$$A_3 = AB_2 = \underset{\textstyle -5 \;\; -1 \;\; -7 \;\; -9}{\begin{Vmatrix} -5 & 2 & 0 & -2 \\ 1 & 0 & -2 & -4 \\ -1 & -7 & -3 & 4 \\ 0 & 4 & -2 & -7 \end{Vmatrix}}, \; p_3 = \frac{1}{3} \operatorname{tr} A_3 = -5, \; B_3 = A_3 + 5E = \begin{Vmatrix} 0 & 2 & 0 & -2 \\ 1 & 5 & -2 & 4 \\ -1 & -7 & 2 & 4 \\ 0 & 4 & -2 & -2 \end{Vmatrix};$$

$$A_4 = AB_3 = \begin{Vmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{Vmatrix}, \quad p_4 = -2.$$

$$\Delta(\lambda) = \lambda^4 - 4\lambda^3 + 2\lambda^2 + 5\lambda + 2,$$

$$|A| = 2, \quad A^{-1} = \frac{1}{p_4} B_3 = \begin{Vmatrix} 0 & -1 & 0 & 1 \\ -\frac{1}{2} & -\frac{5}{2} & 1 & -2 \\ \frac{1}{2} & \frac{7}{2} & -1 & -2 \\ 0 & -2 & 1 & 1 \end{Vmatrix}.$$

Note. If we wish to determine p_1, p_2, p_3, p_4 and only the first columns of B_1, B_2, B_3, it is sufficient to compute in A_2 the elements of the first column and only the diagonal elements of the remaining columns, in A_3 only the elements of the first column, and in A_4 only the first two elements of the first column.

§ 6. The Minimal Polynomial of a Matrix

1. DEFINITION 1: *A scalar polynomial $f(\lambda)$ is called an annihilating polynomial of the square matrix A if*

$$f(A) = O.$$

An annihilating polynomial $\psi(\lambda)$ of least degree with highest coefficient 1 is called a *minimal polynomial* of A.

By the Hamilton-Cayley Theorem the characteristic polynomial $\Delta(\lambda)$ is an annihilating polynomial of A. However, as we shall show below, it is not, in general, a minimal polynomial.

Let us divide an arbitrary annihilating polynomial $f(\lambda)$ by a minimal polynomial

$$f(\lambda) = \psi(\lambda)\, q(\lambda) + r(\lambda),$$

where the degree of $r(\lambda)$ is less than that of $\psi(\lambda)$. Hence we have:

$$f(A) = \psi(A)\, q(A) + r(A).$$

Since $f(A) = O$ and $\psi(A) = O$, it follows that $r(A) = O$. But the degree of $r(\lambda)$ is less than that of the minimal polynomial $\psi(\lambda)$. Therefore $r(\lambda) \equiv 0$.[10] Hence: *Every annihilating polynomial of a matrix is divisible without remainder by the minimal polynomial.*

Let $\psi_1(\lambda)$ and $\psi_2(\lambda)$ be two minimal polynomials of one and the same matrix. Then each is divisible without remainder by the other, i.e., the polynomials differ by a constant factor. This constant factor must be 1, because the highest coefficients in $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are 1. Thus we have proved the *uniqueness of the minimal polynomial* of a given matrix A.

2. We shall now derive a formula connecting the minimal polynomial with the characteristic polynomial.

We denote by $D_{n-1}(\lambda)$ the greatest common divisor of all the minors of order $n-1$ of the characteristic matrix $\lambda E - A$, i.e., of all the elements of the matrix $B(\lambda) = \| b_{ik}(\lambda) \|_1^n$ (see the preceding section). Then

$$B(\lambda) = D_{n-1}(\lambda)\, C(\lambda), \tag{47}$$

where $C(\lambda)$ is a certain polynomial matrix, the *'reduced' adjoint matrix* of $\lambda E - A$. From (20) and (47) we have:

$$\Delta(\lambda) E = (\lambda E - A)\, C(\lambda)\, D_{n-1}(\lambda). \tag{48}$$

Hence it follows that $\Delta(\lambda)$ is divisible without remainder by $D_{n-1}(\lambda)$:[11]

$$\frac{\Delta(\lambda)}{D_{n-1}(\lambda)} = \psi(\lambda), \tag{49}$$

where $\psi(\lambda)$ is some polynomial. The factor $D_{n-1}(\lambda)$ in (48) may be cancelled on both sides:[12]

$$\psi(\lambda) E = (\lambda E - A)\, C(\lambda). \tag{50}$$

---

[10] Otherwise there would exist an annihilating polynomial of degree less than that of the minimal polynomial.

[11] We could also verify this immediately by expanding the characteristic determinant $\Delta(\lambda)$ with respect to the elements of an arbitrary row.

[12] In this case we have, apart from (50), also the identity (see (20'))

$$\psi(\lambda) E = C(\lambda)(\lambda E - A),$$

i.e., $C(\lambda)$ is at one and the same time the left quotient and right quotient of $\psi(\lambda)E$ on division by $\lambda E - A$.

Since $\psi(\lambda)E$ is divisible on the left without remainder by $\lambda E - A$, it follows by the Generalized Bézout Theorem that

$$\psi(A) = O.$$

Thus, the polynomial $\psi(\lambda)$ defined by (49) is an annihilating polynomial of A. Let us show that it is the minimal polynomial.

We denote the minimal polynomial by $\psi^*(\lambda)$. Then $\psi(\lambda)$ is divisible by $\psi^*(\lambda)$ without remainder:

$$\psi(\lambda) = \psi^*(\lambda)\, \chi(\lambda). \tag{51}$$

Since $\psi^*(A) = O$, by the Generalized Bézout Theorem the matrix polynomial $\psi^*(\lambda)E$ is divisible on the left by $\lambda E - A$ without remainder:

$$\psi^*(\lambda) E = (\lambda E - A)\, C^*(\lambda). \tag{52}$$

From (51) and (52) it follows that

$$\psi(\lambda) E = (\lambda E - A)\, C^*(\lambda)\, \chi(\lambda). \tag{53}$$

The identities (50) and (53) show that $C(\lambda)$ as well as $C^*(\lambda)\chi(\lambda)$ are left quotients of $\psi(\lambda)E$ on division by $\lambda E - A$. By the uniqueness of division

$$C(\lambda) = C^*(\lambda)\, \chi(\lambda).$$

Hence it follows that $\chi(\lambda)$ is a common divisor of all the elements of the polynomial matrix $C(\lambda)$. But, on the other hand, the greatest common divisor of all the elements of the reduced adjoint matrix $C(\lambda)$ is equal to 1, because the matrix was obtained from $B(\lambda)$ by division by $D_{n-1}(\lambda)$. Therefore $\chi(\lambda) = \text{const}$. Since the highest coefficients of $\psi(\lambda)$ and $\psi^*(\lambda)$ are equal, we have in (51) $\chi(\lambda) = 1$, i.e., $\psi(\lambda) = \psi^*(\lambda)$, and this is what we had to prove.

We have established the following formula for the minimal polynomial:

$$\psi(\lambda) = \frac{\Delta(\lambda)}{D_{n-1}(\lambda)}. \tag{54}$$

3. For the reduced adjoint matrix $C(\lambda)$ we have a formula analogous to (31) (p. 84):

$$C(\lambda) = \Psi(\lambda E, A); \tag{55}$$

where the polynomial $\Psi(\lambda, \mu)$ is defined by the equation[13]

---

[13] Formula (55) can be deduced in the same way as (31). On both sides of the identity $\psi(\lambda) - \psi(\mu) = (\lambda - \mu)\Psi(\lambda, \mu)$ we substitute for λ and μ the matrices λE and A and compare the matrix equation so obtained with (50).

$$\Psi(\lambda, \mu) = \frac{\psi(\lambda) - \psi(\mu)}{\lambda - \mu}. \tag{56}$$

Moreover,

$$(\lambda E - A)\, C(\lambda) = \psi(\lambda)\, E. \tag{57}$$

Going over to determinants on both sides of (57), we obtain

$$\Delta(\lambda)\, |C(\lambda)| = [\psi(\lambda)]^n. \tag{58}$$

Thus, $\Delta(\lambda)$ is divisible without remainder by $\psi(\lambda)$ and some power of $\psi(\lambda)$ is divisible without remainder by $\Delta(\lambda)$, i.e., the sets of all the distinct roots of the polynomials $\Delta(\lambda)$ and $\psi(\lambda)$ are equal. In other words: *All the distinct characteristic values of A are roots of $\psi(\lambda)$.*

If

$$\begin{gathered}\Delta(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s} \\ (\lambda_i \neq \lambda_j \text{ for } i \neq j;\ n_i > 0,\ i,\ j = 1, 2, \ldots, s),\end{gathered} \tag{59}$$

then

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_s)^{m_s}, \tag{60}$$

where

$$0 < m_k \leqq n_k \qquad (k = 1, 2, \ldots, s). \tag{61}$$

4. We mention one further property of the matrix $C(\lambda)$. Let λ_0 be an arbitrary characteristic value of $A = \| a_{ik} \|_1^n$. Then $\psi(\lambda_0) = 0$ and therefore, by (57),

$$(\lambda_0 E - A)\, C(\lambda_0) = O. \tag{62}$$

Note that $C(\lambda_0) \neq O$ always holds, for otherwise all the elements of the reduced adjoint matrix $C(\lambda)$ would be divisible without remainder by $\lambda - \lambda_0$, and this is impossible.

We denote by c an arbitrary non-zero column of $C(\lambda_0)$. Then from (62)

$$(\lambda_0 E - A)\, c = o,$$

i.e.,

$$Ac = \lambda_0 c. \tag{63}$$

In other words, every non-zero column of $C(\lambda_0)$ (and such a column always exists) determines a characteristic vector for $\lambda = \lambda_0$.

Example.

$$A = \begin{Vmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{Vmatrix},$$

$$\Delta(\lambda) = \begin{Vmatrix} \lambda - 3 & 3 & -2 \\ 1 & \lambda - 5 & 2 \\ 1 & -3 & \lambda \end{Vmatrix} = \lambda^3 - 8\lambda^2 + 20\lambda - 16 = (\lambda - 2)^2 (\lambda - 4),$$

$$\delta(\lambda, \mu) = \frac{\Delta(\mu) - \Delta(\lambda)}{\mu - \lambda} = \mu^2 + \mu(\lambda - 8) + \lambda^2 - 8\lambda + 20,$$

$$B(\lambda) = A^2 + (\lambda - 8) A + (\lambda^2 - 8\lambda + 20) E$$

$$= \begin{Vmatrix} 10 & -18 & 12 \\ -6 & 22 & -12 \\ -6 & 18 & -8 \end{Vmatrix} + (\lambda - 8) \begin{Vmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{Vmatrix} + (\lambda^2 - 8\lambda + 20) \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{Vmatrix}$$

$$= \begin{Vmatrix} \lambda^2 - 5\lambda + 6 & -3\lambda + 6 & 2\lambda - 4 \\ -\lambda + 2 & \lambda^2 - 3\lambda + 2 & -2\lambda + 4 \\ -\lambda + 2 & 3\lambda - 6 & \lambda^2 - 8\lambda + 2 \end{Vmatrix}.$$

All the elements of the matrix $B(\lambda)$ are divisible by $D_2(\lambda) = \lambda - 2$. Cancelling this factor, we have:

$$C(\lambda) = \begin{Vmatrix} \lambda - 3 & -3 & 2 \\ -1 & \lambda - 1 & -2 \\ -1 & 3 & \lambda - 6 \end{Vmatrix}$$

and

$$\psi(\lambda) = \frac{\Delta(\lambda)}{\lambda - 2} = (\lambda - 2)(\lambda - 4).$$

In $C(\lambda)$ we substitute for λ the value $\lambda_0 = 2$:

$$C(2) = \begin{Vmatrix} -1 & -3 & 2 \\ -1 & 1 & -2 \\ -1 & 3 & -4 \end{Vmatrix}.$$

The first column gives us the characteristic vector (1, 1, 1,) for $\lambda_0 = 2$. The second column gives us the characteristic vector $(-3, 1, 3)$ for the same characteristic value $\lambda_0 = 2$. The third column is a linear combination of the first two.

Similarly, setting $\lambda_0 = 4$, we find from the first column of the matrix $C(4)$ the characteristic vector $(1, -1, -1)$ corresponding to the characteristic value $\lambda_0 = 4$.

The reader should note that $\psi(\lambda)$ and $C(\lambda)$ could have been determined by a different method.

To begin with, let us find $D_2(\lambda)$. $D_2(\lambda)$ can only have 2 and 4 as its roots. For $\lambda = 4$ the second order minor

$$\begin{vmatrix} 1 & \lambda - 5 \\ 1 & -3 \end{vmatrix} = -\lambda + 2$$

of $\Delta(\lambda)$ does not vanish. Therefore $D_2(4) \neq 0$. For $\lambda = 2$ the columns of $\Delta(\lambda)$ become proportional. Therefore all the minors of order two in $\Delta(\lambda)$

vanish for $\lambda = 2$: $D_2(2) = 0$. Since the minor to be computed is of the first degree, $D_2(\lambda)$ cannot be divisible by $(\lambda - 2)^2$. Therefore

$$D_2(\lambda) = \lambda - 2.$$

Hence

$$\psi(\lambda) = \frac{\Delta(\lambda)}{\lambda - 2} = (\lambda - 2)(\lambda - 4) = \lambda^2 - 6\lambda + 8,$$

$$\psi(\lambda, \mu) = \frac{\psi(\mu) - \psi(\lambda)}{\mu - \lambda} = \mu + \lambda - 6,$$

$$C(\lambda) = \psi(\lambda E, A) = A + (\lambda - 6)E = \left\| \begin{matrix} \lambda - 3 & -3 & 2 \\ -1 & \lambda - 1 & -2 \\ -1 & 3 & \lambda - 6 \end{matrix} \right\|.$$

CHAPTER V

FUNCTIONS OF MATRICES

§ 1. Definition of a Function of a Matrix

1. Let $A = \| a_{ik} \|_1^n$ be a square matrix and $f(\lambda)$ a function of a scalar argument λ. We wish to define what is to be meant by $f(A)$, i.e., we wish to extend the function $f(\lambda)$ to a matrix value of the argument.

We already know the solution of this problem in the simplest special case where $f(\lambda) = \gamma_0 \lambda^l + \gamma_1 \lambda^{l-1} + \cdots + \gamma_l$ is a polynomial in λ. In this case, $f(A) = \gamma_0 A^l + \gamma_1 A^{l-1} + \cdots + \gamma_l E$. Starting from this special case, we shall obtain a definition of $f(A)$ in the general case.

We denote by

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_s)^{m_s} \tag{1}$$

the minimal polynomial[1] of A (where $\lambda_1, \lambda_2, \ldots, \lambda_s$ are all the distinct characteristic values of A). The degree of this polynomial is $m = \sum_{k=1}^{s} m_k$.

Let $g(\lambda)$ and $h(\lambda)$ be two polynomials such that

$$g(A) = h(A). \tag{2}$$

Then the difference $d(\lambda) = g(\lambda) - h(\lambda)$, as an annihilating polynomial for A, is divisible by $\psi(\lambda)$ without remainder; we shall write this as follows:

$$g(\lambda) \equiv h(\lambda) \ \big(\operatorname{mod} \psi(\lambda)\big). \tag{3}$$

Hence by (1)

$$d(\lambda_k) = 0, \quad d'(\lambda_k) = 0, \quad \ldots, \quad d^{(m_k - 1)}(\lambda_k) = 0 \qquad (k = 1, 2, \ldots, s),$$

i.e.,

$$g(\lambda_k) = h(\lambda_k), \quad g'(\lambda_k) = h'(\lambda_k), \ldots, g^{(m_k-1)}(\lambda_k) = h^{(m_k-1)}(\lambda_k) \qquad (k = 1, 2, \ldots, s). \tag{4}$$

---

[1] See Chapter IV, § 6.

The m numbers

$$f(\lambda_k),\quad f'(\lambda_k),\quad \dots,\quad f^{(m_k-1)}(\lambda_k)\qquad (k=1,2,\dots,s) \tag{5}$$

will be called *the values of the function* $f(\lambda)$ *on the spectrum of the matrix* A and the set of all these values will be denoted symbolically by $f(\Lambda_A)$. If for a function $f(\lambda)$ the values (5) exist (i.e., have meaning), then we shall say that *the function* $f(\lambda)$ *is defined on the spectrum of the matrix* A.

Equation (4) shows that the polynomials $g(\lambda)$ and $h(\lambda)$ have the same values on the spectrum of A. In symbols:

$$g(\Lambda_A)=h(\Lambda_A).$$

Our argument is reversible: from (4) follows (3) and therefore (2).

Thus, given a matrix A, the values of the polynomial $g(\lambda)$ on the spectrum of A determine the matrix $g(A)$ completely, i.e., all polynomials $g(\lambda)$ that assume the same values on the spectrum of A have one and the same matrix value $g(A)$.

We postulate that the definition of $f(A)$ in the general case be subject to the same principle: *The values of the function* $f(\lambda)$ *on the spectrum of the matrix* A *must determine* $f(A)$ *completely, i.e., all functions* $f(\lambda)$ *having the same values on the spectrum of* A *must have the same matrix value* $f(A)$.

But then it is obvious that for the general definition of $f(A)$ it is sufficient to look for a polynomial[2] $g(\lambda)$ that assumes the same values on the spectrum of A as $f(\lambda)$ does and to set:

$$f(A)=g(A).$$

We are thus led to the following definition:

DEFINITION 1: *If the function* $f(\lambda)$ *is defined on the spectrum of the matrix* A, *then*

$$f(\Lambda_A)=g(\Lambda_A),$$

where $g(\lambda)$ *is an arbitrary polynomial that assumes on the spectrum of* A *the same values as does* $f(\lambda)$:

$$f(A)=g(A).$$

Among all the polynomials with complex coefficients that assume on the spectrum of A the same values as $f(\lambda)$ there is one and only one polynomial

---

[2] It will be proved in § 2 that such an interpolation polynomial always exists and an algorithm for the computation of the coefficients of the interpolation polynomial of least degree will be given.

$r(\lambda)$ that is of degree less than m.[3] This polynomial $r(\lambda)$ is uniquely determined by the interpolation conditions:

$$r(\lambda_k)=f(\lambda_k),\quad r'(\lambda_k)=f'(\lambda_k),\quad \dots,\quad r^{(m_k-1)}(\lambda_k)=f^{(m_k-1)}(\lambda_k) \tag{6}$$
$$(k=1,2,\dots,s).$$

The polynomial $r(\lambda)$ is called the *Lagrange-Sylvester interpolation polynomial* for $f(\lambda)$ on the spectrum of A. Definition 1 can also be formulated as follows:

DEFINITION 1′: *Let* $f(\lambda)$ *be a function defined on the spectrum of a matrix* A *and* $r(\lambda)$ *the corresponding Lagrange-Sylvester interpolation polynomial. Then*

$$f(A)=r(A).$$

Note. If the minimal polynomial $\psi(\lambda)$ of a matrix A has no multiple roots[4] (in (1) $m_1=m_2=\dots=m_s=1$; $s=m$), then for $f(A)$ to have a meaning it is sufficient that $f(\lambda)$ be defined at the characteristic values $\lambda_1,\lambda_2,\dots,\lambda_m$. But if $\psi(\lambda)$ has multiple roots, then for some characteristic values the derivatives of $f(\lambda)$ up to a certain order (see (6)) must be defined as well.

Example 1: Let us consider the matrix[5]

$$H=\left\|\begin{matrix}\overbrace{\begin{matrix}0 & 1 & 0 & \dots & 0\end{matrix}}^{n}\\ \begin{matrix}0 & 0 & 1 & \dots & 0\end{matrix}\\ \dots\dots\dots\dots\\ \begin{matrix}0 & 0 & 0 & \dots & 1\end{matrix}\\ \begin{matrix}0 & 0 & 0 & \dots & 0\end{matrix}\end{matrix}\right\|.$$

Its minimal polynomial is λ^n. Therefore the values of $f(\lambda)$ on the spectrum of H are the numbers $f(0), f'(0), \dots, f^{(n-1)}(0)$, and the polynomial $r(\lambda)$ is of the form

$$r(\lambda)=f(0)+\frac{f'(0)}{1!}\lambda+\dots+\frac{f^{(n-1)}(0)}{(n-1)!}\lambda^{n-1}.$$

Therefore

---

[3] This polynomial is obtained from any other polynomial having the same spectral values by taking the remainder on division by $\psi(\lambda)$ of that polynomial.

[4] In Chapter VI it will be shown that A is a matrix of simple structure (see Chapter III, § 8) in this case, and this case only.

[5] The properties of the matrix H were worked out in the example on pp. 13-14.

$$f(H)=f(0)E+\frac{f'(0)}{1!}H+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}H^{n-1}=\begin{Vmatrix} f(0) & \frac{f'(0)}{1!} & \cdot & \cdot & \cdot & \frac{f^{(n-1)}(0)}{(n-1)!} \\ 0 & f(0) & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \frac{f'(0)}{1!} \\ \cdot & & & & \cdot & \\ 0 & 0 & \cdot & \cdot & \cdot & f(0) \end{Vmatrix}$$

Example 2: Let us consider the matrix

$$J=\begin{Vmatrix} \overbrace{\lambda_0 & 1 & 0 & \ldots & 0}^{n} \\ 0 & \lambda_0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_0 \end{Vmatrix}.$$

Note that $J=\lambda_0E+H$, so that $J-\lambda_0E=H$. The minimal polynomial of J is clearly $(\lambda-\lambda_0)^n$. The interpolation polynomial $r(\lambda)$ of $f(\lambda)$ is given by the equation

$$r(\lambda)=f(\lambda_0)+\frac{f'(\lambda_0)}{1!}(\lambda-\lambda_0)+\cdots+\frac{f^{(n-1)}(\lambda_0)}{(n-1)!}(\lambda-\lambda_0)^{n-1}$$

Therefore

$$f(J)=r(J)=f(\lambda_0)E+\frac{f'(\lambda_0)}{1!}H+\cdots+\frac{f^{(n-1)}(\lambda_0)}{(n-1)!}H^{n-1}$$

$$=\begin{Vmatrix} f(\lambda_0) & \frac{f'(\lambda_0)}{1!} & \cdot & \cdot & \cdot & \frac{f^{(n-1)}(\lambda_0)}{(n-1)!} \\ 0 & f(\lambda_0) & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \frac{f'(\lambda_0)}{1!} \\ \cdot & & & & \cdot & \\ 0 & 0 & \cdot & \cdot & \cdot & f(\lambda_0) \end{Vmatrix}.$$

2. We mention two properties of functions of matrices.

1. *If two matrices A and B are similar and T transforms A into B,*

$$B=T^{-1}AT,$$

then the matrices $f(A)$ and $f(B)$ are also similar and T transforms $f(A)$ into $f(B)$,

$$f(B)=T^{-1}f(A)T.$$

For two similar matrices have equal minimal polynomials,[6] so that $f(\lambda)$ assumes the same values on the spectrum of A and of B. Therefore there exists an interpolation polynomial $r(\lambda)$ such that $f(A)=r(A)$ and $f(B)=r(B)$. But then it follows[6] from the equation $r(B)=T^{-1}r(A)T$ that

$$f(B)=T^{-1}f(A)T.$$

2. *If A is a quasi-diagonal matrix*

$$A=\{A_1, A_2, \ldots, A_u\},$$

then

$$f(A)=\{f(A_1), f(A_2), \ldots, f(A_u)\}.$$

Let us denote by $r(\lambda)$ the Lagrange-Sylvester interpolation polynomial of $f(\lambda)$ on the spectrum of A. Then it is easy to see that

$$f(A)=r(A)=\{r(A_1), r(A_2), \ldots, r(A_u)\}. \tag{7}$$

On the other hand, the minimal polynomial $\psi(\lambda)$ of A is an annihilating polynomial for each of the matrices $A_1, A_2, \ldots, A_n$. Therefore it follows from the equation

$$f(A_A)=r(A_A)$$

that

$$f(A_{A_1})=r(A_{A_1}), \ldots, f(A_{A_u})=r(A_{A_u}).$$

Therefore

$$f(A_1)=r(A_1), \ldots, f(A_u)=r(A_u),$$

and equation (7) can be written as follows:

$$f(A)=\{f(A_1), f(A_2), \ldots, f(A_u)\}. \tag{8}$$

Example 1: If the matrix A is of simple structure

$$A=T\{\lambda_1, \lambda_2, \ldots, \lambda_n\}T^{-1},$$

then

$$f(A)=T\{f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)\}T^{-1}.$$

$f(A)$ has meaning if the function $f(\lambda)$ is defined at $\lambda_1, \lambda_2, \ldots, \lambda_n$.

---

[6] From $B=T^{-1}AT$ it follows that $B^k=T^{-1}A^kT$ $(k=0, 1, 2, \ldots)$. Hence for every polynomial $g(\lambda)$ we have $g(B)=T^{-1}g(A)T$. Therefore it follows from $g(A)=O$ that $g(B)=O$, and vice versa.

Example 2: Let J be a matrix of the following quasi-diagonal form

$$J=\left\|\begin{array}{ccc} \overbrace{\begin{matrix}\lambda_1 & 1 & 0 & \dots & 0\\ 0 & \lambda_1 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1\\ 0 & 0 & 0 & \dots & \lambda_1\end{matrix}}^{\nu_1} & & \\ & \ddots & \\ & & \overbrace{\begin{matrix}\lambda_u & 1 & 0 & \dots & 0\\ 0 & \lambda_u & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots \lambda_u & 1\\ 0 & 0 & 0 & \dots 0 & \lambda_u\end{matrix}}^{\nu_u}\end{array}\right\|$$

All the elements in the non-diagonal blocks are zero. By (8) (see also the example on pp. 12-13),

$$f(J)=\left\|\begin{array}{ccc} \begin{matrix}f(\lambda_1) & \frac{f'(\lambda_1)}{1!} & \dots & \frac{f^{(\nu_1-1)}(\lambda_1)}{(\nu_1-1)!}\\ 0 & f(\lambda_1) & \ddots & \vdots\\ \vdots & & \ddots & \frac{f'(\lambda_1)}{1!}\\ 0 & 0 & \dots & f(\lambda_1)\end{matrix} & & \\ & \ddots & \\ & & \begin{matrix}f(\lambda_u) & \frac{f'(\lambda_u)}{1!} & \dots & \frac{f^{(\nu_u-1)}(\lambda_u)}{(\nu_u-1)!}\\ 0 & (f\,\lambda_u) & \ddots & \vdots\\ \vdots & & \ddots & \frac{f'(\lambda_u)}{1!}\\ 0 & 0 & \dots & f(\lambda_u)\end{matrix}\end{array}\right\|$$

Here, as in the matrix J, all the elements in the non-diagonal blocks are also zero.[7]

---

[7] It will be established later (Chapter VI, § 6 or Chapter VII, § 7) that an arbitrary matrix $A=\|a_{ik}\|_1^n$ is always similar to some matrix of the form J : $A=TJT^{-1}$. Therefore (see 1, on p. 98) we always have $f(A)=Tf(J)T^{-1}$.

§ 2. The Lagrange-Sylvester Interpolation Polynomial

1. To begin with, we consider the case in which the characteristic equation $|\lambda E-A|=0$ has no multiple roots. The roots of this equation—the characteristic values of the matrix A—will be denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$. Then

$$\psi(\lambda)=|\lambda E-A|=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n),$$

and condition (6) can be written as follows:

$$r(\lambda_k)=f(\lambda_k)\qquad(k=1,\,2,\,\dots,\,n).$$

In this case, $r(\lambda)$ is the ordinary Lagrange interpolation polynomial for the function $f(\lambda)$ at the points $\lambda_1, \lambda_2, \dots, \lambda_n$:

$$r(\lambda)=\sum_{k=1}^{n}\frac{(\lambda-\lambda_1)\cdots(\lambda-\lambda_{k-1})(\lambda-\lambda_{k+1})\cdots(\lambda-\lambda_n)}{(\lambda_k-\lambda_1)\cdots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})\cdots(\lambda_k-\lambda_n)}f(\lambda_k).$$

By Definition 1′

$$f(A)=r(A)=\sum_{k=1}^{n}\frac{(A-\lambda_1E)\cdots(A-\lambda_{k-1}E)(A-\lambda_{k+1}E)\cdots(A-\lambda_nE)}{(\lambda_k-\lambda_1)\cdots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})\cdots(\lambda_k-\lambda_n)}f(\lambda_k).$$

2. Let us assume now that the characteristic polynomial has multiple roots, but that the minimal polynomial, which is a divisor of the characteristic polynomial, has only simple roots:[8]

$$\psi(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_m).$$

In this case (as in the preceding one) all the exponents m_k in (1) are equal to 1, and the equation (6) takes the form

$$r(\lambda_k)=f(\lambda_k)\qquad(k=1,\,2,\,\dots,\,m).$$

$r(\lambda)$ is again the ordinary Lagrange interpolation polynomial and

$$f(A)=\sum_{k=1}^{m}\frac{(A-\lambda_1E)\cdots(A-\lambda_{k-1}E)(A-\lambda_{k+1}E)\cdots(A-\lambda_mE)}{(\lambda_k-\lambda_1)\cdots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})\cdots(\lambda_k-\lambda_m)}f(\lambda_k).$$

3. We now consider the general case:

$$\psi(\lambda)=(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_s)^{m_s}\qquad(m_1+m_2+\cdots+m_s=m).$$

We represent the rational function $\frac{r(\lambda)}{\psi(\lambda)}$, where the degree of $r(\lambda)$ is less than the degree of $\psi(\lambda)$, as a sum of partial fractions:

---

[8] See footnote 4.

$$\frac{r(\lambda)}{\psi(\lambda)}=\sum_{k=1}^{s}\left[\frac{\alpha_{k1}}{(\lambda-\lambda_k)^{m_k}}+\frac{\alpha_{k2}}{(\lambda-\lambda_k)^{m_k-1}}+\cdots+\frac{\alpha_{k\,m_k}}{\lambda-\lambda_k}\right], \tag{9}$$

where α_{kj} $(j=1,2,\ldots,m_k;\ k=1,2,\ldots,s)$ are certain constants.

In order to determine the numerators α_{kj} of the partial fractions we multiply both sides of (9) by $(\lambda-\lambda_k)^{m_k}$ and denote by $\psi_k(\lambda)$ the polynomial $\frac{\psi(\lambda)}{(\lambda-\lambda_k)^{m_k}}$. Then we obtain:

$$\frac{r(\lambda)}{\psi_k(\lambda)}=\alpha_{k1}+\alpha_{k2}(\lambda-\lambda_k)+\cdots+\alpha_{k\,m_k}(\lambda-\lambda_k)^{m_k-1}+$$
$$+(\lambda-\lambda_k)^{m_k}\varrho_k(\lambda) \qquad (k=1,2,\ldots,s), \tag{10}$$

where $\varrho_k(\lambda)$ is a rational function, regular for $\lambda=\lambda_k$.[9]

Hence

$$\left.\begin{array}{c}\alpha_{k1}=\left[\dfrac{r(\lambda)}{\psi_k(\lambda)}\right]_{\lambda=\lambda_k},\\ \alpha_{k2}=\left[\dfrac{r(\lambda)}{\psi_k(\lambda)}\right]'_{\lambda=\lambda_k}=r(\lambda_k)\left[\dfrac{1}{\psi_k(\lambda)}\right]'_{\lambda=\lambda_k}+r'(\lambda_k)\dfrac{1}{\psi_k(\lambda_k)},\ \ldots (k=1,2,\ldots s).\end{array}\right\} \tag{11}$$

Formulas (11) show that the numerators α_{kj} on the right-hand side of (9) are expressible in terms of the values of the polynomial $r(\lambda)$ on the spectrum of A, and these values are known: they are equal to the corresponding values of the function $f(\lambda)$ and its derivatives. Therefore

$$\alpha_{k1}=\frac{f(\lambda_k)}{\psi_k(\lambda_k)},\ \alpha_{k2}=f(\lambda_k)\left[\frac{1}{\psi_k(\lambda)}\right]'_{\lambda=\lambda_k}+f'(\lambda_k)\frac{1}{\psi_k(\lambda_k)},\ldots \tag{12}$$
$$(k=1,2,\ldots,s).$$

Formulas (12) may be abbreviated as follows:

$$\alpha_{kj}=\frac{1}{(j-1)!}\left[\frac{f(\lambda)}{\psi_k(\lambda)}\right]^{(j-1)}_{\lambda=\lambda_k} \qquad (j=1,2,\ldots,m_k;\quad k=1,2,\ldots,s). \tag{13}$$

When all the α_{kj} have been found, we can determine $r(\lambda)$ from the following formula, which is obtained from (9) by multiplying both sides by $\psi(\lambda)$:

$$r(\lambda)=\sum_{k=1}^{s}\left[\alpha_{k1}+\alpha_{k2}(\lambda-\lambda_k)+\cdots+\alpha_{k\,m_k}(\lambda-\lambda_k)^{m_k-1}\right]\psi_k(\lambda). \tag{14}$$

In this formula the expression in brackets that multiplies $\psi_k(\lambda)$ is, by (13), equal to the sum of the first m_k terms of the Taylor expansion of $f(\lambda)$ in powers of $(\lambda-\lambda_k)$.

---

[9] I.e., that does not become infinite for $\lambda=\lambda_k$.

Note. The Lagrange-Sylvester interpolation polynomial can be obtained by a limiting process from the Lagrange interpolation polynomial.

Let

$$\psi(\lambda)=(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_s)^{m_s} \quad \left(m=\sum_{k=1}^{s}m_k\right).$$

We denote the Lagrange interpolation polynomial constructed for the m points

$$\lambda_1^{(1)},\ \lambda_1^{(2)},\ \ldots,\ \lambda_1^{(m_1)};\ \lambda_2^{(1)},\ \lambda_2^{(2)},\ \ldots,\ \lambda_2^{(m_2)};\ \ldots;\ \lambda_s^{(1)},\ \lambda_s^{(2)},\ \ldots,\ \lambda_s^{(m_s)}$$

by

$$L(\lambda)=L\begin{pmatrix}\lambda_1^{(1)},\ \ldots,\ \lambda_1^{(m_1)};\ \ldots;\ \lambda_s^{(1)},\ \ldots,\ \lambda_s^{(m_s)}; & \\ & \lambda\\ f(\lambda_1^{(1)}),\ \ldots,\ f(\lambda_1^{(m_1)});\ \ldots;\ f(\lambda_s^{(1)}),\ \ldots,\ f(\lambda_s^{(m_s)}); & \end{pmatrix}.$$

Then it is not difficult to show that the required Lagrange-Sylvester polynomial is determined by the formula

$$r(\lambda)=\lim_{\substack{\lambda_1^{(1)},\ldots,\lambda_1^{(m_1)}\to\lambda_1\\ \cdots\cdots\cdots\cdots\\ \lambda_s^{(1)},\ldots,\lambda_s^{(m_s)}\to\lambda_s}} L(\lambda).$$

Example:

$$\psi(\lambda)=(\lambda-\lambda_1)^2(\lambda-\lambda_2)^3 \qquad (m=5).$$

Then

$$\frac{r(\lambda)}{\psi(\lambda)}=\frac{\alpha}{(\lambda-\lambda_1)^2}+\frac{\beta}{\lambda-\lambda_1}+\frac{\gamma}{(\lambda-\lambda_2)^3}+\frac{\delta}{(\lambda-\lambda_2)^2}+\frac{\varepsilon}{\lambda-\lambda_2}.$$

Hence

$$r(\lambda)=[\alpha+\beta(\lambda-\lambda_1)](\lambda-\lambda_2)^3+[\gamma+\delta(\lambda-\lambda_2)+\varepsilon(\lambda-\lambda_2)^2](\lambda-\lambda_1)^2$$

and therefore

$$r(A)=[\alpha E+\beta(A-\lambda_1E)](A-\lambda_2E)^3+[\gamma E+\delta(A-\lambda_2E)+\varepsilon(A-\lambda_2E)^2](A-\lambda_1E)^2.$$

α, β, γ, δ, and ε can be found from the following formulas:

$$\alpha=\frac{f(\lambda_1)}{(\lambda_1-\lambda_2)^3},\quad \beta=-\frac{3}{(\lambda_1-\lambda_2)^4}f(\lambda_1)+\frac{1}{(\lambda_1-\lambda_2)^3}f'(\lambda_1),$$
$$\gamma=\frac{f(\lambda_2)}{(\lambda_2-\lambda_1)^2},\quad \delta=-\frac{2}{(\lambda_2-\lambda_1)^3}f(\lambda_2)+\frac{1}{(\lambda_2-\lambda_1)^2}f'(\lambda_2),$$
$$\varepsilon=\frac{3}{(\lambda_2-\lambda_1)^4}f(\lambda_2)-\frac{2}{(\lambda_2-\lambda_1)^3}f'(\lambda_2)+\frac{1}{2}\frac{1}{(\lambda_2-\lambda_1)^2}f''(\lambda_2).$$

§ 3. Other Forms of the Definition of $f(A)$. The Components of the Matrix A

1. Let us return to the formula (14) for $r(\lambda)$. When we substitute in (14) the expressions (12) for the coefficients α and combine the terms that contain one and the same value of the function $f(\lambda)$ or of one of its derivatives, we represent $r(\lambda)$ in the form

$$r(\lambda)=\sum_{k=1}^{s}\left[f(\lambda_k)\,\varphi_{k1}(\lambda)+f'(\lambda_k)\,\varphi_{k2}(\lambda)+\cdots+f^{(m_k-1)}(\lambda_k)\,\varphi_{k\,m_k}(\lambda)\right]. \qquad (15)$$

Here $\varphi_{kj}(\lambda)$ $(j=1,2,\ldots,m_k;\ k=1,2,\ldots,s)$ are easily computable polynomials in λ of degree less than m. These polynomials are completely determined when $\psi(\lambda)$ is given and do not depend on the choice of the function $f(\lambda)$. The number of these polynomials is equal to the number of values of the function $f(\lambda)$ on the spectrum of A, i.e., equal to m (m is the degree of the minimal polynomial $\psi(\lambda)$). The functions $\varphi_{kj}(\lambda)$ represent the Lagrange-Sylvester interpolation polynomial for the function whose values on the spectrum of A are all equal to zero with the exception of $f^{(j-1)}(\lambda_k)$, which is equal to 1.

All the polynomials $\varphi_{kj}(\lambda)$ $(j=1,2,\ldots,m_k;\ k=1,2,\ldots,s)$ are linearly independent. For suppose that

$$\sum_{k=1}^{s}\sum_{j=1}^{m_k}c_{kj}\varphi_{kj}(\lambda)=0.$$

Let us determine the interpolation polynomial $r(\lambda)$ from the m conditions:

$$r^{(j-1)}(\lambda_k)=c_{kj} \qquad (j=1,2,\ldots,m_k;\ k=1,2,\ldots,s). \qquad (16)$$

Then by (15) and (16)

$$r(\lambda)=\sum_{k=1}^{s}\sum_{j=1}^{m_k}c_{kj}\varphi_{kj}(\lambda)=0$$

and, therefore, by (16)

$$c_{kj}=0 \qquad (j=1,2,\ldots,m_k;\ k=1,2,\ldots,s).$$

From (15) we deduce the *fundamental formula for* $f(A)$:

$$f(A)=\sum_{k=1}^{s}\left[f(\lambda_k)\,Z_{k1}+f'(\lambda_k)\,Z_{k2}+\cdots+f^{(m_k-1)}(\lambda_k)\,Z_{km_k}\right], \qquad (17)$$

where

$$Z_{kj}=\varphi_{kj}(A) \quad (j=1,2,\ldots,m_k;\ k=1,2,\ldots,s). \qquad (18)$$

The matrices Z_{kj} are completely determined when A is given and do not depend on the choice of the function $f(\lambda)$. On the right-hand side of (17) the function $f(\lambda)$ is represented only by its values on the spectrum of A.

The matrices Z_{kj} $(j=1,2,\ldots,m_k;\ k=1,2,\ldots,s)$ will be called the *constituent matrices* or *components* of the given matrix A.

The components Z_{kj} are linearly independent.

For suppose that

$$\sum_{k=1}^{s}\sum_{j=1}^{m_k}c_{kj}Z_{kj}=O.$$

Then by (18)

$$\chi(A)=O, \qquad (19)$$

where

$$\chi(\lambda)=\sum_{k=1}^{s}\sum_{j=1}^{m_k}c_{kj}\varphi_{kj}(\lambda). \qquad (20)$$

Since by (20) the degree of $\chi(\lambda)$ is less than m, the degree of the minimal polynomial $\psi(\lambda)$, it follows from (19) that

$$\chi(\lambda)=0.$$

But then, since the m functions $\varphi_{kj}(\lambda)$ are linearly independent, (20) implies that

$$c_{kj}=0 \qquad (j=1,2,\ldots,m_k;\ k=1,2,\ldots,s),$$

and this is what we had to prove.

2. From the linear independence of the constituent matrices Z_{kj} it follows, among other things, that none of these matrices can be zero. Let us also note that any two components Z_{kj} are permutable among each other and with A, because they are all scalar polynomials in A.

The formula (17) for $f(A)$ is particularly convenient to use when it is necessary to deal with several functions of one and the same matrix A, or when the function $f(\lambda)$ depends not only on λ, but also on some parameter t. In the latter case, the components Z_{kj} on the right-hand side of (17) do not depend on t, and the parameter t enters only into the scalar coefficients of the matrices.

In the example at the end of § 2, where $\psi(\lambda)=(\lambda-\lambda_1)^2(\lambda-\lambda_2)^3$, we may represent $r(\lambda)$ in the form

$$r(\lambda)=f(\lambda_1)\,\varphi_{11}(\lambda)+f'(\lambda_1)\,\varphi_{12}(\lambda)+f(\lambda_2)\,\varphi_{21}(\lambda)+f'(\lambda_2)\,\varphi_{22}(\lambda)+f''(\lambda_2)\,\varphi_{23}(\lambda),$$

where

$$\varphi_{11}(\lambda) = \left(\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\right)^3\left[1-\frac{3(\lambda-\lambda_1)}{\lambda_1-\lambda_2}\right], \qquad \varphi_{12}(\lambda) = \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)^3}{(\lambda_1-\lambda_2)^3},$$

$$\varphi_{21}(\lambda) = \left(\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\right)^2\left[1-\frac{2(\lambda-\lambda_2)}{\lambda_2-\lambda_1}+\frac{3(\lambda-\lambda_2)^2}{(\lambda_2-\lambda_1)^2}\right],$$

$$\varphi_{22}(\lambda) = \frac{(\lambda-\lambda_1)^2(\lambda-\lambda_2)}{(\lambda_2-\lambda_1)^2}\left[1-\frac{2(\lambda-\lambda_2)}{\lambda_2-\lambda_1}\right],$$

$$\varphi_{23}(\lambda) = \frac{(\lambda-\lambda_1)^2(\lambda-\lambda_2)^2}{2(\lambda_2-\lambda_1)^2}.$$

Therefore

$$f(A) = f(\lambda_1)Z_{11} + f'(\lambda_1)Z_{12} + f(\lambda_2)Z_{21} + f'(\lambda_2)Z_{22} + f''(\lambda_2)Z_{23}$$

where

$$Z_{11} = \varphi_{11}(A) = \frac{1}{(\lambda_1-\lambda_2)^3}(A-\lambda_2E)^3\left[E-\frac{3}{\lambda_1-\lambda_2}(A-\lambda_1E)\right],$$

$$Z_{12} = \varphi_{12}(A) = \frac{1}{(\lambda_1-\lambda_2)^3}(A-\lambda_1E)(A-\lambda_2E)^3, \;\ldots.$$

3. When the matrix A is given and its components have actually to be found, we can set in the fundamental formula (17) $f(\mu) = \frac{1}{\lambda-\mu}$, where λ is a parameter. Then we obtain

$$(\lambda E - A)^{-1} = \frac{C(\lambda)}{\psi(\lambda)} = \sum_{k=1}^{s}\left[\frac{Z_{k1}}{\lambda-\lambda_k} + \frac{1!\,Z_{k2}}{(\lambda-\lambda_k)^2} + \cdots + \frac{(m_k-1)!\,Z_{km_k}}{(\lambda-\lambda_k)^{m_k}}\right], \tag{21}$$

where $C(\lambda)$ is the reduced adjoint matrix of $\lambda E - A$ (Chapter IV, § 6).[10]

The matrices $(j-1)!\,Z_{kj}$ are the numerators of the partial fractions in the decomposition (21), and by analogy with (9) they may be expressed by the values of $C(\lambda)$ on the spectrum of A by formulas similar to (11):

$$(m_k-1)!\,Z_{km_k} = \frac{C(\lambda_k)}{\psi_k(\lambda)}, \qquad (m_k-2)!\,Z_{k\,m_k-1} = \left[\frac{C(\lambda)}{\psi_k(\lambda)}\right]'_{\lambda=\lambda_k}, \quad \ldots \;.$$

Hence

$$Z_{kj} = \frac{1}{(j-1)!\,(m_k-j)!}\left[\frac{C(\lambda)}{\psi_k(\lambda)}\right]^{(m_k-j)}_{\lambda=\lambda_k} \quad (j=1,2,\ldots,m_k;\; k=1,2,\ldots,s). \tag{22}$$

When we replace the constituent matrices in (17) by their expressions (22), we can represent the fundamental formula (17) in the form

---

[10] For $f(\mu) = \frac{1}{\lambda-\mu}$ we have $f(A) = (\lambda E - A)^{-1}$. For $f(A) = r(A)$, where $r(\mu)$ is the Lagrange-Sylvester interpolation polynomial. From the fact that $f(\mu)$ and $r(\mu)$ coincide on the spectrum of A it follows that $(\lambda-\mu)\,r(\mu)$ and $(\lambda-\mu)\,f(\mu) = 1$ coincide on this spectrum. Hence $(\lambda E - A)\,r(A) = (\lambda E - A)\,f(A) = E$.

$$f(A) = \sum_{k=1}^{s}\frac{1}{(m_k-1)!}\left[\frac{C(\lambda)}{\psi_k(\lambda)}f(\lambda)\right]^{(m_k-1)}_{\lambda=\lambda_k}. \tag{23}$$

Example 1:[11]

$$A = \left\|\begin{array}{rrr|r} 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{array}\right\|, \quad \lambda E - A = \left\|\begin{array}{ccc} \lambda-2 & 1 & -1 \\ 0 & \lambda-1 & -1 \\ 1 & -1 & \lambda-1 \end{array}\right\|.$$

In this case $\Delta(\lambda) = |\lambda E - A| = (\lambda-1)^2(\lambda-2)$. Since the minor of the element in the first row and second column of $\lambda E - A$ is equal to 1, we have $D_2(\lambda) = 1$ and, therefore,

$$\psi(\lambda) = \Delta(\lambda) = (\lambda-1)^2(\lambda-2) = \lambda^3 - 4\lambda^2 + 5\lambda - 2,$$

$$\Psi(\lambda,\mu) = \frac{\psi(\mu)-\psi(\lambda)}{\mu-\lambda} = \mu^2 + (\lambda-4)\mu + \lambda^2 - 4\lambda + 5$$

and

$$C(\lambda) = \Psi(\lambda E, A) = A^2 + (\lambda-4)A + (\lambda^2-4\lambda+5)E$$

$$= \left\|\begin{array}{rrr|r} 3 & -2 & 2 & 3 \\ -1 & 2 & 2 & 3 \\ -3 & 3 & 1 & 1 \end{array}\right\| + (\lambda-4)\left\|\begin{array}{rrr} 2 & -1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right\| + (\lambda^2-4\lambda+5)\left\|\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right\|.$$

The fundamental formula has in this case the form

$$f(A) = f(1)Z_{11} + f'(1)Z_{12} + f(2)Z_{21}. \tag{24}$$

Setting $f(\mu) = \frac{1}{\lambda-\mu}$, we find:

$$(\lambda E - A)^{-1} = \frac{C(\lambda)}{\psi(\lambda)} = \frac{Z_{11}}{\lambda-1} + \frac{Z_{12}}{(\lambda-1)^2} + \frac{Z_{21}}{\lambda-2};$$

hence

$$Z_{11} = -C(1) - C'(1), \qquad Z_{12} = -C(1), \qquad Z_{21} = C(2).$$

We now use the above expression for $C(\lambda)$, compute Z_{11}, Z_{12}, Z_{21}, and substitute the results obtained in (24):

---

[11] The elements of the sum column are printed in italics and are used for checking the computation. When we multiply the rows of A into the sum column of B we obtain the sum column of AB.

$$f(A)=f(1)\begin{vmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 1 & -1 & 1\end{vmatrix}+f'(1)\begin{vmatrix}1 & -1 & 1\\ 1 & -1 & 1\\ 0 & 0 & 0\end{vmatrix}+f(2)\begin{vmatrix}0 & 0 & 0\\ -1 & 1 & 0\\ -1 & 1 & 0\end{vmatrix}$$

$$=\begin{vmatrix}f(1)+f'(1) & -f'(1) & f'(1)\\ f(1)+f'(1)-f(2) & -f'(1)+f(2) & f'(1)\\ f(1)-f(2) & -f(1)+f(2) & f(1)\end{vmatrix}. \qquad (25)$$

Example 2: Let us show that we can determine $f(A)$ starting only from the fundamental formula. Again let

$$A=\begin{vmatrix}2 & -1 & 1\\ 0 & 1 & 1\\ -1 & 1 & 1\end{vmatrix}, \quad \psi(\lambda)=(\lambda-1)^2(\lambda-2).$$

Then

$$f(A)=f(1)Z_1+f'(1)Z_2+f(2)Z_3. \qquad (24')$$

In $(24')$ we substitute for $f(\lambda)$ in succession $1, \lambda-1, (\lambda-1)^2$:

$$\boldsymbol{Z_1+Z_3=E}=\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix},$$

$$\boldsymbol{Z_2+Z_3=A-E}=\begin{vmatrix}1 & -1 & 1\\ 0 & 0 & 1\\ -1 & 1 & 0\end{vmatrix}\begin{matrix}1\\ 1,\\ 0\end{matrix}$$

$$\boldsymbol{Z_3=(A-E)^2}=\begin{vmatrix}0 & 0 & 0\\ -1 & 1 & 0\\ -1 & 1 & 0\end{vmatrix}\begin{matrix}0\\ 0.\\ 0\end{matrix}$$

Computing the third equation from the first two term by term, we can determine all the Z. Substituting in $(24')$, we obtain the expression for $f(A)$.

4. The examples we have analyzed illustrate three methods of practical computation of $f(A)$. In the first method, we found the interpolation polynomial $r(\lambda)$ and put $f(A)=r(A)$. In the second method, we made use of the decomposition (21) and expressed the components Z_{kj} in (17) by the values of the reduced adjoint matrix $C(\lambda)$ on the spectrum of A. In the third method, we started from the fundamental formula (17) and substituted in succession certain simple polynomials for $f(\lambda)$; from the linear equations so obtained we determined the constituent matrices Z_{kj}.

The third method is perhaps the most convenient for practical purposes. In the general case it can be stated as follows:

In (17) we substitute for $f(\lambda)$ successively certain polynomials $g_1(\lambda)$, $g_2(\lambda), \ldots, g_m(\lambda)$:

$$g_i(A)=\sum_{k=1}^{s}\left[g_i(\lambda_k)Z_{k1}+g_i'(\lambda_k)Z_{k2}+\cdots+g_i^{(m_k-1)}(\lambda_k)Z_{k\,m_k}\right]$$
$$(i=1, 2, \ldots, m). \qquad (26)$$

From the m equations (26) we determine the matrices Z_{kj} and substitute the expressions so obtained in (17).

The result of eliminating Z_{kj} from the $(m+1)$ equations (26) and (17) can be written in the form

$$\begin{vmatrix}f(A) & f(\lambda_1)\ldots f^{(m_1-1)}(\lambda_1)\ldots f(\lambda_s)\ldots f^{(m_s-1)}(\lambda_s)\\ g_1(A) & g_1(\lambda_1)\ldots g_1^{(m_1-1)}(\lambda_1)\ldots g_1(\lambda_s)\ldots g_1^{(m_s-1)}(\lambda_s)\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ g_m(A) & g_m(\lambda_1)\ldots g_m^{(m_1-1)}(\lambda_1)\ldots g_m(\lambda_s)\ldots g_m^{(m_s-1)}(\lambda_s)\end{vmatrix}=0.$$

Expanding this determinant with respect to the elements of the first column, we obtain the required expression for $f(A)$. As the factor of $f(A)$ we have here the determinant $\Delta=|g_i^{(j)}(\lambda_k)|$ (in the i-th row of Δ there are found the values of the polynomial $g_i(\lambda)$ on the spectrum of A; $i=1, 2, \ldots, m$). In order to determine $f(A)$ we must have $\Delta \neq 0$. This will be so if no linear combination[12] of the polynomials vanishes completely on the spectrum of A, i.e., is divisible by $\psi(\lambda)$.

The condition $\Delta \neq 0$ is always satisfied when the degrees of the polynomial $g_1(\lambda), g_2(\lambda), \ldots, g_m(\lambda)$ are $0, 1, \ldots, m-1$, respectively.[13]

5. In conclusion, we mention that high powers of a matrix A^n can be conveniently computed by formula (17) by setting $f(\lambda)$ equal to λ^n.[14]

Example: Given the matrix $A=\begin{Vmatrix}5 & -4\\ 4 & -3\end{Vmatrix}$ it is required to compute the elements of A^{100}. The minimal polynomial of the matrix is $\psi(\lambda)=(\lambda-1)^2$.

---

[12] With coefficients not all equal to zero.

[13] In the last example, $m=3$, $g_1(\lambda)=1$, $g_2(\lambda)=\lambda-1$, $g_3(\lambda)=(\lambda-1)^2$.

[14] Formula (17) may also be used to compute the inverse matrix A^{-1}, by setting $f(\lambda)=\dfrac{1}{\lambda}$ or, what is the same, by setting $\lambda=0$ in (21).

The fundamental formula is

$$f(A)=f(1)Z_1+f'(1)Z_2.$$

Replacing $f(\lambda)$ successively by 1 and $\lambda-1$, we obtain:

$$Z_1=E,\quad Z_2=A-E.$$

Therefore

$$f(A)=f(1)E+f'(1)(A-E).$$

Setting $f(\lambda)=\lambda^{100}$, we find

$$A^{100}=E+100(A-E)=\begin{Vmatrix}1&0\\0&1\end{Vmatrix}+100\begin{Vmatrix}4&-4\\4&-4\end{Vmatrix}=\begin{Vmatrix}401&-400\\400&-399\end{Vmatrix}.$$

§ 4. Representation of Functions of Matrices by means of Series

1. Let $A=\|a_{ik}\|_1^n$ be a matrix with the minimal polynomial (1):

$$\psi(\lambda)=(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_s)^{m_s}\quad\left(m=\sum_{k=1}^{s}m_k\right).$$

Furthermore, let $f(\lambda)$ be a function and let $f_1(\lambda), f_2(\lambda), \ldots, f_p(\lambda), \ldots$ be a sequence of functions defined on the spectrum of A.

We shall say that *the sequence of functions $f_p(\lambda)$ converges for $p\to\infty$ to some limit on the spectrum of A* if the limits

$$\lim_{p\to\infty}f_p(\lambda_k),\quad \lim_{p\to\infty}f'_p(\lambda_k),\quad \ldots,\quad \lim_{p\to\infty}f_p^{(m_k-1)}(\lambda_k)\qquad (k=1,2,\ldots,s)$$

exist.

We shall say that *the sequence of functions $f_p(\lambda)$ converges for $p\to\infty$ to the function $f(\lambda)$ on the spectrum of A*, and we shall write

$$\lim_{p\to\infty}f_p(\Lambda_A)=f(\Lambda_A)$$

if

$$\lim_{p\to\infty}f_p(\lambda_k)=f(\lambda_k),\quad \lim_{p\to\infty}f'_p(\lambda_k)=f'(\lambda_k),\quad\ldots,\quad \lim_{p\to\infty}f_p^{(m_k-1)}(\lambda_k)=f^{(m_k-1)}(\lambda_k)$$
$$(k=1,2,\ldots,s).$$

The fundamental formula

$$f(A)=\sum_{k=1}^{s}\left[f(\lambda_k)Z_{k1}+f'(\lambda_k)Z_{k2}+\cdots+f^{(m_k-1)}(\lambda_k)Z_{km_k}\right]$$

expresses $f(A)$ in terms of the values of $f(\lambda)$ on the spectrum of A. If we regard the matrix as a vector in a space $\boldsymbol{R}_{n^2}$ of dimension n^2, then it follows from the fundamental formula, by the linear independence of the matrices Z_{kj}, that all the $f(A)$ (for given A) form an m-dimensional subspace of $\boldsymbol{R}_{n^2}$ with basis Z_{kj} $(j=1,2,\ldots,m_k;\ k=1,2,\ldots,s)$. In this basis the 'vector' $f(A)$ has as its coordinates the m values of the function $f(\lambda)$ on the spectrum of A.

These considerations make the following theorem perfectly obvious:

THEOREM 1: *A sequence of matrices $f_p(A)$ converges for $p\to\infty$ to some limit if and only if the sequence $f_p(\lambda)$ converges for $p\to\infty$ on the spectrum of A to a limit, i.e., the limits*

$$\lim_{p\to\infty}f_p(A)\quad \textit{and}\quad \lim_{p\to\infty}f_p(\Lambda_A)$$

always exist simultaneously. Moreover, the equation

$$\lim_{p\to\infty}f_p(\Lambda_A)=f(\Lambda_A)\tag{27}$$

implies that

$$\lim_{p\to\infty}f_p(A)=f(A)\tag{28}$$

and conversely.

Proof. 1) If the values of $f_p(\lambda)$ converge on the spectrum of A for $p\to\infty$ to limit values, then from the formulas

$$f_p(A)=\sum_{k=1}^{s}\left[f_p(\lambda_k)Z_{k1}+f'_p(\lambda_k)Z_{k2}+\cdots+f_p^{(m_k-1)}(\lambda_k)Z_{km_k}\right]\tag{29}$$

there follows the existence of the limit $\lim\limits_{p\to\infty}f_p(A)$. On the basis of this formula and of (17) we deduce (28) from (27).

2) Suppose, conversely, that $\lim\limits_{p\to\infty}f_p(A)$ exists. Since the m constituent matrices Z are linearly independent, we can express, by (29), the m values of $f_p(\lambda)$ on the spectrum of A (as a linear form) by the m elements of the matrix $f_p(A)$. Hence the existence of the limit $\lim\limits_{p\to\infty}f_p(\Lambda_A)$ follows, and (27) holds in the presence of (28).

According to this theorem, if a sequence of polynomials $g_p(\lambda)$ $(p=1,2,3,\ldots)$ converges to the function $f(\lambda)$ on the spectrum of A, then

$$\lim_{p\to\infty}g_p(A)=f(A).$$

2. This formula underlines the naturalness and generality of our definition of $f(A)$. $f(A)$ is always obtained from the $g_p(A)$ by passing to the limit $p\to\infty$, provided only that the sequence of polynomials $g_p(\lambda)$ converges to $f(\lambda)$ on the spectrum of A. The latter condition is necessary for the existence of the limit $\lim\limits_{p\to\infty}g_p(A)$.

We shall say that *the series* $\sum_{p=0}^{\infty} u_p(\lambda)$ *converges on the spectrum of* A *to the function* $f(\lambda)$ and we shall write

$$f(\Lambda_A) = \sum_{p=0}^{\infty} u_p(\Lambda_A), \tag{30}$$

if all the functions occurring here are defined on the spectrum of A and the following equations hold:

$$f(\lambda_k) = \sum_{p=0}^{\infty} u_p(\lambda_k), \quad f'(\lambda_k) = \sum_{p=0}^{\infty} u_p'(\lambda_k), \quad \ldots, \quad f^{(m_k-1)}(\lambda_k) = \sum_{p=0}^{\infty} u_p^{(m_k-1)}(\lambda_k)$$
$$(k = 1, 2, \ldots, s),$$

where the series on the right-hand sides of these equations converge. In other words, if we set

$$s_p(\lambda) = \sum_{q=0}^{p} u_q(\lambda) \qquad (p = 0, 1, 2, \ldots),$$

then (30) is equivalent to

$$f(\Lambda_A) = \lim_{p \to \infty} s_p(\Lambda_A). \tag{31}$$

It is obvious that the theorem just proved can be stated in the following equivalent form:

THEOREM 1′: *The series* $\sum_{p=0}^{\infty} u_p(A)$ *converges to a matrix if and only if the series* $\sum_{p=0}^{\infty} u_p(\lambda)$ *converges on the spectrum of* A. *Moreover, the equation*

$$f(\Lambda_A) = \sum_{p=0}^{\infty} u_p(\Lambda_A)$$

implies that

$$f(A) = \sum_{p=0}^{\infty} u_p(A),$$

and conversely.

3. Suppose a power series is given with the circle of convergence $|\lambda - \lambda_0| < R$ and the sum $f(\lambda)$:

$$f(\lambda) = \sum_{p=0}^{\infty} \alpha_p (\lambda - \lambda_0)^p \qquad (|\lambda - \lambda_0| < R). \tag{32}$$

Since a power series may be differentiated term by term any number of times within the circle of convergence, (32) converges on the spectrum of any matrix whose characteristic values lie within the circle of convergence. Thus we have:

THEOREM 2: *If the function* $f(\lambda)$ *can be expanded in a power series in the circle* $|\lambda - \lambda_0| < r$,

$$f(\lambda) = \sum_{p=0}^{\infty} \alpha_p (\lambda - \lambda_0)^p, \tag{33}$$

then this expansion remains valid when the scalar argument λ *is replaced by a matrix* A *whose characteristic values lie within the circle of convergence.*

Note. In this theorem we may allow a characteristic value λ_k of A to fall on the circumference of the circle of convergence; but we must then postulate in addition that the series (33), differentiated $m_k - 1$ times term by term, should converge at the point $\lambda = \lambda_k$. It is well known that this already implies the convergence of the j times differentiated series (33) at the point λ_k to $f^{(j)}(\lambda_k)$ for $j = 0, 1, \ldots, m_k - 1$.

The theorem just proved leads, for example, to the following expansions:[15]

$$e^A = \sum_{p=0}^{\infty} \frac{A^p}{p!}, \quad \cos A = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} A^{2p}, \quad \sin A = \sum_{p=0}^{\infty} (-1)^p \frac{A^{2p+1}}{(2p+1)!},$$

$$\cosh A = \sum_{p=0}^{\infty} \frac{A^{2p}}{(2p)!}, \quad \sinh A = \sum_{p=0}^{\infty} \frac{A^{2p+1}}{(2p+1)!},$$

$$(E - A)^{-1} = \sum_{p=0}^{\infty} A^p \qquad (|\lambda_k| < 1;\ k = 1, 2, \ldots, s),$$

$$\ln A = \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p} (A - E)^p \qquad (|\lambda_k - 1| < 1;\ k = 1, 2, \ldots, s)$$

(by $\ln \lambda$ we mean here the so-called principal value of the many-valued function $\operatorname{Ln} \lambda$, i.e., that branch for which $\operatorname{Ln} 1 = 0$).

Let $G(u_1, u_2, \ldots, u_l)$ be a polynomial in $u_1, u_2, \ldots, u_l$; let $f_1(\lambda), f_2(\lambda), \ldots, f_l(\lambda)$ be functions of λ defined on the spectrum of the matrix A, and let

$$g(\lambda) \equiv G[f_1(\lambda), f_2(\lambda), \ldots, f_l(\lambda)].$$

Then from

$$g(\Lambda_A) = 0 \tag{34}$$

there follows:

$$G[f_1(A), f_2(A), \ldots, f_l(A)] = O. \tag{35}$$

---

[15] The expansions in the first two rows hold for an arbitrary matrix A.

For let us denote by $f_1(\lambda), f_2(\lambda), \ldots, f_l(\lambda)$ the Lagrange-Sylvester interpolation polynomials for $r_1(\lambda), r_2(\lambda), \ldots, r_l(\lambda)$, and let us set:

$$G[f_1(A), f_2(A), \ldots, f_l(A)] = G[r_1(A), r_2(A), \ldots, r_l(A)] = h(A) = O,$$

Then (34) implies

$$h(\lambda) = G[r_1(\lambda), r_2(\lambda), \ldots, r_l(\lambda)],$$

Hence it follows that

$$h(\Lambda_A) = 0. \tag{36}$$

and this is what we had to show.

This result allows us to extend identities between functions of a scalar variable to matrix values of the argument.

For example, from

$$\cos^2\lambda + \sin^2\lambda = 1$$

we obtain for an arbitrary matrix A

$$\cos^2 A + \sin^2 A = E$$

(in this case $G(u_1, u_2) = u_1^2 + u_2^2 - 1$, $f_1(\lambda) = \cos\lambda$, and $f_2(\lambda) = \sin\lambda$).

Similarly, for every matrix A

$$e^A e^{-A} = E,$$

i.e.,

$$e^{-A} = (e^A)^{-1}$$

Further, for every matrix A

$$e^{iA} = \cos A + i \sin A$$

Let A be a non-singular matrix ($|A| \neq 0$). We denote by $\sqrt{\lambda}$ the single-valued branch of the many-valued function $\sqrt{\lambda}$ that is defined in a domain not containing the origin and containing all the characteristic values of A. Then $\sqrt{A}$ has a meaning. From $(\sqrt{\lambda})^2 - \lambda = 0$ it now follows that

$$(\sqrt{A})^2 = A.$$

Let $f(\lambda) = \frac{1}{\lambda}$ and let $A = \| a_{ik} \|_1^n$ be a non-singular matrix. Then $f(\lambda)$ is defined as the spectrum of A, and in the equation

$$\lambda f(\lambda) = 1$$

we can therefore replace λ by A:

i.e.,[16]

$$A \cdot f(A) = E,$$

$$f(A) = A^{-1}.$$

Denoting by $r(\lambda)$ the interpolation polynomial for the function $1/\lambda$ we may represent the inverse matrix A^{-1} in the form of a polynomial in A:

---

[16] We have already made use of this on p. 109. See footnote 10.

$$A^{-1} = r(A).$$

Let us consider a rational function $\varrho(\lambda) = \frac{g(\lambda)}{h(\lambda)}$, where $g(\lambda)$ and $h(\lambda)$ are co-prime polynomials in λ. This function is defined on the spectrum of A if and only if the characteristic values of A are not roots of $h(\lambda)$, i.e.,[17] if $|h(A)| \neq 0$. Under this assumption we may replace λ by A in the identity

$$\varrho(\lambda)\, h(\lambda) = g(\lambda),$$

obtaining:

$$\varrho(A)\, h(A) = g(A).$$

Hence

$$\varrho(A) = g(A)\, [h(A)]^{-1} = [h(A)]^{-1} g(A). \tag{37}$$

Notes. 1) If $\boldsymbol{A}$ is a linear operator in an n-dimensional space $\boldsymbol{R}$, then $f(\boldsymbol{A})$ is defined exactly like $f(A)$:

$$f(\boldsymbol{A}) = r(\boldsymbol{A}),$$

where $r(\lambda)$ is the Lagrange-Sylvester interpolation polynomial for $f(\lambda)$ on the spectrum of the operator $\boldsymbol{A}$ (the spectrum of $\boldsymbol{A}$ is determined by the minimal annihilating polynomial $\psi(\lambda)$ of $\boldsymbol{A}$).

According to this definition, if the matrix $A = \| a_{ik} \|_1^n$ corresponds to the operator $\boldsymbol{A}$ in some basis of the space, then in the same basis the matrix $f(A)$ corresponds to the operator $f(\boldsymbol{A})$. All the statements of this chapter in which there occurs a matrix A remain valid after replacement of the matrix A by the operator $\boldsymbol{A}$.

2) We can also define[18] a function of a matrix $f(A)$ starting from the characteristic polynomial

$$\Delta(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)^{n_k}$$

instead of the minimal polynomial

$$\psi(\lambda) = \prod_{k=1}^{s} (\lambda - \lambda_k)^{m_k}.$$

---

[17] See (25) on p. 84.

[18] See, for example, MacMillan, W. D., *Dynamics of Rigid Bodies* (New York, 1936).

We have then to set $f(A) = g(A)$, where $g(\lambda)$ is an interpolation polynomial of degree less than n modulo $\Delta(\lambda)$ of the function $f(\lambda)$.[19] The formulas (17), (21), and (23) are to be replaced by the following[20]

$$f(A) = \sum_{k=1}^{s} \left[f(\lambda_k) \widehat{Z}_{k1} + f'(\lambda_k) \widehat{Z}_{k2} + \cdots + f^{(n_k-1)}(\lambda_k) \widehat{Z}_{kn_k} \right], \tag{17'}$$

$$(\lambda E - A)^{-1} = \frac{B(\lambda)}{\Delta(\lambda)} = \sum_{k=1}^{s} \left[\frac{\widehat{Z}_{k1}}{\lambda - \lambda_k} + \frac{1!\widehat{Z}_{k2}}{(\lambda - \lambda_k)^2} + \cdots + \frac{(n_k - 1)!\widehat{Z}_{kn_k}}{(\lambda - \lambda_k)^{n_k-1}} \right], \tag{21'}$$

$$f(A) = \sum_{k=1}^{s} \frac{1}{(n_k - 1)!} \left[\frac{B(\lambda)}{\Delta_k(\lambda)} f(\lambda) \right]_{\lambda = \lambda_k}^{(n_k-1)}, \tag{23'}$$

where

$$\Delta_k(\lambda) = \frac{\Delta(\lambda)}{(\lambda - \lambda_k)^{n_k}} \quad (k = 1, 2, \ldots, s).$$

However, in (17′) the values $f^{(m_k)}(\lambda_k), f^{(m_k+1)}(\lambda_k), \ldots, f^{(n_k-1)}(\lambda_k)$ occur only fictitiously, because a comparison of (21) with (21′) yields:

$$\widehat{Z}_{k1} = Z_{k1}, \ \ldots, \ \widehat{Z}_{km_k}, \ \widehat{Z}_{km_k-1} = \ldots = \widehat{Z}_{kn_k} = O.$$

§ 5. Application of a Function of a Matrix to the Integration of a System of Linear Differential Equations with Constant Coefficients

1. We begin by considering a system of homogeneous linear differential equations of the first order with constant coefficients:

$$\left. \begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n, \end{aligned} \right\} \tag{38}$$

where t is the independent variable, $x_1, x_2, \ldots, x_n$ are unknown functions of t, and a_{ik} $(i, k = 1, 2, \ldots, n)$ are complex numbers.

We introduce the square matrix $A = \| a_{ik} \|_1^n$ of the coefficients and the column matrix $x = (x_1, x_2, \ldots, x_n)$. Then the system (38) can be written in the form of a single matrix differential equation

---

[19] The polynomial $g(\lambda)$ is not uniquely determined by the equation $f(A) = g(A)$ and the condition 'degree less than n.'

[20] The special case of (23′) in which $f(\lambda) \equiv \lambda^n$ is sometimes called Perron's formula (see [40], pp. 25-27).

$$\frac{dx}{dt} = Ax. \tag{39}$$

Here, and in what follows, we mean by the *derivative of a matrix* that matrix which is obtained from the given one by replacing all its elements by their derivatives. Therefore $\frac{dx}{dt}$ is the column matrix with the elements $\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}$.

We shall seek a solution of the system of differential equations satisfying the following initial conditions:

$$x_1 |_{t=0} = x_{10}, \ x_2 |_{t=0} = x_{20}, \ \ldots, \ x_n |_{t=0} = x_{n0}$$

or, briefly,

$$x |_{t=0} = x_0. \tag{40}$$

Let us expand the unknown column x into a MacLaurin series in powers of t:

$$x = x_0 + \dot{x}_0 t + \ddot{x}_0 \frac{t^2}{2!} + \cdots \quad \left(\dot{x}_0 = \frac{dx}{dt} \Big|_{t=0}, \ \ddot{x}_0 = \frac{d^2x}{dt^2} \Big|_{t=0}, \ \ldots \right). \tag{41}$$

Then by successive differentiations we find from (39):

$$\frac{d^2x}{dt^2} = A \frac{dx}{dt} = A^2x, \quad \frac{d^3x}{dt^3} = A \frac{d^2x}{dt^2} = A^3x, \quad \ldots . \tag{42}$$

Substituting the value $t = 0$ in (39) and (42), we obtain:

$$\dot{x}_0 = Ax_0, \quad \ddot{x}_0 = A^2x_0, \ \ldots .$$

Now the series (41) can be written as follows:

$$x = x_0 + tAx_0 + \frac{t^2}{2!}A^2x_0 + \cdots = e^{At}x_0. \tag{43}$$

By direct substitution in (39) we see[21] that (43) is a solution of the differential equation (39). Setting $t = 0$ in (43), we find:

$$x |_{t=0} = x_0.$$

Thus, the formula (43) gives us the solution of the given system of differential equations satisfying the initial conditions (40).

Let us set $f(\lambda) = e^{\lambda t}$ in (17). Then

$$e^{At} = \| q_{ik}(t) \|_1^n = \sum_{k=1}^{s} (Z_{k1} + Z_{k2}t + \cdots + Z_{km_k}t^{m_k-1}) e^{\lambda_k t}. \tag{44}$$

---

[21] $\frac{d}{dt}(e^{At}) = \frac{d}{dt}\left(E + At + \frac{A^2t^2}{2!} + \cdots\right) = A + A^2t + \frac{A^3t^3}{2!} + \cdots = Ae^{At}$.

The solution (43) may then be written in the following form:

$$\left.\begin{aligned} x_1 &= q_{11}(t)\,x_{10} + q_{12}(t)\,x_{20} + \cdots + q_{1n}(t)\,x_{n0} \\ x_2 &= q_{21}(t)\,x_{10} + q_{22}(t)\,x_{20} + \cdots + q_{2n}(t)\,x_{n0} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= q_{n1}(t)\,x_{10} + q_{n2}(t)\,x_{20} + \cdots + q_{nn}(t)\,x_{n0}\,, \end{aligned}\right\} \tag{45}$$

where $x_{10}, x_{20}, \ldots, x_{n0}$ are constants equal to the initial values of the unknown functions $x_1, x_2, \ldots, x_n$.

Thus, the integration of the given system of differential equations reduces to the computation of the elements of the matrix e^{At}.

If $t = t_0$ is taken as the initial value of the argument, then (43) is to be replaced by the formula

$$x = e^{A(t-t_0)}\,x_0\,. \tag{46}$$

Example.

$$\begin{aligned} \frac{dx_1}{dt} &= 3x_1 - x_2 + x_3\,, \\ \frac{dx_2}{dt} &= 2x_1 \qquad - x_3\,, \\ \frac{dx_3}{dt} &= x_1 \quad - x_2 + 2x_3\,. \end{aligned}$$

The coefficient matrix is

$$A = \begin{Vmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{Vmatrix}.$$

We form the characteristic determinant

$$\Delta(\lambda) = -\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2.$$

The greatest common divisor of the minors of order 2 is $D_2(\lambda) = 1$. Therefore

$$\psi(\lambda) = \Delta(\lambda) = (\lambda - 1)(\lambda - 2)^2.$$

The fundamental formula is

$$f(A) = f(1)\,Z_1 + f(2)\,Z_2 + f'(2)\,Z_3\,.$$

For $f(\lambda)$ we choose in succession 1, $\lambda - 2$, $(\lambda - 2)^2$. We obtain:

$$Z_1 + Z_2 = E = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

$$-Z_1 + Z_3 = A - 2E = \begin{Vmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{Vmatrix}\begin{matrix} 1 \\ 1, \\ 0 \end{matrix}$$

$$Z_1 = (A - 2E)^2 = \begin{Vmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{Vmatrix}\begin{matrix} 0 \\ 0. \\ 0 \end{matrix}$$

Hence we determine Z_1, Z_2, Z_3 and substitute in the fundamental formula

$$f(A) = f(1)\begin{Vmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{Vmatrix} + f(2)\begin{Vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{Vmatrix} + f'(2)\begin{Vmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{Vmatrix}.$$

If we now replace $f(\lambda)$ by $e^{\lambda t}$, we obtain:

$$e^{At} = e^t\begin{Vmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{Vmatrix} + e^{2t}\begin{Vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{Vmatrix} + te^{2t}\begin{Vmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{Vmatrix} = \begin{Vmatrix} (1+t)\,e^{2t} & -te^{2t} & te^{2t} \\ -e^t + (1+t)\,e^{2t} & e^t - te^{2t} & te^{2t} \\ -e^t + e^{2t} & e^t - e^{2t} & e^{2t} \end{Vmatrix}.$$

Thus

$$\begin{aligned} x_1 &= C_1(1+t)\,e^{2t} - C_2te^{2t} + C_3te^{2t}\,, \\ x_2 &= C_1[-e^t + (1+t)\,e^{2t}] + C_2(e^t - te^{2t}) + C_3te^{2t}\,, \\ x_3 &= C_1(-e^t + e^{2t}) + C_2(e^t - e^{2t}) + C_3e^{2t}\,, \end{aligned}$$

where

$$C_1 = x_{10}\,, \quad C_2 = x_{20}\,, \quad C_3 = x_{30}\,.$$

2. We now consider a system of inhomogeneous linear differential equations with constant coefficients:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1(t) \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2(t) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n(t)\,, \end{aligned}\right\} \tag{47}$$

where $f_i(t)$ $(i=1,2,\dots,n)$ are continuous functions in the interval $t_0 \leqq t \leqq t_1$. Denoting by $f(t)$ the column matrix with the elements $f_1(t), f_2(t), \dots, f_n(t)$ and again setting $A = \| a_{ik} \|_1^n$, we write the system (47) as follows:

$$\frac{dx}{dt} = Ax + f(t). \tag{48}$$

We replace x by a new column z of unknown functions, connected with x by the relation

$$x = e^{At}z \tag{49}$$

Differentiating (49) term by term and substituting the expression for $\frac{dx}{dt}$ in (48) we find:[22]

$$e^{At}\frac{dz}{dt} = f(t). \tag{50}$$

Hence[23]

$$z(t) = c + \int_{t_0}^{t} e^{-A\tau} f(\tau)\, d\tau \tag{51}$$

and so by (49)

$$x = e^{At}\left[c + \int_{t_0}^{t} e^{-A\tau} f(\tau)\, d\tau\right] = e^{At}c + \int_{t_0}^{t} e^{A(t-\tau)} f(\tau)\, d\tau; \tag{52}$$

where c is a column with arbitrary constant elements.

When we give to the argument t in (52) the value t_0 we find $c = e^{-At_0}x_0$; so that (52) can be written in the following form:

$$x = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)} f(\tau)\, d\tau. \tag{53}$$

---

[22] See footnote 21.

[23] If a matrix function of a scalar argument is given, $B(\tau) = \| b_{ik}(\tau) \|$ $(i=1,2,\dots,m;\ k=1,2,\dots,n;\ t_1 \leqq \tau \leqq t_2)$, then the *integral* $\int_{t_1}^{t_2} B(\tau)\, d\tau$ is defined in the natural way:

$$\int_{t_1}^{t_2} B(\tau)\, d\tau = \left\| \int_{t_1}^{t_2} b_{ik}(\tau)\, d\tau \right\| \qquad (i=1,2,\dots,m;\ k=1,2,\dots,n).$$

Setting $e^{At} = \| q_{ik}(t) \|_1^n$, we can write the solution (53) in expanded form:

$$\left.\begin{aligned} x_1 &= q_{11}(t-t_0)\,x_{10} + \cdots + q_{1n}(t-t_0)\,x_{n0} + \\ &\quad + \int_{t_0}^{t} [q_{11}(t-\tau) f_1(\tau) + \cdots + q_{1n}(t-\tau) f_n(\tau)]\, d\tau \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_n &= q_{n1}(t-t_0)\,x_{10} + \cdots + q_{nn}(t-t_0)\,x_{n0} + \\ &\quad + \int_{t_0}^{t} [q_{n1}(t-\tau) f_1(\tau) + \cdots + q_{nn}(t-\tau) f_n(\tau)]\, d\tau. \end{aligned}\right\} \tag{54}$$

3. As an example we consider *the motion of a heavy material point in a vacuum near the surface of the earth, taking the motion of the earth into account.* It is known[24] that in this case the acceleration of the point relative to the earth is determined by the constant force of gravity $m\boldsymbol{g}$ and the inertial Coriolis force $-2m\boldsymbol{\omega} \times \boldsymbol{v}$ ($\boldsymbol{v}$ is the velocity of the point relative to the earth, $\boldsymbol{\omega}$ the constant angular velocity of the earth). Therefore the differential equation of motion of the point has the form[25]

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{g} - 2\boldsymbol{\omega} \times \boldsymbol{v}. \tag{55}$$

We define a linear operator $\boldsymbol{A}$ in three-dimensional euclidean space by the equation

$$\boldsymbol{A}\boldsymbol{x} = -2\boldsymbol{\omega} \times \boldsymbol{x} \tag{56}$$

and write instead of (55)

$$\frac{d\boldsymbol{v}}{dt} = \boldsymbol{A}\boldsymbol{v} + \boldsymbol{g}. \tag{57}$$

Comparing (57) with (48), we easily find from (53):

$$\boldsymbol{v} = e^{\boldsymbol{A}t}\boldsymbol{v}_0 + \int_0^t e^{\boldsymbol{A}\tau}\, d\tau \cdot \boldsymbol{g} \qquad (\boldsymbol{v}_0 = \boldsymbol{v}|_{t=0}).$$

Integrating term by term, we determine the radius vector of the motion of the point:

$$\boldsymbol{r} = \boldsymbol{r}_0 + \int_0^t e^{\boldsymbol{A}\tau}\, d\tau\, \boldsymbol{v}_0 + \int_0^t\!\!\int_0^\tau e^{\boldsymbol{A}\sigma}\, d\sigma\, d\tau\, \boldsymbol{g}, \tag{58}$$

where

$$\boldsymbol{r}_0 = \boldsymbol{r}|_{t=0} \quad \text{and} \quad \boldsymbol{v}_0 = \boldsymbol{v}|_{t=0}.$$

---

[24] See A. Sommerfeld, *Lectures on Theoretical Physics*, Vol. I (Mechanics), § 30.

[25] Here the symbol × denotes the vector product.

Substituting for e^{At} the series

$$E + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots$$

and replacing A by its expression from (56), we have:

$$r = r_0 + v_0 t + \frac{1}{2}gt^2 - \omega \times \left(v_0 t^2 + \frac{1}{3}gt^3\right) + \omega \times \left[\omega \times \left(\frac{2}{3}v_0 t^3 + \frac{1}{6}gt^4\right)\right] + \cdots .$$

Considering that the angular velocity ω is small (for the earth, $\omega \approx 7.3 \times 10^{-5}$ sec^{-1}), we neglect the terms containing the second and higher powers of ω; for the additional displacement of the point due to the motion of the earth we then obtain the approximate formula

$$d = -\omega \times \left(v_0 t^2 + \frac{1}{3}gt^3\right).$$

Returning to the exact solution (58), let us compute e^{At}. As a preliminary we establish that the minimal polynomial of the operator A has the form

$$\psi(\lambda) = \lambda(\lambda^2 + 4\omega^2).$$

For we find from (56)

$$A^2 x = 4\omega \times (\omega \times x) = 4(\omega x)\,\omega - 4\omega^2 x,$$
$$A^3 x = -2\omega \times A^2 x = 8\omega^2(\omega \times x).$$

Hence and from (56) it follows that the operators E, A, A^2 are linearly independent and that

$$A^3 + 4\omega^2 A = O.$$

The minimal polynomial $\psi(\lambda)$ has the simple roots $0, 2\omega i, -2\omega i$. The Lagrange interpolation formula for e^{At} has the form

$$1 + \frac{\sin 2\omega t}{2\omega}\lambda + \frac{1 - \cos 2\omega t}{4\omega^2}\lambda^2.$$

Then

$$e^{At} = E + \frac{\sin 2\omega t}{2\omega}A + \frac{1 - \cos 2\omega t}{4\omega^2}A^2.$$

Substituting this expression for e^{At} in (58) and replacing the operator A by its expression from (56), we find

$$r = r_0 + v_0 t + g\frac{t^2}{2} - \omega \times \left(\frac{1 - \cos 2\omega t}{2\omega^2}v_0 + \frac{2\omega t - \sin 2\omega t}{4\omega^3}g\right) +$$
$$+ \omega \times \left[\omega \times \left(\frac{2\omega t - \sin 2\omega t}{2\omega^3}v_0 + \frac{-1 + 2\omega^2 t^2 + \cos 2\omega t}{4\omega^4}g\right)\right]. \tag{59}$$

Let us consider the special case $v_0 = o$. When we expand the triple vector product we obtain:

$$r = r_0 + g\frac{t^2}{2} + \frac{2\omega t - \sin 2\omega t}{4\omega^3}(g \times \omega) + \frac{\cos 2\omega t - 1 + 2\omega^2 t^2}{4\omega^3}(g \sin\varphi\, \omega - \omega g),$$

where φ is the geographical latitude of the point whose motion we are considering. The term

$$\frac{2\omega t - \sin 2\omega t}{4\omega^3}(g \times \omega)$$

represents the eastward displacement perpendicular to the plane of the meridian, and the last term on the right-hand side of the last formula gives the displacement in the meridian plane perpendicular to, and away from, the earth's axis.

4. Suppose now that the following system of linear differential equations of the second order is given:

$$\left.\begin{aligned}
&\frac{d^2x_1}{dt^2} + a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0\\
&\frac{d^2x_2}{dt^2} + a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{d^2x_n}{dt^2} + a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0,
\end{aligned}\right\} \tag{60}$$

where the a_{ik} $(i, k = 1, 2, \ldots, n)$ are constant coefficients. Introducing again the column $x = (x_1, x_2, \ldots, x_n)$ and the square matrix $A = \| a_{ik} \|_1^n$, we rewrite (60) in matrix form

$$\frac{d^2x}{dt^2} + Ax = O. \tag{60'}$$

We consider, to begin with, the case in which $|A| \neq 0$. If $n = 1$, i.e., if x and A are scalars and $A \neq 0$, the general solution of the equation (60) can be written in the form

$$x = \cos(\sqrt{A}\,t)\,x_0 + (\sqrt{A})^{-1}\sin(\sqrt{A}\,t)\,\dot{x}_0 \tag{61}$$

where $x_0 = x\,|_{t=0}$ and $\dot{x}_0 = \frac{dx}{dt}\Big|_{t=0}$.

By direct verification we see that (61) is a solution of (60) for arbitrary n, where x is a column and A a non-singular square matrix.[26] Here we use the formulas

---

[26] By $\sqrt{A}$ we mean a matrix whose square is equal to A. $\sqrt{A}$, we know, exists when $|A| \neq 0$ (see p. 114).

$$\left.\begin{aligned}\cos(\sqrt{A}\,t) &= E - \frac{1}{2!}At^2 + \frac{1}{4!}A^2t^4 - \cdots,\\ (\sqrt{A})^{-1}\sin(\sqrt{A}\,t) &= Et - \frac{1}{3!}At^3 + \frac{1}{5!}A^2t^5 - \cdots.\end{aligned}\right\} \qquad (62)$$

Formula (61) comprises all solutions of the system (60) or (60′), as the initial values x_0 and $\dot{x}_0$ may be chosen arbitrarily.

The right-hand sides of the formulas (62) have a meaning even when $|A| = 0$. Therefore (61) is the general solution of the given system of differential equations also when $|A| = 0$, provided only that the functions $\cos(\sqrt{A}t)$ and $(\sqrt{A})^{-1}\sin(\sqrt{A}t)$, which are part of this expression, are interpreted as the right-hand sides of the formulas (62).

We leave it to the reader to verify that the general solution of the inhomogeneous system

$$\frac{d^2x}{dt^2} + Ax = f(t) \qquad (63)$$

satisfying the initial conditions $x_{t=0} = x_0$ and $\left.\frac{dx}{dt}\right|_{t=0} = \dot{x}_0$ can be written in the form

$$x = \cos(\sqrt{A}\,t)\,x_0 + (\sqrt{A})^{-1}\sin(\sqrt{A}\,t)\,\dot{x}_0 + \\ + (\sqrt{A})^{-1}\int_0^t \sin[\sqrt{A}\,(t-\tau)]\,f(\tau)\,d\tau. \qquad (64)$$

If $t = t_0$ is taken as the initial time, then in (61) and (64) $\cos(\sqrt{A}t)$ and $\sin(\sqrt{A}t)$ must be replaced by $\cos\left(\sqrt{A}\,(t-t_0)\right)$ and $\sin\left(\sqrt{A}\,(t-t_0)\right)$, and $\int_0^t$ by $\int_{t_0}^t$.

In the special case

$$f(t) = h\sin(pt+\alpha)$$

(h is a constant column, and p and α are numbers), (64) can be replaced by:

$$x = \cos(\sqrt{A}\,t)\,c + (\sqrt{A})^{-1}\sin(\sqrt{A}\,t)\,d + (A - p^2E)^{-1}\,h\sin(pt+\alpha),$$

where c and d are columns with arbitrary constant elements. This formula has meaning when p^2 is not a characteristic value of the matrix A ($|A - p^2E| \neq 0$).

§ 6. Stability of Motion in the Case of a Linear System

1. Let $x_1, x_2, \ldots, x_n$ be parameters that characterize the displacement of 'perturbed' motion of a given mechanical system from an original motion,[27] and suppose that these parameters satisfy a system of differential equations of the first order:

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \ldots, x_n, t) \qquad (i = 1, 2, \ldots, n); \qquad (65)$$

the independent variable t in these equations is the time, and the right-hand sides $f_i(x_1, x_2, \ldots, x_n, t)$ are continuous functions of the variables $x_1, \ldots, x_n$ in some domain containing the point $x_1 = 0, x_2 = 0, \ldots, x_n = 0$) for all $t \geqq t_0$ (t_0 is the initial time).

We now introduce the definition of stability of motion according to Lyapunov.[28]

The motion to be investigated is called *stable* if for every $\varepsilon > 0$ we can find a $\delta > 0$ such that for arbitrary initial values of the parameters $x_{10}, x_{20}, \ldots, x_{n0}$ (for $t = t_0$) with moduli less than δ the parameters $x_1, x_2, \ldots, x_n$ remain of moduli less than ε for the whole time of the motion ($t \geqq t_0$), i.e., if for every $\varepsilon > 0$ we can find a $\delta > 0$ such that from

$$|x_{i0}| < \delta \qquad (i = 1, 2, \ldots, n) \qquad (66)$$

it follows that

$$|x_i(t)| < \varepsilon \qquad (t \geqq t_0). \qquad (67)$$

If, in addition, for some $\delta > 0$ we always have $\lim_{t\to+\infty} x_i(t) = 0$ ($i = 1, 2, \ldots, n$) as long as $|x_{i0}| < \delta$ ($i = 1, 2, \ldots, n$), then the motion is called *asymptotically stable.*

We now consider a linear system, i.e., that special case when (65) is a system of linear homogeneous differential equations

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} p_{ik}(t)\,x_k, \qquad (68)$$

where the $p_{ik}(t)$ are continuous functions for $t \geqq t_0$ ($i, k = 1, 2, \ldots, n$).

In matrix form the system (68) can be written as follows:

---

[27] In these parameters, the motion to be studied is characterized by constant zero values $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. Therefore in the mathematical treatment of the problem we speak of the 'stability' of the zero solution of the system (65) of differential equations.

[28] See [14], p. 13; [9], pp. 10-11; or [36], pp. 11-12. See also [3].

$$\frac{dx}{dt} = P(t)\,x, \tag{68'}$$

where x is the column matrix with the elements $x_1, x_2, \ldots, x_n$ and $P(t) = \| p_{ik}(t) \|_1^n$ is the coefficient matrix.

We denote by

$$q_{1j}(t),\ q_{2j}(t),\ \ldots,\ q_{nj}(t) \qquad (j = 1, 2, \ldots, n) \tag{69}$$

n linearly independent solutions of (68).[29] The matrix $Q(t) = \| q_{ij} \|_1^n$ whose columns are these solutions is called an *integral matrix* of the system (68).

Every solution of the system of linear homogeneous differential equations is obtained as a linear combination of n linearly independent solutions with constant coefficients:

$$x_i = \sum_{j=1}^{n} c_j q_{ij}(t) \qquad (t = 1, 2, \ldots, n),$$

or in matrix form,

$$x = Q(t)\,c, \tag{70}$$

where c is the column matrix whose elements are arbitrary constants $c_1, c_2, \ldots, c_n$.

We now choose the special integral matrix for which

$$Q(t_0) = E; \tag{71}$$

in other words, in the choice of n linearly independent solutions of (69) we shall start from the following special initial conditions:[30]

$$q_{ij}(t_0) = \delta_{ij} = \begin{cases} 0 & (i \neq j), \\ 1 & (i = j) \end{cases} \qquad (i, j = 1, 2, \ldots, n).$$

Then setting $t = t_0$ in (70), we find from (71):

$$x_0 = c,$$

and therefore formula (70) assumes the form

$$x = Q(t)\,x_0 \tag{72}$$

or, in expanded form,

$$x_i = \sum_{j=1}^{n} q_{ij}(t)\,x_{j0} \qquad (i = 1, 2, \ldots, n). \tag{72'}$$

---

[29] Here the second subscript j denotes the number of the solution.

[30] Arbitrary initial conditions determine uniquely a certain solution of a given system.

We consider three cases:

1. $Q(t)$ is a *bounded matrix* in the interval $(t_0, +\infty)$, i.e., there exists a number M such that

$$|q_{ij}(t)| \leqq M \qquad (t \geqq t_0;\ i, j = 1, 2, \ldots, n).$$

In this case it follows from (72') that

$$|x_i(t)| \leqq nM \max |x_{j0}|.$$

The condition of stability is satisfied. (It is sufficient to take $\delta < \frac{\varepsilon}{nM}$ in (66) and (67).) *The motion characterized by the zero solution $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$ is stable.*

2. $\lim\limits_{t \to +\infty} Q(t) = O$. In this case the matrix $Q(t)$ is bounded in the interval $(t_0, +\infty)$ and therefore, as we have already explained, the motion is stable. Moreover, it follows from (72) that

$$\lim_{t \to +\infty} x(t) = o.$$

for every x_0. *The motion is asymptotically stable.*

3. $Q(t)$ is an *unbounded matrix* in the interval $(t_0, +\infty)$. This means that at least one of the functions $q_{ij}(t)$, say $q_{hk}(t)$, is not bounded in the interval. We take the initial conditions $x_{10} = 0$, $x_{20} = 0$, $\ldots$, $x_{k-1,0} = 0$, $x_{k0} \neq 0$, $x_{k+1,0} = 0, \ldots, x_{n0} = 0$. Then

$$x_h(t) = q_{hk}(t)\,x_{k0}.$$

However small in modulus x_{k0} may be, the function $x_h(t)$ is unbounded. The condition (67) is not satisfied for any δ. *The motion is unstable.*

2. We now consider the special case where the coefficients in the system (68) are constants:

$$P(t) = P = \text{const.} \tag{73}$$

We have then (see § 5)

$$x = e^{P(t-t_0)}\,x_0. \tag{74}$$

Comparing (74) with (72), we find that in this case

$$Q(t) = e^{P(t-t_0)}. \tag{75}$$

We denote by

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_s)^{m_s}$$

the minimal polynomial of the coefficient matrix P.

For the investigation of the integral matrix (75) we apply formula (17) on p. 104. In this case $f(\lambda)=e^{\lambda(t-t_0)}$ (t is regarded as a parameter), $f^{(j)}(\lambda_k)=(t-t_0)^j e^{\lambda_k(t-t_0)}$. Formula (17) yields

$$e^{P(t-t_0)}=\sum_{k=1}^{s}[Z_{k1}+Z_{k2}(t-t_0)+\cdots+Z_{km_k}(t-t_0)^{m_k-1}]\,e^{\lambda_k(t-t_0)}. \qquad (76)$$

We consider three cases:

1. $\operatorname{Re}\lambda_k\leqq 0$ $(k=1,2,\ldots,s)$; and moreover, for all λ_k with $\operatorname{Re}\lambda_k=0$ the corresponding $m_k=1$ (i.e., pure imaginary characteristic values are simple roots of the minimal polynomial).
2. $\operatorname{Re}\lambda_k<0$ $(k=1,2,\ldots,s)$.
3. For some k we have $\operatorname{Re}\lambda_k>0$; or $\operatorname{Re}\lambda_k=0$, but $m_k>1$.

From the formula (76) it follows that in the first case the matrix $Q(t)=e^{P(t-t_0)}$ is bounded in the interval $(t_0,+\infty)$, in the second case $\lim\limits_{t\to+\infty}e^{P(t-t_0)}=0$, and in the third case the matrix $e^{P(t-t_0)}$ is not bounded in the interval $(t_0,+\infty)$.[31]

Therefore in the first case the motion ($x_1=0$, $x_2=0$, ..., $x_n=0$) is stable, in the second case it is asymptotically stable, and in the third case it is unstable.

---

[31] Special consideration is only required in the case when in (76) for $e^{P(t-t_0)}$ there occur several terms of maximal growth (for $t\to+\infty$), i.e., with maximal $\operatorname{Re}\lambda_k=\alpha_0$ and (for the given $\operatorname{Re}\lambda_k=\alpha_0$) maximal value $m_k=m_0$. The expression (76) can be represented in the form

$$e^{P(t-t_0)}=e^{\alpha_0(t-t_0)}(t-t_0)^{m_0-1}\Big[\sum_{j=1}^{r}Z_{kjm_0}e^{i\beta_j(t-t_0)}+(*)\Big],$$

where $\beta_1,\beta_2,\ldots,\beta_r$ are distinct real numbers and $(*)$ denotes a matrix that tends to zero as $t\to+\infty$. From this representation it follows that the matrix $e^{P(t-t_0)}$ is not bounded for $\alpha_0+m_0-1>0$, because the matrix $\sum\limits_{j=1}^{r}Z_{kjm_0}e^{i\beta_j(t-t_0)}$ cannot converge for $t\to+\infty$. We can see this by showing that

$$f(t)=\sum_{j=1}^{r}c_je^{i\beta_jt},$$

where c_j are complex numbers and β_j real and distinct numbers, can converge to zero for $t\to+\infty$ only when $f(t)\equiv 0$. But, in fact, it follows from $\lim\limits_{t\to+\infty}f(t)=0$ that

$$\sum_{j=1}^{r}|c_j|^2=\lim_{T\to+\infty}\frac{1}{T}\int_0^T|f(t)|^2\,dt=0$$

and therefore

$$c_1=c_2=\cdots=c_n=0.$$

The results of the investigation may be formulated in the form of the following theorem:[32]

THEOREM 3: *The zero solution of the linear system* (68) *for* $P=\text{const.}$ *is stable in the sense of Lyapunov if*

1) *the real parts of all the characteristic values of* P *are negative or zero,*

2) *those characteristic values whose real part is zero, i.e., the pure imaginary characteristic values (if any such exist), are simple roots of the minimal polynomial of* P*;*

and it is unstable if at least one of the conditions 1), 2) *is violated.*

The zero solution of the linear system (68) *is asymptotically stable if and only if all the characteristic values of* P *have negative real parts.*

The considerations above enable us to make a statement about the nature of the integral matrix $e^{P(t-t_0)}$ in the general case of arbitrary characteristic values of the constant matrix P.

THEOREM 4: *The integral matrix* $e^{P(t-t_0)}$ *of the linear system* (68) *for* $P=\text{const.}$ *is always representable in the form*

$$e^{P(t-t_0)}=Z_-(t)+Z_0+Z_+(t),$$

where

1) $\lim\limits_{t\to+\infty}Z_-(t)=0$,

2) Z_0 *is either constant or is a bounded matrix in the interval* $(t_0,+\infty)$ *that does not have a limit for* $t\to+\infty$,

3) $Z_+(t)=0$ *or* $Z_+(t)$ *is an unbounded matrix in the interval* $(t_0,+\infty)$.

Proof. On the right-hand side of (76) we divide all the summands into three groups. We denote by $Z_-(t)$ the sum of all the terms containing the factors $e^{\lambda_k(t-t_0)}$, with $\operatorname{Re}\lambda_k<0$. We denote by Z_0 the sum of all those matrices Z_{k1} for which $\operatorname{Re}\lambda_k=0$. We denote by $Z_+(t)$ the sum of all the remaining terms. It is easy to see that $Z_-(t)$, $Z_0(t)$, and $Z_+(t)$ have the properties 1), 2), 3) of the theorem.

---

[32] On the question of sharpening the criteria of stability and instability for quasi-linear systems (i.e., of non-linear systems that become linear after neglecting the non-linear terms), see further Chapter XIV, § 3.

CHAPTER VI

EQUIVALENT TRANSFORMATIONS OF POLYNOMIAL MATRICES. ANALYTIC THEORY OF ELEMENTARY DIVISORS

The first three sections of this chapter deal with the theory of equivalent polynomial matrices. On the basis of this, we shall develop, in the next three sections, the analytical theory of elementary divisors, i.e., the theory of the reduction of a constant (non-polynomial) square matrix A to a normal form $\tilde{A}$ $(A = T\tilde{A}T^{-1})$. In the last two sections of the chapter two methods for the construction of the transforming matrix T will be given.

§ 1. Elementary Transformations of a Polynomial Matrix

1. Definition 1: *A polynomial matrix, or λ-matrix, is a rectangular matrix $A(\lambda)$ whose elements are polynomials in λ:*

$$A(\lambda) = \| a_{ik}(\lambda) \| = \| a_{ik}^{(0)}\lambda^l + a_{ik}^{(1)}\lambda^{l-1} + \cdots + a_{ik}^{(l)} \| \qquad \begin{array}{l}(i = 1, 2, \ldots, m; \\ \ k = 1, 2, \ldots, n);\end{array}$$

here l is the largest of the degrees of the polynomials $a_{ik}(\lambda)$.

Setting

$$A_j = \| a_{ik}^{(j)} \| \qquad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n;\ j = 0, 1, \ldots, l),$$

we may represent the polynomial matrix $A(\lambda)$ in the form of a matrix polynomial in λ, i.e., in the form of a polynomial in λ with matrix coefficients:

$$A(\lambda) = A_0\lambda^l + A_1\lambda^{l-1} + \cdots + A_{l-1}\lambda + A_l .$$

We introduce the following *elementary operations* on a polynomial matrix $A(\lambda)$:

1. Multiplication of any row, for example the i-th, by a number $c \neq 0$.

2. Addition to any row, for example the i-th, of any other row, for example the j-th, multiplied by any arbitrary polynomial $b(\lambda)$.

3. Interchange of any two rows, for example the i-th and the j-th.

We leave it to the reader to verify that the operations 1., 2., 3. are equivalent to a multiplication of the polynomial matrix $A(\lambda)$ on the *left* by the following square matrices of order m, respectively:[1]

$$\left.\begin{array}{c}
S' = \begin{array}{c} \begin{array}{ccccc} & & (i) & & \end{array} \\ \left\| \begin{array}{ccccc} 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ \vdots & & c & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{array} \right\| \end{array}, \quad
S'' = \begin{array}{c} \begin{array}{ccccccc} & & (i) & & (j) & & \end{array} \\ \left\| \begin{array}{ccccccc} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ \vdots & & 1 & \cdots & b(\lambda) & \cdots & \vdots \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{array} \right\| \end{array}, \\[2ex]
S''' = \begin{array}{c} \begin{array}{ccccccc} & & (i) & & (j) & & \end{array} \\ \left\| \begin{array}{ccccccc} 1 & & \vdots & & \vdots & & 0 \\ & \ddots & \vdots & & \vdots & & \\ \cdots & \cdots & 0 & \cdots & 1 & \cdots & \cdots \\ & & \vdots & \ddots & \vdots & & \\ \cdots & \cdots & 1 & \cdots & 0 & \cdots & \cdots \\ & & \vdots & & \vdots & \ddots & \\ 0 & & \vdots & & \vdots & & 1 \end{array} \right\| \end{array};
\end{array}\right\} \qquad (1)$$

in other words, as the result of applying the operations 1., 2., 3. the matrix $A(\lambda)$ is transformed into $S' \cdot A(\lambda)$, $S'' \cdot A(\lambda)$, and $S''' \cdot A(\lambda)$, respectively. The operations of type 1., 2., 3. are therefore called *left elementary* operations.

In the same way we define the *right elementary* operations on a polynomial matrix (these are performed not on the rows, but on the columns);[2] the matrices (of order n) corresponding to them are:

---

[1] In the matrices (1) all the elements that are not shown are 1 on the main diagonal and 0 elsewhere.

[2] See footnote 1.

$$T' = \left\Vert \begin{array}{ccccc} 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & c & \cdots & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{array} \right\Vert \begin{array}{c} \\ \\ (i), \\ \\ \\ \end{array} \qquad T'' = \left\Vert \begin{array}{cccccc} 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ \vdots & & 1 & \cdots & \cdots & \vdots \\ \vdots & & \vdots & \ddots & & \vdots \\ \vdots & & b(\lambda) & \cdots & \cdots & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{array} \right\Vert \begin{array}{c} \\ \\ (i) \\ \\ (j) \\ \\ \\ \end{array},$$

$$T''' = \left\Vert \begin{array}{cccccc} 1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & & \vdots \\ \vdots & & 0 & \cdots & 1 \cdots & \vdots \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ \vdots & & 1 & \cdots & 0 & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{array} \right\Vert \begin{array}{c} \\ \\ (i) \\ \\ (j) \\ \\ \\ \end{array}.$$

The result of applying a right elementary operation is equivalent to multiplying the matrix $A(\lambda)$ on the *right* by the corresponding matrix T.

Note that T' and T''' coincide with S' and S''' and that T'' coincides with S'' when the indices i and j are interchanged in these matrices. The matrices of type S', S'', S''' (or, what is the same, T', T'', T''') will be called *elementary matrices.*

The determinant of every elementary matrix does not depend on λ and is different from zero. Therefore each left (right) elementary operation has an inverse operation which is also a left (right) elementary operation.[3]

DEFINITION 2: *Two polynomial matrices* $A(\lambda)$ *and* $B(\lambda)$ *are called* 1) *left-equivalent,* 2) *right-equivalent,* 3) *equivalent if one of them can be obtained from the other by means of* 1) *left-elementary,* 2) *right elementary,* 3) *left and right elementary operations, respectively.*[4]

---

[3] It follows from this that if a matrix $B(\lambda)$ is obtained from $A(\lambda)$ by means of left (right; left and right) elementary operations, then $A(\lambda)$ can, conversely, be obtained from $B(\lambda)$ by means of elementary operations of the same type. The left elementary operations form a group, as do the right elementary operations.

[4] From the definition it follows that only matrices of the same dimensions can be left-equivalent, right-equivalent, or simply equivalent.

Let $B(\lambda)$ be obtained from $A(\lambda)$ by means of the left elementary operations corresponding to $S_1, S_2, \ldots, S_p$. Then

$$B(\lambda) = S_p S_{p-1} \cdots S_1 A(\lambda). \tag{2}$$

Denoting the product $S_p S_{p-1} \cdots S_1$ by $P(\lambda)$, we write (2) in the form

$$B(\lambda) = P(\lambda) A(\lambda), \tag{3}$$

where $P(\lambda)$, like each of the matrices $S_1, S_2, \ldots, S_p$ has a *constant*[5] non-zero determinant.

In the next section we shall prove that every square λ-matrix $P(\lambda)$ with a constant non-zero determinant can be represented in the form of a product of elementary matrices. Therefore (3) is equivalent to (2) and signifies left equivalence of the matrices $A(\lambda)$ and $B(\lambda)$.

In the case of right equivalence of the polynomial matrices $A(\lambda)$ and $B(\lambda)$ we shall have instead of (3) the equation

$$B(\lambda) = A(\lambda) Q(\lambda) \tag{3'}$$

and in the case of (two-sided) equivalence the equation

$$B(\lambda) = P(\lambda) A(\lambda) Q(\lambda). \tag{3''}$$

Here again, $P(\lambda)$ and $Q(\lambda)$ are matrices with non-zero determinants, independent of λ.

Thus, Definition 2 can be replaced by an equivalent definition.

DEFINITION 2′: *Two rectangular* λ*-matrices* $A(\lambda)$ *and* $B(\lambda)$ *are called* 1) *left-equivalent,* 2) *right-equivalent,* 3) *equivalent if*

1) $B(\lambda) = P(\lambda)A(\lambda)$,
2) $B(\lambda) = A(\lambda)Q(\lambda)$,
3) $B(\lambda) = P(\lambda)A(\lambda)Q(\lambda)$,

respectively, where $P(\lambda)$ *and* $Q(\lambda)$ *are polynomial square matrices with constant non-zero determinants.*

2. All the concepts introduced above are illustrated in the following important example.

We consider a system of m linear homogeneous differential equations of order l with constant coefficients, where $x_1, x_2, \ldots, x_n$ are n unknown functions of the independent variable t:

$$\left.\begin{array}{c} a_{11}(D)x_1 + a_{12}(D)x_2 + \cdots + a_{1n}(D)x_n = 0 \\ a_{21}(D)x_1 + a_{22}(D)x_2 + \cdots + a_{2n}(D)x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}(D)x_1 + a_{m2}(D)x_2 + \cdots + a_{mn}(D)x_n = 0; \end{array}\right\} \tag{4}$$

---

[5] I.e., independent of λ.

here

$$a_{ik}(D)=a^{(0)}_{ik}D^{l}+a^{(1)}_{ik}D^{l-1}+\cdots+a^{(l)}_{ik}\quad(i=1,2,\ldots,m;\ k=1,2,\ldots,n)$$

is a polynomial in D with constant coefficients; $D=\frac{d}{dt}$ is the differential operator.

The matrix of operator coefficients

$$A(D)=\|a_{ik}(D)\|\qquad(i=1,2,\ldots,m;\ k=1,2,\ldots,n)$$

is a polynomial matrix, or D-matrix.

Clearly, the left elementary operation 1. on the matrix $A(D)$ signifies term-by-term multiplication of the i-th differential equation of the system by the number $c\neq 0$. The left elementary operation 2. signifies the term-by-term addition to the i-th equation of the j-th equation which has previously been subjected to the differential operator $b(D)$. The left elementary operation 3. signifies an interchange of the i-th and j-th equation.

Thus, if we replace in (4) the matrix $A(D)$ of operator coefficients by a left-equivalent matrix $B(D)$, we obtain a deduced system of equations. Since, conversely, by the same reasoning, the original system is a consequence of the new system, the two systems of equations are equivalent.[6]

It is not difficult in this example to interpret the right elementary operations as well. The first of them signifies the introduction of a new unknown function $\bar{x}_i=\frac{1}{c}x_i$ for the unknown function x_i; the second signifies the introduction of a new unknown function $\bar{x}_j=x_j+b(D)x_i$ (instead of x_j); the third signifies the interchange of the terms in the equations that contain x_i and x_j (i.e., $\bar{x}_i=x_j$, $\bar{x}_j=x_i$).

§ 2. Canonical Form of a λ-Matrix

1. To begin with, we shall examine what comparatively simple form we can obtain for a rectangular polynomial matrix $A(\lambda)$ by means of left elementary operations only.

Let us assume that the first column of $A(\lambda)$ contains elements not identically equal to zero. Among them we choose a polynomial of least degree and by a permutation of the rows we make it into the element $a_{11}(\lambda)$. Then we divide $a_{i1}(\lambda)$ by $a_{11}(\lambda)$; we denote quotient and remainder by $q_{i1}(\lambda)$ and $r_{i1}(\lambda)$ $(i=2,\ldots,m)$:

---

[6] Here it is assumed that the unknown functions $x_1, x_2, \ldots, x_n$ are such that their derivatives of all orders, as far as they occur in the transformations, exist. With this restriction, two systems of equations with left-equivalent matrices $A(D)$ and $B(D)$ have the same solutions.

$$a_{i1}(\lambda)=a_{11}(\lambda)q_{i1}(\lambda)+r_{i1}(\lambda)\qquad(i=2,\ldots,m).$$

Now we subtract from the i-th row the first row multiplied by $q_{i1}(\lambda)$ $(i=2,\ldots,m)$. If not all the remainders $r_{i1}(\lambda)$ are identically equal to zero, then we choose one of them that is not equal to zero and is of least degree and put it into the place of $a_{11}(\lambda)$ by a permutation of the rows. As the result of all these operations, the degree of the polynomial $a_{11}(\lambda)$ is reduced.

Now we repeat this process. Since the degree of the polynomial $a_{11}(\lambda)$ is finite, this must come to an end at some stage—i.e., at this stage all the elements $a_{21}(\lambda), a_{31}(\lambda), \ldots, a_{m1}(\lambda)$ turn out to be identically equal to zero.

Next we take the element $a_{22}(\lambda)$ and apply the same procedure to the rows numbered $2, 3, \ldots, m$, achieving $a_{32}(\lambda)=\ldots=a_{m2}(\lambda)=0$. Continuing still further, we finally reduce the matrix $A(\lambda)$ to the following form:

$$\underset{(m\leqq n)}{\begin{Vmatrix} b_{11}(\lambda) & b_{12}(\lambda) & \ldots & b_{1m}(\lambda) & \ldots & b_{1n}(\lambda)\\ 0 & b_{22}(\lambda) & \ldots & b_{2m}(\lambda) & \ldots & b_{2n}(\lambda)\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \ldots & b_{mm}(\lambda) & \ldots & b_{mn}(\lambda)\end{Vmatrix}},\qquad \underset{(m\geqq n)}{\begin{Vmatrix} b_{11}(\lambda) & b_{12}(\lambda) & \ldots & b_{1n}(\lambda)\\ 0 & b_{22}(\lambda) & \ldots & b_{2n}(\lambda)\\ \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \ldots & b_{nn}(\lambda)\\ 0 & 0 & \ldots & 0\\ \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \ldots & 0\end{Vmatrix}}. \qquad (5)$$

If the polynomial $b_{22}(\lambda)$ is not identically equal to zero, then by applying a left elementary operation of the second type we can make the degree of the element $b_{12}(\lambda)$ less than the degree of $b_{22}(\lambda)$ (if $b_{22}(\lambda)$ is of degree zero, then $b_{12}(\lambda)$ becomes identically equal to zero). In the same way, if $b_{33}(\lambda)\equiv 0$, then by left elementary operations of the second type we make the degrees of the elements $b_{13}(\lambda)$, $b_{23}(\lambda)$ less than the degree of $b_{33}(\lambda)$ without changing the elements $b_{12}(\lambda)$, etc.

We have established the following theorem:

Theorem 1: *An arbitrary rectangular polynomial matrix of dimension $m\times n$ can always be brought into the form (5) by means of left elementary operations, where the polynomials $b_{1k}(\lambda), b_{2k}(\lambda), \ldots, b_{k-1,k}(\lambda)$ are of degree less than that of $b_{kk}(\lambda)$, provided $b_{kk}(\lambda)\not\equiv 0$, and are all identically equal to zero if $b_{kk}(\lambda)=\text{const.}\neq 0$ $(k=2,3,\ldots,\min(m,n))$.*

Similarly, we prove

Theorem 2: *An arbitrary rectangular polynomial matrix of dimension $m\times n$ can always be brought into the form*

$$\left\|\begin{matrix} c_{11}(\lambda) & 0 & \dots & 0 & 0 \dots 0 \\ c_{21}(\lambda) & c_{22}(\lambda) & \dots & 0 & 0 \dots 0 \\ \dots & \dots & \dots & \dots & \dots \\ c_{m1}(\lambda) & c_{m2}(\lambda) & \dots & c_{mm}(\lambda) & 0 \dots 0 \end{matrix}\right\|, \quad \left\|\begin{matrix} c_{11}(\lambda) & 0 & \dots & 0 \\ c_{21}(\lambda) & c_{22}(\lambda) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{n1}(\lambda) & c_{n2}(\lambda) & \dots & c_{nn}(\lambda) \\ \dots & \dots & \dots & \dots \\ c_{m1}(\lambda) & c_{m2}(\lambda) & \dots & c_{mn}(\lambda) \end{matrix}\right\| \tag{6}$$
$$(m \leqq n) \qquad\qquad (m \geqq n)$$

by means of right elementary operations, where the polynomials $c_{k1}(\lambda)$, $c_{k2}(\lambda)$, $\dots$, $c_{k,k-1}(\lambda)$ *are of degree less than that of* $c_{kk}(\lambda)$, *provided* $c_{kk}(\lambda) \not\equiv 0$, *and all are identically equal to zero if* $c_{kk}(\lambda) = \text{const.} \neq 0$ $(k = 2, 3, \dots, \min(m, n))$.

2. From Theorems 1 and 2 we deduce the corollary:

COROLLARY: *If the determinant of a square polynomial matrix* $P(\lambda)$ *does not depend on* λ *and is different from zero, then the matrix can be represented in the form of a product of a finite number of elementary matrices.*

For by Theorem 1 the matrix $P(\lambda)$ can be brought into the form

$$\left\|\begin{matrix} b_{11}(\lambda) & b_{12}(\lambda) & \dots & b_{1n}(\lambda) \\ 0 & b_{22}(\lambda) & \dots & b_{2n}(\lambda) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & b_{nn}(\lambda) \end{matrix}\right\| \tag{7}$$

by left elementary operations, where n is the order of $P(\lambda)$. Since in the application of elementary operations to a square polynomial matrix the determinant of the matrix is only multiplied by constant non-zero factors, the determinant of the matrix (7), like that of $P(\lambda)$, does not depend on λ and is different from 0, i.e.,

$$b_{11}(\lambda)\, b_{22}(\lambda) \cdots b_{nn}(\lambda) = \text{const.} \neq 0.$$

Hence

$$b_{kk}(\lambda) = \text{const.} \neq 0 \quad (k = 1, 2, \dots, n).$$

But then, also by Theorem 1, the matrix (7) has the diagonal form $\| b_k \delta_{ik} \|_1^n$ and can therefore be reduced to the unit matrix E by means of left elementary operations of type 1. But then, conversely, the unit matrix E can be transformed into $P(\lambda)$ by means of the left elementary operations whose matrices are $S_1, S_2, \dots, S_p$. Therefore

$$P(\lambda) = S_p S_{p-1} \cdots S_1 E = S_p S_{p-1} \cdots S_1 .$$

As we pointed out on p. 133, from this corollary there follows the equivalence of the two Definitions 2 and 2′ of equivalence of polynomial matrices.

3. Let us return to our example of the system of differential equations (4). We apply Theorem 1 to the matrix $\| a_{ik}(D) \|$ of operator coefficients. As we have shown on p. 135, the system (4) is then replaced by an equivalent system

$$\left.\begin{aligned} b_{11}(D)\,x_1 + b_{12}(D)\,x_2 + \cdots + b_{1s}(D)\,x_s &= -b_{1,s+1}(D)\,x_{s+1} - \cdots - b_{1n}(D)\,x_n , \\ b_{22}(D)\,x_2 + \cdots + b_{2s}(D)\,x_s &= -b_{2,s+1}(D)\,x_{s+1} - \cdots - b_{2n}(D)\,x_n , \\ \dots\dots\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ b_{ss}(D)\,x_s &= -b_{s,s+1}(D)\,x_{s+1} - \cdots - b_{sn}(D)\,x_n \end{aligned}\right\} \tag{4′}$$

where $s = \min(m, n)$. In this system we may choose the functions x_{s+1}, $\dots$, x_n arbitrarily, after which the functions x_s, $x_{s-1}, \dots, x_1$ can be determined successively; however, at each stage of this process only one differential equation with one unknown function has to be integrated.

4. We now pass on to establishing the 'canonical' form into which a rectangular matrix $A(\lambda)$ can be brought by applying to it both left and right elementary operations.

Among all the elements $a_{ik}(\lambda)$ of $A(\lambda)$ that are not identically equal to zero we choose one which has the least degree in λ and by suitable permutations of the rows and columns we make this element into $a_{11}(\lambda)$. Then we find the quotients and remainders of the polynomials $a_{i1}(\lambda)$ and $a_{1k}(\lambda)$ on division by $a_{11}(\lambda)$:

$$a_{i1}(\lambda) = a_{11}(\lambda)\, q_{i1}(\lambda) + r_{i1}(\lambda), \quad a_{1k}(\lambda) = a_{11}(\lambda)\, q_{1k}(\lambda) + r_{1k}(\lambda)$$
$$(i = 2, 3, \dots, m;\ k = 2, 3, \dots, n).$$

If at least one of the remainders $r_{i1}(\lambda)$, $r_{1k}(\lambda)$ $(i = 2, \dots, m;\ k = 2, \dots, n)$, for example $r_{1k}(\lambda)$, is not identically equal to zero, then by subtracting from the k-th column the first column multiplied by $q_{1k}(\lambda)$, we replace $a_{1k}(\lambda)$ by the remainder $r_{1k}(\lambda)$, which is of smaller degree than $a_{11}(\lambda)$. Then we can again reduce the degree of the element in the top left corner of the matrix by putting in its place an element of smaller degree in λ.

But if all the remainders $r_{21}(\lambda), \dots, r_{m1}(\lambda)$; $r_{12}(\lambda), \dots, r_{1n}(\lambda)$ are identically equal to zero, then by subtracting from the i-th row the first multiplied by $q_{i1}(\lambda)$ $(i = 2, \dots, m)$ and from the k-th column the first multiplied by $q_{1k}(\lambda)$ $(k = 2, \dots, n)$, we reduce our polynomial matrix to the form

$$\left\|\begin{matrix} a_{11}(\lambda) & 0 & \dots & 0 \\ 0 & a_{22}(\lambda) & \dots & a_{2n}(\lambda) \\ \dots & \dots & \dots & \dots \\ 0 & a_{m2}(\lambda) & \dots & a_{mn}(\lambda) \end{matrix}\right\|.$$

If at least one of the elements $a_{ik}(\lambda)$ $(i=2,\dots,m;\ k=2,\dots,n)$ is not divisible without remainder by $a_{11}(\lambda)$, then by adding to the first column that column which contains such an element we arrive at the preceding case and can therefore again replace the element $a_{11}(\lambda)$ by a polynomial of smaller degree.

Since the original element $a_{11}(\lambda)$ had a definite degree and since the process of reducing this degree cannot be continued indefinitely, we must, after a finite number of elementary operations, obtain a matrix of the form

$$\begin{Vmatrix} a_1(\lambda) & 0 & \dots & 0 \\ 0 & b_{22}(\lambda) & \dots & b_{2n}(\lambda) \\ \dots & \dots & \dots & \dots \\ 0 & b_{m2}(\lambda) & \dots & b_{mn}(\lambda) \end{Vmatrix}, \tag{8}$$

in which all the elements $b_{ik}(\lambda)$ are divisible without remainder by $a_1(\lambda)$. If among these elements $b_{ik}(\lambda)$ there is one not identically equal to zero, then continuing the same reduction process on the rows numbered $2,\dots,m$ and the columns $2,\dots,n$, we reduce the matrix (8) to the form

$$\begin{Vmatrix} a_1(\lambda) & 0 & 0 & \dots & 0 \\ 0 & a_2(\lambda) & 0 & \dots & 0 \\ 0 & 0 & c_{33}(\lambda) & \dots & c_{3n}(\lambda) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & c_{m3}(\lambda) & \dots & c_{mn}(\lambda) \end{Vmatrix},$$

where $a_2(\lambda)$ is divisible without remainder by $a_1(\lambda)$ and all the polynomials $c_{ik}(\lambda)$ are divisible without remainder by $a_2(\lambda)$. Continuing the process further, we finally arrive at a matrix of the form

$$\begin{Vmatrix} a_1(\lambda) & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_2(\lambda) & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_s(\lambda) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{Vmatrix}, \tag{9}$$

where the polynomials $a_1(\lambda), a_2(\lambda), \dots, a_s(\lambda)$ $(s \leqq \min(m,n))$ are not identically equal to zero and each is divisible by the preceding one.

By multiplying the first s rows by suitable non-zero numerical factors, we can arrange that the highest coefficients of the polynomials $a_1(\lambda)$, $a_2(\lambda), \dots, a_s(\lambda)$ are equal to 1.

DEFINITION 3: *A rectangular polynomial matrix is called a canonical diagonal matrix if it is of the form* (9), *where* 1) *the polynomials* $a_1(\lambda)$, $a_2(\lambda), \dots, a_s(\lambda)$ *are not identically equal to zero and* 2) *each of the polynomials* $a_2(\lambda), \dots, a_s(\lambda)$ *is divisible by the preceding. Moreover, it is assumed that the highest coefficients of all the polynomials* $a_1(\lambda), a_2(\lambda), \dots,$ $a_s(\lambda)$ *are equal to* 1.

Thus, we have proved that: *An arbitrary rectangular polynomial matrix* $A(\lambda)$ *is equivalent to a canonical diagonal matrix.* In the next section we shall prove that: *The polynomials* $a_1(\lambda), a_2(\lambda), \dots, a_s(\lambda)$ *are uniquely determined by the given matrix* $A(\lambda)$; and we shall set up formulas that connect these polynomials with the elements of $A(\lambda)$.

§ 3. Invariant Polynomials and Elementary Divisors of a Polynomial Matrix

1. We introduce the concept of invariant polynomials of a λ-matrix $A(\lambda)$.

Let $A(\lambda)$ be a polynomial matrix of rank r, i.e., the matrix has minors of order r not identically equal to zero, but all the minors of order greater than r are identically equal to zero in λ. We denote by $D_j(\lambda)$ the greatest common divisor of all the minors of order j in $A(\lambda)$ $(j=1,2,\dots,r)$.[7] Then it is easy to see that in the series

$$D_r(\lambda),\quad D_{r-1}(\lambda),\dots,D_1(\lambda),\quad D_0(\lambda)\equiv 1$$

each polynomial is divisible by the preceding one.[8] The corresponding quotients will be denoted by $i_1(\lambda), i_2(\lambda), \dots, i_r(\lambda)$:

$$i_1(\lambda)=\frac{D_r(\lambda)}{D_{r-1}(\lambda)},\quad i_2(\lambda)=\frac{D_{r-1}(\lambda)}{D_{r-2}(\lambda)},\dots,\ i_r(\lambda)=\frac{D_1(\lambda)}{D_0(\lambda)}=D_1(\lambda). \tag{10}$$

DEFINITION 4: *The polynomials* $i_1(\lambda), i_2(\lambda), \dots, i_r(\lambda)$ *defined by* (10) *are called the invariant polynomials of the rectangular matrix* $A(\lambda)$.

The term 'invariant polynomial' is explained by the following arguments. Let $A(\lambda)$ and $B(\lambda)$ be two equivalent polynomial matrices. Then they are obtained from one another by means of elementary operations. But an easy verification shows immediately that the elementary operations

---

[7] We take the highest coefficient in $D_j(\lambda)$ to be 1 $(j=1,2,\dots,r)$.

[8] If we apply the Bézout decomposition with respect to the elements of any row to an arbitrary minor of order j, then every term in the decomposition is divisible by $D_{j-1}(\lambda)$; therefore every minor of order j, and hence $D_j(\lambda)$, is divisible by $D_{j-1}(\lambda)$ $(j=2,3,\dots,r)$.

change neither the rank of $A(\lambda)$ nor the polynomials $D_1(\lambda), D_2(\lambda), \ldots, D_r(\lambda)$. For when we apply to the identity (3″) the formula that expresses a minor of a product of matrices by the minors of the factors (see p. 12), we obtain for an arbitrary minor of $B(\lambda)$ the expression

$$B\begin{pmatrix} j_1 & j_2 \ldots j_p \\ k_1 & k_2 \ldots k_p \end{pmatrix}; \lambda\Big) \\ = \sum_{\substack{1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leqq m \\ 1 \leqq \beta_1 < \beta_2 < \cdots < \beta_p \leqq m}} P\begin{pmatrix} j_1 & j_2 \ldots j_p \\ \alpha_1 & \alpha_2 \ldots \alpha_p \end{pmatrix} A\begin{pmatrix} \alpha_1 & \alpha_2 \ldots \alpha_p \\ \beta_1 & \beta_2 \ldots \beta_p \end{pmatrix}; \lambda\Big) Q\begin{pmatrix} \beta_1 & \beta_2 \ldots \beta_p \\ k_1 & k_2 \ldots k_p \end{pmatrix}$$

$$(p = 1, 2, \ldots, \min(m, n)).$$

Hence it follows that all the minors of order r or greater of the matrix $B(\lambda)$ are zero, so that we have for the rank r^* of $B(\lambda)$:

$$r^* \leqq r.$$

Moreover, it follows from the same formula that $D_p^*(\lambda)$, the greatest common divisor of all the minors of order p of $B(\lambda)$, is divisible by $D_p(\lambda)$ $(p = 1, 2, \ldots, \min(m, n))$. But the matrices $A(\lambda)$ and $B(\lambda)$ can exchange roles. Therefore $r \leqq r^*$ and $D_p(\lambda)$ is divisible by $D_p^*(\lambda)$ $(p = 1, 2, \ldots, \min(m, n))$. Hence[9]

$$r = r^*, \quad D_1^*(\lambda) = D_1(\lambda), \quad D_2^*(\lambda) = D_2(\lambda), \ldots, D_r^*(\lambda) = D_r(\lambda).$$

Since elementary operations do not change the polynomials $D_1(\lambda), D_2(\lambda), \ldots, D_r(\lambda)$, they also leave the polynomials $i_1(\lambda), i_2(\lambda), \ldots, i_r(\lambda)$ defined by (10) unchanged.

Thus, the polynomials $i_1(\lambda), i_2(\lambda), \ldots, i_r(\lambda)$ remain *invariant* on transition from one matrix to another equivalent one.

If the polynomial matrix has the canonical diagonal form (9), then it is easy to see that for this matrix

$$D_1(\lambda) = a_1(\lambda), \quad D_2(\lambda) = a_1(\lambda)\, a_2(\lambda), \quad \ldots, \quad D_r(\lambda) = a_1(\lambda)\, a_2(\lambda) \cdots a_r(\lambda).$$

But then, by (10), the diagonal polynomials in (9) $a_1(\lambda), a_2(\lambda), \ldots, a_r(\lambda)$ coincide with the invariant polynomials

$$i_1(\lambda) = a_r(\lambda), \; i_2(\lambda) = a_{r-1}(\lambda), \; \ldots, \; i_r(\lambda) = a_1(\lambda). \tag{11}$$

Here $i_1(\lambda), i_2(\lambda), \ldots, i_r(\lambda)$ are at the same time the invariant polynomials of the original matrix $A(\lambda)$, because it is equivalent to (9).

The results obtained can be stated in the form of the following theorem.

---

[9] The highest coefficients in $D_p(\lambda)$ and $D_p^*(\lambda)$ $(p = 1, 2, \ldots, r)$ are 1.

THEOREM 3: *The rectangular polynomial matrix $A(\lambda)$ is always equivalent to a canonical diagonal matrix*

$$\left\| \begin{matrix} i_r(\lambda) & 0 & \ldots & 0 & 0 \ldots 0 \\ 0 & i_{r-1}(\lambda) & \ldots & 0 & 0 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & i_1(\lambda) & 0 \ldots 0 \\ 0 & 0 & & 0 & 0 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & 0 & 0 \ldots 0 \end{matrix} \right\|. \tag{12}$$

Moreover, r must here be the rank of $A(\lambda)$ and $i_1(\lambda), i_2(\lambda), \ldots, i_r(\lambda)$ the invariant polynomials of $A(\lambda)$ defined by (10).

COROLLARY 1: *Two rectangular matrices of the same dimension $A(\lambda)$ and $B(\lambda)$ are equivalent if and only if they have the same invariant polynomials.*

The sufficiency of the condition was explained above. The necessity follows from the fact that two polynomial matrices having the same invariant polynomials are equivalent to one and the same canonical diagonal matrix and, therefore, to each other. Thus: *The invariant polynomials form a complete system of invariants of a λ-matrix.*

COROLLARY 2: *In the sequence of invariant polynomials*

$$i_1(\lambda) = \frac{D_r(\lambda)}{D_{r-1}(\lambda)}, \quad i_2(\lambda) = \frac{D_{r-1}(\lambda)}{D_{r-2}(\lambda)}, \quad \ldots, \quad i_r(\lambda) = \frac{D_1(\lambda)}{D_0(\lambda)} \qquad (D_0(\lambda) \equiv 1) \tag{13}$$

every polynomial from the second onwards divides the preceding one.

This statement does not follow immediately from (13). It does follow from the fact that the polynomials $i_1(\lambda), i_2(\lambda), \ldots, i_r(\lambda)$ coincide with the polynomials $a_r(\lambda), a_{r-1}(\lambda), \ldots, a_1(\lambda)$ of the canonical diagonal matrix (9).

2. We now indicate a method of computing the invariant polynomials of a quasi-diagonal λ-matrix if the invariant polynomials of the matrices in the diagonal blocks are known.

THEOREM 4: *If in a quasi-diagonal rectangular matrix*

$$C(\lambda) = \left\| \begin{matrix} A(\lambda) & O \\ O & B(\lambda) \end{matrix} \right\|$$

every invariant polynomial of $A(\lambda)$ divides every invariant polynomial of $B(\lambda)$, then the set of invariant polynomials of $C(\lambda)$ is the union of the invariant polynomials of $A(\lambda)$ and $B(\lambda)$.

Proof. We denote by $i_1'(\lambda), i_2'(\lambda), \ldots, i_r'(\lambda)$ and $i_1''(\lambda), i_2''(\lambda), \ldots, i_q''(\lambda)$, respectively, the invariant polynomials of the λ-matrices $A(\lambda)$ and $B(\lambda)$. Then[10]

$$A(\lambda) \sim \{i_r'(\lambda), \ldots, i_1'(\lambda), 0, \ldots, 0\}, \quad B(\lambda) \sim \{i_q''(\lambda), \ldots, i_1''(\lambda), 0, \ldots, 0\}$$

and therefore

$$C(\lambda) \sim \{i_r'(\lambda), \ldots, i_1'(\lambda), \quad i_q''(\lambda), \ldots, i_1''(\lambda), 0, \ldots, 0\}. \tag{14}$$

The λ-matrix on the right-hand side of this relation is of canonical diagonal form. By Theorem 3 the diagonal elements of this matrix that are not identically equal to zero then form a complete system of invariants of the polynomial matrix $C(\lambda)$. This proves the theorem.

In order to determine the invariant polynomials of $C(\lambda)$ in the general case of arbitrary invariant polynomials of $A(\lambda)$ and $B(\lambda)$ we make use of the important concept of elementary divisors.

We decompose the invariant polynomials $i_1(\lambda), i_2(\lambda), \ldots, i_r(\lambda)$ into irreducible factors over the given number field F:[11]

$$\begin{array}{l} i_1(\lambda) = [\varphi_1(\lambda)]^{c_1} [\varphi_2(\lambda)]^{c_2} \cdots [\varphi_s(\lambda)]^{c_s}, \\ i_2(\lambda) = [\varphi_1(\lambda)]^{d_1} [\varphi_2(\lambda)]^{d_2} \cdots [\varphi_s(\lambda)]^{d_s}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ i_r(\lambda) = [\varphi_1(\lambda)]^{l_1} [\varphi_2(\lambda)]^{l_2} \cdots [\varphi_s(\lambda)]^{l_s}. \end{array} \quad \left(\begin{array}{c} c_k \geqq d_k \geqq \cdots \geqq l_k \geqq 0; \\ k = 1, 2, \ldots, s \end{array}\right) \tag{15}$$

Here $\varphi_1(\lambda), \varphi_2(\lambda), \ldots, \varphi_s(\lambda)$ are all the distinct factors irreducible over F (and with highest coefficient 1) that occur in $i_1(\lambda), i_2(\lambda), \ldots, i_r(\lambda)$.

DEFINITION 5: *All the powers among* $[\varphi_1(\lambda)]^{c_1}, \ldots, [\varphi_s(\lambda)]^{l_s}$ *in* (15), *as far as they are distinct from* 1, *are called the elementary divisors of the matrix* $A(\lambda)$ *in the field* F.[12]

THEOREM 5: *The set of elementary divisors of the rectangular quasi-diagonal matrix*

$$C(\lambda) = \left\| \begin{array}{cc} A(\lambda) & O \\ O & B(\lambda) \end{array} \right\|$$

is always obtained by combining the elementary divisors of $A(\lambda)$ *with those of* $B(\lambda)$.

---

[10] The symbol $\sim$ denotes here the equivalence of matrices; and braces $\{\ \}$, a diagonal rectangular matrix of the form (12).

[11] Some of the exponents $c_k, d_k, \ldots, l_k$ $(k = 1, 2, \ldots, s)$ may be equal to zero.

[12] The formulas (15) enable us to define not only the elementary divisors of $A(\lambda)$ in the field F in terms of the invariant polynomials but also, conversely, the invariant polynomials in terms of the elementary divisors.

Proof. We decompose the invariant polynomials of $A(\lambda)$ and $B(\lambda)$ into irreducible factors over F:[13]

$$\begin{array}{ll} i_1'(\lambda) = [\varphi_1(\lambda)]^{c_1'} [\varphi_2(\lambda)]^{c_2'} \cdots [\varphi_s(\lambda)]^{c_s'}, & i_1''(\lambda) = [\varphi_1(\lambda)]^{c_1''} [\varphi_2(\lambda)]^{c_2''} \cdots [\varphi_s(\lambda)]^{c_s''}, \\ i_2'(\lambda) = [\varphi_1(\lambda)]^{d_1'} [\varphi_2(\lambda)]^{d_2'} \cdots [\varphi_s(\lambda)]^{d_s'}, & i_2''(\lambda) = [\varphi_1(\lambda)]^{d_1''} [\varphi_2(\lambda)]^{d_2''} \cdots [\varphi_s(\lambda)]^{d_s''}, \\ \dots\dots\dots\dots\dots\dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ i_r'(\lambda) = [\varphi_1(\lambda)]^{h_1'} [\varphi_2(\lambda)]^{h_2'} \cdots [\varphi_s(\lambda)]^{h_s'}, & i_q''(\lambda) = [\varphi_1(\lambda)]^{g_1''} [\varphi_2(\lambda)]^{g_2''} \cdots [\varphi_s(\lambda)]^{g_s''}. \end{array}$$

We denote by

$$c_1 \geqq d_1 \geqq \cdots \geqq l_1 > 0, \tag{16}$$

all the non-zero numbers among $c_1', d_1', \ldots, h_1', c_1'', d_1'', \ldots, g_1''$.

Then the matrix $C(\lambda)$ is equivalent to the matrix (14), and by a permutation of rows and of columns the latter can be brought into 'diagonal' form

$$\{[\varphi_1(\lambda)]^{c_1} \cdot (*), [\varphi_1(\lambda)]^{d_1} \cdot (*), \ldots, [\varphi_1(\lambda)]^{l_1} \cdot (*), (**), \ldots, (**)\} \tag{17}$$

where we have denoted by $(*)$ polynomials that are prime to $\varphi_1(\lambda)$ and by $(**)$ polynomials that are either prime to $\varphi_1(\lambda)$ or identically equal to zero. From the form of the matrix (17) we deduce immediately the following decomposition of the polynomials $D_r(\lambda)$, $D_{r-1}(\lambda), \ldots$ and $i_1(\lambda), i_2(\lambda), \ldots$ of the matrix $C(\lambda)$:

$$D_r(\lambda) = [\varphi_1(\lambda)]^{c_1 + d_1 + \cdots + l_1} \cdot (*), \quad D_{r-1}(\lambda) = [\varphi_1(\lambda)]^{d_1 + \cdots + l_1} \cdot (*), \ldots,$$

$$i_1(\lambda) = [\varphi_1(\lambda)]^{c_1} (*), \quad i_2(\lambda) = [\varphi_1(\lambda)]^{d_1} (*), \ldots .$$

Hence it follows that $[\varphi_1(\lambda)]^{c_1}$, $[\varphi_1(\lambda)]^{d_1}, \ldots, [\varphi_1(\lambda)]^{l_1}$, i.e., all the powers

$$[\varphi_1(\lambda)]^{c_1'}, \ldots, [\varphi_1(\lambda)]^{h_1'}, [\varphi_1(\lambda)]^{c_1''}, \ldots, [\varphi_1(\lambda)]^{g_1''},$$

as far as they are distinct from 1, are elementary divisors of $C(\lambda)$.

The elementary divisors of $C(\lambda)$ that are powers of $\varphi_2(\lambda)$ are determined similarly, etc. This completes the proof of the theorem.

Note. The theory of equivalence for integral matrices (i.e., matrices whose elements are integers) can be constructed along similar lines. Here in 1., 2. (see pp. 130-31) $c = \pm 1$, $b(\lambda)$ is to be replaced by an integer, and in (3), (3′), (3″), in place of $P(\lambda)$ and $Q(\lambda)$ there are integral matrices with determinants equal to ± 1.

---

[13] If any irreducible polynomial $\varphi_k(\lambda)$ occurs as a factor in some invariant polynomials, but not in others, then in the latter we write $\varphi_k(\lambda)$ with a zero exponent.

3. Suppose now that $A = \Vert a_{ik} \Vert_1^n$ is a matrix with elements in the field F. We form its characteristic matrix

$$\lambda E - A = \left\Vert \begin{matrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{n1} & -a_{n2} & \dots & \lambda - a_{nn} \end{matrix} \right\Vert . \tag{18}$$

The characteristic matrix is a λ-matrix of rank n. Its invariant polynomials

$$i_1(\lambda) = \frac{D_n(\lambda)}{D_{n-1}(\lambda)}, \quad i_2(\lambda) = \frac{D_{n-1}(\lambda)}{D_{n-2}(\lambda)}, \ \dots, \ i_n(\lambda) = \frac{D_1(\lambda)}{D_0(\lambda)} \ (D_0(\lambda) \equiv 1), \tag{19}$$

are called the *invariant polynomials of the matrix* A and the corresponding elementary divisors in F are called the *elementary divisors of the matrix* A *in the field* F. A knowledge of the invariant polynomials (and, hence, of the elementary divisors) of A enables us to investigate its structure. Therefore practical methods of computing the invariant polynomials of a matrix are of interest. The formulas (19) give an algorithm for computing these polynomials, but for large n this algorithm is very cumbrous.

Theorem 3 gives another method of computing invariant polynomials, based on the reduction of the characteristic matrix (18) to canonical diagonal form by means of elementary operations.

Example:

$$A = \left\Vert \begin{matrix} 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 6 & 1 & 2 & 1 \\ -14 & -5 & -1 & 0 \end{matrix} \right\Vert , \qquad \lambda E - A = \left\Vert \begin{matrix} \lambda - 3 & -1 & 0 & 0 \\ 4 & \lambda + 1 & 0 & 0 \\ -6 & -1 & \lambda - 2 & -1 \\ 14 & 5 & 1 & \lambda \end{matrix} \right\Vert .$$

In the characteristic matrix $\lambda E - A$ we add to the fourth row the third multiplied by λ:

$$\left\Vert \begin{matrix} \lambda - 3 & -1 & 0 & 0 \\ 4 & \lambda + 1 & 0 & 0 \\ -6 & -1 & \lambda - 2 & -1 \\ 14 - 6\lambda & 5 - \lambda & \lambda^2 - 2\lambda + 1 & 0 \end{matrix} \right\Vert .$$

Now adding to the first three columns the fourth, multiplied by -6, -1, and $\lambda - 2$, respectively, we obtain

$$\left\Vert \begin{matrix} \lambda - 3 & -1 & 0 & 0 \\ 4 & \lambda + 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 14 - 6\lambda & 5 - \lambda & \lambda^2 - 2\lambda + 1 & 0 \end{matrix} \right\Vert .$$

We add to the first column the second multiplied by $\lambda - 3$:

$$\left\Vert \begin{matrix} 0 & -1 & 0 & 0 \\ \lambda^2 - 2\lambda + 1 & \lambda + 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -\lambda^2 + 2\lambda - 1 & 5 - \lambda & \lambda^2 - 2\lambda + 1 & 0 \end{matrix} \right\Vert .$$

To the second and fourth rows we add the first multiplied by $\lambda + 1$ and $5 - \lambda$, respectively; we obtain

$$\left\Vert \begin{matrix} 0 & -1 & 0 & 0 \\ \lambda^2 - 2\lambda + 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -\lambda^2 + 2\lambda - 1 & 0 & \lambda^2 - 2\lambda + 1 & 0 \end{matrix} \right\Vert .$$

To the second row we add the fourth; then we multiply the first and third rows by -1. After permuting some rows and columns we obtain:

$$\left\Vert \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\lambda - 1)^2 & 0 \\ 0 & 0 & 0 & (\lambda - 1)^2 \end{matrix} \right\Vert .$$

The matrix has two elementary divisors $(\lambda - 1)^2$ and $(\lambda - 1)^2$.

§ 4. Equivalence of Linear Binomials

1. In the preceding sections we have considered rectangular λ-matrices. In the present section we consider two square λ-matrices $A(\lambda)$ and $B(\lambda)$ of order n in which all the elements are of degree not higher than 1 in λ. These polynomial matrices may be represented in the form of matrix binomials:

$$A(\lambda) = A_0\lambda + A_1, \ B(\lambda) = B_0\lambda + B_1 .$$

We shall assume that these binomials are of degree 1 and regular, i.e., that $|A_0| \neq 0$, $|B_0| \neq 0$ (see p. 76).

The following theorem gives a criterion for the equivalence of such binomials:

THEOREM 6: *If two regular binomials of the first degree* $A_0\lambda + A_1$ and $B_0\lambda + B_1$ *are equivalent, then they are strictly equivalent, i.e., in the identity*

$$B_0\lambda + B_1 = P(\lambda)(A_0\lambda + A_1)Q(\lambda) \tag{20}$$

the matrices $P(\lambda)$ *and* $Q(\lambda)$*—with constant non-zero determinants—can be replaced by constant non-singular matrices* P *and* Q:[14]

$$B_0\lambda + B_1 = P(A_0\lambda + A_1)Q . \tag{21}$$

---

[14] The identity (21) is equivalent to the two matrix equations: $B_0 = PA_0Q$ and $B_1 = PA_1Q$.

Proof. Since the determinant of $P(\lambda)$ does not depend on λ and is different from zero,[15] the inverse matrix $M(\lambda) = P^{-1}(\lambda)$ is also a polynomial matrix. With the help of this matrix we write (20) in the form

$$M(\lambda)(B_0\lambda + B_1) = (A_0\lambda + A_1)Q(\lambda). \tag{22}$$

Regarding $M(\lambda)$ and $Q(\lambda)$ as matrix polynomials, we divide $M(\lambda)$ on the left by $A_0\lambda + A_1$ and $Q(\lambda)$ on the right by $B_0\lambda + B_1$:

$$M(\lambda) = (A_0\lambda + A_1)S(\lambda) + M, \tag{23}$$

$$Q(\lambda) = T(\lambda)(B_0\lambda + B_1) + Q; \tag{24}$$

here M and Q are constant square matrices (independent of λ) of order n. We substitute these expressions for $M(\lambda)$ and $Q(\lambda)$ in (22). After a few small transformations, we obtain

$$(A_0\lambda + A_1)[T(\lambda) - S(\lambda)](B_0\lambda + B_1) = M(B_0\lambda + B_1) - (A_0\lambda + A_1)Q. \tag{25}$$

The difference in the brackets must be identically equal to zero; for otherwise the product on the left-hand side of (25) would be of degree $\geqq 2$, while the polynomial on the right-hand side of the equation is of degree not higher than 1. Therefore

$$S(\lambda) = T(\lambda); \tag{26}$$

But then we obtain from (25):

$$M(B_0\lambda + B_1) = (A_0\lambda + A_1)Q. \tag{27}$$

We shall now show that M is a non-singular matrix. For this purpose we divide $P(\lambda)$ on the left by $B_0\lambda + B_1$:

$$P(\lambda) = (B_0\lambda + B_1)U(\lambda) + P. \tag{28}$$

From (22), (23), and (28) we deduce:

$$\begin{aligned} E = M(\lambda)P(\lambda) &= M(\lambda)(B_0\lambda + B_1)U(\lambda) + M(\lambda)P \\ &= (A_0\lambda + A_1)Q(\lambda)U(\lambda) + (A_0\lambda + A_1)S(\lambda)P + MP \\ &= (A_0\lambda + A_1)[Q(\lambda)U(\lambda) + S(\lambda)P] + MP. \end{aligned} \tag{29}$$

---

[15] The equivalence of the binomials $A_0\lambda + A_1$ and $B_0\lambda + B_1$ means that an identity (20) exists in which $|P(\lambda)| = \text{const.} \neq 0$ and $|Q(\lambda)| = \text{const.} \neq 0$. However, in this case the last relations follow from (20) itself. For the determinants of regular binomials of the first degree are of degree n:

$$|A_0\lambda + A_1| = |A_0|\lambda^n + \ldots, |B_0\lambda + B_1| = |B_0|\lambda^n + \ldots; |A_0| \neq 0, |B_0| \neq 0.$$

Therefore it follows from

$$|B_0\lambda + B_1| = |P(\lambda)||A_0\lambda + A_1||Q(\lambda)|$$

that

$$|P(\lambda)| = \text{const.} \neq 0, \ |Q(\lambda)| = \text{const.} \neq 0.$$

Since the last term of this chain of equations must be of degree zero in λ (because it is equal to E), the expression in brackets must be identically equal to zero. But then from (29)

$$MP = E, \tag{30}$$

so that $|M| \neq 0$ and $M^{-1} = P$.

Multiplying both sides of (27) on the left by P, we obtain:

$$B_0\lambda + B_1 = P(A_0\lambda + A_1)Q.$$

The fact that P is non-singular follows from (30). That P and Q are non-singular also follows directly from (21), since this identity implies

$$B_0 = PA_0Q$$

and therefore

$$|P||A_0|\,|Q| = |B_0| \neq 0.$$

This completes the proof of the theorem.

Note. From the proof it follows (see (24) and (28)) that the constant matrices P and Q by which we have replaced the λ-matrices $P(\lambda)$ and $Q(\lambda)$ in (20) can be taken as the left and right remainders, respectively, of $P(\lambda)$ and $Q(\lambda)$ on division by $B_0\lambda + B_1$.

§ 5. A Criterion for Similarity of Matrices

1. Let $A \| a_{ik} \|_1^n$ be a matrix with numerical elements from the field F. Its characteristic matrix $\lambda E - A$ is a λ-matrix of rank n and therefore has n invariant polynomials (see § 3)

$$i_1(\lambda), i_2(\lambda), \ldots, i_n(\lambda).$$

The following theorem shows that these invariant polynomials determine the original matrix A to within similarity transformations.

THEOREM 7: *Two matrices $A = \| a_{ik} \|_1^n$ and $B = \| b_{ik} \|_1^n$ are similar ($B = T^{-1}AT$) if and only if they have the same invariant polynomials or, what is the same, the same elementary divisors in the field* F.

Proof. The condition is *necessary.* For if the matrices A and B are similar, then there exists a non-singular matrix T such that

$$B = T^{-1}AT.$$

Hence

$$\lambda E - B = T^{-1}(\lambda E - A)T.$$

This equation shows that the characteristic matrices $\lambda E - A$ and $\lambda E - B$ are equivalent and therefore have the same invariant polynomials.

The condition is *sufficient*. Suppose that the characteristic matrices $\lambda E - A$ and $\lambda E - B$ have the same invariant polynomials. Then these λ-matrices are equivalent (see Corollary 1 to Theorem 3) and there exist, in consequence, two polynomial matrices $P(\lambda)$ and $Q(\lambda)$ such that

$$\lambda E - B = P(\lambda)(\lambda E - A)Q(\lambda). \tag{31}$$

Applying Theorem 6 to the matrix binomials $\lambda E - A$ and $\lambda E - B$, we may replace in (31) the λ-matrices $P(\lambda)$ and $Q(\lambda)$ by constant matrices P and Q:

$$\lambda E - B = P(\lambda E - A)Q; \tag{32}$$

moreover, P and Q may be taken (see the Note on p. 147) as the left remainder and the right remainder, respectively, of $P(\lambda)$ and $Q(\lambda)$ on division by $\lambda E - B$, i.e., by the Generalized Bézout Theorem, we may set:[16]

$$P = \hat{P}(B),\ Q = Q(B) \tag{33}$$

Equating coefficients of the powers of λ on both sides of (32), we obtain:

$$B = PAQ, \quad E = PQ,$$

i.e.,

$$B = T^{-1}AT,$$

where

$$T = Q = P^{-1}$$

This proves the theorem.

2. *Note.* We have incidentally established the following result, which we state separately:

SUPPLEMENT TO THEOREM 7. *If* $A = \| a_{ik} \|_1^n$ *and* $B = \| b_{ik} \|_1^n$ *are two similar matrices,*

$$B = T^{-1}AT, \tag{34}$$

then we can choose as the transforming matrix T the matrix

$$T = Q(B) = [\hat{P}(B)]^{-1}, \tag{35}$$

where $P(\lambda)$ and $Q(\lambda)$ are polynomial matrices in the identity

$$\lambda E - B = P(\lambda)(\lambda E - A)Q(\lambda)$$

which connects the equivalent characteristic matrices $\lambda E - A$ and $\lambda E - B$; in (35) *$Q(B)$ denotes the right value of the matrix polynomial $Q(\lambda)$, and $\hat{P}(B)$ the left value of $P(\lambda)$, when the argument is replaced by B.*

---

[16] We recall that $\hat{P}(B)$ is the left value of the polynomial $P(\lambda)$ and $Q(B)$ the right value of $Q(\lambda)$, when λ is replaced by B (see p. 81).

§ 6. The Normal Forms of a Matrix

1. Let

$$g(\lambda) = \lambda^m + \alpha_1\lambda^{m-1} + \cdots + \alpha_{m-1}\lambda + \alpha_m$$

be a polynomial with coefficients in F.

We consider the square matrix of order m

$$L = \left\| \begin{matrix} 0 & 0 & . & . & . & 0 & -\alpha_m \\ 1 & 0 & . & . & . & 0 & -\alpha_{m-1} \\ 0 & 1 & . & . & . & 0 & -\alpha_{m-2} \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 1 & -\alpha_1 \end{matrix} \right\|. \tag{36}$$

It is not difficult to verify that $g(\lambda)$ is the characteristic polynomial of L:

$$|\lambda E - L| = \left| \begin{matrix} \lambda & 0 & 0 & . & . & . & 0 & \alpha_m \\ -1 & \lambda & 0 & . & . & . & 0 & \alpha_{m-1} \\ 0 & -1 & \lambda & . & . & . & 0 & \alpha_{m-2} \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & -1 & \alpha_1 + \lambda \end{matrix} \right| = g(\lambda).$$

On the other hand, the minor of the element α_m in the characteristic determinant is equal to ± 1. Therefore $D_{m-1}(\lambda) = 1$ and $i_1(\lambda) = \frac{D_m(\lambda)}{D_{m-1}(\lambda)} = D_m(\lambda) = g(\lambda)$, $i_2(\lambda) = \ldots = i_m(\lambda) = 1$.

Thus, L has a single invariant polynomial different from 1, namely $g(\lambda)$.

We shall call L the *companion matrix* of the polynomial $g(\lambda)$.

Let $A = \| a_{ik} \|_1^n$ be a matrix with the invariant polynomials

$$i_1(\lambda),\ i_2(\lambda),\ \ldots,\ i_t(\lambda),\ i_{t+1}(\lambda) = 1,\ \ldots,\ i_n(\lambda) = 1. \tag{37}$$

Here the polynomials $i_1(\lambda), i_2(\lambda), \ldots, i_t(\lambda)$ have positive degrees and, from the second onwards, each divides the preceding one. We denote the companion matrices of these polynomials by $L_1, L_2, \ldots, L_t$.

Then the quasi-diagonal matrix of order n

$$L_{\mathrm{I}} = \{ L_1, L_2, \ldots, L_t \} \tag{38}$$

has the polynomials (37) as its invariant polynomials (see Theorem 4 on p. 141). Since the matrices A and L_{I} have the same invariant polynomials, they are similar, i.e., there always exists a non-singular matrix U ($|U| \neq 0$) such that

$$A = UL_{\mathrm{I}}U^{-1}. \tag{I}$$

The matrix L_{I} is called the *first natural normal form* of the matrix A. This normal form is characterized by: 1) the quasi-diagonal form (38), 2) the special structure of the diagonal blocks (36), and 3) the additional condition: in the sequence of characteristic polynomials of the diagonal blocks every polynomial from the second onwards divides the preceding one.[17]

2. We now denote by

$$\chi_1(\lambda), \chi_2(\lambda), \dots, \chi_u(\lambda) \tag{39}$$

the elementary divisors of $A = \| a_{ik} \|_1^n$ in the number field F. The corresponding companion matrices will be denoted by

$$L^{(1)}, L^{(2)}, \dots, L^{(u)}.$$

Since $\chi_j(\lambda)$ is the only elementary divisor of $L^{(j)}$ $(j = 1, 2, \dots, u)$,[18] the quasi-diagonal matrix

$$L_{\mathrm{II}} = \{ L^{(1)}, L^{(2)}, \dots, L^{(u)} \} \tag{40}$$

has, by Theorem 5, the polynomials (39) as its elementary divisors.

The matrices A and L_{II} have the same elementary divisors in F. Therefore the matrices are similar, i.e., there always exists a non-singular matrix V $(|V| \neq 0)$ such that

$$A = VL_{\mathrm{II}}V^{-1}. \tag{II}$$

The matrix L_{II} is called the *second natural normal form* of the matrix A. This normal form is characterized by: 1) the quasi-diagonal form (40), 2) the special structure of the diagonal blocks (36), and 3) the additional condition: the characteristic polynomial of each diagonal block is a power of an irreducible polynomial over F.

Note. The elementary divisors of a matrix A, in contrast to the invariant polynomials, are essentially connected with the given number field F. If we choose instead of the original field F another number field (which also contains the elements of the given matrix A), then the elementary divisors may change. Together with the elementary divisors, the second natural normal form of a matrix also changes.

---

[17] From the conditions 1), 2), 3) it follows automatically that the characteristic polynomials of the diagonal blocks in L_{I} are the invariant polynomials of the matrix L_{I} and, hence, of A.

[18] $\chi_j(\lambda)$ is the only invariant polynomial of $L^{(j)}$ and is at the same time a power of a polynomial irreducible over F.

Suppose, for example, that $A = \| a_{ik} \|_1^n$ is a matrix with real elements. The characteristic polynomial of the matrix then has real coefficients. But this polynomial may have complex roots. If F is the field of real numbers, then among the elementary divisors there may also be powers of irreducible quadratic trinomials with real coefficients. If F is the field of complex numbers, then every elementary divisor has the form $(\lambda - \lambda_0)^p$.

3. Let us assume now that the number field F contains not only the elements of A, but also the characteristic values of the matrix.[19] Then the elementary divisors of A have the form[20]

$$(\lambda - \lambda_1)^{p_1}, (\lambda - \lambda_2)^{p_2}, \dots, (\lambda - \lambda_u)^{p_u} \qquad (p_1 + p_2 + \cdots + p_u = n). \tag{41}$$

We consider one of these elementary divisors:

$$(\lambda - \lambda_0)^p$$

and associate with it the following matrix of order p:

$$\left\| \begin{matrix} \lambda_0 & 1 & 0 & \dots & 0 \\ 0 & \lambda_0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda_0 \end{matrix} \right\| = \lambda_0 E^{(p)} + H^{(p)}. \tag{42}$$

It is easy to verify that this matrix has only the one elementary divisor $(\lambda - \lambda_0)^p$. The matrix (42) will be called the *Jordan block* corresponding to the elementary divisor $(\lambda - \lambda_0)^p$.

The Jordan blocks corresponding to the elementary divisors (41) will be denoted by

$$J_1, J_2, \dots, J_u.$$

Then the quasi-diagonal matrix

$$J = \{ J_1, J_2, \dots, J_u \}$$

has the powers (41) as its elementary divisors.

The matrix J can also be written in the form

$$J = \{ \lambda_1 E_1 + H_1, \lambda_2 E_2 + H_2, \dots, \lambda_u E_u + H_u \};$$

where

$$E_k = E^{(p_k)}, \ H_k = H^{(p_k)} \qquad (k = 1, 2, \dots, u).$$

---

[19] This always holds for an arbitrary matrix A if F is the field of complex numbers.

[20] Among the numbers $\lambda_1, \lambda_2, \dots, \lambda_u$ there may be some that are equal.

Since the matrices A and J have the same elementary divisors, they are similar, i.e., there exists a non-singular matrix $T (|T| \neq 0)$ such that

$$A = TJT^{-1} = T\{\lambda_1 E_1 + H_1,\ \lambda_2 E_2 + H_2,\ \ldots,\ \lambda_u E_u + H_u\} T^{-1}. \qquad \text{(III)}$$

The matrix J is called the *Jordan normal form* or simply *Jordan form* of A. The Jordan normal form is characterized by its quasi-diagonal form and by the special structure (42) of the diagonal blocks.

The following scheme describes the Jordan matrix J for the elementary divisors $(\lambda - \lambda_1)^2$, $(\lambda - \lambda_2)^3$, $\lambda - \lambda_3$, $(\lambda - \lambda_4)^2$:

$$J = \begin{Vmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{Vmatrix}. \qquad (43)$$

If (and only if) all the elementary divisors of a matrix A are of the first degree, the Jordan form is a diagonal matrix, and in this case we have:

$$A = T\{\lambda_1, \lambda_2, \ldots, \lambda_n\} T^{-1}. \qquad (44)$$

Thus: *A matrix A has simple structure* (see Chapter III, § 8) *if and only if all its elementary divisors are of the first degree.*[21]

Instead of the Jordan block (42) sometimes the '*lower*' *Jordan block* of order p is used:

$$\begin{Vmatrix} \lambda_0 & 0 & . & . & . & 0 & 0 \\ 1 & \lambda_0 & . & . & . & 0 & 0 \\ 0 & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \lambda_0 & 0 \\ 0 & . & . & . & 0 & 1 & \lambda_0 \end{Vmatrix} = \lambda_0 E^{(p)} + F^{(p)}.$$

This matrix also has the single elementary divisor $(\lambda - \lambda_0)^p$ only. To the elementary divisors (41) there corresponds the *lower Jordan matrix*.[22]

---

[21] The elementary divisors of degree 1 are often called 'linear' or 'simple' elementary divisors.

[22] The matrix J is often called the *upper Jordan matrix*, in contrast to the lower Jordan matrix $J_{(1)}$.

$$J_{(1)} = \{\lambda_1 E_1 + F_1,\ \lambda_2 E_2 + F_2,\ \ldots,\ \lambda_u E_u + F_u\}$$
$$(E_k = E^{(p_k)},\ F_k = F^{(p_k)};\ k = 1, 2, \ldots, u).$$

An arbitrary matrix A having the elementary divisors (41) is always similar to $J_{(1)}$, i.e., there exists a non-singular matrix T_1 $(|T_1| \neq 0)$ such that

$$A = T_1 J_{(1)} T_1^{-1} = T_1\{\lambda_1 E_1 + F_1,\ \lambda_2 E_2 + F_2,\ \ldots,\ \lambda_u E_u + F_u\} T_1^{-1}. \quad \text{(IV)}$$

We also note that if $\lambda_0 \neq 0$, each of the two matrices

$$\lambda_0 (E^{(p)} + H^{(p)}), \qquad \lambda_0 (E^{(p)} + F^{(p)})$$

has only the single elementary divisor $(\lambda - \lambda_0)^p$. Therefore for a *non-singular* matrix A having the elementary divisors (41) we have, apart from (III) and (IV), the representations

$$A = T_2\{\lambda_1 (E_1 + H_1), \lambda_2 (E_2 + H_2), \ldots, \lambda_u (E_u + H_u)\} T_2^{-1}, \qquad \text{(V)}$$
$$A = T_3\{\lambda_1 (E_1 + F_1), \lambda_2 (E_2 + F_2), \ldots, \lambda_u (E_u + F_u)\} T_3^{-1}. \qquad \text{(VI)}$$

§ 7. The Elementary Divisors of the Matrix $f(A)$

1. In this section we consider the following problem:

Given the elementary divisors (in the field of complex numbers) of a matrix $A = \| a_{ik} \|_1^n$ *and given a function* $f(\lambda)$ *defined on the spectrum of* A, *to determine the elementary divisors (in the field of complex numbers) of the matrix* $f(A)$.

The matrix $f(A)$ does not alter if we replace the function $f(\lambda)$ by a polynomial that assumes on the spectrum of A the same values as $f(\lambda)$ (see Chapter V, § 1). Without loss of generality we may therefore assume in what follows that $f(\lambda)$ is a polynomial.

We denote by

$$(\lambda - \lambda_1)^{p_1},\ (\lambda - \lambda_2)^{p_2}, \ldots, (\lambda - \lambda_u)^{p_u}$$

the elementary divisors of A.[23] Thus A is similar to the Jordan matrix

$$A = TJT^{-1},$$

and so

$$f(A) = Tf(J)\,T^{-1}.$$

---

[23] Among the numbers $\lambda_1, \lambda_2, \ldots, \lambda_u$ there may be some that are equal.

Moreover,

$$J = \{ J_1, J_2, \ldots, J_u \}, \quad J_i = \lambda_i E^{(p_i)} + H^{(p_i)} \quad (i = 1, 2, \ldots, u)$$

and

$$f(J) = \{ f(J_1), f(J_2), \ldots, f(J_u) \}, \tag{45}$$

where (see Example 2 on p. 100)

$$f(J_i) = \left\| \begin{matrix} f(\lambda_i) & \frac{f'(\lambda_i)}{1!} & \cdots & \frac{f^{(p_i-1)}(\lambda_i)}{(p_i-1)!} \\ 0 & f(\lambda_i) & \ddots & \vdots \\ \vdots & & \ddots & \frac{f'(\lambda_i)}{1!} \\ 0 & 0 & \cdots & f(\lambda_i) \end{matrix} \right\| . \tag{46}$$

Since the similar matrices $f(A)$ and $f(J)$ have the same elementary divisors, we shall from now on consider $f(J)$ instead of $f(A)$.

2. Let us determine the defect[24] d of $f(A)$ or, what is the same, of $f(J)$. The defect of a quasi-diagonal matrix is equal to the sum of the defects of the various diagonal blocks and the defect of $f(J_i)$ (see 46)) is equal to the smaller of the numbers k_i and p_i, where k_i is the multiplicity of λ_i as a root of $f(\lambda)$,[25] so that

$$f(\lambda_i) = f'(\lambda_i) = \cdots = f^{(k_i-1)}(\lambda_i) = 0, \; f^{(k_i)}(\lambda_i) \neq 0 \qquad (i = 1, 2, \ldots, u).$$

We have thus arrived at the following theorem:

THEOREM 8: *The defect of the matrix $f(A)$, where A has the elementary divisors*

$$(\lambda - \lambda_1)^{p_1}, (\lambda - \lambda_2)^{p_2}, \ldots, (\lambda - \lambda_u)^{p_u} \tag{47}$$

is given by the formula

$$d = \sum_{i=1}^{u} \min(k_i, p_i); \tag{48}$$

---

[24] $d = n - r$, where r is the rank of $f(A)$. If the elementary divisors of a matrix are known, then the defect of the matrix is determined as the number of elementary divisors corresponding to the characteristic value 0, i.e., as the number of elementary divisors of the form λ^{ω}.

[25] k_i may be equal to zero; in that case $f(\lambda_i) \neq 0$.

here k_i is the multiplicity of λ_i as root of $f(\lambda)$ $(i = 1, 2, \ldots, u)$.[26]

As an application of this theorem we shall determine all the elementary divisors of an arbitrary matrix $A = \| a_{ik} \|_1^n$ that corresponds to a characteristic value λ_0:

$$\underbrace{\lambda - \lambda_0, \ldots, \lambda - \lambda_0}_{g_1}; \quad \underbrace{(\lambda - \lambda_0)^2, \ldots, (\lambda - \lambda_0)^2}_{g_2}; \quad \ldots; \quad \underbrace{(\lambda - \lambda_0)^m, \ldots, (\lambda - \lambda_0)^m}_{g_m},$$

where $g_i \geqq 0$ $(i = 1, 2, \ldots, m-1)$, $g_m > 0$, provided the defects

$$d_1, d_2, \ldots, d_m$$

of the matrices

$$A - \lambda_0 E, \; (A - \lambda_0 E)^2, \; \ldots, \; (A - \lambda_0 E)^m$$

are given.

For this purpose we note that $(A - \lambda_0 E)^j = f_j(A)$, where $f_j(\lambda) = (\lambda - \lambda_0)^j$ $(j = 1, 2, \ldots, m)$. In order to determine the defect of $(A - \lambda_0 E)^j$ we have, therefore, to set $k_i = j$ in (48) for the elementary divisors corresponding to the characteristic value λ_0 and $k_i = 0$ for all the other terms $(j = 1, 2, \ldots, m)$. Thus we obtain the formulas

$$\left. \begin{aligned} g_1 + g_2 + g_3 + \cdots + g_m &= d_1, \\ g_1 + 2g_2 + 2g_3 + \cdots + 2g_m &= d_2, \\ g_1 + 2g_2 + 3g_3 + \cdots + 3g_m &= d_3, \\ \cdots\cdots\cdots\cdots&\cdots\cdots \\ g_1 + 2g_2 + 3g_3 + \cdots + mg_m &= d_m. \end{aligned} \right\} \tag{49}$$

Hence[27]

$$g_j = 2d_j - d_{j-1} - d_{j+1} \qquad (j = 1, 2, \ldots, m; \; d_0 = 0, \; d_{m+1} = d_m). \tag{50}$$

3. Let us return to the basic problem of determining the elementary divisors of the matrix $f(A)$. As we have mentioned above, the elementary divisors of $f(A)$ coincide with those of $f(J)$ and the elementary divisors of a quasi-diagonal matrix coincide with those of the diagonal blocks (see Theorem 5). Therefore the problem reduces to finding the elementary divisors of a matrix C of regular triangular form:

---

[26] In the general case, where $f(\lambda)$ is not a polynomial, then $\min(k_i, p_i)$ in (48) has to be interpreted as the number p_i if

$$f(\lambda_i) = f'(\lambda_i) = \ldots = f^{(p_i)}(\lambda_i) = 0$$

and as the number $k_i \leqq p_i$ if

$$f(\lambda_i) = f'(\lambda_i) = \ldots = f^{(k_i-1)}(\lambda_i) = 0, \quad f^{(k_i)}(\lambda_i) \neq 0$$

$(i = 1, 2, \ldots, u)$.

[27] The number m is characterized by the fact that $d_{m-1} < d_m = d_{m+j}$ $(j = 1, 2, \ldots)$.

$$C=\sum_{k=0}^{p-1} a_k H^k = \begin{Vmatrix} a_0 & a_1 & \cdots & a_{p-1} \\ 0 & a_0 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & a_1 \\ 0 & 0 & \cdots & a_0 \end{Vmatrix}. \tag{51}$$

We consider separately two cases:

1. $a_1 \neq 0$. The characteristic polynomial of C is obviously equal to

$$D_p(\lambda)=(\lambda-a_0)^p.$$

Since $D_{p-1}(\lambda)$ divides $D_p(\lambda)$ without remainder, we have

$$D_{p-1}(\lambda)=(\lambda-a_0)^g \quad (g\leqq p).$$

Here $D_{p-1}(\lambda)$ denotes the greatest common divisor of the minors of order $p-1$ in the characteristic matrix

$$\lambda E - C = \begin{Vmatrix} \lambda-a_0 & -a_1 & \cdots & -a_{p-1} \\ 0 & \lambda-a_0 & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & -a_1 \\ \overset{+}{0} & 0 & \cdots & \lambda-a_0 \end{Vmatrix}.$$

It is easy to see that when the minor of the zero element marked by '+' is expanded, every term contains at least one factor $\lambda-a_0$, except the product of the elements on the main diagonal, which is $(-a_1)^{p-1}$ and is therefore in our case different from zero. But since $D_{p-1}(\lambda)$ must be a power of $\lambda-a_0$, we see that $g=0$. But then it follows from

$$D_p(\lambda)=(\lambda-a_0)^p, \quad D_{p-1}(\lambda)=1$$

that C has only the one elementary divisor $(\lambda-a_0)^p$.

2. $a_1=\ldots=a_{k-1}=0$, $a_k\neq 0$. In this case,

$$C=a_0E+a_kH^k+\cdots+a_{p-1}H^{p-1}.$$

Therefore for the positive integer j the defect of the matrix

$$(C-a_0E)^j=a_k^jH^{kj}+\cdots$$

is given by

$$d_j=\begin{cases} kj, & \text{when} \quad kj\leqq p,\\ p, & \text{when} \quad kj>p.\end{cases}$$

We set

$$p=qk+h \quad (0\leqq h<k). \tag{52}$$

Then[28]

$$d_1=k,\ d_2=2k,\ \ldots,\ d_q=qk,\ d_{q+1}=p. \tag{53}$$

Therefore we have by (50)

$$g_1=\cdots=g_{q-1}=0,\quad g_q=k-h,\ g_{q+1}=h.$$

Thus, the matrix C has the elementary divisors

$$\underbrace{(\lambda-a_0)^{q+1},\ \ldots,\ (\lambda-a_0)^{q+1}}_{h},\quad \underbrace{(\lambda-a_0)^{q},\ \ldots,\ (\lambda-a_0)^{q}}_{k-h}, \tag{54}$$

where the integers $q>0$ and $h\geqq 0$ are determined by (52).

4. Now we are in a position to ascertain what elementary divisors the matrix $f(J)$ has (see (45) and (46)). To each elementary divisor of A

$$(\lambda-\lambda_0)^p$$

there corresponds in $f(J)$ the diagonal cell

$$f(\lambda_0E+H)=\sum_{i=0}^{p-1}\frac{f^{(i)}(\lambda_0)}{i!}H^i=\begin{Vmatrix} f(\lambda_0) & \frac{f'(\lambda_0)}{1!} & \cdots & \frac{f^{(p-1)}(\lambda_0)}{(p-1)!} \\ 0 & f(\lambda_0) & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & & \frac{f'(\lambda_0)}{1!} \\ \vdots & & & \\ 0 & 0 & \cdots & f(\lambda_0) \end{Vmatrix}. \tag{55}$$

Clearly the problem reduces to finding the elementary divisors of a cell of the form (55). But the matrix (55) is of the regular triangular form (51), where

$$a_0=f(\lambda_0),\ a_1=f'(\lambda_0),\ a_2=\frac{f''(\lambda_0)}{2!},\ \ldots.$$

Thus we arrive at the theorem:

---

[28] In this case the number $q+1$ plays the role of m in (49) and (50).

THEOREM 9: *The elementary divisors of the matrix $f(A)$ are obtained from those of A in the following way: To an elementary divisor*

$$(\lambda - \lambda_0)^p \tag{56}$$

of A for $p = 1$ or for $p > 1$ and $f'(\lambda_0) \neq 0$ there corresponds a single elementary divisor

$$\left(\lambda - f(\lambda_0)\right)^p \tag{57}$$

of $f(A)$; for $p > 1$ and $f(\lambda_0) = \ldots = f^{(k-1)}(\lambda_0) = 0,\ f^{(k)}(\lambda_0) \neq 0\ (k < p)$ to the elementary divisor (56) *of A there correspond the following elementary divisors of $f(A)$:*

$$\underbrace{\left(\lambda - f(\lambda_0)\right)^{q+1}, \ldots, \left(\lambda - f(\lambda_0)\right)^{q+1}}_{h},\quad \underbrace{\left(\lambda - f(\lambda_0)\right)^{q}, \ldots, \left(\lambda - f(\lambda_0)\right)^{q}}_{k-h}, \tag{58}$$

where

$$p = qk + h, \quad 0 \leqq q, \quad 0 \leqq h < k;$$

finally, for $p > 1$, $f'(\lambda_0) = \ldots = f^{(p-1)}(\lambda_0) = 0$, to the elementary divisor (56) *there correspond p elementary divisors of the first degree of $f(A)$:*[29]

$$\lambda - f(\lambda_0), \ldots, \lambda - f(\lambda_0). \tag{59}$$

We note the following special cases of this theorem.

1. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic values of A, then $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$ are the characteristic values of $f(A)$.* (In both sequences each characteristic value is repeated as often as its multiplicity as a root of the characteristic equation indicates.)[30]

2. *If the derivative $f'(\lambda)$ is not zero on the spectrum of A,*[31] *then in going from A to $f(A)$ the elementary divisors are not 'split up' i.e., if A has the elementary divisors*

$$(\lambda - \lambda_1)^{p_1},\ (\lambda - \lambda_2)^{p_2}, \ldots, (\lambda - \lambda_n)^{p_n},$$

then $f(A)$ has the elementary divisors

$$\left(\lambda - f(\lambda_1)\right)^{p_1},\ \left(\lambda - f(\lambda_2)\right)^{p_2}, \ldots, \left(\lambda - f(\lambda_n)\right)^{p_n}.$$

---

[29] (57) is obtained from (58) by setting $k = 1$; (59) is obtained from (58) by setting $k = p$ or $k > p$.

[30] Statement 1. was established separately in Chapter IV, p. 84.

[31] I.e., $f'(\lambda_i) \neq 0$ for those λ_i that are multiple roots of the minimal polynomial.

§ 8. A General Method of Constructing the Transforming Matrix

In many problems in the theory of matrices and its applications it is sufficient to know the normal form into which a given matrix $A = \| a_{ik} \|_1^n$ can be carried by similarity transformations. The normal form is completely determined by the invariant polynomials of the characteristic matrix $\lambda E - A$. To find the latter, we can use the defining formulas (see (10) on p. 139) or the reduction of the characteristic matrix $\lambda E - A$ to canonical diagonal form by elementary transformations.

In some problems, however, it is necessary to know not only the normal form $\tilde{A}$ of the given matrix A, but also a non-singular transforming matrix T.

1. An immediate method of determining T consists in the following. The equation

$$A = T\tilde{A}T^{-1}$$

can be written as:

$$AT - T\tilde{A} = O.$$

This matrix equation in T is equivalent to a system of n^2 linear homogeneous equations in the n^2 unknown coefficients of T. The determination of a transforming matrix reduces to the solution of this system of n^2 equations. Moreover, we have to choose from the set of all solutions one for which $|T| \neq 0$. The existence of such a solution is certain, since A and $\tilde{A}$ have the same invariant polynomials.[32]

Note that whereas the normal form is uniquely determined by the matrix A,[33] for the transforming matrix T we always have an innumerable set of values that are given by

$$T = UT_1, \tag{60}$$

where T_1 is one of the transforming matrices and U is an arbitrary matrix that is permutable with A.[34]

---

[32] From this fact follows the similarity of $\tilde{A}$ and A.

[33] This statement is unconditionally true as regards the first natural normal form. As far as the second normal form or the Jordan normal form is concerned, they are uniquely determined to within the order of the diagonal blocks.

[34] The formula (60) may be replaced by

$$T = T_1V_1,$$

where V is an arbitrary matrix permutable with $\tilde{A}$.

The method proposed above for determining a transforming matrix T is simple enough in concept but of little use in practice, since it requires a great many computations (even for $n=4$ we have to solve 16 linear equations).

2. We proceed to explain a more efficient method of constructing the transforming matrix T. This method is based on the Supplement to Theorem 7 (p. 148). According to this, we can choose as the transforming matrix

$$T=Q(\tilde{A}), \tag{61}$$

provided

$$\lambda E-\tilde{A}=P(\lambda)(\lambda E-A)Q(\lambda).$$

The latter equation expresses the equivalence of the characteristic matrices $\lambda E-A$ and $\lambda E-\tilde{A}$. Here $P(\lambda)$ and $Q(\lambda)$ are polynomial matrices with constant non-zero determinants.

For the actual process of finding $Q(\lambda)$ we reduce the two λ-matrices $\lambda E-A$ and $\lambda E-\tilde{A}$ to canonical form by means of the corresponding elementary transformations

$$\{i_n(\lambda), i_{n-1}(\lambda), \dots, i_1(\lambda)\}=P_1(\lambda)(\lambda E-A)Q_1(\lambda) \tag{62}$$

$$\{i_n(\lambda), i_{n-1}(\lambda), \dots, i_1(\lambda)\}=P_2(\lambda)(\lambda E-\tilde{A})Q_2(\lambda) \tag{63}$$

where

$$Q_1(\lambda)=T_1T_2,\dots,T_{p_1},\quad Q_2(\lambda)=T_1^*T_2^*\dots T_{p_2}^*, \tag{64}$$

and where $T_1,\dots,T_{p_1}, T_1^*,\dots,T_{p_2}^*$ are the elementary matrices corresponding to the elementary operations on the columns of the λ-matrices $\lambda E-A$ and $\lambda E-\tilde{A}$. From (62), (63), and (64) it follows that

$$\lambda E-\tilde{A}=P(\lambda)(\lambda E-A)Q(\lambda),$$

where

$$Q(\lambda)=Q_1(\lambda)Q_2^{-1}(\lambda)=T_1T_2\cdots T_{p_1}T_{p_2}^{*-1}T_{p_2-1}^{*-1}\cdots T_1^{*-1}. \tag{65}$$

We can compute the matrix $Q(\lambda)$ by applying successively to the columns of the unit matrix E the elementary operations with the matrices $T_1,\dots,T_{p_1}, T_{p_2}^{*-1},\dots,T_1^{*-1}$. After this (in accordance with (61)) we replace the argument λ in $Q(\lambda)$ by the matrix $\tilde{A}$.

Example.

$$A=\begin{Vmatrix} 1 & -1 & 1 & -1\\ -3 & 3 & -5 & 4\\ 8 & -4 & 3 & -4\\ 15 & -10 & 11 & -11 \end{Vmatrix}.$$

Let us introduce a symbolic notation for the right elementary operations and the corresponding matrices (see pp. 130-131):

$$T'=[(c)\,i],\ T''=[i+(b(\lambda))\,j],\ T'''=[ij].$$

In transforming the characteristic matrix $\lambda E-A$ into normal diagonal form we shall at the same time keep a record of the elementary right operations to be performed, i.e., the operations on the columns:

$$\lambda E-A=\begin{Vmatrix} \lambda-1 & 1 & -1 & 1\\ 3 & \lambda-3 & 5 & -4\\ -8 & 4 & \lambda-3 & 4\\ -15 & 10 & -11 & \lambda+11 \end{Vmatrix},\quad \begin{Vmatrix} 0 & 0 & 0 & 1\\ 4\lambda-1 & \lambda+1 & 1 & -4\\ -4\lambda-4 & 0 & \lambda+1 & 4\\ -\lambda^2-10\lambda-4 & -\lambda-1 & \lambda & \lambda+11 \end{Vmatrix},$$

$$\begin{Vmatrix} 1 & 0 & 0 & 0\\ -4 & \lambda+1 & 1 & 4\lambda-1\\ 4 & 0 & \lambda+1 & -4\lambda-4\\ \lambda+11 & -\lambda-1 & \lambda & -\lambda^2-10\lambda-4 \end{Vmatrix};$$

$$\begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & \lambda+1 & 1 & 4\lambda-1\\ 0 & 0 & \lambda+1 & -4\lambda-4\\ 0 & -\lambda-1 & \lambda & -\lambda^2-10\lambda-4 \end{Vmatrix},\quad \begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -\lambda^2-2\lambda-1 & \lambda+1 & -4\lambda^2-7\lambda-3\\ 0 & -\lambda^2-2\lambda-1 & \lambda & -5\lambda^2-9\lambda-4 \end{Vmatrix},$$

$$\begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & \lambda+1 & -\lambda^2-2\lambda-1 & -4\lambda^2-7\lambda-3\\ 0 & \lambda & -\lambda^2-2\lambda-1 & -5\lambda^2-9\lambda-4 \end{Vmatrix};$$

$$\begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lambda^2+2\lambda+1 & 4\lambda^2+7\lambda+3\\ 0 & 0 & \lambda^2+2\lambda+1 & 5\lambda^2+9\lambda+4 \end{Vmatrix},\quad \begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lambda^2+2\lambda+1 & -\lambda^2-3\lambda-2\\ 0 & 0 & \lambda^2+2\lambda+1 & -\lambda-1 \end{Vmatrix},$$

$$\begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -\lambda^2-2\lambda-1 & \lambda+1\\ 0 & 0 & \lambda^2+2\lambda+1 & -\lambda^2-3\lambda-2 \end{Vmatrix};$$

$$\begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lambda+1 & -\lambda^2-2\lambda-1\\ 0 & 0 & -\lambda^2-3\lambda-2 & \lambda^2+2\lambda+1 \end{Vmatrix},\quad \begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lambda+1 & 0\\ 0 & 0 & -\lambda^2-3\lambda-2 & -(\lambda+1)^2 \end{Vmatrix},$$

$$\begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lambda+1 & 0\\ 0 & 0 & 0 & (\lambda+1)^3 \end{Vmatrix}.$$

Here

$$Q_1(\lambda)=[1+(1-\lambda)\,4]\,[2-4]\,[3+4]\,[14]\,[2-(\lambda+1)\,3]\,[4+(1-4\lambda)\,3]\,[23]\times$$
$$\times[4-(5)\,3]\,[43]\,[4+(\lambda+1)\,3].$$

We have found the invariant polynomials $(\lambda+1)^3$, $(\lambda+1)$, 1, and 1 of A. The matrix has two elementary divisors, $(\lambda+1)^3$ and $(\lambda+1)$. Therefore the Jordan normal form is

$$J=\begin{Vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}.$$

By elementary operations we bring the matrix $\lambda E-J$ into normal diagonal form

$$\lambda E-J=\begin{Vmatrix} \lambda+1 & -1 & 0 & 0 \\ 0 & \lambda+1 & -1 & 0 \\ 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & \lambda+1 \end{Vmatrix}, \quad \begin{Vmatrix} \lambda+1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & (\lambda+1)^2 & \lambda+1 & 0 \\ 0 & 0 & 0 & \lambda+1 \end{Vmatrix},$$

$$\begin{Vmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ (\lambda+1)^3 & (\lambda+1)^2 & \lambda+1 & 0 \\ 0 & 0 & 0 & \lambda+1 \end{Vmatrix};$$

$$\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ (\lambda+1)^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda+1 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (\lambda+1)^3 \\ 0 & 0 & \lambda+1 & 0 \end{Vmatrix},$$

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & (\lambda+1)^3 \end{Vmatrix}.$$

Here

$$Q_2(\lambda)=[2+(\lambda+1)3]\ [1+(\lambda+1)2]\ [12]\ [23]\ [34].$$

Therefore

$$\begin{aligned} Q(\lambda) &= Q_1(\lambda)\,Q_2^{-1}(\lambda) \\ &= [1+(1-\lambda)4]\,[2-4]\,[3+4]\,[14]\,[2-(\lambda+1)3]\,[4+(1-4\lambda)3]\,[23]\,[4-(5)3]\times \\ &\quad \times[43]\,[4+(\lambda+1)3]\,[34]\,[23]\,[12]\,[1-(\lambda+1)2]\,[2-(\lambda+1)3]. \end{aligned}$$

We apply these elementary operations successively to the unit matrix E:

$$E=\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-\lambda & -1 & 1 & 1 \end{Vmatrix},$$

$$\begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1-\lambda \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda-1 & 1 & 0 \\ 1 & -\lambda-2 & 1 & 1-\lambda \end{Vmatrix};$$

$$\begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda-1 & 1 & 1-4\lambda \\ 1 & -\lambda-2 & 1 & 2-5\lambda \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -\lambda-1 & 1-4\lambda \\ 1 & 1 & -\lambda-2 & 2-5\lambda \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -5 \\ 0 & 1 & -\lambda-1 & \lambda+6 \\ 1 & 1 & -\lambda-2 & 12 \end{Vmatrix};$$

$$\begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 1 \\ 0 & 1 & \lambda+6 & -\lambda-1 \\ 1 & 1 & 12 & -\lambda-2 \end{Vmatrix}, \quad \begin{Vmatrix} 0 & 0 & 1 & \lambda+1 \\ 0 & 0 & -5 & -5\lambda-4 \\ 0 & 1 & \lambda+6 & \lambda^2+6\lambda+5 \\ 1 & 1 & 12 & 11\lambda+10 \end{Vmatrix}, \quad \begin{Vmatrix} \lambda+1 & 0 & 0 & 1 \\ -5\lambda-4 & 0 & 0 & -5 \\ \lambda^2+6\lambda+5 & 0 & 1 & \lambda+6 \\ 11\lambda+10 & 1 & 1 & 12 \end{Vmatrix},$$

$$\begin{Vmatrix} \lambda+1 & 0 & 0 & 1 \\ -5\lambda-4 & 0 & 0 & -5 \\ \lambda^2+6\lambda+5 & -\lambda-1 & 1 & \lambda+6 \\ 10\lambda+9 & -\lambda & 1 & 12 \end{Vmatrix}.$$

Thus

$$Q(\lambda)=\begin{Vmatrix} \lambda+1 & 0 & 0 & 1 \\ -5\lambda-4 & 0 & 0 & -5 \\ \lambda^2+6\lambda+5 & -\lambda-1 & 1 & \lambda+6 \\ 10\lambda+9 & -\lambda & 1 & 12 \end{Vmatrix}$$

$$=\begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}\lambda^2+\begin{Vmatrix} 1 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ 6 & -1 & 0 & 1 \\ 10 & -1 & 0 & 0 \end{Vmatrix}\lambda+\begin{Vmatrix} 1 & 0 & 0 & 1 \\ -4 & 0 & 0 & -5 \\ 5 & -1 & 1 & 6 \\ 9 & 0 & 1 & 12 \end{Vmatrix}.$$

Observing that

$$J^2=\begin{Vmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

we have

$$T=Q(J)=\begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}\cdot\begin{Vmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}+$$

$$+\begin{Vmatrix} 1 & 0 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ 6 & -1 & 0 & 1 \\ 12 & -1 & 0 & 0 \end{Vmatrix}\cdot\begin{Vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}+\begin{Vmatrix} 1 & 0 & 0 & 1 \\ -4 & 0 & 0 & -5 \\ 5 & -1 & 1 & 6 \\ 9 & 0 & 1 & 12 \end{Vmatrix}$$

$$=\begin{Vmatrix} 0 & 1 & 0 & 1 \\ 1 & -5 & 0 & -5 \\ 0 & 4 & 1 & 5 \\ -1 & 11 & 0 & 12 \end{Vmatrix}.$$

Check:

$$AT=\left\|\begin{matrix}0 & -1 & 1 & -1\\ -1 & 6 & -5 & 5\\ 0 & -4 & 3 & -5\\ 1 & -12 & 11 & -12\end{matrix}\right\|,\quad TJ=\left\|\begin{matrix}0 & -1 & 1 & -1\\ -1 & 6 & -5 & 5\\ 0 & -4 & 3 & -5\\ 1 & -12 & 11 & -12\end{matrix}\right\|,$$

i.e., $AT=TJ$.

$$|T|=\left|\begin{matrix}0 & 1 & 0 & 1\\ 1 & -5 & 0 & -5\\ 0 & 4 & 1 & 5\\ -1 & 11 & 0 & 12\end{matrix}\right|=-1\neq 0.$$

Therefore

$$A=TJT^{-1}.$$

§ 9. Another Method of Constructing a Transforming Matrix

1. We shall now explain another method of constructing a transforming matrix which often leads to fewer computations than the method of the preceding section. However, we shall apply this second method only when the Jordan normal form and the elementary divisors

$$(\lambda-\lambda_1)^{p_1},\ (\lambda-\lambda_2)^{p_2},\dots \tag{66}$$

of the given matrix A are known.

Let $A=TJT^{-1}$, where

$$J=\{\lambda_1E^{(p_1)}+H^{(p_1)},\ \lambda_2E^{(p_2)}+H^{(p_2)},\dots\}=\left\|\begin{array}{ccc}\overbrace{\begin{matrix}\lambda_1 & 1 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & 1\\ 0 & \cdots & & \lambda_1\end{matrix}}^{p_1} & & \\ & \overbrace{\begin{matrix}\lambda_2 & 1 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & 1\\ 0 & \cdots & & \lambda_2\end{matrix}}^{p_2} & \\ & & \ddots\end{array}\right\|.$$

Then denoting the k-th column of T by t_k $(k=1,2,\dots,n)$, we replace the matrix equation

$$AT=TJ$$

by the equivalent system of equations

$$At_1=\lambda_1t_1,\ At_2=\lambda_1t_2+t_1,\dots,A\,t_{p_1}=\lambda_1t_{p_1}+t_{p_1-1} \tag{67}$$

$$At_{p_1+1}=\lambda_2t_{p_1+1},\ At_{p_1+2}=\lambda_2t_{p_1+2}+t_{p_1+1},\dots,\ At_{p_1+p_2}=\lambda_2t_{p_1+p_2}+t_{p_1+p_2-1} \tag{68}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

which we rewrite as follows:

$$(A-\lambda_1E)\,t_1=O,\ (A-\lambda_1E)\,t_2=t_1,\dots,(A-\lambda_1E)\,t_{p_1}=t_{p_1-1} \tag{67'}$$

$$(A-\lambda_2E)\,t_{p_1+1}=O,\ (A-\lambda_2E)\,t_{p_1+2}=t_{p_1+1},\dots,(A-\lambda_2E)\,t_{p_1+p_2}=t_{p_1+p_2-1} \tag{68'}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Thus, all the columns of T are split into 'Jordan chains' of columns: $[t_1,t_2,\dots,t_{p_1}]$, $[t_{p_1+1},t_{p_1+2},\dots,t_{p_1+p_2}]$, ...

To every Jordan block of J (or, what is the same, to every elementary divisor (66)) there corresponds its Jordan chain of columns. Each Jordan chain of columns is characterized by a system of equations of type (67), (68), etc.

The task of finding a transforming matrix T reduces to that of finding the Jordan chains that would give in all n linearly independent columns.

We shall show that these Jordan chains of columns can be determined by means of the reduced adjoint matrix $C(\lambda)$ (see Chapter IV, § 6).

For the matrix $C(\lambda)$ we have the identity

$$(\lambda E-A)\,C(\lambda)=\psi(\lambda)\,E. \tag{69}$$

where $\psi(\lambda)$ is the minimal polynomial of A.

Let

$$\psi(\lambda)=(\lambda-\lambda_0)^m\chi(\lambda)\qquad(\chi(\lambda_0)\neq 0).$$

We differentiate the identity (69) term by term $m-1$ times:

$$\left.\begin{matrix}(\lambda E-A)\,C'(\lambda)+C(\lambda)=\psi'(\lambda)\,E\\ (\lambda E-A)\,C''(\lambda)+2C'(\lambda)=\psi''(\lambda)\,E\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ (\lambda E-A)\,C^{(m-1)}(\lambda)+(m-1)\,C^{(m-2)}(\lambda)=\psi^{(m-1)}(\lambda)\,E.\end{matrix}\right\} \tag{70}$$

Substituting λ_0 for λ in (69) and (70) and observing that the right-hand sides are zero, we obtain

$$(A-\lambda_0E)\,C=O,\ (A-\lambda_0E)\,D=C,\ (A-\lambda_0E)\,F=D,\dots,(A-\lambda_0E)K=G; \tag{71}$$

where

$$\left.\begin{matrix}C=C(\lambda_0),\ D=\frac{1}{1!}C'(\lambda_0),\ F=\frac{1}{2!}C''(\lambda_0),\dots,G=\frac{1}{(m-2)!}C^{(m-2)}(\lambda_0)\\ K=\frac{1}{(m-1)!}C^{(m-1)}(\lambda_0).\end{matrix}\right\} \tag{72}$$

In (71) we replace the matrices (72) by their k-th columns ($k=1, 2, \ldots, n$). We obtain:

$$(A-\lambda_0E)\,C_k=o\,,\ (A-\lambda_0E)\,D_k=C_k\,,\ldots,(A-\lambda_0E)\,K_k=G_k \qquad (73)$$
$$(k=1\,,\,2,\,\ldots,\,n)\,.$$

Since $C=C(\lambda_0)\neq O$.[35] we can find a k ($\leqq n$) such that

$$C_k\neq o\,. \qquad (74)$$

Then the m columns

$$C_k\,,\,D_k\,,\,F_k\,,\,\ldots,\,G_k\,,\,K_k \qquad (75)$$

are linearly independent. For let

$$\gamma C_k+\delta D_k+\ldots+\varkappa K_k=o\,. \qquad (76)$$

Multiplying both sides of (76) successively by $A-\lambda_0E, \ldots, (A-\lambda_0E)^{m-1}$, we obtain

$$\delta C_k+\ldots+\varkappa G_k=o\,,\ldots,\varkappa C_k=o\,. \qquad (77)$$

From (76) and (77) we find by (74):

$$\gamma=\delta=\ldots=\varkappa=0\,.$$

Since the linearly independent columns (75) satisfy the system of equations (73), they form a Jordan chain of vectors corresponding to the elementary divisor $(\lambda-\lambda_0)^m$ (compare (73) with (67')).

If $C_k=o$ for some k, but $D_k\neq o$, then the columns $D_k, \ldots, G_k, K_k$ form a Jordan chain of $m-1$ vectors, etc.

2. We shall now show first of all how to construct a transforming matrix T in the case where the elementary divisors of A are pairwise co-prime:

$$(\lambda-\lambda_1)^{m_1}, \ldots, (\lambda-\lambda_s)^{m_s},$$
$$(\lambda_i\neq\lambda_j \text{ for } i\neq j\,;\, i, j=1, 2, \ldots, s)\,.$$

With the elementary divisor $(\lambda-\lambda_j)^{m_j}$ we associate the Jordan chain of columns

$$C^{(j)},\,D^{(j)},\ \ldots\ ,G^{(j)},\,K^{(j)},$$

constructed as indicated above. Then

$$(A-\lambda_jE)\,C^{(j)}=o,\ (A-\lambda_jE)\,D^{(j)}=C^{(j)},\ldots,(A-\lambda_jE)\,K^{(j)}=G^{(j)}\,. \qquad (78)$$

When we give to j the values $1, 2, \ldots, s$, we obtain s Jordan chains containing n columns in all. These columns are linearly independent.

---

[35] From $C(\lambda_0)=O$ it would follow that all the elements of $C(\lambda)$ have a common divisor of positive degree, in contradiction to the definition of $C(\lambda)$.

For, suppose that

$$\sum_{j=1}^{s}[\gamma_jC^{(j)}+\delta_jD^{(j)}+\ldots+\varkappa_jK^{(j)}]=o\,. \qquad (79)$$

We multiply both sides of (79) on the left by

$$(A-\lambda_1E)^{m_1}\cdots(A-\lambda_{j-1}E)^{m_{j-1}}(A-\lambda_jE)^{m_j-1}(A-\lambda_{j+1}E)^{m_{j+}}\cdots(A-\lambda_sE)^{m_s} \qquad (80)$$

and obtain

$$\varkappa_j=0\,.$$

Replacing m_j-1 successively by m_j-2, $m_j-3, \ldots$ in (80), we find:

$$\gamma_j=\delta_j=\ldots=\varkappa_j=0 \quad (j=1\,,\,2,\,\ldots,\,s)\,,$$

and this is what we had to prove.

We define the matrix T by the formula

$$T=(C^{(1)}\,,\,D^{(1)},\,\ldots,\,K^{(1)}\,;\,C^{(2)},\,D^{(2)},\,\ldots,\,K^{(2)}\,;\,\ldots\,;\,C^{(s)}\,,\,D^{(s)},\,\ldots,\,K^{(s)})\,. \qquad (81)$$

Example.

$$\begin{array}{l}
\qquad\begin{array}{rrrr} \mathit{8} & \mathit{3} & -\mathit{10} & -\mathit{3} \end{array}\\
A=\left\|\begin{array}{rrrr} 3 & -1 & -4 & 2\\ 2 & 3 & -2 & -4\\ 2 & -1 & -3 & 2\\ 1 & 2 & -1 & -3 \end{array}\right\|;\\
\qquad\begin{array}{rrrr} \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\\ 3 & 2 & 2 & 1\\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\\ 1 & 4 & 0 & 2 \end{array}
\end{array}$$

$\psi(\lambda)=\Delta(\lambda)=(\lambda-1)^2(\lambda+1)^2=\lambda^4-2\lambda^2+1\,.$

elementary divisors: $(\lambda-1^2)\,,\,(\lambda+1)^2\,,$

$$\Psi(\lambda,\mu)=\frac{\psi(\mu)-\psi(\lambda)}{\mu-\lambda}=\mu^3+\lambda\mu^2+(\lambda^2-2)\,\mu+\lambda^3-2\lambda,$$

$$C(\lambda)=\Psi(\lambda E\,,\,A)=A^3+\lambda A^2+(\lambda^2-2)\,A+(\lambda^3-2\lambda)\,E\,.$$

We make up the first column $C_1(\lambda)$:

$$C_1(\lambda)=[A^3]_1+\lambda\,[A^2]_1+(\lambda^2-2)\,A_1+(\lambda^3-2\lambda)\,E_1\,.$$

For the computation of the first column of A^2 we multiply all the rows of A into the first column of A. We obtain:[36] $[A^2]_1=(1, 4, 0, 2)$. Multiplying all the rows of A into this column, we find: $[A^3]_1=(3, 6, 2, 3)$.

Therefore

$$C_1(\lambda)=\left\|\begin{array}{c}3\\6\\2\\3\end{array}\right\|+\lambda\left\|\begin{array}{c}1\\4\\0\\2\end{array}\right\|+(\lambda^2-2)\left\|\begin{array}{c}3\\2\\2\\1\end{array}\right\|+(\lambda^3-2\lambda)\left\|\begin{array}{c}1\\0\\0\\0\end{array}\right\|=\left\|\begin{array}{c}\lambda^3+3\lambda^2-\lambda-3\\2\lambda^2+4\lambda+2\\2\lambda^2-2\\\lambda^2+2\lambda+1\end{array}\right\|$$

---

[36] The columns into which we multiply the rows are written underneath the rows of A. The elements of the row of column-sums are set up in italics, for checking.

Hence $C_1(1) = (0, 8, 0, 4)$ and $C_1'(1) = (8, 8, 4, 4)$. As $C_1(-1) = (0, 0, 0, 0)$, we pass on to the second column and, proceeding as before, we find: $C_2(-1) = (-4, 0, -4, 0)$ and $C_2'(-1) = (4, -4, 4, -4)$. We set up the matrix:

$$(C_1(1), C_1'(1); C_2(-1), C_2'(-1)) = \left\| \begin{matrix} 0 & 8 & -4 & 4 \\ 8 & 8 & 0 & -4 \\ 0 & 4 & -4 & 4 \\ 4 & 4 & 0 & -4 \end{matrix} \right\|.$$

We cancel[37] 4 in the first two columns and -4 in the last two columns.

$$T = \left\| \begin{matrix} 0 & 2 & 1 & -1 \\ 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{matrix} \right\|.$$

We leave it to the reader to verify that

$$AT = T \cdot \left\| \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{matrix} \right\|$$

3. Coming now to the general case, we shall investigate the Jordan chains of vectors corresponding to a characteristic value λ_0 for which there are p elementary divisors $(\lambda - \lambda_0)^m$, q elementary divisors $(\lambda - \lambda_0)^{m-1}$, r elementary divisors $(\lambda - \lambda_0)^{m-2}$, etc.

As a preliminary to this, we establish some properties of the matrices

$$C = C(\lambda_0),\ D = C'(\lambda_0),\ F = \frac{1}{2!} C''(\lambda_0),\ \ldots,\ K = \frac{1}{(m-1)!} C^{(m-1)}(\lambda_0). \tag{82}$$

1. *The matrices* (82) *can be represented in the form of polynomials in* A:

$$C = h_1(A),\ D = h_2(A),\ \ldots,\ K = h_m(A), \tag{83}$$

where

$$h_i(\lambda) = \frac{\psi(\lambda)}{(\lambda - \lambda_0)^i} \quad (i = 1, 2, \ldots, m). \tag{84}$$

For

$$C(\lambda) = \Psi(\lambda E, A),$$

where

$$\Psi(\lambda, \mu) = \frac{\psi(\mu) - \psi(\lambda)}{\mu - \lambda}.$$

---

[37] A Jordan chain remains a Jordan chain when all its columns are multiplied by a number $c \neq 0$.

Therefore

$$\frac{1}{k!} C^{(k)}(\lambda_0) = \frac{1}{k!} \Psi^{(k)}(\lambda_0 E, A), \tag{85}$$

where

$$\begin{aligned} \frac{1}{k!} \psi^{(k)}(\lambda_0, \mu) &= \frac{1}{k!} \left[\frac{\partial^k}{\partial \lambda^k} \Psi(\lambda, \mu) \right]_{\lambda = \lambda_0} \\ &= \frac{1}{k!} \left[\frac{\partial^k}{\partial \lambda^k} \frac{\psi(\mu)}{\mu - \lambda} \right]_{\lambda = \lambda_0} = \frac{\psi(\mu)}{(\mu - \lambda_0)^{k+1}} = h_{k+1}(\mu). \end{aligned} \tag{86}$$

(83) follows from (82), (85), and (86).

2. *The matrices* (82) *have the ranks*

$$p,\ 2p + q,\ 3p + 2q + r,\ \ldots.$$

This property of the matrices (82) follows immediately from 1. and Theorem 8 (§ 7). if we equate the rank to $n - d$ and use formula (48) for the defect of a function on A (p. 154).

3. *In the sequence of matrices* (82) *every column of each matrix is a linear combination of the columns of every following matrix.*

Let us take two matrices $h_i(A)$ and $h_k(A)$ in (82) (see 1.). Suppose that $i < k$. Then it follows from (84) that:

$$h_i(A) = h_k(A)\,(A - \lambda_0 E)^{k-i}.$$

Hence the j-th column y_j $(j = 1, 2, \ldots, n)$ of $h_i(A)$ is expressed linearly by the columns $z_1, z_2, \ldots, z_n$ of $h_k(A)$:

$$y_j = \sum_{g=1}^{n} \alpha_g z_g,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the elements of the j-th column of $(A - \lambda_0 E)^{k-i}$.

4. *Without changing the basic formulas* (71) *we may replace any column in* C *by an arbitrary linear combination of all the columns, provided we make the corresponding replacements in* $D, \ldots, K$.

We now proceed to the construction of the Jordan chains of columns for the elementary divisors

$$\underbrace{(\lambda - \lambda_0)^m, \ldots, (\lambda - \lambda_0)^m}_{p};\ \underbrace{(\lambda - \lambda_0)^{m-1}, \ldots, (\lambda - \lambda_0)^{m-1}}_{q};\ \ldots.$$

Using the properties 2. and 4., we transform the matrix C into the form

$$C = (C_1, C_2, \ldots, C_p; o, o, \ldots, o); \tag{87}$$

where the columns $C_1, C_2, \ldots, C_p$ are linearly independent. Now

$$D = (D_1, D_2, \ldots, D_p;\ D_{p+1}, \ldots, D_n).$$

By 3., for every i $(1 \leqq i \leqq p)$ C_i is a linear combination of the columns $D_1, D_2, \ldots, D_n$:

$$C_i = \alpha_1 D_1 + \cdots + \alpha_p D_p + \alpha_{p+1} D_{p+1} + \cdots + \alpha_n D_n. \tag{88}$$

We multiply both sides of this equation by $A - \lambda_0 E$. Observing (see (73)) that

$$(A - \lambda_0 E)\, C_i = o \quad (i = 1, 2, \ldots, p), \quad (A - \lambda_0 E)\, D_j = C_j \quad (j = 1, 2, \ldots, n),$$

we obtain by (87)

$$o = \alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_p C_p;$$

hence in (88)

$$\alpha_1 = \cdots = \alpha_p = 0.$$

Therefore the columns $C_1, C_2, \ldots, C_p$ are linearly independent combinations of the columns $D_{p+1}, \ldots, D_n$. Therefore by 4. and 2., we can, without changing the matrix C, take the columns $C_1, \ldots, C_p$ instead of $D_{p+1}, \ldots, D_{2p}$ and zeros instead of $D_{2p+q+1}, \ldots, D_n$.

Then the matrix D assumes the form

$$D = (D_1, \ldots, D_p;\ C_1, C_2, \ldots, C_p;\ D_{2p+1}, \ldots, D_{2p+q};\ o, o, \ldots, o). \tag{89}$$

In the same way, preserving the forms (87) and (89) of the matrices C and D, we can represent the next matrix F in the form

$$\left.\begin{aligned} F = (&F_1, \ldots, F_p;\ D_1, \ldots, D_p;\ F_{2p+1}, F_{2p+q};\ C_1, \ldots, C_p;\\ &D_{2p+1}, \ldots, D_{2p+q}, F_{3p+2q+1}, \ldots, F_{3p+2q+r}, o, \ldots, o), \end{aligned}\right\} \tag{90}$$

etc.

Formulas (73) gives us the Jordan chains

$$\left.\begin{aligned} &\underbrace{\overbrace{(C_1, D_1, \ldots, K_1)}^{m}, \ldots, \overbrace{(C_p, D_p, \ldots, K_p)}^{m};}_{p}\\ &\underbrace{\overbrace{(D_{2p+1}, F_{2p+1}, \ldots, K_{2p+1},)}^{m-1} \ldots, \overbrace{(D_{2p+q}, F_{2p+q}, \ldots, K_{2p+q})}^{m-1};\ \ldots}_{q} \end{aligned}\right\} \tag{91}$$

These Jordan chains are linearly independent. For all the columns C_i in (91) are linearly independent, because they form p linearly independent columns of C. All the columns C_i, D_j in (91) are independent, because they form $2p + q$ independent columns in D, etc.; finally, *all the columns in* (91) *are independent, because they form* $n_0 = mp + (m-1)q + \ldots$ *independent columns in* K. The number of columns in (91) is equal to the sum of the exponents of the elementary divisors corresponding to the given characteristic value λ_0.

Suppose that the matrix $A = \| a_{ik} \|_1^n$ has s distinct characteristic values λ_j

$$(j = 1, 2, \ldots, s;$$

$$\Delta(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$$

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_s)^{m_s}\ ).$$

For each characteristic value λ_j we form its system of independent Jordan chains (91); the number of columns in this system is equal to n_j $(j = 1, 2, 3, \ldots, s)$. All the chains so obtained contain $n = n_1 + n_2 + \ldots + n_s$ columns.

These n columns are linearly independent and form one of the required transforming matrices T.

The proof of the linear independence of these n columns proceeds as follows.

Every linear combination of these n columns can be represented in the form

$$\sum_{j=1}^{s} H_j = o, \tag{92}$$

where H_j is a linear combination of columns in the Jordan chains (91) corresponding to the characteristic value λ_j $(j = 1, 2, \ldots, s)$. But every column in the Jordan chain corresponding to the characteristic value λ_j satisfies the equation

$$(A - \lambda_j E)^{m_j} x = o.$$

Therefore

$$(A - \lambda_j E)^{m_j} H_j = o. \tag{93}$$

We take a fixed number j $(1 \leqq j \leqq s)$ and construct the Lagrange-Sylvester interpolation polynomial $r(\lambda)$ (See Chapter V, §§ 1, 2) with the following values on the spectrum of the matrix:

$$r(\lambda_i) = r'(\lambda_i) = \ldots = r^{(m_i - 1)}(\lambda_i) = 0 \text{ for } i \neq j$$

and

$$r(\lambda_j) = 1,\ r'(\lambda_j) = \cdots = r^{(m_j - 1)}(\lambda_j) = 0.$$

Then, for every $i \neq j$, $r(\lambda)$ is divisible by $(\lambda - \lambda_1)^{m_i}$ without remainder; therefore by (93),

$$r(A)\, H_i = o \qquad (i \neq j). \tag{94}$$

In exactly the same way, the difference $r(\lambda) - 1$ is divisible by $(\lambda - \lambda_j)^{m_j}$ without remainder; therefore

$$r(A)\, H_j = H_j. \tag{95}$$

Multiplying both sides of (92) by $r(A)$, we obtain from (94) and (95):

$$H_j = o.$$

This is valid for every $j = 1, 2, \ldots, s$. But H_j is a linear combination of independent columns corresponding to one and the same characteristic value λ_j $(j = 1, 2, \ldots, s)$. Therefore all the coefficients in the linear combination H_j $(j = 1, 2, \ldots, s)$, and hence all the coefficients in (92), are equal to zero.

Note. Let us point out some transformations on the columns of the matrix T under which it is transformed into the same Jordan form (with the same arrangement of the Jordan diagonal blocks):

I. *Multiplication of all the columns of an arbitrary Jordan chain by a non-zero number.*

II. *Addition to each column (beginning with the second) of a Jordan chain of the preceding column of the same chain, multiplied by one and the same arbitrary number.*

III. *Addition to all the columns of a Jordan chain of the corresponding columns of another chain containing the same or a larger number of columns and corresponding to the same characteristic value.*

Example 1.

$$A = \begin{Vmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 & -3 \\ 0 & 0 & -1 & 2 & -2 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 2 \end{Vmatrix};$$

$$\Delta(\lambda) = (\lambda - 1)^4 (\lambda + 1),$$

$$\psi(\lambda) = (\lambda - 1)^2 (\lambda + 1) = \lambda^3 - \lambda^2 - \lambda + 1.$$

Elementary divisors of the matrix A

$$(\lambda - 1)^2,\ (\lambda - 1)^2,\ \lambda + 1.$$

$$J = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{Vmatrix},$$

$$\Psi(\lambda, \mu) = \frac{\psi(\mu) - \psi(\lambda)}{\mu - \lambda} = \mu^2 + (\lambda - 1)\mu + \lambda^2 - \lambda - 1,$$

$$C(\lambda) = \Psi(\lambda E, A) = A^2 + (\lambda - 1) A + (\lambda^2 - \lambda - 1) E.$$

Let us compute successively the column of A^2 and the corresponding columns of $C(\lambda)$, $C(1)$, $C'(\lambda)$, $C'(1)$, $C(-1)$. We must obtain two linearly independent columns of $C(1)$ and one non-zero column of $C(-1)$.

$$C(\lambda) = \begin{Vmatrix} 1 & 0 & 0 & 2 & \ast \\ 0 & 1 & 0 & 2 & \ast \\ 0 & 0 & 1 & 0 & \ast \\ 2 & -2 & 2 & -1 & \ast \\ 2 & -2 & 2 & -2 & \ast \end{Vmatrix} + (\lambda - 1) \begin{Vmatrix} 1 & 0 & 0 & 1 & \ast \\ 0 & 1 & -2 & 3 & \ast \\ 0 & 0 & -1 & 2 & \ast \\ 1 & -1 & 1 & 0 & \ast \\ 1 & -1 & 1 & -1 & \ast \end{Vmatrix} + (\lambda^2 - \lambda - 1) \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

$$C(+1) = \begin{Vmatrix} 0 & 0 & 0 & 2 & \ast \\ 0 & 0 & 0 & 2 & \ast \\ 0 & 0 & 0 & 0 & \ast \\ 2 & -2 & 2 & -2 & \ast \\ 2 & -2 & 2 & -2 & \ast \end{Vmatrix},\ C'(\lambda) = \begin{Vmatrix} 1 & 0 & 0 & 1 & \ast \\ 0 & 1 & -2 & 3 & \ast \\ 0 & 0 & -1 & 2 & \ast \\ 1 & -1 & 1 & 0 & \ast \\ 1 & -1 & 1 & -1 & \ast \end{Vmatrix} + (2\lambda - 1) \begin{Vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix},$$

$$C'(+1) = \begin{Vmatrix} 2 & \ast & \ast & 1 & \ast \\ 0 & \ast & \ast & 3 & \ast \\ 0 & \ast & \ast & 2 & \ast \\ 1 & \ast & \ast & 1 & \ast \\ 1 & \ast & \ast & -1 & \ast \end{Vmatrix},\ C(-1) = \begin{Vmatrix} 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 4 & \ast & \ast \\ 0 & 0 & 4 & \ast & \ast \\ 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & \ast & \ast \end{Vmatrix}.$$

Therefore[38]

$$T = (C_1(+1), C_1'(+1), C_4(+1), C_4'(+1), C_3(-1)) = \begin{Vmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 2 & 4 \\ 2 & 1 & -2 & 1 & 0 \\ 2 & 1 & -2 & -1 & 0 \end{Vmatrix}.$$

The matrix T can be simplified a little. We

1) Divide the fifth column by 4;
2) Add the first column to the third and the second to the fourth;
3) Subtract the third column from the fourth;
4) Divide the first and second columns by 2;
5) Subtract the first column, multiplied by ½, from the second.

Then we obtain the matrix

$$T_1 = \begin{Vmatrix} 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

We leave it to the reader to verify that $AT_1 = T_1 J$ and $|\,T_1\,| \neq 0$.

Example 2.

$$A = \begin{Vmatrix} 1 & -1 & 1 & -1 \\ -3 & 3 & -5 & 4 \\ 8 & -4 & 3 & -4 \\ 15 & -10 & 11 & -11 \end{Vmatrix};$$

$$\Delta(\lambda) = (\lambda + 1)^4,$$

$$\psi(\lambda) = (\lambda + 1)^3.$$

Elementary divisors: $(\lambda + 1)^3,\ \lambda + 1.$

---

[38] Here the subscript denotes the number of the column; for example, $C_3(-1)$ denotes the third column of $C(-1)$.

$$J=\left\|\begin{matrix}-1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{matrix}\right\|.$$

We form the polynomials

$$h_1(\lambda)=\frac{\psi(\lambda)}{\lambda+1}=(\lambda+1)^2,\ h_2(\lambda)=\frac{\psi(\lambda)}{(\lambda+1)^2}=\lambda+1,\ h_3(\lambda)=\frac{\psi(\lambda)}{(\lambda+1)^3}=1$$

and the matrices[39]

$$C=h_1(A)=(A+E)^2,\ \ D=h_2(A)=A+E,\ \ F=E:$$

$$C=\left\|\begin{matrix}0 & 0 & 0 & 0\\ 2 & -1 & 1 & -1\\ 0 & 0 & 0 & 0\\ -2 & 1 & -1 & 1\end{matrix}\right\|,\ D=\left\|\begin{matrix}2 & -1 & 1 & -1\\ -3 & 4 & -5 & 4\\ 8 & -4 & 4 & -4\\ 15 & -10 & 11 & -10\end{matrix}\right\|,\ F=\left\|\begin{matrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{matrix}\right\|.$$

For the first three columns of T we take the third column of these matrices: $T=(C_3, D_3, F_3, *)$. In the matrices C, D, F, we subtract twice the third column from the first and we add the third column to the second and to the fourth. We obtain

$$\tilde{C}=\left\|\begin{matrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\end{matrix}\right\|,\ \tilde{D}=\left\|\begin{matrix}0 & 0 & 1 & 0\\ 7 & -1 & -5 & -1\\ 0 & 0 & 4 & 0\\ -7 & 1 & 11 & 1\end{matrix}\right\|,\ \tilde{F}=\left\|\begin{matrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ -2 & 1 & 1 & 1\\ 0 & 0 & 0 & 1\end{matrix}\right\|.$$

In the matrices $\tilde{D}$, $\tilde{F}$, we add the fourth column, multiplied by 7, to the first and subtract the fourth column from the second. We obtain

$$\tilde{C}=\left\|\begin{matrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\end{matrix}\right\|,\ \tilde{\tilde{D}}=\left\|\begin{matrix}0 & 0 & 1 & 0\\ 0 & 0 & -5 & -1\\ 0 & 0 & 4 & 0\\ 0 & 0 & 11 & 1\end{matrix}\right\|,\ \tilde{\tilde{F}}=\left\|\begin{matrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 5 & 0 & 1 & 1\\ 7 & -1 & 0 & 1\end{matrix}\right\|.$$

For the last column of T we take the first column of $\tilde{\tilde{F}}$. Then we have

$$T=(C_3,\ D_3,\ F_3,\ \tilde{\tilde{F}}_1)=\left\|\begin{matrix}0 & 1 & 0 & 1\\ 1 & -5 & 0 & 0\\ 0 & 4 & 1 & 5\\ -1 & 11 & 0 & 7\end{matrix}\right\|.$$

As a check, we can verify that $AT=TJ$ and that $|T|\neq 0$.

CHAPTER VII

THE STRUCTURE OF A LINEAR OPERATOR IN AN *n*-DIMENSIONAL SPACE

(*Geometrical Theory of Elementary Divisors*)

The analytic theory of elementary divisors expounded in the preceding chapter has enabled us to determine for every square matrix a similar matrix having 'normal' or 'canonical' form. On the other hand, we have seen in Chapter III that the behaviour of a linear operator in an n-dimensional space with respect to various bases is given by means of a class of similar matrices. The existence of a matrix of normal form in such a class is closely connected with important and deep properties of a linear operator in an n-dimensional space. The study of these properties is the object of the present chapter. The investigation of the structure of a linear operator will lead us, independently of the contents of the preceding chapter, to the theory of transformations of a matrix to a normal form. Therefore the contents of this chapter may be called *the geometrical theory of elementary divisors.*[1]

§ 1. The Minimal Polynomial of a Vector and a Space (with Respect to a Given Linear Operator)

1. We consider an n-dimensional vector space $\boldsymbol{R}$ over the field F and a linear operator $\boldsymbol{A}$ in this space.

Let $\boldsymbol{x}$ be an arbitrary vector of $\boldsymbol{R}$. We form the sequence of vectors

$$\boldsymbol{x},\ \boldsymbol{A}\boldsymbol{x},\ \boldsymbol{A}^2\boldsymbol{x},\ \ldots. \tag{1}$$

Since the space is finite-dimensional, there is an integer p $(0 \leqq p \leqq n)$ such that the vectors $\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}, \ldots, \boldsymbol{A}^{p-1}\boldsymbol{x}$ are linearly independent, while $\boldsymbol{A}^p\boldsymbol{x}$ is a linear combination of these vectors with coefficients in F:

---

[1] The account of the geometric theory of elementary divisors to be given here is based on our paper [167]. For other geometrical constructions of the theory of elementary divisors, see [22], §§ 96-99 and also [53].

$$A^p \boldsymbol{x} = -\gamma_1 A^{p-1}\boldsymbol{x} - \gamma_2 A^{p-2}\boldsymbol{x} - \cdots - \gamma_p \boldsymbol{x}. \tag{2}$$

We form the monic polynomial $\varphi(\lambda) = \lambda^p + \gamma_1 \lambda^{p-1} + \cdots + \gamma_{p-1}\lambda + \gamma_p$. (A *monic* polynomial is a polynomial in which the coefficient of the highest power of the variable is unity.) Then (2) can be written:

$$\varphi(A)\boldsymbol{x} = \boldsymbol{o}. \tag{3}$$

Every polynomial $\varphi(\lambda)$ for which (3) holds will be called an *annihilating polynomial for the vector* $\boldsymbol{x}$.[2] But it is easy to see that of all the monic annihilating polynomials of $\boldsymbol{x}$ the one we have constructed is of least degree. This polynomial will be called the *minimal annihilating polynomial of* $\boldsymbol{x}$ or simply the *minimal polynomial of* $\boldsymbol{x}$.

Note that every annihilating polynomial $\tilde{\varphi}(\lambda)$ of $\boldsymbol{x}$ is divisible by the minimal polynomial $\varphi(\lambda)$.

For let

$$\tilde{\varphi}(\lambda) = \varphi(\lambda)\varkappa(\lambda) + \varrho(\lambda),$$

where $\varkappa(\lambda)$, $\varrho(\lambda)$ are quotient and remainder on dividing $\tilde{\varphi}(\lambda)$ by $\varphi(\lambda)$. Then

$$\tilde{\varphi}(A)\boldsymbol{x} = \varkappa(A)\varphi(A)\boldsymbol{x} + \varrho(A)\boldsymbol{x} = \varrho(A)\boldsymbol{x}$$

and therefore $\varrho(A)\boldsymbol{x} = \boldsymbol{o}$. But the degree of $\varrho(\lambda)$ is less than that of the minimal polynomial $\varphi(\lambda)$. Hence $\varrho(\lambda) \equiv 0$.

From what we have proved it follows, in particular, that every vector $\boldsymbol{x}$ has only one minimal polynomial.

2. We choose a basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ in $\boldsymbol{R}$. We denote by $\varphi_1(\lambda), \varphi_2(\lambda), \ldots, \varphi_n(\lambda)$ the minimal polynomials of the basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ and by $\psi(\lambda)$ the least common multiple of these polynomials ($\psi(\lambda)$ is taken with highest coefficient 1). Then $\psi(\lambda)$ is an annihilating polynomial for all the basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Since every vector $\boldsymbol{x} \in \boldsymbol{R}$ is representable in the form $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + \cdots + x_n\boldsymbol{e}_n$, we have

$$\psi(A)\boldsymbol{x} = x_1\psi(A)\boldsymbol{e}_1 + x_2\psi(A)\boldsymbol{e}_2 + \cdots + x_n\psi(A)\boldsymbol{e}_n = \boldsymbol{o},$$

i.e.,

$$\psi(A) = O. \tag{4}$$

The polynomial $\psi(\lambda)$ is called *an annihilating polynomial for the whole space* $\boldsymbol{R}$. Let $\tilde{\psi}(\lambda)$ be an arbitrary annihilating polynomial for the whole space $\boldsymbol{R}$. Then $\tilde{\psi}(\lambda)$ is an annihilating polynomial for the basis vectors

---

[2] Of course, the phrase 'with respect to the given operator A' is tacitly understood. For the sake of brevity, this circumstance is not mentioned in the definition, because throughout this entire chapter we shall deal with a single operator A.

$\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Therefore $\tilde{\psi}(\lambda)$ must be a common multiple of the minimal polynomials $\varphi_1(\lambda), \varphi_2(\lambda), \ldots, \varphi_n(\lambda)$ of these vectors and must therefore be divisible without remainder by their *least* common multiple $\psi(\lambda)$. Hence it follows that, of all the annihilating polynomials for the whole space $\boldsymbol{R}$, the one we have constructed, $\psi(\lambda)$, has the least degree and it is monic. This polynomial is uniquely determined by the space $\boldsymbol{R}$ and the operator A and is called the *minimal polynomial of the space* $\boldsymbol{R}$.[3] The uniqueness of the minimal polynomial of the space $\boldsymbol{R}$ follows from the statement proved above: *every annihilating polynomial* $\tilde{\psi}(\lambda)$ *of the space* $\boldsymbol{R}$ *is divisible by the minimal polynomial* $\psi(\lambda)$. Although the construction of the minimal polynomial $\psi(\lambda)$ was associated with a definite basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$, the polynomial $\psi(\lambda)$ itself does not depend on the choice of this basis (this follows from the uniqueness of the minimal polynomial for the space $\boldsymbol{R}$).

Finally we mention that the minimal polynomial of the space $\boldsymbol{R}$ annihilates every vector $\boldsymbol{x}$ of $\boldsymbol{R}$ so that *the minimal polynomial of the space is divisible by the minimal polynomial of every vector in the space.*

§ 2. Decomposition into Invariant Subspaces with Co-Prime Minimal Polynomials

1. If some collection of vectors $\boldsymbol{R}'$ forming part of $\boldsymbol{R}$ has the property that the sum of any two vectors of $\boldsymbol{R}'$ and the product of any vector of $\boldsymbol{R}'$ by a number $\alpha \in F$ always belongs to $\boldsymbol{R}'$, then that manifold $\boldsymbol{R}'$ is itself a vector space, a *subspace* of $\boldsymbol{R}$.

If two subspaces $\boldsymbol{R}'$ and $\boldsymbol{R}''$ of $\boldsymbol{R}$ are given and if it is known that

1. $\boldsymbol{R}'$ and $\boldsymbol{R}''$ have no vector in common except the null vector, and
2. every vector $\boldsymbol{x}$ of $\boldsymbol{R}$ can be represented in the form of a sum

$$\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{x}'' \quad (\boldsymbol{x}' \in \boldsymbol{R}', \boldsymbol{x}'' \in \boldsymbol{R}''), \tag{5}$$

then we shall say that the space $\boldsymbol{R}$ is *decomposed* into the two subspaces $\boldsymbol{R}'$ and $\boldsymbol{R}''$ and shall write:

$$\boldsymbol{R} = \boldsymbol{R}' + \boldsymbol{R}'' \tag{6}$$

Note that the condition 1. implies the uniqueness of the representation (5). For if for a certain vector $\boldsymbol{x}$ we had two distinct representations in the form of a sum of terms from $\boldsymbol{R}'$ and $\boldsymbol{R}''$, (5) and

$$\boldsymbol{x} = \tilde{\boldsymbol{x}}' + \tilde{\boldsymbol{x}}'' \quad (\tilde{\boldsymbol{x}}' \in \boldsymbol{R}', \tilde{\boldsymbol{x}}'' \in \boldsymbol{R}'') \tag{7}$$

then, subtracting (7) from (5) term by term, we would obtain:

---

[3] If in some basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ a matrix $A = \| a_{ik} \|_1^n$ then the annihilating or minimal polynomial of the space $\boldsymbol{R}$ (with respect to A) is the annihilating or minimal polynomial of the matrix A, and vice versa. Compare with Chapter IV, § 6.

$$\boldsymbol{x}' - \tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{x}}'' - \boldsymbol{x}'',$$

i.e., equality of the non-null vectors $\boldsymbol{x}' - \tilde{\boldsymbol{x}}' \in \boldsymbol{R}'$ and $\tilde{\boldsymbol{x}}'' - \boldsymbol{x}'' \in \boldsymbol{R}''$, which, by 1., is impossible.

Thus, condition 1. may be replaced by the requirement that the representation (5) be unique. In this form, the definition of decomposition immediately extends to an arbitrary number of subspaces.

Let

$$\boldsymbol{R} = \boldsymbol{R}' + \boldsymbol{R}''$$

and let $\boldsymbol{e}_1', \boldsymbol{e}_2', \ldots, \boldsymbol{e}_{n'}'$ and $\boldsymbol{e}_1'', \boldsymbol{e}_2'', \ldots, \boldsymbol{e}_{n''}''$ be bases of $\boldsymbol{R}'$ and $\boldsymbol{R}''$, respectively. Then the reader can easily prove that all these $n' + n''$ vectors are linearly independent and form a basis of $\boldsymbol{R}$, so that a basis of the whole space is formed from bases of the subspaces. It follows, in particular, that $n = n' + n''$.

Example 1. Suppose that in a three-dimensional space three directions, not parallel to one and the same plane, are given. Since every vector in the space can be split, uniquely, into components in these three directions, we have

$$\boldsymbol{R} = \boldsymbol{R}' + \boldsymbol{R}'' + \boldsymbol{R}''',$$

where $\boldsymbol{R}$ is the set of all the vectors of one space, $\boldsymbol{R}'$ the set of all vectors parallel to the first direction, $\boldsymbol{R}''$ to the second, and $\boldsymbol{R}'''$ to the third. In this case, $n = 3$ and $n' = n'' = n''' = 1$.

Example 2. Suppose that in a three-dimensional space a plane and a line intersecting the plane are given. Then

$$\boldsymbol{R} = \boldsymbol{R}' + \boldsymbol{R}'',$$

where $\boldsymbol{R}$ is the set of all vectors of our space, $\boldsymbol{R}'$ the set of all vectors parallel to the given plane, and $\boldsymbol{R}''$ the set of all vectors parallel to the given line. In this example, $n = 3$, $n' = 2$, $n'' = 1$.

2. A subspace $\boldsymbol{R}' \subset \boldsymbol{R}$ is called *invariant* with respect to the operator $\boldsymbol{A}$ if $\boldsymbol{A}\boldsymbol{R}' \subset \boldsymbol{R}'$, i.e. if $\boldsymbol{x} \in \boldsymbol{R}'$ implies $\boldsymbol{A}\boldsymbol{x} \in \boldsymbol{R}'$. In other words, the operator $\boldsymbol{A}$ carries a vector of an invariant subspace into a vector of the same subspace.

In what follows we shall carry out a decomposition of the whole space into subspaces invariant with respect to $\boldsymbol{A}$. The decomposition reduces the study of the behavior of an operator in the whole space to the study of its behavior in the various component subspaces.

We shall now prove the following theorem:

THEOREM 1 (First Theorem on the Decomposition of a Space into Invariant Subspaces): *If for a given operator $\boldsymbol{A}$ the minimal polynomial $\psi(\lambda)$ of the space is represented over* F *in the form of a product of two co-prime polynomials $\psi_1(\lambda)$ and $\psi_2(\lambda)$ (with highest coefficients* 1)

$$\psi(\lambda) = \psi_1(\lambda)\,\psi_2(\lambda), \tag{8}$$

then the whole space $\boldsymbol{R}$ splits into two invariant subspaces $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$

$$\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2, \tag{9}$$

whose minimal polynomials are $\psi_1(\lambda)$ and $\psi_2(\lambda)$, respectively.

Proof. We denote by $\boldsymbol{I}_1$ the set of all vectors $\boldsymbol{x} \in \boldsymbol{R}$ satisfying the equation $\psi_1(\boldsymbol{A})\,\boldsymbol{x} = \boldsymbol{o}$. $\boldsymbol{I}_2$ is similarly defined by the equation $\psi_2(\boldsymbol{A})\,\boldsymbol{x} = \boldsymbol{o}$. $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ so defined are subspaces of $\boldsymbol{R}$.

Since $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are co-prime, it follows that there exist polynomials $\chi_1(\lambda)$ and $\chi_2(\lambda)$ (with coefficients in F) such that

$$1 = \psi_1(\lambda)\,\chi_1(\lambda) + \psi_2(\lambda)\,\chi_2(\lambda). \tag{10}$$

Now let $\boldsymbol{x}$ be an arbitrary vector of $\boldsymbol{R}$. In (10) we replace λ by $\boldsymbol{A}$ and we apply both sides of the operator equation so obtained to the vector $\boldsymbol{x}$:

$$\boldsymbol{x} = \psi_1(\boldsymbol{A})\,\chi_1(\boldsymbol{A})\,\boldsymbol{x} + \psi_2(\boldsymbol{A})\,\chi_2(\boldsymbol{A})\,\boldsymbol{x}, \tag{11}$$

i.e.,

$$\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{x}'', \tag{12}$$

where

$$\boldsymbol{x}' = \psi_2(\boldsymbol{A})\,\chi_2(\boldsymbol{A})\,\boldsymbol{x}, \quad \boldsymbol{x}'' = \psi_1(\boldsymbol{A})\,\chi_1(\boldsymbol{A})\,\boldsymbol{x}. \tag{13}$$

Furthermore,

$$\psi_1(\boldsymbol{A})\,\boldsymbol{x}' = \psi(\boldsymbol{A})\,\chi_2(\boldsymbol{A})\,\boldsymbol{x} = \boldsymbol{o}, \quad \psi_2(\boldsymbol{A})\,\boldsymbol{x}'' = \psi(\boldsymbol{A})\,\chi_1(\boldsymbol{A})\,\boldsymbol{x} = \boldsymbol{o},$$

i.e.,

$$\boldsymbol{x}' \in \boldsymbol{I}_1, \text{ and } \boldsymbol{x}'' \in \boldsymbol{I}_2.$$

$\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ have only the null vector in common. For if $\boldsymbol{x}_0 \in \boldsymbol{I}_1$ and $\boldsymbol{x}_0 \in \boldsymbol{I}_2$, i.e., $\psi_1(\boldsymbol{A})\,\boldsymbol{x}_0 = \boldsymbol{o}$ and $\psi_2(\boldsymbol{A})\,\boldsymbol{x}_0 = \boldsymbol{o}$, then by (11)

$$\boldsymbol{x}_0 = \chi_1(\boldsymbol{A})\,\psi_1(\boldsymbol{A})\,\boldsymbol{x}_0 + \chi_2(\boldsymbol{A})\,\psi_2(\boldsymbol{A})\,\boldsymbol{x}_0 = \boldsymbol{o}.$$

Thus we have proved that $\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2$.

Now suppose that $\boldsymbol{x} \in \boldsymbol{I}_1$. Then $\psi_1(\boldsymbol{A})\boldsymbol{x} = \boldsymbol{o}$. Multiplying both sides of this equation by $\boldsymbol{A}$ and reversing the order of $\boldsymbol{A}$ and $\psi_1(\boldsymbol{A})$, we obtain $\psi_1(\boldsymbol{A})\boldsymbol{A}\boldsymbol{x} = \boldsymbol{o}$, i.e., $\boldsymbol{A}\boldsymbol{x} \in \boldsymbol{I}_1$. This proves that the subspace $\boldsymbol{I}_1$ is invariant with respect to $\boldsymbol{A}$. The invariance of the subspace $\boldsymbol{I}_2$ is proved similarly.

We shall now show that $\psi_1(\lambda)$ is the minimal polynomial of $\boldsymbol{I}_1$. Let $\tilde{\psi}_1(\lambda)$ be an arbitrary annihilating polynomial for $\boldsymbol{I}_1$, and $\boldsymbol{x}$ an arbitrary vector of $\boldsymbol{R}$. Using the decomposition (12) already established, we write:

$$\tilde{\psi}_1(\boldsymbol{A})\,\psi_2(\boldsymbol{A})\,\boldsymbol{x} = \psi_2(\boldsymbol{A})\,\tilde{\psi}_1(\boldsymbol{A})\,\boldsymbol{x}' + \tilde{\psi}_1(\boldsymbol{A})\,\psi_2(\boldsymbol{A})\,\boldsymbol{x}'' = \boldsymbol{o}.$$

Since $\boldsymbol{x}$ is an arbitrary vector of $\boldsymbol{R}$, it follows that the product $\tilde{\psi}_1(\lambda)\psi_2(\lambda)$ is an annihilating polynomial for $\boldsymbol{R}$ and is therefore divisible by $\psi(\lambda) = \psi_1(\lambda)\psi_2(\lambda)$ without remainder; in other words, $\psi_1(\lambda)$ is divisible by $\tilde{\psi}_1(\lambda)$. But $\tilde{\psi}_1(\lambda)$ is an arbitrary annihilating polynomial for $\boldsymbol{I}_1$ and $\psi_1(\lambda)$ is a particular one of the annihilating polynomials (by the definition of $\boldsymbol{I}_1$). Hence $\psi_1(\lambda)$ is the minimal polynomial of $\boldsymbol{I}_1$. In exactly the same way it is shown that $\psi_2(\lambda)$ is the minimal polynomial for the invariant subspace $\boldsymbol{I}_2$.

This completes the proof of the theorem.

Let us decompose $\psi(\lambda)$ into irreducible factors over F:

$$\psi(\lambda) = [\varphi_1(\lambda)]^{c_1}[\varphi_2(\lambda)]^{c_2}\cdots[\varphi_s(\lambda)]^{c_s} \tag{14}$$

(here $\varphi_1(\lambda), \varphi_2(\lambda), \ldots, \varphi_s(\lambda)$ are distinct irreducible polynomials over F with highest coefficient 1). Then by the theorem we have

$$\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2 + \ldots + \boldsymbol{I}_s, \tag{15}$$

where $\boldsymbol{I}_k$ is an invariant subspace with the minimal polynomial $[\varphi_k(\lambda)]^{c_k}$ $(k = 1, 2, \ldots, s)$.

Thus, the theorem reduces the study of the behaviour of a linear operator in an arbitrary space to the study of the behaviour of this operator in a space where the minimal polynomial is a power of an irreducible polynomial over F. We shall take advantage of this to prove the following important theorem:

THEOREM 2: *In a vector space there always exists a vector whose minimal polynomial coincides with the minimal polynomial of the whole space.*

We consider first the special case where the minimal polynomial of the space $\boldsymbol{R}$ is a power of an irreducible polynomial $\varphi(\lambda)$:

$$\psi(\lambda) = [\varphi(\lambda)]^l.$$

In $\boldsymbol{R}$ we choose a basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. The minimal polynomial of $\boldsymbol{e}_i$ is a divisor of $\psi(\lambda)$ and is therefore representable in the form $[\varphi(\lambda)]^{l_i}$, where $l_i \leqq l$ $(i = 1, 2, \ldots, n)$.

But the minimal polynomial of the space is the least common multiple of the minimal polynomials of the basis vectors, so that $\psi(\lambda)$ is the largest of the powers $[\varphi(\lambda)]^{l_i}$ $(i = 1, 2, \ldots, n)$. In other words, $\psi(\lambda)$ coincides with the minimal polynomial of one of the basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$.

Turning now to the general case, we prove the following preliminary lemma:

LEMMA: *If the minimal polynomials of the vectors* $\boldsymbol{e}'$ *and* $\boldsymbol{e}''$ *are co-prime, then the minimal polynomial of the sum vector* $\boldsymbol{e}' + \boldsymbol{e}''$ *is equal to the product of the minimal polynomials of the constituent vectors.*

Proof. Let $\chi_1(\lambda)$ and $\chi_2(\lambda)$ be the minimal polynomials of the vectors $\boldsymbol{e}'$ and $\boldsymbol{e}''$. By assumption, $\chi_1(\lambda)$ and $\chi_2(\lambda)$ are co-prime. Let $\chi(\lambda)$ be an arbitrary annihilating polynomial of the vector $\boldsymbol{e} = \boldsymbol{e}' + \boldsymbol{e}''$. Then

$$\chi_2(\boldsymbol{A})\,\chi(\boldsymbol{A})\,\boldsymbol{e}' = \chi_2(\boldsymbol{A})\,\chi(\boldsymbol{A})\,\boldsymbol{e} - \chi(\boldsymbol{A})\,\chi_2(\boldsymbol{A})\,\boldsymbol{e}'' = \boldsymbol{o},$$

i.e., $\chi_2(\lambda)\chi(\lambda)$ is an annihilating polynomial of $\boldsymbol{e}'$. Therefore $\chi_2(\lambda)\chi(\lambda)$ is divisible by $\chi_1(\lambda)$, and since $\chi_1(\lambda)$ and $\chi_2(\lambda)$ are co-prime, $\chi(\lambda)$ is divisible by $\chi_1(\lambda)$. It is proved similarly that $\chi(\lambda)$ is divisible by $\chi_2(\lambda)$. But $\chi_1(\lambda)$ and $\chi_2(\lambda)$ are co-prime. Therefore $\chi(\lambda)$ is divisible by the product $\chi_1(\lambda)\,\chi_2(\lambda)$. Thus, every annihilating polynomial of the vector $\boldsymbol{e}$ is divisible by $\chi_1(\lambda)\,\chi_2(\lambda)$. Therefore $\chi_1(\lambda)\chi_2(\lambda)$ is the minimal polynomial of the vector $\boldsymbol{e} = \boldsymbol{e}' + \boldsymbol{e}''$.

We now return to Theorem 2. For the proof in the general case we use the decomposition (15). Since the minimal polynomials of the subspaces $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_s$ are powers of irreducible polynomials, our assertion is already proved for these subspaces. Therefore there exist vectors $\boldsymbol{e}' \in \boldsymbol{I}_1, \boldsymbol{e}'' \in \boldsymbol{I}_2, \ldots, \boldsymbol{e}^{(s)} \in \boldsymbol{I}_s$ whose minimal polynomials are $[\varphi_1(\lambda)]^{c_1}$, $[\varphi_2(\lambda)]^{c_2}, \ldots, [\varphi_s(\lambda)]^{c_s}$, respectively. By the lemma, the minimal polynomial of the vector $\boldsymbol{e} = \boldsymbol{e}' + \boldsymbol{e}'' + \cdots + \boldsymbol{e}^{(s)}$ is equal to the product

$$[\varphi_1(\lambda)]^{c_1}\,[\varphi_2(\lambda)]^{c_2}\cdots[\varphi_s(\lambda)]^{c_s},$$

i.e., to the minimal polynomial of the space $\boldsymbol{R}$.

§ 3. Congruence. Factor Space

1. Suppose given a subspace $\boldsymbol{I} \subset \boldsymbol{R}$. We shall say that two vectors $\boldsymbol{x}, \boldsymbol{y}$ of $\boldsymbol{R}$ are *congruent* modulo $\boldsymbol{I}$ and shall write $\boldsymbol{x} \equiv \boldsymbol{y}\ (\mathrm{mod}\ \boldsymbol{I})$ if and only if $\boldsymbol{y} - \boldsymbol{x} \in \boldsymbol{I}$. It is easy to verify that the concept of congruence so introduced has the following properties:

For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}$

1. $\mathbf{x} \equiv \mathbf{x} \pmod{\mathbf{I}}$ (reflexivity of congruence).
2. From $\mathbf{x} \equiv \mathbf{y} \pmod{\mathbf{I}}$ it follows that $\mathbf{y} \equiv \mathbf{x} \pmod{\mathbf{I}}$ (symmetry of congruence).
3. From $\mathbf{x} \equiv \mathbf{y} \pmod{\mathbf{I}}$ and $\mathbf{y} \equiv \mathbf{z} \pmod{\mathbf{I}}$ it follows that $\mathbf{x} \equiv \mathbf{z} \pmod{\mathbf{I}}$ (transitivity of congruence).

The presence of these three properties enables us to make use of congruence to divide all the vectors of the space into classes, by assigning vectors that are pairwise congruent $(\mathrm{mod}\ \mathbf{I})$ to the same class (vectors of distinct classes are incongruent $(\mathrm{mod}\ \mathbf{I})$). The class containing the vector $\mathbf{x}$ will be denoted by $\bar{\mathbf{x}}$.[5] The subspace $\mathbf{I}$ is one of these classes, namely $\bar{\mathbf{o}}$. Note that to every congruence $\mathbf{x} \equiv \mathbf{y} \pmod{\mathbf{I}}$ there corresponds the equality[6] of the associated classes: $\bar{\mathbf{x}} = \bar{\mathbf{y}}$.

It is elementary to prove that congruences may be added term by term and multiplied by a number of F:

1. From

$$\mathbf{x} \equiv \mathbf{x}' \text{ and } \mathbf{y} \equiv \mathbf{y}' \qquad (\mathrm{mod}\ \mathbf{I})$$

it follows that

$$\mathbf{x} + \mathbf{y} \equiv \mathbf{x}' + \mathbf{y}' \qquad (\mathrm{mod}\ \mathbf{I}).$$

2. From

$$\mathbf{x} \equiv \mathbf{x}' \qquad (\mathrm{mod}\ \mathbf{I})$$

it follows that

$$\alpha\mathbf{x} \equiv \alpha\mathbf{x}' \qquad (\mathrm{mod}\ \mathbf{I})\ (\alpha \in \mathrm{F}).$$

These properties of congruence show that the operations of addition and multiplication by a number of F do not 'break up' the classes. If we take two classes $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ and add elements $\mathbf{x}, \mathbf{x}', \ldots$ of the first class to arbitrary elements $\mathbf{y}, \mathbf{y}', \ldots$ of the second class, then all the sums so obtained belong to one and the same class, which we call the *sum* of the classes $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ and denote by $\bar{\mathbf{x}} + \bar{\mathbf{y}}$. Similarly, if all the vectors $\mathbf{x}, \mathbf{x}', \ldots$ of the class $\mathbf{x}$ are multiplied by a number $\alpha \in \mathrm{F}$, then the products belong to one class, which we denote by $\alpha\bar{\mathbf{x}}$.

Thus, in the manifold $\bar{\mathbf{R}}$ of all classes $\bar{\mathbf{x}}, \bar{\mathbf{y}}, \ldots$ two operations are introduced: 'addition' and 'multiplication by a number of F.' It is easy to verify that these operations have the properties set forth in the definition of a vector space (Chapter III, § 1). Therefore $\bar{\mathbf{R}}$, as well as $\mathbf{R}$, is a vector space over the field F. We shall say that $\bar{\mathbf{R}}$ is a *factor space* of $\mathbf{R}$. If $n, m, \bar{n}$ are the dimensions of the spaces $\mathbf{R}, \mathbf{I}, \bar{\mathbf{R}}$, respectively, then $\bar{n} = n - m$.

2. All the concepts introduced in this section can be illustrated very well by the following example.

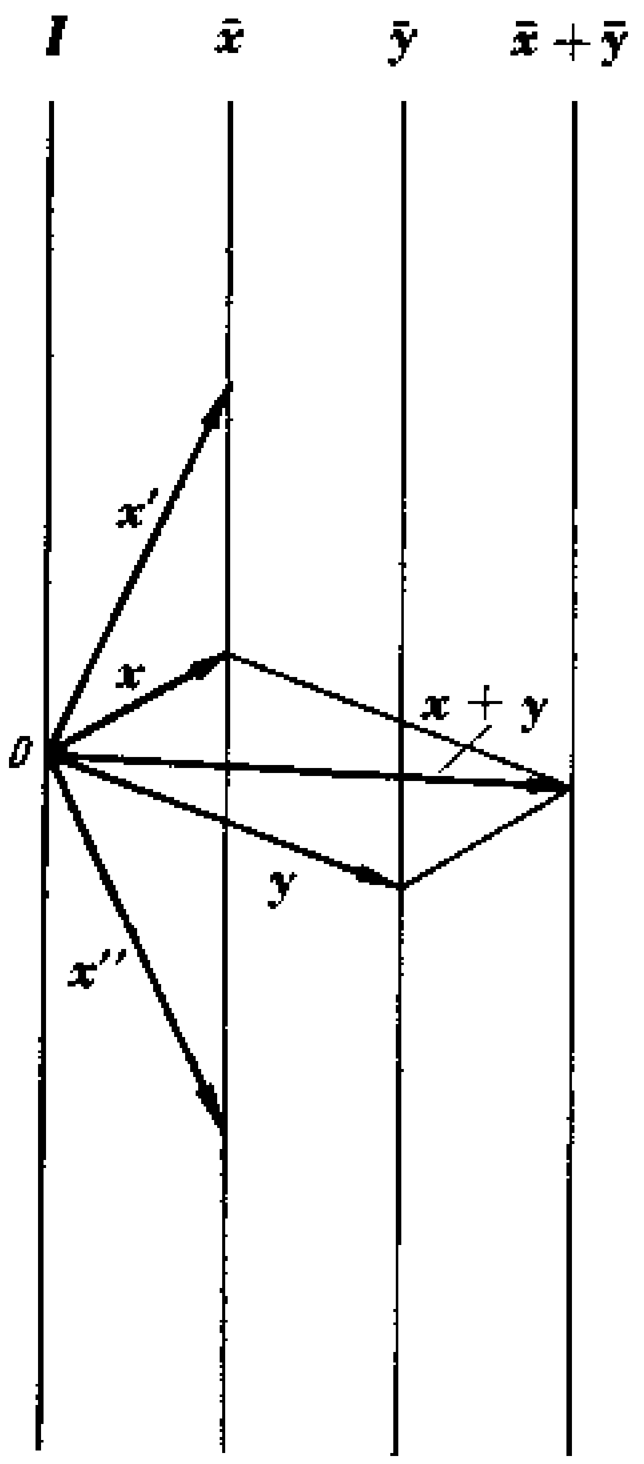

Fig. 4

Example. Let $\mathbf{R}$ be the set of all vectors of a three-dimensional space and F the field of real numbers. For greater clarity, we shall represent vectors in the form of directed segments beginning at a point O. Let $\mathbf{I}$ be a straight line passing through O (more accurately: the set of vectors that lie along some line passing through O; Fig. 4.).

The congruence $\mathbf{x} \equiv \mathbf{x}' \pmod{\mathbf{I}}$ signifies that the vectors $\mathbf{x}$ and $\mathbf{x}'$ differ by a vector of $\mathbf{I}$, i.e., the segment containing the end-points of $\mathbf{x}$ and $\mathbf{x}'$ is parallel to $\mathbf{I}$. Therefore the class $\bar{\mathbf{x}}$ is represented by the line passing through the end-point of $\mathbf{x}$ and parallel to $\mathbf{I}$ (more accurately: by the 'bundle' of vectors starting from O whose end-points lie on that line). 'Bundles' may be added and multiplied by a real number (by adding and multiplying the vectors that occur in the bundles). These 'bundles' are also the elements of the factor space $\bar{\mathbf{R}}$. In this example, $n = 3$, $m = 1$, $\bar{n} = 2$.

We obtain another example by taking for $\mathbf{I}$ a plane passing through O. In this example, $n = 3$, $m = 2$, $\bar{n} = 1$.

Now let $\mathbf{A}$ be a linear operator in $\mathbf{R}$. Let us assume that $\mathbf{I}$ *is an invariant subspace with respect to* $\mathbf{A}$. The reader will easily prove that from $\mathbf{x} \equiv \mathbf{x}'$ $(\mathrm{mod}\ \mathbf{I})$ it follows that $\mathbf{A}\mathbf{x} \equiv \mathbf{A}\mathbf{x}' \pmod{\mathbf{I}}$, so that the operator $\mathbf{A}$ can be applied to both sides of a congruence. In other words, if the operator $\mathbf{A}$ is applied to all vectors $\mathbf{x}, \mathbf{x}', \ldots$ of a class $\bar{\mathbf{x}}$, then the vectors $\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x}', \ldots$ also belong to one class, which we denote by $\mathbf{A}\bar{\mathbf{x}}$. The linear operator $\mathbf{A}$ carries classes into classes and is, thus, a linear operator in $\bar{\mathbf{R}}$.

We shall say that the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ are *linearly dependent modulo* $\mathbf{I}$ if there exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_p$ in F, not all equal to zero, such that

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_p\mathbf{x}_p \equiv \mathbf{o} \quad (\mathrm{mod}\ \mathbf{I}). \tag{16}$$

---

[5] Since each class contains an infinite set of vectors, there is, by this condition, an infinite number of ways of designating the class.

[6] That is, identity.

Note that not only the concept of linear dependence of vectors, but also all the concepts, statements, and reasonings, in the preceding sections of this chapter can be repeated word for word with the symbol '$=$' replaced throughout by the symbol '$\equiv \pmod{\boldsymbol{I}}$,' where $\boldsymbol{I}$ is some fixed subspace invariant with respect to $\boldsymbol{A}$.

Thus, we can introduce the concepts of an annihilating polynomial and of the minimal polynomial of a vector or a space $(\operatorname{mod} \boldsymbol{I})$. All these concepts will be called 'relative,' in contrast to the 'absolute' concepts that were introduced earlier (and that hold for the symbol '$=$').

The reader should observe that *the relative minimal polynomial (of a vector or a space) is a divisor of the absolute one*. For example, let $\sigma_1(\lambda)$ be the relative minimal polynomial of a vector $\boldsymbol{x}$ and $\sigma(\lambda)$ the corresponding absolute minimal polynomial.

Then

$$\sigma(\boldsymbol{A})\boldsymbol{x} = \boldsymbol{o},$$

and hence it follows that also

$$\sigma(\boldsymbol{A})\boldsymbol{x} \equiv \boldsymbol{o} \quad (\operatorname{mod} \boldsymbol{I}).$$

Therefore $\sigma(\lambda)$ is a relative annihilating polynomial of $\boldsymbol{x}$ and as such is divisible by the relative minimal polynomial $\sigma_1(\lambda)$.

Side by side with the 'absolute' statements of the preceding sections we have 'relative' statements. For example, we have the statement: 'In every space there always exists a vector whose relative minimal polynomial coincides with the relative minimal polynomial of the whole space.'

The truth of all 'relative' statements depends on the fact that by operating with congruences modulo $\boldsymbol{I}$ we deal essentially with equalities—however not in the space $\boldsymbol{R}$, but in the space $\overline{\boldsymbol{R}}$.

§ 4. Decomposition of a Space into Cyclic Invariant Subspaces

1. Let $\sigma(\lambda) = \lambda^p + \alpha_1 \lambda^{p-1} + \cdots + \alpha_{p-1}\lambda + \alpha_p$ be the minimal polynomial of a vector $\boldsymbol{e}$. Then the vectors

$$\boldsymbol{e}, \boldsymbol{A}\boldsymbol{e}, \ldots, \boldsymbol{A}^{p-1}\boldsymbol{e} \tag{17}$$

are linearly independent, and

$$\boldsymbol{A}^p\boldsymbol{e} = -\alpha_p\boldsymbol{e} - \alpha_{p-1}\boldsymbol{A}\boldsymbol{e} - \cdots - \alpha_1\boldsymbol{A}^{p-1}\boldsymbol{e}. \tag{18}$$

The vectors (17) form a basis of a p-dimensional subspace $\boldsymbol{I}$. We shall call this subspace *cyclic* in view of the special character of the basis (17) and of (18).[7] The operator $\boldsymbol{A}$ carries the first vector of (17) into the second, the second into the third, etc. The last basis vector is carried by $\boldsymbol{A}$ into a linear combination of the basis vectors in accordance with (18). Thus, $\boldsymbol{A}$ carries every basis vector into a vector of $\boldsymbol{I}$ and hence an arbitrary vector of $\boldsymbol{I}$ into another vector of $\boldsymbol{I}$. In other words, *a cyclic subspace is always invariant with respect to* $\boldsymbol{A}$.

Every vector $\boldsymbol{x} \in \boldsymbol{I}$ is representable in the form of a linear combination of the basis vectors (17), i.e., in the form

$$\boldsymbol{x} = \chi(\boldsymbol{A})\boldsymbol{e}, \tag{19}$$

where $\chi(\lambda)$ is a polynomial in λ of degree $\leqq p-1$ with coefficients in F. By forming all possible polynomials $\chi(\lambda)$ of degree $\leqq p-1$ with coefficients in F we obtain all the vectors of $\boldsymbol{I}$, each once only, i.e., for only one polynomial $\chi(\lambda)$. In view of the basis (17) or the formula (19) we shall say that the vector $\boldsymbol{e}$ *generates* the subspace.

Note that *the minimal polynomial of the generating vector* $\boldsymbol{e}$ *is also the minimal polynomial of the whole subspace* $\boldsymbol{I}$.

2. We are now ready to establish the fundamental proposition of the whole theory, according to which the space $\boldsymbol{R}$ splits into cyclic subspaces.

Let $\psi_1(\lambda) = \psi(\lambda) = \lambda^m + \alpha_1\lambda^{m-1} + \cdots + \alpha_m$ be the minimal polynomial of the space $\boldsymbol{R}$. Then there exists a vector $\boldsymbol{e}$ in the space for which this polynomial is minimal (Theorem 2, p. 180). Let $\boldsymbol{I}_1$ denote the cyclic subspace with the basis

$$\boldsymbol{e}, \boldsymbol{A}\boldsymbol{e}, \ldots, \boldsymbol{A}^{m-1}\boldsymbol{e}. \tag{20}$$

If $n = m$, then $\boldsymbol{R} = \boldsymbol{I}_1$. Suppose that $n > m$ and that the polynomial

$$\psi_2(\lambda) = \lambda^p + \beta_1\lambda^{p-1} + \cdots + \beta_p$$

is the minimal polynomial of $\boldsymbol{R}$ $(\operatorname{mod} \boldsymbol{I}_1)$. By the remark at the end of § 3, $\psi_2(\lambda)$ is a divisor of $\psi_1(\lambda)$, i.e., there exists a polynomial $\varkappa(\lambda)$ such that

$$\psi_1(\lambda) = \psi_2(\lambda)\varkappa(\lambda). \tag{21}$$

---

[7] It would be more accurate to call this subspace: cyclic with respect to the linear operator $\boldsymbol{A}$. But since the whole theory is built up with reference to a single operator $\boldsymbol{A}$, the words 'with respect to the linear operator $\boldsymbol{A}$' are omitted for the sake of brevity (see the similar remark in footnote 2, p. 176).

Moreover, in $\boldsymbol{R}$ there exists a vector $\boldsymbol{g}^*$ whose relative minimal polynomial is $\psi_2(\lambda)$. Then

$$\psi_2(\boldsymbol{A})\,\boldsymbol{g}^* \equiv \boldsymbol{o} \qquad (\operatorname{mod} \boldsymbol{I}_1), \tag{22}$$

i.e., there exists a polynomial $\chi(\lambda)$ of degree $\leqq m-1$ such that

$$\psi_2(\boldsymbol{A})\,\boldsymbol{g}^* = \chi(\boldsymbol{A})\,\boldsymbol{e}. \tag{23}$$

We apply the operator $\varkappa(\boldsymbol{A})$ to both sides of the equation. Then by (21) we obtain on the left $\psi_1(\boldsymbol{A})\boldsymbol{g}^*$, i.e. zero, because $\psi_1(\lambda)$ is the absolute minimal polynomial of the space; therefore

$$\varkappa(\boldsymbol{A})\,\chi(\boldsymbol{A})\,\boldsymbol{e} = \boldsymbol{o}.$$

This equation shows that the product $\varkappa(\lambda)\,\chi(\lambda)$ is an annihilating polynomial of the vector $\boldsymbol{e}$ and is therefore divisible by the minimal polynomial $\psi_1(\lambda) = \varkappa(\lambda)\,\psi_2(\lambda)$, so that $\chi(\lambda)$ is divisible by $\psi_2(\lambda)$:

$$\chi(\lambda) = \varkappa_1(\lambda)\,\psi_2(\lambda), \tag{24}$$

where $\varkappa_1(\lambda)$ is a polynomial. Using this decomposition of $\chi(\lambda)$, we may rewrite (23) as follows:

$$\psi_2(\boldsymbol{A})\,[\boldsymbol{g}^* - \varkappa_1(\boldsymbol{A})\,\boldsymbol{e}] = \boldsymbol{o}. \tag{25}$$

We now introduce the vector

$$\boldsymbol{g} = \boldsymbol{g}^* - \varkappa_1(\boldsymbol{A})\,\boldsymbol{e}. \tag{26}$$

Then (25) can be written as follows:

$$\psi_2(\boldsymbol{A})\,\boldsymbol{g} = \boldsymbol{o}. \tag{27}$$

The last equation shows that $\psi_2(\lambda)$ is an absolute annihilating polynomial of the vector $\boldsymbol{g}$ and is therefore divisible by the absolute minimal polynomial of $\boldsymbol{g}$. On the other hand, we have from (26):

$$\boldsymbol{g} \equiv \boldsymbol{g}^* \quad (\operatorname{mod} \boldsymbol{I}_1). \tag{28}$$

Hence $\psi_2(\lambda)$, being the relative minimal polynomial of $\boldsymbol{g}^*$, is the same for $\boldsymbol{g}$ as well. Comparing the last two statements, we deduce that $\psi_2(\lambda)$ is simultaneously the relative and the absolute minimal polynomial of $\boldsymbol{g}$.

From the fact that $\psi_2(\lambda)$ is the absolute minimal polynomial of $\boldsymbol{g}$ it follows that the subspace $\boldsymbol{I}_2$ with the basis

$$\boldsymbol{g},\ \boldsymbol{A}\boldsymbol{g},\ \ldots,\ \boldsymbol{A}^{p-1}\boldsymbol{g} \tag{29}$$

is cyclic.

From the fact that $\psi_2(\lambda)$ is the relative minimal polynomial of $\boldsymbol{g}$ $(\operatorname{mod} \boldsymbol{I}_1)$ it follows that the vectors (29) are linearly independent $(\operatorname{mod} \boldsymbol{I}_1)$, i.e., no linear combination with coefficients not all zero can be equal to a linear combination of the vectors (20). Since the latter are themselves linearly independent, our last statement asserts the linear independence of the $m+p$ vectors

$$\boldsymbol{e},\ \boldsymbol{A}\boldsymbol{e},\ \ldots,\ \boldsymbol{A}^{m-1}\boldsymbol{e};\ \boldsymbol{g},\ \boldsymbol{A}\boldsymbol{g},\ \ldots,\ \boldsymbol{A}^{p-1}\boldsymbol{g}. \tag{30}$$

The vectors (30) form a basis of the invariant subspace $\boldsymbol{I}_1 + \boldsymbol{I}_2$ of dimension $m+p$.

If $n = m+p$, then $\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2$. If $n > m+p$, we consider $\boldsymbol{R}$ (mod $\boldsymbol{I}_1 + \boldsymbol{I}_2$) and continue our process of separating cyclic subspaces. Since the whole space $\boldsymbol{R}$ is of finite dimension n, this process must come to an end with some subspace $\boldsymbol{I}_t$, where $t \leqq n$.

We have arrived at the following theorem:

THEOREM 3 (Second Theorem on the Decomposition of a Space into Invariant Subspaces): *Relative to a given linear operator* $\boldsymbol{A}$ *the space can always be split into cyclic subspaces* $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_t$ *with minimal polynomials* $\psi_1(\lambda), \psi_2(\lambda), \ldots, \psi_t(\lambda)$

$$\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2 + \ldots + \boldsymbol{I}_t \tag{31}$$

such that $\psi_1(\lambda)$ *coincides with the minimal polynomial* $\psi(\lambda)$ *of the whole space and that each* $\psi_i(\lambda)$ *is divisible by* $\psi_{i-1}(\lambda)$ $(i = 2, 3, \ldots, t)$.

3. We now mention some properties of cyclic spaces. Let $\boldsymbol{R}$ be a cyclic n-dimensional space and $\psi(\lambda) = \lambda^m + \ldots$ its minimal polynomial. Then it follows from the definition of a cyclic space that $m = n$.[1] Conversely, suppose that $\boldsymbol{R}$ is an arbitrary space and that it is known that $m = n$. Applying the proof of the decomposition theorem, we represent $\boldsymbol{R}$ in the form (31). But the dimension of the cyclic subspace $\boldsymbol{I}_1$ is m, because its minimal polynomial coincides with the minimal polynomial of the whole space. Since $m = n$ by assumption, we have $\boldsymbol{R} = \boldsymbol{I}_1$, i.e., $\boldsymbol{R}$ is a cyclic space.

Thus we have established the following *criterion for cyclicity of a space*:

THEOREM 4: *A space is cyclic if and only if its dimension is equal to the degree of its minimal polynomial.*

Next, suppose that we have a decomposition of a cyclic space $\boldsymbol{R}$ into two invariant subspaces $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$:

$$\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2. \tag{32}$$

We denote the dimensions of $\boldsymbol{R}$, $\boldsymbol{I}_1$, and $\boldsymbol{I}_2$ by n, n_1, and n_2, their minimal polynomials by $\psi(\lambda)$, $\psi_1(\lambda)$, and $\psi_2(\lambda)$, and the degrees of these minimal polynomials by m, m_1, and m_2, respectively. Then

$$m_1 \leqq n_1, \quad m_2 \leqq n_2. \tag{33}$$

We add these inequalities term by term:

$$m_1 + m_2 \leqq n_1 + n_2. \tag{34}$$

Since $\psi(\lambda)$ is the least common multiple of $\psi_1(\lambda)$ and $\psi_2(\lambda)$, we have

$$m \leqq m_1 + m_2. \tag{35}$$

Moreover, it follows from (32) that

$$n = n_1 + n_2. \tag{36}$$

(34), (35), and (36) give us a chain of relations

$$m \leqq m_1 + m_2 \leqq n_1 + n_2 = n. \tag{37}$$

But since the space $\boldsymbol{R}$ is cyclic, the extreme numbers of this chain, m and n, are equal. Therefore we have equality in the middle terms, i.e.,

$$m = m_1 + m_2 = n_1 + n_2.$$

From the fact that $m = m_1 + m_2$ we deduce that $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are co-prime.

Bearing (33) in mind, we find from $m_1 + m_2 = n_1 + n_2$ that

$$m_1 = n_1 \quad \text{and} \quad m_2 = n_2. \tag{38}$$

These equations mean that the subspaces $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are cyclic.

Thus we have arrived at the following proposition:

THEOREM 5: *A cyclic space can only split into invariant subspaces that* 1. *are also cyclic and* 2. *have co-prime minimal polynomials.*

The same arguments (in the opposite order) show that Theorem 5 has a converse:

THEOREM 6: *If a space is split into invariant subspaces that* 1. *are cyclic and* 2. *have co-prime minimal polynomials, then the space itself is cyclic.*

Suppose now that $\boldsymbol{R}$ is a cyclic space and that its minimal polynomial is a power of an irreducible polynomial over F: $\psi(\lambda) = [\varphi(\lambda)]^c$. In this case, the minimal polynomial of every invariant subspace of $\boldsymbol{R}$ must also be a power of this irreducible polynomial $\varphi(\lambda)$. Therefore the minimal polynomials of any two invariant subspaces cannot be co-prime. But then, by what we have proved, $\boldsymbol{R}$ cannot split into invariant subspaces.

Suppose, conversely, that some space $\boldsymbol{R}$ is known not to split into invariant subspaces. Then $\boldsymbol{R}$ is a cyclic space, for otherwise, by the second decomposition theorem, it could be split into cyclic subspaces; moreover, the minimal polynomial of $\boldsymbol{R}$ must be a power of an irreducible polynomial, because otherwise $\boldsymbol{R}$ could be split into invariant subspaces, by the first decomposition theorem.

Thus we have reached the following conclusion:

THEOREM 7: *A space does not split into invariant subspaces if and only if* 1. *it is cyclic and* 2. *its minimal polynomial is a power of an irreducible polynomial over* F.

We now return to the decomposition (31) and split the minimal polynomials $\psi_1(\lambda), \psi_2(\lambda), \ldots, \psi_t(\lambda)$ of the cyclic subspaces $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_t$ into irreducible factors over F:

$$\left.\begin{array}{l}
\psi_1(\lambda) = [\varphi_1(\lambda)]^{c_1}[\varphi_2(\lambda)]^{c_2}\cdots[\varphi_s(\lambda)]^{c_s}, \\
\psi_2(\lambda) = [\varphi_1(\lambda)]^{d_1}[\varphi_2(\lambda)]^{d_2}\cdots[\varphi_s(\lambda)]^{d_s}, \\
\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
\psi_t(\lambda) = [\varphi_1(\lambda)]^{l_1}[\varphi_2(\lambda)]^{l_2}\cdots[\varphi_s(\gamma)]^{l_s}
\end{array}\right\} \tag{39}$$

$$(c_k \geqq d_k \geqq \ldots \geqq l_k \geqq 0;\text{[8]}\ k = 1,, 2, \ldots, s).$$

To $\boldsymbol{I}_1$ we apply the first decomposition theorem. Then we obtain

$$\boldsymbol{I}_1 = \boldsymbol{I}_1' + \boldsymbol{I}_1'' + \cdots + \boldsymbol{I}_1^{(s)};$$

where $\boldsymbol{I}_1', \boldsymbol{I}_1'', \ldots, \boldsymbol{I}_1^{(s)}$ are cyclic subspaces with the minimal polynomials $[\varphi_1(\lambda)]^{c_1}, [\varphi_2(\lambda)]^{c_2}, \ldots, [\varphi_s(\lambda)]^{c_s}$. Similarly we decompose the spaces $\boldsymbol{I}_2, \ldots, \boldsymbol{I}_t$. In this way we obtain a decomposition of the whole space $\boldsymbol{R}$ into cyclic subspaces with the minimal polynomials $[\varphi_k(\lambda)]^{c_k}$, $[\varphi_k(\lambda)]^{d_k}, \ldots, [\varphi_k(\lambda)]^{l_k}$ $(k = 1, 2, \ldots, s)$. (Here we neglect the powers whose exponents are zero.) From Theorem 7 it follows that these cyclic subspaces are indecomposable (into invariant subspaces). We have thus arrived at the following theorem:

THEOREM 8 (Third Theorem on the Decomposition of a Space into Invariant Subspaces): *A space can always be split into cyclic invariant subspaces*

$$\boldsymbol{R} = \boldsymbol{I}' + \boldsymbol{I}'' + \cdots + \boldsymbol{I}^{(u)}. \tag{40}$$

such that the minimal polynomial of each of these cyclic subspaces is a power of an irreducible polynomial.

This theorem gives the decomposition of a space into indecomposable invariant subspaces.

---

[8] Some of the exponents $d_k, \ldots, l_k$ for $k > 1$ may be equal to zero.

Note. Theorem 8 (the third decomposition theorem) has been proved by applying the first two decomposition theorems. But it can also be obtained by other means, namely, as an immediate (and almost trivial) corollary of Theorem 7.

For if the space $\boldsymbol{R}$ splits at all, then it can always be split into indecomposable invariant subspaces:

$$\boldsymbol{R} = \boldsymbol{I}' + \boldsymbol{I}'' + \ldots + \boldsymbol{I}^{(u)}. \tag{40}$$

By Theorem 7, each of the constituent subspaces is cyclic and has as its minimal polynomial a power of an irreducible polynomial over F.

§ 5. The Normal Form of a Matrix

1. Let $\boldsymbol{I}_1$ be an m-dimensional invariant subspace of $\boldsymbol{R}$. In $\boldsymbol{I}_1$ we take an arbitrary basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_m$ and complement it to a basis

$$\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_m, \boldsymbol{e}_{m+1}, \ldots, \boldsymbol{e}_n$$

of $\boldsymbol{R}$. Let us see what the matrix A of the operator $\boldsymbol{A}$ looks like in this basis. We remind the reader that the k-th column of A consists of the coordinates of the vector $\boldsymbol{A}\boldsymbol{e}_k$ $(k = 1, 2, \ldots, n)$. For $k \leqq m$ the vector $\boldsymbol{A}\boldsymbol{e}_k \in \boldsymbol{I}_1$ (by the invariance of $\boldsymbol{I}_1$) and the last $n - m$ coordinates of $\boldsymbol{A}\boldsymbol{e}_k$ are zero. Therefore A has the following form

$$A = \left\| \begin{array}{cc} \overbrace{A_1}^{m} & \overbrace{A_3}^{n-m} \\ O & A_2 \end{array} \right\| \begin{array}{l} \}\, m \\ \}\, n-m \end{array}, \tag{41}$$

where A_1 and A_2 are square matrices of orders m and $n - m$, respectively, and A_3 is a rectangular matrix. The fact that the fourth 'block' is zero expresses the invariance of the subspace $\boldsymbol{I}_1$. The matrix A_1 gives the operator $\boldsymbol{A}$ in $\boldsymbol{I}_1$ (with respect to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_m$).

Let us assume now that $\boldsymbol{e}_{m+1}, \ldots, \boldsymbol{e}_n$ is the basis of some invariant subspace $\boldsymbol{I}_2$, so that $\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2$ and a basis of the whole space is formed from the two parts that are the bases of the invariant subspaces $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$. Then obviously the block A_3 in (41) is also equal to zero and the matrix A has the quasi-diagonal form

$$A = \left\| \begin{array}{cc} A_1 & O \\ O & A_2 \end{array} \right\| = \{A_1, A_2\}, \tag{42}$$

where A_1 and A_2 are, respectively, square matrices of orders m and $n - m$ which give the operator in the subspaces $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ (with respect to the bases $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_m$ and $\boldsymbol{e}_{m+1}, \ldots, \boldsymbol{e}_n$). It is not difficult to see that, conversely, to a quasi-diagonal form of the matrix there always corresponds a decomposition of the space into invariant subspaces (and the basis of the whole space is formed from the bases of these subspaces).

2. By the second decomposition theorem, we can split the whole space $\boldsymbol{R}$ into cyclic subspaces $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_t$:

$$\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2 + \cdots + \boldsymbol{I}_t. \tag{43}$$

In the sequence of minimal polynomials of these subspaces $\psi_1(\lambda), \psi_2(\lambda), \ldots, \psi_t(\lambda)$ each factor is a divisor of the proceeding one (from which it follows automatically that the first polynomial is the minimal polynomial of the whole space).

Let

$$\begin{aligned} \psi_1(\lambda) &= \lambda^m + \alpha_1\lambda^{m-1} + \cdots + \alpha_m, \\ \psi_2(\lambda) &= \lambda^p + \beta_1\lambda^{p-1} + \cdots + \beta_p, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \psi_t(\lambda) &= \lambda^v + \varepsilon_1\lambda^{v-1} + \cdots + \varepsilon_v. \end{aligned} \qquad (m \geqq p \geqq \ldots \geqq v). \tag{44}$$

We denote by $\boldsymbol{e}, \boldsymbol{g}, \ldots, \boldsymbol{l}$ generating vectors of the subspaces $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_t$ and we form a basis of the whole space from the following bases of the cyclic subspaces:

$$\boldsymbol{e}, \boldsymbol{A}\boldsymbol{e}, \ldots, \boldsymbol{A}^{m-1}\boldsymbol{e}; \boldsymbol{g}, \boldsymbol{A}\boldsymbol{g}, \ldots, \boldsymbol{A}^{p-1}\boldsymbol{g}; \ldots; \boldsymbol{l}, \boldsymbol{A}\boldsymbol{l}, \ldots, \boldsymbol{A}^{v-1}\boldsymbol{l}. \tag{45}$$

Let us see what the matrix L_{I} corresponding to $\boldsymbol{A}$ in this basis looks like.

As we have explained at the beginning of this section, the matrix L_{I} must have quasi-diagonal form

$$L_{\mathrm{I}} = \begin{pmatrix} L_1 & O & \ldots & O \\ O & L_2 & \ldots & O \\ \cdot & \cdot & \cdot & \cdot \\ O & O & \ldots & L_t \end{pmatrix}. \tag{46}$$

The matrix L_1 corresponds to the operator $\boldsymbol{A}$ in $\boldsymbol{I}_1$ with respect to the basis $\boldsymbol{e}_1 = \boldsymbol{e}, \boldsymbol{e}_2 = \boldsymbol{A}\boldsymbol{e}, \ldots, \boldsymbol{e}_m = \boldsymbol{A}^{m-1}\boldsymbol{e}$. By applying the rule for the formation of the matrix for a given operator in a given basis (Chapter III, p. 67), we find

$$L_1 = \begin{Vmatrix} 0 & 0 & \dots & 0 & -\alpha_m \\ 1 & 0 & \dots & 0 & -\alpha_{m-1} \\ 0 & 1 & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & 0 & -\alpha_2 \\ 0 & 0 & \dots & 1 & -\alpha_1 \end{Vmatrix}. \tag{47}$$

Similarly

$$L_2 = \begin{Vmatrix} 0 & 0 & \dots & 0 & -\beta_p \\ 1 & 0 & \dots & 0 & -\beta_{p-1} \\ 0 & 1 & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & 0 & -\beta_2 \\ 0 & 0 & \dots & 1 & -\beta_1 \end{Vmatrix}. \tag{48}$$

Computing the characteristic polynomials of the matrices $L_1, L_2, \dots, L_t$, we find:

$$|\lambda E - L_1| = \psi_1(\lambda),\ |\lambda E - L_2| = \psi_2(\lambda),\ \dots,\ |\lambda E - L_t| = \psi_t(\lambda)$$

(for cyclic subspaces the characteristic polynomial of an operator $\boldsymbol{A}$ coincides with the minimal polynomial of the subspace relative to this operator).

The matrix L_{I} corresponds to the operator $\boldsymbol{A}$ in the 'canonical' basis (45). If A is the matrix corresponding to $\boldsymbol{A}$ in an arbitrary basis, then A is similar to L_{I}, i.e., there exists a non-singular matrix T such that

$$A = T L_{\mathrm{I}} T^{-1}. \tag{49}$$

Of the matrix L_{I} we shall say that it has the *first natural normal form*. This form is characterized by

1) The quasi-diagonal form;
2) The special structure of the diagonal blocks (47), (48), etc.
3) The additional condition: the characteristic polynomial of each diagonal block is divisible by the characteristic polynomial of the following block.

If we start not from the second, but from the third decomposition theorem, then in exactly the same way we would obtain a matrix L_{II} corresponding to the operator $\boldsymbol{A}$ in the appropriate basis—a matrix having the *second natural normal form*, which is characterized by

1) The quasi-diagonal form

$$L_{\mathrm{II}} = \{L^{(1)}, L^{(2)}, \dots, L^{(u)}\};$$

2) The special structure of the diagonal blocks (47), (48), etc.;
3) The additional condition: the characteristic polynomial of each block is a power of an irreducible polynomial over F.

3. In the following section we shall show that in the class of similar matrices corresponding to one and the same operator there is one and only one matrix having the first normal form,[9] and one and only one [10] having the second normal form. Moreover, we shall give an algorithm for the computation of the polynomials $\psi_1(\lambda), \psi_2(\lambda), \dots, \psi_t(\lambda)$ from the elements of the matrix A. Knowledge of these polynomials enables us to write out all the elements of the matrices L_{I} and L_{II} similar to A and having the first and second natural normal forms, respectively.

§ 6. Invariant Polynomials. Elementary Divisors

1. We[11] denote by $D_p(\lambda)$ the greatest common divisor of all the minors of order p of the characteristic matrix $A_\lambda = \lambda E - A$ $(p = 1, 2, \dots, n)$.[12] Since in the sequence

$$D_n(\lambda),\ D_{n-1}(\lambda),\ \dots,\ D_1(\lambda)$$

each polynomial is divisible by the following, the formulas

$$i_1(\lambda) = \frac{D_n(\lambda)}{D_{n-1}(\lambda)},\ i_2(\lambda) = \frac{D_{n-1}(\lambda)}{D_{n-2}(\lambda)},\ \dots,\ i_n(\lambda) = \frac{D_1(\lambda)}{D_0(\lambda)}\quad (D_0(\lambda) \equiv 1) \tag{50}$$

define n polynomials whose product is equal to the characteristic polynomial

$$\Delta(\lambda) = |\lambda E - A| = D_n(\lambda) = i_1(\lambda)\, i_2(\lambda) \cdots i_n(\lambda). \tag{51}$$

We split the polynomials $i_p(\lambda)$ $(p = 1, 2, \dots, n)$ into irreducible factors over F:

$$i_p(\lambda) = [\varphi_1(\lambda)]^{\gamma_p} [\varphi_2(\lambda)]^{\delta_p} \cdots \quad (p = 1, 2, \dots, n); \tag{52}$$

where $\varphi_1(\lambda)$, $\varphi_2(\lambda), \dots$ are distinct irreducible polynomials over F.

---

[9] This does not mean that there exists only one canonical basis of the form (45). There may be many canonical bases, but to all of them there corresponds one and the same matrix L_{I}.

[10] To within the order of the diagonal blocks.

[11] In subsection 1. of the present section we repeat the basic concepts of Chapter VI, § 3 for the characteristic matrix that were there established for an arbitrary polynomial matrix.

[12] We always take the highest coefficient of the greatest common divisor as 1.

The polynomials $i_1(\lambda), i_2(\lambda), \ldots, i_n(\lambda)$ are called the *invariant polynomials*, and all the non-constant powers among $[\varphi_1(\lambda)]^{\gamma_p}$, $[\varphi_2(\lambda)]^{\delta_p}, \ldots$ are called the *elementary divisors*, of the characteristic matrix $A_\lambda = \lambda E - A$ or, simply, of A.

The product of all the elementary divisors, like the product of all the invariant polynomials, is equal to the characteristic polynomial $\Delta(\lambda) = |\lambda E - A|$.

The name 'invariant polynomial' is justified by the fact that two similar matrices A and $\tilde{A}$,

$$\tilde{A} = T^{-1} A T, \tag{53}$$

always have identical invariant polynomials

$$i_p(\lambda) = \tilde{i}_p(\lambda) \quad (p = 1, 2, \ldots, n). \tag{54}$$

For it follows from (53) that

$$\tilde{A}_\lambda = \lambda E - \tilde{A} = T^{-1}(\lambda E - A)\, T = T^{-1} A_\lambda T. \tag{55}$$

Hence (see Chapter I, § 2) we obtain a relation between the minors of the similar matrices A_λ and $\tilde{A}_\lambda$:

$$\tilde{A}_\lambda \begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ k_1 & k_2 & \ldots & k_p \end{pmatrix} = \sum_{\substack{\alpha_1 < \alpha_2 < \cdots < \alpha_p \\ \beta_1 < \beta_2 < \cdots < \beta_p}} T^{-1} \begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ \alpha_1 & \alpha_2 & \ldots & \alpha_p \end{pmatrix} A_\lambda \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_p \\ \beta_1 & \beta_2 & \ldots & \beta_p \end{pmatrix} T \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_p \\ k_1 & k_2 & \ldots & k_p \end{pmatrix} \tag{56}$$

$$(p = 1, 2, \ldots, n).$$

This equation shows that every common divisor of all the minors of order p of A_λ is a common divisor of all the minors of order p of $\tilde{A}_\lambda$, and vice versa (since A and $\tilde{A}$ can interchange places). Hence it follows that $D_p(\lambda) = \tilde{D}_p(\lambda)$ $(p = 1, 2, \ldots, n)$ and that (54) holds.

Since all the matrices representing a given operator A in various bases are similar and therefore have the same invariant polynomials and the same elementary divisors, we can speak of the invariant polynomials and the elementary divisors of an operator A.

2. We choose now for $\tilde{A}$ the matrix L_I having the first natural normal form and we compute the invariant polynomials of A starting from the form of the matrix $\tilde{A}_\lambda = \lambda E - \tilde{A}$ (in (57) this matrix is written out for the case $m = 5,\ p = 4,\ q = 4,\ r = 3$):

$$\left\| \begin{array}{ccccc:cccc:cccc:ccc}
\lambda & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & \lambda & 0 & 0 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & \lambda & 0 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & \lambda & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \alpha_1+\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & \beta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & \lambda & 0 & \beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & \lambda & \beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \beta_1+\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & \gamma_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \lambda & 0 & \gamma_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \lambda & \gamma_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \gamma_1+\lambda & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & \varepsilon_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \lambda & \varepsilon_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \varepsilon_1+\lambda
\end{array} \right\| . \tag{57}$$

Using Laplace's Theorem, we find

$$D_n(\lambda) = |\lambda E - \tilde{A}| = |\lambda E - L_1|\,|\lambda E - L_2| \cdots |\lambda E - L_t| = \psi_1(\lambda)\psi_2(\lambda)\cdots\psi_t(\lambda). \tag{58}$$

Now let us find $D_{n-1}(\lambda)$. We consider the minor of the element α_m. This minor, apart from a factor ± 1, is equal to

$$|\lambda E - L_2| \cdots |\lambda E - L_t| = \psi_2(\lambda) \cdots \psi_t(\lambda). \tag{59}$$

We shall show that this minor of order $n - 1$ is a divisor of all the other minors of order $n - 1$, so that

$$D_{n-1}(\lambda) = \psi_2(\lambda) \cdots \psi_t(\lambda). \tag{60}$$

For this purpose we first take the minor of an element outside the diagonal blocks and show that it vanishes. To obtain this minor we have to suppress one row and one column in the matrix (57). The lines crossed out in this case intersect two distinct diagonal blocks, so that in each of these blocks only one line is crossed out. Suppose, for example, that in the j-th diagonal block one of the rows is crossed out. In the minor we take that vertical strip which contains this diagonal block. In this strip, which has s columns, all the rows except $s - 1$ rows consist entirely of zeros (we have denoted the order of A_j by s). Expanding the determinant of order $n - 1$ by Laplace's Theorem with respect to the minors of order s in this strip, we see that it is equal to zero.

Now we take the minor of an element inside one of the diagonal blocks. In this case the lines crossed out 'mutilate' only one of the diagonal blocks, say the j-th, and the matrix of the minor is again quasi-diagonal. Therefore the minor is equal to

$$\psi_1(\lambda)\cdots\psi_{j-1}(\lambda)\,\psi_{j+1}(\lambda)\cdots\psi_t(\lambda)\,\chi(\lambda), \tag{61}$$

where $\chi(\lambda)$ is the determinant of the 'mutilated' j-th diagonal block. Since $\psi_i(\lambda)$ is divisible by $\psi_{i+1}(\lambda)$ $(i=1,2,\ldots,t-1)$, the product (61) is divisible by (59). Thus, equation (60) can be regarded as proved. By similar arguments we obtain:

$$\left.\begin{aligned} D_{n-2}(\lambda) &= \psi_3(\lambda)\cdots\psi_t(\lambda),\\ &\cdots\cdots\cdots\cdots\cdots\\ D_{n-t+1}(\lambda) &= \psi_t(\lambda),\\ D_{n-t}(\lambda) &= \cdots = D_1(\lambda)=1. \end{aligned}\right\} \tag{62}$$

From (58), (60), and (62) we find:

$$\begin{gathered} \psi_1(\lambda)=\frac{D_n(\lambda)}{D_{n-1}(\lambda)}=i_1(\lambda),\ \ \psi_2(\lambda)=\frac{D_{n-1}(\lambda)}{D_{n-2}(\lambda)}=i_2(\lambda),\ \ \ldots,\\ \psi_t(\lambda)=\frac{D_{n-t+1}(\lambda)}{D_{n-t}(\lambda)}=i_t(\lambda),\\ i_{t+1}(\lambda)=\cdots=i_n(\lambda)=1. \end{gathered} \tag{63}$$

The formulas (63) show that the polynomials $\psi_1(\lambda), \psi_2(\lambda), \ldots, \psi_t(\lambda)$ coincide with the invariant polynomials, other than 1, of the operator $\boldsymbol{A}$ (or the corresponding matrix A).

Let us give three equivalent formulations of the results obtained:

THEOREM 9 (More precise form of the Second Decomposition Theorem): *If $\boldsymbol{A}$ is a linear operator in $\boldsymbol{R}$, then the space $\boldsymbol{R}$ can be decomposed into cyclic subspaces*

$$\boldsymbol{R}=\boldsymbol{I}_1+\boldsymbol{I}_2+\ldots+\boldsymbol{I}_t$$

such that in the sequence of minimal polynomials $\psi_1(\lambda), \psi_2(\lambda), \ldots, \psi_t(\lambda)$ of the subspaces $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_t$, each is divisible by the following. The polynomials $\psi_1(\lambda), \psi_2(\lambda), \ldots, \psi_t(\lambda)$ are uniquely determined: they coincide with the invariant polynomials, other than 1, of the operator $\boldsymbol{A}$.

THEOREM 9′: *For every linear operator $\boldsymbol{A}$ in $\boldsymbol{R}$ there exists a basis in which the matrix L_{I} that gives the operator is of the first natural normal form. This matrix is uniquely determined when the operator $\boldsymbol{A}$ is given: the characteristic polynomials of the diagonal blocks of L_{I} are the invariant polynomials of $\boldsymbol{A}$.*

THEOREM 9″: *In every class of similar matrices (with elements in* F*) there exists one and only one matrix L_{I} having the first natural normal form. The characteristic polynomials of the diagonal blocks of L_{I} coincide with the invariant polynomials (other than* 1*) of every matrix of that class.*

On p. 194 we established that two similar matrices have the same invariant polynomials. Now suppose, conversely, that two matrices A and B with elements in F are known to have the same invariant polynomials. Since the matrix L_{I} is uniquely determined when these polynomials are given, the two matrices A and B are similar to one and the same matrix L_{I} and, therefore, to each other. We thus arrive at the following proposition:

THEOREM 10: *Two matrices with elements in* F *are similar if and only if they have the same invariant polynomials.*[13]

3. The characteristic polynomial $\Delta(\lambda)$ of the operator $\boldsymbol{A}$ coincides with $D_n(\lambda)$, and hence with the product of all invariant polynomials:

$$\Delta(\lambda)=\psi_1(\lambda)\,\psi_2(\lambda)\cdots\psi_t(\lambda). \tag{64}$$

But $\psi_1(\lambda)$ is the minimal polynomial of the whole space with respect to $\boldsymbol{A}$; hence $\psi_1(\boldsymbol{A})=\boldsymbol{O}$ and by (64)

$$\Delta(\boldsymbol{A})=\boldsymbol{O}. \tag{65}$$

Thus we have incidentally obtained the Hamilton-Cayley Theorem (see Chapter IV, § 4):

Every linear operator (every square matrix) satisfies its characteristic equation.

In § 4 by splitting the polynomials $\psi_1(\lambda), \psi_2(\lambda), \ldots, \psi_t(\lambda)$ into irreducible factors over F:

$$\begin{aligned} \psi_1(\lambda)&=[\varphi_1(\lambda)]^{c_1}\,[\varphi_2(\lambda)]^{c_2}\cdots[\varphi_s(\lambda)]^{c_s},\\ \psi_2(\lambda)&=[\varphi_1(\lambda)]^{d_1}\,[\varphi_2(\lambda)]^{d_2}\cdots[\varphi_s(\lambda)]^{d_s},\quad \begin{pmatrix} c_k\geqq d_k\geqq\cdots\geqq l_k,\\ k=1,2,\ldots,s \end{pmatrix},\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \psi_t(\lambda)&=[\varphi_1(\lambda)]^{l_1}\,[\varphi_2(\lambda)]^{l_2}\cdots[\varphi_s(\lambda)]^{l_s} \end{aligned} \tag{66}$$

we were led to the third decomposition theorem. To each power with nonzero exponent on the right-hand sides of (66) there corresponds an invariant subspace in this decomposition.

By (63) all the powers, other than 1, among $[\varphi_k(\lambda)]^{c_k}, \ldots, [\varphi_k(\lambda)]^{l_k}$ $(k=1,2,\ldots,s)$ are the elementary divisors of $\boldsymbol{A}$ (or A) in the field F (see p. 194).

Thus we arrive at the following more precise statement of the third decomposition theorem:

---

[13] Or (what is the same) the same elementary divisors in the field F.

THEOREM 11: *If* $\boldsymbol{A}$ *is a linear operator in a vector space* $\boldsymbol{R}$ *over a field* F, *then* $\boldsymbol{R}$ *can be split into cyclic subspaces whose minimal polynomials are the elementary divisors of* $\boldsymbol{A}$ *in* F.

Let

$$\boldsymbol{R} = \boldsymbol{I}' + \boldsymbol{I}'' + \dots + \boldsymbol{I}^{(u)} \tag{67}$$

be such a decomposition. We denote by $\boldsymbol{e}', \boldsymbol{e}'', \dots, \boldsymbol{e}^{(u)}$ generating vectors of the subspaces $\boldsymbol{I}', \boldsymbol{I}'', \dots, \boldsymbol{I}^{(u)}$ and from the 'cyclic' bases of these subspaces we form a basis of the whole space

$$\boldsymbol{e}', \boldsymbol{Ae}', \dots;\ \boldsymbol{e}'', \boldsymbol{Ae}'', \dots;\ \boldsymbol{e}^{(u)}, \boldsymbol{Ae}^{(u)}, \dots. \tag{68}$$

It is easy to see that the matrix L_{II} corresponding to the operator $\boldsymbol{A}$ in the basis (68) has quasi-diagonal form, like L_{I}:

$$L_{\text{II}} = \{L_1, L_2, \dots, L_u\}. \tag{69}$$

The diagonal blocks $L_1, L_2, \dots, L_u$ are of the same structure as the blocks (47) and (48) of L_{I}. However, the characteristic polynomials of these diagonal blocks are not the invariant polynomials, but the elementary divisors of $\boldsymbol{A}$. The matrix L_{II} has the second natural normal form (see § 5).

We have arrived at another formulation of Theorem 11:

THEOREM 11′: *For every linear operator* $\boldsymbol{A}$ *in* $\boldsymbol{R}$ (*over the field* F) *there exists a basis in which the matrix* L_{II} *giving the operator is of the second natural normal form; the characteristic polynomials of the diagonal blocks are the elementary divisors of* $\boldsymbol{A}$ *in* F.

This theorem also admits a formulation in terms of matrices:

THEOREM 11″: *A matrix* A *with elements in the field* F *is always similar to a matrix* L_{II} *having the second natural normal form in which the characteristic polynomials of the diagonal blocks are the elementary divisors of* A.

Theorem 11 and the associated Theorems 11′ and 11″ have, in a certain sense, a converse.

Let

$$\boldsymbol{R} = \boldsymbol{I}' + \boldsymbol{I}'' + \dots + \boldsymbol{I}^{(u)}$$

be an arbitrary decomposition of a space $\boldsymbol{R}$ into indecomposable invariant subspaces. Then by Theorem 7 the subspaces $\boldsymbol{I}', \boldsymbol{I}'', \dots, \boldsymbol{I}^{(u)}$ are cyclic and their minimal polynomials are powers of irreducible polynomials over F. We may write these powers, after adding powers with zero exponent if necessary, in the form[14]

---

[14] At least one of the numbers $l_1, l_2, \dots, l_s$ is positive.

$$\begin{matrix} [\varphi_1(\lambda)]^{c_1}, [\varphi_2(\lambda)]^{c_2}, \dots, [\varphi_s(\lambda)]^{c_s}, \\ [\varphi_1(\lambda)]^{d_1}, [\varphi_2(\lambda)]^{d_2}, \dots, [\varphi_s(\lambda)]^{d_s}, \\ \dots\dots\dots\dots\dots\dots\dots \\ [\varphi_1(\lambda)]^{l_1}, [\varphi_2(\lambda)]^{l_2}, \dots, [\varphi_s(\lambda)]^{l_s}. \end{matrix} \quad \begin{pmatrix} c_k \geqq d_k \geqq \cdots \geqq l_k \geqq 0, \\ k = 1, 2, \dots, s \end{pmatrix}. \tag{70}$$

We denote the sum of the subspaces whose minimal polynomials are in the first row by $\boldsymbol{I}_1$. Similarly, we introduce $\boldsymbol{I}_2, \dots, \boldsymbol{I}_t$ (t is the number of rows in (70)). By Theorem 6, the subspaces $\boldsymbol{I}_1, \boldsymbol{I}_2, \dots, \boldsymbol{I}_t$ are cyclic and their minimal polynomials $\psi_1(\lambda), \psi_2(\lambda), \dots, \psi_t(\lambda)$ are determined by the formulas (66). Here in the sequence $\psi_1(\lambda), \psi_2(\lambda), \dots, \psi_t(\lambda)$ each polynomial is divisible by the following. But then Theorem 9 is immediately applicable to the decomposition

$$\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2 + \dots + \boldsymbol{I}_t .$$

By this theorem

$$\psi_p(\lambda) = i_p(\lambda) \quad (p = 1, 2, \dots, n),$$

and therefore, by (66), all the powers (70) with non-zero exponent are the elementary divisors of $\boldsymbol{A}$ in the field F. Thus we have the following theorem:

THEOREM 12: *If the vector space* $\boldsymbol{R}$ (*over the field* F) *is split in any way into decomposable invariant subspaces* (*with respect to an operator* $\boldsymbol{A}$), *then the minimal polynomials of these subspaces are all the elementary divisors of* $\boldsymbol{A}$ *in* F.

There is an equivalent formulation in terms of matrices:

THEOREM 12′: *In each class of similar matrices* (*with elements in* F) *there exists only one matrix* (*to within the order of the diagonal blocks*) *having the second normal form* L_{II}; *the characteristic polynomials of its diagonal blocks are the elementary divisors of every matrix of the given class.*

Suppose that the space $\boldsymbol{R}$ is split into two invariant subspaces (with respect to an operator $\boldsymbol{A}$)

$$\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2.$$

When we split $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ into indecomposable subspaces, we obtain at the same time a decomposition of the whole space $\boldsymbol{R}$ into indecomposable subspaces. Hence, bearing Theorem 12 in mind, we obtain:

THEOREM 13: *If the space* $\boldsymbol{R}$ *is split into invariant subspaces with respect to an operator* $\boldsymbol{A}$, *then the elementary divisors of* $\boldsymbol{A}$ *in each of these invariant subspaces, taken in their totality, form a complete system of elementary divisors of* $\boldsymbol{A}$ *in* $\boldsymbol{R}$.

This theorem has the following matrix form:

THEOREM 13′: *A complete system of elementary divisors in* F *of a quasi-diagonal matrix is obtained as the union of the elementary divisors of the diagonal blocks.*

Theorem 13′ is often used for the actual process of finding the elementary divisors of a matrix.

§ 7. The Jordan Normal Form of a Matrix

1. Suppose that all the roots of the characteristic polynomial $\Delta(\lambda)$ of an operator $\boldsymbol{A}$ belong to the field F. This will hold true, in particular, if F is the field of all complex numbers.

In this case, the decomposition of the invariant polynomials into elementary divisors in F will look as follows:

$$\begin{aligned} i_1(\lambda) &= (\lambda-\lambda_1)^{c_1}\ (\lambda-\lambda_2)^{c_2}\ \cdots\ (\lambda-\lambda_s)^{c_s}, \\ i_2(\lambda) &= (\lambda-\lambda_1)^{d_1}\ (\lambda-\lambda_2)^{d_2}\ \cdots\ (\lambda-\lambda_s)^{d_s}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ i_t(\lambda) &= (\lambda-\lambda_1)^{l_1}\ (\lambda-\lambda_2)^{l_2}\ \cdots\ (\lambda-\lambda_s)^{l_s}. \end{aligned} \quad \begin{pmatrix} c_k \geqq d_k \geqq \cdots \geqq l_k \geqq 0, \\ c_k > 0;\ k = 1, 2, \ldots, s \end{pmatrix} \qquad (71)$$

Since the product of all the invariant polynomials is equal to the characteristic polynomial $\Delta(\lambda)$, $\lambda_1, \lambda_2, \ldots, \lambda_s$ in (71) are all the distinct roots of $\Delta(\lambda)$.

We take an arbitrary elementary divisor

$$(\lambda-\lambda_0)^p; \qquad (72)$$

here λ_0 is one of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_s$ and p is one of the (non-zero) exponents $c_k, d_k, \ldots, l_k$ $(k = 1, 2, \ldots, s)$.

To this elementary divisor there corresponds in (67) a definite cyclic subspace $\boldsymbol{I}$, generated by a vector which we denote by $\boldsymbol{e}$. For this vector $(\lambda-\lambda_0)^p$ is the minimal polynomial.

We consider the vectors

$$\boldsymbol{e}_1 = (\boldsymbol{A}-\lambda_0\boldsymbol{E})^{p-1}\boldsymbol{e},\ \ \boldsymbol{e}_2 = (\boldsymbol{A}-\lambda_0\boldsymbol{E})^{p-2}\boldsymbol{e},\ \ldots,\ \boldsymbol{e}_p = \boldsymbol{e}. \qquad (73)$$

The vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_p$ are linearly independent, since otherwise there would be an annihilating polynomial for $\boldsymbol{e}$ of degree less than p, which is impossible. Now we note that

$$(\boldsymbol{A}-\lambda_0\boldsymbol{E})\,\boldsymbol{e}_1 = \boldsymbol{o},\ (\boldsymbol{A}-\lambda_0\boldsymbol{E})\,\boldsymbol{e}_2 = \boldsymbol{e}_1,\ \ldots,\ (\boldsymbol{A}-\lambda_0\boldsymbol{E})\,\boldsymbol{e}_p = \boldsymbol{e}_{p-1} \qquad (74)$$

or

$$\boldsymbol{A}\boldsymbol{e}_1 = \lambda_0\boldsymbol{e}_1,\ \ \boldsymbol{A}\boldsymbol{e}_2 = \lambda_0\boldsymbol{e}_2 + \boldsymbol{e}_1,\ \ldots,\ \ \boldsymbol{A}\boldsymbol{e}_p = \lambda_0\boldsymbol{e}_p + \boldsymbol{e}_{p-1}. \qquad (75)$$

With the help of (75) we can easily write down the matrix corresponding to $\boldsymbol{A}$ in $\boldsymbol{I}$ for the basis (73). This matrix looks as follows:

$$\left\| \begin{matrix} \lambda_0 & 1 & 0 & \ldots & 0 \\ 0 & \lambda_0 & 1 & \ldots & 0 \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & 1 \\ 0 & 0 & 0 & \ldots & \lambda_0 \end{matrix} \right\| = \lambda_0 E^{(p)} + H^{(p)}, \qquad (76)$$

where $E^{(p)}$ is the unit matrix of order p and $H^{(p)}$ the matrix of order p which has 1's along the first superdiagonal and 0's everywhere else.

Linearly independent vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_p$ for which (75) holds form a so-called *Jordan chain of vectors* in $\boldsymbol{I}$. From Jordan chains connected with each subspace $\boldsymbol{I}', \boldsymbol{I}'', \ldots, \boldsymbol{I}^{(u)}$ we form a *Jordan basis* of $\boldsymbol{R}$. If we now denote the minimal polynomials of these subspaces, i.e., the elementary divisors of $\boldsymbol{A}$, by

$$(\lambda-\lambda_1)^{p_1},\ (\lambda-\lambda_2)^{p_2},\ \ldots,\ (\lambda-\lambda_u)^{p_u} \qquad (77)$$

(the numbers $\lambda_1, \lambda_2, \ldots, \lambda_u$ need not all be distinct), then the matrix J corresponding to $\boldsymbol{A}$ in a Jordan basis has the following quasi-diagonal form:

$$J = \{\, \lambda_1 E^{(p_1)} + H^{(p_1)},\ \ \lambda_2 E^{(p_2)} + H^{(p_2)},\ \ldots,\ \lambda_u E^{(p_u)} + H^{(p_u)} \,\}. \qquad (78)$$

We shall say of the matrix J that it is of *Jordan normal form* or simply *Jordan form*. The matrix J can be written down at once when the elementary divisors of $\boldsymbol{A}$ in the field F containing all the characteristic roots of the equation $\Delta(\lambda) = 0$ are known.

Every matrix A is similar to a matrix J of Jordan normal form, i.e., for an arbitrary matrix A there always exists a non-singular matrix $T\,(|\,T\,| \neq 0)$ such that

$$A = TJT^{-1}.$$

If all the elementary divisors of $\boldsymbol{A}$ are of the first degree (and in that case only), the Jordan form is a diagonal matrix and we have:

$$A = T\,\{\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n\}\,T^{-1}.$$

Thus: *A linear operator $\boldsymbol{A}$ has simple structure* (see Chapter III, § 8) *if and only if all the elementary divisors of $\boldsymbol{A}$ are linear.*

Let us number the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_p$ defined by (70) in the reverse order:

$$\boldsymbol{g}_1 = \boldsymbol{e}_p = \boldsymbol{e},\ \boldsymbol{g}_2 = \boldsymbol{e}_{p-1} = (\boldsymbol{A}-\lambda_0\boldsymbol{E})\,\boldsymbol{e},\ \ldots,\ \boldsymbol{g}_p = \boldsymbol{e}_1 = (\boldsymbol{A}-\lambda_0\boldsymbol{E})^{p-1}\,\boldsymbol{e}. \qquad (79)$$

Then

$$(\boldsymbol{A}-\lambda_0\boldsymbol{E})\,\boldsymbol{g}_1=\boldsymbol{g}_2,\quad (\boldsymbol{A}-\lambda_0\boldsymbol{E})\,\boldsymbol{g}_2=\boldsymbol{g}_3,\ \ldots,\quad (\boldsymbol{A}-\lambda_0\boldsymbol{E})\,\boldsymbol{g}_p=\boldsymbol{o}\,;$$

hence

$$\boldsymbol{A}\boldsymbol{g}_1=\lambda_0\boldsymbol{g}_1+\boldsymbol{g}_2,\quad \boldsymbol{A}\boldsymbol{g}_2=\lambda_0\boldsymbol{g}_2+\boldsymbol{g}_3,\ \ldots,\quad \boldsymbol{A}\boldsymbol{g}_p=\lambda_0\boldsymbol{g}_p.$$

The vectors (79) form a basis in the cyclic invariant subspace $\boldsymbol{I}$ that corresponds in (67) to the elementary divisor $(\lambda-\lambda_0)^p$.

In this basis, as is easy to see, to the operator $\boldsymbol{A}$ there corresponds the matrix

$$\left\|\begin{matrix}\lambda_0 & 0 & 0 & \ldots & 0\\ 1 & \lambda_0 & 0 & \ldots & 0\\ 0 & 1 & \lambda_0 & & \cdot\\ \cdot & \cdot & \cdot & & \cdot\\ \cdot & & \cdot & \cdot & \cdot\\ 0 & 0 & \ldots & 1 & \lambda_0\end{matrix}\right\|=\lambda_0E^{(p)}+F^{(p)}.$$

We shall say of the vectors (79) that they form a *lower Jordan chain* of vectors. If we take a lower Jordan chain of vectors in each subspace $\boldsymbol{I}', \boldsymbol{I}''$, $\ldots, \boldsymbol{I}^{(u)}$ of (67), we can form from these chains a *lower Jordan basis* in which to the operator $\boldsymbol{A}$ there corresponds the quasi-diagonal matrix

$$J_1=\{\lambda_1E^{(p_1)}+F^{(p_1)},\ \lambda_2E^{(p_2)}+F^{(p_2)},\ \ldots,\ \lambda_uE^{(p_u)}+F^{(p_u)}\}. \qquad (80)$$

We shall say of the matrix J_1 that it is of *lower Jordan form*. In contrast to (80), we shall sometimes call (78) an *upper Jordan matrix*.

Thus: *Every matrix A is similar to an upper and to a lower Jordan matrix.*

§ 8. Krylov's Method of Transforming the Secular Equation

1. When a matrix $A=\|a_{ik}\|_1^n$ is given, then its characteristic (secular) equation can be written in the form

$$\Delta(\lambda)\equiv(-1)^n\left|\begin{matrix}a_{11}-\lambda & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22}-\lambda & \ldots & a_{2n}\\ \cdot & \cdot & \cdot & \cdot\\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda\end{matrix}\right|=0. \qquad (81)$$

On the left-hand side of this equation is the characteristic polynomial $\Delta(\lambda)$ of degree n. For the direct computation of the coefficients of this polynomial it is necessary to expand the characteristic determinant $|A-\lambda E|$; and for large n this involves very cumbersome computational work, because λ occurs in the diagonal elements of the determinant.[15]

In 1937, A. N. Krylov [251] proposed a transformation of the characteristic determinant as a result of which λ occurs only in the elements of one column (or row).

Krylov's transformation simplifies the computation of the coefficients of the characteristic equation considerably.[16]

In this section we shall give an algebraic method of transforming the characteristic equation which differs somewhat from Krylov's own method.[17]

We consider an n-dimensional vector space $\boldsymbol{R}$ with basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ and the linear operator $\boldsymbol{A}$ in $\boldsymbol{R}$ determined by a given matrix $A=\|a_{ik}\|_1^n$ in this basis. We take an arbitrary vector $\boldsymbol{x}\neq\boldsymbol{o}$ in $\boldsymbol{R}$ and form the sequence of vectors

$$\boldsymbol{x},\ \boldsymbol{A}\boldsymbol{x},\ \boldsymbol{A}^2\boldsymbol{x},\ \ldots\,. \qquad (82)$$

Suppose that the first p vectors $\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}, \ldots, \boldsymbol{A}^{p-1}\boldsymbol{x}$ of this sequence are linearly independent and that the $(p+1)$-st vector $\boldsymbol{A}^p\boldsymbol{x}$ is a linear combination of these p vectors:

$$\boldsymbol{A}^p\boldsymbol{x}=-\alpha_p\boldsymbol{x}-\alpha_{p-1}\,\boldsymbol{A}\boldsymbol{x}-\cdots-\alpha_1\boldsymbol{A}^{p-1}\boldsymbol{x} \qquad (83)$$

or

$$\varphi(\boldsymbol{A})\,\boldsymbol{x}=\boldsymbol{o}, \qquad (84)$$

where

$$\varphi(\lambda)=\lambda^p+\alpha_1\lambda^{p-1}+\cdots+\alpha_p. \qquad (85)$$

All the further vectors in (82) can also be expressed linearly by the first p vectors of the sequence.[18] Thus, in (82) there are p linearly independent

---

[15] We recall that the coefficient of λ^k in $\Delta(\lambda)$ is equal (apart from the sign) to the sum of all the principal minors of order $n-k$ in A $(k=1,2,\ldots,n)$. Thus, even for $n=6$, the direct determination of the coefficient of λ in $\Delta(\lambda)$ would require the computation of six determinants of order 5; that of λ^2 would require fifteen determinants of order 4; etc.

[16] The algebraic analysis of Krylov's method of transforming the secular equation is contained in a number of papers [268], [269], [211], [168], and [149].

[17] Krylov arrived at his method of transformation by starting from a system of n linear differential equations with constant coefficients. Krylov's approach in algebraic form can be found, for example, in [268] and [168] and in § 21 of the book [25].

[18] When we apply the operator $\boldsymbol{A}$ to both sides of (83) we express $\boldsymbol{A}^{p+1}\boldsymbol{x}$ linearly in terms of $\boldsymbol{A}\boldsymbol{x}, \ldots, \boldsymbol{A}^{p-1}\boldsymbol{x}, \boldsymbol{A}^p\boldsymbol{x}$. But $\boldsymbol{A}^p\boldsymbol{x}$, by (83), is expressed linearly in terms of $\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}, \ldots, \boldsymbol{A}^{p-1}\boldsymbol{x}$. Hence we obtain a similar expression for $\boldsymbol{A}^{p+1}\boldsymbol{x}$. By applying the operator $\boldsymbol{A}$ to the expression thus obtained for $\boldsymbol{A}^{p+1}\boldsymbol{x}$, we express $\boldsymbol{A}^{p+2}\boldsymbol{x}$ in terms of $\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}, \ldots, \boldsymbol{A}^{p+1}\boldsymbol{x}$, etc.

vectors and this maximal number of linearly independent vectors in (82) is always realized by the *first* p vectors.

The polynomial $\varphi(\lambda)$ is the minimal (annihilating) polynomial of the vector $\boldsymbol{x}$ with respect to the operator $\boldsymbol{A}$ (see § 1). *The method of Krylov consists in an effective determination of the minimal polynomial* $\varphi(\lambda)$ *of* $\boldsymbol{x}$.

We consider separately two cases: the *regular* case, where $p = n$; and the *singular* case, where $p < n$.

The polynomial $\varphi(\lambda)$ is a divisor of the minimal polynomial $\psi(\lambda)$ of the whole space $\boldsymbol{R}$,[19] and $\psi(\lambda)$ in turn is a divisor of the characteristic polynomial $\Delta(\lambda)$. Therefore $\varphi(\lambda)$ is always a divisor of $\Delta(\lambda)$.

In the regular case, $\varphi(\lambda)$ and $\Delta(\lambda)$ are of the same degree and, since their highest coefficients are equal, they coincide. Thus, *in the regular case*

$$\Delta(\lambda) \equiv \psi(\lambda) \equiv \varphi(\lambda),$$

and therefore *in the regular case Krylov's method is a method of computing the coefficients of the characteristic polynomial* $\Delta(\lambda)$.

In the singular case, as we shall see later, Krylov's method does not enable us to determine $\Delta(\lambda)$, and in this case it only determines the divisor $\varphi(\lambda)$ of $\Delta(\lambda)$.

In explaining Krylov's transformation, we shall denote the coordinates of $\boldsymbol{x}$ in the given basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ by $a, b, \ldots, l$, and the coordinates of the vector $\boldsymbol{A}^k\boldsymbol{x}$ by $a_k, b_k, \ldots, l_k$ $(k = 1, 2, \ldots, n)$.

2. *Regular case*: $p = n$. In this case, the vectors $\boldsymbol{x}, \boldsymbol{Ax}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{x}$ are linearly independent and the equations (83), (84), and (85) assume the form

$$\boldsymbol{A}^n\boldsymbol{x} = -\alpha_n\boldsymbol{x} - \alpha_{n-1}\boldsymbol{Ax} - \cdots - \alpha_1\boldsymbol{A}^{n-1}\boldsymbol{x} \tag{86}$$

or

$$\Delta(\boldsymbol{A})\boldsymbol{x} = \boldsymbol{o}, \tag{87}$$

where

$$\Delta(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n. \tag{88}$$

The condition of linear independence of the vectors $\boldsymbol{x}, \boldsymbol{Ax}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{x}$ may be written analytically as follows (see Chapter III, § 1):

$$M = \begin{vmatrix} a & b & \ldots & l \\ a_1 & b_1 & \ldots & l_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1} & b_{n-1} & \ldots & l_{n-1} \end{vmatrix} \neq 0. \tag{89}$$

We consider the matrix formed from the coordinate vectors $\boldsymbol{x}, \boldsymbol{Ax}, \ldots, \boldsymbol{A}^n\boldsymbol{x}$:

---

[19] $\psi(\lambda)$ is the minimal polynomial of A.

$$\left\| \begin{matrix} a & b & \ldots & l \\ a_1 & b_1 & \ldots & l_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1} & b_{n-1} & \ldots & l_{n-1} \\ a_n & b_n & & l_n \end{matrix} \right\|. \tag{90}$$

In the regular case the rank of this matrix is n. The first n rows of the matrix are linearly independent, and the last, $(n+1)$-st, row is a linear combination of the preceding n.

We obtain the dependence between the rows of (90) when we replace the vector equation (86) by the equivalent system of n scalar equations

$$\left.\begin{array}{l} -\alpha_n a - \alpha_{n-1}a_1 - \cdots - \alpha_1 a_{n-1} = a_n \\ -\alpha_n b - \alpha_{n-1}b_1 - \cdots - \alpha_1 b_{n-1} = b_n \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ -\alpha_n l - \alpha_{n-1}l_1 - \cdots - \alpha_1 l_{n-1} = l_n. \end{array}\right\}. \tag{91}$$

From this system of n linear equations we may determine the unknown coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ uniquely,[20] and substitute their values in (88). This elimination of $\alpha_1, \alpha_2, \ldots, \alpha_n$ from (88) and (91) can be performed symmetrically. For this purpose we rewrite (88) and (91) as follows:

$$\left.\begin{array}{ll} a\alpha_n + a_1\alpha_{n-1} + \cdots + a_{n-1}\alpha_1 + a_n\alpha_0 & = 0 \\ b\alpha_n + b_1\alpha_{n-1} + \cdots + b_{n-1}\alpha_1 + b_n\alpha_0 & = 0 \\ \cdot & \\ l\alpha_n + l_1\alpha_{n-1} + \cdots + l_{n-1}\alpha_1 + l_n\alpha_0 & = 0 \\ 1\alpha_n + \lambda\alpha_{n-1} + \cdots + \lambda^{n-1}\alpha_1 + [\lambda^n - \Delta(\lambda)]\,\alpha_0 & = 0 \end{array}\right\} \quad (\alpha_0 = 1).$$

Since this system of $n+1$ equations in the $n+1$ unknown $\alpha_0, \alpha_2, \ldots, \alpha_n$ has a non-zero solution $(\alpha_0 = 1)$, its determinant must vanish:

$$\begin{vmatrix} a & a_1 & \ldots & a_{n-1} & a_n \\ b & b_1 & \ldots & b_{n-1} & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ l & l_1 & \ldots & l_{n-1} & l_n \\ 1 & \lambda & \ldots & \lambda^{n-1} & \lambda^n - \Delta(\lambda) \end{vmatrix} = 0. \tag{92}$$

Hence we determine $\Delta(\lambda)$ after a preliminary transposition of the determinant (92) with respect to the main diagonal:

---

[20] By (89), the determinant of this system is different from zero .

$$M\Delta(\lambda) = \begin{vmatrix} a & b & \dots & l & 1 \\ a_1 & b_1 & \dots & l_1 & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n-1} & b_{n-1} & \dots & l_{n-1} & \lambda^{n-1} \\ a_n & b_n & \dots & l_n & \lambda^n \end{vmatrix}, \tag{93}$$

where the constant factor M is determined by (89) and differs from zero.

The identity (93) represents Krylov's transformation. In Krylov's determinant on the right-hand side of the identity, λ occurs only in the elements of the last column; the remaining elements of the determinant do not depend on λ.

Note. In the regular case, the whole space $\boldsymbol{R}$ is cyclic (with respect to $\boldsymbol{A}$). If we choose the vectors $\boldsymbol{x}, \boldsymbol{Ax}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{x}$ as a basis, then in this basis the operator $\boldsymbol{A}$ corresponds to a matrix $\tilde{A}$ having the natural normal form

$$\tilde{A} = \begin{Vmatrix} 0 & 0 & \dots & 0 & -\alpha_n \\ 1 & 0 & \dots & 0 & -\alpha_{n-1} \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & 0 & -\alpha_2 \\ 0 & \dots & & 1 & -\alpha_1 \end{Vmatrix}. \tag{94}$$

The transition from the original basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ to the basis $\boldsymbol{x}, \boldsymbol{Ax}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{x}$ is accomplished by means of the non-singular transforming matrix

$$T = \begin{Vmatrix} a & a_1 & \dots & a_{n-1} \\ b & b_1 & \dots & b_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ l & l_1 & \dots & l_{n-1} \end{Vmatrix}, \tag{95}$$

and then

$$A = T\tilde{A}T^{-1}. \tag{96}$$

3. *Singular case*: $p < n$. In this case, the vectors $\boldsymbol{x}, \boldsymbol{Ax}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{x}$ are linearly dependent, so that

$$M = \begin{vmatrix} a & b & \dots & l \\ a_1 & b_1 & \dots & l_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1} & b_{n-1} & \dots & l_{n-1} \end{vmatrix} = 0.$$

Now (93) had been deduced under the assumption $M \neq 0$. But both sides of this equation are rational integral functions of λ and of the parameters $a, b, \dots, l$.[21] Therefore it follows by a 'continuity' argument that (93) also holds for $M = 0$. But then, when Krylov's determinant is expanded, all the coefficients turn out to be zero. Thus in the singular case ($p < n$) the formula (93) goes over into the trivial identity $0 = 0$.

Let us consider the matrix formed from the coordinates of the vectors $\boldsymbol{x}, \boldsymbol{Ax}, \dots, \boldsymbol{A}^p\boldsymbol{x}$

$$\begin{Vmatrix} a & b & \dots & l \\ a_1 & b_1 & \dots & l_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{p-1} & b_{p-1} & \dots & l_{p-1} \\ a_p & b_p & \dots & l_p \end{Vmatrix}. \tag{97}$$

This matrix is of rank p and the first p rows are linearly independent, but the last, $(p+1)$-st, row is a linear combination of the first p rows with the coefficients $-\alpha_p, -\alpha_{p-1}, \dots, -\alpha_1$ (see (83)). From the n coordinates $a, b, \dots, l$ we can choose p coordinates $c, f, \dots, h$ such that the determinant formed from the coordinates of the vectors $\boldsymbol{x}, \boldsymbol{Ax}, \dots, \boldsymbol{A}^{p-1}\boldsymbol{x}$ is different from zero:

$$M^* = \begin{vmatrix} c & f & \dots & h \\ c_1 & f_1 & \dots & h_1 \\ \cdot & \cdot & \cdot & \cdot \\ c_{p-1} & f_{p-1} & \dots & h_{p-1} \end{vmatrix}. \tag{98}$$

Furthermore, it follows from (83) that:

$$\left.\begin{array}{l} -\alpha_p c - \alpha_{p-1}c_1 - \cdots - \alpha_1 c_{p-1} = c_p \\ -\alpha_p f - \alpha_{p-1}f_1 - \cdots - \alpha_1 f_{p-1} = f_p \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -\alpha_p h - \alpha_{p-1}h_1 - \cdots - \alpha_1 h_{p-1} = h_p. \end{array}\right\} \tag{99}$$

From this system of equations the coefficients $\alpha_1, \alpha_2, \dots, \alpha_p$ of the polynomial $\varphi(\lambda)$ (the minimal polynomial of $\boldsymbol{x}$) are uniquely determined. In exact analogy with the regular case (however, with the value n replaced by p and the letters $a, b, \dots, l$ by $c, f, \dots, h$), we may eliminate $\alpha_1, \alpha_2, \dots, \alpha_p$ from (85) and (99) and obtain the following formula for $\varphi(\lambda)$:

---

[21] $a_i = a_{11}^{(i)}a + a_{12}^{(i)}b + \cdots + a_{1n}^{(i)}l, \quad b_i = a_{21}^{(i)}a + a_{22}^{(i)}b + \cdots + a_{2n}^{(i)}l$, etc. ($i = 1, 2, \dots, n$), where $a_{jk}^{(i)}$ ($j, k = 1, 2, \dots, n$) are the elements of A^i ($i = 1, 2, \dots, n$).

$$M^*\varphi(\lambda)=\begin{vmatrix} c & f & \dots & h & 1\\ c_1 & f_1 & \dots & h_1 & \lambda\\ \dots & \dots & \dots & \dots & \dots\\ c_{p-1} & f_{p-1} & \dots & h_{p-1} & \lambda^{p-1}\\ c_p & f_p & \dots & h_p & \lambda^p \end{vmatrix}. \tag{100}$$

4. Let us now clarify the problem: for what matrices $A=\|a_{ik}\|_1^n$ and for what choice of the original vector $\boldsymbol{x}$ or, what is the same, of the initial parameters $a, b, \dots, l$ the regular case holds.

We have seen that in the regular case

$$\Delta(\lambda)\equiv\psi(\lambda)\equiv\varphi(\lambda).$$

The fact that the characteristic polynomial $\Delta(\lambda)$ coincides with the minimal polynomial $\psi(\lambda)$ means that in the matrix $A=\|a_{ik}\|_1^n$ there are no two elementary divisors with one and the same characteristic value, i.e., all the elementary divisors are co-prime in pairs. In the case where A is a matrix of simple structure, this requirement is equivalent to the condition that the characteristic equation of A have no multiple roots.

The fact that the polynomials $\psi(\lambda)$ and $\varphi(\lambda)$ coincide means that for $\boldsymbol{x}$ we have chosen a vector that generates (by means of $\boldsymbol{A}$) the whole space $\boldsymbol{R}$. Such a vector always exists, by Theorem 2 of § 2.

But if the condition $\Delta(\lambda)\equiv\psi(\lambda)$ is not satisfied, then however we choose the vector $\boldsymbol{x}\neq\boldsymbol{o}$, we do not obtain $\Delta(\lambda)$, since the polynomial $\varphi(\lambda)$ obtained by Krylov's method is a divisor of $\psi(\lambda)$ which in this case does not coincide with $\Delta(\lambda)$ but is only a factor of it. By varying the vector $\boldsymbol{x}$ we may obtain for $\varphi(\lambda)$ every divisor of $\psi(\lambda)$.[22]

The results we have reached can be stated in the form of the following theorem:

THEOREM 14: *Krylov's transformation gives an expression for the characteristic polynomial $\Delta(\lambda)$ of the matrix $A=\|a_{ik}\|_1^n$ in the form of the determinant* (93) *if and only if two conditions are satisfied:*

1. *The elementary divisors of A are co-prime in pairs.*

2. *The initial parameters $a, b, \dots, l$ are the coordinates of a vector $\boldsymbol{x}$ that generates the whole n-dimensional space (by means of the operator $\boldsymbol{A}$ corresponding to the matrix A).*[23]

---

[22] See, for example, [168], p. 48.

[23] In analytical form, this condition means that the columns $\boldsymbol{x}, \boldsymbol{Ax}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{x}$ are linearly independent, where $\boldsymbol{x}=(a, b, \dots, l)$.

In general, the Krylov transformation leads to some divisor $\varphi(\lambda)$ of the characteristic polynomial $\Delta(\lambda)$. This divisor $\varphi(\lambda)$ is the minimal polynomial of the vector $\boldsymbol{x}$ with the coordinates $a, b, \dots, l$ (where $a, b, \dots, l$ are the initial parameters in the Krylov transformation).

5. Let us show how to find the coordinates of a characteristic vector $\boldsymbol{y}$ for an arbitrary characteristic value λ_0 which is a root of the polynomial $\varphi(\lambda)$ obtained by Krylov's method.[24]

We shall seek a vector $\boldsymbol{y}\neq\boldsymbol{o}$ in the form

$$\boldsymbol{y}=\xi_1\boldsymbol{x}+\xi_2\boldsymbol{Ax}+\cdots+\xi_p\boldsymbol{A}^{p-1}\boldsymbol{x}. \tag{101}$$

Substituting this expression for $\boldsymbol{y}$ in the vector equation

$$\boldsymbol{Ay}=\lambda_0\boldsymbol{y}$$

and using (83), we obtain

$$\begin{aligned}\xi_1\boldsymbol{Ax}+\xi_2\boldsymbol{A}^2\boldsymbol{x}+\cdots+\xi_{p-1}\boldsymbol{A}^{p-1}\boldsymbol{x}+\xi_p(-\alpha_p\boldsymbol{x}-\alpha_{p-1}\boldsymbol{Ax}-\cdots-\alpha_1\boldsymbol{A}^{p-1}\boldsymbol{x})&\\ =\lambda_0(\xi_1\boldsymbol{x}+\xi_2\boldsymbol{Ax}+\cdots+\xi_p\boldsymbol{A}^{p-1}\boldsymbol{x}).&\end{aligned} \tag{102}$$

Hence, among other things, it follows that $\xi_p\neq 0$, because the equation $\xi_p=0$ would yield by (102) a linear dependence among the vectors $\boldsymbol{x}, \boldsymbol{Ax}, \dots, \boldsymbol{A}^{p-1}\boldsymbol{x}$. In what follows we set $\xi_p=1$. Then we obtain from (102):

$$\left.\begin{aligned}\xi_p=1,\ \xi_{p-1}=\lambda_0\xi_p+\alpha_1,\ \xi_{p-2}=\lambda_0\xi_{p-1}+\alpha_2, \dots, \xi_1=\lambda_0\xi_2+\alpha_{p-1},\\ 0=\lambda_0\xi_1+\alpha_p.\end{aligned}\right\} \tag{103}$$

The first of these equations determine for us in succession the values $\xi_p, \xi_{p-1}, \dots, \xi_1$ (the coordinates of $\boldsymbol{y}$ in the 'new' basis $\boldsymbol{x}, \boldsymbol{Ax}, \dots, \boldsymbol{A}^{p-1}\boldsymbol{x}$); the last equation is a consequence of the preceding ones and of the relation $\lambda_0^p+\alpha_1\lambda_0^{p-1}+\dots+\alpha_p=0$.

The coordinates $a', b', \dots, l'$ of the vector $\boldsymbol{y}$ in the original basis may be found from the following formulas, which follow from (101):

$$\left.\begin{aligned}a'&=\xi_1 a+\xi_2 a_1+\cdots+\xi_p a_{p-1}\\ b'&=\xi_1 b+\xi_2 b_1+\cdots+\xi_p b_{p-1}\\ &\dots\dots\dots\dots\dots\\ l'&=\xi_1 l+\xi_2 l_1+\cdots+\xi_p l_{p-1}.\end{aligned}\right\} \tag{104}$$

Example 1.

We recommend to the reader the following scheme of computations.

---

[24] The following arguments hold both in the regular case $p=n$ and the singular case $p<n$.

Under the given matrix A we write the row of the coordinates of $\boldsymbol{x}$: $a, b, \ldots, l$. These numbers are given arbitrarily (with only one condition: at least one is different from zero). Under the row $a, b, \ldots, l$ we write the row $a_1, b_1, \ldots, l_1$, i.e., the coordinates of the vector $\boldsymbol{Ax}$. The numbers $a_1, b_1, \ldots, l_1$ are obtained by multiplying the row $a, b, \ldots, l$ successively into the rows of the given matrix A. For example, $a_1 = a_{11}a + a_{12}b + \ldots + {}_{1n}l$, $b_1 = a_{21}a + a_{22}b + \ldots + a_{2n}l$, etc. Under the row $a_1, b_1, \ldots, l_1$ we write the row $a_2, b_2, \ldots, l_2$, etc. Each of the rows, beginning with the second, is determined by multiplying the preceding row successively into the rows of the given matrix.

Above the given matrix we write the sum row as a check.

$$\begin{array}{r|rrrr|r|r}
 & \mathit{8} & \mathit{3} & -\mathit{10} & -\mathit{3} & & \\
 & 3 & -1 & -4 & 2 & & \\
A = & 2 & 3 & -2 & -4 & & \\
 & 2 & -1 & -3 & 2 & & \\
 & 1 & 2 & -1 & -3 & & \\
\hline
\boldsymbol{x} = \boldsymbol{e}_1 + \boldsymbol{e}_2 & 1 & 1 & 0 & 0 & -1 & 1 \\
\boldsymbol{Ax} & 2 & 5 & 1 & 3 & -1 & -1 \\
\boldsymbol{A}^2\boldsymbol{x} & 3 & 5 & 2 & 2 & 1 & -1 \\
\boldsymbol{A}^3\boldsymbol{x} & 0 & 9 & -1 & 5 & 1 & 1 \\
\boldsymbol{A}^4\boldsymbol{x} & 5 & 9 & 4 & 4 & & \\
\boldsymbol{y} & 0 & 8 & 0 & 4 & & \\
 & 0 & 2 & 0 & 1 & & \\
\boldsymbol{z} & -4 & 0 & -4 & 0 & & \\
 & 1 & 0 & 1 & 0 & &
\end{array}.$$

The given case is regular, because

$$\boldsymbol{M} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 2 & 5 & 1 & 3 \\ 3 & 5 & 2 & 2 \\ 0 & 9 & -1 & 5 \end{vmatrix} = -16 \neq 0.$$

Krylov's determinant has the form

$$-16\,\Delta(\lambda) = \begin{vmatrix} 1 & 1 & 0 & 0 & 1 \\ 2 & 5 & 1 & 3 & \lambda \\ 3 & 5 & 2 & 2 & \lambda^2 \\ 0 & 9 & -1 & 5 & \lambda^3 \\ 5 & 9 & 4 & 4 & \lambda^4 \end{vmatrix}.$$

Expanding this determinant and cancelling -16 we find:

$$\Delta(\lambda) = \lambda^4 - 2\lambda^2 + 1 = (\lambda - 1)^2 (\lambda + 1)^2.$$

We denote by

$$\boldsymbol{y} = \xi_1 \boldsymbol{x} + \xi_2 \boldsymbol{Ax} + \xi_3 \boldsymbol{A}^2\boldsymbol{x} + \xi_4 \boldsymbol{A}^3\boldsymbol{x}$$

a characteristic vector of A corresponding to the characteristic value $\lambda_0 = 1$. We find the numbers $\xi_1, \xi_2, \xi_3, \xi_4$ by the formulas (103):

$$\xi_4 = 1,\ \xi_3 = 1 \cdot \lambda_0 + 0 = 1,\ \xi_2 = 1 \cdot \lambda_0 - 2 = -1,\ \xi_1 = -1 \cdot \lambda_0 + 0 = -1.$$

The control equation $-1 \cdot \lambda_0 + 1 = 0$ is, of course, satisfied.

We place the numbers $\xi_1, \xi_2, \xi_3, \xi_4$ in a vertical column parallel to the columns of $\boldsymbol{x}, \boldsymbol{Ax}, \boldsymbol{A}^2\boldsymbol{x}, \boldsymbol{A}^3\boldsymbol{x}$. Multiplying the column $\xi_1, \xi_2, \xi_3, \xi_4$ into the columns a_1, a_2, a_3, a_4, we obtain the first coordinate a' of the vector $\boldsymbol{y}$ in the original basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$; similarly we obtain b', c', d'. As coordinates of $\boldsymbol{y}$ we find (after cancelling by 4): 0, 2, 0, 1. Similarly, we determine the coordinates 1, 0, 1, 0 of a characteristic vector $\boldsymbol{z}$ for the characteristic value $\lambda_0 = -1$.

Furthermore, by (94) and (95),

$$A = T\tilde{A}T^{-1}$$

where

$$\tilde{A} = \left\| \begin{matrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{matrix} \right\|, \qquad T = \left\| \begin{matrix} 1 & 2 & 3 & 0 \\ 1 & 5 & 5 & 9 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 2 & 5 \end{matrix} \right\|.$$

Example 2. We consider the same matrix A, but as initial parameters we take the numbers $a = 1$, $b = 0$, $c = 0$, $d = 0$.

$$\begin{array}{r|rrrr}
 & \mathit{8} & \mathit{3} & -\mathit{10} & -\mathit{3} \\
 & 3 & -1 & -4 & 2 \\
A = & 2 & 3 & -2 & -4 \\
 & 2 & -1 & -3 & 2 \\
 & 1 & 2 & -1 & -3 \\
\boldsymbol{x} = \boldsymbol{e}_1 & 1 & 0 & 0 & 0 \\
\boldsymbol{Ax} & 3 & 2 & 2 & 1 \\
\boldsymbol{A}^2\boldsymbol{x} & 1 & 4 & 0 & 2 \\
\boldsymbol{A}^3\boldsymbol{x} & 3 & 6 & 2 & 3
\end{array}$$

But in this case

$$\boldsymbol{M} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 \\ 1 & 4 & 0 & 2 \\ 3 & 6 & 2 & 3 \end{vmatrix} = 0$$

and $p = 3$. We have a singular case to deal with.

Taking the first three coordinates of the vectors $\boldsymbol{x}$, $\boldsymbol{Ax}$, $\boldsymbol{A}^2\boldsymbol{x}$, $\boldsymbol{A}^3\boldsymbol{x}$, we write the Krylov determinant in the form

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 3 & 2 & 2 & \lambda \\ 1 & 4 & 0 & \lambda^2 \\ 3 & 6 & 2 & \lambda^3 \end{vmatrix}.$$

Expanding this determinant and cancelling -8, we obtain:

$$\varphi(\lambda) = \lambda^3 - \lambda^2 - \lambda + 1 = (\lambda - 1)^2 (\lambda + 1).$$

Hence we find three characteristic values: $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -1$. The fourth characteristic value can be obtained from the condition that the sum of all the characteristic values must be equal to the trace of the matrix. But $\operatorname{tr} A = 0$. Hence $\lambda_4 = -1$.

These examples show that in applying Krylov's method, when we write down successively the rows of the matrix

$$\left\| \begin{matrix} a & b & \dots & l \\ a_1 & b_1 & \dots & l_1 \\ a_2 & b_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{matrix} \right\| \tag{105}$$

it is necessary to watch the rank of the matrix obtained so that we stop after the first row (the $(p+1)$-st from above) that is a linear combination of the preceding ones. The determination of the rank is connected with the computation of certain determinants. Moreover, after obtaining Krylov's determinant in the form (93) or (100), in order to expand it with respect to the elements of the last column we have to compute a certain number of determinants of order $p - 1$ (in the regular case, of order $n - 1$).

Instead of expanding Krylov's determinant we can determine the coefficients $\alpha_1, \alpha_2, \dots$ directly from the system of equations (91) (or (99)) by applying any efficient method of solution to the system—for example, the elimination method. This method can be applied immediately to the matrix

$$\left\| \begin{matrix} a & b & \dots & l & 1 \\ a_1 & b_1 & \dots & l_1 & \lambda \\ a_2 & b_2 & \dots & l_2 & \lambda^2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{matrix} \right\| \tag{106}$$

by using it in parallel with the computation of the corresponding rows by Krylov's method. We shall then discover at once a row of the matrix (105) that depends on the preceding ones, without computing any determinant.

Let us explain this in some detail. In the first row of (106) we take an arbitrary element $c \neq 0$ and we use it to make the element c_1 under it into zero, by subtracting from the second row the first row multiplied by c_1/c. Next we take an element $f_1{}^* \neq 0$ in the second row and by means of c and $f_1{}^*$ we make the elements c_2 and f_2 into zero, etc.[25] As a result of such a transformation, the element in the last column of (106) is replaced by a polynomial of degree k, $g_k(\lambda) = \lambda^k + \cdots$ $(k = 0, 1, 2, \dots)$

Since under our transformation the rank of the matrix formed from the first k rows for any k and the first n columns of (106) does not change, the $(p+1)$-st row of the matrix must, after the transformation, have the form

$$0, 0, \dots, 0, g_p(\lambda).$$

Our transformation does not change the value of the Krylov determinant

$$\begin{vmatrix} c & f & \dots h & 1 \\ c_1 & f_1 & \dots h_1 & \lambda \\ \dots & \dots & \dots & \dots \\ c_{p-1} & f_{p-1} & \dots h_{p-1} & \lambda^{p-1} \\ c_p & f_p & \dots h_p & \lambda^p \end{vmatrix} = M^* \varphi(\lambda).$$

Therefore

$$M^* \varphi(\lambda) = c f_1^* \cdots g_p(\lambda), \tag{107}$$

i.e.,[26] $g_p(\lambda)$ is the required polynomial $\varphi(\lambda) : g_p(\lambda) \equiv \varphi(\lambda)$.

We recommend the following simplification. After obtaining the k-th transformed row of (106)

$$a_{k-1}^*, b_{k-1}^*, \dots, l_{k-1}^*, g_{k-1}(\lambda), \tag{108}$$

one should obtain the following $(k+1)$-st row by multiplying $a_{k-1}^*, b_{k-1}^*, \dots, l_{k-1}^*$ (and not the original $a_{k-1}, b_{k-1}, \dots, l_{k-1}$) into the rows of the given matrix.[27] Then we find the $(k+1)$-st row in the form

$$\tilde{a}_k, \tilde{b}_k, \dots, \tilde{l}_k, \lambda g_{k-1}(\lambda),$$

and after subtracting the preceding rows, we obtain:

---

[25] The elements $c, f_1{}^*, \dots$ must not belong to the last column containing the powers of λ.

[26] We recall that the highest coefficients of $\varphi(\lambda)$ and $g_p(\lambda)$ are 1.

[27] The simplification consists in the fact that in the row of (108) to be transformed $k - 1$ elements are equal to zero. Therefore it is simple to multiply such a row into the rows of A.

$$a_k^*,\ b_k^*,\ \dots,\ l_k^*,\ g_k(\lambda).$$

The slight modification of Krylov's method that we have recommended (its combination with the elimination method) enables us to find at once the polynomial $\varphi(\lambda)$ that we are interested in (in the regular case, $\Delta(\lambda)$) without computing any determinants or solving any auxiliary system of equations.[28]

Example.

$$\begin{array}{rrrrrr}
 & \mathit{4} & \mathit{4} & \mathit{1} & \mathit{5} & \mathit{0} \\
A= & \left\| \begin{array}{r} 1 \\ 1 \\ -1 \\ 1 \\ 2 \end{array} \right. & \begin{array}{r} 1 \\ 2 \\ 2 \\ -2 \\ 1 \end{array} & \begin{array}{r} -1 \\ -1 \\ 3 \\ 1 \\ -1 \end{array} & \begin{array}{r} 1 \\ 0 \\ -1 \\ 2 \\ 3 \end{array} & \left. \begin{array}{r} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{array} \right\|
\end{array}$$

$$\begin{array}{rrrrr:l}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & -1 & 0 & \lambda \\
0 & 2 & 3 & -4 & -2 & \lambda^2 \quad [2-4\lambda] \\
0 & -2 & 3 & 0 & 0 & \lambda^2-4\lambda+2 \\
-5 & -7 & 5 & 7 & -5 & \lambda^3-4\lambda^2+2\lambda \quad [5+7\lambda] \\
-5 & 0 & 5 & 0 & 0 & \lambda^3-4\lambda^2+9\lambda+5 \\
-10 & -10 & 20 & 0 & -15 & \lambda^4-4\lambda^3+9\lambda^2+5\lambda \quad [15-5(\lambda^2-4\lambda+2) \\
 & & & & & \qquad\qquad -2(\lambda^3-4\lambda^2+9\lambda+5)] \\
0 & 0 & -5 & 0 & 0 & \lambda^4-6\lambda^3+12\lambda^2+7\lambda-5 \\
5 & 5 & -15 & -5 & 5 & \lambda^5-6\lambda^4+12\lambda^3+7\lambda^2-5\lambda \quad [-5-5\lambda+(\lambda^3 \\
 & & & & & \quad -4\lambda^2+9\lambda+5)-2(\lambda^4-6\lambda^3+12\lambda^2+7\lambda-5)] \\
0 & 0 & 0 & 0 & 0 & \underbrace{\lambda^5-8\lambda^4+25\lambda^3-21\lambda^2-15\lambda+10}_{\Delta(\lambda)}
\end{array}$$

---

[28] Apart from the method of Krylov, we have acquainted the reader in Chapter IV with the method of D. K. Faddeev for the computation of the coefficients of the characteristic polynomial. Faddeev's method involves more computations than Krylov's but it is more general, being without singular cases. We also refer the reader to the very effective method of A. M. Danilevskii [131]; see also the expository paper [376] and the book [15], § 24. See also [5] and [194].

CHAPTER VIII

MATRIX EQUATIONS

In this chapter we consider certain types of matrix equations that occur in various problems in the theory of matrices and its applications.

§ 1. The Equation $AX = XB$

1. Suppose that the equation

$$AX = XB \tag{1}$$

is given, where A and B are square matrices (in general of different orders)

$$A = \| a_{ij} \|_1^m, \qquad B = \| b_{kl} \|_1^n$$

and where X is an unknown rectangular matrix of dimension $m \times n$:

$$X = \| x_{jk} \| \qquad (j = 1, 2, \dots, m; \quad k = 1, 2, \dots, n).$$

We write down the elementary divisors of A and B (in the field of complex numbers):

$$\begin{aligned}
&(A): (\lambda-\lambda_1)^{p_1}, \quad (\lambda-\lambda_2)^{p_2}, \ \dots, \ (\lambda-\lambda_u)^{p_u} \quad (p_1+p_2+\cdots+p_u=m), \\
&(B): (\lambda-\mu_1)^{q_1}, \quad (\lambda-\mu_2)^{q_2}, \ \dots, \ (\lambda-\mu_v)^{q_v} \quad (q_1+q_2+\cdots+q_v=n).
\end{aligned}$$

In accordance with these elementary divisors we reduce A and B to Jordan normal form

$$A = U\tilde{A}U^{-1}, \qquad B = V\tilde{B}V^{-1}, \tag{2}$$

where U and V are square non-singular matrices of orders m and n, respectively, and $\tilde{A}$ and $\tilde{B}$ are the Jordan matrices:

$$\begin{aligned}
\tilde{A} &= \{ \lambda_1 E^{(p_1)} + H^{(p_1)}, \quad \lambda_2 E^{(p_2)} + H^{(p_2)}, \quad \dots, \quad \lambda_u E^{(p_u)} + H^{(p_u)} \}, \\
\tilde{B} &= \{ \mu_1 E^{(q_1)} + H^{(q_1)}, \quad \mu_2 E^{(q_2)} + H^{(q_2)}, \quad \dots, \quad \mu_v E^{(q_v)} + H^{(q_v)} \}.
\end{aligned} \tag{3}$$

Replacing A and B in (1) by their expressions given in (2), we obtain:

$$U\tilde{A}U^{-1}X = XV\tilde{B}V^{-1}.$$

We multiply both sides of this equation on the left by U^{-1} and on the right by V:

$$\tilde{A}U^{-1}XV = U^{-1}XV\tilde{B}. \tag{4}$$

When we introduce in place of X a new unknown matrix $\tilde{X}$ (of the same dimension $m \times n$)

$$\tilde{X} = U^{-1}XV, \tag{5}$$

we can write equation (4) as follows:

$$\tilde{A}\tilde{X} = \tilde{X}\tilde{B}. \tag{6}$$

We have thus replaced the matrix equation (1) by the equation (6), of the same form, in which the given matrices have Jordan normal form.

We partition $\tilde{X}$ into blocks corresponding to the quasi-diagonal form of the matrices $\tilde{A}$ and $\tilde{B}$:

$$\tilde{X} = (X_{\alpha\beta}) \qquad (\alpha = 1, 2, \ldots, u; \quad \beta = 1, 2, \ldots, v)$$

(here $X_{\alpha\beta}$ is a rectangular matrix of dimension $p_\alpha \times q_\beta$ $(\alpha = 1, 2, \ldots, u;$ $\beta = 1, 2, \ldots, v)$).

Using the rule for multiplying a partitioned matrix by a quasi-diagonal one (see p. 42), we carry out the multiplication of the matrices on the left-hand and right-hand sides of (6). Then this equation breaks up into uv matrix equations

$$[\lambda_\alpha E^{(p_\alpha)} + H^{(p_\alpha)}]\, X_{\alpha\beta} = X_{\alpha\beta}\,[\mu_\beta E^{(q_\beta)} + H^{(q_\beta)}]$$
$$(\alpha = 1, 2, \ldots, u; \quad \beta = 1, 2, \ldots, v),$$

which we rewrite as follows:

$$(\mu_\beta - \lambda_\alpha)\, X_{\alpha\beta} = H_\alpha X_{\alpha\beta} - X_{\alpha\beta} G_\beta \quad (\alpha = 1, 2, \ldots, u; \quad \beta = 1, 2, \ldots, v); \tag{7}$$

we have used here the abbreviations

$$H_\alpha = H^{(p_\alpha)} \quad G_\beta = H^{(q_\beta)} \qquad (\alpha = 1, 2, \ldots, u; \quad \beta = 1, 2, \ldots, v). \tag{8}$$

Let us take one of the equations (7). Two cases can occur:

1. $\lambda_\alpha \neq \mu_\beta$. We iterate equation (7) $r - 1$ times:[1]

---

[1] We multiply both sides of (7) by $\mu_\beta - \lambda_\alpha$ and in each term of the right-hand side we replace $(\mu_\beta - \lambda_\alpha)\, X_{\alpha\beta}$ by $H_\alpha X_{\alpha\beta} - X_{\alpha\beta}G_\beta$. This process is repeated $r - 1$ times.

$$(\mu_\beta - \lambda_\alpha)^r X_{\alpha\beta} = \sum_{\sigma + \tau = r} (-1)^\tau \binom{r}{\tau} H_\alpha^\sigma X_{\alpha\beta} G_\beta^\tau. \tag{9}$$

Note that, by (8),

$$H_\alpha^{p_\alpha} = G_\beta^{q_\beta} = O \tag{10}$$

If in (9) we take $r \geqq p_\alpha + q_\beta - 1$, then in each term of the sum on the right-hand side of (9) at least one of the relations

$$\sigma \geqq p_\alpha, \quad \tau \geqq q_\beta$$

is satisfied, so that by (10) either $H_\alpha^\sigma = O$ or $G_\beta^\tau = O$. Moreover, since in this case $\lambda_\alpha \neq \mu_\beta$, we find from (9):

$$X_{\alpha\beta} = O. \tag{11}$$

2. $\lambda_\alpha = \mu_\beta$. In this case equation (7) assumes the form

$$H_\alpha X_{\alpha\beta} = X_{\alpha\beta} G_\beta. \tag{12}$$

In the matrices H_α and G_β the elements of the first superdiagonal are equal to 1, and all the remaining elements are zero. Taking this specific structure of H_α and G_β into account and setting

$$X_{\alpha\beta} = \| \xi_{ik} \| \qquad (i = 1, 2, \ldots, p_\alpha; \quad k = 1, 2, \ldots, q_\beta),$$

we replace the matrix equation (12) by the following equivalent system of scalar equations:[2]

$$\xi_{i+1,k} = \xi_{i,k-1}\, (\xi_{i0} = \xi_{p_\alpha+1,k} = 0; \quad i = 1, 2, \ldots, p_\alpha; \;\; k = 1, 2, \ldots, q_\beta). \tag{13}$$

The equations (13) have this meaning:

1) In the matrix $X_{\alpha\beta}$ the elements of every line parallel to the main diagonal are equal;

2) $$\xi_{21} = \xi_{31} = \cdots = \xi_{p_\alpha 1} = \xi_{p_\alpha 2} = \cdots = \xi_{p_\alpha, q_\beta - 1} = 0.$$

Let $p_\alpha = q_\beta$. Then $X_{\alpha\beta}$ is a square matrix. From 1) and 2) it follows that in $X_{\alpha\beta}$ all the elements below the main diagonal are zero, all the elements in the main diagonal are equal to a certain number $c_{\alpha\beta}$, all the elements of the first superdiagonal are equal to a number $c'_{\alpha\beta}$, etc.; i.e.,

---

[2] From the structure of the matrices H_α and G_β it follows that the product $H_\alpha X_{\alpha\beta}$ is obtained from $X_{\alpha\beta}$ by shifting all the rows one place upwards and filling the last row with zeros; similarly, $X_{\alpha\beta}G_\beta$ is obtained from $X_{\alpha\beta}$ by shifting all the columns one place to the right and filling the first column with zeros (see Chapter I, p. 14). To simplify the notation we do not write the additional indices α, β in ξ_{ik}.

$$X_{\alpha\beta} = \left\| \begin{array}{cccc} c_{\alpha\beta} & c'_{\alpha\beta} & \cdots & c_{\alpha\beta}^{(p_\alpha-1)} \\ 0 & c_{\alpha\beta} & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & c'_{\alpha\beta} \\ 0 & \cdots & 0 & c_{\alpha\beta} \end{array} \right\| = T_{p_\alpha}; \qquad (14)$$

$$(p_\alpha = q_\beta)$$

here $c_{\alpha\beta}, c'_{\alpha\beta}, \ldots, c_{\alpha\beta}^{(p_\alpha-1)}$ are arbitrary parameters (the equations (12) do not impose any restrictions on the values of these parameters).

It is easy to see that for $p_\alpha < q_\beta$

$$X_{\alpha\beta} = (\overbrace{0}^{q_\beta - p_\alpha}, \quad T_{p_\alpha}) \qquad (15)$$

and for $p_\alpha > q_\beta$

$$X_{\alpha\beta} = \begin{pmatrix} T_{q_\beta} \\ 0 \end{pmatrix} \}\, p_\alpha - q_\beta . \qquad (16)$$

We shall say of the matrices (14), (15), and (16) that they have *regular* upper triangular form. The number of arbitrary parameters in $X_{\alpha\beta}$ is equal to the smaller of the numbers p_α and q_β. The scheme below shows the structure of the matrices $X_{\alpha\beta}$ for $x_\alpha = \mu_\beta$ (the arbitrary parameters are here denoted by a, b, c, and d):

$$X_{\alpha\beta} = \left\| \begin{array}{cccc} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{array} \right\|, \quad X_{\alpha\beta} = \left\| \begin{array}{ccccc} 0 & 0 & a & b & c \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & a \end{array} \right\|, \quad X_{\alpha\beta} = \left\| \begin{array}{ccc} a & b & c \\ 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right\|$$

$$(p_\alpha = q_\beta = 4) \qquad (p_\alpha = 3, \ q_\beta = 5) \qquad (p_\alpha = 5, \ q_\beta = 3)$$

In order to subsume case 1 also in the count of arbitrary parameters in X, we denote by $d_{\alpha\beta}(\lambda)$ the greatest common divisor of the elementary divisors $(\lambda - \lambda_\alpha)^{p_\alpha}$ and $(\lambda - \mu_\beta)^{q_\beta}$ and by $\delta_{\alpha\beta}$ the degree of the polynomial $d_{\alpha\beta}(\lambda)$ $(\alpha = 1, 2, \ldots, u;\ \beta = 1, 2, \ldots, v)$. In case 1, we have $\delta_{\alpha\beta} = 0$; in case 2, $\delta_{\alpha\beta} = \min(p_\alpha, q_\beta)$. Thus, in both cases the number of arbitrary parameters in $X_{\alpha\beta}$ is equal to $\delta_{\alpha\beta}$. The number of arbitrary parameters in $\tilde{X}$ is determined by the formula.

$$N = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \delta_{\alpha\beta} .$$

In what follows it will be convenient to denote the general solution of (6) by $X_{\tilde{A}\tilde{B}}$ (so far we have denoted it by $\tilde{X}$).

The results obtained in this section can be stated in the form of the following theorem:

THEOREM 1: *The general solution of the matrix equation*

$$AX = XB$$

where

$$A = \| a_{ik} \|_1^m = U\tilde{A}U^{-1} = U\{\lambda_1 E^{(p_1)} + H^{(p_1)}, \ \ldots, \ \lambda_u E^{(p_u)} + H^{(p_u)}\} U^{-1},$$
$$B = \| b_{ik} \|_1^n = V\tilde{B}V^{-1} = V\{\mu_1 E^{(q_1)} + H^{(q_1)}, \ \ldots, \ \mu_v E^{(q_v)} + H^{(q_v)}\} V^{-1}$$

is given by the formula

$$X = UX_{\tilde{A}\tilde{B}}V^{-1}. \qquad (17)$$

Here $X_{\tilde{A}\tilde{B}}$ is the general solution of the equation

$$\tilde{A}\tilde{X} = \tilde{X}\tilde{B}$$

and has the following structure:

$X_{\tilde{A}\tilde{B}}$ is decomposed into blocks

$$X_{\tilde{A}\tilde{B}} = (\overbrace{X_{\alpha\beta}}^{q_\beta})\,\}\, p_\alpha \qquad (\alpha = 1, 2, \ \ldots, \ u; \ \beta = 1, 2, \ \ldots, \ v);$$

if $\lambda_\alpha \neq \mu_\beta$, then the null matrix stands in the place $X_{\alpha\beta}$, but if $\lambda_\alpha = \mu_\beta$, then an arbitrary regular upper triangular matrix stands in the place $X_{\alpha\beta}$.

$X_{\tilde{A}\tilde{B}}$, and therefore also X, depends linearly on N arbitrary parameters $c_1, c_2, \ldots, c_N$

$$X = \sum_{j=1}^{N} c_j X_j, \qquad (18)$$

where N is determined by the formula

$$N = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \delta_{\alpha\beta} \qquad (19)$$

(here $\delta_{\alpha\beta}$ denotes the degree of the greatest common divisor of $(\lambda - \lambda_\alpha)^{p_\alpha}$ and $(\lambda - \mu_\beta)^{q_\beta}$).

Note that the matrices $X_1, X_2, \ldots, X_N$ that occur in (18) are solutions of the original equation (1) (X_j is obtained from X by giving to the parameter c_j the value 1 and to the remaining parameters the value 0; $j = 1, 2, \ldots, N$). These solutions are linearly independent, since otherwise for certain values of the parameters $c_1, c_2, \ldots, c_N$, not all zero, the matrix X, and therefore $X_{\tilde{A}\tilde{B}}$, would be the null matrix, which is impossible. Thus (18) shows that every solution of the original equation is a linear combination of N linearly independent solutions.

If the matrices A and B do not have common characteristic values (if the characteristic polynomials $|\lambda E - A|$ and $|\lambda E - B|$ are co-prime), *then* $N = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \delta_{\alpha\beta} = 0$, *and so* $X = O$, *i.e., in this case the equation* (1) *has only the trivial solution* $X = O$.

Note. Suppose that the elements of A and B belong to some number field F. Then we cannot say that the elements of U, V, and $X_{\tilde{A}\tilde{B}}$ that occur in (17) also belong to F. The elements of these matrices may be taken in an extension field F_1 which is obtained from F by adjoining the roots of the characteristic equations $|\lambda E - A| = 0$ and $|\lambda E - B| = 0$. We always have to deal with such an extension of the ground field when we use the reduction of given matrices to Jordan normal form.

However, the matrix equation (1) is equivalent to a system of mn linear homogeneous equations, where the unknown are the elements x_{jk} ($j = 1, 2, 3, \dots, m$; $k = 1, 2, \dots, n$) of the required matrix X:

$$\sum_{j=1}^{m} a_{ij} x_{jk} = \sum_{h=1}^{n} x_{ih} b_{hk} \qquad (i = 1, 2, \dots, m;\ k = 1, 2, \dots, n). \tag{20}$$

What we have shown is that this system has N linearly independent solutions, where N is determined by (19). But it is well known that fundamental linearly independent solutions can be chosen in the ground field F to which the coefficients of (20) belong. Thus, in (18) the matrices $X_1, X_2, \dots, X_N$ can be so chosen that their elements lie in F. If we then give to the arbitrary parameters in (18) all possible values in F, we obtain all the matrices X with elements in F that satisfy the equation (1).[3]

§ 2. The Special Case $A = B$. Commuting Matrices

1. Let us consider the special case of the equation (1):

$$AX = XA, \tag{21}$$

where $A = \| a_{ik} \|_1^n$ is a given matrix and $X = \| x_{ik} \|_1^n$ an unknown matrix. We have come to a problem of Frobenius: to determine all the matrices X that commute with a given matrix A.

We reduce A to Jordan normal form:

$$A = U\tilde{A}U^{-1} = U\{\lambda_1 E^{(p_1)} + H^{(p_1)}, \dots, \lambda_u E^{(p_u)} + H^{(p_u)}\} U^{-1}. \tag{22}$$

---

[3] The matrices $A = \| a_{ij} \|_1^m$ and $B = \| b_{kl} \|_1^n$ determine a linear operator $\hat{F}(X) = AX - XB$ in the space of rectangular matrices X of dimension $m \times n$. A treatment of operators of this type is contained in the paper [179].

Then when we set in (17) $V = U$, $\tilde{B} = \tilde{A}$ and denote $X_{\tilde{A}\tilde{A}}$ simply by $X_{\tilde{A}}$, we obtain all solutions of (21), i.e., all matrices that commute with A, in the following form:

$$X = UX_{\tilde{A}}U^{-1}, \tag{23}$$

where $X_{\tilde{A}}$ denotes an arbitrary matrix permutable with $\tilde{A}$. As we have explained in the preceding section, $X_{\tilde{A}}$ is split into u^2 blocks

$$X_{\tilde{A}} = (X_{\alpha\beta})_1^u$$

corresponding to the splitting of the Jordan matrix $\tilde{A}$ into blocks; $X_{\alpha\beta}$ is either the null matrix or an arbitrary regular upper triangular matrix, depending on whether $\lambda_\alpha \neq \lambda_\beta$ or $\lambda_\alpha = \lambda_\beta$.

As an example, we write down the elements of $X_{\tilde{A}}$ in the case where A has the following elementary divisors:

$$(\lambda - \lambda_1)^4,\ (\lambda - \lambda_1)^3,\ (\lambda - \lambda_2)^2,\ \lambda - \lambda_2 \quad (\lambda_1 \neq \lambda_2).$$

In this case $X_{\tilde{A}}$ has the following form:

$$\left\| \begin{array}{cccc|ccc|cc|c}
a & b & c & d & e & f & g & 0 & 0 & 0 \\
0 & a & b & c & 0 & e & f & 0 & 0 & 0 \\
0 & 0 & a & b & 0 & 0 & e & 0 & 0 & 0 \\
0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & h & k & l & m & p & q & 0 & 0 & 0 \\
0 & 0 & h & k & 0 & m & p & 0 & 0 & 0 \\
0 & 0 & 0 & h & 0 & 0 & m & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & r & s & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & z
\end{array} \right\| \qquad (a, b, \dots, z \text{ are arbitrary parameters}).$$

The number of parameters in $X_{\tilde{A}}$ is equal to N, where $N = \sum_{\alpha,\beta=1}^{u} \delta_{\alpha\beta}$; here $\delta_{\alpha\beta}$ denotes the degree of the greatest common divisor of the polynomials $(\lambda - \lambda_\alpha)^{p_\alpha}$ and $(\lambda - \lambda_\beta)^{p_\beta}$.

Let us bring the invariant polynomials of A into the discussion: $i_1(\lambda)$, $i_2(\lambda), \dots, i_t(\lambda)$; $i_{t+1}(\lambda) = \dots = i_n(\lambda) = 1$. We denote the degrees of these polynomials by $n_1 \geqq n_2 \geqq \dots \geqq n_t > n_{t+1} = \dots = 0$. Since each invariant polynomial is a product of certain co-prime elementary divisors, the formula for N can be written as follows:

$$N = \sum_{g,j=1}^{t} \varkappa_{gj}, \tag{24}$$

where $\varkappa_{gj}$ is the degree of the greatest common divisor of $i_g(\lambda)$ and $i_j(\lambda)$ $(g, j = 1, 2, \ldots, t)$. But the greatest common divisor of $i_g(\lambda)$ and $i_j(\lambda)$ is one of these polynomials and therefore $\varkappa_{gj} = \min(n_g, n_j)$. Hence we obtain:

$$N = n_1 + 3n_2 + \cdots + (2t-1)n_t.$$

N is the number of linearly independent matrices that commute with A (we may assume that the elements of these matrices belong to the ground field F containing the elements of A; see the remark at the end of the preceding section). We have arrived at the following theorem:

THEOREM 2: *The number of linearly independent matrices that commute with the matrix* $A = \| a_{ik} \|_1^n$ *is given by the formula*

$$N = n_1 + 3n_2 + \ldots + (2t-1)n_t, \tag{25}$$

where $n_1, n_2, \ldots, n_t$ *are the degrees of the non-constant invariant polynomials* $i_1(\lambda), i_2(\lambda), \ldots, i_t(\lambda)$ *of* A.

Note that

$$n = n_1 + n_2 + \ldots + n_t. \tag{26}$$

From (25) and (26) it follows that

$$N \geqq n, \tag{27}$$

where the equality sign holds if and only if $t = 1$, i.e., if all the elementary divisors of A are co-prime in pairs.

2. Let $g(\lambda)$ be an arbitrary polynomial in λ. Then $g(A)$ is permutable with A. There arises the converse question: when can every matrix that is permutable with A be expressed as a polynomial in A? Every matrix that commutes with A would then be a linear combination of the linearly independent matrices

$$E, A, A^2, \ldots, A^{n_1-1}.$$

Hence $N = n_1 \leqq n$; on comparing this with (27), we obtain: $N = n_1 = n$.

COROLLARY 1 TO THEOREM 2: *All the matrices that are permutable with* A *can be expressed as polynomials in* A *if and only if* $n_1 = n$, *i.e., if all the elementary divisors of* A *are co-prime in pairs.*

3. The polynomials in a matrix that commutes with A also commute with A. We raise the question: when can all the matrices that commute with A be expressed in the form of polynomials in one and the same matrix C? Let us consider the case in which they can be so expressed. Then since by the Hamilton-Cayley Theorem the matrix C satisfies its characteristic equation, every matrix that commutes with C must be expressible linearly by the matrices

$$E, C, C^2, \ldots, C^{n-1}.$$

Therefore in this case $N \leqq n$. Comparing this with (27), we find that $N = n$. Hence from (25) and (26) we also have $n_1 = n$.

COROLLARY 2 TO THEOREM 2: *All the matrices that are permutable with* A *can be expressed in the form of polynomials in one and the same matrix* C *if and only if* $n_1 = n$, *i.e. if and only if all the elementary divisors of* $\lambda E - A$ *are co-prime. In this case all the matrices that are permutable with* A *can be represented in the form of polynomials in* A.

4. We mention a very important property of permutable matrices.

THEOREM 3: *If two matrices* $A = \| a_{ik} \|_1^n$ *and* $B = \| b_{ik} \|_1^n$ *are permutable and if one of them, say* A, *has quasi-diagonal form*

$$A = \{\overset{s_1}{A_1}, \overset{s_2}{A_2}\}, \tag{28}$$

where the matrices A_1 *and* A_2 *do not have characteristic values in common, then the other matrix also has the same quasi-diagonal form*

$$B = \{\overset{s_1}{B_1}, \overset{s_2}{B_2}\}. \tag{29}$$

Proof. We split B into blocks corresponding to the quasi-diagonal form (28):

$$B = \begin{pmatrix} \overset{s_1}{B_1} & \overset{s_2}{X} \\ Y & B_2 \end{pmatrix}.$$

From the relation $AB = BA$ we obtain four matrix equations:

$$1.\ A_1B_1 = B_1A_1, \quad 2.\ A_1X = XA_2, \quad 3.\ A_2Y = YA_1, \quad 4.\ A_2B_2 = B_2A_2. \tag{30}$$

As we explained in § 1 (p. 220), the second and third of the equations in (30) only have the solutions $X = O$, $Y = O$, since A_1 and A_2 have no characteristic values in common. This proves our statement. The first and fourth of the equations in (30) express the permutability of A_1 and B_1 and of A_2 and B_2.

In geometrical language, this theorem runs as follows:

THEOREM 3′: *If*

$$\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2$$

is a decomposition of the whole space $\boldsymbol{R}$ *into invariant subspaces* $\boldsymbol{I}_1$ *and* $\boldsymbol{I}_2$ *with respect to an operator* $\boldsymbol{A}$ *and if the minimal polynomials of these subspaces (with respect to* $\boldsymbol{A}$*) are co-prime, then* $\boldsymbol{I}_1$ *and* $\boldsymbol{I}_2$ *are invariant with respect to any linear operator* $\boldsymbol{B}$ *that commutes with* $\boldsymbol{A}$.

Let us also give a geometrical proof of this statement. We denote by $\psi_1(\lambda)$ and $\psi_2(\lambda)$ the minimal polynomials of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ with respect to $\boldsymbol{A}$. From the fact that they are co-prime it follows that all the vectors of $\boldsymbol{R}$ that satisfy the equation $\psi_1(\boldsymbol{A})\boldsymbol{x} = \boldsymbol{o}$ belong to $\boldsymbol{I}_1$ and all the vectors that satisfy $\psi_2(\boldsymbol{A})\boldsymbol{x} = \boldsymbol{o}$ belong to $\boldsymbol{I}_2$.[4] Let $\boldsymbol{x}_1 \in \boldsymbol{I}_1$. Then $\psi_1(\boldsymbol{A})\boldsymbol{x}_1 = \boldsymbol{o}$. The permutability of $\boldsymbol{A}$ and $\boldsymbol{B}$ implies that of $\psi_1(\boldsymbol{A})$ and $\boldsymbol{B}$, so that

$$\psi_1(\boldsymbol{A})\,\boldsymbol{B}\boldsymbol{x}_1 = \boldsymbol{B}\psi_1(\boldsymbol{A})\,\boldsymbol{x}_1 = \boldsymbol{o}\,,$$

i.e., $\boldsymbol{B}\boldsymbol{x}_1 \in \boldsymbol{I}_1$. The invariance of $\boldsymbol{I}_2$ with respect to $\boldsymbol{B}$ is proved similarly.

This theorem leads to a number of corollaries:

COROLLARY 1: *If the linear operators* $\boldsymbol{A}, \boldsymbol{B}, \ldots, \boldsymbol{L}$ *are pairwise permutable, then the whole space* $\boldsymbol{R}$ *can be split into subspaces invariant with respect to all the operators* $\boldsymbol{A}, \boldsymbol{B}, \ldots, \boldsymbol{L}$

$$\boldsymbol{R} = \boldsymbol{I}_1 + \boldsymbol{I}_2 + \ldots + \boldsymbol{I}_w$$

such that the minimal polynomial of each of these subspaces with respect to any one of the operators $\boldsymbol{A}, \boldsymbol{B}, \ldots, \boldsymbol{L}$ *is a power of an irreducible polynomial.*

As a special case of this we obtain:

COROLLARY 2: *If the linear operators* $\boldsymbol{A}, \boldsymbol{B}, \ldots, \boldsymbol{L}$ *are pairwise permutable and all the characteristic values of these operators belong to the ground field, then the whole space* $\boldsymbol{R}$ *can be split into subspaces* $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_w$*, invariant with respect to all the operators such that each operator* $\boldsymbol{A}, \boldsymbol{B}, \ldots, \boldsymbol{L}$ *has equal characteristic values in each of them.*

Finally, we mention a further special case of this statement:

COROLLARY 3: *If* $\boldsymbol{A}, \boldsymbol{B}, \ldots, \boldsymbol{L}$ *are pairwise permutable operators of simple structure* (see Chapter III, § 8), *then a basis of the space can be formed from common characteristic vectors of these operators.*

We also give the matrix form of the last statement:

Permutable matrices of simple structure can be brought into diagonal form simultaneously by a similarity transformation.

---

[4] See Theorem 1 of Chapter VII (p. 179).

§ 3. The Equation $AX - XB = C$

1. Suppose that the matrix equation

$$AX - XB = C \tag{31}$$

is given, where $A = \| a_{ij} \|_1^m$ and $B = \| b_{kl} \|_1^n$ are given square matrices of order m and n and where $C = \| c_{jk} \|$ and $X = \| x_{jk} \|$ are a given and an unknown rectangular matrix, respectively, of dimension $m \times n$. The equation (31) is equivalent to a system of mn scalar equations in the elements of X:

$$\sum_{j=1}^{m} a_{ij}x_{jk} - \sum_{l=1}^{n} x_{il}b_{lk} = c_{ik} \qquad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)\,. \tag{31′}$$

The corresponding homogeneous system of equations

$$\sum_{j=1}^{m} a_{ij}x_{jk} - \sum_{l=1}^{n} x_{il}b_{lk} = 0 \qquad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)\,,$$

can be written in matrix form as follows:

$$AX - XB = O. \tag{32}$$

Thus, if (32) only has the trivial solution $X = O$, then (31) has a unique solution. But we have established in § 1 that the only solution of (32) is the trivial one if and only if A and B do not have common characteristic values. Therefore, *if the matrices* A *and* B *do not have characteristic values in common, then* (31) *has a unique solution; but if the matrices* A *and* B *have characteristic values in common, then two cases may arise depending on the 'constant' term* C*: either the equation* (31) *is contradictory, or it has an infinite number of solutions given by the formula*

$$X = X_0 + X_1,$$

where X_0 *is a fixed particular solution of* (31) *and* X_1 *the general solution of the homogeneous equation* (32) (the structure of X_1 was described in § 1).

§ 4. The Scalar Equation $f(X) = O$

1. To begin with, let us consider the equation

$$g(X) = O, \tag{33}$$

where

$$g(\lambda) = (\lambda - \lambda_1)^{\alpha_1}(\lambda - \lambda_2)^{\alpha_2} \cdots (\lambda - \lambda_h)^{\alpha_h}$$

is a given polynomial in the variable λ and X is an unknown square matrix of order n. Since the minimal polynomial of X, i.e., the first invariant polynomial $i_1(\lambda)$, must be a divisor of $g(\lambda)$, the elementary divisors of X must have the following form:

$$(\lambda-\lambda_{j_1})^{p_{j_1}}, (\lambda-\lambda_{j_2})^{p_{j_2}}, \dots, (\lambda-\lambda_{j_\nu})^{p_{j_\nu}} \quad \begin{pmatrix} j_1, j_2, \dots, j_\nu = 1, 2, \dots, h, \\ p_{j_1} \leqq a_{j_1}, p_{j_2} \leqq a_{j_2}, \dots, p_{j_\nu} \leqq a_{j_\nu}, \\ p_{j_1} + p_{j_2} + \dots + p_{j_\nu} = n \end{pmatrix}$$

(among the indices $j_1, j_2, \dots, j_\nu$ there may be some that are equal; n is the given order of the unknown matrix X).

We represent X in the form

$$X = T\{\lambda_{j_1}E^{(p_{j_1})} + H^{(p_{j_1})}, \dots, \lambda_{j_\nu}E^{(p_{j_\nu})} + H^{(p_{j_\nu})}\}\, T^{-1}, \tag{34}$$

where T is an arbitrary non-singular matrix of order n. The set of solutions of the equation (33) with a given order of the unknown matrix splits, by formula (34), into a finite number of classes of similar matrices.

Example 1. Let the equation

$$X^m = O \tag{35}$$

be given.

If a certain power of a matrix is the null matrix, then the matrix is called *nilpotent*. The least exponent for which the power of the matrix is the null matrix is called the *index of nilpotency*.

Obviously, the solutions of (35) are all the nilpotent matrices with an index of nilpotency $\mu \leqq m$. The formula that comprises all the solutions of a given order n looks as follows (T is an arbitrary non-singular matrix):

$$X = T\{H^{(p_1)}, H^{(p_2)}, \dots, H^{(p_\nu)}\}\, T^{-1} \quad \begin{pmatrix} p_1, p_2, \dots, p_\nu \leqq m, \\ p_1 + p_2 + \dots + p_\nu = n \end{pmatrix}. \tag{36}$$

Example 2. Let the equation

$$X^2 = X \tag{37}$$

be given.

A matrix satisfying this equation is called *idempotent*. The elementary divisors of an idempotent matrix can only be λ or $\lambda - 1$. Therefore an idempotent matrix can be described as a matrix of simple structure (i.e., reducible to diagonal form) with characteristic values 0 or 1. The formula comprising all the idempotent matrices of a given order n has the form

$$X = T\{\underbrace{1, 1, \dots, 1, 0, \dots, 0}_{n}\}T^{-1}, \tag{38}$$

where T is an arbitrary non-singular matrix of order n.

2. Let us now consider the more general equation

$$f(X) = O, \tag{39}$$

where $f(\lambda)$ is a regular function of λ in some domain G of the complex plane. We shall require of the unknown solution $X = \| x_{ik} \|_1^n$ that its characteristic values belong to G and that their multiplicities be as follows:

$$\begin{array}{ll} \text{Zeros:} & \lambda_1, \lambda_2, \dots, \\ \text{Multiplicities:} & a_1, a_2, \dots . \end{array}$$

As in the preceding case, every elementary divisor of X must have the form

$$(\lambda - \lambda_i)^{p_i} \quad (p_i \leqq a_i),$$

and therefore

$$X = T\{\lambda_{j_1}E^{(p_{j_1})} + H^{(p_{j_1})}, \dots, \lambda_{j_\nu}E^{(p_{j_\nu})} + H^{(p_{j_\nu})}\}\, T^{-1} \tag{40}$$

$$(j_1, j_2, \dots, j_\nu = 1, 2, \dots;\ p_{j_1} \leqq a_{j_1}, p_{j_2} \leqq a_{j_2}, \dots, p_{j_\nu} \leqq a_{j_\nu};$$
$$p_{j_1} + p_{j_2} + \dots + p_{j_\nu} = n)$$

(T is an arbitrary non-singular matrix).

§ 5. Matrix Polynomial Equations

1. Let us consider the equations

$$A_0X^m + A_1X^{m-1} + \dots + A_m = O, \tag{41}$$

$$Y^mA_0 + Y^{m-1}A_1 + \dots + A_m = O, \tag{42}$$

where $A_0, A_1, \dots, A_m$ are given square matrices of order n and X, Y are unknown square matrices of the same order. The equation (33) investigated in the preceding section is a very special—one could almost say, trivial—case of (41) and (42) and is obtained by setting $A_i = a_iE$, where a_i is a number and $i = 1, 2, \dots, m$.

The following theorem establishes a connection between (41), (42), and (33).

THEOREM 4: *Every solution of the matrix equation*

$$A_0X^m + A_1X^{m-1} + \cdots + A_m = O$$

satisfies the scalar equation

$$g(X) = O, \tag{43}$$

where

$$g(\lambda) \equiv |A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_m|. \tag{44}$$

The same scalar equation is satisfied by every solution Y of the matrix equation

$$Y^mA_0 + Y^{m-1}A_1 + \cdots + A_m = O.$$

Proof. We denote by $F(\lambda)$ the matrix polynomial

$$F(\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_m.$$

Then the equations (41) and (42) can be written as follows (see p. 81):

$$F(X) = O, \quad \widehat{F}(Y) = O.$$

By the generalized Bézout Theorem (Chapter IV, § 3), if X and Y are solutions of these equations, the matrix polynomial $F(\lambda)$ is divisible on the right by $\lambda E - X$ and on the left by $\lambda E - Y$:

$$F(\lambda) = Q(\lambda)(\lambda E - X) = (\lambda E - Y)Q_1(\lambda).$$

Hence

$$g(\lambda) = |F(\lambda)| = |Q(\lambda)|\Delta(\lambda) = |Q_1(\lambda)|\Delta_1(\lambda), \tag{45}$$

where $\Delta(\lambda) = |\lambda E - X|$ and $\Delta_1(\lambda) = |\lambda E - Y|$ are the characteristic polynomials of X and Y. By the Hamilton-Cayley Theorem (Chapter IV, § 4),

$$\Delta(X) = O, \quad \Delta(Y) = O.$$

Therefore (45) implies that

$$g(X) = g(Y) = O,$$

and the theorem is proved.

Note that the Hamilton-Cayley Theorem is a special case of this theorem. For every square matrix A, when substituted for λ, satisfies the equation

$$\lambda E - A = O.$$

Therefore, by the theorem just proved,

$$\Delta(A) = O,$$

where $\Delta(\lambda) = |\lambda E - A|$.

2. Theorem 4 can be generalized as follows:

THEOREM 5:[5] *If $X_0, X_1, \ldots, X_m$ are pairwise permutable square matrices of order n that satisfy the matrix equation*

$$A_0X_0 + A_1X_1 + \cdots + A_mX_m = O \tag{46}$$

($A_0, A_1, \ldots, A_m$ are given square matrices of order n), then the same matrices $X_0, X_1, \ldots, X_m$ satisfy the scalar equation

$$g(X_0, X_1, \ldots, X_m) = O, \tag{47}$$

where

$$g(\xi_0, \xi_1, \ldots, \xi_m) = |A_0\xi_0 + A_1\xi_1 + \cdots + A_m\xi_m|. \tag{48}$$

Proof. We set[6]

$$F(\xi_0, \xi_1, \ldots, \xi_m) = \|f_{ik}(\xi_0, \xi_1, \ldots, \xi_m)\|_1^n = A_0\xi_0 + A_1\xi_1 + \cdots + A_m\xi_m.$$

$\xi_0, \xi_1, \ldots, \xi_m$ are scalar variables.

We denote by $\widehat{F}(\xi_0, \xi_1, \ldots, \xi_m) = \|\widehat{f}_{ik}(\xi_0, \xi_1, \ldots, \xi_m)\|_1^n$ the adjoint matrix of F ($\widehat{f}_{ik}$ is the algebraic complement of f_{ki} in the determinant $|F(\xi_0, \xi_1, \ldots, \xi_m)| = |f_{ik}|_1^n$ $(i, k = 1, 2, \ldots, n)$). Then every element $\widehat{f}_{ik}$ $(i, k = 1, 2, \ldots, n)$ of $\widehat{F}$ is a homogeneous polynomial in $\xi_0, \xi_1, \ldots, \xi_m$ of degree $m - 1$, so that $\widehat{F}$ can be represented in the form

$$\widehat{F} = \sum_{j_0 + j_1 + \cdots + j_m = n-1} F_{j_0 j_1 \cdots j_m} \xi_0^{j_0}\xi_1^{j_1} \cdots \xi_m^{j_m},$$

where $F_{j_0 j_1 \cdots j_m}$ are certain constant matrices of order n.

From the definition of $\widehat{F}$ there follows the identity

$$\widehat{F}F = g(\xi_0, \xi_1, \ldots, \xi_m)E.$$

We write this in the following form:

$$\sum_{j_0 + j_1 + \cdots + j_{m-1} = n-1} F_{j_0 j_1 \cdots j_m}(A_0\xi_0 + A_1\xi_1 + \cdots + A_m\xi_m)\xi_0^{j_0}\xi_1^{j_1} \cdots \xi_m^{j_m} = g(\xi_0, \xi_1, \ldots, \xi_m)E. \tag{49}$$

The transition from the left-hand side of (49) to the right-hand side is accomplished by removing the parentheses and collecting similar terms. In this process *we have to permute the variables $\xi_0, \xi_1, \ldots, \xi_m$ among each other, but we do not have to permute the variables $\xi_0, \xi_1, \ldots, \xi_m$ with the matrix coefficients A_i and $F_{j_0 j_1 \cdots j_m}$.* Therefore the equation (49) is not violated when we substitute for the variables $\xi_0, \xi_1, \ldots, \xi_m$ the pairwise permutable matrices $X_0, X_1, \ldots, X_m$:

---

[5] See [318].

[6] The $f_{ik}(\xi_0, \xi_1, \ldots, \xi_m)$ are linear forms in $\xi_0, \xi_1, \ldots, \xi_m$ $(i, k = 1, 2, \ldots, n)$.

$$\sum_{i_0+i_1+\cdots+i_m=n-1} F_{i_0 i_1 \cdots i_m}(A_0X_0 + A_1X_1 + \cdots + A_mX_m)\, X_0^{i_0}X_1^{i_1}\cdots X_m^{i_m}$$
$$= g(X_0, X_1, \ldots, X_m). \qquad (50)$$

But, by assumption,

$$A_0X_0 + A_1X_1 + \cdots + A_mX_m = O.$$

Therefore we find from (50):

$$g(X_0, X_1, \ldots, X_m) = O,$$

and this is what we had to prove.

Note 1. Theorem 5 remains valid if (46) is replaced by

$$X_0A_0 + X_1A_1 + \cdots + X_mA_m = O. \qquad (51)$$

For we can apply Theorem 5 to the equation

$$A_0'X_0 + A_1'X_1 + \cdots + A_m'X_m = O$$

and then go over term by term to the transposed matrices.

Note 2. Theorem 4 is obtained as a special case of Theorem 5, when we take for $X_0, X_1, \ldots, X_m$

$$X^m, X^{m-1}, \ldots, X, E.$$

3. We have shown that every solution of (41) satisfies the scalar equation (of degree $\leqq mn$)

$$g(\lambda) = 0.$$

But the set of matrix solutions of this equation with a given order n splits into a finite number of classes of similar matrices (see § 4). Therefore all the solutions of (41) have to be looked for among the matrices of the form

$$T_iD_iT_i^{-1} \qquad (52)$$

(here D_i are well-defined matrices; if we wish, we may assume that the D_i have Jordan normal form. T_i are arbitrary non-singular matrices of order n; $i = 1, 2, \ldots, n$). In (41) we substitute for X the matrix (52) and choose T_i such that the equation (41) is satisfied. For each T_i we obtain a linear equation

$$A_0T_iD_i^m + A_1T_iD_i^{m-1} + \cdots + A_mT_i = O \qquad (i = 1, 2, \ldots, n). \qquad (53)$$

A natural method of finding solutions T_i of (53) is to replace the matrix equation by a system of linear homogeneous scalar equations in the elements of the required matrix T_i. Each non-singular solution T_i of (53), when substituted in (52), yields a solution of the given equation (41). Similar arguments may be applied to the equation (42).

In the following two sections we shall consider special cases of (41) connected with the extraction of m-th roots of a matrix.

§ 6. The Extraction of m-th Roots of a Non-Singular Matrix

1. In this section and the following, we deal with the equation

$$X^m = A, \qquad (54)$$

where A is a given matrix and X an unknown matrix (both of order n) and m is a given positive integer.

In this section we consider the case $|A| \neq 0$ (A is non-singular). All the characteristic values of A are different from zero in this case (since $|A|$ is the product of these characteristic values).

We denote by

$$(\lambda - \lambda_1)^{p_1}, (\lambda - \lambda_2)^{p_2}, \ldots, (\lambda - \lambda_u)^{p_u} \qquad (55)$$

the elementary divisors of A and reduce A to Jordan normal form:[7]

$$A = U\tilde{A}U^{-1} = U\{\lambda_1E_1 + H_1, \ldots, \lambda_uE_u + H_u\}\,U^{-1}. \qquad (56)$$

Since the characteristic values of the unknown matrix X, when raised to the m-th power, give the characteristic values of A, all the characteristic values of X are also different from zero. Therefore the derivative of $f(\lambda) = \lambda^m$ does not vanish on these characteristic values. But then (see Chapter VI, p. 158) the elementary divisors of X do not 'decompose' when X is raised to the m-th power. From what we have said, it follows that the elementary divisors of X are:

$$(\lambda - \xi_1)^{p_1}, (\lambda - \xi_2)^{p_2}, \ldots, (\lambda - \xi_u)^{p_u}. \qquad (57)$$

where $\xi_j^m = \lambda_j$, i.e., ξ_j is one of the m-th roots of λ_j ($\xi_j = \sqrt[m]{\lambda_j}$; $j = 1, 2, \ldots, u$).

We now determine $\sqrt[m]{\lambda_jE_j + H_j}$ in the following way. In the λ-plane we take a circle, with center λ_j, not containing the origin. In this circle we have m distinct branches of the function $\sqrt[m]{\lambda}$. These branches can be distinguished from one another by the value they assume at the center λ_j of the circle. We denote by $\sqrt[m]{\lambda}$ that branch whose value at λ_j coincides with the characteristic value ξ_j of the unknown matrix X, and starting from this branch we define the matrix function $\sqrt[m]{\lambda_jE_j + H_j}$ by means of the series

---

[7] Here $E_j = E^{(p_j)}$ and $H_j = H^{(p_j)}$ $(j = 1, 2, \ldots, u)$.

$$\sqrt[m]{\lambda_j E_j + H_j} = \lambda_j^{\frac{1}{m}} E_j + \frac{1}{m} \lambda_j^{\frac{1}{m}-1} H_j + \frac{1}{2!} \frac{1}{m}\left(\frac{1}{m}-1\right) \lambda_j^{\frac{1}{m}-2} H_j^2 + \cdots, \tag{58}$$

which breaks off.

Since the derivative of the function $\sqrt[m]{\lambda}$ at λ_j is not zero, the matrix (58) has only one elementary divisor $(\lambda - \xi_j)^{p_j}$, where $\xi_j = \sqrt[m]{\lambda}$ (here $j = 1, 2, 3, \ldots, u$). Hence it follows that the quasi-diagonal matrix

$$\{\sqrt[m]{\lambda_1 E_1 + H_1}, \sqrt[m]{\lambda_2 E_2 + H_2}, \ldots, \sqrt[m]{\lambda_u E_u + H_u}\}$$

has the elementary divisors (57), i.e., the same elementary divisors as the unknown matrix X. Therefore there exists a non-singular matrix T ($|T| \neq 0$) such that

$$X = T\,\{\sqrt[m]{\lambda_1 E_1 + H_1}, \sqrt[m]{\lambda_2 E_2 + H_2}, \ldots, \sqrt[m]{\lambda_u E_u + H_u}\}\, T^{-1}. \tag{59}$$

In order to determine T, we note that if on both sides of the identity

$$(\sqrt[m]{\lambda})^m = \lambda$$

we substitute the matrix $\lambda_j E_j + H_j$ $(j = 1, 2, \ldots, u)$ in place of λ, we obtain:

$$(\sqrt[m]{\lambda_j E_j + H_j})^m = \lambda_j E_j + H_j \quad (j = 1, 2, \ldots, u).$$

Now from (54) and (59) it follows that

$$A = T\,\{\lambda_1 E_1 + H_1, \lambda_2 E_2 + H_2, \ldots, \lambda_u E_u + H_u\}\, T^{-1}. \tag{60}$$

Comparing (56) and (60) we find:

$$T = U X_{\tilde{A}}, \tag{61}$$

where $X_{\tilde{A}}$ is an arbitrary non-singular matrix permutable with $\tilde{A}$ (the structure of $X_{\tilde{A}}$ is described in detail in § 2).

When we substitute in (59) for T the expression $U X_{\tilde{A}}$ we obtain a formula that comprises all the solutions of the equation (54):

$$X = U X_{\tilde{A}}\,\{\sqrt[m]{\lambda_1 E_1 + H_1}, \sqrt[m]{\lambda_2 E_2 + H_2}, \ldots, \sqrt[m]{\lambda_u E_u + H_u}\}\, X_{\tilde{A}}^{-1} U^{-1}. \tag{62}$$

The multivalence of the right-hand side of this formula has a discrete as well as a continuous character: the discrete (in this case finite) character arises from the choice of the distinct branches of the function $\sqrt[m]{\lambda}$ in the various blocks of the quasi-diagonal matrix (for $\lambda_i = \lambda_k$ the branches of $\sqrt[m]{\lambda}$ in the j-th and k-th diagonal blocks may even be distinct); the continuous character arises from the arbitrary parameters contained in $X_{\tilde{A}}$.

All solutions of (54) will be called *m-th roots of* A and will be denoted by the many-valued symbol $\sqrt[m]{A}$. We point out that $\sqrt[m]{A}$ is, in general, not a function of the matrix A (i.e., is not representable in the form of a polynomial in A).

Note. If all the elementary divisors of A are co-prime in pairs, i.e., if the numbers $\lambda_1, \lambda_2, \ldots, \lambda_u$ are all distinct, then the matrix $X_{\tilde{A}}$ has quasi-diagonal form

$$X_{\tilde{A}} = \{X_1, X_2, \ldots, X_u\},$$

where X_j is permutable with $\lambda_j E_j + H_j$ and therefore permutable with every function of $\lambda_j E_j + H_j$ and, in particular, with $\sqrt[m]{\lambda_j E_j + H_j}$ $(j = 1, 2, \ldots, u)$. Therefore in this case (62) assumes the form

$$X = U\,\{\sqrt[m]{\lambda_1 E_1 = H_1}, \sqrt[m]{\lambda_2 E_2 + H_2}, \ldots, \sqrt[m]{\lambda_u E_u + H_u}\}\, U^{-1}.$$

Thus, if the elementary divisors of A are co-prime in pairs, then in the formula for $X = \sqrt[m]{A}$ only a discrete multivalence occurs. In this case every value of $\sqrt[m]{A}$ can be represented as a polynomial in A.

2. *Example.* Suppose it is required to find all square roots of

$$A = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

i.e., all solutions of the equation

$$X^2 = A.$$

In this case A has already the Jordan normal form. Therefore in (62) we can set $A = \tilde{A}$, $U = E$. The matrix $X_{\tilde{A}}$ in this case looks as follows:

$$X_{\tilde{A}} = \begin{Vmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{Vmatrix},$$

where a, b, c, d, and e are arbitrary parameters.

The formula (62), which gives all the required solutions X, now assumes the following form:

$$X = \begin{Vmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{Vmatrix} \begin{Vmatrix} \varepsilon & \frac{\varepsilon}{2} & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \eta \end{Vmatrix} \begin{Vmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{Vmatrix}^{-1} \quad (\varepsilon^2 = \eta^2 = 1). \tag{63}$$

Without changing X we may multiply $X_{\tilde{A}}$ in (62) by a scalar so that $|X_{\tilde{A}}| = 1$. Then this leads to the equation $a^2 e = 1$; and hence $e = a^{-2}$.

Let us compute the elements of $X_{\tilde{A}}^{-1}$. For this purpose we write down the linear transformation with the matrix coefficients of $X_{\tilde{A}}$:

$$\begin{aligned} y_1 &= ax_1 + bx_2 + cx_3, \\ y_2 &= ax_2, \\ y_3 &= dx_2 + a^{-2}x_3. \end{aligned}$$

We solve this system of equations with respect to x_1, x_2, x_3. Then we obtain the transformation with the inverse matrix $X_{\tilde{A}}^{-1}$:

$$\begin{aligned} x_1 &= a^{-1}y_1 - (a^{-2}b - cd)\, y_2 - acy_3, \\ x_2 &= a^{-1}y_2, \\ x_3 &= -a\, dy_2 + a^2 y_3. \end{aligned}$$

Hence we find:

$$X_{\tilde{A}}^{-1} = \left\| \begin{matrix} a & b & c \\ 0 & a & 0 \\ 0 & d & a^{-2} \end{matrix} \right\|^{-1} = \left\| \begin{matrix} a^{-1} & cd - a^{-2}b & -ac \\ 0 & a^{-1} & 0 \\ 0 & -ad & a^2 \end{matrix} \right\|.$$

The formula (63) yields:

$$X = \left\| \begin{matrix} \varepsilon & (\varepsilon - \eta)\, a\, cd + \frac{\varepsilon}{2} & a^2 c\, (\eta - \varepsilon) \\ 0 & \varepsilon & 0 \\ 0 & (\varepsilon - \eta)\, da^{-1} & \eta \end{matrix} \right\|$$

$$= \left\| \begin{matrix} \varepsilon & (\varepsilon - \eta)\, vw + \frac{\varepsilon}{2} & (\eta - \varepsilon)\, v \\ 0 & \varepsilon & 0 \\ 0 & (\varepsilon - \eta)\, w & \eta \end{matrix} \right\| \quad (v = a^2 c,\ w = a^{-1} d). \tag{64}$$

The solution X depends on two arbitrary parameters u and w and two arbitrary signs ε and η.

§ 7. The Extraction of *m*-th Roots of a Singular Matrix

1. We pass on to the discussion of the case where $|A| = 0$ (A is a singular matrix).

As in the first case, we reduce A to the Jordan normal form:

$$A = U\, \{\lambda_1 E^{(p_1)} + H^{(p_1)}, \ldots, \lambda_u E^{(p_u)} + H^{(p_u)};\ H^{(q_1)}, H^{(q_2)}, \ldots, H^{(q_t)}\}\, U^{-1}; \tag{65}$$

here we have denoted by $(\lambda - \lambda_1)^{p_1}, \ldots, (\lambda - \lambda_u)^{p_u}$ the elementary divisors of A that correspond to non-zero characteristic values, and by $\lambda^{q_1}, \lambda^{q_2}, \ldots, \lambda^{q_t}$ the elementary divisors with characteristic value zero.

Then

$$A = U\, \{A_1, A_2\}\, U^{-1}. \tag{66}$$

where

$$A_1 = \{\lambda_1 E^{(p_1)} + H^{(p_1)}, \ldots, \lambda_u E^{(p_u)} + H^{(p_u)}\},\ A_2 = \{H^{(q_1)}, H^{(q_2)}, \ldots, H^{(q_t)}\}. \tag{67}$$

Note that A_1 is a non-singular matrix ($|A_1| \neq 0$) and A_2 a nilpotent matrix with index of nilpotency $\mu = \max\,(q_1, q_2, \ldots, q_t)$ ($A_2^\mu = O$).

The original equation (54) implies that A commutes with the unknown matrix X and therefore the similar matrices

$$U^{-1} A U = \{A_1, A_2\} \quad \text{and} \quad U^{-1} X U \tag{68}$$

also commute.

As we have shown in § 2 (Theorem 3), from the permutability of the matrices (68) and the fact that A_1 and A_2 do not have characteristic values in common, it follows that the second matrix in (68) has a corresponding quasi-diagonal form

$$U^{-1} X U = \{X_1, X_2\}. \tag{69}$$

When we replace the matrices A and X in (54) by the similar matrices

$$\{A_1, A_2\} \quad \text{and} \quad \{X_1, X_2\},$$

we replace (54) by two equations:

$$X_1^m = A_1, \tag{70}$$

$$X_2^m = A_2. \tag{71}$$

Since $|A_1| \neq 0$, the results of the preceding section are applicable to (70). Therefore we find X_1 by the formula (62):

$$X_1 = X_{A_1} \{\sqrt[m]{\lambda_1 E^{(p_1)} + H^{(p_1)}}, \ldots, \sqrt[m]{\lambda_u E^{(p_u)} + H^{(p_u)}}\}\, X_{A_1}^{-1}. \tag{72}$$

Thus it remains to consider the equation (71), i.e., to find all m-th roots of the nilpotent matrix A_2, which already has the Jordan normal form

$$A_2 = \{H^{(q_1)}, H^{(q_2)}, \ldots, H^{(q_t)}\}; \tag{73}$$

$\mu = \max\,(q_1, q_2, \ldots, q_t)$ is the index of nilpotency of A_2.

From $A_2^\mu = O$ and (71) we find

$$X_2^{m\mu} = O.$$

The last equation shows that the required matrix X_2 is also nilpotent with an index of nilpotency ν, where $m(\mu - 1) < \nu \leq m\mu$. We reduce X_2 to the Jordan form:

$$X_2 = T\{H^{(v_1)}, H^{(v_2)}, \ldots, H^{(v_s)}\}\, T^{-1} \tag{74}$$
$$(v_1, v_2, \ldots, v_s \leqq v).$$

Now we raise both sides of (74) to the m-th power. We obtain:

$$A_2 = X_2^m = T\{[H^{(v_1)}]^m, [H^{(v_2)}]^m, \ldots, [H^{(v_s)}]^m\}\, T^{-1}. \tag{75}$$

2. Let us now clarify the question of what elementary divisors the matrix $[H^{(v)}]^m$ has.[8] We denote by $\boldsymbol{H}$ the linear operator given by $H^{(v)}$ in a v-dimensional vector space with the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_v$. Then from the form of the matrix $H^{(v)}$ (in $H^{(v)}$ all the elements of the first superdiagonal are equal to 1 and all the remaining elements are 0) it follows that

$$\boldsymbol{He}_1 = \boldsymbol{o},\ \boldsymbol{He}_2 = \boldsymbol{e}_1,\ \ldots,\ \boldsymbol{He}_v = \boldsymbol{e}_{v-1}. \tag{76}$$

These equations show that the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_v$ form a Jordan chain for $\boldsymbol{H}$, corresponding to the elementary divisor λ^v.

We write (76) as follows:

$$\boldsymbol{He}_j = \boldsymbol{e}_{j-1} \qquad (j = 1, 2, \ldots, v;\ \boldsymbol{e}_0 = \boldsymbol{o}).$$

Obviously,

$$\boldsymbol{H}^m\boldsymbol{e}_j = \boldsymbol{e}_{j-m} \qquad (j = 1, 2, \ldots, v;\ \boldsymbol{e}_0 = \boldsymbol{e}_{-1} = \cdots \boldsymbol{e}_{-m+1} = \boldsymbol{o}). \tag{77}$$

We express v in the form

$$v = km + r \qquad (r < m),$$

where k and r are non-negative integers. We arrange the basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_v$ in the following way:

$$\begin{array}{lll} \boldsymbol{e}_1, & \boldsymbol{e}_2, & \ldots,\ \boldsymbol{e}_m, \\ \boldsymbol{e}_{m+1}, & \boldsymbol{e}_{m+2}, & \ldots,\ \boldsymbol{e}_{2m}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ \boldsymbol{e}_{(k-1)m+1}, & \boldsymbol{e}_{(k-1)m+2}, & \ldots,\ \boldsymbol{e}_{km}, \\ \boldsymbol{e}_{km+1}, & \ldots,\ \boldsymbol{e}_{km+r}. & \end{array} \tag{78}$$

This table has m columns: the first r columns contain $k + 1$ vectors each, the remaining ones k vectors. The equation (77) shows that the vectors of each column form a Jordan chain with respect to the operator $\boldsymbol{H}^m$. If instead of numbering the vectors (78) by rows we number them by columns, we obtain a new basis in which the matrix of the operator $\boldsymbol{H}^m$ has the following Jordan normal form:[9]

$$\{\underbrace{H^{(k+1)}, \ldots, H^{(k+1)}}_{r},\ \underbrace{H^{(k)}, \ldots, H^{(k)}}_{m-r}\};$$

and therefore

$$[H^{(v)}]^m = P_{v,m}\{\underbrace{H^{(k+1)}, \ldots, H^{(k+1)}}_{r},\ \underbrace{H^{(k)}, \ldots, H^{(k)}}_{m-r}\}\, P_{v,m}^{-1}, \tag{79}$$

where the matrix $P_{v,m}$ (describing the transition from the one basis to the other) has the following form (see Chapter III, § 4):

$$P_{v,m} = \left\| \begin{array}{cccccc} \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad}^{m}} & & \\ 1 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 1 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & & \\ 0 & 1 & \ldots & 0 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right\| \begin{array}{l} \\ \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} m. \\ \\ \\ \end{array} \tag{80}$$

The matrix $H^{(v)}$ has the single elementary divisor λ^v. When $H^{(v)}$ is raised to the *m-th* power, this elementary divisor 'falls apart.' As (79) shows, $[H^{(v)}]^m$ has the elementary divisors:

$$\underbrace{\lambda^{k+1}, \ldots, \lambda^{k+1}}_{r},\ \underbrace{\lambda^k, \ldots, \lambda^k}_{m-r}.$$

Turning now to (75), we set:

$$v_i = k_i m + r_i \quad (0 \leqq r_i < m,\ k_i \geqq 0; \quad i = 1, 2, \ldots, s). \tag{81}$$

Then, by (79), equation (75) can be written as follows:

$$\begin{aligned} A_2 = X_2^m = TP\{&\underbrace{H^{(k_1+1)}, \ldots, H^{(k_1+1)}}_{r_1}, \underbrace{H^{(k_1)}, \ldots, H^{(k_1)}}_{m-r_1}, \\ &\underbrace{H^{(k_2+1)}, \ldots, H^{(k_2+1)}}_{r_2}, H^{(k_2)}, \ldots\}\, P^{-1}\, T^{-1}. \end{aligned} \tag{82}$$

where
$$P = \{P_{v_1,m}, P_{v_2,m}, \ldots, P_{v_s,m}\}$$

---

[8] This question is answered by Theorem 9 of Chapter VI (p. 158). Here we are compelled to use another method of investigating the problem, because we have to find not only the elementary divisor of the matrix $[H^{(v)}]^m$, but also a matrix $p_{v,m}$ transforming $[H^{(v)}]^m$ into Jordan form.

[9] In the case $k = 0$, the blocks $\underbrace{H^{(k)}, \ldots, H^{(k)}}_{m-r}$ are absent, and the matrix has the form $\{\underbrace{H^{(1)}, \ldots, H^{(1)}}_{r}\}$.

Comparing (82) with (73), we see that the blocks

$$H^{(k_1+1)}, \dots, H^{(k_1+1)}, H^{(k_1)}, \dots, H^{(k_1)}, H^{(k_2+1)}, \dots, H^{(k_2+1)}, \dots \tag{83}$$

must coincide, apart from the order, with the blocks

$$H^{(q_1)}, H^{(q_2)}, \dots, H^{(q_t)}. \tag{84}$$

3. Let us call a system of elementary divisors $\lambda^{v_1}, \lambda^{v_2}, \dots, \lambda^{v_s}$ *admissible* for X_2 if after raising of the matrix to the m-th power these elementary divisors split and generate the given system of elementary divisors of A_2: $\lambda^{q_1}, \lambda^{q_2}, \lambda^{q_3}, \dots, \lambda^{q_t}$. The number of admissible systems of elementary divisors is always finite, because

$$\max(v_1, v_2, \dots, v_s) \leqq m\mu, \quad v_1 + v_2 + \cdots + v_s = n_2 \tag{85}$$

(n_2 is the order of A_2).

In every concrete case the admissible systems of elementary divisors for X_2 can easily be determined by a finite number of trials.

Let us show that for each admissible system of elementary divisors $\lambda^{v_1}, \lambda^{v_2}, \dots, \lambda^{v_s}$ form a corresponding solution of (71) and let us determine all these solutions. In this case there exists a transforming matrix Q such that

$$\{H^{(k_1+1)}, \dots, H^{(k_1+1)}, H^{(k_1)}, \dots, H^{(k_1)}, H^{(k_2+1)}, \dots\} = Q^{-1} A_2 Q. \tag{86}$$

The matrix Q describes the permutation of the blocks in the quasi-diagonal matrix that brings about the proper renumbering of the basis vectors. Therefore Q can be regarded as known. Using (86), we obtain from (82):

$$A_2 = TPQ^{-1}A_2QP^{-1}T^{-1}.$$

Hence

$$TPQ^{-1} = X_{A_2},$$

or

$$T = X_{A_2}QP^{-1}, \tag{87}$$

where X_{A_2} is an arbitrary matrix that commutes with A_2.

Substituting (87) for T in (74), we have

$$X_2 = X_{A_2}QP^{-1}\{H^{(v_1)}, H^{(v_2)}, \dots, H^{(v_s)}\}PQ^{-1}X_{A_2}^{-1}. \tag{88}$$

From (69), (72), and (88) we obtain a general formula which comprises all the solutions:

$$X = U\{X_{A_1}, X_{A_2}QP^{-1}\}\{\sqrt[m]{\lambda_1 E^{(p_1)} + H^{(p_1)}}, \dots, \sqrt[m]{\lambda_u E^{(p_u)} + H^{(p_u)}},$$
$$H^{(v_1)}, \dots, H^{(v_s)}\} \cdot \{X_{A_1}^{-1}, PQ^{-1}X_{A_2}^{-1}\}U^{-1}. \tag{89}$$

We draw the reader's attention to the fact that the m-th root of a singular matrix does not always exist. Its existence is bound up with the existence of a system of admissible elementary divisors for X_2.

It is easy to see, for example, that the equation

$$X^m = H^{(p)}$$

has no solution for $m > 1$, $p > 1$.

Example. Suppose it is required to extract the square root of

$$A = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix},$$

i.e., to find all the solutions of the equation

$$X^2 = A.$$

In this case, $A = A_2$, $X = X_2$, $m = 2$, $t = 2$, $q_1 = 2$, and $q_2 = 1$. The matrix X can only have the one elementary divisor λ^3. Therefore $s = 1$, $v_1 = 3$, $k_1 = 1$, $r_1 = 1$ and (see (80))

$$P = P_{3,2} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix} = P^{-1}, Q = E.$$

Moreover, as in the example on page 233, in (88) we may set:

$$X_{A_2} = \begin{Vmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & a^{-2} \end{Vmatrix}, \quad X_{A_2}^{-1} = \begin{Vmatrix} a^{-1} & cd - a^{-2}b & -ac \\ 0 & a^{-1} & 0 \\ 0 & -ad & a^2 \end{Vmatrix}.$$

From this formula we obtain

$$X = X_2 = X_{A_2}P^{-1}H^{(3)}PX_{A_2}^{-1} = \begin{Vmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 \end{Vmatrix},$$

where $\alpha = ca^{-1} - a^2d$ and $\beta = a^3$ are arbitrary parameters.

§ 8. The Logarithm of a Matrix

1. We consider the matrix equation

$$e^X = A. \tag{90}$$

All the solutions of this equation are called (natural) *logarithms* of A and are denoted by $\ln A$.

The characteristic values λ_j of A are connected with the characteristic values ξ_j of X by the formula $\lambda_j = e^{\xi_j}$; therefore, if the equation (90) has a solution, then all the characteristic values of A are different from zero, and A is non-singular ($|A| \neq 0$). Thus, the condition $|A| \neq 0$ is necessary for the existence of solutions of the equation (90). Below, we shall see that this condition is also sufficient.

Suppose, then, that $|A| \neq 0$. We write down the elementary divisors of A:

$$(\lambda-\lambda_1)^{p_1},\ (\lambda-\lambda_2)^{p_2},\ \ldots,\ (\lambda-\lambda_u)^{p_u}$$

$$(\lambda_1\lambda_2 \cdots \lambda^u \neq 0, \quad p_1+p_2+\cdots+p_u=n). \tag{91}$$

Corresponding to these elementary divisors we reduce A to the Jordan normal form:

$$\begin{aligned} A &= U\tilde{A}U^{-1} \\ &= U\{\lambda_1E^{(p_1)}+H^{(p_1)},\ \lambda_2E^{(p_2)}+H^{(p_2)},\ \ldots,\ \lambda_uE^{(p_u)}+H^{(p_u)}\}\,U^{-1}. \end{aligned} \tag{92}$$

Since the derivative of the function e^{ξ} is different from zero for all values of ξ, we know (see Chapter VI, p. 158) that in the transition from X to $A=e^X$ the elementary divisors do not split, so that X has the elementary divisors

$$(\lambda-\xi_1)^{p_1},\ (\lambda-\xi_2)^{p_2},\ \ldots,\ (\lambda-\xi_u)^{p_u}, \tag{93}$$

where $e^{\xi_j}=\lambda_j$ $(j=1,2,\ldots,u)$, i.e., ξ_j is one of the values of $\ln\lambda_j$ $(j=1,2,3,\ldots,u)$.

In the plane of the complex variable λ we draw a circle with center at λ_j and with radius less than $|\lambda_j|$ and we denote by $f_j(\lambda)=\ln\lambda$ that branch of the function $\ln\lambda$ in this circle which at λ_j assumes the value equal to the characteristic value ξ_j of X $(j=1,2,\ldots,u)$. After this, we set:

$$\ln(\lambda_jE^{(p_j)}+H^{(p_j)}) = f_j(\lambda_jE^{(p_j)}+H^{(p_j)}) = \ln\lambda_jE^{(p_j)}+\lambda_j^{-1}H^{(p_j)}+\cdots. \tag{94}$$

Since the derivative of $\ln\lambda$ vanishes nowhere (in the finite part of the λ-plane), the matrix (94) has only the one elementary divisor $(\lambda-\xi_j)^{p_j}$. Therefore the quasi-diagonal matrix

$$\{\ln(\lambda_1E^{(p_1)}+H^{(p_1)}),\ \ln(\lambda_2E^{(p_2)}+H^{(p_2)}),\ \ldots,\ \ln(\lambda_uE^{(p_u)}+H^{(p_u)})\} \tag{95}$$

has the same elementary divisors as the unknown matrix X. Therefore there exists a matrix T ($|T| \neq 0$) such that

$$X = T\{\ln(\lambda_1E^{(p_1)}+H^{(p_1)}),\ \ldots,\ \ln(\lambda_uE^{(p_u)}+H^{(p_u)})\}\,T^{-1}. \tag{96}$$

In order to determine T, we note that

$$A = e^X = T\{\lambda_1E^{(p_1)}+H^{(p_1)},\ \ldots,\ \lambda_uE^{(p_u)}+H^{(p_u)}\}\,T^{-1}. \tag{97}$$

Comparing (97) and (92), we find:

$$T = UX_{\tilde{A}}, \tag{98}$$

where $X_{\tilde{A}}$ is an arbitrary matrix that commutes with $\tilde{A}$. Substituting the expression for T from (98) into (96), we obtain a general formula that comprises all the logarithms of the matrix:

$$\begin{aligned} X = UX_{\tilde{A}}\{&\ln(\lambda_1E^{(p_1)}+H^{(p_1)}), \\ &\ln(\lambda_2E^{(p_2)}+H^{(p_2)}),\ \ldots,\ \ln(\lambda_uE^{(p_u)}+H^{(p_u)})\}\,X_{\tilde{A}}^{-1}U^{-1}. \end{aligned} \tag{99}$$

Note. If all the elementary divisors of A are co-prime, then on the right-hand side of (99) the factors $X_{\tilde{A}}$ and $X_{\tilde{A}}^{-1}$ can be omitted (see a similar remark on p. 233).

CHAPTER IX

LINEAR OPERATORS IN A UNITARY SPACE

§ 1. General Considerations

In Chapters III and VII we studied linear operators in an arbitrary n-dimensional vector space. All the bases of such a space are of equal standing. To a given linear operator there corresponds in each basis a certain matrix. The matrices corresponding to one and the same operator in the various bases are similar. Thus, the study of linear operators in an n-dimensional vector space enables us to bring out those properties of matrices that are inherent in an entire class of similar matrices.

At the beginning of this chapter we shall introduce a metric into an n-dimensional space by assigning in a special way to each pair of vectors a certain number, the 'scalar product' of the two vectors. By means of the scalar product we shall define the 'length' of a vector and the cosine of the 'angle' between two vectors. This metrization leads to a unitary space if the ground field F is the field of all complex numbers and to a euclidean space if F is the field of all real numbers.

In the present chapter we shall study the properties of linear operators that are connected with the metric of the space. All the bases of the space are by no means of equal standing with respect to the metric. However, this does hold true of all *orthonormal* bases. The transition from one orthonormal basis to another in a unitary space is brought about by means of a special—namely, unitary—transformation (in a euclidean space, an orthogonal transformation). Therefore all the matrices that correspond to one and the same linear operator in two distinct bases of a unitary (euclidean) space are unitarily (orthogonally) similar. Thus, by studying linear operators in an n-dimensional metrized space we study the properties of matrices that remain invariant under transition from a given matrix to a unitarily—or orthogonally—similar one. This will lead in a natural way to the investigation of properties of special classes of matrices (normal, hermitian, unitary, symmetric, skew-symmetric, orthogonal matrices).

§ 2. Metrization of a Space

1. We consider a vector space $\boldsymbol{R}$ over the field of complex numbers. To every pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of $\boldsymbol{R}$ given in a definite order let a certain complex number be assigned, the so-called *scalar product*, or *inner product*, of the vectors, denoted by $(\boldsymbol{xy})$ or $(\boldsymbol{x}, \boldsymbol{y})$. Suppose further that the 'scalar multiplication' has the following properties:

For arbitrary vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ of $\boldsymbol{R}$ and an arbitrary complex number α, let[1]

$$\left.\begin{array}{ll} 1. & (\boldsymbol{xy}) = \overline{(\boldsymbol{yx})}, \\ 2. & (\alpha\boldsymbol{x}, \boldsymbol{y}) = \alpha(\boldsymbol{xy}), \\ 3. & (\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{xz}) + (\boldsymbol{yz}). \end{array}\right\} \quad (1)$$

Then we shall say that a *hermitian metric* is introduced in $\boldsymbol{R}$.

Note that 1., 2., and 3. have the following consequences for arbitrary $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\boldsymbol{R}$:

$$\begin{array}{ll} 2'. & (\boldsymbol{x}, \alpha\boldsymbol{y}) = \bar{\alpha}(\boldsymbol{xy}), \\ 3'. & (\boldsymbol{x}, \boldsymbol{y}+\boldsymbol{z}) = (\boldsymbol{xy}) + (\boldsymbol{xz}). \end{array}$$

From 1. we deduce that for every vector $\boldsymbol{x}$ the scalar product $(\boldsymbol{x}\,\boldsymbol{x})$ is a real number. This number is called the *norm* of $\boldsymbol{x}$ and is denoted by $\mathrm{N}\boldsymbol{x}$: $\mathrm{N}\boldsymbol{x} = (\boldsymbol{x}, \boldsymbol{x})$.

If for every vector $\boldsymbol{x}$ of $\boldsymbol{R}$

$$4. \quad \mathrm{N}\boldsymbol{x} = (\boldsymbol{x}\,\boldsymbol{x}) \geqq 0, \qquad (2)$$

then the hermitian metric is called *positive semi-definite*. And if, moreover,

$$5. \quad \mathrm{N}\boldsymbol{x} = (\boldsymbol{x}\,\boldsymbol{x}) > 0 \text{ for } \boldsymbol{x} \neq \boldsymbol{o}, \qquad (3)$$

then the hermitian metric is called *positive definite*.

DEFINITION 1: *A vector space $\boldsymbol{R}$ with a positive-definite hermitian metric will be called a unitary space.*[2]

In this chapter we shall consider finite-dimensional unitary spaces.[3]

By the *length* of the vector $\boldsymbol{x}$ we mean[4] ${}_{+}\sqrt{\mathrm{N}\boldsymbol{x}} = {}_{+}\sqrt{(\boldsymbol{x}, \boldsymbol{x})} = |\boldsymbol{x}|$. From 2. and 5. it follows that every vector other than the null vector has a positive

---

[1] A number with a bar over it denotes the complex conjugate of the number.

[2] The study of n-dimensional vector spaces with an arbitrary (not positive-definite) metric is taken up in the paper [319].

[3] In §§ 2-7 of this chapter, wherever it is not expressly stated that the space is finite-dimensional, all the arguments remain valid for infinite-dimensional spaces.

[4] The symbol ${}_{+}\sqrt{\ }$ denotes the non-negative (arithmetical) value of the root.

length and that the null vector has length 0. A vector $\mathbf{x}$ is called *normalized* (or is said to be a *unit vector*) if $|\mathbf{x}| = 1$. To normalize an arbitrary vector $\mathbf{x} \neq \mathbf{o}$ it is sufficient to multiply it by any complex number λ for which $|\lambda| = \frac{1}{|\mathbf{x}|}$.

By analogy with the ordinary three-dimensional vector spaces, two vectors $\mathbf{x}$ and $\mathbf{y}$ are called *orthogonal* (in symbols: $\mathbf{x} \perp \mathbf{y}$) if $(\mathbf{xy}) = 0$. In this case it follows from 1., 3., and 3'. that

$$\mathrm{N}(\mathbf{x}+\mathbf{y}) = (\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) = (\mathbf{xx}) + (\mathbf{yy}) = \mathrm{N}\mathbf{x} + \mathrm{N}\mathbf{y},$$

i.e. (the theorem of Pythagoras!),

$$|\mathbf{x}+\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 \quad (\mathbf{x} \perp \mathbf{y}).$$

Let $\boldsymbol{R}$ be a unitary space of finite dimension n. We consider an arbitrary basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ of $\boldsymbol{R}$. Let us denote by x_i and y_i $(i = 1, 2, \dots, n)$ the coordinates of the vectors $\mathbf{x}$ and $\mathbf{y}$ in this basis:

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i, \quad \mathbf{y} = \sum_{i=1}^{n} y_i \mathbf{e}_i.$$

Then by 2., 3., 2'., and 3'.,

$$(\mathbf{xy}) = \sum_{i,k=1}^{n} h_{ik} x_i \bar{y}_k, \tag{4}$$

where

$$h_{ik} = (\mathbf{e}_i \mathbf{e}_k) \quad (i, k = 1, 2, \dots, n). \tag{5}$$

In particular,

$$\mathrm{N}\mathbf{x} = (\mathbf{xx}) = \sum_{i,k=1}^{n} h_{ik} x_i \bar{x}_k. \tag{6}$$

From 1. and (5) we deduce

$$h_{ki} = \bar{h}_{ik} \quad (i, k = 1, 2, \dots, n). \tag{7}$$

2. A form $\sum_{i,k=1}^{n} h_{ik} x_i \bar{x}_k$, where $h_{ki} = \bar{h}_{ik}$ $(i, k = 1, 2, \dots, n)$ is called *hermitian*.[5] Thus, the norm of a vector, i.e., the square of its length, is a hermitian form in its coordinates. Hence the name 'hermitian metric.' The form on the right-hand side of (6) is, by 4., *non-negative*:

$$\sum_{i,k=1}^{n} h_{ik} x_i \bar{x}_k \geqq 0 \tag{8}$$

for all values of the variables $x_1, x_2, \dots, x_n$. By the additional condition 5., the form is in fact *positive definite*, i.e., the equality sign in (8) only holds when all the x_i are zero $(i = 1, 2, \dots, n)$.

---

[5] In accordance with this, the expression on the right-hand side of (4) is called a hermitian bilinear form (in $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$).

Definition 2: *A system of vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m$ is called orthonormal if*

$$(\mathbf{e}_i \mathbf{e}_k) = \delta_{ik} = \begin{cases} 0, & \text{for } i \neq k, \\ 1, & \text{for } i = k \end{cases} \quad (i, k = 1, 2, \dots, m). \tag{9}$$

When $m = n$, where n is the dimension of the space, we obtain an *orthonormal basis* of the space.

In § 7 we shall prove that *every n-dimensional space has an orthonormal basis.*

Let x_i and y_i $(i = 1, 2, \dots, n)$ be the coordinates of $\mathbf{x}$ and $\mathbf{y}$ in an orthonormal basis. Then by (4), (5), and (9)

$$\left.\begin{aligned} (\mathbf{xy}) &= \sum_{i=1}^{n} x_i \bar{y}_i, \\ \mathrm{N}\mathbf{x} = (\mathbf{xx}) &= \sum_{i=1}^{n} |x_i|^2. \end{aligned}\right\} \tag{10}$$

Let us take an arbitrary fixed basis in an n-dimensional space $\boldsymbol{R}$. In this basis every metrization of the space is connected with a certain positive-definite hermitian form $\sum_{i,k=1}^{n} h_{ik} x_i \bar{x}_k$; and conversely, by (4), every such form determines a certain positive-definite hermitian metric in $\boldsymbol{R}$. However, *these metrics do not all give essentially different unitary n-dimensional spaces.* For let us take two such metrics with the respective scalar products $(\mathbf{xy})$ and $(\mathbf{xy})'$. We determine orthonormal bases in $\boldsymbol{R}$ with respect to these metrices: $\mathbf{e}_i$ and $\mathbf{e}_i'$ $(i = 1, 2, \dots, n)$. Let the vector $\mathbf{x}$ in $\boldsymbol{R}$ be mapped onto the vector $\mathbf{x}'$ in $\boldsymbol{R}$, where $\mathbf{x}'$ is the vector whose coordinates in the basis $\mathbf{e}_i'$ are the same as the coordinates of $\mathbf{x}$ in the basis $\mathbf{e}_i$ $(i = 1, 2, \dots, n)$. $(\mathbf{x} \rightarrow \mathbf{x}'.)$ This mapping is *affine*.[6] Moreover, by (10),

$$(\mathbf{xy}) = (\mathbf{x}'\mathbf{y}')'.$$

Therefore: *To within an affine transformation of the space all positive definite hermitian metrizations of an n-dimensional vector space coincide.*

If the field F is the field of real numbers, then a metric satisfying the postulates 1., 2., 3., 4., and 5. is called *euclidean.*

Definition 3: *A vector space $\boldsymbol{R}$ over the field of real numbers with a positive euclidean metric is called a euclidean space.*

If x_i and y_i $(i = 1, 2, \dots, n)$ are the coordinates of the vectors $\mathbf{x}$ and $\mathbf{y}$ in some basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ of an n-dimensional euclidean space, then

---

[6] I.e., the operator $\boldsymbol{A}$ that maps the vector $\mathbf{x}$ of $\boldsymbol{R}$ onto the vector $\mathbf{x}'$ of $\boldsymbol{R}'$ is linear and non-singular.

$$(\mathbf{x}\mathbf{y})=\sum_{i,k=1}^{n} s_{ik}x_iy_k, \qquad \mathrm{N}\mathbf{x}=|\mathbf{x}|^2=\sum_{i,k=1}^{n} s_{ik}x_ix_k.$$

Here $s_{ik}=s_{ki}$ $(i,\ k=1,\ 2,\ \ldots,\ n)$ are real numbers.[7] The expression $\sum_{i,k=1}^{n} s_{ik}x_ix_k$ is called a *quadratic form* in $x_1, x_2, \ldots, x_n$. From the fact that the metric is positive definite it follows that the quadratic form $\sum_{i,k=1}^{n} s_{ik}x_ix_k$, which gives this metric analytically, is *positive definite*, i.e., $\sum_{i,k=1}^{n} s_{ik}x_ix_k>0$ if $\sum_{i=1}^{n} x_i^2>0$.

In an orthonormal basis

$$(\mathbf{x}\mathbf{y})=\sum_{i=1}^{n} x_iy_i, \qquad \mathrm{N}\mathbf{x}=|\mathbf{x}|^2=\sum_{i=1}^{n} x_i^2. \tag{11}$$

For $n=3$ we obtain the well-known formulas for the scalar product of two vectors and for the square of the length of a vector in a three-dimensional euclidean space.

§ 3. Gram's Criterion for Linear Dependence of Vectors

1. Suppose that the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ of a unitary or of a euclidean space $\boldsymbol{R}$ are linearly dependent, i.e., that there exist numbers[8] $c_1, c_2, \ldots, c_m$ not all zero, such that

$$c_1\mathbf{x}_1+c_2\mathbf{x}_2+\cdots+c_m\mathbf{x}_m=\mathbf{o}. \tag{12}$$

When we perform the scalar multiplication by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ in succession on both sides of this equation, we obtain

$$\left.\begin{array}{l}(\mathbf{x}_1\mathbf{x}_1)\,\bar c_1+(\mathbf{x}_1\mathbf{x}_2)\,\bar c_2+\cdots+(\mathbf{x}_1\mathbf{x}_m)\,\bar c_m=0\\ (\mathbf{x}_2\mathbf{x}_1)\,\bar c_1+(\mathbf{x}_2\mathbf{x}_2)\,\bar c_2+\cdots+(\mathbf{x}_2\mathbf{x}_m)\,\bar c_m=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ (\mathbf{x}_m\mathbf{x}_1)\,\bar c_1+(\mathbf{x}_m\mathbf{x}_2)\,\bar c\ +\cdots+(\mathbf{x}_m\mathbf{x}_m)\,\bar c_m=0.\end{array}\right\} \tag{13}$$

Regarding $\bar c_1, \bar c_2, \ldots, \bar c_m$ as a non-zero solution of the system (13) of linear homogeneous equations with the determinant

---

[7] $s_{ik}=(\mathbf{e}_i\mathbf{e}_k)$ $(i, k=1, 2, \ldots, n)$.

[8] In the case of a euclidean space, $c_1, c_2, \ldots, c_m$ are real numbers.

$$G(\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m)=\left|\begin{array}{cccc}(\mathbf{x}_1\mathbf{x}_1) & (\mathbf{x}_1\mathbf{x}_2) & \ldots & (\mathbf{x}_1\mathbf{x}_m)\\ (\mathbf{x}_2\mathbf{x}_1) & (\mathbf{x}_2\mathbf{x}_2) & \ldots & (\mathbf{x}_2\mathbf{x}_m)\\ \cdot & \cdot & \cdot & \cdot\\ (\mathbf{x}_m\mathbf{x}_1) & (\mathbf{x}_m\mathbf{x}_2) & \ldots & (\mathbf{x}_m\mathbf{x}_m)\end{array}\right|, \tag{14}$$

we conclude that this determinant must vanish:

$$G(\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_m)=0.$$

$G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ is called the *Gramian* of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$.

Suppose, conversely, that the Gramian (14) is zero. Then the system of equations (13) has a non-zero solution $\bar c_1, \bar c_2, \ldots, \bar c_m$. Equations (13) can be written as follows:

$$\left.\begin{array}{l}(\mathbf{x}_1,\ c_1\mathbf{x}_1+c_2\mathbf{x}_2+\cdots+c_m\mathbf{x}_m)=0\\ (\mathbf{x}_2,\ c_1\mathbf{x}_1+c_2\mathbf{x}_2+\cdots+c_m\mathbf{x}_m)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ (\mathbf{x}_m,\ c_1\mathbf{x}_1+c_2\mathbf{x}_2+\cdots+c_m\mathbf{x}_m)=0.\end{array}\right\} \tag{13'}$$

Multiplying these equations by $c_1, c_2, \ldots, c_m$ respectively, and then adding, we obtain:

$$\mathrm{N}(c_1\mathbf{x}_1+c_2\mathbf{x}_2+\cdots+c_m\mathbf{x}_m)=0;$$

and since the metric is positive definite

$$c_1\mathbf{x}_1+c_2\mathbf{x}_2+\cdots+c_m\mathbf{x}_m=\mathbf{o},$$

i.e., the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly dependent.

Thus we have proved:

THEOREM 1: *The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly independent if and only if their Gramian is not equal to zero.*

We note the following property of the Gramian:

If any principal minor of the Gramian is zero, then the Gramian is zero.

For a principal minor is the Gramian of part of the vectors. When this principal minor vanishes, it follows that these vectors are linearly dependent and then the whole system of vectors is dependent.

2. *Example.* Let $f_1(t), f_2(t), \ldots, f_n(t)$ be n complex functions of a real argument t, sectionally continuous in the closed interval $[\alpha, \beta]$. It is required to determine conditions under which they are linearly dependent. For this purpose, we introduce a positive-definite metric into the space of functions sectionally continuous in $[\alpha, \beta]$ by setting

$$(f, g) = \int_{\alpha}^{\beta} f(t)\,\overline{g(t)}\,dt.$$

Then Gram's criterion (Theorem 1) applied to the given function yields the required condition:

$$\begin{vmatrix} \int_{\alpha}^{\beta} f_1(t)\overline{f_1(t)}\,dt & \ldots & \int_{\alpha}^{\beta} f_1(t)\overline{f_n(t)}\,dt \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot \\ \int_{\alpha}^{\beta} f_n(t)\overline{f_1(t)}\,dt & \ldots & \int_{\alpha}^{\beta} f_n(t)\overline{f_n(t)}\,dt \end{vmatrix} = 0.$$

§ 4. Orthogonal Projection

1. Let $\mathbf{x}$ be an arbitrary vector in a unitary or euclidean space $\mathbf{R}$ and $\mathbf{S}$ an m-dimensional subspace with a basis $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$. We shall show that $\mathbf{x}$ can be represented (and moreover, represented uniquely) in the form

$$\mathbf{x} = \mathbf{x}_S + \mathbf{x}_N, \tag{15}$$

where

$$\mathbf{x}_S \in \mathbf{S} \text{ and } \mathbf{x}_N \perp \mathbf{S}$$

(the symbol $\perp$ denotes orthogonality of vectors; orthogonality to a subspace means orthogonality to every vector of the subspace); *$\mathbf{x}_S$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{S}$, $\mathbf{x}_N$ the projecting vector.*

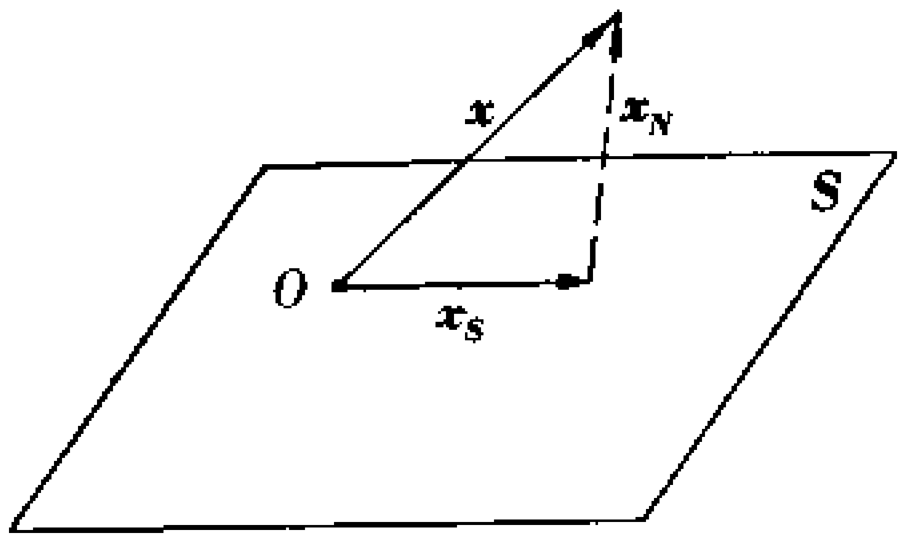

Fig. 5

Example. Let $\mathbf{R}$ be a three-dimensional euclidean vector space and $m = 2$. Let all vectors originate at a fixed point O. Then $\mathbf{S}$ is a plane passing through O; $\mathbf{x}_S$ is the orthogonal projection of $\mathbf{x}$ onto the plane $\mathbf{S}$; $\mathbf{x}_N$ is the perpendicular dropped from the endpoint of $\mathbf{x}$ onto the plane $\mathbf{S}$ (Fig. 5); and $h = |\mathbf{x}_N|$ is the distance of the endpoint of $\mathbf{x}$ from $\mathbf{S}$.

To establish the decomposition (15), we represent the required $\mathbf{x}_S$ in the form

$$\mathbf{x}_S = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m, \tag{16}$$

where $c_1, c_2, \ldots, c_m$ are complex numbers.[9]

---

[9] In the case of a euclidean space, $c_1, c_2, \ldots, c_m$ are real numbers.

To determine these numbers we shall start from the relations

$$(\mathbf{x} - \mathbf{x}_S, \mathbf{x}_k) = 0 \qquad (k = 1, 2, \ldots, m). \tag{17}$$

When we substitute in (17) for $\mathbf{x}_S$ its expression (16), we obtain:

$$\left.\begin{array}{r} (\mathbf{x}_1\mathbf{x}_1)\,c_1 + \cdots + (\mathbf{x}_m\mathbf{x}_1)\,c_m + (\mathbf{x}\mathbf{x}_1)\cdot(-1) = 0 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ (\mathbf{x}_1\mathbf{x}_m)\,c_1 + \cdots + (\mathbf{x}_m\mathbf{x}_m)\,c_m + (\mathbf{x}\mathbf{x}_m)\cdot(-1) = 0 \\ \mathbf{x}_1c_1 + \cdots + \quad \mathbf{x}_mc_m + \quad \mathbf{x}_S\cdot(-1) = \mathbf{o}. \end{array}\right\} \tag{18}$$

Regarding this as a system of linear homogeneous equations with the non-zero solution $c_1, c_2, \ldots, c_m, -1$, we equate the determinant of the system to zero and obtain (after transposition with respect to the main diagonal):[10]

$$\begin{vmatrix} (\mathbf{x}_1\mathbf{x}_1) \ldots (\mathbf{x}_1\mathbf{x}_m) & \mathbf{x}_1 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot \\ (\mathbf{x}_m\mathbf{x}_1) \ldots (\mathbf{x}_m\mathbf{x}_m) & \mathbf{x}_m \\ (\mathbf{x}\mathbf{x}_1) \ldots (\mathbf{x}\mathbf{x}_m) & \mathbf{x}_S \end{vmatrix} = \mathbf{o}. \tag{19}$$

When we separate from this determinant the term containing $\mathbf{x}_S$, we obtain (in a readily understandable notation):

$$\mathbf{x}_S = -\frac{\begin{vmatrix} & & \mathbf{x}_1 \\ & G & \cdot \\ & & \cdot \\ & & \cdot \\ & & \mathbf{x}_m \\ (\mathbf{x}\mathbf{x}_1) & \ldots\ (\mathbf{x}\mathbf{x}_m) & 0 \end{vmatrix}}{G}, \tag{20}$$

where $G = G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$ is the Gramian of the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ (in virtue of the linear independence of these vectors, $G \neq 0$). From (15) and (20), we find:

$$\mathbf{x}_N = \mathbf{x} - \mathbf{x}_S = \frac{\begin{vmatrix} & & \mathbf{x}_1 \\ & G & \cdot \\ & & \cdot \\ & & \cdot \\ & & \mathbf{x}_m \\ (\mathbf{x}\mathbf{x}_1) & \ldots\ (\mathbf{x}\mathbf{x}_m) & \mathbf{x} \end{vmatrix}}{G}. \tag{21}$$

---

[10] The determinant on the left-hand side of (19) is a vector whose i-th coordinate is obtained by replacing all the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$, $\mathbf{x}_S$ in the last column by their i-th coordinates $(i = 1, 2, \ldots, n)$; the coordinates are taken in an arbitrary basis. To justify the transition from (18) to (19), it is sufficient to replace the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$, $\mathbf{x}_S$ by their i-th coordinates.

The formulas (20) and (21) express the projection $\boldsymbol{x}_S$ of $\boldsymbol{x}$ onto the subspace $\boldsymbol{S}$ and the projecting vector $\boldsymbol{x}_N$ in terms of the given vector $\boldsymbol{x}$ and the basis of $\boldsymbol{S}$.

2. We draw attention to another important formula. We denote by h the length of the vector $\boldsymbol{x}_N$. Then, by (15) and (21),

$$h^2 = (\boldsymbol{x}_N \boldsymbol{x}_N) = (\boldsymbol{x}_N \boldsymbol{x}) = \frac{\begin{vmatrix} & & (\boldsymbol{x}_1\boldsymbol{x}) \\ & G & \vdots \\ & & (\boldsymbol{x}_m\boldsymbol{x}) \\ (\boldsymbol{x}\boldsymbol{x}_1) & \ldots\ (\boldsymbol{x}\boldsymbol{x}_m) & (\boldsymbol{x}\ \boldsymbol{x}) \end{vmatrix}}{G},$$

i.e.,

$$h^2 = \frac{G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x})}{G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)}. \tag{22}$$

The quantity h can also be interpreted in the following way:

Let the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}$ issue from a single point and construct on these vectors as edges an $(m+1)$-dimensional parallelepiped. Then h is the height of this parallelepiped measured from the end of the edge $\boldsymbol{x}$ to the base $\boldsymbol{S}$ that passes through the edges $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$.

Let $\boldsymbol{y}$ be an arbitrary vector of $\boldsymbol{S}$ and $\boldsymbol{x}$ an arbitrary vector of $\boldsymbol{R}$. If all vectors start from the origin of coordinates of an n-dimensional point space, then $|\boldsymbol{x}-\boldsymbol{y}|$ and $|\boldsymbol{x}-\boldsymbol{x}_S|$ are equal to the value of the slant height and the height respectively from the endpoint of $\boldsymbol{x}$ to the hyperplane $\boldsymbol{S}$.[11] Therefore, when we set down that *the height is shorter than the slant height*, we have:[12]

$$h = |\boldsymbol{x}-\boldsymbol{x}_S| \leqq |\boldsymbol{x}-\boldsymbol{y}|$$

(with equality only for $\boldsymbol{y}=\boldsymbol{x}_S$). Thus, among all vectors $\boldsymbol{y} \in \boldsymbol{S}$ the vector $\boldsymbol{x}_S$ deviates the least from the given vector $\boldsymbol{x} \in \boldsymbol{R}$. The quantity $h = \sqrt{\mathrm{N}(\boldsymbol{x}-\boldsymbol{x}_S)}$ is the *mean-square error* in the approximation $\boldsymbol{x} \approx \boldsymbol{x}_S$.[13]

§ 5. The Geometrical Meaning of the Gramian and Some Inequalities

1. We consider arbitrary vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$. Let us assume, to begin with, that they are linearly independent. In this case the Gramian formed from any of these vectors is different from zero. Then, when we set, in accordance with (22),

---

[11] See the example on p. 248.

[12] $\mathrm{N}(\boldsymbol{x}-\boldsymbol{y}) = \mathrm{N}(\boldsymbol{x}_N + \boldsymbol{x}_S - \boldsymbol{y}) = \mathrm{N}\boldsymbol{x}_N + \mathrm{N}(\boldsymbol{x}_S - \boldsymbol{y}) \geqq \mathrm{N}(\boldsymbol{x}_N) = h^2$.

[13] As regards the application of metrized functional spaces to problems of approximation of functions, see [1].

$$\frac{G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{p+1})}{G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p)} = h_p^2 > 0 \qquad (p=1, 2, \ldots, m-1), \tag{23}$$

and multiply these inequalities and the inequality

$$G(\boldsymbol{x}_1) = (\boldsymbol{x}_1\boldsymbol{x}_1) > 0, \tag{24}$$

we obtain

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m) > 0.$$

Thus: *The Gramian of linearly independent vectors is positive; that of linearly dependent vectors is zero. Negative Gramians do not exist.*

Let us use the abbreviation $G_p = G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p)$ $(p=1, 2, \ldots, m)$. Then, from (23) and (24), we have

$$\sqrt{G_1} = |\boldsymbol{x}_1| = V_1,$$
$$\sqrt{G_2} = V_1 h_1 = V_2,$$

where V_2 is the area of the parallelogram spanned by $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Further,

$$\sqrt{G_3} = V_2 h_2 = V_3,$$

where V_3 is the volume of the parallelepiped spanned by $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$. Continuing further, we find:

$$\sqrt{G_4} = V_3 h_3 = V_4$$

and, in general,

$$\sqrt{G_m} = V_{m-1} h_{m-1} = V_m. \tag{25}$$

It is natural to call V_m the volume of the m-dimensional parallelepiped spanned by the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$.[14]

We denote by $x_{1k}, x_{2k}, \ldots, x_{nk}$ the coordinates of $\boldsymbol{x}_k$ $(k=1, 2, \ldots, m)$ in an orthonormal basis of $\boldsymbol{R}$ and set

$$X = \| x_{ik} \| \quad (i=1, 2, \ldots, n;\ k=1, 2, \ldots, m).$$

Then, in consequence of (10),

$$G_m = |X^{\mathsf{T}} \overline{X}|$$

and therefore (see formula (25)),

$$V_m^2 = G_m = \sum_{1 \leqq i_1 < i_2 < \cdots < i_m \leqq n} \bmod \begin{vmatrix} x_{i_1 1} & x_{i_1 2} & \ldots & x_{i_1 m} \\ x_{i_2 1} & x_{i_2 2} & \ldots & x_{i_2 m} \\ \ldots & \ldots & \ldots & \ldots \\ x_{i_m 1} & x_{i_m 2} & \ldots & x_{i_m m} \end{vmatrix}^2. \tag{26}$$

---

[14] Formula (25) gives an inductive definition of the volume of an m-dimensional parallelepiped.

This equation has the following geometric meaning:

The square of the volume of a parallelepiped is equal to the sum of the squares of the volumes of its projections on all the m-dimensional coordinate subspaces. In particular, for $m=n$, it follows from (26) that

$$V_n = \operatorname{mod} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}. \qquad (27)$$

The formulas (20), (21), (22), (26), and (27) solve a number of fundamental metrical problems of n-dimensional unitary and n-dimensional euclidean analytical geometry.

2. Let us return to the decomposition (15). This has the immediate consequence:

$$(\boldsymbol{x}\boldsymbol{x}) = (\boldsymbol{x}_S + \boldsymbol{x}_N,\ \boldsymbol{x}_S + \boldsymbol{x}_N) = (\boldsymbol{x}_S,\ \boldsymbol{x}_S) + (\boldsymbol{x}_N,\ \boldsymbol{x}_N) \geqq (\boldsymbol{x}_N \boldsymbol{x}_N) = h^2,$$

which, in conjunction with (22), gives an inequality (for arbitrary vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}$)

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m, \boldsymbol{x}) \leqq G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m) G(\boldsymbol{x}); \qquad (28)$$

the equality sign holds if and only if $\boldsymbol{x}$ is orthogonal to $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$.

From this we easily obtain the so-called *Hadamard inequality*

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m) \leqq G(\boldsymbol{x}_1) G(\boldsymbol{x}_2) \cdots G(\boldsymbol{x}_m), \qquad (29)$$

where the equality sign holds if and only if the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ are pairwise orthogonal. The inequality (29) expresses the following fact, which is geometrically obvious:

The volume of a parallelepiped does not exceed the product of the lengths of its edges and is equal to it only when the parallelepiped is rectangular.

Hadamard's inequality can be put into its usual form by setting $m=n$ in (29) and introducing the determinant Δ formed from the coordinates $x_{1k}, x_{2k}, \ldots, x_{nk}$ of the vectors $\boldsymbol{x}_k$ $(k=1, 2, \ldots, n)$ in some orthonormal basis:

$$\Delta = \begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \cdots & \cdots & \cdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}.$$

Then it follows from (27) and (29) that

$$|\Delta|^2 \leqq \sum_{i=1}^{n} |x_{i1}|^2 \sum_{i=1}^{n} |x_{i2}|^2 \cdots \sum_{i=1}^{n} |x_{in}|^2. \qquad (29')$$

3.[15] We now turn to the inequality

$$G(\boldsymbol{x}_{1S}, \boldsymbol{x}_{2S}, \ldots, \boldsymbol{x}_{mS}) \leqq G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m) \qquad (30)$$

If $G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m) \neq 0$, then the equality sign holds in (30) if and only if $\boldsymbol{x}_{iN} = 0$ $(i=1, 2, \ldots, m)$. If $G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m) = 0$, then (30) implies, of course, that $G(\boldsymbol{x}_{1S}, \boldsymbol{x}_{2S}, \ldots, \boldsymbol{x}_{mS}) = 0$.

In virtue of (25), the inequality (30) expresses the following geometric fact.

The volume of the orthogonal projection of a parallelepipep onto a subspace **S** *does not exceed the volume of the given parallelepiped; these volumes are equal if and only if the projecting parallelepiped lies in* **S** *or has zero volume.*

We prove (30) by induction on m.

The first step $(m=1)$ is trivial and yields the inequality

$$G(\boldsymbol{x}_{1S}) \leqq G(\boldsymbol{x}_1),$$

i.e., $|\boldsymbol{x}_{1S}| \leqq |\boldsymbol{x}_1|$ (see Fig. 5 on page 248).

We write the volume $\sqrt{G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)}$ of our parallelepiped as the product of the 'base' $\sqrt{G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{m-1})}$ by the distance h of the vertex of $\boldsymbol{x}_m$ from the base:

$$\sqrt{G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{m-1})} \cdot h = \sqrt{G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)}. \qquad (31)$$

If we now go over on the left-hand side of (31) from the vectors $\boldsymbol{x}_i$ to their projections $\boldsymbol{x}_{iS}$ $(i=1, 2, \ldots, m)$, then the first factor cannot increase, by the induction hypothesis, nor the second, by a simple geometric argument. But the product so obtained is the volume $\sqrt{G(\boldsymbol{x}_{1S}, \boldsymbol{x}_{2S}, \ldots, \boldsymbol{x}_{mS})}$ of the parallelepiped projected onto the subspace **S**. Hence

$$\sqrt{G(\boldsymbol{x}_{1S}, \boldsymbol{x}_{2S}, \ldots, \boldsymbol{x}_{mS})} \leqq \sqrt{G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m)}.$$

and by squaring both sides, we obtain (30).

Our condition for the equality sign to hold follows immediately from the proof.

---

[15] Subsections 3 and 4 have been modified in accordance with a correction published by the author in 1954 (Uspehi Mat. Nauk, vol. 9, no. 3).

4. Now we shall establish a generalization of Hadamard's inequality which comprises both the inequalities (28) and (29):

$$G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) \leqq G(\mathbf{x}_1, \ldots, \mathbf{x}_p)\, G(\mathbf{x}_{p+1}, \ldots, \mathbf{x}_m), \qquad (32)$$

where the equality sign holds if and only if each vector $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ is orthogonal to each of the vectors $\mathbf{x}_{p+1}, \ldots, \mathbf{x}_m$ or one of the determinants $G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p)$, $G(\mathbf{x}_{p+1}, \ldots, \mathbf{x}_m)$ vanishes.

The inequality (32) has the following geometric meaning:

The volume of a parallepiped does not exceed the product of the volumes of two complementary 'faces' and is equal to this product if and only if these faces are orthogonal or at least one of them has volume zero.

Let us prove the inequality (32). Let $p < m$. If $G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) = 0$, then (32) holds with the equality sign. Let $G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \neq 0$. Then the p vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ are linearly independent and form a basis of a p-dimensional subspace $\mathbf{T}$ of $\mathbf{R}$. The set of all vectors $\mathbf{y}$ of $\mathbf{R}$ that are orthogonal to $\mathbf{T}$ are easily seen also to form a subspace of $\mathbf{R}$ (the so-called orthogonal complement of $\mathbf{T}$; for details, see § 8 of this Chapter). We denote it by $\mathbf{S}$, and then $\mathbf{R} = \mathbf{T} + \mathbf{S}$.

Since every vector of $\mathbf{S}$ is orthogonal to every vector of $\mathbf{T}$, we can go over, in the Gramian $G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)$, whose square represents a certain volume, from the vectors $\mathbf{x}_{p+1}, \ldots, \mathbf{x}_m$ to their projections $\mathbf{x}_{p+1S}, \ldots, \mathbf{x}_{mS}$ onto the *subspace* $\mathbf{S}$:

$$G(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{x}_{p+1}, \ldots, \mathbf{x}_m) = G(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{x}_{p+1S}, \ldots, \mathbf{x}_{mS}).$$

The same arguments show that the Gramian on the right-hand side of this equation can be split:

$$G(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{x}_{p+1S}, \ldots, \mathbf{x}_{mS}) = G(\mathbf{x}_1, \ldots, \mathbf{x}_p)\, G(\mathbf{x}_{p+1S}, \ldots, \mathbf{x}_{mS}).$$

If we now go back from the projections to the original vectors and use (30), then we obtain

$$G(\mathbf{x}_1, \ldots, \mathbf{x}_p)\, G(\mathbf{x}_{p+1S}, \ldots, \mathbf{x}_{mS}) \leqq G(\mathbf{x}_1, \ldots, \mathbf{x}_p)\, G(\mathbf{x}_{p+1}, \ldots, \mathbf{x}_m).$$

The equality sign holds in two cases: 1. When $G(\mathbf{x}_{p+1}, \ldots, \mathbf{x}_m) = 0$, for then it is obvious that $G(\mathbf{x}_{p+1S}, \ldots, \mathbf{x}_{mS}) = 0$; and 2. When $\mathbf{x}_{iS} = \mathbf{x}_i$ $(i = 1, 2, 3, \ldots, m)$, i.e., when the vectors $\mathbf{x}_{p+1}, \ldots, \mathbf{x}_m$ belong to $\mathbf{S}$ or, what is the same, each vector $\mathbf{x}_{p+1}, \ldots, \mathbf{x}_m$ is orthogonal to every vector $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ (the case $G(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) = 0$ has been considered at the beginning of the proof). By combining the last three relations we obtain the generalized Hadamard inequality (32) and the conditions for the equality sign to hold. This completes the proof.

5. The generalized Hadamard inequality (32) can also be put into analytic form.

Let $\sum_{i,k=1}^{n} h_{ik}x_i\bar{x}_k$ be an arbitrary positive-definite hermitian form. By regarding $x_1, x_2, \ldots, x_n$ as the coordinates, in a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, of a vector $\mathbf{x}$ in an n-dimensional space $\mathbf{R}$, we take $\sum_{i,k=1}^{n} h_{ik}x_i\bar{x}_k$ as the fundamental metric form of $\mathbf{R}$ (see p. 244). Then $\mathbf{R}$ becomes a unitary space. We apply the generalized Hadamard inequality to the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$:

$$G(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) \leqq G(\mathbf{e}_1, \ldots, \mathbf{e}_p)\, G(\mathbf{e}_{p+1}, \ldots, \mathbf{e}_n).$$

Setting $H = \| h_{ik} \|_1^n$ and noting that $(\mathbf{e}_i\mathbf{e}_k) = h_{ik}$ $(i, k = 1, 2, \ldots, n)$, we can rewrite the latter inequality as follows:

$$H\begin{pmatrix}1 & 2 \ldots n\\ 1 & 2 \ldots n\end{pmatrix} \leqq H\begin{pmatrix}1 & 2 \ldots p\\ 1 & 2 \ldots p\end{pmatrix} H\begin{pmatrix}p+1 \ldots n\\ p+1 \ldots n\end{pmatrix} \qquad (p < n). \qquad (33)$$

Here the equality sign holds if and only if $h_{ik} = h_{ki} = 0$ $(i = 1, 2, \ldots, p;$ $k = p+1, \ldots, n)$.

The inequality (33) holds for the coefficient matrix $H = \| h_{ik} \|_1^n$ of an arbitrary positive-definite hermitian form. In particular, (33) holds if H is the real coefficient matrix of a positive-definite quadratic form $\sum_{i,k=1}^{n} h_{ik}\, x_i\, x_k$.[16]

6. We remind the reader of *Schwarz's inequality*:†

For arbitrary vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}$

$$|(\mathbf{x}\mathbf{y})|^2 \leqq \mathrm{N}\mathbf{x}\mathrm{N}\mathbf{y}, \qquad (34)$$

and the equality sign holds only if the vectors $\mathbf{x}$ and $\mathbf{y}$ differ only by a scalar factor

The validity of Schwarz's inequality follows easily from the inequality established above

$$G(\mathbf{x}, \mathbf{y}) = \begin{vmatrix}(\mathbf{x}\mathbf{x}) & (\mathbf{x}\mathbf{y})\\ (\mathbf{y}\mathbf{x}) & (\mathbf{y}\mathbf{y})\end{vmatrix} \geqq 0.$$

By analogy with the scalar product of vectors in a three-dimensional euclidean space, we can introduce in an n-dimensional unitary space the

---

[16] An analytical approach to the generalized Hadamard inequality can be found in the book [17], § 8.

† In the Russian literature, this is known as Bunyakovskii's inequality.

'angle' θ between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ by defining[17]

$$\cos^2\theta = \frac{|(\boldsymbol{x}\boldsymbol{y})|^2}{N\boldsymbol{x}N\boldsymbol{y}}.$$

From Schwarz's inequality it follows that θ is real.

§ 6. Orthogonalization of a Sequence of Vectors

1. The smallest subspace containing the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p$ will be denoted by $[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p]$. This subspace consists of all possible linear combinations $c_1\boldsymbol{x}_2 + c_2\boldsymbol{x}_2 + \cdots + c_p\boldsymbol{x}_p$ of the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p$($c_1, c_2, c_3, \ldots, c_p$ are complex numbers.)[18] If $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p$ are linearly independent, then they form a basis of $[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p]$. In that case, the subspace is p-dimensional.

Two sequences of vectors

$$\mathbf{X}\colon \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots,$$
$$\mathbf{Y}\colon \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots$$

containing an equal number of vectors, finite or infinite, will be called *equivalent* if for all p

$$[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p] \equiv [\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_p] \quad (p = 1, 2, \ldots).$$

A sequence of vectors

$$\mathbf{X}\colon \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$$

will be called *non-degenerate* if for every p the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p$ are linearly independent.

A sequence of vectors is called *orthogonal* if any two vectors of the sequence are orthogonal.

By *orthogonalization* of a sequence of vectors we mean a process of replacing the sequence by an equivalent orthogonal sequence.

THEOREM 2: *Every non-degenerate sequence of vectors can be orthogonalized. The orthogonalizing process leads to vectors that are uniquely determined to within scalar multiples.*

---

[17] In the case of a euclidean space, the angle θ between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by the formula

$$\cos\theta = \frac{(\boldsymbol{x}\boldsymbol{y})}{|\boldsymbol{x}||\boldsymbol{y}|}.$$

[18] In the case of a euclidean space, these numbers are real.

Proof. 1) Let us prove the second part of the theorem first. Suppose that two orthogonalizing sequences $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots (\mathbf{Y})$ and $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots (\mathbf{Z})$ are equivalent to one and the same non-degenerate sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots (\mathbf{X})$. Then $\mathbf{Y}$ and $\mathbf{Z}$ are equivalent to each other. Therefore for every p there exist numbers $c_{p1}, c_{p2}, \ldots, c_{pp}$ such that

$$\boldsymbol{z}_p = c_{p1}\boldsymbol{y}_1 + c_{p2}\boldsymbol{y}_2 + \cdots + c_{p,p-1}\boldsymbol{y}_{p-1} + c_{pp}\boldsymbol{y}_p \qquad (p = 1, 2, \ldots).$$

When we form the scalar products of both sides of this equation by $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{p-1}$ and take account of the orthogonality of $\mathbf{Y}$ and of the relation

$$\boldsymbol{z}_p \perp [\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_{p-1}] \equiv [\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{p-1}],$$

we obtain $c_{p1} = c_{p2} = \cdots = c_{p\,p-1} = 0$, and therefore

$$\boldsymbol{z}_p = c_{pp}\boldsymbol{y}_p \qquad (p = 1, 2, \ldots).$$

2) A concrete form of the orthogonalizing process for an arbitrary non-degenerate sequence of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots (\mathbf{X})$ is given by the following construction.

Let

$$\mathbf{S}_p \equiv [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p], \quad G_p = G(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p) \quad (p = 1, 2, \ldots).$$

We project the vector $\boldsymbol{x}_p$ orthogonally onto the subspace $\mathbf{S}_{p-1}$ $(p = 1, 2, \ldots)$:[19]

$$\boldsymbol{x}_p = \boldsymbol{x}_{pS_{p-1}} + \boldsymbol{x}_{pN}, \; \boldsymbol{x}_{pS_{p-1}} \in \mathbf{S}_{p-1}, \; \boldsymbol{x}_{pN} \perp \mathbf{S}_{p-1} \qquad (p = 1, 2, \ldots).$$

We set

$$\boldsymbol{y}_p = \lambda_p \boldsymbol{x}_{pN} \qquad (p = 1, 2, \ldots;\ \boldsymbol{x}_{1N} = \boldsymbol{x}_1),$$

where λ_p $(p = 1, 2, \ldots)$ are arbitrary non-zero numbers.

Then it is easily seen that

$$\mathbf{Y}\colon \boldsymbol{y}_1, \boldsymbol{y}_2, \ldots$$

is an orthogonal sequence equivalent to $\mathbf{X}$. This proves Theorem 2.

By (21)

$$\boldsymbol{x}_{pN} = \frac{\begin{vmatrix} & & & \boldsymbol{x}_1 \\ & G & & \vdots \\ & & & \boldsymbol{x}_{p-1} \\ (\boldsymbol{x}_p\boldsymbol{x}_1) & \ldots & (\boldsymbol{x}_p\boldsymbol{x}_{p-1}) & \boldsymbol{x}_p \end{vmatrix}}{G_{p-1}} \qquad (p = 1, 2, \ldots;\ G_0 = 1).$$

---

[19] For $p = 1$ we set $\boldsymbol{x}_{1\mathbf{S}_0} = \boldsymbol{o}$ and $\boldsymbol{x}_{1N} = \boldsymbol{x}_1$.

Setting $\lambda_p = G_{p-1}$ $(p = 1, 2, \ldots; G_0 = 1)$, we obtain the following formulas for the vectors of the orthogonalized sequence:

$$\boldsymbol{y}_1 = \boldsymbol{x}_1,\ \boldsymbol{y}_2 = \begin{vmatrix} (\boldsymbol{x}_1\boldsymbol{x}_1) & \boldsymbol{x}_1 \\ (\boldsymbol{x}_2\boldsymbol{x}_1) & \boldsymbol{x}_2 \end{vmatrix}, \ldots, \boldsymbol{y}_p = \begin{vmatrix} (\boldsymbol{x}_1\boldsymbol{x}_1) & \ldots (\boldsymbol{x}_1\boldsymbol{x}_{p-1}) & \boldsymbol{x}_1 \\ \cdots & \cdots & \cdots \\ (\boldsymbol{x}_{p-1}\boldsymbol{x}_1) & \ldots (\boldsymbol{x}_{p-1}\boldsymbol{x}_{p-1}) & \boldsymbol{x}_{p-1} \\ (\boldsymbol{x}_p\boldsymbol{x}_1) & \ldots (\boldsymbol{x}_p\boldsymbol{x}_{p-1}) & \boldsymbol{x}_p \end{vmatrix}, \ldots . \tag{35}$$

By (22),

$$\mathrm{N}\boldsymbol{y}_p = G_{p-1}^2 \mathrm{N}\boldsymbol{x}_{pN} = G_{p-1}^2 \cdot \frac{G_p}{G_{p-1}} = G_{p-1}G_p \qquad (p = 1, 2, \ldots;\ G_0 = 1). \tag{36}$$

Therefore, setting

$$\boldsymbol{z}_p = \frac{\boldsymbol{y}_p}{\sqrt{G_{p-1}G_p}} \qquad (p = 1, 2, \ldots), \tag{37}$$

we obtain an orthogonal sequence $\boldsymbol{Z}$ equivalent to the given sequence $\boldsymbol{X}$.

Example. In the space of real functions that are sectionally continuous in the interval $[-1, +1]$, we define the scalar product

$$(f, g) = \int_{-1}^{+1} f(x)g(x)dx.$$

We consider the non-degenerate sequence of 'vectors'

$$1, x, x^2, x^3, \ldots .$$

We orthogonalize this sequence by the formulas (35):

$$y_0 \equiv 1, \quad y_m = \begin{vmatrix} \frac{1}{1} & 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 \ldots 1 \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} \ldots x \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} & 0 \ldots x^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots x^m \end{vmatrix} \qquad (m = 1, 2, \ldots).$$

These orthogonal polynomials coincide, apart from constant factors, with the well-known Legendre polynomials:[20]

$$P_0(x) = 1, P_m(x) = \frac{1}{2^m m!} \frac{d^m(x^2-1)^m}{dx^m} \qquad (m = 1, 2, \ldots).$$

The same sequence of powers $1, x, x^2, \ldots$ in a different metric

---

[20] See [12], p. 77ff.

$$(f, g) = \int_a^b f(x)g(x)\tau(x)dx$$

(where $\tau(x) \geqq 0$ for $a \leqq x \leqq b$) gives another sequence of orthogonal polynomials.

For example, if $a = -1$, $b = 1$ and $\tau(x) = \frac{1}{\sqrt{1-x^2}}$, then we obtain the Tchebyshev (Chebyshev) polynomials:

$$T_n(x) = \frac{1}{2^{n-1}} \cos(n \arccos x).$$

For $a = -\infty$, $b = +\infty$ and $\tau(x) = e^{-x^2}$ we obtain the hermitian polynomials, etc.[21]

2. We shall now take note of the so-called Bessel inequality for an orthonormal sequence of vectors $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots$ $(\boldsymbol{Z})$. Let $\boldsymbol{x}$ be an arbitrary vector. We denote by ξ_p the projection of $\boldsymbol{x}$ onto $\boldsymbol{z}_p$:

$$\xi_p = (\boldsymbol{x}\boldsymbol{z}_p) \qquad (p = 1, 2, \ldots).$$

Then the projection of $\boldsymbol{x}$ onto the subspace $\boldsymbol{S}_p = [\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_p]$ can be represented in the form (see (20))

$$\boldsymbol{x}_{S_p} = \xi_1\boldsymbol{z}_1 + \xi_2\boldsymbol{z}_2 + \cdots + \xi_p\boldsymbol{z}_p \qquad (p = 1, 2, \ldots).$$

But $\mathrm{N}\boldsymbol{x}_{S_p} = |\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_p|^2 \leq \mathrm{N}\boldsymbol{x}$. Therefore, for every p,

$$|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_p|^2 \leqq \mathrm{N}\boldsymbol{x}. \tag{38}$$

This is *Bessel's inequality*.

In the case of a space of finite dimension n, this inequality has a completely obvious geometrical meaning. For $p = n$ it goes over into the theorem of Pythagoras

$$|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2 = |\boldsymbol{x}|^2.$$

In the case of an infinite-dimensional space and an infinite sequence $\boldsymbol{Z}$, it follows from (38) that the series $\sum_{k=1}^{\infty} |\xi_k|^2$ converges and that

$$\sum_{k=1}^{\infty} |\xi_k|^2 \leq \mathrm{N}\boldsymbol{x} = |\boldsymbol{x}|^2.$$

Let us form the series

---

[21] For further details see [12], Chapter II, § 9.

$$\sum_{k=1}^{\infty} \xi_k \boldsymbol{z}_k .$$

For every p the p-th partial sum of this series,

$$\xi_1 \boldsymbol{z}_1 + \xi_2 \boldsymbol{z}_2 + \cdots + \xi_p \boldsymbol{z}_p ,$$

is the projection $\boldsymbol{x}_{S_p}$ of $\boldsymbol{x}$ onto the subspace

$$S_p = [\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_v]$$

and is therefore the best approximation to the vector $\boldsymbol{x}$ in this subspace:

$$\mathrm{N}\,(\boldsymbol{x} - \sum_{k=1}^{p} \xi_k \boldsymbol{z}_k) \leqq \mathrm{N}\,(\boldsymbol{x} - \sum_{k=1}^{p} c_k \boldsymbol{z}_k) .$$

where $c_1, c_2, \ldots, c_p$ are arbitrary complex numbers. Let us calculate the corresponding mean-square-deviation δ_p:

$$\delta_p^2 = \mathrm{N}\,(\boldsymbol{x} - \sum_{k=1}^{p} \xi_k \boldsymbol{z}_k) = (\boldsymbol{x} - \sum_{k=1}^{p} \xi_k \boldsymbol{z}_k ,\ \ \boldsymbol{x} - \sum_{k=1}^{p} \xi_k \boldsymbol{z}_k) = \mathrm{N}\boldsymbol{x} - \sum_{k=1}^{p} |\, \xi_k |^2 .$$

Hence

$$\lim_{p \to \infty} \delta_p^2 = \mathrm{N}\boldsymbol{x} - \sum_{k=1}^{\infty} |\, \xi_k |^2 .$$

If

$$\lim_{p \to \infty} \delta_p = 0 ,$$

then we say that the series $\sum_{k=1}^{\infty} \xi_k \boldsymbol{z}_k$ *converges in the mean* (or converges with respect to the norm) to the vector $\boldsymbol{x}$.

In this case we have an equality for the vector $\boldsymbol{x}$ in $\boldsymbol{R}$ (the theorem of Pythagoras in an infinite-dimensional space!):

$$\mathrm{N}\boldsymbol{x} = |\, \boldsymbol{x} |^2 = \sum_{k=1}^{\infty} |\, \xi_k |^2 . \tag{39}$$

If for every vector $\boldsymbol{x}$ of $\boldsymbol{R}$ the series $\sum_{k=1}^{\infty} \xi_k \boldsymbol{z}_k$ converges in the mean to $\boldsymbol{x}$, then the orthonormal sequence of vectors $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots$ is called *complete*. In this case, when we replace $\boldsymbol{x}$ in (39) by $\boldsymbol{x} + \boldsymbol{y}$ and use (39) three times, for $\mathrm{N}(\boldsymbol{x} + \boldsymbol{y})$, $\mathrm{N}\boldsymbol{x}$, and $\mathrm{N}\boldsymbol{y}$, then we easily obtain:

$$(\boldsymbol{x}\boldsymbol{y}) = \sum_{k=1}^{\infty} \xi_k \bar{\eta}_k \quad [\xi_k = (\boldsymbol{x}\boldsymbol{z}_k),\ \eta_k = (\boldsymbol{y}\boldsymbol{z}_k);\ k = 1, 2, \ldots] . \tag{40}$$

Example. We consider the space of all complex functions $f(t)$ (t is a real variable) that are sectionally continuous in the closed interval $[0, 2\pi]$. Let us define the norm of $f(t)$ by

$$\mathrm{N}f = \int_0^{2\pi} |\, f(t) |^2 \, dt .$$

Correspondingly, we have the formula

$$(f, g) = \int_0^{2\pi} f(t) \overline{g(t)} \, dt$$

for the scalar product of two functions $f(t)$ and $g(t)$.

We take the infinite sequence of functions

$$\frac{1}{\sqrt{2\pi}} e^{ikt} \qquad (k = 0, \pm 1, \pm 2, \ldots) .$$

These functions form an orthogonal sequence, because

$$\int_0^{2\pi} e^{i\mu t} e^{-i\nu t} \, dt = \int_0^{2\pi} e^{i(\mu - \nu)t} \, dt = \begin{cases} 0, & \text{for } \mu \neq \nu , \\ 2\pi, & \text{for } \mu = \nu . \end{cases}$$

The series

$$\sum_{k=-\infty}^{\infty} f_k e^{ikt} \qquad \left(f_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) \, e^{-ikt} \, dt ; \quad (k = 0, \pm 1, \pm 2, \ldots \right)$$

converges in the mean to $f(t)$ in the interval $[0, 2\pi]$. This series is called the *Fourier series* of $f(t)$ and the coefficients f_k ($k = 0, \pm 1, \pm 2, \ldots$) are called the *Fourier coefficients* of $f(t)$.

In the theory of Fourier series it is proved that the system of functions e^{ikt} ($k = 0, \pm 1, \pm 2, \ldots$) is complete.[22]

The condition of completeness gives *Parseval's equality* (see (40))

$$\int_0^{2\pi} f(t) \overline{g(t)} \, dt = \sum_{k=-\infty}^{+\infty} \frac{1}{2\pi} \int_0^{2\pi} f(t) \, e^{-ikt} \, dt \int_0^{2\pi} g(t) \, e^{ikt} \, dt .$$

If $f(t)$ is a real function, then f_0 is real, and f_k and f_{-k} are conjugate complex numbers. Setting

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) \, e^{-ikt} \, dt = \frac{1}{2} (a_k + i b_k) ,$$

---

[22] See, for example, [12], Chapter II.

where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,dt, \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,dt \quad (k=0,1,2,\ldots).$$

we have

$$f_k e^{ikt} + f_{-k}e^{-ikt} = a_k\cos kt + b_k\sin kt \quad (k=1,2,\ldots).$$

Therefore, for a real function $f(t)$ the Fourier series assumes the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos kt + b_k\sin kt) \quad \left(\begin{array}{ll} a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,dt, & \\ & k=0,1,2,\ldots \\ b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,dt, & \end{array}\right).$$

§ 7. Orthonormal Bases

1. A basis of any finite-dimensional subspace $\boldsymbol{S}$ in a unitary or a euclidean space $\boldsymbol{R}$ is a non-degenerate sequence of vectors and therefore—by Theorem 2 of the preceding section—can be orthogonalized and normalized. Thus: *Every finite-dimensional subspace* $\boldsymbol{S}$ (*and, in particular, the whole space* $\boldsymbol{R}$ *if it is finite-dimensional*) *has an orthonormal basis.*

Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ be an orthonormal basis of $\boldsymbol{R}$. We denote by $x_1, x_2, x_3, \ldots, x_n$ the coordinates of an arbitrary vector $\boldsymbol{x}$ in this basis:

$$\boldsymbol{x} = \sum_{k=1}^{n} x_k\boldsymbol{e}_k.$$

Multiplying both sides of this equation on the right by $\boldsymbol{e}_k$ and taking into account that the basis is orthonormal, we easily find:

$$x_k = (\boldsymbol{x}\boldsymbol{e}_k) \qquad (k=1,2,\ldots,n);$$

i.e., *in an orthonormal basis the coordinates of a vector are equal to its projections onto the corresponding basis vectors*:

$$\boldsymbol{x} = \sum_{k=1}^{n}(\boldsymbol{x}\boldsymbol{e}_k)\,\boldsymbol{e}_k. \tag{41}$$

Let $x_1, x_2, \ldots, x_n$ and $x'_1, x'_2, \ldots, x'_n$ be the coordinates of one and the same vector $\boldsymbol{x}$ in two different orthonormal bases $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ and $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \ldots, \boldsymbol{e}'_n$ of a unitary space $\boldsymbol{R}$. The formulas for the coordinate transformation have the form

$$x_i = \sum_{k=1}^{n} u_{ik}x'_k \qquad (i=1,2,\ldots,n). \tag{42}$$

Here the coefficients $u_{1k}, u_{2k}, \ldots, u_{nk}$ that form the k-th column of the matrix $U = \| u_{ik} \|_1^n$ are easily seen to be the coordinates of the vector $\boldsymbol{e}'_k$ in the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Therefore, when we write down the condition for the basis $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \ldots, \boldsymbol{e}'_n$ to be orthonormal in terms of coordinates (see (10)), we obtain the relations

$$\sum_{i=1}^{n} u_{ik}\bar{u}_{il} = \delta_{kl} = \begin{cases} 1, & \text{for } k=l, \\ 0, & \text{for } k\neq l. \end{cases} \tag{43}$$

A transformation (42) in which the coefficients satisfy the conditions (43) is called *unitary* and the corresponding matrix U is called a *unitary matrix*. Thus: *In an n-dimensional unitary space the transition from one orthonormal basis to another is effected by a unitary coordinate transformation.*

Let $\boldsymbol{R}$ be an n-dimensional euclidean space. The transition from one orthonormal basis of $\boldsymbol{R}$ to another is effected by a coordinate transformation

$$x_i = \sum_{k=1}^{n} v_{ik}x'_k \qquad (i=1,2,\ldots,n) \tag{44}$$

whose coefficients are connected by the relation

$$\sum_{i=1}^{n} v_{ik}v_{il} = \delta_{kl} \quad (k, l=1,2,\ldots,n). \tag{45}$$

Such a coordinate transformation is called *orthogonal* and the corresponding matrix V is called an *orthogonal matrix*.

2. We note an interesting matrix method of writing the orthogonalizing process. Let $A = \| a_{ik} \|_1^n$ be an arbitrary non-singular matrix ($|A| \neq 0$) with complex elements. We consider a unitary space $\boldsymbol{R}$ with an orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ and define the linearly independent vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$ by the equations

$$\boldsymbol{a}_k = \sum_{i=1}^{n} a_{ik}\boldsymbol{e}_i \qquad (k=1,2,\ldots,n).$$

Let us perform the orthogonalizing process on the vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$. The orthonormal basis of $\boldsymbol{R}$ so obtained we shall denote by $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. Suppose we have

$$\boldsymbol{u}_i = \sum_{i=1}^{n} u_{ik}\boldsymbol{e}_i \qquad (k=1,2,\ldots,n).$$

Then

$$[\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p] = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p] \quad (p=1, 2, \dots, n),$$

i.e.,

$$\begin{aligned} \mathbf{a}_1 &= c_{11}\mathbf{u}_1, \\ \mathbf{a}_2 &= c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2, \\ &\dots\dots\dots\dots\dots \\ \mathbf{a}_n &= c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots + c_{nn}\mathbf{u}_n, \end{aligned}$$

where the c_{ik} $(i, k = 1, 2, \dots, n;\ i \leqq k)$ are certain complex numbers.

Setting $c_{ik} = 0$ for $i > k$, we have:

$$\mathbf{a}_k = \sum_{p=1}^{n} c_{pk}\mathbf{u}_p \qquad (k = 1, 2, \dots, n).$$

When we go over to coordinates and introduce the upper triangular matrix $C = \| c_{ik} \|_1^n$ and the unitary matrix $U = \| u_{ik} \|_1^n$, we obtain

$$a_{ik} = \sum_{p=1}^{n} u_{ip}c_{pk} \qquad (i, k = 1, 2, \dots, n),$$

or

$$A = UC. \tag{*}$$

According to this formula: *Every non-singular matrix $A = \| a_{ik} \|_1^n$ can be represented in the form of a product of a unitary matrix U and an upper triangular matrix C.*

Since the orthogonalizing process determines the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ uniquely, apart from scalar multipliers $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ $(|\varepsilon_i| = 1;\ i = 1, 2, \dots, n)$, the factors U and C in (*) are uniquely determined apart from a diagonal factor $M = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$:

$$U = U_1 M_1 \qquad C = M^{-1}C_1.$$

This can also be shown directly.

Note 1. If A is a real matrix, the factors U and C in (*) can be chosen to be real. In this case, U is an orthogonal matrix.

Note 2. The formula (*) also remains valid for a singular matrix A $(|A| = 0)$. This can be seen by setting $A = \lim_{m\to\infty} A_m$, where $|A_m| \neq 0$ $(m = 1, 2, \dots)$.

Then $A_m = U_m C_m$ $(m = 1, 2, \dots)$. When we select from the sequence $\{U_m\}$ a convergent subsequence $\{U_{m_p}\}$ $(\lim_{p\to\infty} U_{m_p} = U)$ and proceed to the limit, then we obtain from the equation $A_{m_p} = U_{m_p} C_{m_p}$ for $p \to \infty$ the required decomposition $A = UC$. However, in the case $|A| = 0$ the factors U and C are no longer uniquely determined to within a diagonal factor M.

Note 3. Instead of (*) we can also obtain a formula

$$A = DW, \tag{**}$$

where D is a lower triangular matrix and W a unitary matrix. For when we apply the formula (*) that was established above to the transposed matrix A^{T}

$$A^{\mathsf{T}} = UC$$

and then set $W = U^{\mathsf{T}}$, $D = C^{\mathsf{T}}$, we obtain (**).[23]

§ 8. The Adjoint Operator

1. Let $\mathbf{A}$ be a linear operator in an n-dimensional unitary space.

DEFINITION 4: *A linear operator $\mathbf{A}^*$ is called adjoint to the operator $\mathbf{A}$ if and only if for any two vectors $\mathbf{x}, \mathbf{y}$ of $\mathbf{R}$*

$$(\mathbf{Ax}, \mathbf{y}) = (\mathbf{x}, \mathbf{A}^*\mathbf{y}). \tag{46}$$

We shall show that for every linear operator $\mathbf{A}$ there exists one and only one adjoint operator $\mathbf{A}^*$. To prove this, we take an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ in $\mathbf{R}$. Then (see (41)) the required operator $\mathbf{A}^*$ and an arbitrary vector $\mathbf{y}$ of $\mathbf{R}$ must satisfy the equation

$$\mathbf{A}^*\mathbf{y} = \sum_{k=1}^{n} (\mathbf{A}^*\mathbf{y}, \mathbf{e}_k)\,\mathbf{e}_k.$$

By (46) this can be rewritten as follows:

$$\mathbf{A}^*\mathbf{y} = \sum_{k=1}^{n} (\mathbf{y}, \mathbf{Ae}_k)\,\mathbf{e}_k. \tag{47}$$

We now take (47) as the definition of an operator $\mathbf{A}^*$.

It is easy to verify that the operator $\mathbf{A}^*$ so defined is linear and satisfies (46) for arbitrary vectors $\mathbf{x}$ and $\mathbf{y}$ of $\mathbf{R}$. Moreover, (47) determines the operator $\mathbf{A}^*$ *uniquely*. Thus the existence and uniqueness of the adjoint operator $\mathbf{A}^*$ is established.

Let $\mathbf{A}$ be a linear operator in a unitary space and let $A = \| a_{ik} \|_1^n$ be the corresponding matrix in an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Then, by applying the formula (41) to the vector $\mathbf{Ae}_k = \sum_{i=1}^{n} a_{ik}\,\mathbf{e}_i$, we obtain

$$a_{ik} = (\mathbf{Ae}_k, \mathbf{e}_i) \quad (i, k = 1, 2, \dots, n). \tag{48}$$

---

[23] From the fact that U is unitary it follows that U^{T} is unitary, since the condition (43), written in matrix form $U^{\mathsf{T}}\overline{U} = E$, implies that $U\overline{U}^{\mathsf{T}} = E$.

Now let $A^* = \| a^*_{ik} \|_1^n$ be the matrix corresponding to $\boldsymbol{A}^*$ in the same basis. Then, by (48),

$$a^*_{ik} = (\boldsymbol{A}^*\boldsymbol{e}_k, \boldsymbol{e}_i) \quad (i, k = 1, 2, \ldots, n). \tag{49}$$

From (48) and (49) it follows by (46) that

$$a^*_{ik} = \bar{a}_{ki} \quad (i, k = 1, 2, \ldots, n),$$

i.e.,

$$A^* = \bar{A}^{\mathsf{T}}.$$

The matrix A^* is the complex conjugate of the transpose of A. This matrix will be called the *adjoint* of A. (This is not to be confused with the adjoint of a matrix as defined on p. 82.)

Thus: *In an orthonormal basis adjoint matrices correspond to adjoint operators.*

The following properties of the adjoint operator follow from its definition:

1. $(\boldsymbol{A}^*)^* = \boldsymbol{A}$,
2. $(\boldsymbol{A} + \boldsymbol{B})^* = \boldsymbol{A}^* + \boldsymbol{B}^*$,
3. $(\alpha \boldsymbol{A})^* = \bar{\alpha} \boldsymbol{A}^*$ (α a scalar),
4. $(\boldsymbol{AB})^* = \boldsymbol{B}^*\boldsymbol{A}^*$.

2. We shall now introduce an important concept. Let $\boldsymbol{S}$ be an arbitrary subspace of $\boldsymbol{R}$. We denote by $\boldsymbol{T}$ the set of all vectors $\boldsymbol{y}$ of $\boldsymbol{R}$ that are orthogonal to $\boldsymbol{S}$. It is easy to see that $\boldsymbol{T}$ is a subspace of $\boldsymbol{R}$ and that every vector $\boldsymbol{x}$ of $\boldsymbol{R}$ can be represented uniquely in the form of a sum $\boldsymbol{x} = \boldsymbol{x}_S + \boldsymbol{x}_T$, where $\boldsymbol{x}_S \in \boldsymbol{S}$, $\boldsymbol{x}_T \in \boldsymbol{T}$, so that we have the resolution

$$\boldsymbol{R} = \boldsymbol{S} + \boldsymbol{T}, \quad \boldsymbol{S} \perp \boldsymbol{T}.$$

We obtain this resolution by applying the decomposition (15) to the arbitrary vector $\boldsymbol{x}$ of $\boldsymbol{R}$. $\boldsymbol{T}$ is called the *orthogonal complement* of $\boldsymbol{S}$. Obviously, $\boldsymbol{S}$ is the orthogonal complement of $\boldsymbol{T}$. We write $\boldsymbol{S} \perp \boldsymbol{T}$, meaning by this that each vector of $\boldsymbol{S}$ is orthogonal to every vector of $\boldsymbol{T}$.

Now we can formulate the fundamental property of the adjoint operator:

5. *If a subspace $\boldsymbol{S}$ is invariant with respect to $\boldsymbol{A}$, then the orthogonal complement $\boldsymbol{T}$ of the subspace is invariant with respect to $\boldsymbol{A}^*$.*

For let $\boldsymbol{x} \in \boldsymbol{S}$, $\boldsymbol{y} \in \boldsymbol{T}$. Then it follows from $\boldsymbol{Ax} \in \boldsymbol{S}$ that $(\boldsymbol{Ax}, \boldsymbol{y}) = 0$ and hence by (46) that $(\boldsymbol{x}, \boldsymbol{A}^*\boldsymbol{y}) = 0$. Since $\boldsymbol{x}$ is an arbitrary vector of $\boldsymbol{S}$, $\boldsymbol{A}^*\boldsymbol{y} \in \boldsymbol{T}$, and this is what we had to prove.

We introduce the following definition:

DEFINITION 5: *Two systems of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m$ and $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_m$ are called bi-orthogonal if*

$$(\boldsymbol{x}_i \boldsymbol{y}_k) = \delta_{ik} \quad (i, k = 1, 2, \ldots, m), \tag{50}$$

where δ_{ik} is the Kronecker symbol.

Now we shall prove the following proposition:

6. *If $\boldsymbol{A}$ is a linear operator of simple structure, then the adjoint operator $\boldsymbol{A}^*$ is also of simple structure, and complete systems of characteristic vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ and $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n$ of $\boldsymbol{A}$ and $\boldsymbol{A}^*$ can be chosen such that they are bi-orthogonal:*

$$\boldsymbol{A}\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i, \quad \boldsymbol{A}^*\boldsymbol{y}_i = \mu_i \boldsymbol{y}_i, \quad (\boldsymbol{x}_i \boldsymbol{y}_k) = \delta_{ik} \quad (i, k = 1, 2, \ldots, n).$$

For let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be a complete system of characteristic vectors of $\boldsymbol{A}$. We use the notation

$$\boldsymbol{S}_k = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{k-1}, \boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_n] \quad (k = 1, 2, \ldots, n).$$

Consider the one-dimensional orthogonal complement $\boldsymbol{T}_k = [\boldsymbol{y}_k]$ to the $(n-1)$-dimensional subspace $\boldsymbol{S}_k$ $(k = 1, 2, \ldots, n)$. Then $\boldsymbol{T}_k$ is invariant with respect to $\boldsymbol{A}^*$:

$$\boldsymbol{A}^*\boldsymbol{y}_k = \mu_k \boldsymbol{y}_k, \quad \boldsymbol{y}_k \neq \boldsymbol{o} \quad (k = 1, 2, \ldots, n).$$

From $\boldsymbol{S}_k \perp \boldsymbol{y}_k$ it follows that $(\boldsymbol{x}_k \boldsymbol{y}_k) \neq 0$, because otherwise the vector $\boldsymbol{y}_k$ would have to be the null vector. Multiplying $\boldsymbol{x}_k, \boldsymbol{y}_k$ $(k = 1, 2, \ldots, n)$ by suitable numerical factors we obtain

$$(\boldsymbol{x}_i \boldsymbol{y}_k) = \delta_{ik} \quad (i, k = 1, 2, \ldots, n).$$

From the bi-orthogonality of the systems $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ and $\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n$ it follows that the vectors of each system are linearly independent.

We mention one further proposition:

7. *If the operators $\boldsymbol{A}$ and $\boldsymbol{A}^*$ have a common characteristic vector, then the corresponding characteristic values are complex conjugates.*

For let $\boldsymbol{Ax} = \lambda \boldsymbol{x}$ and $\boldsymbol{A}^*\boldsymbol{x} = \mu \boldsymbol{x}$ $(\boldsymbol{x} \neq \boldsymbol{o})$. Then, setting $\boldsymbol{y} = \boldsymbol{x}$ in (46), we have $\lambda(\boldsymbol{x}, \boldsymbol{x}) = \bar{\mu}(\boldsymbol{x}, \boldsymbol{x})$ and hence $\lambda = \bar{\mu}$.

§ 9. Normal Operators in a Unitary Space

1. DEFINITION 6. *A linear operator $\boldsymbol{A}$ is called normal if it commutes with its adjoint*:

$$\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A}. \tag{51}$$

DEFINITION 7. *A linear operator $\boldsymbol{H}$ is called hermitian if it is equal to its adjoint*:

$$\boldsymbol{H}^* = \boldsymbol{H}. \tag{52}$$

DEFINITION 8. *A linear operator $\boldsymbol{U}$ is called unitary if it is inverse to its adjoint*:

$$\boldsymbol{U}\boldsymbol{U}^* = \boldsymbol{E} \tag{53}$$

Note that a unitary operator can be regarded as an isometric operator in a hermitian space, i.e., as an operator preserving the metric.

For suppose that for arbitrary vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of $\boldsymbol{R}$

$$(\boldsymbol{U}\boldsymbol{x}, \boldsymbol{U}\boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y}). \tag{54}$$

Then by (46)

$$(\boldsymbol{U}^*\boldsymbol{U}\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y})$$

and therefore, since $\boldsymbol{y}$ is arbitrary,

$$\boldsymbol{U}^*\boldsymbol{U}\boldsymbol{x} = \boldsymbol{x},$$

i.e., $\boldsymbol{U}^*\boldsymbol{U} = \boldsymbol{E}$, or $\boldsymbol{U}^* = \boldsymbol{U}^{-1}$. Conversely, (53) implies (54).

From (53) and (54) it follows that 1. the product of two unitary operators is itself a unitary operator, 2. the unit operator $\boldsymbol{E}$ is unitary, and 3. the inverse of a unitary operator is also unitary. Therefore the set of all unitary operators is a group.[24] This is called the *unitary group*.

Hermitian operators and unitary operators are special cases of a normal operator.

2. We have

THEOREM 3: *Every linear operator $\boldsymbol{A}$ can be represented in the form*

$$\boldsymbol{A} = \boldsymbol{H}_1 + i\boldsymbol{H}_2, \tag{55}$$

where $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are hermitian operators (the 'hermitian components' of $\boldsymbol{A}$). *The hermitian components are uniquely determined by $\boldsymbol{A}$. The operator $\boldsymbol{A}$ is normal if and only if its hermitian components $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are permutable.*

---

[24] See footnote 13 on p. 18.

Proof. Suppose that (55) holds. Then

$$\boldsymbol{A}^* = \boldsymbol{H}_1 - i\boldsymbol{H}_2. \tag{56}$$

From (55) and (56) we have:

$$\boldsymbol{H}_1 = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^*), \quad \boldsymbol{H}_2 = \frac{1}{2i}(\boldsymbol{A} - \boldsymbol{A}^*). \tag{57}$$

Conversely, the formulas (57) define hermitian operators $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ connected with $\boldsymbol{A}$ by (55).

Now let $\boldsymbol{A}$ be a normal operator: $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A}$. Then it follows from (57) that $\boldsymbol{H}_1\boldsymbol{H}_2 = \boldsymbol{H}_2\boldsymbol{H}_1$. Conversely, from $\boldsymbol{H}_1\boldsymbol{H}_2 = \boldsymbol{H}_2\boldsymbol{H}_1$ it follows by (55) and (56) that $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A}$. This completes the proof.

The representation of an arbitrary linear operator $\boldsymbol{A}$ in the form (55) is an analogue to the representation of a complex number z in the form $x_1 + ix_2$, where x_1 and x_2 are real.

Suppose that in some orthonormal basis the operators $\boldsymbol{A}$, $\boldsymbol{H}$, and $\boldsymbol{U}$ correspond to the matrices A, H, and U. Then the operator equations

$$\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A}, \quad \boldsymbol{H}^* = \boldsymbol{H}, \quad \boldsymbol{U}\boldsymbol{U}^* = \boldsymbol{E} \tag{58}$$

correspond to the matrix equations

$$AA^* = A^*A, \quad H^* = H, \quad UU^* = E. \tag{59}$$

Therefore we define a matrix as *normal* if it commutes with its adjoint, as *hermitian* if it is equal to its adjoint, and finally as *unitary* if it is inverse to its adjoint.

Then: *In an orthonormal basis a normal* (*hermitian, unitary*) *operator corresponds to a normal* (*hermitian, unitary*) *matrix.*

A hermitian matrix $H = \| h_{ik} \|_1^n$ is, by (59), characterized by the following relation among its elements:

$$h_{ki} = \bar{h}_{ik} \quad (i, k = 1, 2, \ldots, n);$$

i.e., a hermitian matrix is always the coefficient matrix of some hermitian form (see § 1).

A unitary matrix $U = \| u_{ik} \|_1^n$ is, by (59), characterized by the following relations among its elements:

$$\sum_{j=1}^{n} u_{ij}\bar{u}_{kj} = \delta_{ik} \quad (i, k = 1, 2, \ldots, n). \tag{60}$$

Since $UU^* = E$ implies that $U^*U = E$, from (60) there follow the equivalent relations:

$$\sum_{j=1}^{n} u_{ji}\bar{u}_{jk} = \delta_{ik} \quad (i, k = 1, 2, \ldots, n). \tag{61}$$

Equation (60) expresses the 'orthonormality' of the rows and equation (61) that of the columns of the matrix $U = \| u_{ik} \|_1^n$.[25]

A unitary matrix is the coefficient matrix of some unitary transformation (see § 7).

§ 10. The Spectra of Normal, Hermitian, and Unitary Operators

1. As a preliminary, we establish a property of permutable operators in the form of a lemma.

LEMMA 1: *Permutable operators* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ $(\boldsymbol{AB} = \boldsymbol{BA})$ *always have a common characteristic vector.*

Proof. Let $\boldsymbol{x}$ be a characteristic vector of $\boldsymbol{A}$: $\boldsymbol{Ax} = \lambda\boldsymbol{x}$, $\boldsymbol{x} \neq \boldsymbol{o}$. Then, since $\boldsymbol{A}$ and $\boldsymbol{B}$ are permutable,

$$\boldsymbol{AB}^k\boldsymbol{x} = \lambda\boldsymbol{B}^k\boldsymbol{x} \quad (k = 0, 1, 2, \ldots). \tag{62}$$

Suppose that in the sequence of vectors

$$\boldsymbol{x}, \boldsymbol{Bx}, \boldsymbol{B}^2\boldsymbol{x}, \ldots$$

the first p are linearly independent, while the $(p+1)$-th vector $\boldsymbol{B}^p\boldsymbol{x}$ is a linear combination of the preceding ones. Then $\boldsymbol{S} \equiv [\boldsymbol{x}, \boldsymbol{Bx}, \ldots, \boldsymbol{B}^{p-1}\boldsymbol{x}]$ is a subspace invariant with respect to $\boldsymbol{B}$, so that in this subspace $\boldsymbol{S}$ there exists a characteristic vector $\boldsymbol{y}$ of $\boldsymbol{B}$: $\boldsymbol{By} = \mu\boldsymbol{y}$, $\boldsymbol{y} \neq \boldsymbol{o}$. On the other hand, (62) shows that the vectors $\boldsymbol{x}, \boldsymbol{Bx}, \ldots, \boldsymbol{B}^{p-1}\boldsymbol{x}$ are characteristic vectors of $\boldsymbol{A}$ corresponding to one and the same characteristic value λ. Therefore every linear combination of these vectors, and in particular $\boldsymbol{y}$, is a characteristic vector of $\boldsymbol{A}$ corresponding to λ. Thus we have proved the existence of a common characteristic vector of the operators $\boldsymbol{A}$ and $\boldsymbol{B}$.

Let $\boldsymbol{A}$ be an arbitrary normal operator in an n-dimensional hermitian space $\boldsymbol{R}$. In that case $\boldsymbol{A}$ and $\boldsymbol{A}^*$ are permutable and therefore have a common characteristic vector $\boldsymbol{x}_1$. Then (see § 8, 7.)

---

[25] Thus, orthonormality of the columns of the matrix U is a consequence of the orthonormality of the rows, and vice versa.

$$\boldsymbol{Ax}_1 = \lambda_1\boldsymbol{x}_1, \quad \boldsymbol{A}^*\boldsymbol{x}_1 = \bar{\lambda}_1\boldsymbol{x}_1 \qquad (\boldsymbol{x}_1 \neq \boldsymbol{o}).$$

We denote by $\boldsymbol{S}_1$ the one-dimensional subspace containing the vector $\boldsymbol{x}_1$ $(\boldsymbol{S}_1 = [\boldsymbol{x}_1])$ and by $\boldsymbol{T}_1$ the orthogonal complement of $\boldsymbol{S}_1$ in $\boldsymbol{R}$:

$$\boldsymbol{R} = \boldsymbol{S}_1 + \boldsymbol{T}_1, \quad \boldsymbol{S}_1 \perp \boldsymbol{T}_1.$$

Since $\boldsymbol{S}_1$ is invariant with respect to $\boldsymbol{A}$ and $\boldsymbol{A}^*$, $\boldsymbol{T}_1$ is also invariant with respect to these operators (see § 8, 5.). Therefore, by Lemma 1, the permutable operators $\boldsymbol{A}$ and $\boldsymbol{A}^*$ have a common characteristic vector $\boldsymbol{x}_2$ in $\boldsymbol{T}_1$:

$$\boldsymbol{Ax}_2 = \lambda_2\boldsymbol{x}_2, \quad \boldsymbol{A}^*\boldsymbol{x}_2 = \bar{\lambda}_2\boldsymbol{x}_2 \qquad (\boldsymbol{x}_2 \neq \boldsymbol{o}).$$

Obviously, $\boldsymbol{x}_1 \perp \boldsymbol{x}_2$. Setting $\boldsymbol{S}_2 = [\boldsymbol{x}_1, \boldsymbol{x}_2]$ and

$$\boldsymbol{R} = \boldsymbol{S}_2 + \boldsymbol{T}_2, \quad \boldsymbol{S}_2 \perp \boldsymbol{T}_2,$$

we establish in a similar way the existence of a common characteristic vector $\boldsymbol{x}_3$ of $\boldsymbol{A}$ and $\boldsymbol{A}^*$ in $\boldsymbol{T}_3$. Obviously $\boldsymbol{x}_1 \perp \boldsymbol{x}_3$ and $\boldsymbol{x}_2 \perp \boldsymbol{x}_3$. Continuing this process, we obtain n pairwise orthogonal common characteristic vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ of $\boldsymbol{A}$ and $\boldsymbol{A}^*$:

$$\left.\begin{array}{l} \boldsymbol{Ax}_k = \lambda_k\boldsymbol{x}_k, \quad \boldsymbol{A}^*\boldsymbol{x}_k = \lambda_k\boldsymbol{x}_k \quad (\boldsymbol{x}_k \neq \boldsymbol{o}), \\ \qquad (\boldsymbol{x}_i\boldsymbol{x}_k) = 0, \quad \text{for} \quad i \neq k \end{array} \quad (i, k = 1, 2, \ldots, n). \right\} \tag{63}$$

The vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ can be normalized without violating (63).

Thus we have proved that a normal operator always has a complete orthonormal system of characteristic vectors.[26]

Since $\lambda_k = \lambda_l$ always implies that $\bar{\lambda}_k = \bar{\lambda}_l$, it follows from (63) that:

1. *If* $\boldsymbol{A}$ *is a normal operator, every characteristic vector of* $\boldsymbol{A}$ *is a characteristic vector of the adjoint operator* $\boldsymbol{A}^*$, i.e., if $\boldsymbol{A}$ is a normal operator, then $\boldsymbol{A}$ and $\boldsymbol{A}^*$ have the same characteristic vectors.

Suppose now, conversely, that a linear operator $\boldsymbol{A}$ has a complete orthonormal system of characteristic vectors:

$$\boldsymbol{Ax}_k = \lambda_k\boldsymbol{x}_k, \quad (\boldsymbol{x}_i\boldsymbol{x}_k) = \delta_{ik} \qquad (i, k = 1, 2, \ldots, n).$$

We shall show that $\boldsymbol{A}$ is then a normal operator. For let us set:

$$\boldsymbol{y}_l = \boldsymbol{A}^*\boldsymbol{x}_l - \bar{\lambda}_l\boldsymbol{x}_l.$$

Then

$$\begin{aligned}(\boldsymbol{x}_k\boldsymbol{y}_l) &= (\boldsymbol{x}_k, \boldsymbol{A}^*\boldsymbol{x}_l) - \lambda_l(\boldsymbol{x}_k\boldsymbol{x}_l) = (\boldsymbol{Ax}_k, \boldsymbol{x}_l) - \lambda_l(\boldsymbol{x}_k\boldsymbol{x}_l) \\ &= (\lambda_k - \lambda_l)\,\delta_{kl} = 0 \qquad (k, l = 1, 2, \ldots, n).\end{aligned}$$

Hence it follows that

---

[26] Here, and in what follows, we mean by a complete orthonormal system of vectors an orthonormal system of n vectors, where n is the dimension of the space.

$$\boldsymbol{y}_l = \boldsymbol{A}^*\boldsymbol{x}_l - \bar{\lambda}_l\boldsymbol{x}_l = \boldsymbol{o} \qquad (l=1,\ 2,\ \ldots,\ n),$$

i.e., that (63) holds.

But then

$$\boldsymbol{A}\boldsymbol{A}^*\boldsymbol{x}_k = \lambda_k\bar{\lambda}_k\boldsymbol{x}_k \text{ and } \boldsymbol{A}^*\boldsymbol{A}\boldsymbol{x}_k = \lambda_k\bar{\lambda}_k\boldsymbol{x}_k \qquad (k=1,\ 2,\ \ldots,\ n),$$

or

$$\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A}.$$

Thus we have obtained the following 'internal' (spectral) characterization of a normal operator $\boldsymbol{A}$ (apart from the 'external' one: $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A}$):

THEOREM 4: *A linear operator is normal if and only if it has a complete orthonormal system of characteristic values.*

In particular, we have shown that a normal operator is always of simple structure.

Let $\boldsymbol{A}$ be a normal operator with the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Using the Lagrange interpolation formula, we define two polynomials $p(\lambda)$ and $q(\lambda)$ by the conditions

$$p(\lambda_k) = \bar{\lambda}_k, \quad q(\bar{\lambda}_k) = \lambda_k \qquad (k=1,\ 2,\ \ldots,\ n).$$

Then by (63)

$$\boldsymbol{A}^* = p(\boldsymbol{A}), \quad \boldsymbol{A} = q(\boldsymbol{A}^*); \tag{64}$$

i.e.:

2. *If $\boldsymbol{A}$ is a normal operator, then each of the operators $\boldsymbol{A}$ and $\boldsymbol{A}^*$ can be represented as a polynomial in the other; these two polynomials are determined by the characteristic values of $\boldsymbol{A}$.*

Let $\boldsymbol{S}$ be an invariant subspace of $\boldsymbol{R}$ for a normal operator $\boldsymbol{A}$ and let $\boldsymbol{R} = \boldsymbol{S} + \boldsymbol{T}, \boldsymbol{S} \perp \boldsymbol{T}$. Then by § 8, 5., the subspace $\boldsymbol{T}$ is invariant with respect to $\boldsymbol{A}^*$. But $\boldsymbol{A} = q(\boldsymbol{A}^*)$, where $q(\lambda)$ is a polynomial. Therefore $\boldsymbol{T}$ is also invariant with respect to $\boldsymbol{A}$. Thus:

3. *If $\boldsymbol{S}$ is an invariant subspace with respect to a normal operator $\boldsymbol{A}$ and $\boldsymbol{T}$ is the orthogonal complement of $\boldsymbol{S}$, then $\boldsymbol{T}$ is also an invariant subspace for $\boldsymbol{A}$.*

2. Let us now discuss the spectrum of a hermitian operator. Since a hermitian operator $\boldsymbol{H}$ is a special form of a normal operator, by what we have proved it has a complete orthonormal system of characteristic vectors:

$$\boldsymbol{H}\boldsymbol{x}_k = \lambda_k\boldsymbol{x}_k, \quad (\boldsymbol{x}_k\boldsymbol{x}_l) = \delta_{kl} \qquad (k,\ l=1,\ 2,\ \ldots,\ n). \tag{65}$$

From $\boldsymbol{H}^* = \boldsymbol{H}$ it follows that

$$\bar{\lambda}_k = \lambda_k \qquad (k=1,\ 2.\ \ldots,\ n), \tag{66}$$

i.e., all the characteristic values of a hermitian operator $\boldsymbol{H}$ are real.

It is not difficult to see that, conversely, a normal operator with real characteristic values is always hermitian. For from (65), (66), and

$$\boldsymbol{H}^*\boldsymbol{x}_k = \lambda_k\boldsymbol{x}_k \qquad (k=1,\ 2,\ \ldots,\ n)$$

it follows that

$$\boldsymbol{H}^*\boldsymbol{x}_k = \boldsymbol{H}\boldsymbol{x}_k \qquad (k=1,\ 2,\ \ldots,\ n),$$

i.e.,

$$\boldsymbol{H}^* = \boldsymbol{H}.$$

We have obtained the following 'internal' characterization of a hermitian operator (apart from the 'external' one: $\boldsymbol{H}^* = \boldsymbol{H}$):

THEOREM 5: *A linear operator $\boldsymbol{H}$ is hermitian if and only if it has a complete orthonormal system of characteristic vectors with real characteristic values.*

Let us now discuss the spectrum of a unitary operator. Since a unitary operator $\boldsymbol{U}$ is normal, it has a complete orthonormal system of characteristic vectors

$$\boldsymbol{U}\boldsymbol{x}_k = \lambda_k\boldsymbol{x}_k, \quad (\boldsymbol{x}_k\boldsymbol{x}_l) = \delta_{kl} \qquad (k,\ l=1,\ 2,\ \ldots,\ n), \tag{67}$$

where

$$\boldsymbol{U}^*\boldsymbol{x}_k = \bar{\lambda}_k\boldsymbol{x}_k \qquad (k=1,\ 2,\ \ldots,\ n). \tag{68}$$

From $\boldsymbol{U}\boldsymbol{U}^* = \boldsymbol{E}$ we find:

$$\lambda_k\bar{\lambda}_k = 1. \tag{69}$$

Conversely, from (67), (68), and (69) it follows that $\boldsymbol{U}\boldsymbol{U}^* = \boldsymbol{E}$. Thus, among the normal operators a unitary operator is distinguished by the fact that all its characteristic values have modulus 1.

We have thus obtained the following 'internal' characterization of a unitary operator (apart from the 'external' one: $\boldsymbol{U}\boldsymbol{U}^* = \boldsymbol{E}$):

THEOREM 6: *A linear operator is unitary if and only if it has a complete orthonormal system of characteristic vectors with characteristic values of modulus* 1.

Since in an orthonormal basis a normal (hermitian, unitary) matrix corresponds to a normal (hermitian, unitary) operator, we obtain the following propositions:

THEOREM 4′: *A matrix A is normal if and only if it is unitarily similar to a diagonal matrix:*

$$A = U\,\|\lambda_i\delta_{ik}\|_1^n U^{-1} \quad (U^* = U^{-1}). \tag{70}$$

THEOREM 5′: *A matrix H is hermitian if and only if it is unitarily similar to a diagonal matrix with real diagonal elements:*

$$H = U \| \lambda_i \delta_{ik} \|_1^n U^{-1} \quad (U^* = U^{-1};\ \lambda_i = \bar{\lambda}_i;\ i = 1, 2, \ldots, n). \tag{71}$$

THEOREM 6′: *A matrix U is unitary if and only if it is unitarily similar to a diagonal matrix with diagonal elements of modulus* 1:

$$U = U_1 \| \lambda_i \delta_{ik} \|_1^n U_1^{-1} \quad (U_1^* = U_1^{-1};\ |\lambda_i| = 1;\ i = 1, 2, \ldots, n). \tag{72}$$

§ 11. Positive-Semidefinite and Positive-Definite Hermitian Operators

1. We introduce the following definition:

DEFINITION 9: *A hermitian operator $\boldsymbol{H}$ is called positive semidefinite if for every vector $\boldsymbol{x}$ of $\boldsymbol{R}$*

$$(\boldsymbol{Hx},\ \boldsymbol{x}) \geqq 0,$$

and positive definite if for every vector $\boldsymbol{x} \neq \boldsymbol{o}$ of $\boldsymbol{R}$

$$(\boldsymbol{Hx},\ \boldsymbol{x}) > 0.$$

If a vector $\boldsymbol{x}$ is given by its coordinates $x_1, x_2, \ldots, x_n$ in an arbitrary orthonormal basis, then $(\boldsymbol{Hx}, \boldsymbol{x})$, as is easy to see, is a hermitian form in the variables $x_1, x_2, \ldots, x_n$; and to a positive-semidefinite (positive-definite) operator there corresponds a positive-semidefinite (positive-definite) hermitian form (see § 1).

We choose an orthonormal basis $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ of characteristic vectors of $\boldsymbol{H}$:

$$\boldsymbol{Hx}_k = \lambda_k \boldsymbol{x}_k, \quad (\boldsymbol{x}_k \boldsymbol{x}_l) = \delta_{kl} \qquad (k, l = 1, 2, \ldots, n). \tag{73}$$

Then, setting $\boldsymbol{x} = \sum_{k=1}^{n} \xi_k \boldsymbol{x}_k$, we have

$$(\boldsymbol{Hx},\ \boldsymbol{x}) = \sum_{k=1}^{n} \lambda_k |\xi_k|^2 \qquad (k = 1, 2, \ldots, n).$$

Hence we easily deduce the 'internal' characterizations of positive-semidefinite and positive-definite operators:

THEOREM 7: *A hermitian operator is positive semidefinite (positive definite) if and only if all its characteristic values are non-negative (positive).*

From what we have shown, it follows that a positive-definite hermitian operator is non-singular and positive semidefinite.

Let $\boldsymbol{H}$ be a positive-semidefinite hermitian operator. The equation (73) holds for $\boldsymbol{H}$ with $\lambda_k \geqq 0$ $(k = 1, 2, \ldots, n)$. We set $\varrho_k = \sqrt{\lambda_k} \geqq 0$ $(k = 1, 2, 3, \ldots, n)$ and define a linear operator $\boldsymbol{F}$ by the equation

$$\boldsymbol{Fx}_k = \varrho_k \boldsymbol{x}_k \qquad (k = 1, 2, \ldots, n). \tag{74}$$

Then $\boldsymbol{F}$ is also a positive-semidefinite operator and

$$\boldsymbol{F}^2 = \boldsymbol{H}. \tag{75}$$

We shall call the positive-semidefinite hermitian operator $\boldsymbol{F}$ connected with $\boldsymbol{H}$ by (75) the *arithmetical square root* of $\boldsymbol{H}$ and shall denote it by

$$\boldsymbol{F} = \sqrt{\boldsymbol{H}}.$$

If $\boldsymbol{H}$ is positive definite, then $\boldsymbol{F}$ is also positive definite.

We define the Lagrange interpolation polynomial $g(\lambda)$ by the equations

$$g(\lambda_k) = \varrho_k (= \sqrt{\lambda_k}) \quad (k = 1, 2, \ldots, n). \tag{76}$$

Then from (73), (74), and (76) it follows that:

$$\boldsymbol{F} = g(\boldsymbol{H}). \tag{77}$$

The latter equation shows that $\sqrt{\boldsymbol{H}}$ *is a polynomial in* $\boldsymbol{H}$ *and is uniquely determined when the positive-semidefinite hermitian operator* $\boldsymbol{H}$ *is given* (the coefficients of $g(\lambda)$ depend on the characteristic values of $\boldsymbol{H}$).

2. Examples of positive-semidefinite hermitian operators are $\boldsymbol{AA}^*$ and $\boldsymbol{A}^*\boldsymbol{A}$, where $\boldsymbol{A}$ is an arbitrary linear operator in the given space. Indeed, for an arbitrary vector $\boldsymbol{x}$,

$$(\boldsymbol{AA}^*\boldsymbol{x}, \boldsymbol{x}) = (\boldsymbol{A}^*\boldsymbol{x}, \boldsymbol{A}^*\boldsymbol{x}) \geqq 0,$$
$$(\boldsymbol{A}^*\boldsymbol{Ax}, \boldsymbol{x}) = (\boldsymbol{Ax}, \boldsymbol{Ax}) \geqq 0.$$

If $\boldsymbol{A}$ is non-singular, then $\boldsymbol{AA}^*$ and $\boldsymbol{A}^*\boldsymbol{A}$ are positive-definite hermitian operators.

The operators $\boldsymbol{AA}^*$ and $\boldsymbol{A}^*\boldsymbol{A}$ are sometimes called the *left norm* and *right norm* of $\boldsymbol{A}$. $\sqrt{\boldsymbol{AA}^*}$ and $\sqrt{\boldsymbol{A}^*\boldsymbol{A}}$ are called the *left modulus* and *right modulus* of $\boldsymbol{A}$.

For a normal operator the left and right norms, and hence the left and right moduli, are equal.[27]

---

[27] For a detailed study of normal operators, see [168]. In this paper necessary and sufficient conditions for the product of two normal operators to be normal are established.

§ 12. Polar Decomposition of a Linear Operator in a Unitary Space. Cayley's Formulas

1. We shall prove the following theorem:[28]

THEOREM 8: *Every linear operator* A *in a unitary space can be represented in the forms*

$$A = HU, \tag{78}$$

$$A = U_1H_1, \tag{79}$$

where H, H_1 *are positive-semidefinite hermitian operators and* U, U_1 *are unitary operators.* A *is normal if and only if in* (78) (*or* (79)) *the factors* H *and* U (H_1 *and* U_1) *are permutable.*

Proof. From (78) and (79) it follows that H and H_1 are the left and right moduli, respectively, of A.

For

$$AA^* = HUU^*H = H^2,\ A^*A = H_1U_1^*U_1H_1 = H_1^2.$$

Note that it is sufficient to establish (78), since by applying this decomposition to A^* we obtain $A^* = HU$ and hence

$$A = U^{-1}H,$$

i.e., the decomposition (79) for A.

We begin by establishing (78) in the special case where A is non-singular ($|A| \neq 0$). We set:

$$H = \sqrt{AA^*}\ (\text{here } |H|^2 = |A|^2 \neq 0),\ U = H^{-1}A$$

and verify that U is unitary:

$$UU^* = H^{-1}AA^*H^{-1} = H^{-1}H^2H^{-1} = E.$$

Note that in this case not only the first factor H in (78), but also the second factor U is uniquely determined by the non-singular operator A.

We now consider the general case where A may be singular.

First of all we observe that a complete orthonormal system of characteristic vectors of the right norm of A is always transformed by A into an orthogonal system of vectors. For let

$$A^*Ax_k = \varrho_k^2x_k \quad [(x_kx_l) = \delta_{kl},\ \varrho_k \geqq 0;\ k, l = 1, 2, \ldots, n].$$

Then

$$(Ax_k, Ax_l) = (A^*Ax_k, x_l) = \varrho_k^2 \cdot (x_kx_l) = 0 \quad (k \neq l).$$

---

[28] See [168], p. 77.

Here

$$|Ax_k|^2 = (Ax_k, Ax_k) = \varrho_k^2 \quad (k = 1, 2, \ldots, n).$$

Therefore there exists an orthonormal system of vectors $z_1, z_2, \ldots, z_n$ such that

$$Ax_k = \varrho_kz_k \quad [(z_kz_l) = \delta_{kl};\ k, l = 1, 2, \ldots, n]. \tag{80}$$

We define linear operators H and U by the equations

$$Ux_k = z_k, \quad Hz_k = \varrho_kz_k. \tag{81}$$

From (80) and (81) we find:

$$A = HU.$$

Here H is, by (81), a positive-semidefinite hermitian operator, because it has a complete orthonormal system of characteristic vectors $z_1, z_2, \ldots, z_n$ with non-negative characteristic values $\varrho_1, \varrho_2, \ldots, \varrho_n$; and U is a unitary operator, because it carries the orthonormal system of vectors $x_1, x_2, \ldots, x_n$ into the orthonormal system $z_1, z_2, \ldots, z_n$.

Thus we can take it as proved that an arbitrary linear operator A has decompositions (78) and (79), that the hermitian factors H and H_1 are always uniquely determined by x (they are the left and right moduli of A, respectively) and that the unitary factors U and U_1 are uniquely determined only when A is non-singular.

From (78) we find easily:

$$AA^* = H^2, \quad A^*A = U^{-1}H^2U. \tag{82}$$

If A is a normal operator ($AA^* = A^*A$), then it follows from (82) that

$$H^2U = UH^2. \tag{83}$$

Since $H = \sqrt{H^2} = g(H^2)$ (see § 11), (83) shows that U and H commute. Conversely, if H and U commute, then it follows from (82) that A is normal. This completes the proof of the theorem.[29]

---

[29] If the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\varrho_1, \varrho_2, \ldots, \varrho_n$ of the linear operator A and its left modulus $H = \sqrt{AA^*}$ (by (82) $\varrho_1, \varrho_2, \ldots, \varrho_n$ are also the characteristic values of the right modulus $H_1 = \sqrt{A^*A}$) are so numbered that

$$|\lambda_1| \geqq |\lambda_2| \geqq \cdots \geqq |\lambda_n|, \quad \varrho_1 \geqq \varrho_2 \geqq \cdots \geqq \varrho_n,$$

then (see [379], or [153] and [296]) the following *inequality of Weyl* holds:

$$|\lambda_1| \leqq \varrho_1, \quad |\lambda_1| + |\lambda_2| \leqq \varrho_1 + \varrho_2, \quad \ldots, \quad |\lambda_1| + \cdots + |\lambda_n| \leqq \varrho_1 + \cdots + \varrho_n.$$

It is hardly necessary to mention that together with the operator equations (78) and (79) the corresponding matrix equations hold.

The decompositions (78) and (79) are analogues to the representation of a complex number z in the form $z = ru$, where $r = |z|$ and $|u| = 1$.

2. Now let $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n$ be a complete orthonormal system of characteristic vectors of the arbitrary unitary operator $\boldsymbol{U}$. Then

$$\boldsymbol{U}\boldsymbol{x}_k = e^{if_k}\boldsymbol{x}_k, \quad (\boldsymbol{x}_k\boldsymbol{x}_l) = \delta_{kl} \quad (k, l = 1, 2, \dots, n). \tag{84}$$

where the f_k $(k = 1, 2, \dots, n)$ are real numbers. We define a hermitian operator $\boldsymbol{F}$ by the equations

$$\boldsymbol{F}\boldsymbol{x}_k = f_k\boldsymbol{x}_k \quad (k = 1, 2, \dots, n). \tag{85}$$

From (84) and (85) it follows that:[30]

$$\boldsymbol{U} = e^{i\boldsymbol{F}}. \tag{86}$$

Thus, a unitary operator $\boldsymbol{U}$ is always representable in the form (86), where $\boldsymbol{F}$ is a hermitian operator. Conversely, if $\boldsymbol{F}$ is a hermitian operator, then $\boldsymbol{U} = e^{i\boldsymbol{F}}$ is unitary.

The decompositions (78) and (79) together with (86) give the following equations:

$$\boldsymbol{A} = \boldsymbol{H}e^{i\boldsymbol{F}}, \tag{87}$$

$$\boldsymbol{A} = e^{i\boldsymbol{F}_1}\boldsymbol{H}_1 \tag{88}$$

where $\boldsymbol{H}, \boldsymbol{F}, \boldsymbol{H}_1$, and $\boldsymbol{F}_1$ are hermitian operators, with $\boldsymbol{H}$ and $\boldsymbol{H}_1$ positive semidefinite.

The decompositions (87) and (88) are analogues to the representation of a complex number z in the form $z = re^{i\varphi}$, where $r \geqq 0$ and φ are real numbers.

Note. In (86), the operator $\boldsymbol{F}$ is not uniquely determined by $\boldsymbol{U}$. For $\boldsymbol{F}$ is defined by means of the numbers f_k $(k = 1, 2, \dots, n)$ and we can add to each of these numbers an arbitrary multiple of 2π without changing the original equations (84). By choosing these multiples of 2π suitably we can assume that $e^{if_k} = e^{if_l}$ always implies that $f_k = f_l$ $(1 \leqq k, l \leqq n)$. Then we can determine the interpolation polynomial $g(\lambda)$ by the equations

$$g(e^{if_k}) = f_k \quad (k = 1, 2, \dots, n). \tag{89}$$

---

[30] $e^{i\boldsymbol{F}} = r(\boldsymbol{F})$, where $r(\lambda)$ is the Lagrange interpolation polynomial for the function $e^{i\lambda}$ at the places $f_1, f_2, \dots, f_n$.

From (84), (85), and (89) it follows that

$$\boldsymbol{F} = g(\boldsymbol{U}) = g(e^{i\boldsymbol{F}}). \tag{90}$$

Similarly we can normalize the choice of $\boldsymbol{F}_1$ so that

$$\boldsymbol{F}_1 = h(\boldsymbol{U}_1) = h(e^{i\boldsymbol{F}_1}), \tag{91}$$

where $h(\lambda)$ is a polynomial.

By (90) and (91), the permutability of $\boldsymbol{H}$ and $\boldsymbol{U}$ ($\boldsymbol{H}_1$ and $\boldsymbol{U}_1$) implies that of $\boldsymbol{H}$ and $\boldsymbol{F}$ ($\boldsymbol{H}_1$ and $\boldsymbol{F}_1$), and vice versa. Therefore, by Theorem 8, $\boldsymbol{A}$ is normal if and only if in (87) $\boldsymbol{H}$ and $\boldsymbol{F}$ (or, in (88), $\boldsymbol{H}_1$ and $\boldsymbol{F}_1$) are permutable, provided the characteristic values of $\boldsymbol{F}$ (or $\boldsymbol{F}_1$) are suitably normalized.

The formula (86) is based on the fact that the functional dependence

$$\mu = e^{if} \tag{92}$$

carries n arbitrary numbers $f_1, f_2, \dots, f_n$ on the real axis into certain numbers $\mu_1, \mu_2, \dots, \mu_n$ on the unit circle $|\mu| = 1$, and vice versa.

The transcendental dependence (92) can be replaced by the rational dependence

$$\mu = \frac{1 + if}{1 - if}, \tag{93}$$

which carries the real axis $f = \bar{f}$ into the circle $|\mu| = 1$; here the point at infinity on the real axis goes over into the point $\mu = -1$. From (93), we find:

$$f = i\,\frac{1 - \mu}{1 + \mu}. \tag{94}$$

Repeating the arguments which have led us to the formula (86), we obtain from (93) and (94) the pair of inverse formulas:

$$\left.\begin{aligned} \boldsymbol{U} &= (\boldsymbol{E} + i\boldsymbol{F})(\boldsymbol{E} - i\boldsymbol{F})^{-1}, \\ \boldsymbol{F} &= i(\boldsymbol{E} - \boldsymbol{U})(\boldsymbol{E} + \boldsymbol{U})^{-1}, \end{aligned}\right\} \tag{95}$$

We have thus obtained *Cayley's formulas*. These formulas establish a one-to-one correspondence between arbitrary hermitian operators $\boldsymbol{F}$ and those unitary operators $\boldsymbol{U}$ that do not have the characteristic value -1.[31]

---

[31] The exceptional value -1 can be replaced by any number μ_0 ($|\mu_0| = 1$). For this purpose, we have to take instead of (93) a fractional-linear function mapping the real axis $f = \bar{f}$ onto the circle $|\mu| = 1$ and carrying the point $f = \infty$ into $\mu = \mu_0$. The formulas (94) and (95) can be modified correspondingly.

The formulas (86), (87), (88), and (95) are obviously valid when we replace all the operators by the corresponding matrices.

§ 13. Linear Operators in a Euclidean Space

1. We consider an n-dimensional euclidean space R. Let A be a linear operator in R.

DEFINITION 10: *The linear operator A^{T} is called the transposed operator of A (or the transpose of A) if for any two vectors x and y of R:*

$$(Ax, y) = (x, A^{\mathsf{T}}y). \tag{96}$$

The existence and uniqueness of the transposed operator is established in exactly the same way as was done in § 8 for the adjoint operator in a unitary space.

The transposed operator has the following properties:

1. $(A^{\mathsf{T}})^{\mathsf{T}} = A$,
2. $(A + B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}}$,
3. $(\alpha A)^{\mathsf{T}} = \alpha A^{\mathsf{T}}$ (α a real number),
4. $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$.

We introduce a number of definitions.

DEFINITION 11: *A linear operator A is called normal if*

$$AA^{\mathsf{T}} = A^{\mathsf{T}}A.$$

DEFINITION 12: *A linear operator S is called symmetric if*

$$S^{\mathsf{T}} = S.$$

DEFINITION 13: *A symmetric operator S is called positive semidefinite if for every vector x of R*

$$(Sx, x) \geqq 0.$$

DEFINITION 14: *A symmetric operator S is called positive definite if for every vector $x \neq o$ of R*

$$(Sx, x) > 0.$$

DEFINITION 15: *A linear operator K is called skew-symmetric if*

$$K^{\mathsf{T}} = -K.$$

An arbitrary linear operator A can always be represented uniquely in the form

$$A = S + K, \tag{97}$$

where S is symmetric and K is skew-symmetric.

For it follows from (97) that

$$A^{\mathsf{T}} = S - K. \tag{98}$$

From (97) and (98) we have:

$$S = \frac{1}{2}(A + A^{\mathsf{T}}), \quad K = \frac{1}{2}(A - A^{\mathsf{T}}). \tag{99}$$

Conversely, (99) defines a symmetric operator S and a skew-symmetric operator K for which (97) holds.

S and K are called respectively the *symmetric component* and the *skew-symmetric component* of A.

DEFINITION 16: *An operator Q is called orthogonal if it preserves the metric of the space, i.e., if for any two vectors x, y of R*

$$(Qx, Qy) = (x, y). \tag{100}$$

By (96), equation (100) can be written as: $(x, Q^{\mathsf{T}}Qy) = (x, y)$. Hence

$$Q^{\mathsf{T}}Q = E. \tag{101}$$

Conversely, (101) implies (100) (for arbitrary vectors x, y).[32] From (101) it follows that: $|Q|^2 = 1$, i.e.,

$$|Q| = \pm 1.$$

We shall call Q an orthogonal operator *of the first kind* (or *proper*) if $|Q| = 1$ and *of the second kind* (or *improper*) if $|Q| = -1$.

Symmetric, skew-symmetric, and orthogonal operators are special forms of a normal operator.

We consider an arbitrary orthonormal basis in the given euclidean space. Suppose that in this basis A corresponds to the matrix $A = \| a_{ik} \|_1^n$ (here all the a_{ik} are real numbers). The reader will have no difficulty in showing that the transposed operator A^{T} corresponds in this basis to the transposed matrix $A^{\mathsf{T}} = \| a^{\mathsf{T}}_{ik} \|_1^n$, where $a^{\mathsf{T}}_{ik} = a_{ki}$ $(i, k = 1, 2, \ldots, n)$. Hence it follows that in an orthonormal basis a normal operator A corresponds to a normal

---

[32] The orthogonal operators in a euclidean space form a group, the so-called orthogonal group.

matrix A $(AA^{\mathsf{T}} = A^{\mathsf{T}}A)$, a symmetric operator $\mathbf{S}$ to a symmetric matrix $S = \| s_{ik} \|_1^n$ $(S^{\mathsf{T}} = S)$, a skew-symmetric operator K to a skew-symmetric matrix $K = \| k_{ij} \|_1^n$ $(K^{\mathsf{T}} = -K)$ and, finally, an orthogonal operator $\mathbf{Q}$ to an orthogonal matrix $Q = \| q_{ik} \|_1^n$ $(QQ^{\mathsf{T}} = E)$.[33]

Just as was done in § 8 for the adjoint operator, we can here make the following statement for the transposed operator:

If a subspace $\mathbf{S}$ *of* $\mathbf{R}$ *is invariant with respect to a linear operator* $\mathbf{A}$, *then the orthogonal complement* $\mathbf{T}$ *of* $\mathbf{S}$ *in* $\mathbf{R}$ *is invariant with respect to* A^{T}.

2. For the study of linear operators in a euclidean space $\mathbf{R}$, we extend $\mathbf{R}$ to a unitary space $\tilde{\mathbf{R}}$. This extension is made in the following way:

1. The vectors of $\mathbf{R}$ are called 'real' vectors.

2. We introduce 'complex' vectors $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are real, i.e., $\mathbf{x} \in \mathbf{R}$, $\mathbf{y} \in \mathbf{R}$.

3. The operations of addition of complex vectors and of multiplication by a complex number are defined in the natural way. Then the set of all complex vectors forms an n-dimensional vector space $\tilde{\mathbf{R}}$ over the field of complex numbers which contains $\mathbf{R}$ as a subspace.

4. In $\tilde{\mathbf{R}}$ we introduce a hermitian metric such that in $\mathbf{R}$ it coincides with the existing euclidean metric. The reader can easily verify that the required hermitian metric is given in the following way:

If $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ $(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in \mathbf{R})$, then

$$(\mathbf{z}\mathbf{w}) = (\mathbf{x}\mathbf{u}) + (\mathbf{y}\mathbf{v}) + i\,[(\mathbf{y}\mathbf{u}) - (\mathbf{x}\mathbf{v})].$$

Setting $\bar{\mathbf{z}} = \mathbf{x} - i\mathbf{y}$ and $\mathbf{w} = \mathbf{u} - i\mathbf{v}$, we have

$$(\bar{\mathbf{z}}\,\bar{\mathbf{w}}) = (\overline{\mathbf{z}\mathbf{w}}).$$

If we choose a real basis, i.e., a basis of $\mathbf{R}$, then $\tilde{\mathbf{R}}$ will be the set of all vectors with complex coordinates and $\mathbf{R}$ the set of all vectors with real coordinates in this basis.

Every linear operator $\mathbf{A}$ in $\mathbf{R}$ extends uniquely to a linear operator in $\tilde{\mathbf{R}}$:

$$\mathbf{A}(\mathbf{x} + i\mathbf{y}) = \mathbf{A}\mathbf{x} + i\mathbf{A}\mathbf{y}.$$

3. Among all the linear operators of $\tilde{\mathbf{R}}$ those that are obtainable as the result of such an extension of operators of $\mathbf{R}$ can be characterized by the fact that they carry $\mathbf{R}$ into $\mathbf{R}$ $(\mathbf{A}\mathbf{R} \subset \mathbf{R})$. These operators are called *real.*

---

[33] The papers [138], [262a], [170b] are devoted to the study of the structure of orthogonal matrices. Orthogonal matrices, like orthogonal operators, are called proper and improper according as $| Q | = +1$ or $| Q | = -1$.

In a real basis real operators are determined by real matrices, i.e., matrices with real elements.

A real operator $\mathbf{A}$ carries conjugate complex vectors $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, $\bar{\mathbf{z}} = \mathbf{x} - i\mathbf{y}$ $(\mathbf{x}, \mathbf{y} \in \mathbf{R})$ into conjugate complex vectors:

$$\mathbf{A}\mathbf{z} = \mathbf{A}\mathbf{x} + i\mathbf{A}\mathbf{y},\ \mathbf{A}\bar{\mathbf{z}} = \mathbf{A}\mathbf{x} - i\mathbf{A}\mathbf{y} \quad (\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y} \in \mathfrak{R}).$$

The secular equation of a real operator has real coefficients, so that when it has a root λ of multiplicity p it also has the root $\bar{\lambda}$ with the multiplicity p. From $\mathbf{A}\mathbf{z} = \lambda\mathbf{z}$ it follows that $\mathbf{A}\bar{\mathbf{z}} = \bar{\lambda}\bar{\mathbf{z}}$, i.e., to conjugate characteristic values there correspond conjugate characteristic vectors.

The two-dimensional space $[\mathbf{z},\ \mathbf{z}]$ has a real basis:

$$\mathbf{x} = \frac{1}{2}(\mathbf{z} + \bar{\mathbf{z}}),\ \ \mathbf{y} = \frac{1}{2i}(\mathbf{z} - \bar{\mathbf{z}}).$$

We shall call the plane in $\mathbf{R}$ spanned by this basis an *invariant plane* of $\mathbf{A}$ corresponding to the pair of characteristic values λ, $\bar{\lambda}$.

Let $\lambda = \mu + i\nu$. Then it is easy to see that

$$\mathbf{A}\mathbf{x} = \mu\mathbf{x} - \nu\mathbf{y},$$

$$\mathbf{A}\mathbf{y} = \nu\mathbf{x} + \mu\mathbf{y}.$$

We consider a real operator $\mathbf{A}$ of simple structure with the characteristic values:

$$\lambda_{2k-1} = \mu_k + i\nu_k,\ \lambda_{2k} = \mu_k - i\nu_k,\ \lambda_l = \mu_l\ (k = 1, 2, \ldots, q;\ l = 2q + 1, \ldots, n),$$

where μ_k, ν_k, μ_l are real and $\nu_k \neq 0$ $(k = 1, 2, \ldots, q)$.

Then the characteristic vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_n$ corresponding to these characteristic values can be chosen such that

$$\mathbf{z}_{2k-1} = \mathbf{x}_k + i\mathbf{y}_k, \quad \mathbf{z}_{2k} = \mathbf{x}_k - i\mathbf{y}_k, \quad \mathbf{z}_l = \mathbf{x}_l \tag{102}$$

$$(k = 1, 2, \ldots, q;\ l = 2q + 1, \ldots, n).$$

The vectors

$$\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2, \ldots, \mathbf{x}_q, \mathbf{y}_q, \mathbf{x}_{2q+1}, \ldots, \mathbf{x}_n \tag{103}$$

form a basis of the euclidean space $\mathbf{R}$. Here

---

[34] If to the characteristic value λ of the real operator $\mathbf{A}$ there correspond the linearly independent characteristic vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_p$, then to the characteristic value $\bar{\lambda}$ there correspond the linearly independent characteristic vectors $\bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2, \ldots, \bar{\mathbf{z}}_p$.

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x}_k &= \mu_k \boldsymbol{x}_k - \nu_k \boldsymbol{y}_k, \\ \boldsymbol{A}\boldsymbol{y}_k &= \nu_k \boldsymbol{x}_k + \mu_k \boldsymbol{y}_k, \\ \boldsymbol{A}\boldsymbol{x}_l &= \mu_l \boldsymbol{x}_l \end{aligned} \qquad \begin{pmatrix} k=1,\,2,\,\ldots,\,q \\ l=2q+1,\ldots,n \end{pmatrix}. \tag{104}$$

In the basis (103) there corresponds to the operator $\boldsymbol{A}$ the real quasi-diagonal matrix

$$\left\{ \left\| \begin{matrix} \mu_1 & \nu_1 \\ -\nu_1 & \mu_1 \end{matrix} \right\|, \ldots, \left\| \begin{matrix} \mu_q & \nu_q \\ -\nu_q & \mu_q \end{matrix} \right\|, \mu_{2q+1}, \ldots, \mu_n \right\}. \tag{105}$$

Thus: *For every operator* $\boldsymbol{A}$ *of simple structure in a euclidean space there exists a basis in which* $\boldsymbol{A}$ *corresponds to a matrix of the form* (105). Hence it follows that: *A real matrix of simple structure is real-similar to a canonical matrix of the form* (105):

$$A = T \left\{ \left\| \begin{matrix} \mu_1 & \nu_1 \\ -\nu_1 & \mu_1 \end{matrix} \right\|, \ldots, \left\| \begin{matrix} \mu_q & \nu_q \\ -\nu_q & \mu_q \end{matrix} \right\|, \mu_{2q+1}, \ldots, \mu_n \right\} T^{-1} \quad (T = \overline{T}). \tag{106}$$

The transposed operator $\boldsymbol{A}^{\mathsf{T}}$ of $\boldsymbol{A}$ in $\boldsymbol{R}$ upon extension becomes the adjoint operator $\boldsymbol{A}^*$ of $\boldsymbol{A}$ in $\tilde{\boldsymbol{R}}$. Therefore: *Normal, symmetric, skew-symmetric, and orthogonal operators in* $\boldsymbol{R}$ *after the extension become normal, hermitian, hermitian multiplied by i, and unitary real operators in* $\tilde{\boldsymbol{R}}$.

It is easy to show that *for a normal operator* $\boldsymbol{A}$ *in a euclidean space a canonical basis can be chosen as an orthonormal basis* (103) *for which* (104) *holds.*[35] *Therefore a real normal matrix is always real-similar and orthogonally-similar to a matrix of the form* (105):

$$A = Q \left\{ \left\| \begin{matrix} \mu_1 & \nu_1 \\ -\nu_1 & \mu_1 \end{matrix} \right\|, \ldots, \left\| \begin{matrix} \mu_q & \nu_q \\ -\nu_q & \mu_q \end{matrix} \right\|, \mu_{2q+1}, \ldots, \mu_n \right\} Q^{-1} \tag{107}$$
$$(Q = Q^{\mathsf{T}\,-1} = \overline{Q}).$$

All the characteristic values of a symmetric operator $\boldsymbol{S}$ in a euclidean space are real, since after the extension the operator becomes hermitian. For a symmetric operator $\boldsymbol{S}$ we must set $q=0$ in (104). Then we obtain:

$$\boldsymbol{S}\boldsymbol{x}_l = \mu_l \boldsymbol{x}_l \quad [(\boldsymbol{x}_k \boldsymbol{x}_l) = \delta_{kl};\ k, l = 1, 2, \ldots, n]. \tag{108}$$

A symmetric operator $\boldsymbol{S}$ *in a euclidean space always has an othonormal system of characteristic vectors wih real characteristic values.*[36] Therefore:

---

[35] The orthonormality of the basis (102) in the hermitian metric implies the orthonormality of the basis (103) in the corresponding euclidean metric.

[36] The symmetric operator S is positive semidefinite if in (108) all $\mu_l \geqq 0$ and positive definite if all $\mu_l > 0$.

A real symmetric matrix is always real-similar and orthogonally-similar to a diagonal matrix:

$$S = Q\{\mu_1, \mu_2, \ldots, \mu_n\} Q^{-1} \quad (Q = Q^{\mathsf{T}-1} = \overline{Q}). \tag{109}$$

All the characteristic values of a skew-symmetric operator $\boldsymbol{K}$ in a euclidean space are pure imaginary (after the extension the operator is i times a hermitian operator). For a skew-symmetric operator we must set in (104):

$$\mu_1 = \mu_2 = \cdots = \mu_q = \mu_{2q+1} = \cdots = \mu_n = 0$$

then the formulas assume the form

$$\begin{aligned} \boldsymbol{K}\boldsymbol{x}_k &= -\nu_k \boldsymbol{y}_k, \\ \boldsymbol{K}\boldsymbol{y}_k &= \nu_k \boldsymbol{x}_k, \\ \boldsymbol{K}\boldsymbol{x}_l &= \boldsymbol{o} \end{aligned} \qquad (k = 1, 2, \ldots, q;\ l - 2q + 1, \ldots, n). \tag{110}$$

Since $\boldsymbol{K}$ is a normal operator, the basis (103) can be assumed to be orthonormal. Thus: *Every real skew-symmetric matrix is real-similar and orthogonally-similar to a canonical skew-symmetric matrix*:

$$K = Q \left\{ \left\| \begin{matrix} 0 & \nu_1 \\ -\nu_1 & 0 \end{matrix} \right\|, \ldots, \left\| \begin{matrix} 0 & \nu_q \\ -\nu_q & 0 \end{matrix} \right\|, 0, \ldots, 0 \right\} Q^{-1} \quad (Q = Q^{\mathsf{T}-1} = \overline{Q}). \tag{111}$$

All the characteristic values of an orthogonal operator $\boldsymbol{Q}$ in a euclidean space are of modulus 1 (upon extension the operator becomes unitary). Therefore in the case of an orthogonal operator we must set in (104):

$$\mu_k^2 + \nu_k^2 = 1,\ \mu_l = \pm 1 \qquad (k = 1, 2, \ldots, q;\ l = 2q + 1, \ldots, n).$$

For this basis (103) can be assumed to be orthonormal. The formulas (104) can be represented in the form

$$\begin{aligned} \boldsymbol{Q}\boldsymbol{x}_k &= \boldsymbol{x}_k \cos\varphi_k - \boldsymbol{y}_k \sin\varphi_k, \\ \boldsymbol{Q}\boldsymbol{y}_k &= \boldsymbol{x}_k \sin\varphi_k + \boldsymbol{y}_k \cos\varphi_k, \\ \boldsymbol{Q}\boldsymbol{x}_l &= \pm \boldsymbol{x}_l \end{aligned} \qquad \begin{pmatrix} k = 1, 2, \ldots, q, \\ l = 2q + 1, \ldots, n \end{pmatrix}. \tag{112}$$

From what we have shown, it follows that: *Every real orthogonal matrix is real-similar and orthogonally-similar to a canonical orthogonal matrix*:

$$Q = Q_1 \left\{ \left\| \begin{matrix} \cos\varphi_1 & \sin\varphi_1 \\ -\sin\varphi_1 & \cos\varphi_1 \end{matrix} \right\|, \ldots, \left\| \begin{matrix} \cos\varphi_q & \sin\varphi_q \\ -\sin\varphi_q & \cos\varphi_q \end{matrix} \right\|, \pm 1, \ldots, \pm 1 \right\} Q_1^{-1} \tag{113}$$
$$(Q_1 = Q_1^{\mathsf{T}-1} = \overline{Q}_1).$$

Example. We consider an arbitrary finite rotation around the point O in a three-dimensional space. It carries a directed segment $\overrightarrow{OA}$ into a directed segment $\overrightarrow{OB}$ and can therefore be regarded as an operator $\boldsymbol{Q}$ in a three-dimensional vector space (formed by all possible segments $\overrightarrow{OA}$). This operator is linear and orthogonal. Its determinant is $+1$, since $\boldsymbol{Q}$ does not change the orientation of the space.

Thus, $\boldsymbol{Q}$ is a proper orthogonal operator. For this operator the formulas (112) look as follows:

$$\begin{aligned}\boldsymbol{Q}\boldsymbol{x}_1 &= \boldsymbol{x}_1 \cos\varphi - \boldsymbol{y}_1 \sin\varphi\,,\\ \boldsymbol{Q}\boldsymbol{y}_1 &= \boldsymbol{x}_1 \sin\varphi + \boldsymbol{y}_1 \cos\varphi\,,\\ \boldsymbol{Q}\boldsymbol{x}_2 &= \pm\,\boldsymbol{x}_2\,.\end{aligned}$$

From the equation $|\boldsymbol{Q}| = 1$ it follows that $\boldsymbol{Q}\boldsymbol{x}_2 = \boldsymbol{x}_2$. This means that all the points on the line through O in the direction of $\boldsymbol{x}_2$ remain fixed. Thus we have obtained the *Theorem of Euler-D'Alembert*:

Every finite rotation of a rigid body around a fixed point can be obtained as a finite rotation by an angle φ around some fixed axis passing through that point.

§ 14. Polar Decomposition of an Operator and the Cayley Formulas in a Euclidean Space

1. In § 12 we established the polar decomposition of a linear operator in a unitary space. In exactly the same way we obtain the polar decomposition of a linear operator in a euclidean space.

THEOREM 9. *Every linear operator $\boldsymbol{A}$ is representable in the form of a product*[37]

$$\boldsymbol{A} = \boldsymbol{S}\boldsymbol{Q} \tag{114}$$

$$\boldsymbol{A} = \boldsymbol{Q}_1\boldsymbol{S}_1 \tag{115}$$

where $\boldsymbol{S}, \boldsymbol{S}_1$ are positive-semidefinite symmetric and $\boldsymbol{Q}, \boldsymbol{Q}_1$ are orthogonal operators: here $\boldsymbol{S} = \sqrt{\boldsymbol{A}\boldsymbol{A}^{\mathsf{T}}} = g(\boldsymbol{A}\boldsymbol{A}^{\mathsf{T}})$, $\boldsymbol{S}_1 = \sqrt{\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A}} = h(\boldsymbol{A}^{\mathsf{T}}\boldsymbol{A})$, where $g(\lambda)$ and $h(\lambda)$ are real polynomials.

$\boldsymbol{A}$ is a normal operator if and only if $\boldsymbol{S}$ and $\boldsymbol{Q}$ ($\boldsymbol{S}_1$ and $\boldsymbol{Q}_1$) are permutable.

Similar statements hold for matrices.

---

[37] As in Theorem 8, the operators $\boldsymbol{S}$ and $\boldsymbol{S}_1$ are uniquely determined by $\boldsymbol{A}$. If $\boldsymbol{A}$ is non-singular, then the orthogonal factors $\boldsymbol{Q}$ and $\boldsymbol{Q}_1$ are also uniquely determined.

Let us point out the geometrical content of the formulas (114) and (115). We let the vectors of an n-dimensional euclidean point space issue from the origin of the coordinate system. Then every vector is the radius vector of some point of the space. The orthogonal transformation realized by the operator $\boldsymbol{Q}$ (or $\boldsymbol{Q}_1$) is a 'rotation' in this space, because it preserves the euclidean metric and leaves the origin of the coordinate system fixed.[38] The symmetric operator $\boldsymbol{S}$ (or $\boldsymbol{S}_1$) represents a 'dilatation' of the n-dimensional space (i.e., a 'stretching' along n mutually perpendicular directions with stretching factors $\varrho_1, \varrho_2, \ldots, \varrho_n$ that are, in general, distinct ($\varrho_1, \varrho_2, \ldots, \varrho_n$ are arbitrary non-negative numbers)). According to the formulas (114) and (115), every linear homogeneous transformation of an n-dimensional euclidean space can be obtained by carrying out in succession some rotation and some dilatation (in any order).

2. Just as was done in the preceding section for a unitary operator, we now consider some representations of an orthogonal operator in a euclidean space $\boldsymbol{R}$.

Let $\boldsymbol{K}$ be an arbitrary skew-symmetric operator ($\boldsymbol{K}^{\mathsf{T}} = -\boldsymbol{K}$) and let

$$\boldsymbol{Q} = e^{\boldsymbol{K}}. \tag{116}$$

Then $\boldsymbol{Q}$ is a proper orthogonal operator. For

$$\boldsymbol{Q}^{\mathsf{T}} = e^{\boldsymbol{K}^{\mathsf{T}}} = e^{-\boldsymbol{K}} = \boldsymbol{Q}^{-1}$$

and

$$|\boldsymbol{Q}| = 1.^{39}$$

Let us show that every proper orthogonal operator is representable in the form (116). For this purpose we take the corresponding orthogonal matrix Q. Since $|Q| = 1$, we have, by (113),[40]

---

[38] For $\boldsymbol{Q}| = 1$ this is a proper rotation; but for $|\boldsymbol{Q}| = -1$ it is a combination of a rotation and a reflection in a coordinate plane.

[39] If $k_1, k_2, \ldots, k_n$ are the characteristic values of $\boldsymbol{K}$, then $\mu_1 = e^{k_1}, \mu_2 = e^{k_2}, \ldots, \mu_n = e^{k_n}$ are the characteristic values of $\boldsymbol{Q} = e^{\boldsymbol{K}}$; moreover

$$|\boldsymbol{Q}| = \mu_1\mu_2\cdots\mu_n = e^{\sum_{i=1}^{n} k_i} = 1\,,$$

since

$$\sum_{i=1}^{n} k_i = 0.$$

[40] Among the characteristic values of a proper orthogonal matrix Q there is an even number equal to -1. The diagonal matrix $\begin{Vmatrix} -1 & 0 \\ 0 & -1 \end{Vmatrix}$ can be written in the form $\begin{Vmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{Vmatrix}$ for $\varphi = \pi$.

$$Q = Q_1 \left\{ \begin{Vmatrix} \cos\varphi_1 & \sin\varphi_1 \\ -\sin\varphi_1 & \cos\varphi_1 \end{Vmatrix}, \dots, \begin{Vmatrix} \cos\varphi_p & \sin\varphi_q \\ -\sin\varphi_p & \cos\varphi_q \end{Vmatrix}, +1, \dots, +1 \right\} Q_1^{-1} \quad (117)$$

$$(Q_1 = (Q_1^{\mathsf{T}})^{-1} = \overline{Q}_1).$$

We define the skew-symmetric matrix K by the equation

$$K = Q_1 \left\{ \begin{Vmatrix} 0 & \varphi_1 \\ -\varphi_1 & 0 \end{Vmatrix}, \dots, \begin{Vmatrix} 0 & \varphi_q \\ -\varphi_q & 0 \end{Vmatrix}, 0, \dots, 0 \right\} Q_1^{-1}. \quad (118)$$

Since

$$\exp\left\{ \begin{Vmatrix} 0 & \varphi \\ -\varphi & 0 \end{Vmatrix} \right\} = \begin{Vmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{Vmatrix},$$

it follows from (117) and (118) that

$$Q = e^{K}. \quad (119)$$

The matrix equation (119) implies the operator equation (116).

In order to represent an improper orthogonal operator we introduce a special operator $\boldsymbol{W}$ which is defined in an orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ by the equations

$$\boldsymbol{W}\boldsymbol{e}_1 = \boldsymbol{e}_1, \dots, \boldsymbol{W}\boldsymbol{e}_{n-1} = \boldsymbol{e}_{n-1}, \boldsymbol{W}\boldsymbol{e}_n = -\boldsymbol{e}_n. \quad (120)$$

$\boldsymbol{W}$ is an improper orthogonal operator. If $\boldsymbol{Q}$ is an arbitrary improper orthogonal operator then $\boldsymbol{W}^{-1}\boldsymbol{Q}$ and $\boldsymbol{Q}\boldsymbol{W}^{-1}$ are proper and therefore representable in the form $e^{\boldsymbol{K}}$ and $e^{\boldsymbol{K}_1}$, where $\boldsymbol{K}$ and $\boldsymbol{K}_1$ are skew-symmetric operators. Hence we obtain the formulas for an improper orthogonal operator

$$\boldsymbol{Q} = \boldsymbol{W}e^{\boldsymbol{K}} = e^{\boldsymbol{K}_1}\boldsymbol{W}. \quad (121)$$

The basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ in (120) can be chosen such that it coincides with the basis $\boldsymbol{x}_k, \boldsymbol{y}_k, \boldsymbol{x}_l$ $(k = 1, 2, \dots, q; l = 2q+1, \dots, n)$ in (110) and (112). The operator $\boldsymbol{W}$ so defined is permutable with $\boldsymbol{K}$; therefore the two formulas (121) merge into one

$$\boldsymbol{Q} = \boldsymbol{W}e^{\boldsymbol{K}} \quad (\boldsymbol{W} = \boldsymbol{W}^{\mathsf{T}} = \boldsymbol{W}^{-1}; \quad \boldsymbol{K}^{\mathsf{T}} = -\boldsymbol{K}, \boldsymbol{W}\boldsymbol{K} = \boldsymbol{K}\boldsymbol{W}). \quad (122)$$

Let us now turn to the Cayley formulas, which establish a connection between orthogonal and skew-symmetric operators in a euclidean space. The formula

$$\boldsymbol{Q} = (\boldsymbol{E} - \boldsymbol{K})(\boldsymbol{E} + \boldsymbol{K})^{-1}, \quad (123)$$

as is easily verified, carries the skew-symmetric operator $\boldsymbol{K}$ into the orthogonal operator $\boldsymbol{Q}$. (123) enables us to express $\boldsymbol{K}$ in terms of $\boldsymbol{Q}$:

$$\boldsymbol{K} = (\boldsymbol{E} - \boldsymbol{Q})(\boldsymbol{E} + \boldsymbol{Q})^{-1}. \quad (124)$$

The formulas (123) and (124) establish a one-to-one correspondence between the skew-symmetric operators and those orthogonal operators that do not have the characteristic value -1. Instead of (123) and (124) we can take the formulas

$$\boldsymbol{Q} = -(\boldsymbol{E} - \boldsymbol{K})(\boldsymbol{E} + \boldsymbol{K})^{-1}, \quad (125)$$

$$\boldsymbol{K} = (\boldsymbol{E} + \boldsymbol{Q})(\boldsymbol{E} - \boldsymbol{Q})^{-1}. \quad (126)$$

In this case the number $+1$ plays the role of the exceptional value.

3. The polar decomposition of a real matrix in accordance with Theorem 9 enables us to obtain the fundamental formulas (107), (109), (111), and (113) without embedding the euclidean space in a unitary space, as was done above. This second approach to the fundamental formulas is based on the following theorem:

THEOREM 10: *If two real normal matrices are similar,*

$$B = T^{-1}AT \quad (AA^{\mathsf{T}} = A^{\mathsf{T}}A, BB^{\mathsf{T}} = B^{\mathsf{T}}B, A = \overline{A}, B = \overline{B}), \quad (127)$$

then they are real-similar and orthogonally-similar:

$$B = Q^{-1}AQ \quad (Q = \overline{Q} = Q^{\mathsf{T}-1}). \quad (128)$$

Proof: Since the normal matrices A and B have the same characteristic values, there exists a polynomial $g(\lambda)$ (see 2. on p. 272) such that

$$A^{\mathsf{T}} = g(A), B^{\mathsf{T}} = g(B).$$

Therefore the equation

$$g(B) = T^{-1}g(A)T,$$

which is a consequence of (127), can be written as follows:

$$B^{\mathsf{T}} = T^{-1}A^{\mathsf{T}}T. \quad (129)$$

When we go over to the transposed matrices in this equation, we obtain:

$$B = T^{\mathsf{T}}AT^{\mathsf{T}-1}. \quad (130)$$

A comparison of (127) with (130) shows that

$$TT^{\mathsf{T}}A = ATT^{\mathsf{T}}. \quad (131)$$

Now we make use of the polar decomposition of T:

$$T = SQ, \tag{132}$$

where $S = \sqrt{TT^{\mathsf{T}}} = h(TT^{\mathsf{T}})$ ($h(\lambda)$ a polynomial) is symmetric and Q is real and orthogonal. Since A, by (131), is permutable with TT^{T}, it is also permutable with $S = h(TT^{\mathsf{T}})$. Therefore, when we substitute the expression for T from (132) in (127), we have:

$$B = Q^{-1}S^{-1}ASQ = Q^{-1}AQ\,.$$

This completes the proof.

Let us consider the real canonical matrix

$$\left\{\left\|\begin{matrix} \mu_1 & \nu_1 \\ -\nu_1 & \mu_1 \end{matrix}\right\|, \ldots, \left\|\begin{matrix} \mu_q & \nu_q \\ -\nu_q & \mu_q \end{matrix}\right\|, \quad \mu_{2q+1}, \ldots, \mu_n\right\}. \tag{133}$$

The matrix (133) is normal and has the characteristic values $\mu_1 \pm i\nu_1, \ldots, \mu_q \pm i\nu_q, \mu_{2q+1}, \ldots, \mu_n$. Since normal matrices are of simple structure, every normal matrix having the same characteristic values is similar (and by Theorem 10 real-similar and orthogonally-similar) to the matrix (133). Thus we arrive at the formula (107).

The formulas (109), (111), and (113) are obtained in exactly the same way.

§ 15. Commuting Normal Operators

In § 10 we have shown that two commuting operators $\boldsymbol{A}$ and $\boldsymbol{B}$ in an n-dimensional unitary space $\boldsymbol{R}$ always have a common characteristic vector. By mathematical induction we can show that this statement is true not only for two, but for any finite number, of commuting operators. For given m pairwise commuting operators $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m$ the first $m-1$ of which have a common characteristic vector $\boldsymbol{x}$, by repeating verbatim the argument of Lemma 1 (p. 270) (for $\boldsymbol{A}$ we take any $\boldsymbol{A}_i$ $(i = 1, 2, \ldots, m-1)$ and for $\boldsymbol{B}$ we take $\boldsymbol{A}_m$), we obtain a vector $\boldsymbol{y}$ which is a common characteristic vector of $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m$.

This statement is even true for an infinite set of commuting operators, because such a set can only contain a finite number ($\leqq n^2$) of linearly independent operators, and a common characteristic value of the latter is a common characteristic value of all the operators of the given set.

2. Now suppose that an arbitrary finite or infinite set of pairwise commuting *normal* operators $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ is given. They all have a common characteristic vector $\boldsymbol{x}_1$. We denote by $\boldsymbol{T}_1$ the $(n-1)$-dimensional subspace consisting of all vectors of $\boldsymbol{R}$ that are orthogonal to $\boldsymbol{x}_1$. By § 10, 3. (p. 272), the subspace $\boldsymbol{T}_1$ is invariant with respect to $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$. Therefore all these operators have a common characteristic vector $\boldsymbol{x}_2$ in $\boldsymbol{T}_1$. We consider the orthogonal complement $\boldsymbol{T}_2$ of the plane $[\boldsymbol{x}_1, \boldsymbol{x}_2]$ and select in it a vector $\boldsymbol{x}_3$, etc. Thus we obtain an orthogonal system $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ of common characteristic vectors of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$. These vectors can be normalized. Hence we have proved:

THEOREM 11: *If a finite or infinite set of pairwise commuting normal operators $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ in a unitary space $\boldsymbol{R}$ is given, then all these operators have a complete orthonormal system of common characteristic vectors $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n$:*

$$\boldsymbol{A}\boldsymbol{z}_i = \lambda_i \boldsymbol{z}_i,\ \boldsymbol{B}\boldsymbol{z}_i = \lambda_i' \boldsymbol{z}_i,\ \boldsymbol{C}\boldsymbol{z}_i = \lambda_i'' \boldsymbol{z}_i, \ldots\ [(\boldsymbol{z}_i \boldsymbol{z}_k) = \delta_{ik};\ i, k = 1, 2, \ldots, n]. \tag{134}$$

In matrix form, this theorem reads as follows:

THEOREM 11′: *If a finite or infinite set of pairwise commuting normal matrices $A, B, C, \ldots$ is given, then all these matrices can be carried by one and the same unitary transformation into diagonal form, i.e., there exists a unitary matrix U such that*

$$\left.\begin{aligned} A = U\{\lambda_1, \ldots, \lambda_n\}U^{-1}, \quad B = U\{\lambda_1', \ldots, \lambda_n'\}U^{-1}, \\ C = U\{\lambda_1'', \ldots, \lambda_n''\}U^{-1}, \ldots \quad (U = U^{*-1}). \end{aligned}\right\} \tag{135}$$

Now suppose that commuting normal operators in a euclidean space $\boldsymbol{R}$ are given. We denote by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ the linearly independent ones among them (their number is finite). We embed $\boldsymbol{R}$ (under preservation of the metric) in a unitary space $\tilde{\boldsymbol{R}}$, as was done in § 13. Then by Theorem 11, the operators $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ have a complete orthonormal system of common characteristic vectors $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n$ in $\tilde{\boldsymbol{R}}$, i.e., (134) is satisfied.

We consider an arbitrary linear combination of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$:

$$\boldsymbol{P} = \alpha\boldsymbol{A} + \beta\boldsymbol{B} + \gamma\boldsymbol{C} + \cdots.$$

For arbitrary real values $\alpha, \beta, \gamma, \ldots$ $\boldsymbol{P}$ is a real ($\boldsymbol{P}\boldsymbol{R} \subset \boldsymbol{R}$) normal operator in $\tilde{\boldsymbol{R}}$ and

$$\left.\begin{aligned} \boldsymbol{P}\boldsymbol{z}_j = \Lambda_j \boldsymbol{z}_j, \quad \Lambda_j = \alpha\lambda_j + \beta\lambda_j' + \gamma\lambda_j'' + \cdots \\ [(\boldsymbol{z}_j \boldsymbol{z}_k) = \delta_{jk}; \quad j, k = 1, 2, \ldots, n]. \end{aligned}\right\} \tag{136}$$

The characteristic values Λ_j $(j = 1, 2, \ldots, n)$ of $\boldsymbol{P}$ are linear forms in $\alpha, \beta, \gamma, \ldots$. Since $\boldsymbol{P}$ is real, these forms can be split into pairs of complex conjugates and real ones; with a suitable numbering of the characteristic vectors, we have

$$A_{2k-1}=M_k+iN_k,\ A_{2k}=M_k-iN_k,\ A_l=M_l \qquad (137)$$
$$(k=1,\ 2,\ \ldots,\ q;\quad l=2q+1,\ \ldots,\ n),$$

where M_k, N_k, and M_l are real linear forms in $\alpha, \beta, \gamma, \ldots$.

We may assume that in (136) the corresponding vectors $\boldsymbol{z}_{2k-1}$ and $\boldsymbol{z}_{2k}$ are complex conjugates, and the $\boldsymbol{z}_l$ real:

$$\boldsymbol{z}_{2k-1}=\boldsymbol{x}_k+i\boldsymbol{y}_k,\ \boldsymbol{z}_{2k}=\boldsymbol{x}_k-i\boldsymbol{y}_k,\ \boldsymbol{z}_l=\boldsymbol{x}_l \qquad (138)$$
$$(k=1,\ 2,\ \ldots,\ q;\quad l=2q+1,\ \ldots,\ n).$$

But then, as is easy to see, the real vectors

$$\boldsymbol{x}_k,\ \boldsymbol{y}_k,\ \boldsymbol{x}_l \quad (k=1,\ 2,\ \ldots,\ q;\ l=2q+1,\ \ldots,\ n) \qquad (139)$$

form an orthonormal basis of $\boldsymbol{R}$. In this canonical basis we have:[41]

$$\begin{matrix}\boldsymbol{P}\boldsymbol{x}_k=M_k\boldsymbol{x}_k-N_k\boldsymbol{y}_k,\\ \boldsymbol{P}\boldsymbol{y}_k=N_k\boldsymbol{x}_k+M_k\boldsymbol{y}_k,\\ \boldsymbol{P}\boldsymbol{x}_l=M_l\boldsymbol{x}_l\end{matrix}\quad \begin{pmatrix}k=1,\ 2,\ \ldots,\ q,\\ l=2q+1,\ \ldots,\ n\end{pmatrix}. \qquad (140)$$

Since all the operators of the given set are obtained from $\boldsymbol{P}$ for special values of $\alpha, \beta, \gamma, \ldots$ the basis (139), which does not depend on these parameters, is a common canonical basis for all the operators. Thus we have proved:

THEOREM 12: *If an arbitrary set of commuting normal linear operators in a euclidean space* **R** *is given, then all these operators have a common orthonormal canonical basis* $\boldsymbol{x}_k$, $\boldsymbol{y}_k$, $\boldsymbol{x}_l$:

$$\left.\begin{matrix}\boldsymbol{A}\boldsymbol{x}_k=\mu_k\boldsymbol{x}_k-\nu_k\boldsymbol{y}_k, & \boldsymbol{B}\boldsymbol{x}_k=\mu'_k\boldsymbol{x}_k-\nu'_k\boldsymbol{y}_k,\ \ldots,\\ \boldsymbol{A}\boldsymbol{y}_k=\nu_k\boldsymbol{x}_k+\mu_k\boldsymbol{y}_k, & \boldsymbol{B}\boldsymbol{y}_k=\nu'_k\boldsymbol{x}_k+\mu'_k\boldsymbol{y}_k,\ \ldots,\\ \boldsymbol{A}\boldsymbol{x}_l=\mu_l\boldsymbol{x}_l; & \boldsymbol{B}\boldsymbol{x}_l=\mu'_l\boldsymbol{x}_l,\ \ldots.\end{matrix}\right\} \qquad (141)$$

We give the matrix form of Theorem 12:

THEOREM 12′: *Every set of commuting normal real matrices* $A, B, C, \ldots$ *can be carried by one and the same real orthogonal transformation* Q *into canonical form*

$$\left.\begin{matrix}A=Q\left\{\left\|\begin{matrix}\mu_1 & \nu_1\\ -\nu_1 & \mu_1\end{matrix}\right\|,\ \ldots,\ \left\|\begin{matrix}\mu_q & \nu_q\\ -\nu_q & \mu_q\end{matrix}\right\|,\ \mu_{2q+1},\ \ldots,\ \mu_n\right\}Q^{-1},\\ B=Q\left\{\left\|\begin{matrix}\mu'_1 & \nu'_1\\ -\nu'_1 & \mu'_1\end{matrix}\right\|,\ \ldots,\ \left\|\begin{matrix}\mu'_q & \nu'_q\\ -\nu'_q & \mu'_q\end{matrix}\right\|,\ \mu'_{2q+1},\ \ldots,\ \mu'_n\right\}Q^{-1},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{matrix}\right\} \qquad (142)$$

---

[41] The equation (140) follows from (136), (137), and (138).

Note. If one of the operators $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ (matrices $A, B, C, \ldots$)—say $\boldsymbol{A}$ (A)—is symmetric, then in the corresponding formulas (141) ((142)) all the ν are zero. In the case of skew-symmetry, all the μ are zero. In the case where $\boldsymbol{A}$ is an orthogonal operator (A an orthogonal matrix), we have $\mu_k=\cos\varphi_k$, $\nu_k=\sin\varphi_k$, $\mu_l=\pm 1$ ($k=1, 2, \ldots, q$; $l=2q+1, \ldots, n$).

CHAPTER X

QUADRATIC AND HERMITIAN FORMS

§ 1. Transformation of the Variables in a Quadratic Form

1. A *quadratic form* is a homogeneous polynomial of the second degree in n variables $x_1, x_2, \ldots, x_n$. A quadratic form always has a representation

$$\sum_{i,k=1}^{n} a_{ik}x_i x_k \qquad (a_{ik}=a_{ki};\ i,k=1,2,\ldots,n),$$

where $A = \| a_{ik} \|_1^n$ is a symmetric matrix.

If we denote the column matrix $(x_1, x_2, \ldots, x_n)$ by x and denote the quadratic form by

$$A(x,x)=\sum_{i,k=1}^{n} a_{ik}x_ix_k, \tag{1}$$

then we can write:[1]

$$A(x,x)=x^{\mathsf T}Ax. \tag{2}$$

If $A=\|a_{ik}\|_1^n$ is a real symmetric matrix, then the form (1) is called *real*. In this chapter we shall mainly be concerned with real quadratic forms.

The determinant $|A|=|a_{ik}|_1^n$ is called the *discriminant* of the quadratic form $A(x,x)$. The form is called *singular* if its discriminant is zero.

To every quadratic form there corresponds a *bilinear form*

$$A(x,y)=\sum_{i,k=1}^{n} a_{ik}x_iy_k. \tag{3}$$

or

$$A(x,y)=x^{\mathsf T}Ay \quad (x=(x_1,\ldots,x_n),\ y=(y_1,\ldots,y_n)). \tag{4}$$

If $x^1, x^2, \ldots, x^l, y^1, y^2, \ldots, y^m$ are column matrices and $c_1, c_2, \ldots, c_l$, $d_1, d_2, \ldots, d_m$ are scalars, then by the bilinearity of $A(x,y)$ (see (4)),

---

[1] The sign $^{\mathsf T}$ denotes transposition. In (2) the quadratic form is represented as a product of three matrices: the row $x^{\mathsf T}$, the square matrix A, and the column x.

$$A\Big(\sum_{i=1}^{l} c_ix^i,\ \sum_{j=1}^{m} d_jy^j\Big)=\sum_{i=1}^{l}\sum_{j=1}^{m} c_id_jA(x^i,y^j). \tag{5}$$

If $\boldsymbol{A}$ is an operator in an n-dimensional euclidean space and if in some orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ this symmetric operator corresponds to the matrix $A=\|a_{ik}\|_1^n$, then for arbitrary vectors

$$\boldsymbol{x}=\sum_{i=1}^{n} x_i\boldsymbol{e}_i,\qquad \boldsymbol{y}=\sum_{i=1}^{n} y_i\boldsymbol{e}_i$$

we have the identity[2]

$$A(x,y)=(\boldsymbol{Ax},\boldsymbol{y})=(\boldsymbol{x},\boldsymbol{Ay}).$$

In particular,

$$A(x,x)=(\boldsymbol{Ax},\boldsymbol{x})=(\boldsymbol{x},\boldsymbol{Ax}),$$

where

$$a_{ik}=(\boldsymbol{Ae}_i,\boldsymbol{e}_k)\qquad(i,k=1,2,\ldots,n).$$

2. Let us see how the coefficient matrix of the form changes under a transformation of the variables:

$$x_i=\sum_{k=1}^{n} t_{ik}\xi_k\ (i=1,2,\ldots,n). \tag{6}$$

In matrix notation, this transformation looks as follows:

$$x=T\xi. \tag{6'}$$

Here x, ξ are column matrices: $x=(x_1,x_2,\ldots,x_n)$ and $\xi=(\xi_1,\xi_2,\ldots,\xi_n)$; and T is the transforming matrix: $T=\|t_{ik}\|_1^n$.

Substituting the expression for x in (2), we obtain from (6'):

$$A(x,x)=\xi^{\mathsf T}T^{\mathsf T}AT\xi=\xi^{\mathsf T}\tilde{A}\xi=\tilde{A}(\xi,\xi),$$

where

$$\tilde{A}=T^{\mathsf T}AT. \tag{7}$$

The formula (7) expresses the coefficient matrix $\tilde{A}=\|\tilde{a}_{ik}\|_1^n$ of the transformed form $\tilde{A}(\xi,\xi)=\sum_{i,k=1}^{n}\tilde{a}_{ik}\xi_i\xi_k$ in terms of the coefficient matrix of the original form $A=\|a_{ik}\|_1^n$ and the transformation matrix $T=\|t_{ik}\|_1^n$.

It follows from (7) that under a transformation the discriminant of the form is multiplied by the square of the determinant of the transformation:

---

[2] In $A(x,y)$, the parentheses form part of the notation; in $(\boldsymbol{Ax},\boldsymbol{y})$ and $(\boldsymbol{x},\boldsymbol{Ay})$, they denote the scalar product.

$$|\tilde{A}|=|A|\,|T|^2. \tag{8}$$

In what follows we shall make use exclusively of non-singular transformations of the variables ($|T|\neq 0$). Under such transformations, as is clear from (7), the rank of the coefficient matrix remains unchanged (the rank of A is the same as that of $\tilde{A}$).[3] The rank of the coefficient matrix is usually called the *rank of the quadratic form*.

DEFINITION 1: *Two symmetric matrices A and $\tilde{A}$ connected as in formula* (7), *with* $|T|\neq 0$, *are called congruent.*

Thus, a whole class of congruent symmetric matrices is associated with every quadratic form. As mentioned above, all these matrices have one and the same rank, the rank of the form. The rank is an invariant for the given class of matrices. In the real case, a second invariant is the so-called 'signature' of the quadratic form. We shall now proceed to introduce this concept.

§ 2. Reduction of a Quadratic Form to a Sum of Squares. The Law of Inertia

1. A real quadratic form $A(x, x)$ can be represented in an infinite number of ways in the form

$$A(x, x)=\sum_{i=1}^{r} a_i X_i^2, \tag{9}$$

where $a_i\neq 0$ $(i=1, 2, \dots, r)$ and

$$X_i=\sum_{k=1}^{n} \alpha_{ki}x_k \qquad (i=1, 2, \dots, r)$$

are *linearly independent* real linear forms in the variables $x_1, x_2, \dots, x_n$ (so that $r\leqq n$).

Let us consider a non-singular transformation of the variables under which the first r of the new variables $\xi_1, \xi_2, \dots, \xi_n$ are connected with $x_1, x_2, \dots, x_n$ by the formulas[4]

$$\xi_i=X_i \qquad (i=1, 2, \dots, r)$$

Then, in the new variables,

---

[3] See p. 17.

[4] We obtain the necessary transformation by adjoining to the system of linear forms $X_1, \dots, X_r$ such linear forms $X_{r+1}, \dots, X_n$ that the forms X_j $(j=1, 2, \dots, n)$ are linearly independent and then setting $\xi_j=X_j$ $(j=1, 2, \dots, n)$.

$$A(x, x)=\tilde{A}(\xi, \xi)=\sum_{i=1}^{r} a_i\xi_i^2$$

and therefore $\tilde{A}=\{a_1, a_2, \dots, a_r, 0, \dots, 0\}$. But the rank of $\tilde{A}$ is r. Hence: *The number of squares in the representation* (9) *is always equal to the rank of the form.*

2. We shall show that not only is the total number of squares invariant in the various representations of $A(x, x)$ in the form (9), but also so is the number of positive[5] (and, hence, the number of negative) squares.

THEOREM 1 (The Law of Inertia for Quadratic Forms): *In a representation of a real quadratic form $A(x, x)$ as a sum of independent squares*[6]

$$A(x, x)=\sum_{i=1}^{r} a_i X_i^2, \tag{9}$$

the number of positive and the number of negative squares are independent of the choice of the representation.

Proof. Let us assume that we have, in addition to (9), another representation of $A(x, x)$ in the form of a sum of independent squares

$$A(x, x)=\sum_{i=1}^{r} b_i Y_i^2$$

and that

$$a_1>0, a_2>0, \dots, a_g>0, a_{g+1}<0, \dots, a_r<0,$$
$$b_1>0, b_2>0, \dots, b_h>0, b_{h+1}<0, \dots, b_r<0.$$

Suppose that $g\neq h$, say $g<h$. Then in the identity

$$\sum_{i=1}^{r} a_i X_i^2=\sum_{i=1}^{r} b_i Y_i^2 \tag{10}$$

we give to the variables $x_1, x_2, \dots, x_n$ values that satisfy the system of $r-(h-g)$ equations

$$X_1=0, \quad X_2=0, \ \dots, X_g=0, \quad Y_{h+1}=0, \ \dots \ , Y_r=0, \tag{11}$$

---

[5] By the number of positive (negative) squares in (9) we mean the number of positive (or negative) a_i.

[6] By a sum of independent squares we mean a sum of the form (9) in which all $a_i\neq 0$ and the forms $X_1, X_2, \dots, X_r$ are linearly independent.

and for which at least one of the forms $X_{g+1}, \ldots, X_r$ does not vanish.[7] For these values of the variables the left-hand side of the identity is

$$\sum_{j=g+1}^{r} a_j X_j^2 < 0,$$

and the right-hand side is

$$\sum_{k=1}^{h} b_k Y_k^2 \geqq 0.$$

Thus, the assumption $g \neq h$ has led to a contradiction, and the theorem is proved.

DEFINITION 2: *The difference σ between the number π of positive squares and the number ν of negative squares in the representation of $A(x, x)$ is called the signature of the form $A(x, x)$.* (Notation: $\sigma = \sigma[A(x, x)]$).

The rank r and the signature σ determine the numbers π and ν uniquely, since

$$r = \pi + \nu, \quad \sigma = \pi - \nu.$$

Note that in (9) the positive factor $\sqrt{|a_i|}$ can be absorbed into the form X_i $(i = 1, 2, \ldots, r)$. Then (9) assumes the form

$$A(x, x) = X_1^2 + X_2^2 + \cdots + X_\pi^2 - X_{\pi+1}^2 - \cdots - X_r^2. \tag{12}$$

Setting[8] $\xi_i = X_i$ $(i = 1, 2, \ldots, r)$, we reduce $A(x, x)$ to the canonical form

$$\tilde{A}(\xi, \xi) = \xi_1^2 + \xi_2^2 + \cdots + \xi_\pi^2 - \xi_{\pi+1}^2 - \cdots - \xi_r^2. \tag{13}$$

Hence we deduce from Theorem 1 that: *Every real symmetric matrix A is congruent to a diagonal matrix in which the diagonal elements are $+1$, -1, or 0:*

$$A = T^{\mathsf{T}} \{ \underbrace{+1, \ldots, +1}_{\pi}, \underbrace{-1, \ldots, -1}_{\nu}, 0, \ldots, 0 \} T. \tag{14}$$

In the next section we shall give a rule for determining the signature from the coefficients of the quadratic form.

---

[7] Such values exist, since otherwise the equations $X_{g+1} = 0, \ldots, X_r = 0$ and hence all the equations $X_1 = 0, X_2 = 0, \ldots, X_r = 0$ would be consequences of the $r - (h - g)$ equations (11). This is impossible, because the linear forms $X_1, X_2, \ldots, X_r$ are independent.

[8] See footnote 4.

§ 3. The Methods of Lagrange and Jacobi of Reducing a Quadratic Form to a Sum of Squares

It follows from the preceding section that in order to determine the rank and the signature of a form it is sufficient to reduce it in any way to a sum of independent squares.

We shall describe here two reduction methods: that of Lagrange and that of Jacobi.

1. *Lagrange's Method.* Let a quadratic form

$$A(x, x) = \sum_{i, k=1}^{n} a_{ik} x_i x_k$$

be given.

We consider two cases:

1) For some g $(1 \leqq g \leqq n)$ the diagonal coefficient a_{gg} is not equal to zero. Then we set

$$A(x, x) = \frac{1}{a_{gg}} \left(\sum_{k=1}^{n} a_{gk} x_k \right)^2 + A_1(x, x) \tag{15}$$

and convince ourselves by direct verification that the quadratic form $A_1(x, x)$ does not contain the variable x_g. This method of separating out a square form in a quadratic form is always applicable when there is a non-zero diagonal element in the matrix $A = \| a_{ik} \|_1^n$.

2) $a_{gg} = 0$ and $a_{hh} = 0$, but $a_{gh} \neq 0$. Then we set:

$$A(x, x) = \frac{1}{2a_{hg}} \left[\sum_{k=1}^{n} (a_{gk} + a_{hk}) x_k \right]^2 - \frac{1}{2a_{hg}} \left[\sum_{k=1}^{n} (a_{gk} - a_{hk}) x_k \right]^2 + A_2(x, x). \tag{16}$$

The forms

$$\sum_{k=1}^{n} a_{gk} x_k, \qquad \sum_{k=1}^{n} a_{hk} x_k \tag{17}$$

are linearly independent, since the first contains x_h but not x_g, and the second contains x_g but not x_h. Therefore, in (16), the forms within the brackets are linearly independent (as sum and difference, respectively, of the independent linear forms (17)).

Therefore we have separated out two independent squares in $A(x, x)$. Each of these squares contains x_g and x_h, whereas $A_2(x, x)$ does not contain these variables, as is easy to verify.

By successive application of a combination of the methods 1) and 2), we can always reduce the form $A(x, x)$ by means of rational operations to a sum of squares. Moreover, the squares so obtained are linearly independent, since at each stage the square that is separated out contains an unknown that does not occur in the subsequent squares.

Note that the basic formulas (15) and (16) can be written as follows

$$A(x, x) = \frac{1}{4a_{gg}}\left(\frac{\partial A}{\partial x_g}\right)^2 + A_1(x, x), \tag{15'}$$

$$A(x, x) = \frac{1}{8a_{gh}}\left[\left(\frac{\partial A}{\partial x_g} + \frac{\partial A}{\partial x_h}\right)^2 - \left(\frac{\partial A}{\partial x_g} - \frac{\partial A}{\partial x_h}\right)^2\right] + A_2(x, x). \tag{16'}$$

Example.

$$A(x, x) = 4x_1^2 + x_2^2 + x_3^2 + x_4^2 - 4x_1x_2 - 4x_1x_3 + 4x_1x_4 + 4x_2x_3 - 4x_2x_4.$$

We apply formula (15′) with $g = 1$:

$$\begin{aligned} A(x, x) &= \frac{1}{16}(8x_1 - 4x_2 - 4x_3 + 4x_4)^2 + A_1(x, x) \\ &= (2x_1 - x_2 - x_3 + x_4)^2 + A_1(x, x), \end{aligned}$$

where

$$A_1(x, x) = 2x_2x_3 - 2x_2x_4 + 2x_3x_4.$$

We apply formula (16′) with $g = 2$ and $h = 3$:

$$\begin{aligned} A_1(x, x) &= \frac{1}{8}(2x_2 + 2x_3)^2 - \frac{1}{8}(2x_3 - 2x_2 - 4x_4)^2 + A_2(x, x) \\ &= \frac{1}{2}(x_2 + x_3)^2 - \frac{1}{2}(x_3 - x_2 - 2x_4)^2 + A_2(x, x), \end{aligned}$$

where

$$A_2(x, x) = 2x_4^2.$$

Finally,

$$A(x, x) = (2x_1 - x_2 - x_3 + x_4)^2 + \frac{1}{2}(x_2 + x_3)^2 - \frac{1}{2}(x_3 - x_2 - 2x_4)^2 + 2x_4^2,$$

$$r = 4, \quad \sigma = 2.$$

2. *Jacobi's Method.* We denote the rank of $A(x, x) = \sum_{i,k=1}^{n} a_{ik}x_ix_k$ by r and assume that

$$D_k = A\begin{pmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{pmatrix} \neq 0 \qquad (k = 1, 2, \dots, r).$$

Then the symmetric matrix $A = \| a_{ik} \|_1^n$ can be reduced to the form

$$G = \begin{Vmatrix} g_{11} & g_{12} & \dots & \dots & g_{1n} \\ 0 & g_{22} & \dots & \dots & g_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots g_{rr} & \dots & g_{rn} \\ 0 & 0 & \dots 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots 0 & \dots & 0 \end{Vmatrix} \tag{18}$$

by Gauss's elimination algorithm (see Chapter II, § 1).

The elements of G are expressed in terms of the elements of A by the well-known formulas[9]

$$g_{pq} = \frac{A\begin{pmatrix} 1 & 2 & \dots & p-1 & p \\ 1 & 2 & \dots & p-1 & q \end{pmatrix}}{A\begin{pmatrix} 1 & 2 & \dots & p-1 \\ 1 & 2 & \dots & p-1 \end{pmatrix}} \quad (q = p, p+1, \dots, n;\ p = 1, 2, \dots, r). \tag{19}$$

In particular,

$$g_{pp} = \frac{D_p}{D_{p-1}} \qquad (p = 1, 2, \dots, r;\ D_0 = 1). \tag{20}$$

In Chapter II, § 4 (formula (55) on page 41) we have shown that

$$A = G^{\mathsf{T}}\hat{D}G, \tag{21}$$

where $\hat{D}$ is the diagonal matrix:

$$\hat{D} = \left\{\frac{1}{D_1}, \frac{D_1}{D_2}, \dots, \frac{D_{r-1}}{D_r}, 0, \dots, 0\right\} = \left\{\frac{1}{g_{11}}, \frac{1}{g_{22}}, \dots, \frac{1}{g_{rr}}, 0, \dots, 0\right\}. \tag{22}$$

Without infringing (21) we may replace some of the zeros in the last $n - r$ rows of G by arbitrary elements. By such a replacement we can make G into a non-singular upper triangular matrix

$$T = \begin{Vmatrix} g_{11} & g_{12} & \dots & \dots & g_{1n} \\ 0 & g_{22} & \dots & \dots & g_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 \dots & g_{rr} & \dots & g_{rn} \\ 0 & 0 \dots & 0 & * \dots & * \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \dots & * \end{Vmatrix} \qquad (|T| \neq 0). \tag{23}$$

---

[9] See Chapter II, § 2.

The equation (21) can then be rewritten:

$$A = T^{\mathsf{T}} \widehat{D} T. \tag{24}$$

From this equation it follows that the quadratic form[10]

$$\widehat{D}(\xi, \xi) = \sum_{k=1}^{r} \frac{D_{k-1}}{D_k} \xi_k^2 = \sum_{k=1}^{r} \frac{\xi_k^2}{g_{kk}}$$
$$(\xi = (\xi_1, \xi_2, \ldots, \xi_n); \quad D_0 = 1)$$

goes over into the form $A(x, x)$ under the transformation

$$\xi = Tx$$

Since

$$\xi_k = X_k, \quad X_k \equiv g_{kk} x_k + g_{k,k+1} x_{k+1} + \cdots + g_{kn} x_n \qquad (k = 1, \ldots, r), \tag{25}$$

we have *Jacobi's Formula*[11]

$$A(x, x) = \sum_{k=1}^{r} \frac{D_{k-1}}{D_k} X_k^2 = \sum_{k=1}^{r} \frac{X_k^2}{g_{kk}} \qquad (D_0 = 1). \tag{26}$$

This formula gives a representation of $A(x, x)$ in the form of a sum of r independent squares.[12]

Jacobi's formula is often given in another form.

Instead of X_k $(k = 1, 2, \ldots, r)$, the linearly independent forms

$$Y_k = D_{k-1} X_k \qquad (k = 1, 2, \ldots, r; \; D_0 = 1) \tag{27}$$

are introduced. Then Jacobi's formula (26) can be written as:

$$A(x, x) = \sum_{k=1}^{r} \frac{Y_k^2}{D_{k-1} D_k}. \tag{28}$$

Here

$$Y_k = c_{kk} x_k + c_{k,k+1} x_{k+1} + \cdots + c_{kn} x_n \qquad (k = 1, 2, \ldots, r) \tag{29}$$

where

---

[10] We regard $\widehat{D}(\xi, \xi)$ as a quadratic form in the n variables $\xi_1, \xi_2, \ldots, \xi_n$.

[11] Another approach to Jacobi's formula, which does not depend on (21), can be found, for example, in [17], pp. 43-44.

[12] The independence of the squares in Jacobi's formula follows from the fact that the form $A(x, x)$ is of rank r. But we can also convince ourselves directly of the independence of the forms $X_1, X_2, \ldots, X_r$. For, according to (20), $g_{kk} = \dfrac{D_k}{D_{k-1}} \neq 0$ and therefore X_k contains the variable x_k, which does not occur in the forms $X_{k+1}, \ldots, X_r$ $(k = 1, 2, 3, \ldots, r)$. Hence $X_1, X_2, \ldots, X_r$ are linearly independent forms.

$$c_{kq} = A\begin{pmatrix} 1 & 2 \ldots & k-1 & k \\ 1 & 2 \ldots & k-1 & q \end{pmatrix} \qquad (q = k, k+1, \ldots, n; \; k = 1, 2, \ldots, r). \tag{30}$$

Example.

$$A(x, x) = x_1^2 + 3x_2^2 - 3x_4^2 - 4x_1x_2 + 2x_1x_3 - 2x_1x_4 - 6x_2x_3 + 8x_2x_4 + 2x_3x_4.$$

We reduce the matrix

$$A = \begin{Vmatrix} 1 & -2 & 1 & -1 \\ -2 & 3 & -3 & 4 \\ 1 & -3 & 0 & 1 \\ -1 & 4 & 1 & -3 \end{Vmatrix}$$

to the Gaussian form

$$G = \begin{Vmatrix} 1 & -2 & 1 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Hence $r = 2$, $g_{11} = 1$, $g_{22} = -1$.

Jacobi's formula (26) yields:

$$A(x, x) = (x_1 - 2x_2 + x_3 - x_4)^2 - (-x_2 - x_3 + 2x_4)^2.$$

Jacobi's formula (28) yields the following theorem:

THEOREM 2 (Jacobi). *If for the quadratic form*

$$A(x, x) = \sum_{i,k=1}^{n} a_{ik} x_i x_k$$

of rank r the inequality

$$D_k = A\begin{pmatrix} 1 & 2 \ldots k \\ 1 & 2 \ldots k \end{pmatrix} \neq 0 \qquad (k = 1, 2, \ldots, r), \tag{31}$$

holds, then the number π of positive squares and the number ν of negative squares of $A(x, x)$ coincide, respectively, with the number P of permanences of sign and the number V of variations of sign in the sequence

$$1, D_1, D_2, \ldots, D_r, \tag{32}$$

i.e., $\pi = P(1, D_1, D_2, \ldots, D_r)$, $\nu = V(1, D_1, D_2, \ldots, D_r)$, *and the signature*

$$\sigma = r - 2V(1, D_1, D_2, \ldots, D_r). \tag{33}$$

Note 1. *If in the sequence* $1, D_1, \ldots, D_r \neq 0$ *there are zeros, but not three in succession, then the signature can be determined by the use of the formula*

$$\sigma = r - 2V(1, D_1, D_2, \ldots, D_r)$$

omitting the zero D_k *provided* $D_{k-1}D_{k+1} \neq 0$, *and setting*

$$V(D_{k-1}, D_k, D_{k+1}, D_{k+2}) = \begin{cases} 1, \text{ when } \dfrac{D_{k+2}}{D_{k-1}} < 0, \\ 2, \text{ when } \dfrac{D_{k+2}}{D_{k-1}} > 0 \end{cases} \tag{34}$$

if $D_k = D_{k+1} = 0$.

We state this rule without proof.[13]

Note 2. *When three consecutive zeros occur in* $D_1, D_2, \ldots, D_{r-1}$, *then the signature of the quadratic form cannot be immediately determined by Jacobi's Theorem.* In this case, the signs of the non-zero D_k do not determine the signature of the form. This is shown by the following example:

$$A(x, x) = 2a_1x_1x_4 + a_2x_2^2 + a_3x_3^2 \qquad (a_1a_2a_3 \neq 0).$$

Here

$$D_1 = D_2 = D_3 = 0, \quad D_4 = -a_1^2a_2a_3 \neq 0.$$

But

$$\nu = \begin{cases} 1, \text{ when } a_2 > 0, a_3 > 0, \\ 3, \text{ when } a_2 < 0, a_3 < 0. \end{cases}$$

In both cases, $D_4 < 0$.

Note 3. *If* $D_1 \neq 0, \ldots, D_{r-1} \neq 0$, *but* $D_r = 0$, *then the signs of* $D_1, D_2, \ldots, D_{r-1}$ *do not determine the signature of the form.* As a corroborating example, we can take the form

$$ax_1^2 + ax_2^2 + bx_3^2 + 2ax_1x_2 + 2ax_2x_3 + 2ax_1x_3 = a(x_1 + x_2 + x_3)^2 + (b - a)x_3^2.$$

§ 4. Positive Quadratic Forms

1. In this section we deal with the special, but important, class of positive quadratic forms.

DEFINITION 3: *A real quadratic form* $A(x, x) = \sum_{i,k=1}^{n} a_{ik}x_ix_k$ *is called positive (negative) semidefinite if for arbitrary real values of the variables:*

$$A(x, x) \geqq 0 \qquad (\leqq 0). \tag{35}$$

---

[13] This rule was found in the case of a single zero D_k by Gundenfinger and for two successive zeros D_k by Frobenius [162].

DEFINITION 4: *A real quadratic form* $A(x, x) = \sum_{i,k=1}^{n} a_{ik}x_ix_k$ *is called positive (negative) definite if for arbitrary values of the variables, not all zero,* $(x \neq o)$

$$A(x, x) > 0 \qquad (< 0). \tag{36}$$

The class of positive (negative) definite forms is part of the class of positive (negative) semidefinite forms.

Let $A(x, x)$ be a positive-semidefinite form. We represent it in the form of a sum of linearly independent squares:

$$A(x, x) = \sum_{i=1}^{r} a_iX_i^2, \tag{37}$$

In this representation, all the squares must be positive:

$$a_i > 0 \qquad (i = 1, 2, \ldots, r). \tag{38}$$

For if any a_i were negative, then we could select values of $x_1, x_2, \ldots, x_n$ for which

$$X_1 = \cdots = X_{i-1} = X_{i+1} = \cdots = X_r = 0, \quad X_i \neq 0.$$

But then $A(x, x)$ would have a negative value for these values of the variables, and by assumption this is impossible. It is clear that, conversely, it follows from (37) and (38) that the form $A(x, x)$ is positive semidefinite.

Thus, a positive semidefinite quadratic form is characterized by the equations $\sigma = r$ $(\pi = r, \nu = 0)$.

Now let $A(x, x)$ be a positive-definite form. Then $A(x, x)$ is also positive semidefinite. Therefore it is representable in the form (37), where all the a_i $(i = 1, 2, \ldots, r)$ are positive. From the positive definiteness it follows that $r = n$. For if $r < n$, we could find values of $x_1, x_2, \ldots, x_n$, not all zero, such that all the X_i would be zero. But then by (37) $A(x, x) = 0$ for $x \neq o$, and this contradicts (36).

It is easy to see that, conversely, if in (37) $r = n$ and all the $a_1, a_2, \ldots, a_n$ are positive, then $A(x, x)$ is a positive-definite form.

In other words: *A positive-semidefinite form is positive definite if and only if it is not singular.*

2. The following theorem gives a criterion for positive definiteness in the form of inequalities which the coefficients of the form must satisfy. We shall use the notation of the preceding section for the sequence of the principal minors of A:

$$D_1 = a_{11}, \quad D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \ldots, \quad D_n = \begin{vmatrix} a_{11} & a_{12} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \ldots & \ldots\ldots\ldots \\ a_{n1} & a_{n2} \ldots a_{nn} \end{vmatrix}.$$

THEOREM 3: *A quadratic form is positive definite if and only if*

$$D_1 > 0, D_2 > 0, \ldots, D_n > 0. \tag{39}$$

Proof. The sufficiency of the conditions (39) follows immediately from Jacobi's formula (28). The necessity of (39) is established as follows. From the fact that $A(x, x) = \sum_{i,k=1}^{n} a_{ik}x_ix_k$ is positive definite, it follows that the 'restricted' forms[14]

$$A_p(x, x) = \sum_{i,k=1}^{p} a_{ik}x_ix_k \qquad (p = 1, 2, \ldots, n)$$

are also positive definite. But then all these forms must be non singular, i.e.,

$$D_p = |A_p| \neq 0 \qquad (p = 1, 2, \ldots, n).$$

We are now in a position to apply Jacobi's formula (28) (for $r = n$). Since all the squares on the right-hand side of the formula must be positive, we have

$$D_1 > 0, D_1D_2 > 0, D_2D_3 > 0, \ldots, D_{n-1}D_n > 0.$$

Hence the inequality (39) follows, and the theorem is proved.

Since every principal minor of A can be brought into the top left corner by a suitable numbering of the variables, we have the

COROLLARY: *In a positive-definite quadratic form $A(x, x) = \sum_{i,k=1}^{n} a_{ik}x_ix_k$, all the principal minors of the coefficient matrix are positive:*[15]

$$A\begin{pmatrix} i_1 i_2 \ldots i_p \\ i_1 i_2 \ldots i_p \end{pmatrix} > 0 \quad (1 \leqq i_1 < i_2 < \cdots < i_p \leqq n;\ p = 1, 2, \ldots, n).$$

Note. If the successive principal minors are non-negative,

$$D_1 \geqq 0, D_2 \geqq 0, \ldots, D_n \geqq 0, \tag{40}$$

---

[14] The form $A_p(x, x)$ is obtained from $A(x, x)$ if we set in the latter

$$x_{p+1} = \cdots = x_n = 0 \quad (p = 1, 2, \ldots, n).$$

[15] Thus, when the successive principal minors of a real symmetric matrix are positive, all the remaining principal minors are then also positive.

it does *not* follow that $A(x, x)$ is positive semidefinite. For, the form

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

in which $a_{11} = a_{12} = 0$, $a_{22} < 0$ satisfies (40), but is not positive semidefinite.

However, we have the following theorem.

THEOREM 4: *A quadratic form $A(x, x) = \sum_{i,k=1}^{n} a_{ik}x_ix_k$ is positive semidefinite if and only if all the principal minors of its coefficient matrix are non-negative:*

$$A\begin{pmatrix} i_1 i_2 \ldots i_p \\ i_1 i_2 \ldots i_p \end{pmatrix} \geqq 0 \quad (1 \leqq i_1 < i_2 < \cdots < i_p \leqq n;\ p = 1, 2, \ldots, n). \tag{41}$$

Proof. We introduce the auxiliary form

$$A_\varepsilon(x, x) = A(x, x) + \varepsilon \sum_{i=1}^{n} x_i^2 \qquad (\varepsilon < 0).$$

Obviously $\lim_{\varepsilon \to 0} A_\varepsilon(x, x) = A(x, x)$.

The fact that $A(x, x)$ is positive semidefinite implies that $A_\varepsilon(x, x)$ is positive definite, so that we have the inequality (cf. Corollary to Theorem 3):

$$A_\varepsilon\begin{pmatrix} i_1 i_2 \ldots i_p \\ i_1 i_2 \ldots i_p \end{pmatrix} > 0 \quad (1 \leqq i_1 < i_2 < \cdots < i_p \leqq n;\ p = 1, 2, \ldots, n).$$

Proceeding to the limit for $\varepsilon \to 0$, we obtain (41).

Suppose, conversely, that (41) holds. Then we have

$$A_\varepsilon\begin{pmatrix} i_1 i_2 \ldots i_p \\ i_1 i_2 \ldots i_p \end{pmatrix} = \varepsilon^p + \cdots \geqq \varepsilon^p > 0 \quad (1 \leqq i_1 < i_2 < \cdots < i_p \leqq n;\ p = 1, 2, \ldots, n).$$

But then (by Theorem 3), $A_\varepsilon(x, x)$ is positive definite

$$A_\varepsilon(x, x) > 0 \quad (x \neq o).$$

Proceeding to the limit for $\varepsilon \to 0$ we obtain:

$$A(x, x) \geqq 0.$$

This completes the proof.

The conditions for a form to be negative semidefinite and negative definite are obtained from (39) and (41), respectively, when these inequalities are applied to $-A(x, x)$.

THEOREM 5: *A quadratic form $A(x, x)$ is negative definite if and only if the following inequalities hold:*

$$D_1 < 0, D_2 > 0, D_3 < 0, \ldots, (-1)^n D_n > 0. \tag{42}$$

THEOREM 6: *A quadratic form $A(x, x)$ is negative semidefinite if and only if the following inequalities hold:*

$$(-1)^p A\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ i_1 & i_2 & \cdots & i_p \end{pmatrix} \geqq 0 \quad (1 \leqq i_1 < i_2 < \cdots < i_p \leqq n;\ p = 1, 2, \ldots, n). \tag{43}$$

§ 5. Reduction of a Quadratic Form to Principal Axes

1. We consider an arbitrary real quadratic form

$$A(x, x) = \sum_{i, k=1}^{n} a_{ik} x_i x_k .$$

Its coefficient matrix $A = \| a_{ik} \|_1^n$ is real and symmetric. Therefore (see Chapter IX, § 13) it is orthogonally similar to a real diagonal matrix Λ, i.e., there exists a real orthogonal matrix Q such that

$$\Lambda = Q^{-1} A Q \quad (\Lambda = \| \lambda_i \delta_{ik} \|_1^n, \quad QQ^{\top} = E). \tag{44}$$

Here $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic values of A.

Since for an orthogonal matrix $Q^{-1} = Q^{\top}$, it follows from (43) that under the orthogonal transformation of the variables

$$x = Q\xi \qquad (QQ^{\top} = E) \tag{45}$$

or, in greater detail,

$$x_i = \sum_{k=1}^{n} q_{ik} \xi_k \quad \left(\sum_{j=1}^{n} q_{ij} q_{kj} = \delta_{ik};\ i, k = 1, 2, \ldots, n \right), \tag{45'}$$

the form $A(x, x)$ goes over into

$$\Lambda(\xi, \xi) = \sum_{i=1}^{n} \lambda_i \xi_i^2 . \tag{46}$$

THEOREM 7: *Every real quadratic form $A(x, x) = \sum_{i, k=1}^{n} a_{ik} x_i x_k$ can be reduced to the canonical form (46) by an orthogonal transformation, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic values of $A = \| a_{ik} \|_1^n$.*

The reduction of the quadratic form $A(x, x)$ to the canonical form (46) is called *reduction to principal axes.* The reason for this name is that the equation of a central hypersurface of the second order

$$\sum_{i, k=1}^{n} a_{ik} x_i x_k = c \ (c = \text{const.} \neq 0) \tag{47}$$

under the orthogonal transformation (45′) of the variables assumes the canonical form

$$\sum_{i=1}^{n} \varepsilon_i \frac{\xi_i^2}{a_i^2} = 1 \quad \left(\frac{\varepsilon_i}{a_i^2} = \frac{\lambda_i}{c}; \quad \varepsilon_i = \pm 1; \quad i = 1, 2, \ldots, n \right). \tag{48}$$

If we regard $x_1, x_2, \ldots, x_n$ as coordinates in an orthonormal basis in an n-dimensional euclidean space, then $\xi_1, \xi_2, \ldots, \xi_n$ are the coordinates in a new orthonormal basis of the same space, and the 'rotation'[16] of the axes is brought about by the orthogonal transformation (45). The new coordinate axes are axes of symmetry of the central surface (47) and are usually called its *principal axes.*

2. It follows from (46) that *the rank r of $A(x, x)$ is equal to the number of non-zero characteristic values of A and the signature σ is equal to the difference between the number of positive and the number of negative characteristic values of A.*

Hence, in particular, we have the following proposition:

If under a continuous change of the coefficients of a quadratic form the rank remains unchanged, then the signature also remains unchanged.

Here we have started from the fact that a continuous change of the coefficients produces a continuous change of the characteristic values. The signature can only change when some characteristic value changes sign. But then at some intermediate stage this characteristic value must pass through zero, and this results in a change of the rank of the form.

---

[16] If $|Q| = -1$, then (45) is a combination of a rotation with a reflection (see p. 287). However, the reduction to principal axes can always be effected by a proper orthogonal matrix ($|Q| = 1$). This follows from the fact that, without changing the canonical form, we can perform the additional transformation

$$\xi_i = \xi_i^{\top} \quad (i = 1, 2, \ldots, n-1), \quad \xi_n = -\xi_n^{\top}.$$

§ 6. Pencils of Quadratic Forms

1. In the theory of small oscillations it is necessary to consider simultaneously two quadratic forms one of which gives the potential, and the other the kinetic energy, of the system. The second form is always positive definite.

The study of a system of two such forms is the object of this section.

Two real quadratic forms

$$A(x, x) = \sum_{i, k=1}^{n} a_{ik} x_i x_k \quad \text{and} \quad B(x, x) = \sum_{i, k=1}^{n} b_{ik} x_i x_k$$

determine the *pencil* of forms $A(x, x) - \lambda B(x, x)$ (λ is a parameter).

If the form $B(x, x)$ is positive definite, the pencil $A(x, x) - \lambda B(x, x)$ is then called *regular*.

The equation

$$| A - \lambda B | = 0$$

is called the *characteristic equation of the pencil of forms* $A(x,x) - \lambda B(x,x)$.

We denote by λ_0 some root of this equation. Since the matrix $A - \lambda_0 B$ is singular, there exists a column $z = (z_1, z_2, \dots, z_n) \neq o$ such that $(A - \lambda_0 B) z = o$, or

$$Az = \lambda_0 Bz \quad (z \neq o).$$

The number λ_0 will be called a *characteristic value of the pencil* $A(x, x) - \lambda B(x, x)$ and z a corresponding *principal column* or '*principal vector*' of the pencil. The following theorem holds:

THEOREM 8: *The characteristic equation*

$$| A - \lambda B | = 0$$

of a regular pencil of forms $A(x, x) - \lambda B(x, x)$ *always has n real roots* λ_k *with the corresponding principal vectors* $z^k = (z_{1k}, z_{2k}, \dots, z_{nk})$ $(k = 1, 2, \dots, n)$:

$$Az^k = \lambda_k B z^k \qquad (k = 1, 2, \dots, n). \tag{49}$$

These principal vectors z^k *can be chosen such that the relations*

$$B(z^i, z^k) = \delta_{ik} \qquad (i, k = 1, 2, \dots, n) \tag{50}$$

are satisfied.

Proof. We observe that (49) can be written as:

$$B^{-1} A z^k = \lambda_k z^k \ (k = 1, 2, \dots, n). \tag{51}$$

Thus, our theorem states that the matrix

$$D = B^{-1} A \tag{52}$$

1. has simple structure, 2. has real characteristic values, and 3. has characteristic columns (vectors) $z^1, z^2, \dots, z^n$ corresponding to these characteristic values and satisfying the relations (50).[17]

In order to prove these three statements, we introduce an n-dimensional vector space $\boldsymbol{R}$ over the field of real numbers. In this space we fix a basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ and introduce a scalar product of two arbitrary vectors

$$\boldsymbol{x} = \sum_{i=1}^{n} x_i \boldsymbol{e}_i, \qquad \boldsymbol{y} = \sum_{i=1}^{n} y_i \boldsymbol{e}_i$$

by means of the positive-definite bilinear form $B(x, y)$:

$$(\boldsymbol{xy}) = B(x, y) = \sum_{i, k=1}^{n} b_{ik} x_i y_k = x^{\mathsf{T}} B y \tag{53}$$

and hence the square of the length of a vector $\boldsymbol{x}$ by means of the form $B(x, x)$:

$$(\boldsymbol{xx}) = B(x, x) = x^{\mathsf{T}} B x\,, \tag{53'}$$

where x and y are columns $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$.

It is easy to verify that the metric so introduced satisfies the postulates 1.-5. (p. 243) and is, therefore, euclidean.

We have obtained an n-dimensional euclidean space $\boldsymbol{R}$, but the original basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ is, in general, not orthonormal. To the matrices A, B, and $D = B^{-1} A$ there correspond in this basis linear operators in $\boldsymbol{R}$: $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{D} = \boldsymbol{B}^{-1} \boldsymbol{A}$.[18]

---

[17] If D were a symmetric matrix, then the properties 1. and 2. would follow immediately from properties of a symmetric operator (Chapter IX, p. 284). However, D, as a product of two symmetric matrices, is not necessarily itself symmetric, since $D = B^{-1} A$ and $D^{\mathsf{T}} = AB^{-1}$.

[18] Since the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ is not orthonormal, the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ to which, in this basis, the symmetric matrices A and B correspond, are not necessarily symmetric themselves.

We shall show that $\boldsymbol{D}$ is a symmetric operator in $\boldsymbol{R}$ (see Chapter IX, § 13).[19] Indeed, for arbitrary vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with the coordinate columns $x=(x_1, x_2, \ldots, x_n)$ and $y=(y_1, y_2, \ldots, y_n)$ we have, by (52) and (53),

$$(\boldsymbol{Dx},\ \boldsymbol{y}) = (Dx)^{\mathsf{T}}By = x^{\mathsf{T}}D^{\mathsf{T}}By = x^{\mathsf{T}}AB^{-1}By = x^{\mathsf{T}}Ay$$

and

$$(\boldsymbol{x},\ \boldsymbol{Dy}) = x^{\mathsf{T}}BDy = x^{\mathsf{T}}BB^{-1}Ay = x^{\mathsf{T}}Ay,$$

i.e.,

$$(\boldsymbol{Dx},\ \boldsymbol{y}) = (\boldsymbol{x},\ \boldsymbol{Dy}).$$

The symmetric operator $\boldsymbol{D}=\boldsymbol{B}^{-1}\boldsymbol{A}$ has real characteristic values $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ and a complete orthonormal system of characteristic vectors $\boldsymbol{z}^1, \boldsymbol{z}^2, \boldsymbol{z}^3, \ldots, \boldsymbol{z}^n$ (see p. 284, Chapter IX):

$$\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{z}^k = \lambda_k\boldsymbol{z}^k \qquad (k=1,\ 2,\ \ldots,\ n), \tag{54}$$

$$(\boldsymbol{z}^i\boldsymbol{z}^k) = \delta_{ik} \qquad (i,\ k=1,\ 2,\ \ldots,\ n). \tag{54'}$$

Let $z^k = (z_{1k}, z_{2k}, \ldots, z_{nk})$ be the coordinate column of $\boldsymbol{z}^k$ $(k=1, 2, \ldots, n)$ in the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Then the equations (54) can be written in the form (51) or (49) and the relations (54′), by (53), yield the equation (50).

This completes the proof.

Note that it follows from (50) that the columns $z^1, z^2, \ldots, z^n$ are linearly independent. For suppose that

$$\sum_{k=1}^{n} c_k z^k = 0. \tag{55}$$

Then for every i $(1 \leqq i \leqq n)$, by (50),

$$0 = B\left(z^i, \sum_{k=1}^{n} c_k z^k\right) = \sum_{k=1}^{n} c_k B(z^i,\ z^k) = c_i.$$

Then all the c_i $(i=1, 2, \ldots, n)$ in (55) are zero and there is no linear dependence among the columns $z^1, z^2, \ldots, z^n$.

A square matrix formed from principal columns $z^1, z^2, \ldots, z^n$ satisfying the relations (50)

$$Z = (z^1,\ z^2,\ \ldots,\ z^n) = \| z_{ik} \|_1^n$$

will be called a *principal matrix* for the pencil of forms $A(x, x) - \lambda B(x, x)$.

---

[19] Hence D is similar to some symmetric matrix.

The principal matrix Z is non-singular ($|Z| \neq 0$), because its columns are linearly independent.

The equation (50) can be written as follows:

$${z^i}^{\mathsf{T}}Bz^k = \delta_{ik} \qquad (i,\ k=1,\ 2,\ \ldots,\ n). \tag{56}$$

Moreover, when we multiply both sides of (49) on the left by the row matrix ${z^i}^{\mathsf{T}}$, we obtain:

$${z^i}^{\mathsf{T}}Az^k = \lambda_k {z^i}^{\mathsf{T}}Bz^k = \lambda_k\delta_{ik} \qquad (i,\ k=1,\ 2,\ \ldots,\ n). \tag{57}$$

By introducing the principal matrix $Z=(z^1, z^2, \ldots, z^n)$, we can represent (56) and (57) in the form

$$Z^{\mathsf{T}}AZ = \| \lambda_k\delta_{ik} \|_1^n, \qquad Z^{\mathsf{T}}BZ = E. \tag{58}$$

The formulas (58) show that the non-singular transformation

$$x = Z\xi \tag{59}$$

reduces the quadratic forms $A(x, x)$ and $B(x, x)$ simultaneously to sums of squares:

$$\sum_{k=1}^{n} \lambda_k \xi_k^2 \ \text{ and } \ \sum_{k=1}^{n} \xi_k^2. \tag{60}$$

This property of (59) characterizes a principal matrix Z. For suppose that the transformation (59) reduces the forms $A(x, x)$ and $B(x, x)$ simultaneously to the canonical forms (60). Then (58) holds, and hence (56) and (57) holds for Z. (58) implies that Z is non-singular ($|Z| \neq 0$). We rewrite (57) as follows:

$${z^i}^{\mathsf{T}}(Az^k - \lambda_k Bz^k) = o \qquad (i=1,\ 2,\ \ldots,\ n), \tag{61}$$

where k has an arbitrary fixed value $(1 \leqq k \leqq n)$. The system of equations (61) can be contracted into the single equation

$$Z^{\mathsf{T}}(Az_k - \lambda_k Bz^k) = O;$$

hence, since Z^{T} is non-singular,

$$Az^k - \lambda_k Bz^k = O;$$

i.e., for every k (49) holds. Therefore Z is a principal matrix. Thus we have proved the following theorem:

THEOREM 9: *If $Z = \| z_{ik} \|_1^n$ is a principal matrix of a regular pencil of forms $A(x, x) - \lambda B(x, x)$, then the transformation*

$$x = Z\xi \tag{62}$$

reduces the forms $A(x, x)$ and $B(x, x)$ simultaneously to sums of squares

$$\sum_{k=1}^{n} \lambda_k \xi_k^2, \quad \sum_{k=1}^{n} \xi_k^2, \tag{63}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic values of the pencil $A(x, x) - \lambda B(x, x)$ corresponding to the columns $z^1, z^2, \ldots, z^n$ of Z.

Conversely, if some transformation (62) simultaneously reduces $A(x, x)$ and $B(x, x)$ to the form (63), then $Z = \| z_{ik} \|_1^n$ is a principal matrix of the regular pencil of forms $A(x, x) - \lambda B(x, x)$.

Sometimes the characteristic property of the transformation (62) formulated in Theorem 9 is used for the construction of a principal matrix and the proof of Theorem 8.[20] For this purpose, we first of all carry out a transformation of variables $x = Ty$ that reduces the form $B(x, x)$ to the 'unit' sum of squares $\sum_{k=1}^{n} y_k^2$ (which is always possible, since $B(x, x)$ is positive definite). Then $A(x, x)$ is carried into a certain form $A_1(y, y)$. Now the form $A_1(y, y)$ is reduced to the form $\sum_{k=1}^{n} \lambda_k \xi_k^2$ by an orthogonal transformation $y = Q\xi$ (reduction to principal axes!). Then, obviously,[21] $\sum_{k=1}^{n} y_k^2 = \sum_{k=1}^{n} \xi_k^2$. Thus the transformation $x = Z\xi$, where $Z = TQ$, reduces the two given forms to (63). Afterwards it turns out (as we have shown on p. 313) that the columns $z^1, z^2, \ldots, z^n$ of Z satisfy the relations (49) and (50).

In the special case where $B(x, x)$ is the unit form, i.e., $B(x, x) = \sum_{k=1}^{n} x_k^2$, so that $B = E$, the characteristic equation of the pencil $A(x, x) - \lambda B(x, x)$ coincides with the characteristic equation of A, and the principal vectors of the pencil are characteristic vectors of A. In this case the relations (50) can be written as follows:

$$z^{i\mathsf{T}} z^k = \delta_{ik} \ (i, k = 1, 2, \ldots, n)$$

and they express the orthonormality of the columns $z^1, z^2, \ldots, z^n$.

---

[20] See [17], pp. 56-57.

[21] An orthogonal transformation does not alter a sum of squares of the variables, because $(Qx)^{\mathsf{T}} Qx = x^{\mathsf{T}} x$.

2. Theorems 8 and 9 admit of an intuitive geometric interpretation. We introduce a euclidean space $\boldsymbol{R}$ with the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ and the fundamental metric form $B(x, x)$ just as was done for the proof of Theorem 8. In $\boldsymbol{R}$ we consider a central hypersurface of the second order whose equation is

$$A(x, x) = \sum_{i, k=1}^{n} a_{ik} x_i x_k = c. \tag{64}$$

After the coordinate transformation $x = Z\xi$, where $Z = \| z_{ik} \|_1^n$ is a principal matrix of the pencil $A(x, x) - \lambda B(x, x)$, the new basis vectors are the vectors $z^1, z^2, \ldots, z^n$ whose coordinates in the old basis form the columns of Z, i.e., the principal vectors of the pencil. These vectors form an orthonormal basis in which the equation of the hypersurface (64) has the form

$$\sum_{k=1}^{n} \lambda_k \xi_k^2 = c. \tag{65}$$

Therefore the principal vectors $z^1, z^2, \ldots, z^n$ of the pencil coincide in direction with the principal axes of the hypersurface (64), and the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the pencil determine the lengths of the semi-axes:

$$\lambda_k = \pm \frac{c}{a_k^2} \ (k = 1, 2, \ldots, n).$$

Thus, the task of determining the characteristic values and the principal vectors of a regular pencil of forms $A(x, x) - \lambda B(x, x)$ is equivalent to the task of reducing the equation (64) of a central hypersurface of the second order to principal axes, provided the equation of the hypersurface is given in a general skew coordinate system[22] in which the 'unit sphere' has the equation $B(x, x) = 1$.

Example. Given the equation of a surface of the second order

$$2x^2 - 2y^2 - 3z^2 - 10yz + 2xz - 4 = 0 \tag{66}$$

in a general skew coordinate system in which the equation of the unit sphere is

$$2x^2 + 3y^2 + 2z^2 + 2xz = 1, \tag{67}$$

it is required to reduce equation (66) to principal axes.

In this case

$$A = \begin{Vmatrix} 2 & 0 & 1 \\ 0 & -2 & -5 \\ 1 & -5 & -3 \end{Vmatrix}, \qquad B = \begin{Vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{Vmatrix}.$$

---

[22] I.e., a skew coordinate system with distinct units of lengths along the axes.

The characteristic equation of the pencil $| A - \lambda B | = 0$ has the form

$$\begin{vmatrix} 2-2\lambda & 0 & 1-\lambda \\ 0 & -2-3\lambda & -5 \\ 1-\lambda & -5 & -3-2\lambda \end{vmatrix} = 0. \tag{68}$$

This equation has three roots: $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = -4$.

We denote the coordinates of a principal vector corresponding to the characteristic value 1 by u, v, w. The values of u, v, w are determined from the system of homogeneous equations whose coefficients are the elements of the determinant (68) for $\lambda = 1$:

$$\begin{aligned} 0 \cdot u + 0 \cdot v + 0 \cdot w &= 0, \\ 0 \cdot u - 5v \quad - 5w \quad &= 0, \\ 0 \cdot u - 5v \quad - 5w \quad &= 0. \end{aligned}$$

In fact we have only one relation

$$v + w = 0.$$

To the characteristic value $\lambda = 1$ there must correspond two orthonormal principal vectors. The coordinates of the first can be chosen arbitrarily, provided they satisfy the relation $v + w = 0$.

We set

$$u = 0,\ v,\ w = -v.$$

We take the coordinates of the second principal vector in the form

$$u',\ v',\ w' = -v'$$

and write down the condition for orthogonality $(B(z^1, z^2) = 0)$:

$$2uu' + 3vv' + 2ww' + uw' + u'w = 0.$$

Hence we find: $u' = 5v'$. Thus, the coordinates of the second principal vector are

$$u' = 5v',\ v',\ w' = -v'.$$

Similarly, by setting $\lambda = -4$ in the characteristic determinant, we find for the corresponding principal vector:

$$u'',\ v'' = -u'',\ w'' = -2u''.$$

The values of v, v', and u'' are determined from the condition that the coordinates of a principal vector must satisfy the equation of the unit sphere $(B(x, x) = 1)$, i.e., (67). Hence we find:

$$v = \frac{1}{\sqrt{5}}, \qquad v' = \frac{1}{3\sqrt{5}}, \qquad u'' = -\frac{1}{3}.$$

Therefore the principal matrix has the form

$$Z = \left\| \begin{matrix} 0 & \frac{\sqrt{5}}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{1}{3\sqrt{5}} & \frac{1}{3} \\ -\frac{1}{\sqrt{5}} & -\frac{1}{3\sqrt{5}} & \frac{2}{3} \end{matrix} \right\|,$$

and the corresponding coordinate transformation $(x = Z\xi)$ reduces the equations (66) and (67) to the canonical form

$$\xi_1^2 + \xi_2^2 - 4\xi_3^2 - 4 = 0, \quad \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$$

The first equation can also be written as follows:

$$\frac{\xi_1^2}{4} + \frac{\xi_2^2}{4} - \frac{\xi_3^2}{1} = 1.$$

This is the equation of a one-sheet hyperboloid of rotation with real semi-axes equal to 2, and an imaginary one equal to 1. The coordinates of the endpoint of the axis of rotation is determined by the third column of Z, i.e., $-1/3, 1/3, 2/3$. The coordinates of the endpoints of the other two orthogonal axes are given by the first and second columns.

§ 7. Extremal Properties of the Characteristic Values of a Regular Pencil of Forms[23]

1. Suppose that two quadratic forms are given

$$A(x, x) = \sum_{i, k=1}^{n} a_{ik} x_i x_k \quad \text{and} \quad B(x, x) = \sum_{i, k=1}^{n} b_{ik} x_i x_k,$$

of which $B(x, x)$ is positive definite. We number the characteristic values of the regular pencil of forms $A(x, x) - \lambda B(x, x)$ in non-descending order:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n. \tag{69}$$

---

[23] In the exposition of this section, we follow the book [17], § 10.

The principal vectors[24] corresponding to these characteristic values are denoted, as before, by $z^1, z^2, \ldots, z^n$:

$$z^k = (z_{1k}, z_{2k}, \ldots, z_{nk}) \quad (k = 1, 2, \ldots, n).$$

Let us determine the least value (minimum) of the ratio of the forms $\frac{A(x,x)}{B(x,x)}$ considering all possible values of the variables, not all equal to zero $(x \neq o)$. For this purpose it is convenient to go over to new variabes $\xi_1, \xi_2, \ldots, \xi_n$ by means of the transformation

$$x = Z\xi \quad (x_i = \sum_{k=1}^{n} z_{ik}\xi_k;\ i = 1, 2, \ldots, n),$$

where $Z = \| z_{ik} \|_1^n$ is a principal matrix of the pencil $A(x,x) - \lambda B(x,x)$. In the new variables the ratio of the forms is represented (see (63)) by

$$\frac{A(x,x)}{B(x,x)} = \frac{\lambda_1\xi_1^2 + \lambda_2\xi_2^2 + \cdots + \lambda_n\xi_n^2}{\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2}. \tag{70}$$

On the real axis we take the n points $\lambda_1, \lambda_2, \ldots, \lambda_n$. We ascribe to these points non-negative masses $m_1 = \xi_1^2, m_2 = \xi_2^2, \ldots, m_n = \xi_n^2$, respectively. Then, by (70), the quotient $\frac{A(x,x)}{B(x,x)}$ is the coordinate of the center of these masses. Therefore

$$\lambda_1 \leqq \frac{A(x,x)}{B(x,x)} \leqq \lambda_n.$$

Let us, for the time being, ignore the second part of the inequality and investigate when the equality sign holds in the first part. For this purpose, we group together the equal characteristic values in (69):

$$\lambda_1 = \cdots = \lambda_{p_1} < \lambda_{p_1+1} = \cdots = \lambda_{p_1+p_2} < \cdots. \tag{71}$$

The center of mass can coincide with the least value λ_1 only if all the masses are zero except at this point, i.e., when

$$\xi_{p_1+1} = \cdots = \xi_n = 0.$$

In this case the corresponding x is a linear combination of the principal columns $z^1, z^2, \ldots, z^{p_1}$.[25] Therefore all these columns correspond to the characteristic value λ_1, so that x is also a principal column (vector) for $\lambda = \lambda_1$.

---

[24] Here we use the term 'principal vector' in the sense of a principal column of the pencil (see p. 310). Throughout this section, having the geometric interpretation in mind, we often call a column, a vector.

[25] From $x = Z\xi$ it follows that $x = \sum_{k=1}^{n} \xi_k z^k$.

We have proved:

THEOREM 10: *The smallest characteristic value of the regular pencil* $A(x,x) - \lambda B(x,x)$ *is the minimum of the ratio of the forms* $A(x,x)$ *and* $B(x,x)$

$$\lambda_1 = \min \frac{A(x,x)}{B(x,x)}, \tag{72}$$

and this minimum is only assumed for principal vectors of the characteristic value λ_1.

2. In order to give an analogous 'minimal' characteristic for the next characteristic value λ_2, we restrict ourselves to all the vectors orthogonal to z^1, i.e., to those that satisfy the equation[26]

$$B(z^1, x) = 0.$$

For these vectors,

$$\frac{A(x,x)}{B(x,x)} = \frac{\lambda_2\xi_2^2 + \cdots + \lambda_n\xi_n^2}{\xi_2^2 + \cdots + \xi_n^2}$$

and therefore

$$\min \frac{A(x,x)}{B(x,x)} = \lambda_2 \ (B(z^1, x) = 0).$$

Here the equality sign holds only for those vectors orthogonal to z^1 that are principal vectors for the characteristic value λ_2.

Proceeding to the subsequent characteristic values, we eventually obtain the following theorem:

THEOREM 11: *For every* p $(1 \leqq p \leqq n)$ *the* p*-th characteristic value* λ_p *in* (69) *is the minimum of the ratio of the forms*

$$\lambda_p = \min \frac{A(x,x)}{B(x,x)}, \tag{73}$$

provided that the variable vector x *is orthogonal to the first* $p - 1$ *orthonormal principal vectors* $z^1. z^2, \ldots, z^{p-1}$:

---

[26] Here, and in what follows, we shall mean by the orthogonality of two vectors (columns) x, y that the equation $B(x,y) = 0$ holds. This is in complete agreement with the geometric interpretation given in the preceding section. We shall regard the quantities $x_1, x_2, \ldots, x_n$ as the coordinates of a vector x in some basis of a euclidean space in which the square of the length (the norm) is given by the positive-definite form $B(x,x) = \sum_{i,k=1}^{n} b_{ik}x_ix_k$. In this metric the vectors $z^1, z^2, \ldots, z^n$ form an orthonormal basis. Therefore, if the vector $x = \sum_{k=1}^{n} \xi_k z^k$ is orthogonal to one of the z^k, then the corresponding $\xi_k = 0$.

$$B(z^1, x) = 0, \ \ldots, \ B(z^{p-1}, x) = 0. \tag{74}$$

Moreover, the minimum is assumed only for those vectors that satisfy the condition (74) *and are at the same time principal vectors for the characteristic value* λ_p.

3. The characterization of λ_p given in Theorem 11 has the disadvantage that it is connected with the preceding principal vectors $z^1, z^2, \ldots, z^{p-1}$ and can therefore be used only when these vectors are known. Moreover, there is a certain arbitrariness in the choice of these vectors.

In order to give a characterization of λ_p $(p = 1, 2, \ldots, n)$ free from these defects, we introduce the concept of *constraint* imposed on the variables $x_1, x_2, \ldots, x_n$.

Suppose that linear forms in the variables $x_1, x_2, \ldots, x_n$ are given:

$$L_k(x) = l_{1k}x_1 + l_{2k}x_2 + \cdots + l_{nk}x_n \quad (k = 1, 2, \ldots, h). \tag{74'}$$

We shall say that the variables $x_1, x_2, \ldots, x_n$ or (what is the same) the vector x is subject to h constraints $L_1, L_2, \ldots, L_h$ if only such values of the variables are considered that satisfy the system of equations

$$L_k(x) = 0 \quad (k = 1, 2, \ldots, h). \tag{74''}$$

Preserving the notation (74′) for arbitrary linear forms we introduce a specialized notation for the 'scalar product' of x with the principal vectors $z^1, z^2, \ldots, z^n$:

$$\widetilde{L}_k(x) = B(z^k, x) \quad (k = 1, 2, \ldots, n).^{27} \tag{75}$$

Furthermore, when the variable vector is subject to the constraints (74″) we shall denote $\min \frac{A(x, x)}{B(x, x)}$ as follows:

$$\mu\left(\frac{A}{B};\ L_1, L_2, \ \ldots, \ L_h\right).$$

In this notation, (73) is written as follows:

$$\lambda_p = \mu\left(\frac{A}{B};\widetilde{L}_1, \ \widetilde{L}_2, \ \ldots, \ \widetilde{L}_{p-1}\right) \quad (p = 1, 2, \ \ldots, n). \tag{76}$$

We consider the constraints

$$L_1(x) = 0, \ \ldots, \ L_{p-1}(x) = 0 \tag{77}$$

and

$$\widetilde{L}_{p+1}(x) = 0, \ \ldots, \ \widetilde{L}_n(x) = 0. \tag{78}$$

---

[27] $\widetilde{L}_k(x) = z^{k\mathsf{T}}Bx = \widetilde{l}_{1k}x_1 + \widetilde{l}_{2k}x_2 + \cdots + \widetilde{l}_{nk}x_n$, where $\widetilde{l}_{1k}, \widetilde{l}_{2k}, \ldots, \widetilde{l}_{nk}$ are the elements of the row matrix $z^{k\mathsf{T}}B$ $(k = 1, 2, \ldots, n)$.

Since the number of constraints (77) and (78) is less than n, there exists a vector $x^{(1)} \neq o$ satisfying all these constraints. Since the constraints (78) express the orthogonality of x to the principal vectors $z^{p+1}, \ldots, z^n$, the corresponding coordinates of $x^{(1)}$ are $\xi_{p+1} = \cdots = \xi_n = 0$. Therefore, by (70),

$$\frac{A(x^{(1)}, x^{(1)})}{B(x^{(1)}, x^{(1)})} = \frac{\lambda_1\xi_1^2 + \cdots + \lambda_p\xi_p^2}{\xi_1^2 + \cdots + \xi_p^2} \leqq \lambda_p.$$

But then

$$\mu\left(\frac{A}{B};\ L_1, L_2, \ \ldots, \ L_{p-1}\right) \leqq \frac{A(x^{(1)}, x^{(1)})}{B(x^{(1)}, x^{(1)})} \leqq \lambda_p.$$

This inequality in conjunction with (76) shows that for variable constraints $L_1, L_2, \ldots, L_{p-1}$ the value of μ remains less than or equal to λ_p and becomes λ_p if the specialized constraints $\widetilde{L}_1, \widetilde{L}_2, \ldots, \widetilde{L}_{p-1}$ are taken.

Thus we have proved:

THEOREM 12: *If we consider the minimum of the ratio of the two forms* $\frac{A(x, x)}{B(x, x)}$ *for* $p - 1$ *arbitrary, but variable, constraints* $L_1, L_2, \ldots, L_{p-1}$, *then the maximum of these minima is equal to* λ_p:

$$\lambda_p = \max \mu\left(\frac{A}{B};\ L_1, L_2, \ \ldots, \ L_{p-1}\right) \quad (p = 1, \ldots, n). \tag{79}$$

Theorem 12 gives a 'maximal-minimal' characterization of $\lambda_1, \lambda_2, \ldots, \lambda_n$ in contrast to the 'minimal' characterization which we discussed in Theorem 11.

4. Note that when in the pencil $A(x, x) - \lambda B(x, x)$ the form $A(x, x)$ is replaced by $-A(x, x)$, all the characteristic values of the pencil change sign, but the corresponding principal vectors remain unchanged. Thus, the characteristic values of the pencil $-A(x, x) - \lambda B(x, x)$ are

$$-\lambda_n \leqq -\lambda_{n-1} \leqq \cdots \leqq -\lambda_1.$$

Moreover, by using the notation

$$\nu\left(\frac{A}{B};\ L_1, L_2, \ \ldots, \ L_h\right) = \max \frac{A(x, x)}{B(x, x)} \tag{80}$$

when the variable vector is subject to the constraints $L_1, L_2, \ldots, L_h$, we can write:

$$\mu\left(-\frac{A}{B};\ L_1, L_2, \ \ldots, \ L_h\right) = -\nu\left(\frac{A}{B};\ L_1, L_2, \ \ldots, \ L_h\right)$$

and

$$\max \mu\left(-\frac{A}{B};\ L_1, L_2, \ \ldots, \ L_h\right) = -\min \nu\left(\frac{A}{B};\ L_1, L_2, \ \ldots, \ L_h\right).$$

Therefore, by applying Theorems 10, 11, and 12 to the ratio $-\frac{A(x, x)}{B(x, x)}$, we obtain instead of (72), (76), and (79) the formulas

$$\lambda_n = \max \frac{A(x,x)}{B(x,x)},$$

$$\left.\begin{aligned} \lambda_{n-p+1} &= \nu\left(\frac{A}{B}; \tilde{L}_n, \tilde{L}_{n-1}, \ldots, \tilde{L}_{n-p+2}\right) \\ \lambda_{n-p+1} &= \min \nu\left(\frac{A}{B}; L_1, L_2, \ldots, L_{p-1}\right), \end{aligned}\right\} \quad (p = 2, \ldots, n).$$

These formulas establish the 'maximal' and the 'minimal-maximal' properties, respectively, of $\lambda_1, \lambda_2, \ldots, \lambda_n$, which we formulate in the following theorem:

THEOREM 13: *Suppose that to the characteristic values*

$$\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$$

of the regular pencil of forms $A(x,x) - \lambda B(x,x)$ there correspond the linearly independent principal vectors of the pencil $z^1, z^2, \ldots, z^n$. Then:

1) *The largest characteristic value λ_n is the maximum of the ratio of the forms $\frac{A(x,x)}{B(x,x)}$:*

$$\lambda_n = \max \frac{A(x,x)}{B(x,x)}, \tag{81}$$

and this maximum is assumed only for principal vectors of the pencil corresponding to the characteristic value λ_n.

2) *The characteristic value p-th from the end λ_{n-p+1} $(2 \leqq p \leqq n)$ is the maximum of the same ratio of the forms*

$$\lambda_{n-p+1} = \max \frac{A(x,x)}{B(x,x)} \tag{82}$$

provided that the variable vector x is subject to the constraints:[28]

$$B(z^n, x) = 0,\ B(z^{n-1}, x) = 0, \ \ldots\ , B(z^{n-p+2}, x) = 0, \tag{83}$$

i.e.,

$$\lambda_{n-p+1} = \nu\left(\frac{A}{B}; \tilde{L}_n, \tilde{L}_{n-1}, \ldots, \tilde{L}_{n-p+2}\right); \tag{84}$$

this maximum is assumed only for principal vectors of the pencil corresponding to the characteristic value λ_{n-p+1} and satisfying the constraints (83).

---

[28] In a euclidean space with a metric form $B(x,x)$, the condition (83) expresses the fact that the vector x is orthogonal to the principal vectors $z^{n-p+2}, \ldots, z^n$. See footnote 26.

3) *If in the maximum of the ratio of the forms $\frac{A(x,x)}{B(x,x)}$ with the constraints*

$$L_1(x) = 0, \ \ldots\ , L_{p-1}(x) = 0 \qquad (2 \leqq p \leqq n)$$

$(2 \leqq p \leqq n)$ *the constraints are varied, then the least value* (*minimum*) *of this maximum is equal to λ_{n-p+1}*:

$$\lambda_{n-p+1} = \min \nu\left(\frac{A}{B}; L_1, L_2, \ldots, L_{p-1}\right). \tag{85}$$

5. Let

$$L_1^0(x) = 0,\ L_2^0(x) = 0, \ \ldots,\ L_h^0(x) = 0. \tag{86}$$

be h *independent* constraints.[29] Then we can express h of the variables $x_1, x_2, \ldots, x_n$ by the remaining variables, which we denote by $v_1, v_2, \ldots, v_{n-h}$. Therefore, when the constraints (86) are imposed, the regular pencil of forms $A(x,x) - \lambda B(x,x)$ goes over into the pencil $A^0(v,v) - \lambda B^0(v,v)$, where $B^0(v,v)$ is again a positive-definite form (only in $n - h$ variables). The regular pencil so obtained has $n - h$ real characteristic values

$$\lambda_1^0 \leqq \lambda_2^0 \leqq \cdots \leqq \lambda_{n-h}^0. \tag{87}$$

Subject to the constraints (86) we can express all the variables in terms of $n - h$ independent ones $v_1, v_2, \ldots, v_{n-h}$ in various ways. However, the characteristic values (87) are independent of this arbitrariness and have completely definite values. This follows, for example, from the maximal-minimal property of the characteristic values

$$\lambda_1^0 = \min \frac{A^0(v,v)}{B^0(v,v)} = \mu\left(\frac{A}{B}; L_1^0, L_2^0, \ldots, L_h^0\right) \tag{88}$$

and, in general,

$$\begin{aligned} \lambda_p^0 &= \max \mu\left(\frac{A^0}{B^0}; L_1, L_2, \ldots, L_{p-1}\right) \\ &= \max \mu\left(\frac{A}{B}; L_1^0, \ldots, L_h^0, L_1, \ldots, L_{p-1}\right), \end{aligned} \tag{89}$$

where in (89) only the constraints $L_1, L_2, \ldots, L_{p-1}$ are allowed to vary.

---

[29] The constraints (86) are independent when the linear forms $L_1^0(x), L_2^0(x), \ldots, L_h^0(x)$ on the left-hand sides of (86) are independent.

The following theorem holds:

THEOREM 14: *If* $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$ *are the characteristic values of the regular pencil of forms* $A(x,x) - \lambda B(x,x)$ *and* $\lambda_1^0 \leqq \lambda_2^0 \leqq \cdots \leqq \lambda_{n-h}^0$ *are the characteristic values of the same pencil subject to h independent constraints, then*

$$\lambda_p \leqq \lambda_p^0 \leqq \lambda_{p+h} \quad (p=1,2,\ldots,n-h). \qquad (90)$$

Proof. The inequality $\lambda_p \leq \lambda_p^0 \, (p=1,2,\ldots,n-h)$ follows easily from (79) and (89). For when new constraints are added, the value of the minimum $\mu\left(\frac{A}{B}; L_1,\ldots,L_{p-1}\right)$ increases or remains the same. Therefore

$$\mu\left(\frac{A}{B}; L_1,\ldots,L_{p-1}\right) \leqq \mu\left(\frac{A}{B}; L_1^0,\ldots,L_h^0; L_1,\ldots,L_{p-1}\right).$$

Hence

$$\lambda_p = \max \mu\left(\frac{A}{B}; L_1,\ldots,L_{p-1}\right) \leq \lambda_p^0 = \max \mu\left(\frac{A}{B}; L_1^0,\ldots,L_h^0, L_1,\ldots,L_{p-1}\right).$$

The second part of the inequality (90) holds in view of the relations

$$\begin{aligned}\lambda_p^0 &= \max \mu\left(\frac{A}{B}; L_1^0,\ldots,L_h^0; L_1,\ldots,L_{p-1}\right)\\ &\leq \max \mu\left(\frac{A}{B}; L_1,\ldots,L_{p-1}, L_p,\ldots,L_{p+h-1}\right) = \lambda_{p+h}.\end{aligned}$$

Here not only are $L_1,\ldots,L_{p-1}$ varied, on the right-hand side, but $L_p,\ldots,L_{p+h-1}$ also; on the left-hand side the latter are replaced by the fixed constraints $L_1^0, L_2^0, \ldots, L_h^0$.

This completes the proof.

6. Suppose that two regular pencils of forms

$$A(x,x) - \lambda B(x,x), \quad \tilde{A}(x,x) - \lambda \tilde{B}(x,x) \qquad (91)$$

are given and that for every $x \neq o$,

$$\frac{A(x,x)}{B(x,x)} \leq \frac{\tilde{A}(x,x)}{\tilde{B}(x,x)}.$$

Then obviously,

$$\max \mu\left(\frac{A}{B}; L_1, L_2,\ldots,L_{p-1}\right) \leqq \max \mu\left(\frac{\tilde{A}}{\tilde{B}}; L_1, L_2,\ldots,L_{p-1}\right)$$
$$(p=1,2,\ldots,n).$$

Therefore, if we denote by $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$ and $\tilde{\lambda}_1 \leqq \tilde{\lambda}_2 \leqq \ldots \leqq \tilde{\lambda}_n$, respectively, the characteristic values of the pencils (91), then we have:

$$\lambda_p \leqq \tilde{\lambda}_p \quad (p=1,2,\ldots,n).$$

Thus, we have proved the following theorem:

THEOREM 15: *If two regular pencils of forms* $A(x,x) - \lambda B(x,x)$ *and* $\tilde{A}(x,x) - \lambda \tilde{B}(x,x)$ *with the characteristic values* $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$ *and* $\tilde{\lambda}_1 \leqq \tilde{\lambda}_2 \leqq \ldots \leqq \tilde{\lambda}_n$ *are given, then the identical relation*

$$\frac{A(x,x)}{B(x,x)} \leqq \frac{\tilde{A}(x,x)}{\tilde{B}(x,x)} \qquad (92)$$

implies that

$$\lambda_p \leqq \tilde{\lambda}_p \quad (p=1,2,\ldots,n). \qquad (93)$$

Let us consider the special case where, in (92), $B(x,x) \equiv \tilde{B}(x,x)$. In this case, the difference $\tilde{A}(x,x) - A(x,x)$ is a positive-semidefinite quadratic form and can therefore be expressed as a sum of independent positive squares:

$$\tilde{A}(x,x) = A(x,x) + \sum_{i=1}^{r} [X_i(x)]^2.$$

Then, when the r independent constraints

$$X_1(x) = 0, \; X_2(x) = 0, \; \ldots, \; X_r(x) = 0$$

are imposed, the forms $A(x,x)$ and $\tilde{A}(x,x)$ coincide, and the pencils $A(x,x) - \lambda B(x,x)$ and $\tilde{A}(x,x) - \lambda B(x,x)$ have the same characteristic values

$$\lambda_1^0 \leq \lambda_2^0 \leqq \cdots \leqq \lambda_{n-r}^0.$$

Applying Theorem 14 to both pencils $A(x,x) - \lambda B(x,x)$ and $\tilde{A}(x,x) - \lambda B(x,x)$, we have:

$$\tilde{\lambda}_p \leq \lambda_p^0 \leq \lambda_{p+r} \quad (p=1,2,\ldots,n-r).$$

In conjunction with the inequality (93), this leads to the following theorem:

THEOREM 16: *If* $\lambda_1 \leqq \lambda_2 \leqq \dots \leqq \lambda_n$ *and* $\tilde{\lambda}_1 \leqq \tilde{\lambda}_2 \leqq \dots \leqq \tilde{\lambda}_n$ *are the characteristic values of two regular pencils of forms* $A(x,x) - \lambda B(x,x)$ *and* $\tilde{A}(x,x) - \lambda B(x,x)$, *where*

$$\tilde{A}(x,x) = A(x,x) + \sum_{i=1}^{r} [X_i(x)]^2,$$

and $X_i(x)$ $(i = 1, 2, \dots, r)$ *are independent linear forms, then the following inequalities hold:*[30]

$$\lambda_p \leqq \tilde{\lambda}_p \leqq \lambda_{p+r} \quad (p = 1, 2, \dots, n). \tag{94}$$

In exactly the same way the following theorem is proved:

THEOREM 17: *If* $\lambda_1 \leqq \lambda_2 \leqq \dots \leqq \lambda_n$ *and* $\tilde{\lambda}_1 \leqq \tilde{\lambda}_2 \leqq \dots \leqq \tilde{\lambda}_n$ *are the characteristic values of the regular pencil of forms* $A(x,x) - \lambda B(x,x)$ *and* $A(x,x) - \lambda\tilde{B}(x,x)$, *where the form* $\tilde{B}(x,x)$ *is obtained from* $B(x,x)$ *by adding r positive squares, then the following inequalities hold:*[31]

$$\lambda_{p-r} \leqq \tilde{\lambda}_p \leqq \lambda_p \quad (p = 1, 2, \dots, n). \tag{95}$$

Note. In Theorems 16 and 17 we can claim that for some p we have, respectively $\lambda_p < \tilde{\lambda}_p$ and $\tilde{\lambda}_p < \lambda_p$, provided of course that $r \neq 0$.[32]

§ 8. Small Oscillations of a System with *n* Degrees of Freedom

The results of the two preceding sections have important applications in the theory of small oscillations of a mechanical system with n degrees of freedom.

1. We consider the free oscillations of a conservative mechanical system with n degrees of freedom near a stable position of equilibrium. We shall give the deviation of the system from the position of equilibrium by means of independent generalized coordinates $q_1, q_2, \dots, q_n$. The position of equilibrium itself corresponds to zero values of these coordinates: $q_1 = 0$, $q_2 = 0, \dots, q_n = 0$. Then the kinetic energy of the system is represented as a quadratic form in the generalized velocities $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n$:[33]

$$T = \sum_{i,k=1}^{n} b_{ik}(q_1, q_2, \dots, q_n)\, \dot{q}_i \dot{q}_k .$$

---

[30] The second parts of these inequalities hold for $p \leqq n - r$ only.

[31] The first parts of the inequalities hold for $p > r$.

[32] See [17], pp. 71-73.

[33] A dot denotes the derivative with respect to time.

Expanding the coefficients $b_{ik}(q_1, q_2, \dots, q_n)$ as power series in $q_1, q_2, \dots, q_n$

$$b_{ik}(q_1, q_2, \dots, q_n) = b_{ik} + \cdots \quad (i, k = 1, 2, \dots, n)$$

and keeping only the constant terms b_{ik}, since the deviations $q_1, q_2, \dots, q_n$ are small, we then have:

$$T = \sum_{i,k=1}^{n} b_{ik} \dot{q}_i \dot{q}_k \quad (b_{ik} = b_{ki};\ i, k = 1, 2, \dots, n).$$

The kinetic energy is always positive, and is zero only for zero velocities $\dot{q}_1 = \dot{q}_2 = \dots = \dot{q}_n = 0$. Therefore $\sum_{i,k=1}^{n} b_{ik}\dot{q}_i\dot{q}_k$ is a positive-definite form.

The potential energy of the system is a function of the coordinates: $P(q_1, q_2, \dots, q_n)$. Without loss of generality, we can take

$$P_0 = P(0, 0, \dots, 0) = 0.$$

Then, expanding the potential energy as a power series in $q_1, q_2, \dots, q_n$, we obtain:

$$P = \sum_{i=1}^{n} a_i q_i + \sum_{i,k=1}^{n} a_{ik} q_i q_k + \cdots .$$

Since in a position of equilibrium the potential energy always has a stationary value, we have

$$a_i = \left.\frac{\partial P}{\partial q_i}\right|_0 = 0 \quad (i = 1, 2, \dots, n).$$

Keeping only the terms of the second order in $q_1, q_2, \dots, q_n$, we have

$$P = \sum_{i,k=1}^{n} a_{ik} q_i q_k \quad (a_{ik} = a_{ki};\ i, k = 1, 2, \dots, n).$$

Thus, the potential energy P and the kinetic energy T are determined by two quadratic forms:

$$P = \sum_{i,k=1}^{n} a_{ik} q_i q_k , \quad T = \sum_{i,k=1}^{n} b_{ik} \dot{q}_i \dot{q}_k , \tag{96}$$

the second of which is positive definite.

We now write down the differential equations of motion in the form of Lagrange's equations of the second kind:[34]

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial P}{\partial q_i} \quad (i = 1, 2, \dots, n). \tag{97}$$

---

[34] See, for example, G. K. Suslow (Suslov), *Theoretische Mechanik*, § 191.

When we substitute for T and P their expressions from (96), we obtain:

$$\sum_{k=1}^{n} b_{ik}\ddot{q}_k + \sum_{k=1}^{n} a_{ik}q_k = 0 \qquad (i = 1, 2, \ldots, n). \tag{98}$$

We introduce the real symmetric matrices

$$A = \| a_{ik} \|_1^n \text{ and } B = \| b_{ik} \|_1^n$$

and the column matrix $q = (q_1, q_2, \ldots, q_n)$ and write the system of equations (98) in the following matrix form:

$$B\ddot{q} + Aq = o. \tag{98'}$$

We shall seek solutions of (98) in the form of harmonic oscillations

$$q_1 = v_1 \sin(\omega t + \alpha),\ q_2 = v_2 \sin(\omega t + \alpha), \ldots, q_n = v_n \sin(\omega t + \alpha),$$

in matrix notation:

$$q = v \sin(\omega t + \alpha). \tag{99}$$

Here $v = (v_1, v_2, \ldots, v_n)$ is the constant-amplitude column (constant-amplitude 'vector'), ω is the frequency, and α is the initial phase of the oscillation.

Substituting the expression (99) for q in (98′) and cancelling $\sin(\omega t + \alpha)$, we obtain:

$$Av = \lambda Bv \qquad (\lambda = \omega^2).$$

But this equation is the same as (49). Therefore the required amplitude vector is a principal vector, and the square of the frequency $\lambda = \omega^2$ is the corresponding characteristic value of the regular pencil of forms $A(x, x) - \lambda B(x, x)$.

We subject the potential energy to an additional restriction by postulating that the function $P(q_1, q_2, \ldots, q_n)$ in a position of equilibrium shall have a strict minimum.[35]

Then, by a theorem of Dirichlet,[36] the position of equilibrium is stable. On the other hand, our assumption means that the quadratic form $P = A(q, q)$ is also positive definite.

By Theorem 8, the regular pencil of forms $A(x, x) - \lambda B(x, x)$ has real characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ and n corresponding principal characteristic vectors $v^1, v^2, \ldots, v_n$ ($v^k = (v_{1k}, v_{2k}, \ldots, v_{nk})$; $k = 1, 2, \ldots, n$) satisfying the condition

---

[35] I.e., that the value of P_0 in the position of equilibrium is less than all other values of the function in some neighborhood of the position of equilibrium.

[36] See G. K. Suslow (Suslov), *Theoretische Mechanik*, § 210.

$$B(v^i, v^k) = \sum_{\mu, \nu = 1}^{n} b_{\mu\nu} v_{\mu i} v_{\nu k} = \delta_{ik} \qquad (i, k = 1, 2, \ldots, n). \tag{100}$$

From the fact that $A(x, x)$ is positive definite it follows that all the characteristic values of the pencil $A(x, x) - \lambda B(x, x)$ are positive:[37]

$$\lambda_k > 0 \qquad (k = 1, 2, \ldots, n).$$

But then there exist n harmonic oscillations[38]

$$v^k \sin(\omega_k t + \alpha_k) \qquad (\omega_k^2 = \lambda_k,\ k = 1, 2, \ldots, n), \tag{101}$$

whose amplitude vectors $v^k = (v_{1k}, v_{2k}, \ldots, v_{nk})$ $(k = 1, 2, \ldots, n)$ satisfy the conditions of 'orthonormality' (100).

Since the equation (98′) is linear, every oscillation can be obtained by a superposition of the harmonic oscillations (101):

$$q = \sum_{k=1}^{n} A_k \sin(\omega_k t + \alpha_k) v^k, \tag{102}$$

where A_k and α_k are arbitrary constants. For, whatever the values of these constants, the expression (102) is a solution of (98′). On the other hand, the arbitrary constants can be made to satisfy the following initial conditions:

$$q|_{t=0} = q_0, \quad \dot{q}|_{t=0} = q_0.$$

For from (102) we find:

$$q_0 = \sum_{k=1}^{n} A_k \sin\alpha_k v^k, \quad \dot{q}_0 = \sum_{k=1}^{n} \omega_k A_k \cos\alpha_k v^k. \tag{103}$$

Since the principal columns $v^1, v^2, \ldots, v^n$ are always linearly independent, the values $A_k \sin\alpha_k$ and $\omega_k \cos\alpha_k$ $(k = 1, 2, \ldots, n)$, and hence the constants A_k and α_k $(k = 1, 2, \ldots, n)$, are uniquely determined from (103).

The solution (102) of our system of differential equations can be written more conveniently:

$$q_i = \sum_{k=1}^{n} A_k \sin(\omega_k t + \alpha_k) v_{ik}. \tag{104}$$

Note that we could also derive the formulas (102) and (104) starting from Theorem 9. For let us consider a non-singular transformation of the

---

[37] This follows, for example, from the representation (63).

[38] Here the initial phases α_k $(k = 1, 2, \ldots, n)$ are arbitrary constants.

variables with the matrix $V = \| v_{ik} \|_1^n$ that reduces the two forms $A(x, x)$ and $B(x, x)$ simultaneously to the canonical form (63). Setting

$$q_i = \sum_{k=1}^{n} v_{ik}\theta_k \qquad (i = 1, 2, \ldots, n) \qquad (105)$$

or, more briefly,

$$q = V\theta \quad (\theta = (\theta_1, \theta_2, \ldots, \theta_n)) \qquad (106)$$

and observing that $\dot{q} = V\dot{\theta}$, we have:

$$P = A(q, q) = \sum_{i=1}^{n} \lambda_k \theta_k^2, \quad T = B(\dot{q}, \dot{q}) = \sum_{k=1}^{n} \dot{\theta}_k^2. \qquad (107)$$

The coordinates $\theta_1, \theta_2, \ldots, \theta_n$ in which the potential and kinetic energies have a representation as in (107) are called *principal coordinates.*

We now make use of Lagrange's equations of the second kind (98) and substitute the expressions (107) for P and T. We obtain:

$$\ddot{\theta}_k + \lambda_k \theta_k = 0 \qquad (k = 1, 2, \ldots, n). \qquad (108)$$

Since $A(q, q)$ is positive definite, all the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive and can be represented in the form

$$\lambda_k = \omega_k^2 \qquad (\omega_k > 0;\ k = 1, 2, \ldots, n). \qquad (109)$$

From (108) and (109), we find:

$$\theta_k = A_k \sin(\omega_k t + \alpha_k) \qquad (k = 1, 2, \ldots, n). \qquad (110)$$

When we substitute these expressions for θ_k in (105), we again obtain the formulas (104) and therefore (102). The values v_{ik} $(i, k = 1, 2, \ldots, n)$ in both methods are the same, because the matrix $V = \| v_{ik} \|_1^n$ in (106) is, by Theorem 9, a principal matrix of the regular pencil of forms $A(x, x) - \lambda B(x, x)$.

2. We also mention a mechanical interpretation of Theorems 14 and 15.

We number the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ of the given mechanical system in non-descending order:

$$0 < \omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_n .$$

The disposition of the corresponding characteristic values $\lambda_k = \omega_k^2$ $(k = 1, 2, 3, \ldots, n)$ of the pencil $A(x, x) - \lambda B(x, x)$ is then also determined:

$$\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n .$$

We impose h independent finite stationary constraints[39] on the given system. Since the deviations $q_1, q_2, \ldots, q_n$ are supposed to be small, these connections can be assumed to be linear in $q_1, q_2, \ldots, q_n$:

$$L_1(q) = 0,\ L_2(q) = 0,\ \ldots,\ L_h(q) = 0.$$

After the constraints are imposed, our system has $n - h$ degrees of freedom. The frequencies of the system,

$$\omega_1^0 \leqq \omega_2^0 \leqq \cdots \leqq \omega_{n-h}^0 ,$$

are connected with the characteristic values $\lambda_1^0 \leqq \lambda_2^0 \leqq \cdots \leqq \lambda_{n-h}^0$ of the pencil $A(x, x) - \lambda B(x, x)$, subject to the constraints $L_1, L_2, \ldots, L_h$, by the relations $\lambda_j^0 = \omega_j^{02}$ $(j = 1, 2, \ldots, n - h)$. Therefore Theorem 14 immediately implies that

$$\omega_j \leqq \omega_j^0 \leqq \omega_{j+h} \qquad (j = 1, 2, \ldots, n - h).$$

Thus: *When h constraints are imposed, the frequencies of a system can only increase, but the value of the new j-th frequency ω_j^0 cannot exceed the value of the previous $(j + h)$-th frequency ω_{j+h}.*

In exactly the same way, we can assert on the basis of Theorem 15 that: *With increasing rigidity of the system, i.e., with an increase of the form $A(q, q)$ for the potential energy (without a change in $B(\dot{q}, \dot{q})$), the frequencies can only increase; and with increasing inertia of the system, i.e., with an increase of the form $B(\dot{q}, \dot{q})$ for the kinetic energy (without a change in $A(q, q)$), the frequencies can only decrease.*

Theorems 16 and 17 lead to an additional sharpening of this proposition.[40]

§ 9. Hermitian Forms[41]

1. All the results of §§ 1-7 of this chapter that were established for quadratic forms can be extended to hermitian forms.

We recall[42] that a *hermitian form* is an expression

---

[39] A finite stationary constraint is expressed by an equation $f(q_1, q_2, \ldots, q_n) = 0$, where $f(q_1, q_2, \ldots, q_n)$ is some function of the generalized coordinates.

[40] The reader can find an account of the oscillatory properties of elastic oscillations of a system with n degrees of freedom in [17], Chapter III.

[41] In the preceding sections, all the numbers and variables were real. In this section, the numbers are complex and the variables assume complex values.

[42] See Chapter IX, § 2.

$$H(x,x)=\sum_{i,k=1}^{n} h_{ik}x_i\bar{x}_k \qquad (h_{ik}=\bar{h}_{ki};\ i,k=1,2,\dots,n). \tag{111}$$

To the hermitian form (111) there corresponds the following *bilinear hermitian form*:

$$H(x,y)=\sum_{i,k=1}^{n} h_{ik}x_i\bar{y}_k; \tag{112}$$

moreover,

$$H(y,x)=\overline{H(x,y)} \tag{113}$$

and, in particular,

$$H(x,x)=\overline{H(x,x)} \tag{113'}$$

i.e., the hermitian form $H(x,x)$ assumes real values only.

The coefficient matrix $H=\| h_{ik} \|_1^n$ of the hermitian form is hermitian, i.e., $H^*=H$.[43]

By means of the matrix $H=\| h_{ik} \|_1^n$ we can represent $H(x,y)$ and, in particular, $H(x,x)$ in the form of a product of three matrices, a row, a square, and a column matrix:[44]

$$H(x,y)=x^{\mathsf{T}}H\bar{y},\quad H(x,x)=x^{\mathsf{T}}H\bar{x}. \tag{114}$$

If

$$x=\sum_{i=1}^{m} c_i u^i,\quad y=\sum_{k=1}^{p} d_k v^k, \tag{115}$$

where u^i, v^k are column matrices and c_i, d_k are complex numbers ($i=1,2,3,\dots,m;\ k=1,2,\dots,p$), then

$$H(x,y)=\sum_{i=1}^{m}\sum_{k=1}^{p} c_i\bar{d}_k H(u^i,v^k). \tag{116}$$

We subject the variables $x_1, x_2, \dots, x_n$ to the linear transformation

$$x_i=\sum_{k=1}^{n} t_{ik}\xi_k \qquad (i=1,2,\dots,n) \tag{117}$$

---

[43] A matrix symbol followed by an asterisk * denotes the matrix that is obtained from the given one by transposition and replacement of all the elements by their complex conjugates ($H^*=\bar{H}^{\mathsf{T}}$).

[44] Here

$$x=(x_1,x_2,\dots,x_n),\ \ \bar{x}=(\bar{x}_1,\bar{x}_2,\dots,\bar{x}_n),\ \ y=(y_1,y_2,\dots,y_n),\ \bar{y}=(\bar{y}_1,\bar{y}_2,\dots,\bar{y}_n);$$

the sign ${}^{\mathsf{T}}$ denotes transposition.

or, in matrix notation,

$$x=T\xi \qquad (T=\| t_{ik} \|_1^n). \tag{117'}$$

After the transformation, $H(x,x)$ assumes the form

$$\tilde{H}(\xi,\xi)=\sum_{i,k=1}^{n} \tilde{h}_{ik}\xi_i\bar{\xi}_k,$$

where the new coefficient matrix $\tilde{H}=\| \tilde{h}_{ik} \|_1^n$ is connected with the old coefficient matrix $H=\| h_{ik} \|_1^n$ by the formula

$$\tilde{H}=T^{\mathsf{T}}H\bar{T}. \tag{118}$$

This is immediately clear when, in the second of the formulas (114), x is replaced by $T\xi$.

If we set $T=\bar{W}$, then we can rewrite (118) as follows:

$$\tilde{H}=W^*HW. \tag{119}$$

From the formula (118) it follows that H and $\tilde{H}$ have the same rank provided the transformation (117) is non-singular ($|T|\neq 0$). The rank of H is called the *rank of the form* $H(x,x)$.

The determinant $|H|$ is called the *discriminant* of $H(x,x)$. From (118) we obtain the formula for the transformation of the discriminant on transition to new variables:

$$|\tilde{H}|=|H|\,|T|\,|\bar{T}|.$$

A hermitian form is called *singular* if its discriminant is zero. Obviously, a singular form remains singular under any transformation of the variables (117).

A hermitian form $H(x,x)$ can be represented in infinitely many ways in the form

$$H(x,x)=\sum_{i=1}^{r} a_i X_i\bar{X}_i, \tag{120}$$

where $a_i\neq 0$ ($i=1,2,\dots,r$) are real numbers and

$$X_i=\sum_{k=1}^{n} a_{ik}x_k \ (i=1,2,\dots,r)$$

are independent complex linear forms in the variables $x_1, x_2, \dots, x_n$.[45]

---

[45] Therefore $r\leqq n$.

We shall call the right-hand side of (120) a *sum of linearly independent squares*[46] and every term in the sum a *positive* or a *negative* square according as $a_i > 0$ or < 0. Just as for quadratic forms, the number r in (120) is equal to the rank of the form $H(x, x)$.

THEOREM 18 (The Law of Inertia for Hermitian Forms): *In the representation of a hermitian form $H(x, x)$ as a sum of linearly independent squares,*

$$H(x, x) = \sum_{i=1}^{r} a_i X_i \overline{X}_i ,$$

the number of positive squares and the number of negative squares do not depend on the choice of the representation.

The proof is completely analogous to the proof of Theorem 1 (p. 297).

The difference σ between the number π of positive squares and the number ν of negative squares in (120) is called the *signature* of the hermitian form $H(x, x)$: $\sigma = \pi - \nu$.

Lagrange's method of reduction of quadratic forms to sums of squares can also be used for hermitian forms, only the fundamental formulas (15) and (16) on p. 299 must then be replaced by the formulas[47]

$$H(x, x) = \frac{1}{h_{gg}} \left| \sum_{k=1}^{n} h_{kg} x_k \right|^2 + H_1(x, x), \tag{121}$$

$$H(x, x) = \frac{1}{2} \left\{ \left| \sum_{k=1}^{n} \left(h_{kf} + \frac{h_{kg}}{h_{fg}} \right) x_k \right|^2 - \left| \sum_{k=1}^{n} \left(h_{kf} - \frac{h_{kg}}{h_{fg}} \right) x_k \right|^2 \right\} + H_2(x, x). \tag{122}$$

Let us proceed to establish Jacobi's formula for a hermitian form $H(x, x) = \sum_{i,k=1}^{n} h_{ik} x_i \overline{x}_k$ of rank r. Here, as in the case of a quadratic form, we assume that

$$D_k = H\begin{pmatrix} 1 & 2 \dots k \\ 1 & 2 \dots k \end{pmatrix} \neq 0 \qquad (k = 1, 2, \dots, r). \tag{123}$$

This inequality enables us to use Theorem 2 of Chapter II (p. 38) on the representation of an arbitrary square matrix in the form of a product of three matrices: a lower triangular matrix F, a diagonal matrix D, and an upper triangular matrix L. We apply this theorem to the matrix $H = \| h_{ik} \|_1^n$

---

[46] This terminology is connected with the fact that $X_i \overline{X}_i$ is the square of the modulus of X_i ($X_i \overline{X}_i = |X_i|^2$).

[47] The formula (121) is applicable when $h_{gg} \neq 0$; and (122), when $h_{ff} = h_{gg} = 0$, $h_{fg} \neq 0$.

and obtain

$$H = F \left\{ D_1, \frac{D_2}{D_1}, \dots, \frac{D_r}{D_{r-1}}, 0, \dots, 0 \right\} L, \tag{124}$$

where $F = \| f_{ik} \|_1^n$, $L = \| l_{ik} \|_1^n$, and

$$f_{jk} = \frac{1}{D_k} H\begin{pmatrix} 1 \dots k-1 & j \\ 1 \dots k-1 & k \end{pmatrix}, \quad l_{kj} = \frac{1}{D_k} H\begin{pmatrix} 1 \dots k-1 & k \\ 1 \dots k-1 & j \end{pmatrix} \tag{125}$$

$$(j = k, k+1, \dots, n; \ k = 1, 2, \dots, r),$$

$$f_{ik} = l_{ki} = 0 \qquad (i < k; \ i, k = 1, 2, \dots, n). \tag{126}$$

Since $H = \| h_{ik} \|_1^n$ is a hermitian matrix, it follows from these equations that

$$f_{ik} = \overline{l}_{ki} \quad \begin{pmatrix} i \geqq k; \ k = 1, 2, \dots, r; \ i = 1, 2, \dots, n, \\ i < k; \ i, k = 1, 2, \dots, n \end{pmatrix}. \tag{127}$$

Since all the elements in the last $n - r$ columns of F and the last $n - r$ rows of L can be chosen arbitrarily,[48] we choose these elements such that 1) the relations (127) hold for all i, k

$$f_{ik} = \overline{l}_{ik} \quad (i, k = 1, 2, \dots, n)$$

and 2) $|F| = |L| \neq 0$. Then

$$F = L^*, \tag{128}$$

and (124) assumes the form

$$H = L^* \left\{ D_1, \frac{D_2}{D_1}, \dots, \frac{D_r}{D_{r-1}}, 0, \dots, 0 \right\} L. \tag{129}$$

Setting

$$T = \| t_{ik} \|_1^n = \overline{L}, \tag{130}$$

we write (129) as follows:

$$H = T^{\mathrm{T}} \left\{ D_1, \frac{D_2}{D_1}, \dots, \frac{D_r}{D_{r-1}}, 0, \dots, 0 \right\} \overline{T} \qquad (|T| \neq 0). \tag{131}$$

A comparison of this formula with (118) shows that the hermitian form

$$\sum_{k=1}^{n} \frac{D_k}{D_{k-1}} \xi_k \overline{\xi}_k \qquad (D_0 = 1) \tag{132}$$

under the transformation of the variables

---

[48] These elements, in fact, drop out of the right-hand side of (124), because the last $n - r$ diagonal elements of D are zero.

$$\xi_i = \sum_{k=1}^{n} t_{ik} x_k \qquad (i = 1, 2, \ldots, n)$$

goes over into $H(x, x)$, i.e., that Jacobi's formula holds:

$$H(x, x) = \sum_{k=1}^{n} \frac{D_k}{D_{k-1}} X_k \overline{X}_k \qquad (D_0 = 1), \tag{133}$$

where

$$X_k = x_k + t_{k,k+1} x_{k+1} + \cdots + t_{kn} x_n \qquad (k = 1, 2, \ldots, r) \tag{134}$$

and

$$t_{kj} = \frac{1}{D_k} H\begin{pmatrix} 1 & 2 \ldots k-1 & j \\ 1 & 2 \ldots k-1 & k \end{pmatrix} \quad (j = k+1, \ldots, n;\ k = 1, 2, \ldots, r). \tag{135}$$

The linear forms $X_1, X_2, \ldots, X_r$ are independent, since X_k contains the variable x_k which does not occur in the subsequent forms $X_{k+1}, \ldots, X_r$.

When we introduce, in place of $X_1, X_2, \ldots, X_r$, the linearly independent forms

$$Y_k = D_k X_k \qquad (k = 1, 2, \ldots, r), \tag{136}$$

we can write Jacobi's formula (133) in the form

$$H(x, x) = \sum_{k=1}^{r} \frac{Y_k \overline{Y}_k}{D_{k-1} D_k} \qquad (D_0 = 1). \tag{137}$$

According to Jacobi's formula (137), the number of negative squares in the representation of $H(x, x)$ is equal to the number of variations of sign in the sequence $1, D_1, D_2, \ldots, D_r$

$$\nu = V(1, D_1, D_2, \ldots, D_r),$$

so that the signature σ of $H(x, x)$ is determined by the formula

$$\sigma = r - 2V(1, D_1, D_2, \ldots, D_r). \tag{138}$$

All the remarks about the special cases that may occur, made for quadratic forms (§ 3), automatically carry over to hermitian forms.

DEFINITION 5: *A hermitian form* $H(x, x) = \sum_{i,k=1}^{n} h_{ik} x_i \overline{x}_k$ *is called positive (negative) semidefinite if for arbitrary values of the variables* $x_1, x_2, x_3, \ldots, x_n$, *not all equal to zero,*

$$H(x, x) \geqq 0 \qquad (\leqq 0).$$

DEFINITION 6: *A hermitian form* $H(x, x) = \sum_{i,k=1}^{n} h_{ik} x_i \overline{x}_k$ *is called positive (negative) definite if for arbitrary values of the variables* $x_1, x_2, \ldots, x_n$, *not all equal to zero,*

$$H(x, x) > 0 \qquad (< 0).$$

THEOREM 19: *A hermitian form* $H(x, x) = \sum_{i,k=1}^{n} h_{ik} x_i \overline{x}_k$ *is positive definite if and only if the following inequalities hold:*

$$D_k = H\begin{pmatrix} 1 & 2 \ldots k \\ 1 & 2 \ldots k \end{pmatrix} > 0 \qquad (k = 1, 2, \ldots, n). \tag{139}$$

THEOREM 20: *A hermitian form* $H(x, x) = \sum_{i,k=1}^{n} h_{ik} x_i \overline{x}_k$ *is positive semidefinite if and only if all the principal minors of* $H = \| h_{ik} \|_1^n$ *are non-negative:*

$$H\begin{pmatrix} i_1 & i_2 \ldots i_p \\ i_1 & i_2 \ldots i_p \end{pmatrix} \geqq 0 \tag{140}$$

$$(i_1, i_2, \ldots, i_p = 1, 2, \ldots, n;\ p = 1, 2, \ldots, n).$$

The proofs of Theorems 19 and 20 are completely analogous to the proofs of Theorems 3 and 4 for quadratic forms.

The conditions for a hermitian form $H(x, x)$ to be negative definite or semidefinite are obtained by applying (139) and (140) to the form $-H(x, x)$.

From Theorem 5′ of Chapter IX (p. 274), we obtain the *Theorem on the reduction of a hermitian form to principal axes:*

THEOREM 21: *Every hermitian form* $H(x, x) = \sum_{i,k=1}^{n} h_{ik} x_i \overline{x}_k$ *can be reduced by a unitary transformation of the variables*

$$x = U\xi \qquad (UU^* = E) \tag{141}$$

to the canonical form

$$\Lambda(\xi, \xi) = \sum_{i=1}^{n} \lambda_i \xi_i \overline{\xi}_i, \tag{142}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the characteristic values of the matrix* $H = \| h_{ik} \|_1^n$.

Theorem 21 follows from the formula

$$H = U \| \lambda_i \delta_{ik} \| U^{-1} = T^{\mathsf{T}} \| \lambda_i \delta_{ik} \| \overline{T} \quad (U^{\mathsf{T}} = \overline{U}^{-1} = T). \tag{143}$$

Let $H(x, x) = \sum_{i,k=1}^{n} h_{ik} x_i \overline{x}_k$ and $G(x, x) = \sum_{i,k=1}^{n} g_{ik} x_i \overline{x}_k$ be two hermitian forms. We shall study the pencil of hermitian forms $H(x, x) - \lambda G(x, x)$

(λ is a real parameter). This pencil is called *regular* if $G(x, x)$ is positive definite. By means of the hermitian matrices $H = \| h_{ik} \|_1^n$ and $G = \| g_{ik} \|_1^n$ we form the equation

$$|H - \lambda G| = 0.$$

This equation is called the *characteristic equation of the pencil of hermitian forms.* Its roots are called the *characteristic values of the pencil.*

If λ_0 is a characteristic value of the pencil, then there exists a column $z = (z_1, z_2, \dots, z_n) \neq o$ such that

$$Hz = \lambda_0 z.$$

We shall call the column z a *principal column* or *principal vector of the pencil* $H(x, x) - \lambda G(x, x)$ corresponding to the characteristic value λ_0.

Then the following theorem holds:

THEOREM 22: *The characteristic equation of a regular pencil of hermitian forms $H(x, x) - \lambda G(x, x)$ has n real roots $\lambda_1, \lambda_2, \dots, \lambda_n$. To these roots there correspond n principal vectors $z^1, z^2, \dots, z^n$ satisfying the conditions of 'orthonormality':*

$$G(z^i, z^k) = \delta_{ik} \qquad (i, k = 1, 2, \dots, n).$$

The proof is completely analogous to the proof of Theorem 8.

All extremal properties of the characteristic values of a regular pencil of quadratic forms remain valid for hermitian forms.

Theorems 10-17 remain valid if the term 'quadratic form' is replaced throughout by the term 'hermitian form.' The proofs of the theorems are then unchanged.

§ 10. Hankel Forms

1. Let $s_0, s_1, \dots, s_{2n-2}$ be a sequence of numbers. We form, by means of these numbers, a quadratic form in n variables

$$S(x, y) = \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k. \tag{144}$$

This is called a *Hankel form.* The matrix $S = \| s_{i+k} \|_0^{n-1}$ corresponding to this form is called a *Hankel matrix.* It has the form

$$S = \left\| \begin{matrix} s_0 & s_1 & s_2 & \dots & s_{n-1} \\ s_1 & s_2 & s_3 & \dots & s_n \\ s_2 & s_3 & s_4 & \dots & s_{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & s_{n+1} & \dots & s_{2n-2} \end{matrix} \right\|.$$

We denote the sequence of principal minors of S by $D_1, D_2, \dots, D_n$:

$$D_p = | s_{i+k} |_0^{p-1} \qquad (p = 1, 2, \dots, n).$$

In this section we shall derive the fundamental results of Frobenius about the rank and signature of real Hankel forms.[49]

We begin by proving two lemmas.

LEMMA 1: *If the first h rows of the Hankel matrix $S = \| s_{i+k} \|_0^{n-1}$ are linearly independent, but the first $h+1$ rows linearly dependent, then*

$$D_h \neq 0.$$

Proof. We denote the first $h+1$ rows of S by $R_1, R_2, \dots, R_h, R_{h+1}$. By assumption, $R_1, R_2, \dots, R_h$ are linearly independent and R_{h+1} is expressed linearly in terms of them:

$$R_{h+1} = \sum_{j=1}^{h} \alpha_j R_{h-j+1}$$

or

$$s_q = \sum_{j=1}^{h} \alpha_j s_{q-j} \qquad (q = h, h+1, \dots, h+n-1). \tag{145}$$

We write down the matrix formed from the first h rows $R_1, R_2, \dots, R_h$ of S:

$$\left\| \begin{matrix} s_0 & s_1 & s_2 & \dots & s_{n-1} \\ s_1 & s_2 & s_3 & \dots & s_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{h-1} & s_h & s_{h+1} & \dots & s_{h+n-2} \end{matrix} \right\|. \tag{146}$$

This matrix is of rank h. On the other hand, by (145) every column of the matrix can be expressed linearly in terms of the preceding h columns and hence in the terms of the first h columns. But since the rank of (146) is h, these first h columns of (146) must then be linearly independent, i.e.,

$$D_h \neq 0.$$

This proves the lemma.

---

[49] See [162].

LEMMA 2: *If in the matrix* $S = \| s_{i+k} \|_0^{n-1}$, *for a certain* h $(< n)$,

$$D_h \neq 0, \quad D_{h+1} = \cdots = D_n = 0 \tag{147}$$

and

$$t_{ik} = \frac{S\begin{pmatrix} 1 \dots h & h+i+1 \\ 1 \dots h & h+k+1 \end{pmatrix}}{S\begin{pmatrix} 1 \dots h \\ 1 \dots h \end{pmatrix}} = \frac{1}{D_h} \left| \begin{array}{cccc} & & & s_{h+k} \\ & D_h & & \vdots \\ & & & s_{2h+k-1} \\ s_{h+i} & \cdots & s_{2h+i-1} & s_{2h+i+k} \end{array} \right|, \tag{148}$$

$$(i, k, = 0, 1, \dots, n-h-1)$$

then the matrix $T = \| t_{ik} \|_0^{n-h-1}$ *is also a Hankel matrix and all its elements above the second diagonal are zero, i.e., there exist numbers* $t_{n-h-1}, \dots, t_{2n-2h-2}$ *such that*

$$t_{ik} = t_{i+k} \quad (i, k = 0, 1, \dots, n-h-1;\, t_0 = t_1 = \cdots = t_{n-h-2} = 0).$$

Proof. We introduce the matrices

$$T_p = \| t_{ik} \|_0^{p-1} \qquad (p = 1, 2, \dots\ n-h).$$

In this notation $T = T_{n-h}$.

We shall show that every T_p $(p = 1, 2, \dots, n-h)$ is a Hankel matrix and that $t_{ik} = 0$ for $i + k \leqq p - 2$. The proof is by induction with respect to p.

For the matrix T_1, our assertion is trivial; for T_2, it is obvious, since

$$T_2 = \left\| \begin{matrix} t_{00} & t_{01} \\ t_{10} & t_{11} \end{matrix} \right\|, \quad t_{01} = t_{10} \quad \text{(because } S \text{ is symmetric) and} \quad t_{00} = \frac{D_{h+1}}{D_h} = 0.$$

Let us assume that our assertion is true for the matrices T_p $(p < n - h)$; we shall show that it is also true for $T_{p+1} = \| t_{ik} \|_0^p$. From the assumption it follows that there exist numbers $t_{p-1}, t_p, \dots, t_{2p-2}$ such that with $t_0 = \dots = t_{p-2} = 0$

$$T_p = \| t_{i+k} \|_0^{p-1}.$$

Here

$$| T_p | = t_{p-1}^p. \tag{149}$$

On the other hand, using Sylvester's determinant identity (see (28) on page 32), we find:

$$| T_p | = \frac{D_{h+p}}{D_h} = 0. \tag{150}$$

Comparing (149) with (150), we obtain

$$t_{p-1} = 0. \tag{151}$$

Furthermore from (148)

$$t_{ik} = s_{2h+i+k} + \frac{1}{D_h} \left| \begin{array}{cccc} & & & s_{h+k} \\ & D_h & & \vdots \\ & & & s_{2h+k-1} \\ s_{h+i} & \cdots & s_{2h+i-1} & 0 \end{array} \right|. \tag{152}$$

By the preceding lemma, it follows from (147) that the $(h+1)$-th row of the matrix $S = \| s_{i+k} \|_0^{n-1}$ is linearly dependent on the first h rows:

$$s_q = \sum_{g=1}^{h} \alpha_g s_{q-g} \quad (q = h, h+1, \dots, h+n-1). \tag{153}$$

Let $i, k \leqq p \leqq i + k \leqq 2p - 1$. Then one of the numbers i or k is less than p. Without loss of generality, we assume that $i < p$. Then, when we expand, by (153), the last column of the determinant of the right-hand side of (152) and use the relations (152) again, we shall have

$$\begin{aligned} t_{ik} &= s_{2h+i+k} + \sum_{g=1}^{h} \frac{\alpha_g}{D_h} \left| \begin{array}{cccc} & & & s_{h+k-g} \\ & D_h & & \vdots \\ & & & s_{2h+k-g-1} \\ s_{h+i} & \cdots & s_{2h+i-1} & 0 \end{array} \right| \\ &= s_{2h+i+k} + \sum_{g=1}^{h} \alpha_g (t_{i,k-g} - s_{2h+i+k-g}). \end{aligned} \tag{154}$$

By the induction hypothesis (151) holds, and since in (154) $i < p$, $k - g < p$ and $i + k - g \leqq 2p - 2$, we have $t_{i,k-g} = t_{i+k-g}$. Therefore, for $i + k < p$ all the $t_{ik} = 0$, and for $p \leqq i + k \leqq 2p - 1$ the value of t_{ik}, by (154), depends on $i + k$ only.

Thus, T_{p+1} is a Hankel matrix, and all its elements $t_0, t_1, \dots, t_{p-1}$ above the second diagonal are zero.

This proves the lemma.

Using Lemma 2, we shall prove the following theorem:

THEOREM 23: *If the Hankel matrix* $S = \| s_{i+k} \|_0^{n-1}$ *has rank r and if for some h* $(< r)$

$$D_h \neq 0, \quad D_{h+1} = \cdots = D_r = 0,$$

then the principal minor of order r formed from the first h and the last r — h rows and columns of S is not zero:

$$D^{(r)} = S\begin{pmatrix} 1 & \dots & h & n-r+h+1 & n-r+h+2 & \dots & n \\ 1 & \dots & h & n-r+h+1 & n-r+h+2 & \dots & n \end{pmatrix} \neq 0.$$

Proof. By the preceding lemma, the matrix

$$T = \| t_{ik} \|_0^{n-h-1} \qquad \left(t_{ik} = \frac{S\begin{pmatrix} 1 & \dots & h & h+i+1 \\ 1 & \dots & h & h+k+1 \end{pmatrix}}{S\begin{pmatrix} 1 & \dots & h \\ 1 & \dots & h \end{pmatrix}} \quad (i, k = 0, 1, \dots, n-h+1) \right)$$

is a Hankel matrix in which all the elements above the second diagonal are zero. Therefore

$$|T| = t_{0,\, n-h-1}^{n-h}.$$

On the other hand,[50] $|T| = \frac{D_n}{D_h} = 0$. Therefore $t_{0,\, n-h-1} = 0$, and the matrix T has the form

$$T = \begin{Vmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & \iddots & u_{n-h-1} \\ \vdots & & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots \\ \vdots & \iddots & & & u_2 \\ 0 & u_{n-h-1} & \cdots & u_2 & u_1 \end{Vmatrix}.$$

The rank of T must be $r - h$.[51] Therefore for $r < n - 1$ in the matrix T the elements $u_{r-h+1} = \dots = u_{n-h+1} = 0$, and

---

[50] By Sylvester's determinant identity (see (28) on p. 32).

[51] From Sylvester's identity it follows that all the minors of T in which the order exceeds $r - h$ are zero. On the other hand, S contains some non-vanishing minors of order r bordering D_h. Hence it follows that the corresponding minor of order $r - h$ of T is different from zero.

$$T = \begin{Vmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \iddots & \vdots \\ \vdots & & & \iddots & & 0 \\ \vdots & & \iddots & & \iddots & u_{r-h} \\ \vdots & \iddots & & \iddots & & \vdots \\ \vdots & & \iddots & & & \vdots \\ 0 & \dots 0 & u_{r-h} & & \cdots & u_1 \end{Vmatrix} \quad (u_{r-h} \neq 0).$$

But then, by Sylvester's identity (see page 32),

$$D^{(r)} = D_h T\begin{pmatrix} n-r+1 & \dots & n-h \\ n-r+1 & \dots & n-h \end{pmatrix} = D_h u_{r-h}^{r-h} \neq 0,$$

and this is what we had to prove.

Let us consider a real[52] Hankel form $S(x, x) = \sum_{i,k=0}^{n-1}{}' s_{i+k} x_i x_k$ of rank r. We denote by π, ν, and σ, respectively, the number of positive and of negative squares and the signature of the form:

$$\pi + \nu = r, \quad \sigma = \pi - \nu = r - 2\nu.$$

By the theorem of Jacobi (p. 303) these values can be determined from the signs of the successive minors

$$D_0 = 1, D_1, D_2, \dots, D_{r-1}, D_r \tag{155}$$

by the formulas

$$\left.\begin{aligned} &\pi = P(1, D_1, \dots D_r), \quad \nu = V(1, D_1, \dots, D_r), \\ &\sigma = P(1, D_1, \dots, D_r) - V(1, D_1, \dots D_r) = r - 2\, V(1, D_1, \dots, D_r). \end{aligned}\right\} \tag{156}$$

These formulas become inapplicable when the last term in (155) or any three consecutive terms are zero (see § 3). However, as Frobenius has shown, for Hankel forms there is a rule that enables us to use the formulas (156) in the general case:

THEOREM 24 (Frobenius): *For a real Hankel form* $S(x, x) = \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k$ *of rank r the values of* π, ν, *and* σ *can be determined by the formulas* (156) *provided that*

---

[52] In the preceding Lemmas 1 and 2 and in Theorem 23, the ground field can be taken as an arbitrary number field—in particular, the field of complex or of real numbers.

1) *for*

$$D_h \neq 0,\ D_{h+1} = \cdots = D_r = 0 \ (h < r) \tag{157}$$

D_r is replaced by $D^{(r)}$, where

$$D^{(r)} = S\begin{pmatrix} 1 \ldots h\, n-r+h+1 \ldots n \\ 1 \ldots h\, n-r+h+1 \ldots n \end{pmatrix} \neq 0;$$

2) *in any group of p consecutive zero determinants*

$$(D_h \neq 0) \quad D_{h+1} = D_{h+2} = \cdots = D_{h+p} = 0 \quad (D_{h+p+1} \neq 0) \tag{158}$$

a sign is attributed to the zero determinants according to the formula

$$\operatorname{sign} D_{h+j} = (-1)^{\frac{j(j-1)}{2}} \operatorname{sign} D_h. \tag{159}$$

The values of P, V, and $P - V$ corresponding to the group (158) *are then:*[53]

$$\begin{array}{c|c|c|} & p \text{ odd} & p \text{ even} \\ \hline P_{h,p} = P(D_h, D_{h+1}, \ldots, D_{h+p+1}) & \frac{p+1}{2} & \frac{p+1+\varepsilon}{2} \\ \hline V_{h,p} = V(D_h, D_{h+1}, \ldots, D_{h+p+1}) & \frac{p+1}{2} & \frac{p+1-\varepsilon}{2} \\ \hline P_{h,p} - V_{h,p} & 0 & \varepsilon \end{array}; \tag{160}$$

$$\varepsilon = (-1)^{\frac{p}{2}} \operatorname{sign} \frac{D_{h+p+1}}{D_h}.$$

Proof. To begin with we consider the case where $D_r \neq 0$. Then the forms $S(x, x) = \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k$ and $S_r(x, x) = \sum_{i,k=0}^{r-1} s_{i+k} x_i x_k$ have not only the same rank r, but also the same signature σ. For let $S(x, x) = \sum_{i=1}^{r} \varepsilon_i Z_i^2$, where the Z_i are real linear forms and $\varepsilon_i = \pm 1$ $(i = 1, 2, \ldots, r)$. We set $x_{r+1} = \ldots = x_{n-1} = 0$. Then the forms $S(x, x)$ and Z_i go over, respectively, into $S_r(x, x)$ and $\hat{Z}_i$ $(i = 1, 2, \ldots, r)$; and $S_r(x, x) = \sum_{i=1}^{r} \varepsilon_i \hat{Z}_i^2$, i.e., $S_r(x, x)$ has the same number of positive and negative squares as $S(x, x)$.[54] Thus the signature of $S_r(x, x)$ is σ.

We now vary the parameters $s_0, s_1, \ldots, s_{2r-2}$ continuously in such a way that for the new parameter values $s_0^*, s_1^*, \ldots, s_{2r-2}^*$ all the terms of the sequence[55]

$$1, D_1^*, D_2^*, \ldots, D_r^* \quad (D_q^* = |\, s_{i+k}^* \,|_0^{q-1};\ q = 1, 2, \ldots, r)$$

are different from zero and that in the process of variation none of the nonzero determinants (155) vanishes.[56]

Since the rank of $S_r(x, x)$ does not change during the variation, its signature also remains unchanged (see p. 309). Therefore

$$\sigma = P(1, D_1^*, \ldots, D_r^*) - V(1, D_1^*, \ldots, D_r^*). \tag{161}$$

If $D_i \neq 0$ for some i, then sign $D_i^* =$ sign D_i. Therefore the whole problem reduces to determining the variations in sign among those D_i^* that correspond to $D_i = 0$. More accurately, for every group of the form (158) we have to determine

$$P(D_h^*, D_{h+1}^*, \ldots, D_{h+p+1}^*) - V(D_h^*, D_{h+1}^*, \ldots, D_{h+p}^*, D_{h+p+1}^*).$$

For this purpose we set:

$$t_{ik} = \frac{1}{D_h} \left| \begin{array}{cc} & s_{h+k} \\ D_h & \vdots \\ & s_{2h+k-1} \\ s_{h+i} \cdots s_{2h+i-1} & s_{2h+i+k} \end{array} \right| \quad (i, k = 0, 1, \ldots, p).$$

By Lemma 2, the matrix $T = \| t_{ik} \|_0^p$ is a Hankel matrix and all its elements above the second diagonal are zero, so that T has the form

---

[53] The formulas (159) and (160) are also applicable to (157), but we have to set $p = r - h - 1$ and interpret D_{h+p+1} not as $D_r = 0$, but as $D^{(r)} \neq 0$.

[54] The linear forms $\hat{Z}_1, \hat{Z}_2, \ldots, \hat{Z}_r$ are linearly independent, because the quadratic form $S(x, x) = \sum_{i=1}^{r} \varepsilon_i \hat{Z}_i^2$ is of rank r $(D_r \neq 0)$.

[55] In this section, the asterisk * does not indicate the adjoint matrix.

[56] Such a variation of the parameter is always possible, because in the space of the parameters $s_0, s_1, \ldots, s_{2r-2}$ an equation of the form $D_i = 0$ determines a certain algebraic hypersurface. If a point lies in some such hypersurfaces, then it can always be approximated by arbitrarily close points that do not lie in these hypersurfaces.

$$T = \left\| \begin{array}{cccccc} 0 & \cdot & \cdot & \cdot & 0 & t_p \\ \cdot & & & & \cdot & * \\ \cdot & & & \cdot & & \cdot \\ \cdot & & \cdot & & & \cdot \\ 0 & \cdot & & & & \cdot \\ t_p & * & \cdot & \cdot & \cdot & * \end{array} \right\| . \qquad (162)$$

We denote the successive minors of T by $\widehat{D}_1, \widehat{D}_2, \ldots, \widehat{D}_{p+1}$:

$$\widehat{D}_q = | t_{ik} |_0^{q-1} \quad (q = 1, 2, \ldots, p+1).$$

Together with T, we consider the matrix

$$T^* = \| t_{ik}^* \|_0^p ,$$

where

$$t_{ik}^* = \frac{1}{D_h^*} \begin{vmatrix} & & & s_{h+k}^* \\ & D_h^* & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & s_{2h+k-1}^* \\ s_{h+i}^* & \cdots & s_{2h+i-1}^* & s_{2h+i+k}^* \end{vmatrix} \qquad (i, k = 0, 1, \ldots, p)$$

and the corresponding determinants

$$\widehat{D}_q^* = | t_{ik}^* |_0^{q-1} \quad (q = 1, 2, \ldots, p+1).$$

By Sylvester's determinant identity,

$$D_{h+q}^* = D_h^* \widehat{D}_q^* \qquad (q = 1, 2, \ldots, p+1).$$

Therefore

$$\begin{aligned} P(D_h^*, D_{h+1}^*, \ldots, D_{h+p+1}^*) - V(D_h^*, D_{h+1}^*, \ldots, D_{h+p+1}^*) \\ = P(1, \widehat{D}_1^*, \ldots, \widehat{D}_{p+1}^*) - V(1, \widehat{D}_1^*, \ldots, \widehat{D}_{p+1}^*) = \widehat{\sigma}^*, \end{aligned} \qquad (163)$$

where $\widehat{\sigma}^*$ is the signature of the form

$$T^*(x, x) = \sum_{i,k=0}^{p} t_{ik}^* x_i x_k .$$

Together with $T^*(x, x)$, we consider the forms

$$T(x, x) = \sum_{i,k=0}^{p} t_{i+k} x_i x_k \quad \text{and} \quad T^{**}(x, x) = t_p (x_0 x_p + x_1 x_{p-1} + \cdots + x_p x_0).$$

The matrix T^{**} is obtained from T (see (162)) when we replace in the latter all the elements above the second diagonal by zeros. We denote the signatures of $T(x, x)$ and $T^{**}(x. x)$ by $\widehat{\sigma}$ and $\widehat{\sigma}^{**}$. Since $T^*(x, x)$ and $T^{**}(x, x)$ are obtained from $T(x, x)$ by variations of the coefficients during which the rank of the form does not change $\left(| T^{**} | = | T | = \frac{D_{h+p+1}}{D_h} \neq 0, | T^* | = \frac{D_{h+p+1}^*}{D_h^*} \neq 0 \right)$, the signatures of $T(x, x)$, $T^*(x, x)$, and $T^{**}(x, x)$ must also be equal:

$$\widehat{\sigma} = \widehat{\sigma}^* = \widehat{\sigma}^{**}. \qquad (164)$$

But

$$T^{**}(x, x) = \begin{cases} 2t_p (x_0 x_{2k-1} + \cdots + x_{k-1} x_k) & \text{for odd } p, \\ t_p [2(x_0 x_{2k} + \cdots + x_{k-1} x_{k+1}) + x_k^2] & \text{for even } p \end{cases}$$

Since every product of the form $x_\alpha x_\beta$ with $\alpha \neq \beta$ can be replaced by a difference of squares $\left(\frac{x_\alpha + x_\beta}{2}\right)^2 - \left(\frac{x_\alpha - x_\beta}{2}\right)^2$, we can obtain a decomposition of $T^{**}(x, x)$ into independent real squares and we have

$$\widehat{\sigma}^{**} = \begin{cases} 0 & \text{for odd } p, \\ \operatorname{sign} t_p & \text{for even } p. \end{cases} \qquad (165)$$

On the other hand, from (162),

$$\frac{D_{h+p+1}}{D_h} = | T | = (-1)^{\frac{p(p+1)}{2}} t_p^{p+1}. \qquad (166)$$

From (163), (164), (165), and (166), it follows that:

$$\begin{aligned} P(D_h^*, D_{h+1}^*, \ldots, D_{h+p+1}^*) - V(D_h^*, D_{h+1}^*, \ldots, D_{h+p+1}^*) \\ = \begin{cases} 0 & \text{for odd } p, \\ \varepsilon & \text{for even } p. \end{cases} \end{aligned} \qquad (167)$$

where

$$\varepsilon = (-1)^{\frac{p}{2}} \operatorname{sign} \frac{D_{h+p+1}}{D_h}.$$

Since

$$P(D_{h+1}^*, D_{h+2}^*, \ldots, D_{h+p+1}^*) + V(D_{h+1}^*, D_{h+2}^*, \ldots, D_{h+p+1}^*) = p + 1, \qquad (168)$$

the table (160) can be deduced from (167) and (168).

Now let $D_r = 0$. Then for some $h < r$

$$D_h \neq 0, \ D_{h+1} = \cdots = D_r = 0.$$

In this case, by Theorem 25,

$$D^{(r)}=S\begin{pmatrix}1\dots h & n-r+h+1\dots n\\ 1\dots h & n-r+h+1\dots n\end{pmatrix}\neq 0.$$

The case to be considered reduces to the preceding case by renumbering the variables in the quadratic form $S(x,x)=\sum_{i,k=0}^{n-1}s_{i+k}x_ix_k$. We set:

$$\tilde{x}_0=x_0,\quad\dots,\quad\tilde{x}_{h-1}=x_{h-1},\quad\tilde{x}_h=x_{n-r+h},\quad\dots,\quad\tilde{x}_{r-1}=x_{n-1},$$
$$\tilde{x}_r=x_h,\quad\dots,\quad\tilde{x}_{n-1}=x_{n-r+h-1}.$$

Then $S(x,x)=\sum_{i,k=0}^{n-1}\tilde{s}_{i+k}\,x_ix_k$.

Starting from the structure of the matrix T on page 346 and using the relations

$$\hat{D}_j=\frac{D_{h+j}}{D_h},\ \hat{\tilde{D}}_j=\frac{\tilde{D}_{h+j}}{D_h}\qquad(j=1,2,\dots,n-h)$$

obtained from Sylvester's determinant identity, we find that the sequence $1,\tilde{D}_1,\tilde{D}_2,\dots,\tilde{D}_n$ is obtained from $1,D_1,D_2,\dots,D_n$ by replacing the single element D_r by $D^{(r)}$.

We leave it to the reader to verify that the table (160) corresponds to the attribution of signs to the zero determinants given by (159).

This completes the proof of the theorem.

Note. It follows from (166) that for odd p (p is the number of zero determinants in the group (158))

$$\operatorname{sign}\frac{D_{h+p+1}}{D_h}=(-1)^{\frac{p+1}{2}}.$$

In particular, for $p=1$ we have $D_hD_{h+2}<0$. In this case, we can omit D_{h+1} in computing $V(1,D_1,\dots,D_r)$, thus obtaining Gundenfinger's rule. In exactly the same way, we obtain Frobenius' rule (see page 304) from (160) for $p=2$.

BIBLIOGRAPHY

BIBLIOGRAPHY

Items in the Russian language are indicated by *

Part A. Textbooks, Monographs, and Surveys

[1] Achieser (Akhieser), N. J., *Theory of Approximation.* New York: Ungar, 1956. [Translated from the Russian.]

[2] Aitken, A. C., *Determinants and matrices.* 9th ed., Edinburgh: Oliver and Boyd, 1956.

[3] Bellman, R., *Stability Theory of Differential Equations.* New York: McGraw-Hill, 1953.

*[4] Bernstein, S. N., *Theory of Probability.* 4th ed., Moscow: Gostekhizdat, 1946.

[5] Bodewig, E., *Matrix Calculus.* 2nd ed., Amsterdam: North Holland, 1959.

[6] Cahen, G., *Éléments du calcul matriciel.* Paris: Dunod, 1955.

*[7] Chebotarëv, N. G., and Meĭman, N. N., *The problem of Routh-Hurwitz for polynomials and integral functions.* Trudy Mat. Inst. Steklov., vol. 26 (1949).

*[8] Chebyshev, P. L., *Complete collected works.* vol. III. Moscow: Izd. AN SSSR, 1948.

*[9] Chetaev, N. G., *Stability of motion.* Moscow: Gostekhizdat, 1946.

[10] Collatz, L., *Eigenwertaufgaben mit technischen Anwendungen.* Leipzig: Akad. Velags., 1949.

[11] ——— *Eigenwertprobleme und ihre numerische Behandlung.* New York: Chelsea, 1948.

[12] Courant, R. and Hilbert, D., *Methods of Mathematical Physics,* vol. I. Trans. and revised from the German original. New York: Interscience, 1953.

*[13] Erugin, N. R., *The method of Lappo-Danilevskiĭ in the theory of linear differential equations.* Leningrad: Leningrad University, 1956.

*[14] Faddeev, D. K. and Sominskiĭ, I. S., *Problems in higher algebra.* 2nd ed., Moscow, 1949; 5th ed. Moscow: Gostekhizdat, 1954.

[15] Faddeeva, V. N., *Computational methods of linear algebra.* New York: Dover Publications, 1959. [Translated from the Russian.]

[16] Frazer, R. A., Duncan, W. J., and Collar, A., *Elementary Matrices and Some Applications to Dynamics and Differential Equations.* Cambridge: Cambridge University Press, 1938.

*[17] Gantmacher (Gantmakher), F. R. and Kreĭn, M. G., *Oscillation matrices and kernels and small vibrations of dynamical systems.* 2nd ed., Moscow: Gostekhizdat, 1950. [A German translation is in preparation.]

[18] Gröbner, W., *Matrizenrechnung.* Munich: Oldenburg, 1956.

[19] Hahn, W., *Theorie und Anwendung der direkten Methode von Lyapunov* (Ergebnisse der Mathematik, Neue Folge, Heft 22). Berlin: Springer, 1959. [Contains an extensive bibliography.]

[20] Ince, E. L., *Ordinary Differential Equations.* New York: Dover, 1948.

[21] Jung, H., *Matrizen und Determinanten. Eine Einführung.* Leipzig, 1953.

[22] Klein, F., *Vorlesungen über höhere Geometrie.* 3rd ed., New York: Chelsea, 1949.

[23] Kowalewski, G., *Einführung in die Determinantentheorie.* 3rd ed., New York: Chelsea, 1949.

*[24] Kreĭn, M. G., *Fundamental propositions in the theory of λ-zone stability of a canonical system of linear differential equations with periodic coefficients.* Moscow: Moscow Academy, 1955.

*[25] Kreĭn, M. G. and Naĭmark, M. A., *The method of symmetric and hermitian forms in the theory of separation of roots of algebraic equations.* Kharkov: GNTI, 1936.

*[26] Kreĭn, M. G. and Rutman, M. A., *Linear operators leaving a cone in a Banach space invariant.* Uspehi Mat. Nauk, vol. 3 no. 1, (1948).

*[27] Kudryavchev, L. D., *On some mathematical problems in the theory of electrical networks.* Uspehi Mat. Nauk, vol. 3 no. 4 (1948).

*[28] Lappo-Danilevskiĭ, I. A., *Theory of functions of matrices and systems of linear differential equations.* Moscow, 1934.

[29] ——— *Mémoires sur la théorie des systemes des équations différentielles linéaires.* 3 vols., Trudy Mat. Inst. Steklov. vols. 6-8 (1934-1936). New York: Chelsea, 1953.

[30] Lefschetz, S., *Differential Equations: Geometric Theory.* New York: Interscience, 1957.

[31] Lichnerowicz, A., *Algèbre et analyse linéaires.* 2nd ed., Paris: Masson, 1956.

[32] Lyapunov (Liapounoff), A. M., *Le Problème général de la stabilité du mouvement* (Annals of Mathematics Studies, No. 17). Princeton: Princeton Univ. Press, 1949.

[33] MacDuffee, C. C., *The Theory of Matrices.* New York: Chelsea, 1946.

[34] ——— *Vectors and matrices.* La Salle: Open Court, 1943.

*[35] Malkin, I. G., *The method of Lyapunov and Poincaré in the theory of non-linear oscillations.* Moscow: Gostekhizdat, 1949.

[36] ——— *Theory of stability of motion.* Moscow: Gostekhizdat, 1952. [A German translation is in preparation.]

[37] Marden, M., *The geometry of the zeros of a polynomial in a complex variable* (Mathematical Surveys, No. 3). New York: Amer. Math. Society, 1949.

*[38] Markov, A. A., *Collected works.* Moscow, 1948.

*[39] Meĭman, N. N., *Some problems in the disposition of roots of polynomials.* Uspehi Mat. Nauk, vol. 4 (1949).

[40] Mirsky, L., *An Introduction to Linear Algebra.* Oxford: Oxford University Press, 1955.

*[41] Naĭmark, Y. I., *Stability of linearized systems.* Leningrad: Leningrad Aeronautical Engineering Academy, 1949.

[42] Parodi, M., *Sur quelques propriétés des valeurs caractéristiques des matrices carrées* (MémorialdesSciences Mathématiques, vol. 118), Paris: Gauthiers-Villars, 1952.

[43] Perlis, S., *Theory of Matrices.* Cambridge. (Mass.): Addison-Wesley, 1952.

[44] Pickert, G., *Normalformen von Matrizen* (Enz. Math. Wiss., Band I, Teil B. Heft 3, Teil I). Leipzig: Teubner, 1953.

*[45] Potapov, V. P., *The multiplicative structure of J-inextensible matrix functions.* Trudy Moscow Mat. Soc., vol. 4 (1955).

*[46] Romanovskiĭ, V. I., *Discrete Markov chains.* Moscow: Gostekhizdat, 1948.

[47] Routh, E. J., *A treatise on the stability of a given state of motion.* London: Macmillan, 1877.

[48] ——— *The advanced part of a Treatise on the Dynamics of a Rigid Body.* 6th ed., London: Macmillan, 1905; repr., New York: Dover, 1959.

[49] Schlesinger, L., *Vorlesungen über lineare Differentialgleichungen.* Berlin, 1908.

[50] ——— *Einführung in die Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage.* Berlin, 1922.

[51] Schmeidler, W., *Vortrage über Determinanten und Matrizen mit Anwendungen in Physik und Technik.* Berlin: Akademie-Verlag, 1949.

[52] Schreier, O. and Sperner, E., *Vorlesungen über Matrizen.* Leipzig: Teubner, 1932. [A slightly revised version of this book appears as Chapter V of [53].]

[53] ——— *Introduction to Modern Algebra and Matrix Theory.* New York: Chelsea, 1958.

[54] Schwerdtfeger, H., *Introduction to Linear Algebra and the Theory of Matrices.* Groningen: Noordhoff, 1950.

[55] Shohat, J. A. and Tamarkin, J. D., *The problem of moments* (Mathematical Surveys, No. 1). New York: Amer. Math. Society, 1943.

[56] Smirnow, W. I. (Smirnov, V. I.), *Lehrgang der höheren Mathematik,* Vol. III. Berlin, 1956. [This is a translation of the 13th Russian edition.]

[57] Specht, W., *Algebraische Gleichungen mit reellen oder komplexen Koeffizienten* (Enz. Math. Wiss., Band I, Teil B, Heft 3, Teil II). Stuttgart: Teubner, 1958.

[58] Stieltjes, T. J., *Oeuvres Complètes.* 2 vols., Groningen: Noordhoff.

[59] Stoll, R. R., *Linear Algebra and Matrix Theory.* New York: McGraw-Hill, 1952.

[60] Thrall, R. M. and Tornheim, L., *Vector spaces and matrices.* New York: Wiley, 1957.

[61] Turnbull, H. W., *The Theory of Determinants, Matrices and Invariants.* London: Blackie, 1950.

[62] Turnbull, H. W. and Aitken, A. C., *An Introduction to the Theory of Canonical Matrices.* London: Blackie, 1932.

[63] Volterra, V. et Hostinsky, B., *Opérations infinitésimales linéaires.* Paris: Gauthiers-Villars, 1938.

[64] Wedderburn, J. H. M., *Lectures on matrices.* New York: Amer. Math. Society, 1934.

[65] Weyl, H., *Mathematische Analyse des Raumproblems.* Berlin, 1923. [A reprint is in preparation: Chelsea, 1960.]

[66] Wintner, A., *Spektraltheorie der unendlichen Matrizen.* Leipzig, 1929.

[67] Zurmühl, R., *Matrizen.* Berlin, 1950.

PART B. Papers

[101] Afriat, S., *Composite matrices,* Quart. J. Math. vol. 5, pp. 81-89 (1954).

*[102] Aizerman (Aisermann), M. A., *On the computation of non-linear functions of several variables in the investigation of the stability of an automatic regulating system,* Avtomat. i Telemeh. vol. 8 (1947).

[103] Aisermann, M. A. and F. R. Gantmacher, *Determination of stability by linear approximation of a periodic solution of a system of differential equations with discontinuous right-hand sides,* Quart. J. Mech. Appl. Math. vol. 11, pp. 385-98 (1958).

[104] Aitken, A. C., *Studies in practical mathematics. The evaluation, with applications, of a certain triple product matrix.* Proc. Roy. Soc. Edinburgh vol. 57, (1936-37).

[105] Amir Moéz Ali, R., *Extreme properties of eigenvalues of a hermitian transformation and singular values of the sum and product of linear transformations*, Duke Math. J. vol. 23, pp. 463-76 (1956).

*[106] Artashenkov, P. V., *Determination of the arbitrariness in the choice of a matrix reducing a system of linear differential equations to a system with constant coefficients.* Vestnik Leningrad. Univ., Ser. Mat., Phys. i Chim., vol. 2, pp. 17-29 (1953).

*[107] Arzhanych, I. S., *Extension of Krylov's method to polynomial matrices*, Dokl. Akad. Nauk SSSR, Vol. 81, pp. 749-52 (1951).

*[108] Azbelev, N. and R. Vinograd, *The process of successive approximations for the computation of eigenvalues and eigenvectors*, Dokl. Akad. Nauk., vol. 83, pp. 173-74 (1952).

[109] Baker, H. F., *On the integration of linear differential equations*, Proc. London Math. Soc., vol. 35, pp. 333-78 (1903).

[110] Barankin, E. W., *Bounds for characteristic roots of a matrix*, Bull. Amer. Math. Soc., vol. 51, pp. 767-70 (1945).

[111] Bartsch, H., *Abschätzungen für die Kleinste charakteristische Zahl einer positivdefiniten hermitschen Matrix*, Z. Angew. Math. Mech., vol. 34, pp. 72-74 (1954).

[112] Bellman, R., *Notes on matrix theory*, Amer. Math. Monthly, vol. 60, pp. 173-75, (1953); vol. 62, pp. 172-73, 571-72, 647-48 (1955); vol. 64, pp. 189-91 (1957).

[113] Bellman, R. and A. Hoffman, *On a theorem of Ostrowski*, Arch. Math., vol. 5, pp. 123-27 (1954).

[114] Bendat, J. and S. Silverman, *Monotone and convex operator functions*, Trans. Amer. Math. Soc., vol. 79, pp. 58-71 (1955).

[115] Berge, C., *Sur une propriété des matrices doublement stochastiques*, C. R. Acad. Sci. Paris, vol. 241, pp. 269-71 (1955).

[116] Birkhoff, G., *On product integration*, J. Math. Phys., vol. 16, pp. 104-32 (1937).

[117] Birkhoff, G. D., *Equivalent singular points of ordinary linear differential equations*, Math. Ann., vol. 74, pp. 134-39 (1913).

[118] Bott, R. and R. Duffin, *On the algebra of networks*, Trans. Amer. Math. Soc., vol. 74, pp. 99-109 (1953).

[119] Brauer, A., *Limits for the characteristic roots of a matrix*, Duke Math. J., vol. 13, pp. 387-95 (1946); vol. 14, pp. 21-26 (1947); vol. 15, pp. 871-77 (1948); vol. 19, pp. 73-91, 553-62 (1952); vol. 22, pp. 387-95 (1955).

[120] ——— *Über die Lage der charakteristischen Wurzeln einer Matrix*, J. Reine Angew. Math., vol. 192, pp. 113-16 (1953).

[121] ——— *Bounds for the ratios of the coordinates of the characteristic vectors of a matrix*, Proc. Nat. Acad. Sci. U.S.A., vol. 41, pp. 162-64 (1955).

[122] ——— *The theorems of Ledermann and Ostrowski on positive matrices*, Duke Math. J., vol. 24, pp. 265-74 (1957).

[123] Brenner, J., *Bounds for determinants*, Proc. Nat. Acad. Sci. U.S.A., vol. 40, pp. 452-54 (1954); Proc. Amer. Math. Soc., vol. 5, pp. 631-34 (1954); vol. 8, pp. 532-34 (1957); C. R. Acad. Sci. Paris, vol. 238, pp. 555-56 (1954).

[124] Bruijn, N., *Inequalities concerning minors and eigenvalues*, Nieuw Arch. Wisk., vol. 4, pp. 18-35 (1956).

[125] Bruijn, N. and G. Szekeres, *On some exponential and polar representatives of matrices*, Nieuw Arch. Wisk., vol. 3, pp. 20-32 (1955).

*[126] Bulgakov, B. V., *The splitting of rectangular matrices*, Dokl. Akad. Nauk SSSR, vol. 85, pp. 21-24 (1952).

[127] Cayley, A., *A memoir on the theory of matrices*, Phil. Trans. London, vol. 148, pp. 17-37 (1857); Coll. Works, vol. 2, pp. 475-96.

[128] Collatz, L., *Einschliessungssatz für die charakteristischen Zahlen von Matrizen*, Math. Z., vol. 48, pp. 221-26 (1942).

[129] ——— *Über monotone systeme linearen Ungleichungen*, J. Reine Angew. Math., vol. 194, pp. 193-94 (1955).

[130] Cremer, L., *Die Verringerung der Zahl der Stabilitätskriterien bei Voraussetzung positiven koeffizienten der charakteristischen Gleichung*, Z. Angew. Math. Mech., vol. 33, pp. 222-27 (1953).

*[131] Danilevskiĭ, A. M., *On the numerical solution of the secular equation*, Mat. Sb., vol. 2, pp. 169-72 (1937).

[132] Diliberto, S., *On systems of ordinary differential operations.* In: *Contributions to the Theory of Non-linear Oscillations*, vol. I, edited by S. Lefschetz (Annals of Mathematics Studies, No. 20). Princeton: Princeton Univ. Press (1950), pp. 1-38.

*[133] Dmitriev, N. A. and E. B. Dynkin, *On the characteristic roots of stochastic matrices*, Dokl. Akad. Nauk SSSR, vol. 49, pp. 159-62 (1945).

*[133a] ——— *Characteristic roots of Stochastic Matrices*, Izv. Akad. Nauk, Ser. Fiz-Mat., vol. 10, pp. 167-94 (1946).

[134] Dobsch, O., *Matrixfunktionen beschränkter Schwankung*, Math. Z., vol. 43, pp. 353-88 (1937).

*[135] Donskaya, L. I., *Construction of the solution of a linear system in the neighborhood of a regular singularity in special cases*, Vestnik Leningrad. Univ., vol. 6 (1952).

*[136] ——— *On the structure of the solution of a system of linear differential equations in the neighbourhood of a regular singularity*, Vestnik Leningrad. Univ., vol. 8, pp. 55-64 (1954).

*[137] Dubnov, Y. S., *On simultaneous invariants of a system of affinors*, Trans. Math. Congress in Moscow 1927, pp. 236-37.

*[138] ——— *On doubly symmetric orthogonal matrices*, Bull. Ass. Inst. Univ. Moscow, pp. 33-35 (1927).

*[139] ——— *On Dirac's matrices*, Uč. zap. Univ. Moscow, vol. 2, pp. 2, 43-48 (1934).

*[140] Dubnov, Y. S. and V. K. Ivanov, On the reduction of the degree of affinor *polynomials*, Dokl. Akad. Nauk SSSR, vol. 41, pp. 99-102 (1943).

[141] Duncan, W., *Reciprocation of triply-partitioned matrices*, J. Roy. Aero. Soc., vol. 60, pp. 131-32 (1956).

[142] Egerváry, E., *On a lemma of Stieltjes on matrices*, Acta. Sci. Math., vol. 15, pp. 99-103 (1954).

[143] ——— *On hypermatrices whose blocks are commutable in pairs and their application in lattice-dynamics*, Acta Sci. Math., vol. 15, pp. 211-22 (1954).

[144] Epstein, M. and H. Flanders, *On the reduction of a matrix to diagonal form*, Amer. Math. Monthly, vol. 62, pp. 168-71 (1955).

*[145] Ershov, A. P., *On a method of inverting matrices*, Dokl. Akad. Nauk SSSR, vol. 100, pp. 209-11 (1955).

[146] Erugin, N. P., *Sur la substitution exposante pour quelques systèmes irreguliers*, Mat. Sb., vol. 42, pp. 745-53 (1935).

*[147] ——— *Exponential substitutions of an irregular system of linear differential equations*, Dokl. Akad. Nauk SSSR, vol. 17, pp. 235-36 (1935).

*[148] ——— *On Riemann's problem for a Gaussian system*, Uč. Zap. Ped. Inst., vol. 28, pp. 293-304 (1939).

*[149] Faddeev, D. K., *On the transformation of the secular equation of a matrix*, Trans. Inst. Eng. Constr., vol. 4, pp. 78-86 (1937).

[150] Faedo, S., *Un nuove problema di stabilità per le equationi algebriche a coefficienti reali*, Ann. Scuola Norm. Sup. Pisa, vol. 7, pp. 53-63 (1953).

*[151] Fage, M. K., *Generalization of Hadamard's determinant inequality*, Dokl. Akad. Nauk SSSR, vol. 54, pp. 765-68 (1946).

*[152] ——— *On symmetrizable matrices*, Uspehi Mat. Nauk, vol. 6, no. 3, pp. 153-56 (1951).

[153] Fan, K., *On a theorem of Weyl concerning eigenvalues of linear transformations*, Proc. Nat. Acad. Sci. U.S.A., vol. 35, pp. 652-55 (1949); vol. 36, pp. 31-35 (1950).

[154] ——— *Maximum properties and inequalities for the eigenvalues of completely continuous operators*, Proc. Nat. Acad. Sci. U.S.A., vol. 37, pp. 760-66 (1951).

[155] ——— *A comparison theorem for eigenvalues of normal matrices*, Pacific J. Math., vol. 5, pp. 911-13 (1955).

[156] ——— *Some inequalities concerning positive-definite Hermitian matrices*, Proc. Cambridge Philos. Soc., vol. 51, pp. 414-21 (1955).

[157] ——— *Topological proofs for certain theorems on matrices with non-negative elements*, Monatsh. Math., vol. 62, pp. 219-37 (1958).

[158] Fan, K. and A. Hoffman, *Some metric inequalities in the space of matrices*, Proc. Amer. Math. Soc., vol. 6, pp. 111-16 (1958).

[159] Fan, K. and G. Pale, *Imbedding conditions for Hermitian and normal matrices*, Canad. J. Math., vol. 9, pp. 298-304 (1957).

[160] Fan, K. and J. Todd, *A determinantal inequality*, J. London Math. Soc., vol. 30, pp. 58-64 (1955).

[161] Frobenius, G., *Über lineare substitutionen und bilineare Formen*, J. Reine Angew. Math., vol. 84, pp. 1-63 (1877).

[162] ——— *Über das Trägheitsgesetz der quadratischen Formen*, S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl., 1894, pp. 241-56, 407-31.

[163] ——— *Über die cogredienten transformationen der bilinearer Formen*, S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl., 1896, pp. 7-16.

[164] ——— *Über die vertauschbaren Matrizen*, S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl., 1896, pp. 601-614.

[165] ——— *Über Matrizen aus positiven Elementen*, S.-B. Deutsch. Akad. Wiss. Berlin. Math-Nat. Kl. 1908, pp. 471-76; 1909, pp. 514-18.

[166] ——— *Über Matrizen aus nicht negativen Elementen*, S.-B. Deutsch. Akad. Wiss. Berlin Math.-Nat. Kl., 1912, pp. 456-77.

*[167] Gantmacher, F. R., *Geometric theory of elementary divisors after Krull*, Trudy Odessa Gos. Univ. Mat., vol. 1, pp. 89-108 (1935).

*[168] ——— *On the algebraic analysis of Krylov's method of transforming the secular equation*, Trans. Second Math. Congress, vol. II, pp. 45-48 (1934).

[169] ——— *On the classification of real simple Lie groups*, Mat. Sb., vol. 5, pp. 217-50 (1939).

*[170] Gantmacher, F. R. and M. G. Kreĭn, *On the structure of an orthogonal matrix*, Trans. Ukrain. Acad. Sci. Phys.-Mat. Kiev (Trudy fiz.-mat. otdela VUAN, Kiev), 1929, pp. 1-8.

*[171] ——— *Normal operators in a hermitian space*, Bull. Phys-Mat. Soc. Univ. Kasan (Izvestiya fiz.-mat. ob-va pri Kazanskom universitete), IV, vol. 1, ser. 3, pp. 71-84 (1929-30).

*[172] ——— *On a special class of determinants connected with Kellogg's integral kernels*, Mat. Sb., vol. 42, pp. 501-8 (1935).

[173] ——— *Sur les matrices oscillatoires et complètement non-négatives*, Compositio Math., vol. 4, pp. 445-76 (1937).

[174] Gantschi, W., *Bounds of matrices with regard to an hermitian metric*, Compositio Math., vol. 12, pp. 1-16 (1954).

*[175] Gelfand, I. M. and V. B. Lidskiĭ, *On the structure of the domains of stability of linear canonical systems of differential equations with periodic coefficients*, Uspehi. Mat. Nauk, vol. 10, no. 1, pp. 3-40 (1955).

[176] Gershgorin, S. A., *Über die Abgrenzung der Eigenwerte einer Matrix*, Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 6, pp. 749-54 (1931).

[177] Goddard, L., *An extension of a matrix theorem of A. Brauer*, Proc. Int. Cong. Math. Amsterdam, 1954, vol. 2, pp. 22-23.

[178] Goheen, H. E., *On a lemma of Stieltjes on matrices*, Amer. Math. Monthly, vol. 56, pp. 328-29 (1949).

*[179] Golubchikov, A. F., *On the structure of the automorphisms of the complex simple Lie groups*, Dokl. Akad. Nauk SSSR, vol. 27, pp. 7-9 (1951).

*[180] Grave, D. A., *Small oscillations and some propositions in algebra*, Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 2, pp. 563-70 (1929).

*[181] Grossman, D. P., *On the problem of a numerical solution of systems of simultaneous linear algebraic equations*, Uspehi Mat. Nauk, vol. 5, no. 3, pp. 87-103 (1950).

[182] Hahn, W., *Eine Bemerkung zur zweiten Methode von Lyapunov*, Math. Nachr., vol. 14, pp. 349-54 (1956).

[183] ——— *Über die Anwendung der Methode von Lyapunov auf Differenzengleichungen*, Math. Ann., vol. 136, pp. 430-41 (1958).

[184] Haynsworth, E., *Bounds for determinants with dominant main diagonal*, Duke Math. J., vol. 20, pp. 199-209 (1953).

[185] ——— *Note on bounds for certain determinants*, Duke Math. J., vol. 24, pp. 313-19 (1957).

[186] Hellmann, O., *Die Anwendung der Matrizanten bei Eigenwertaufgaben*, Z. Angew. Math. Mech., vol. 35, pp. 300-15 (1955).

[187] Hermite, C., *Sur le nombre des racines d'une équation algébrique comprise entre des limites données*, J. Reine Angew. Math., vol. 52, pp. 39-51 (1856).

[188] Hjelmsler, J., *Introduction à la théorie des suites monotones*, Kgl. Danske Vid. Selbsk. Forh. 1914, pp. 1-74.

[189] Hoffman, A. and O. Taussky, *A characterization of normal matrices*, J. Res. Nat. Bur. Standards, vol. 52, pp. 17-19 (1954).

[190] Hoffman, A. and H. Wielandt, *The variation of the spectrum of a normal matrix*, Duke Math. J., vol. 20, pp. 37-39 (1953).

[191] Horn, A., *On the eigenvalues of a matrix with prescribed singular values*, Proc. Amer. Math. Soc., vol. 5, pp. 4-7 (1954).

[192] Hotelling, H., *Some new methods in matrix calculation*, Ann. Math. Statist., vol. 14, pp. 1-34 (1943).

[193] Householder, A. S., *On matrices with non-negative elements*, Monatsh. Math., vol. 62, pp. 238-49 (1958).

[194] Householder, A. S. and F. L. Bauer, *On certain methods for expanding the characteristic polynomial*, Numer. Math., vol. 1, pp. 29-35 (1959).

[195] Hsu, P. L., *On symmetric, orthogonal, and skew-symmetric matrices*, Proc. Edinburgh Math. Soc., vol. 10, pp. 37-44 (1953).

[196] ——— _On a kind of transformation of matrices_, Acta Math. Sinica, vol. 5, pp. 333-47 (1955).

[197] Hua, L.-K., _On the theory of automorphic functions of a matrix variable_, Amer. J. Math., vol. 66, pp. 470-88; 531-63 (1944).

[198] ——— _Geometries of matrices_, Trans. Amer. Math. Soc., vol. 57, pp. 441-90 (1945).

[199] ——— _Orthogonal classification of Hermitian matrices_, Trans. Amer. Math. Soc., vol. 59, pp. 508-23 (1946).

*[200] ——— _Geometries of symmetric matrices over the real field_, Dokl. Akad. Nauk SSSR, vol. 53, pp. 95-98; 195-96 (1946).

*[201] ——— _Automorphisms of the real symplectic group_, Dokl. Akad. Nauk SSSR, vol. 53, pp. 303-306 (1946).

[202] ——— _Inequalities involving determinants_, Acta Math. Sinica, vol. 5, pp. 463-70 (1955).

*[203] Hua, L.-K. and B. A. Rosenfeld, _The geometry of rectangular matrices and their application to the real projective and non-euclidean geometries_, Izv. Higher Ed. SSSR, Matematika, vol. 1, pp. 233-46 (1957).

[204] Hurwitz, A., _Über die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt_, Math. Ann., vol. 46, pp. 273-84 (1895).

[205] Ingraham, M. H., _On the reduction of a matrix to its rational canonical form_, Bull. Amer. Math. Soc., vol. 39, pp. 379-82 (1933).

[206] Ionescu, D., _O identitate importantá si descompunere a unei forme bilineare into sumá de produse_, Gaz. Mat. Ser. Fiz. A. 7, vol. 7, pp. 303-312 (1955).

[207] Ishak, M., _Sur les spectres des matrices_, Sém. P. Dubreil et Ch. Pisot, Fac. Sci. Paris, vol. 9, pp. 1-14 (1955/56).

*[208] Kagan, V. F., _On some number systems arising from Lorentz transformations_, Izv. Ass. Inst. Moscow Univ. 1927, pp. 3-31.

*[209] Karpelevich, F. I., _On the eigenvalues of a matrix with non-negative elements_, Izv. Akad. Nauk SSSR Ser. Mat., vol. 15, pp. 361-83 (1951).

[210] Khan, N. A., _The characteristic roots of a product of matrices_, Quart. J. Math., vol. 7, pp. 138-43 (1956).

*[211] Khlodovskiĭ, I. N., _On the theory of the general case of Krylov's transformation of the secular equation_, Izv. Akad. Nauk, Ser. Fiz.-Mat., vol. 7, pp. 1076-1102 (1933).

*[212] Kolmogorov, A. N., _Markov chains with countably many possible states_, Bull. Univ. Moscow (A), vol. 1:3 (1937).

*[213] Kotelyanskiĭ, D. M., _On monotonic matrix functions of order n_, Trans. Univ. Odessa, vol. 3, pp. 103-114 (1941).

*[214] ——— _On the theory of non-negative and oscillatory matrices_, Ukrain. Mat. Z., vol. 2, pp. 94-101 (1950).

*[215] ——— _On some properties of matrices with positive elements_, Mat. Sb., vol. 31, pp. 497-506 (1952).

*[216] ——— _On a property of matrices of symmetric signs_, Uspehi Mat. Nauk, vol. 8, no. 4, pp. 163-67 (1953).

*[217] ——— _On some sufficient conditions for the spectrum of a matrix to be real and simple_, Mat. Sb., vol. 36, pp. 163-68 (1955).

*[218] ——— _On the influence of Gauss' transformation on the spectra of matrices_, Uspehi Mat. Nauk, vol. 9, no. 3, pp. 117-21 (1954).

*[219] ——— _On the distribution of points on a matrix spectrum_, Ukrain. Mat. Z., vol. 7, pp. 131-33 (1955).

*[220] ——— _Estimates for determinants of matrices with dominant main diagonal_, Izv. Akad. Nauk SSSR, Ser. Mat., vol. 20, pp. 137-44 (1956).

*[221] Kovalenko, K. R. and M. G. Kreĭn, _On some investigations of Lyapunov concerning differential equations with periodic coefficients_, Dokl. Akad. Nauk SSSR, vol. 75, pp. 495-99 (1950).

[222] Kowalewski, G., _Natürliche Normalformen linearer Transformationen_, Leipz. Ber., vol. 69, pp. 325-35 (1917).

*[223] Krasovskiĭ, N. N., _On the stability after the first approximation_, Prikl. Mat. Meh., vol. 19, pp. 516-30 (1955).

*[224] Krasnosel'skiĭ, M. A. and M. G. Kreĭn, _An iteration process with minimal deviations_, Mat. Sb., vol. 31, pp. 315-34 (1952).

[225] Kraus, F., _Über konvexe Matrixfunktionen_, Math. Z., vol. 41, pp. 18-42 (1936).

*[226] Kravchuk, M. F., _On the general theory of bilinear forms_, Izv. Polyt. Inst. Kiev, vol. 19, pp. 17-18 (1924).

*[227] ——— _On the theory of permutable matrices_, Zap. Akad. Nauk Kiev, Ser. Fiz.-Mat., vol. 1:2, pp. 28-33 (1924).

*[228] ——— _On a transformation of quadratic forms_, Zap. Akad. Nauk Kiev, Ser. Fiz.-Mat., vol. 1:2, pp. 87-90 (1924).

*[229] ——— _On quadratic forms and linear transformations_, Zap. Akad. Nauk Kiev, Ser. Fiz.-Mat., vol. 1:3, pp. 1-89 (1924).

*[230] ——— _Permutable sets of linear transformations_, Zap. Agr. Inst. Kiev, vol. 1, pp. 25-58 (1926).

[231] ——— _Über vertauschbare Matrizen_, Rend. Circ. Mat. Palermo, vol. 51, pp. 126-30 (1927).

*[232] ——— _On the structure of permutable groups of matrices_, Trans. Second. Mat. Congress 1934, vol. 2, pp. 11-12.

*[233] Kravchuk, M. F. and Y. S. Gol'dbaum, _On groups of commuting matrices_, Trans. Av. Inst. Kiev, 1929, pp. 73-98; 1936, pp. 12-23.

*[234] ——— _On the equivalence of singular pencils of matrices_, Trans. Av. Inst. Kiev, 1936, pp. 5-27.

*[235] Kreĭn, M. G., _Addendum to the paper 'On the structure of an orthogonal matrix,'_ Trans. Fiz.-Mat. Class. Akad. Nauk Kiev, 1931, pp. 103-7.

*[236] ——— _On the spectrum of a Jacobian form in connection with the theory of torsion oscillations of drums_, Mat. Sb., vol. 40, pp. 455-66 (1933).

*[237] ——— _On a new class of hermitian forms_, Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 9, pp. 1259-75 (1933).

*[238] ——— _On the nodes of harmonic oscillations of mechanical systems of a special type_, Mat. Sb., vol. 41, pp. 339-48 (1934).

[239] ——— _Sur quelques applications des noyaux de Kellog aux problèmes d'oscillations_, Proc. Charkov Mat. Soc. (4), vol. 11, pp. 3-19 (1935).

[240] ——— _Sur les vibrations propres des tiges dont l'une des extrémités est encastrée et l'autre libre_, Proc. Charkov. Mat. Soc. (4), vol. 12, pp. 3-11 (1935).

*[241] ——— _Generalization of some results of Lyapunov on linear differential equations with periodic coefficients_, Dokl. Akad. Nauk SSSR, vol. 73, pp. 445-48 (1950).

*[242] ——— _On an application of the fixed-point principle in the theory of linear transformations of spaces with indefinite metric_, Uspehi Mat. Nauk, vol. 5, no. 2, pp. 180-90 (1950).

*[243] ——— _On an application of an algebraic proposition in the theory of monodromy matrices_, Uspehi Mat. Nauk, vol. 6, no. 1, pp. 171-77 (1951).

*[244] ——— *On some problems concerning Lyapunov's ideas in the theory of stability,* Uspehi Mat. Nauk, vol. 3, no. 3, pp. 166-69 (1948).

*[245] ——— *On the theory of integral matrix functions of exponential type,* Ukrain. Mat. Z., vol. 3, pp. 164-73 (1951).

*[246] ——— *On some problems in the theory of oscillations of Sturm systems,* Prikl. Mat. Meh., vol. 16, pp. 555-68 (1952).

*[247] Kreĭn, M. G. and M. A. Naĭmark (Neumark), *On a transformation of the Bézoutian leading to Sturm's theorem,* Proc. Charkov Mat. Soc., (4), vol. 10, pp. 33-40 (1933).

*[248] ——— *On the application of the Bézoutian to problems of the separation of roots of algebraic equations,* Trudy Odessa Gos. Univ. Mat., vol. 1, pp. 51-69 (1935).

[249] Kronecker, L., *Algebraische Reduction der Schaaren bilinearer Formen,* S.-B. Akad. Berlin 1890, pp. 763-76.

[250] Krull, W., *Theorie und Anwendung der verallgemeinerten Abelschen Gruppen,* S.-B. Akad. Heidelberg 1926, p. 1.

*[251] Krylov, A. N., *On the numerical solution of the equation by which the frequency of small oscillations is determined in technical problems,* Izv. Akad. Nauk SSSR Ser. Fiz.-Mat., vol. 4, pp. 491-539 (1931).

[252] Lappo-Danilevskiĭ, I. A., *Résolution algorithmique des problèmes réguliers de Poincaré et de Riemann,* J. Phys. Mat. Soc. Leningrad, vols. 2:1, pp. 94-120; 121-54 (1928).

[253] ——— *Théorie des matrices satisfaisantes à des systèmes des équations differentielles linéaires a coefficients rationnels arbitraires,* J. Phys. Mat. Soc. Leningrad, vols. 2:2, pp. 41-80 (1928).

*[254] ——— *Fundamental problems in the theory of systems of linear differential equations with arbitrary rational coefficients,* Trans. First Math. Congr., ONTI, 1936, pp. 254-62.

[255] Ledermann, W., *Reduction of singular pencils of matrices,* Proc. Edinburgh Math. Soc., vol. 4, pp. 92-105 (1935).

[256] ——— *Bounds for the greatest latent root of a positive matrix,* J. London Math. Soc., vol. 25, pp. 265-68 (1950).

*[257] Lidskiĭ, V. B., *On the characteristic roots of a sum and a product of symmetric matrices,* Dokl. Akad. Nauk SSSR, vol. 75, pp. 769-72 (1950).

*[258] ——— *Oscillation theorems for canonical systems of differential equations,* Dokl. Akad. Nauk SSSR, vol. 102, pp. 111-17 (1955).

[259] Liénard, and Chipart, *Sur la signe de la partie réelle des racines d'une équation algébrique,* J. Math. Pures Appl. (6), vol. 10, pp. 291-346 (1914).

*[260] Lipin, N. V., *On regular matrices,* Trans. Inst. Eng 8. Transport, vol. 9, p. 105 (1934).

*[261] Livshitz, M. S. and V. P. Potapov, *The multiplication theorem for characteristic matrix functions,* Dokl. Akad. Nauk SSSR, vol. 72, pp. 164-73 (1950).

*[262] Lopshitz, A. M., *Vector solution of a problem on doubly symmetric matrices,* Trans. Math. Congress Moscow, 1927, pp. 186-87.

*[263] ——— *The characteristic equation of lowest degree for affinors and its application to the integration of differential equations,* Trans. Sem. Vectors and Tensors, vols. 2/3 (1935).

*[264] ——— *A numerical method of determining the characteristic roots and characteristic planes of a linear operator,* Trans. Sem. Vectors and Tensors, vol. 7, pp. 233-59 (1947).

*[265] ——— *An extremal theorem for a hyper-ellipsoid and its application to the solution of a system of linear algebraic equations,* Trans. Sem. Vectors and Tensors, vol. 9, pp. 183-97 (1952).

[266] Löwner, K., *Über monotone Matrixfunktionen,* Math. Z., vol. 38, pp. 177-216 (1933); vol. 41, pp. 18-42 (1936).

[267] ——— *Some classes of functions defined by difference on differential inequalities,* Bull. Amer. Math. Soc., vol. 56, pp. 308-19 (1950).

*[268] Lusin, N. N., *On Krylov's method of forming the secular equation,* Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 7, pp. 903-958 (1931).

*[269] ——— *On some properties of the displacement factor in Krylov's method,* Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 8, pp. 596-638; 735-62; 1065-1102 (1932).

*[270] ——— *On the matrix theory of differential equations,* Avtomat. i Telemeh, vol. 5, pp. 3-66 (1940).

*[271] Lyusternik, L. A., *The determination of eigenvalues of functions by an electric scheme,* Električestvo, vol. 11, pp. 67-8 (1946).

*[272] ——— *On electric models of symmetric matrices,* Uspehi Mat. Nauk, vol. 4, no. 2, pp. 198-200 (1949).

*[273] Lyusternik, L. A. and A. M. Prokhorov, *Determination of eigenvalues and functions of certain operators by means of an electrical network,* Dokl. Akad Nauk SSSR, vol. 55, pp. 579-82; Izv. Akad. Nauk SSSR, Ser. Mat., vol. 11, pp. 141-45 (1947).

[274] Marcus, M., *A remark on a norm inequality for square matrices,* Proc. Amer. Math. Soc., vol. 6, pp. 117-19 (1955).

[275] ——— *An eigenvalue inequality for the product of normal matrices,* Amer. Math. Monthly, vol. 63, pp. 173-74 (1956).

[276] ——— *A determinantal inequality of H. P. Robertson,* II, J. Washington Acad. Sci., vol. 47, pp. 264-66 (1957).

[277] ——— *Convex functions of quadratic forms,* Duke Math. J., vol. 24, pp. 321-26 (1957).

[278] Marcus, M. and J. L. McGregor, *Extremal properties of Hermitian matrices,* Canad. J. Math., vol. 8, pp. 524-31 (1956).

[279] Marcus, M. and B. N. Moyls, *On the maximum principle of Ky Fan,* Canad. J. Math., vol. 9, pp. 313-20 (1957).

[280] ——— *Maximum and minimum values for the elementary symmetric functions of Hermitian forms,* J. London Math. Soc., vol. 32, pp. 374-77 (1957).

*[281] Mayants, L. S., *A method for the exact determination of the roots of secular equations of high degree and a numerical analysis of their dependence on the parameters of the corresponding matrices,* Dokl. Akad. Nauk SSSR, vol. 50, pp. 121-24 (1945).

[282] Mirsky, L., *An inequality for positive-definite matrices,* Amer. Math. Monthly, vol. 62, pp. 428-30 (1955).

[283] ——— *The norm of adjugate and inverse matrices,* Arch. Math., vol. 7, pp. 276-77 (1956).

[284] ——— *The spread of a matrix,* Mathematika, vol. 3, pp. 127-30 (1956).

[285] ——— *Inequalities for normal and Hermitian matrices,* Duke Math. J., vol. 24, pp. 591-99 (1957).

[286] Mitrović, D., *Conditions graphiques pour que toutes les racines d'une équation algébrique soient à parties réelles négatives,* C. R. Acad. Sci. Paris, vol. 240, pp. 1177-79 (1955).

[287] Morgenstern, D., *Eine Verschärfung der Ostrowskischen Determinantenabschätzung,* Math. Z., vol. 66, pp. 143-46 (1956).

[288] Motzkin, T. and O. Taussky, *Pairs of matrices with property L.*, Trans. Amer. Math. Soc., vol. 73, pp. 108-14 (1952); vol. 80, pp. 387-401 (1954).

*[289] Neĭgaus (Neuhaus), M. G. and V. B. Lidskiĭ, *On the boundedness of the solutions of linear systems of differential equations with periodic coefficients*, Dokl. Akad. Nauk SSSR, vol. 77, pp. 183-93 (1951).

[290] Neumann, J., *Approximative of matrices of high order*, Portugal. Math., vol. 3, pp. 1-62 (1942).

*[291] Nudel'man, A. A. and P. A. Shvartsman, *On the spectrum of the product of unitary matrices*, Uspehi Mat. Nauk, vol. 13, no. 6, pp. 111-17 (1958).

[292] Okamoto, M., *On a certain type of matrices with an application to experimental design*, Osaka Math. J., vol. 6, pp. 73-82 (1954).

[293] Oppenheim, A., *Inequalities connected with definite Hermitian forms*, Amer. Math. Monthly, vol. 61, pp. 463-66 (1954).

[294] Orlando, L., *Sul problema di Hurwitz relativo alle parti reali delle radici di un' equatione algebrica*, Math. Ann., vol. 71, pp. 233-45 (1911).

[295] Ostrowski, A., *Bounds for the greatest latent root of a positive matrix*, J. London Math. Soc., vol. 27, pp. 253-56 (1952).

[296] ——— *Sur quelques applications des fonctions convexes et concaves au sens de I. Schur*, J. Math. Pures Appl., vol. 31, pp. 253-92 (1952).

[297] ——— *On nearly triangular matrices*, J. Res. Nat. Bur. Standards, vol. 52, pp. 344-45 (1954).

[298] ——— *On the spectrum of a one-parametric family of matrices*, J. Reine Angew. Math., vol. 193, pp. 143-60 (1954).

[299] ——— *Sur les déterminants à diagonale dominante*, Bul. Soc. Math. Belg., vol. 7, pp. 46-51 (1955).

[300] ——— *Note on bounds for some determinants*, Duke Math. J., vol. 22, pp. 95-102 (1955).

[301] ——— *Über Normen von Matrizen*, Math. Z., vol. 63, pp. 2-18 (1955).

[302] ——— *Über die Stetigkeit von charakteristischen Wurzeln in Abhängigkeit von den Matrizenelementen*, Jber. Deutsch. Math. Verein., vol. 60, pp. 40-42 (1957).

*[303] Papkovich, P. F., *On a method of computing the roots of a characteristic determinant*, Prikl. Mat. Meh., vol. 1, pp. 314-18 (1933).

[304] Papulis, A., *Limits on the zeros of a network determinant*, Quart. Appl. Math., vol. 15, pp. 193-94 (1957).

[305] Parodi, M., *Remarques sur la stabilité*, C. R. Acad. Sci. Paris, vol. 228, pp. 51-2; 807-8; 1198-1200 (1949).

[306] ——— *Sur une propriété des racines d'une équation qui intervient en mécanique*, C. R. Acad. Sci. Paris, vol. 241, pp. 1019-21 (1955).

[307] ——— *Sur la localisation des valeurs caractéristiques des matrices dans le plan complexe*, C. R. Acad. Sci. Paris, vol. 242, pp. 2617-18 (1956).

[308] Peano, G., *Intégration par series des équations différentielles linéaires*, Math. Ann., vol. 32, pp. 450-56 (1888).

[309] Penrose, R., *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc., vol. 51, pp. 406-13 (1955).

[310] ——— *On best approximate solutions of linear matrix equations*, Proc. Cambridge Philos. Soc., vol. 52, pp. 17-19 (1956).

[311] Perfect, H., *On matrices with positive elements*, Quart. J. Math., vol. 2, pp. 286-90 (1951).

[312] ——— *On positive stochastic matrices with real characteristic roots*, Proc. Cambridge Philos. Soc., vol. 48, pp. 271-76 (1952).

[313] ——— *Methods of constructing certain stochastic matrices*, Duke Math. J., vol. 20, pp. 395-404 (1953); vol. 22, pp. 305-11 (1955).

[314] ——— *A lower bound for the diagonal elements of a non-negative matrix*, J. London Math. Soc., vol. 31, pp. 491-93 (1956).

[315] Perron, O., *Jacobischer Kettenbruchalgorithmus*, Math. Ann., vol. 64, pp. 1-76 (1907).

[316] ——— *Über Matrizen*, Math. Ann., vol. 64, pp. 248-63 (1907).

[317] ——— *Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzengleichungen*, J. Reine Angew. Math., vol. 161, pp. 41-64 (1929).

[318] Phillips, H. B., *Functions of matrices*, Amer. J. Math., vol. 41, pp. 266-78 (1919).

*[319] Pontryagin, L. S., *Hermitian operators in a space with indefinite metric*, Izv. Akad. Nauk SSSR, Ser. Mat., vol. 8, pp. 243-80 (1944).

*[320] Potapov, V. P., *On holomorphic matrix functions bounded in the unit circle*, Dokl. Akad. Nauk SSSR, vol. 72, pp. 849-53 (1950).

[321] Rasch, G., *Zur Theorie und Anwendung der Produktintegrals*, J. Reine Angew. Math., vol. 171, pp. 65-119 (19534).

[322] Reichardt, H., *Einfarbe Herleitung der Jordanschen Normalform*, Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe, vol. 6, pp. 445-47 (1953/54).

*[323] Rechtman-Ol'shanskaya, P. G., *On a theorem of Markov*, Uspehi Mat. Nauk, vol. 12, no. 3, pp. 181-87 (1957).

[324] Rham, G. de, *Sur un théorème de Stieltjes relatif à certains matrices*, Acad. Serbe Sci. Publ. Inst. Math., vol. 4, pp. 133-54 (1952).

[325] Richter, H., *Über Matrixfunktionen*, Math. Ann., vol. 122, pp. 16-34 (1950).

[326] ——— *Bemerkung zur Norm der Inversen einer Matrix*, Arch. Math., vol. 5, pp. 447-48 (1954).

[327] ——— *Zur Abschätzung von Matrizennormen*, Math. Nachr., vol. 18, pp. 178-87 (1958).

[328] Romanovskiĭ, V. I., *Un théorème sur les zéros des matrices non-négatives*, Bull. Soc. Math. France, vol. 61, pp. 213-19 (1933).

[329] ——— *Recherches sur les chaines de Markoff*, Acta Math., vol. 66, pp. 147-251 (1935).

[330] Roth, W., *On the characteristic polynomial of the product of two matrices*, Proc. Amer. Math. Soc., vol. 5, pp. 1-3 (1954).

[331] ——— *On the characteristic polynomial of the product of several matrices*, Proc. Amer. Math. Soc., vol. 7, pp. 578-82 (1956).

[332] Roy, S., *A useful theorem in matrix theory*, Proc. Amer. Math. Soc., vol. 5, pp. 635-38 (1954).

*[333] Sakhnovich, L. A., *On the limits of multiplicative integrals*, Uspehi Mat. Nauk, vol. 12 no. 3, pp. 205-11 (1957).

*[334] Sarymsakov, T. A., *On sequences of stochastic matrices*, Dokl. Akad. Nauk, vol. 47, pp. 331-33 (1945).

[335] Schneider, H., *An inequality for latent roots applied to determinants with dominant principal diagonal*, J. London Math. Soc., vol. 28, pp. 8-20 (1953).

[336] ——— *A pair of matrices with property P*, J. Amer. Math. Monthly, vol. 62, pp. 247-49 (1955).

[337] ——— *A matrix problem concerning projections*, Proc. Edinburgh Math. Soc., vol. 10, pp. 129-30 (1956).

[338] ——— *The elementary divisors, associated with 0, of a singular M-matrix*, Proc. Edinburgh Math. Soc., vol. 10, pp. 108-22 (1956).

[339] Schoenberg, J., *Über variationsvermindernde lineare transformationen*, Math. Z., vol. 32, pp. 321-28 (1930).

[340] ——— *Zur abzählung der reellen wurzeln algebraischer gleichungen*, Math. Z., vol. 38, p. 546 (1933).

[341] Schoenberg, I. J., and A. Whitney, *A theorem on polygons in n dimensions with application to variation diminishing linear transformations*, Compositio Math., vol. 9, pp. 141-60 (1951).

[342] Schur, I., *Über die charakteristischen wurzeln einer linearen substitution mit einer anwendung auf die theorie der integralgleichungen*, Math. Ann., vol. 66, pp. 488-510 (1909).

*[343] Semendyaev, K. A., *On the determination of the eigenvalues and invariant manifolds of matrices by means of iteration*, Prikl. Matem. Meh., vol. 3, pp. 193-221 (1943).

*[344] Sevast'yanov, B. A., *The theory of branching random processes*, Uspehi Mat. Nauk, vol. 6, no. 6, pp. 46-99 (1951).

*[345] Shiffner, L. M., *The development of the integral of a system of differential equations with regular singularities in series of powers of the elements of the differential substitution*, Trudy Mat. Inst. Steklov. vol. 9, pp. 235-66 (1935).

*[346] ——— *On the powers of matrices*, Mat. Sb., vol. 42, pp. 385-94 (1935).

[347] Shoda, K., *Über mit einer matrix vertauschbare matrizen*, Math. Z., vol. 29, pp. 696-712 (1929).

*[348] Shostak, P. Y., *On a criterion for the conditional definiteness of quadratic forms in n linearly independent variables and on a sufficient condition for a conditional extremum of a function of n variables*, Uspehi Mat. Nauk, vol. 8, no. 4, pp. 199-206 (1954).

*[349] Shreĭder, Y. A., *A solution of systems of linear algebraic equations*, Dokl. Akad. Nauk, vol. 76, pp. 651-55 (1950).

*[350] Shtaerman (Steiermann), I. Y., *A new method for the solution of certain algebraic equations which have application to mathematical physics*, Z. Mat., Kiev, vol. 1, pp. 83-89 (1934); vol. 4, pp. 9-20 (1934).

*[351] Shtaerman (Steiermann), I. Y. and N. I. Akhieser (Achieser), *On the theory of quadratic forms*, Izv. Polyteh., Kiev, vol. 19, pp. 116-23 (1934).

*[352] Shura-Bura, M. R., *An estimate of error in the numerical computation of matrices of high order*, Uspehi Mat. Nauk, vol. 6, no. 4, pp. 121-50 (1951).

*[353] Shvartsman (Schwarzmann), A. P., *On Green's matrices of self-adjoint differential operators*, Proc. Odessa Univ. Matematika, vol. 3, pp. 35-77 (1941).

[354] Siegel, C. L., *Symplectic Geometry*, Amer. J. Math., vol. 65, pp. 1-86 (1943).

*[355] Skal'kina, M. A., *On the preservation of asymptotic stability on transition from differential equations to the corresponding difference equations*, Dokl. Akad. Nauk SSSR, vol. 104, pp. 505-8, (1955).

*[356] Smogorzhevskiĭ, A. S., *Sur les matrices unitaires du type de circulants*, J. Mat. Circle Akad. Nauk Kiev, vol. 1, pp. 89-91 (1932).

*[356a] Smogorzhevskiĭ, A. S. and M. F. Kravchuk, *On orthogonal transformations*, Zap. Inst. Ed. Kiev, vol. 2, pp. 151-56 (1927).

[357] Stenzel, H., *Über die Darstellbarkeit einer Matrix als Produkt von zwei symmetrischen Matrizen*, Math. Z., vol. 15, pp. 1-25 (1922).

[358] Stöhr, A., *Oszillationstheoreme für die Eigenvektoren speziellen Matrizen*, J. Reine Angew. Math., vol. 185, pp. 129-43 (1943).

*[359] Suleĭmanova, K. R., *Stochastic matrices with real characteristic values*, Dokl. Akad. Nauk SSSR, vol. 66, pp. 343-45 (1949).

*[360] ——— *On the characteristic values of stochastic matrices*, Uč. Zap. Moscow Ped. Inst., Ser. 71, Math., vol. 1, pp. 167-97 (1953).

*[361] Sultanov, R. M., *Some properties of matrices with elements in a non-commutative ring*, Trudy Mat. Sectora Akad. Nauk Baku, vol. 2, pp. 11-17 (1946).

*[362] Sushkevich, A. K., *On matrices of a special type*, Uč. Zap. Univ. Charkov, vol. 10, pp. 1-16 (1937).

[363] Sz-Nagy, B., *Remark on S. N. Roy's paper 'A useful theorem in matrix theory,'* Proc. Amer. Math. Soc., vol. 7, p. 1 (1956).

[364] Ta Li, *Die Stabilitätsfrage bei Differenzengleichungen*, Acta Math., vol. 63, pp. 99-141 (1934).

[365] Taussky, O., *Bounds for characteristic roots of matrices*, Duke Math. J., vol. 15, pp. 1043-44 (1948).

[366] ——— *A determinantal inequality of H. P. Robertson*, I, J. Washington Acad. Sci., vol. 47, pp. 263-64 (1957).

[367] ——— *Commutativity in finite matrices*, Amer. Math. Monthly, vol. 64, pp. 229-35 (1957).

[368] Toeplitz, O., *Das algebraische Analogon zu einem Satz von Fejér*, Math. Z., vol. 2, pp. 187-97 (1918).

[369] Turnbull, H. W., *On the reduction of singular matrix pencils*, Proc. Edinburgh Math. Soc., vol. 4, pp. 67-76 (1935).

*[370] Turchaninov, A. S., *On some applications of matrix calculus to linear differential equations*, Uč. Zap. Univ. Odessa, vol. 1, pp. 41-48 (1921).

*[371] Verzhbitskiĭ, B. D., *Some problems in the theory of series compounded from several matrices*, Mat. Sb., vol. 5, pp. 505-12 (1939).

*[372] Vilenkin, N. Y., *On an estimate for the maximal eigenvalue of a matrix*, Uč. Zap. Moscow Ped. Inst., vol. 108, pp. 55-57 (1957).

[373] Vivier, M., *Note sur les structures unitaires et paraunitaires*, C. R. Acad. Sci. Paris, vol. 240, pp. 1039-41 (1955).

[374] Volterra, V., *Sui fondamenti della teoria delle equazioni differenziali lineari*, Mem. Soc. Ital. Sci. (3), vol. 6, pp. 1-104 (1887); vol. 12, pp. 3-68 (1902).

[375] Walker, A. and J. Weston, *Inclusion theorems for the eigenvalues of a normal matrix*, J. London Math. Soc., vol. 24, pp. 28-31 (1949).

[376] Wayland, H., *Expansions of determinantal equations into polynomial form*, Quart. Appl. Math., vol. 2, pp. 277-306 (1945).

[377] Weierstrass, K., *Zur theorie der bilinearen und quadratischen Formen*, Monatsh. Akad. Wiss. Berlin, 1867, pp. 310-38.

[378] Wellstein, J., *Über symmetrische, alternierende und orthogonale Normalformen von Matrizen*, J. Reine Angew. Math., vol. 163, pp. 166-82 (1930).

[379] Weyl, H., *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad. Sci., vol. 35, pp. 408-11 (1949).

[380] Weyr, E., *Zur Theorie der bilinearen Formen*, Monatsh. f. Math. und Physik, vol. 1, pp. 163-236 (1890).

[381] Whitney, A., *A reduction theorem for totally positive matrices*, J. Analyse Math., vol. 2, pp. 88-92 (1952).

[382] Wielandt, H., *Ein Einschliessungssatz für charakteristische Wurzeln normaler Matrizen*, Arch. Math., vol. 1, pp. 348-52 (1948/49).

[383] ——— *Die Einschliessung von Eigenwerten normaler Matrizen*, Math. Ann. vol. 121, pp. 234-41 (1949).

[384] ——— *Unzerlegbare nicht-negative Matrizen*, Math. Z., vol. 52, pp. 642-48 (1950).

[385] ——— *Lineare Scharen von Matrizen mit reellen Eigenwerten*, Math. Z., vol. 53, pp. 219-25 (1950).

[386] ——— *Pairs of normal matrices with property L*, J. Res. Nat. Bur. Standards, vol. 51, pp. 89-90 (1953).

[387] ——— *Inclusion theorems for eigenvalues*, Nat. Bur. Standards, Appl. Math. Sci., vol. 29, pp. 75-78 (1953).

[388] ——— *An extremum property of sums of eigenvalues*, Proc. Amer. Math. Soc., vol. 6, pp. 106-110 (1955).

[389] ——— *On eigenvalues of sums of normal matrices*, Pacific J. Math., vol. 5, pp. 633-38 (1955).

[390] WINTNER, A., *On criteria for linear stability*, J. Math. Mech., vol. 6, pp. 301-9 (1957).

[391] WONG, Y., *An inequality for Minkowski matrices*, Proc. Amer. Math. Soc., vol. 4, pp. 137-41 (1953).

[392] ——— *On non-negative valued matrices*, Proc. Nat. Acad. Sci. U.S.A., vol. 40, pp. 121-24 (1954).

*[393] YAGLOM, I. M., *Quadratic and skew-symmetric bilinear forms in a real symplectic space*, Trudy Sem. Vect. Tens. Anal. Moscow, vol. 8, pp. 364-81 (1950).

*[394] YAKUBOVICH, V. A., *Some criteria for the reducibility of a system of differential equations*, Dokl. Akad. Nauk SSSR, vol. 66, pp. 577-80 (1949).

*[395] ZEITLIN (Tseĭtlin), M. L., *Application of the matrix calculus to the synthesis of relay-contact schemes*, Dokl. Akad. Nauk SSSR, vol. 86, pp. 525-28 (1952).

*[396] ZIMMERMANN (Tsimmerman), G. K., *Decomposition of the norm of a matrix into products of norms of its rows*, Nauč. Zap. Ped. Inst. Nikolaevsk, vol. 4, pp. 130-35 (1953).

INDEX

INDEX

[Numbers in italics refer to Volume Two]

THE THEORY OF MATRICES

BY

F. R. GANTMACHER

VOLUME TWO

CHELSEA PUBLISHING COMPANY
NEW YORK, N. Y.

LIBRARY OF CONGRESS CATALOG CARD NUMBER 59-11779

REPRINTED, 1964

THE PRESENT WORK, PUBLISHED IN TWO VOLUMES, IS AN ENGLISH TRANSLATION, BY K. A. HIRSCH, OF THE RUSSIAN-LANGUAGE BOOK TEORIYA MATRITS BY F. R. GANTMACHER (Гантмахер)

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

THE MATRIX CALCULUS is widely applied nowadays in various branches of mathematics, mechanics, theoretical physics, theoretical electrical engineering, etc. However, neither in the Soviet nor the foreign literature is there a book that gives a sufficiently complete account of the problems of matrix theory and of its diverse applications. The present book is an attempt to fill this gap in the mathematical literature.

The book is based on lecture courses on the theory of matrices and its applications that the author has given several times in the course of the last seventeen years at the Universities of Moscow and Tiflis and at the Moscow Institute of Physical Technology.

The book is meant not only for mathematicians (undergraduates and research students) but also for specialists in allied fields (physics, engineering) who are interested in mathematics and its applications. Therefore the author has endeavoured to make his account of the material as accessible as possible, assuming only that the reader is acquainted with the theory of determinants and with the usual course of higher mathematics within the programme of higher technical education. Only a few isolated sections in the last chapters of the book require additional mathematical knowledge on the part of the reader. Moreover, the author has tried to keep the individual chapters as far as possible independent of each other. For example, Chapter V, *Functions of Matrices*, does not depend on the material contained in Chapters II and III. At those places of Chapter V where fundamental concepts introduced in Chapter IV are being used for the first time, the corresponding references are given. Thus, a reader who is acquainted with the rudiments of the theory of matrices can immediately begin with reading the chapters that interest him.

The book consists of two parts, containing fifteen chapters.

In Chapters I and III, information about matrices and linear operators is developed *ab initio* and the connection between operators and matrices is introduced.

Chapter II expounds the theoretical basis of Gauss's elimination method and certain associated effective methods of solving a system of n linear equations, for large n. In this chapter the reader also becomes acquainted with the technique of operating with matrices that are divided into rectangular 'blocks.'

In Chapter IV we introduce the extremely important 'characteristic' and 'minimal' polynomials of a square matrix, and the 'adjoint' and 'reduced adjoint' matrices.

In Chapter V, which is devoted to functions of matrices, we give the general definition of $f(A)$ as well as concrete methods of computing it—where $f(\lambda)$ is a function of a scalar argument λ and A is a square matrix. The concept of a function of a matrix is used in §§ 5 and 6 of this chapter for a complete investigation of the solutions of a system of linear differential equations of the first order with constant coefficients. Both the concept of a function of a matrix and this latter investigation of differential equations are based entirely on the concept of the minimal polynomial of a matrix and—in contrast to the usual exposition—do not use the so-called theory of elementary divisors, which is treated in Chapters VI and VII.

These five chapters constitute a first course on matrices and their applications. Very important problems in the theory of matrices arise in connection with the reduction of matrices to a normal form. This reduction is carried out on the basis of Weierstrass' theory of elementary divisors. In view of the importance of this theory we give two expositions in this book: an analytic one in Chapter VI and a geometric one in Chapter VII. We draw the reader's attention to §§ 7 and 8 of Chapter VI, where we study effective methods of finding a matrix that transforms a given matrix to normal form. In § 8 of Chapter VII we investigate in detail the method of A. N. Krylov for the practical computation of the coefficients of the characteristic polynomial.

In Chapter VIII certain types of matrix equations are solved. We also consider here the problem of determining all the matrices that are permutable with a given matrix and we study in detail the many-valued functions of matrices $\sqrt[m]{A}$ and $\ln A$.

Chapters IX and X deal with the theory of linear operators in a unitary space and the theory of quadratic and hermitian forms. These chapters do not depend on Weierstrass' theory of elementary divisors and use, of the preceding material, only the basic information on matrices and linear operators contained in the first three chapters of the book. In § 9 of Chapter X we apply the theory of forms to the study of the principal oscillations of a system with n degrees of freedom. In § 11 of this chapter we give an account of Frobenius' deep results on the theory of Hankel forms. These results are used later, in Chapter XV, to study special cases of the Routh-Hurwitz problem.

The last five chapters form the second part of the book [the second volume, in the present English translation]. In Chapter XI we determine normal forms for complex symmetric, skew-symmetric, and orthogonal matrices and establish interesting connections of these matrices with real matrices of the same classes and with unitary matrices.

In Chapter XII we expound the general theory of pencils of matrices of the form $A + \lambda B$, where A and B are arbitrary rectangular matrices of the same dimensions. Just as the study of regular pencils of matrices $A + \lambda B$ is based on Weierstrass' theory of elementary divisors, so the study of singular pencils is built upon Kronecker's theory of minimal indices, which is, as it were, a further development of Weierstrass's theory. By means of Kronecker's theory—the author believes that he has succeeded in simplifying the exposition of this theory—we establish in Chapter XII canonical forms of the pencil of matrices $A + \lambda B$ in the most general case. The results obtained there are applied to the study of systems of linear differential equations with constant coefficients.

In Chapter XIII we explain the remarkable spectral properties of matrices with non-negative elements and consider two important applications of matrices of this class: 1) homogeneous Markov chains in the theory of probability and 2) oscillatory properties of elastic vibrations in mechanics. The matrix method of studying homogeneous Markov chains was developed in the book [46] by V. I. Romanovskii and is based on the fact that the matrix of transition probabilities in a homogeneous Markov chain with a finite number of states is a matrix with non-negative elements of a special type (a 'stochastic' matrix).

The oscillatory properties of elastic vibrations are connected with another important class of non-negative matrices—the 'oscillation matrices.' These matrices and their applications were studied by M. G. Krein jointly with the author of this book. In Chapter XIII, only certain basic results in this domain are presented. The reader can find a detailed account of the whole material in the monograph [17].

In Chapter XIV we compile the applications of the theory of matrices to systems of differential equations with variable coefficients. The central place (§§ 5-9) in this chapter belongs to the theory of the multiplicative integral (Produktintegral) and its connection with Volterra's infinitesimal calculus. These problems are almost entirely unknown in Soviet mathematical literature. In the first sections and in § 11, we study reducible systems (in the sense of Lyapunov) in connection with the problem of stability of motion; we also give certain results of N. P. Erugin. Sections 9-11 refer to the analytic theory of systems of differential equations. Here we clarify an inaccuracy in Birkhoff's fundamental theorem, which is usually applied to the investigation of the solution of a system of differential equations in the neighborhood of a singular point, and we establish a canonical form of the solution in the case of a regular singular point.

In § 12 of Chapter XIV we give a brief survey of some results of the fundamental investigations of I. A. Lappo-Danilevskiĭ on analytic functions of several matrices and their applications to differential systems.

The last chapter, Chapter XV, deals with the applications of the theory of quadratic forms (in particular, of Hankel forms) to the Routh-Hurwitz problem of determining the number of roots of a polynomial in the right half-plane ($\mathrm{Re}\, z > 0$). The first sections of the chapter contain the classical treatment of the problem. In § 5 we give the theorem of A. M. Lyapunov in which a stability criterion is set up which is equivalent to the Routh-Hurwitz criterion. Together with the stability criterion of Routh-Hurwitz we give, in § 11 of this chapter, the comparatively little known criterion of Liénard and Chipart in which the number of determinant inequalities is only about half of that in the Routh-Hurwitz criterion.

At the end of Chapter XV we exhibit the close connection between stability problems and two remarkable theorems of A. A. Markov and P. L. Chebyshev, which were obtained by these celebrated authors on the basis of the expansion of certain continued fractions of special types in series of decreasing powers of the argument. Here we give a matrix proof of these theorems.

This, then, is a brief summary of the contents of this book.

F. R. Gantmacher

PUBLISHERS' PREFACE

THE PUBLISHERS WISH TO thank Professor Gantmacher for his kindness in communicating to the translator new versions of several paragraphs of the original Russian-language book.

The Publishers also take pleasure in thanking the VEB Deutscher Verlag der Wissenschaften, whose many published translations of Russian scientific books into the German language include a counterpart of the present work, for their kind spirit of cooperation in agreeing to the use of their formulas in the preparation of the present work.

No material changes have been made in the text in translating the present work from the Russian except for the replacement of several paragraphs by the new versions supplied by Professor Gantmacher. Some changes in the references and in the Bibliography have been made for the benefit of the English-language reader.

CONTENTS

XIV. APPLICATIONS OF THE THEORY OF MATRICES TO THE INVESTIGATION OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS 113

XV. THE PROBLEM OF ROUTH-HURWITZ AND RELATED QUESTIONS 172

CHAPTER XI

COMPLEX SYMMETRIC, SKEW-SYMMETRIC, AND ORTHOGONAL MATRICES

In Volume I, Chapter IX, in connection with the study of linear operators in a euclidean space, we investigated real symmetric, skew-symmetric, and orthogonal matrices, i.e., real square matrices characterized by the relations†.

$$S^{\mathsf{T}} = S,\ K^{\mathsf{T}} = -K,\ \text{and}\ Q^{\mathsf{T}} = Q^{-1},$$

respectively (here Q^{T} denotes the transpose of the matrix Q). We have shown that in the field of complex numbers all these matrices have linear elementary divisors and we have set up normal forms for them, i.e., 'simplest' real symmetric, skew-symmetric, and orthogonal matrices to which arbitrary matrices of the types under consideration are real-similar and orthogonally similar.

The present chapter deals with the investigation of complex symmetric, skew-symmetric, and orthogonal matrices. We shall clarify the question of what elementary divisors these matrices can have and shall set up normal forms for them. These forms have a considerably more complicated structure than the corresponding normal forms in the real case. As a preliminary, we shall establish in the first section interesting connections between complex orthogonal and unitary matrices on the one hand, and real symmetric, skew-symmetric, and orthogonal matrices on the other hand.

§ 1. Some Formulas for Complex Orthogonal and Unitary Matrices

1. We begin with a lemma:

LEMMA 1:[1] 1. *If a matrix* G *is both hermitian and orthogonal* ($G^{\mathsf{T}} = \overline{G} = G^{-1}$), *then it can be represented in the form*

$$G = Ie^{iK}, \tag{1}$$

where I *is a real symmetric involutory matrix and* K *a real skew-symmetric matrix permutable with it:*

---

[1] See [169], pp. 223-225.

† In this and in the following chapters, a matrix denoted by the letter Q is not necessarily orthogonal.

$$I=\overline{I}=I^{\mathsf{T}},\ I^2=E,\ K=\overline{K}=-K^{\mathsf{T}}. \tag{2}$$

2. *If, in addition, G is a positive-definite hermitian matrix,*[2] *then in* (1) $I=E$ *and*

$$G=e^{iK}. \tag{3}$$

Proof. Let

$$G=S+iT, \tag{4}$$

where S and T are real matrices. Then

$$\overline{G}=S-iT \text{ and } G^{\mathsf{T}}=S^{\mathsf{T}}+iT^{\mathsf{T}}. \tag{5}$$

Therefore the equation $\overline{G}=G^{\mathsf{T}}$ implies that $S=S^{\mathsf{T}}$ and $T=-T^{\mathsf{T}}$, i.e., S is symmetric and T skew-symmetric.

Moreover, when the expressions for G and $\overline{G}$ from (4) and (5) are substituted in the complex equation $G\overline{G}=E$, it breaks up into two real equations:

$$S^2+T^2=E \text{ and } ST=TS. \tag{6}$$

The second of these equations shows that S and T commute.

By Theorem 12′ of Chapter IX (Vol. I, p. 292), the commuting normal matrices S and T can be carried simultaneously into quasi-diagonal form by a real orthogonal transformation. Therefore[3]

$$\left.\begin{aligned} S&=Q\{s_1, s_1, s_2, s_2, \ldots, s_q, s_q, s_{2q+1}, \ldots, s_n\}Q^{-1}, \\ T&=Q\left\{\left\|\begin{matrix} 0 & t_1 \\ -t_1 & 0 \end{matrix}\right\|, \left\|\begin{matrix} 0 & t_2 \\ -t_2 & 0 \end{matrix}\right\|, \ldots, \left\|\begin{matrix} 0 & t_q \\ -t_q & 0 \end{matrix}\right\|, 0, \ldots, 0\right\}Q^{-1} \end{aligned}\right\} \quad (Q=\overline{Q}=Q^{\mathsf{T}-1}) \tag{7}$$

(the numbers s_i and t_i are real). Hence

$$G=S+iT=Q\left\{\left\|\begin{matrix} s_1 & it_1 \\ -it_1 & s_1 \end{matrix}\right\|, \left\|\begin{matrix} s_2 & it_2 \\ -it_2 & s_2 \end{matrix}\right\|, \ldots, \left\|\begin{matrix} s_q & it_q \\ -it_q & s_q \end{matrix}\right\|, s_{2q+1}, \ldots, s_n\right\}Q^{-1}. \tag{8}$$

On the other hand, when we compare the expressions (7) for S and T with the first of the equations (6), we find:

$$s_1^2-t_1^2=1,\quad s_2^2-t_2^2=1, \ldots,\quad s_q^2-t_q^2=1,\quad s_{2q+1}=\pm 1, \ldots,\quad s_n=\pm 1. \tag{9}$$

---

[2] I.e., G is the coefficient matrix of a positive-definite hermitian form (see Vol. I, Chapter X, § 9).

[3] See also the Note following Theorem 12′ of Vol. I, Chapter IX (p. 293).

Now it is easy to verify that a matrix of the type $\left\|\begin{matrix} s & it \\ -it & s \end{matrix}\right\|$ with $s^2-t^2=1$ can always be represented in the form

$$\left\|\begin{matrix} s & it \\ -it & s \end{matrix}\right\|=\varepsilon e^{i\left\|\begin{smallmatrix} 0 & \varphi \\ -\varphi & 0 \end{smallmatrix}\right\|},$$

where

$$|s|=\cosh\varphi, \quad \varepsilon t=\sinh\varphi, \quad \varepsilon=\operatorname{sign} s.$$

Therefore we have from (8) and (9):

$$G=Q\{\pm e^{i\left\|\begin{smallmatrix} 0 & \varphi_1 \\ -\varphi_1 & 0 \end{smallmatrix}\right\|}, \pm e^{i\left\|\begin{smallmatrix} 0 & \varphi_2 \\ -\varphi_2 & 0 \end{smallmatrix}\right\|}, \ldots, \pm e^{i\left\|\begin{smallmatrix} 0 & \varphi_q \\ -\varphi_q & 0 \end{smallmatrix}\right\|}, \pm 1, \ldots, \pm 1\}Q^{-1}, \tag{10}$$

i.e.,

$$G=Ie^{iK},$$

where

$$\left.\begin{aligned} I&=Q\{\pm 1, \pm 1, \ldots, \pm 1\}Q^{-1}, \\ K&=Q\left\{\left\|\begin{matrix} 0 & \varphi_1 \\ -\varphi_1 & 0 \end{matrix}\right\|, \ldots, \left\|\begin{matrix} 0 & \varphi_q \\ -\varphi_q & 0 \end{matrix}\right\|, 0, \ldots, 0\right\}Q^{-1} \end{aligned}\right\} \tag{11}$$

and

$$IK=KI.$$

From (11) there follows the equation (2).

2. If, in addition, it is known that G is a positive-definite hermitian matrix, then we can state that all the characteristic values of G are positive (see Volume I, Chapter IX, p. 270). But by (10) these characteristic values are

$$\pm e^{\varphi_1}, \pm e^{-\varphi_1}, \pm e^{\varphi_2}, \pm e^{-\varphi_2}, \ldots, \pm e^{\varphi_q}, \pm e^{-\varphi_q}, \pm 1, \ldots, \pm 1$$

(here the signs correspond to the signs in (10)).

Therefore in the formula (10) and the first formula of (11), wherever the sign ± occurs, the + sign must hold. Hence

$$I=Q\{1, 1, \ldots, 1\}Q^{-1}=E,$$

and this is what we had to prove.

This completes the proof of the lemma.

By means of the lemma we shall now prove the following theorem:

THEOREM 1: *Every complex orthogonal matrix Q can be represented in the form*

$$Q = Re^{iK}, \tag{12}$$

where R is a real orthogonal matrix and K a real skew-symmetric matrix

$$R = \bar{R} = R^{\top -1}, \quad K = \bar{K} = -K^{\top}. \tag{13}$$

Proof. Suppose that (12) holds. Then

$$Q^* = \bar{Q}^{\top} = e^{iK} R^{\top}$$

and

$$Q^* Q = e^{iK} R^{\top} R e^{iK} = e^{2iK}.$$

By the preceding lemma the required real skew-symmetric matrix K can be determined from the equation

$$Q^* Q = e^{2iK} \tag{14}$$

because the matrix Q^*Q is positive-definite hermitian and orthogonal. After K has been determined from (14) we can find R from (12):

$$R = Qe^{-iK}. \tag{15}$$

Then

$$R^* R = e^{-iK} Q^* Q e^{-iK} = E;$$

i.e., R is unitary. On the other hand, it follows from (15) that R, as the product of two orthogonal matrices, is itself orthogonal: $R^{\top}R = E$. Thus R is at the same time unitary and orthogonal, and hence real. The formula (15) can be written in the form (12).

This proves the theorem.[4]

Now we establish the following lemma:

LEMMA 2: *If a matrix D is both symmetric and unitary $(D = D^{\top} = \bar{D}^{-1})$, then it can be represented in the form*

$$D = e^{iS}, \tag{16}$$

where S is a real symmetric matrix $(S = \bar{S} = S^{\top})$.

---

[4] The formula (12), like the polar decomposition of a complex matrix (in connection with the formulas (87), (88) on p. 278 of Vol. I) has a close connection with the important Theorem of Cartan which establishes a certain representation for the automorphisms of the complex Lie groups; see [169], pp. 232-233.

Proof. We set

$$D = U + iV \quad (U = \bar{U},\ V = \bar{V}). \tag{17}$$

Then

$$\bar{D} = U - iV, \quad D^{\top} = U^{\top} + iV^{\top}.$$

The complex equation $D = D^{\top}$ splits into the two real equations

$$U = U^{\top}, \quad V = V^{\top}.$$

Thus, U and V are real symmetric matrices.

The equation $D\bar{D} = E$ implies:

$$U^2 + V^2 = E, \quad UV = VU. \tag{18}$$

By the second of these equations, U and V commute. When we apply Theorem 12′ (together with the Note) of Chapter IX (Vol. I, pp. 292-3) to them, we obtain:

$$U = Q\{s_1, s_2, \dots, s_n\}Q^{-1}, \quad V = Q\{t_1, t_2, \dots, t_n\}Q^{-1}. \tag{19}$$

Here s_k and t_k $(k = 1, 2, \dots, n)$ are real numbers. Now the first of the equations (18) yields:

$$s_k^2 + t_k^2 = 1 \quad (k = 1, 2, \dots, n).$$

Therefore there exist real numbers φ_k $(k = 1, 2, \dots, n)$ such that

$$s_k = \cos\varphi_k, \quad t_k = \sin\varphi_k \quad (k = 1, 2, \dots, n).$$

Substituting these expressions for s_k and t_k in (19) and using (17), we find:

$$D = Q\{e^{i\varphi_1}, e^{i\varphi_2}, \dots, e^{i\varphi_n}\}Q^{-1} = e^{iS},$$

where

$$S = Q\{\varphi_1, \varphi_2, \dots, \varphi_n\}Q^{-1}. \tag{20}$$

From (20) it follows that $S = \bar{S} = S^{\top}$.

This proves the lemma.

Using the lemma we shall now prove the following theorem:

THEOREM 2: *Every unitary matrix U can be represented in the form*

$$U = Re^{iS}, \tag{21}$$

where R is a real orthogonal matrix and S a real symmetric matrix

$$R = \bar{R} = R^{\top -1}, \ S = \bar{S} = S^{\top}. \tag{22}$$

Proof. From (21) it follows that

$$U^\top = e^{iS} R^\top . \tag{23}$$

Multiplying (21) and (23), we obtain from (22):

$$U^\top U = e^{iS} R^\top R e^{iS} = e^{2iS} .$$

By Lemma 2, the real symmetric matrix S can be determined from the equation

$$U^\top U = e^{2iS} \tag{24}$$

because $U^\top U$ is symmetric and unitary. After S has been determined, we determine R by the equation

$$R = U e^{-iS} . \tag{25}$$

Then

$$R^\top = e^{-iS} U^\top , \tag{26}$$

and so from (24), (25), and (26) it follows that

$$R^\top R = e^{-iS} U^\top U e^{-iS} = E ,$$

i.e., R is orthogonal.

On the other hand, by (25) R is the product of two unitary matrices and is therefore itself unitary. Since R is both orthogonal and unitary, it is real. Formula (25) can be written in the form (21).

This proves the theorem.

§ 2. Polar Decomposition of a Complex Matrix

We shall prove the following theorem:

THEOREM 3: *If $A = \| a_{ik} \|_1^n$ is a non-singular matrix with complex elements, then*

$$A = SQ \tag{27}$$

and

$$A = Q_1 S_1 , \tag{28}$$

where S and S_1 are complex symmetric matrices, Q and Q_1 complex orthogonal matrices. Moreover,

$$S = \sqrt{AA^\top} = f(AA^\top), \quad S_1 = \sqrt{A^\top A} = f_1(A^\top A),$$

where $f(\lambda)$, $f_1(\lambda)$ are polynomials in λ.

The factors S and Q in (27) (Q_1 and S_1 in (28)) are permutable if and only if A and $A^\top$ are permutable.

Proof. It is sufficient to establish (27), for when we apply this decomposition to the matrix $A^\top$ and determine A from the formula thus obtained, we arrive at (28).

If (27) holds, then

$$A = SQ , \quad A^\top = Q^{-1} S$$

and therefore

$$AA^\top = S^2 . \tag{29}$$

Conversely, since $AA^\top$ is non-singular ($|AA^\top| = |A|^2 \neq 0$), the function $\sqrt{\lambda}$ is defined on the spectrum of this matrix[5] and therefore an interpolation polynomial $f(\lambda)$ exists such that

$$\sqrt{AA^\top} = f(AA^\top) . \tag{30}$$

We denote the symmetric matrix (30) by

$$S = \sqrt{AA^\top} .$$

Then (29) holds, and so $|S| \neq 0$. Determining Q from (27)

$$Q = S^{-1} A ,$$

we verify easily that it is an orthogonal matrix. Thus (27) is established.

If the factors S and Q in (27) are permutable, then the matrices

$$A = SQ \quad \text{and} \quad A^\top = Q^{-1} S$$

are permutable, since

$$AA^\top = S^2 , \quad A^\top A = Q^{-1} S^2 Q .$$

Conversely, if $AA^\top = A^\top A$, then

$$S^2 = Q^{-1} S^2 Q ,$$

i.e., Q is permutable with $S^2 = AA^\top$. But then Q is also permutable with the matrix $S = f(AA^\top)$.

Thus the theorem is proved completely.

2. Using the polar decomposition we shall now prove the following theorem:

---

[5] See Vol. I, Chapter V, § 1. We choose a single-valued branch of the function $\sqrt{\lambda}$ in a simply connected domain containing all the characteristic values of $AA^\top$, but not the number 0.

THEOREM 4: *If two complex symmetric or skew-symmetric or orthogonal matrices are similar:*

$$B=T^{-1}AT, \tag{31}$$

then they are orthogonally similar; i.e., there exists an orthogonal matrix Q such that

$$B=Q^{-1}AQ. \tag{32}$$

Proof. From the conditions of the theorem there follows the existence of a polynomial $q(\lambda)$ such that

$$A^{\mathsf{T}}=q(A), \quad B^{\mathsf{T}}=q(B). \tag{33}$$

In the case of symmetric matrices this polynomial $q(\lambda)$ is identically equal to λ and, in the case of skew-symmetric matrices, to $-\lambda$. If A and B are orthogonal matrices, then $q(\lambda)$ is the interpolation polynomial for $1/\lambda$ on the common spectrum of A and B.

Using (33), we conduct the proof of our theorem exactly as we did the proof of the corresponding Theorem 10 of Chapter IX in the real case (Vol. I, p. 289). From (31) we deduce

$$q(B)=T^{-1}q(A)\,T$$

or by (33)

$$B^{\mathsf{T}}=T^{-1}A^{\mathsf{T}}T.$$

Hence

$$B=T^{\mathsf{T}}AT^{\mathsf{T}-1}.$$

Comparing this equation with (31), we easily find:

$$TT^{\mathsf{T}}A=ATT^{\mathsf{T}}. \tag{34}$$

Let us apply the polar decomposition to the non-singular matrix T

$$T=SQ \quad (S=S^{\mathsf{T}}=f(TT^{\mathsf{T}}), \quad Q^{\mathsf{T}}=Q^{-1}).$$

Since by (34) the matrix TT^{T} is permutable with A, the matrix $S=f(TT^{\mathsf{T}})$ is also permutable with A. Therefore, when we substitute the product SQ for T in (31), we have

$$B=Q^{-1}S^{-1}ASQ=Q^{-1}AQ.$$

This completes the proof of the theorem.

§ 3. The Normal Form of a Complex Symmetric Matrix

1. We shall prove the following theorem:

THEOREM 5: *There exists a complex symmetric matrix with arbitrary preassigned elementary divisors.*[6]

Proof. We consider the matrix H of order n in which the elements of the first superdiagonal are 1 and all the remaining elements are zero. We shall show that there exists a symmetric matrix S similar to H:

$$S=THT^{-1}. \tag{35}$$

We shall look for the transforming matrix T starting from the conditions:

$$S=THT^{-1}=S^{\mathsf{T}}=T^{\mathsf{T}-1}H^{\mathsf{T}}T^{\mathsf{T}}.$$

This equation can be rewritten as

$$VH=H^{\mathsf{T}}V, \tag{36}$$

where V is the symmetric matrix connected with T by the equation[7]

$$T^{\mathsf{T}}T=-2iV. \tag{37}$$

Recalling properties of the matrices H and $F=H^{\mathsf{T}}$ (Vol. I, pp. 13-14) we find that every solution V of the matrix equation (36) has the following form:

$$V=\left\|\begin{matrix} 0 & \cdot & \cdot & \cdot & 0 & a_0\\ \cdot & & & \cdot & a_0 & a_1\\ \cdot & & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & & \cdot\\ 0 & a_0 & \cdot & & & \cdot\\ a_0 & a_1 & \cdot & \cdot & \cdot & a_{n-1}\end{matrix}\right\|, \tag{38}$$

where $a_0, a_1, \ldots, a_{n-1}$ are arbitrary complex numbers.

Since it is sufficient for us to find a single transforming matrix T, we set $a_0=1$, $a_1=\ldots=a_{n-1}=0$ in this formula and define V by the equation[8]

$$V=\left\|\begin{matrix} 0 & \ldots & 0 & 1\\ 0 & \ldots & 1 & 0\\ \cdot & \cdot & \cdot & \cdot\\ 1 & \ldots & 0 & 0\end{matrix}\right\|. \tag{39}$$

---

[6] In connection with the contents of the present section as well as the two sections that follow, §§ 4 and 5, see [378].

[7] To simplify the following formulas it is convenient to introduce the factor $-2i$.

[8] The matrix V is both symmetric and orthogonal.

Furthermore, we shall require the transforming matrix T to be symmetric:

$$T = T^{\mathsf{T}}. \tag{40}$$

Then the equation (37) for T can be written as:

$$T^2 = -2iV. \tag{41}$$

We shall now look for the required matrix T in the form of a polynomial in V. Since $V^2 = E$, this can be taken as a polynomial of the first degree:

$$T = \alpha E + \beta V.$$

From (41), taking into account that $V^2 = E$, we find:

$$\alpha^2 + \beta^2 = 0, \quad 2\alpha\beta = -2i.$$

We can satisfy these relations by setting $\alpha = 1$, $\beta = -i$. Then

$$T = E - iV. \tag{42}$$

T is a non-singular symmetrix matrix.[9] At the same time, from (41):

$$T^{-1} = \frac{1}{2} iV^{-1}T = \frac{1}{2} iVT,$$

i.e.,

$$T^{-1} = \frac{1}{2}(E + iV). \tag{43}$$

Thus, a symmetric form S of H is determined by

$$S = THT^{-1} = \frac{1}{2}(E - iV)H(E + iV), \quad V = \begin{Vmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdots & 0 & 0 \end{Vmatrix}. \tag{44}$$

Since S satisfies the equation (36) and $V^2 = E$, the equation (44) can be rewritten as follows:

$$2S = (H + H^{\mathsf{T}}) + i(HV - VH)$$
$$= \begin{Vmatrix} 0 & 1 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{Vmatrix} + i \begin{Vmatrix} 0 & \cdots & 1 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & -1 \\ 1 & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & -1 & \cdots & 0 \end{Vmatrix}. \tag{45}$$

---

[9] The fact that T is non-singular follows, in particular, from (41), because V is non-singular.

The formula (45) determines a symmetric form S of the matrix H.

In what follows, if n is the order of H, $H = H^{(n)}$, then we shall denote the corresponding matrices T, V, and S by $T^{(n)}$, $V^{(n)}$ and $S^{(n)}$.

Suppose that arbitrary elementary divisors are given:

$$(\lambda - \lambda_1)^{p_1}, \quad (\lambda - \lambda_2)^{p_2}, \quad \ldots, \quad (\lambda - \lambda_u)^{p_u}. \tag{46}$$

We form the corresponding Jordan matrix

$$J = \{ \lambda_1 E^{(p_1)} + H^{(p_1)}, \ \lambda_2 E^{(p_2)} + H^{(p_2)}, \ \ldots, \ \lambda_u E^{(p_u)} + H^{(p_u)} \}.$$

For every matrix $H^{(p_j)}$ we introduce the corresponding symmetric form $S^{(p_j)}$. From

$$S^{(p_j)} = T^{(p_j)} H^{(p_j)} \, [T^{(p_j)}]^{-1} \qquad (j = 1, 2, \ldots, u)$$

it follows that

$$\lambda_j E^{(p_j)} + S^{(p_j)} = T^{(p_j)} [\lambda_j E^{(p_j)} + H^{(p_j)}] \, [T^{(p_j)}]^{-1}.$$

Therefore setting

$$\tilde{S} = \{ \lambda_1 E^{(p_1)} + S^{(p_1)}, \quad \lambda_2 E^{(p_2)} + S^{(p_2)}, \quad \ldots, \quad \lambda_u E^{(p_u)} + S^{(p_u)} \}, \tag{47}$$

$$T = \{ T^{(p_1)}, \ T^{(p_2)}, \ \ldots, \ T^{(p_u)} \}, \tag{48}$$

we have:

$$\tilde{S} = TJT^{-1}.$$

$\tilde{S}$ is a symmetric form of J. $\tilde{S}$ is similar to J and has the same elementary divisors (46) as J. This proves the theorem.

COROLLARY 1. *Every square complex matrix* $A = \| a_{ik} \|_1^n$ *is similar to a symmetric matrix.*

Applying Theorem 4, we obtain:

COROLLARY 2. *Every complex symmetric matrix* $S = \| a_{ik} \|_1^n$ *is orthogonally similar to a symmetric matrix with the normal form* $\tilde{S}$, *i.e., there exists an orthogonal matrix* Q *such that*

$$\tilde{S} = QSQ^{-1}. \tag{49}$$

The normal form of a complex symmetric matrix has the quasi-diagonal form

$$\tilde{S} = \{ \lambda_1 E^{(p_1)} + S^{(p_1)}, \quad \lambda_2 E^{(p_2)} + S^{(p_2)}, \quad \ldots, \quad \lambda_u E^{(p_u)} + S^{(p_u)} \}, \tag{50}$$

where the blocks $S^{(p)}$ are defined as follows (see (44), (45)):

$$
\begin{aligned}
S^{(p)} &= \frac{1}{2}\,[E^{(p)} - iV^{(p)}]\; H^{(p)}\,[E^{(p)} + iV^{(p)}] \\
&= \frac{1}{2}\,[H^{(p)} + H^{(p)\mathsf{T}} + i\,(H^{(p)}V^{(p)} - V^{(p)}H^{(p)})] \\
&= \frac{1}{2}\left\{\left\|\begin{matrix} 0 & 1 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{matrix}\right\| + i\left\|\begin{matrix} 0 & \cdots & 1 & 0 \\ \vdots & ⋰ & ⋰ & -1 \\ 1 & ⋰ & ⋰ & \vdots \\ 0 & -1 & \cdots & 0 \end{matrix}\right\|\right\}.
\end{aligned} \tag{51}
$$

§ 4. The Normal Form of a Complex Skew-symmetric Matrix

1. We shall examine what restrictions the skew symmetry of a matrix imposes on its elementary divisors. In this task we shall make use of the following theorem:

THEOREM 6: *A skew-symmetric matrix always has even rank.*

Proof. Let r be the rank of the skew-symmetric matrix K. Then K has r linearly independent rows, say those numbered $i_1, i_2, \ldots, i_r$; all the remaining rows are linear combinations of these r rows. Since the columns of K are obtained from the corresponding rows by multiplying the elements by -1, every column of K is a linear combination of the columns numbered $i_1, i_2, \ldots, i_r$. Therefore every minor of order r of K can be represented in the form

$$\alpha K\begin{pmatrix} i_1 & i_2 & \ldots & i_r \\ i_1 & i_2 & \ldots & i_r \end{pmatrix},$$

where α is a constant.

Hence it follows that

$$K\begin{pmatrix} i_1 & i_2 & \ldots & i_r \\ i_1 & i_2 & \ldots & i_r \end{pmatrix} \neq 0.$$

But a skew-symmetric determinant of odd order is always zero. Therefore r is even, and the theorem is proved.

THEOREM 7: *If λ_0 is a characteristic value of the skew-symmetric matrix K with the corresponding elementary divisors*

$$(\lambda - \lambda_0)^{f_1}, \quad (\lambda - \lambda_0)^{f_2}, \ \ldots, \ (\lambda - \lambda_0)^{f_t},$$

then $-\lambda_0$ is also a characteristic value of K with the same number and the same powers of the corresponding elementary divisors of K

$$(\lambda + \lambda_0)^{f_1}, \quad (\lambda + \lambda_0)^{f_2}, \ \ldots, \ (\lambda + \lambda_0)^{f_t}.$$

2. *If zero is a characteristic value of the skew-symmetric matrix K,*[10] *then in the system of elementary divisors of K all those of even degree corresponding to the characteristic value zero are repeated an even number of times.*

Proof. 1. The transposed matrix K^{T} has the same elementary divisors as K. But $K^{\mathsf{T}} = -K$, and the elementary divisors of $-K$ are obtained from those of K by replacing the characteristic values $\lambda_1, \lambda_2, \ldots$ by $-\lambda_1$, $-\lambda_2, \ldots$. Hence the first part of our theorem follows.

2. Suppose that to the characteristic value zero of K there correspond δ_1 elementary divisors of the form λ, δ_2 of the form λ^2, etc. In general, we denote by δ_p the number of elementary divisors of the form λ^p $(p = 1, 2, \ldots)$. We shall show that $\delta_2, \delta_4, \ldots$ are even numbers.

The defect d of K is equal to the number of linearly independent characteristic vectors corresponding to the characteristic value zero or, what is the same, to the number of elementary divisors of the form $\lambda, \lambda^2, \lambda^3, \ldots$. Therefore

$$d = \delta_1 + \delta_2 + \delta_3 + \cdots. \tag{52}$$

Since, by Theorem 6, the rank of K is even and $d = n - r$, d has the same parity as n. The same statement can be made about the defects $d_3, d_5, \ldots$ of the matrices $K^3, K^5, \ldots$, because odd powers of a skew-symmetric matrix are themselves skew-symmetric. Therefore all the numbers $d_1 = d, d_3, d_5, \ldots$ have the same parity.

On the other hand, when K is raised to the m-th power, every elementary divisor λ^p for $p < m$ splits into p elementary divisors (of the first degree) and for $p \geqq m$ into m elementary divisors.[11] Therefore the number of elementary divisors of the matrices $K, K^3, \ldots$ that are powers of λ are determined by the formulas[12]

$$
\begin{aligned}
d_3 &= \delta_1 + 2\delta_2 + 3(\delta_3 + \delta_4 + \cdots), \\
d_5 &= \delta_1 + 2\delta_2 + 3\delta_3 + 4\delta_4 + 5(\delta_5 + \delta_6 + \cdots), \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{53}
$$

Comparing (52) with (53) and bearing in mind that all the numbers $d_1 = d, d_3, d_5, \ldots$ are of the same parity, we conclude easily that $\delta_2, \delta_4, \ldots$ are even numbers.

This completes the proof of the theorem.

---

[10] I.e., if $|K| = 0$. For odd n we always have $|K| = 0$.

[11] See Vol. I, Chapter VI, Theorem 9, p. 158.

[12] These formulas were introduced (without reference to Theorem 9) in Vol. I, Chapter VI (see formulas (49) on p. 155).

2. THEOREM 8: *There exists a skew-symmetric matrix with arbitrary preassigned elementary divisors subject to the restrictions* 1., 2. *of the preceding theorem.*

Proof. To begin with, we shall find a skew-symmetric form for the quasi-diagonal matrix of order $2p$:

$$J_{\lambda_0}^{(pp)} = \{ \lambda_0 E + H, -\lambda_0 E - H \} \tag{54}$$

having two elementary divisors $(\lambda - \lambda_0)^p$ and $(\lambda + \lambda_0)^p$; here $E = E^{(p)}$, $H = H^{(p)}$.

We shall look for a transforming matrix T such that

$$T J_{\lambda^0}^{(pp)} T^{-1}$$

is skew-symmetric, i.e., such that the following equation holds:

$$T J_{\lambda_0}^{(pp)} T^{-1} + T^{\mathsf{T}-1} [J_{\lambda_0}^{(pp)}]^{\mathsf{T}} T^{\mathsf{T}} = O$$

or

$$W J_{\lambda_0}^{(pp)} + [J_{\lambda_0}^{(pp)}]^{\mathsf{T}} W = O\,, \tag{55}$$

where W is the symmetric matrix connected with T by the equation[13]

$$T^{\mathsf{T}} T = -2\,i W. \tag{56}$$

We dissect W into four square blocks each of order p:

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}.$$

Then (55) can be written as follows:

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} \lambda_0 E + H & O \\ O & -\lambda_0 E - H \end{pmatrix} + \begin{pmatrix} \lambda_0 E + H^{\mathsf{T}} & O \\ O & -\lambda_0 E - H^{\mathsf{T}} \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = O. \tag{57}$$

When we perform the indicated operations on the partitioned matrices on the left-hand side of (57), we replace this equation by four matrix equations:

$$\begin{aligned} &1.\ \ H^{\mathsf{T}} W_{11} + W_{11}(2\,\lambda_0 E + H) = O\,,\\ &2.\ \ H^{\mathsf{T}} W_{12} - W_{12} H = O\,,\\ &3.\ \ H^{\mathsf{T}} W_{21} - W_{21} H = O\,,\\ &4.\ \ H^{\mathsf{T}} W_{22} + W_{22}(2\,\lambda_0 E + H) = O\,. \end{aligned} \tag{58}$$

---

[13] See footnote 7 on p. 9.

The equation $AX - XB = O$, where A and B are square matrices without common characteristic values, has only the trivial solution $X = O$.[14] Therefore the first and fourth of the equations (58) yield: $W_{11} = W_{22} = O$.[15] As regards the second of these equations, it can be satisfied, as we have seen in the proof of Theorem 5, by setting

$$W_{12} = V = \begin{Vmatrix} 0 & . & . & . & 0 & 1 \\ 0 & . & . & . & 1 & 0 \\ . & . & . & . & . & . \\ 1 & . & . & . & 0 & 0 \end{Vmatrix}, \tag{59}$$

since (cf. (36))

$$VH - H^{\mathsf{T}} V = O.$$

From the symmetry of W and V it follows that

$$W_{21} = W_{12}^{\mathsf{T}} = V.$$

The third equation is then automatically satisfied.

Thus,

$$W = \begin{pmatrix} O & V \\ V & O \end{pmatrix} = V^{(2p)}. \tag{60}$$

But then, as has become apparent on page 10, the equation (56) will be satisfied if we set

$$T = E^{(2p)} - i V^{(2p)}. \tag{61}$$

Then

$$T^{-1} = \frac{1}{2}(E^{(2p)} + i V^{(2p)}). \tag{62}$$

Therefore, the required skew-symmetric matrix can be found by the formula[16]

$$\begin{aligned} K_{\lambda^0}^{(pp)} &= \frac{1}{2}[E^{(2p)} - i V^{(2p)}]\, J_{\lambda_0}^{(pp)} [E^{(2p)} + i V^{(2p)}] \\ &= \frac{1}{2}[J_{\lambda_0}^{(pp)} - J_{\lambda_0}^{(pp)\mathsf{T}} + i\,(J_{\lambda_0}^{(pp)} V^{(2p)} - V^{(2p)} J_{\lambda_0}^{(pp)})]\,. \end{aligned} \tag{63}$$

When we substitute for $J_{\lambda_0}^{(pp)}$ and $V^{(2p)}$ the corresponding partitioned matrices from (54) and (60), we find:

---

[14] See Vol. I, Chapter VIII, § 1.

[15] For $\lambda_0 \neq 0$ the equations 1. and 4. have no solutions other than zero. For $\lambda_0 = 0$ there exist other solutions, but we choose the zero solution.

[16] Here we use equations (55) and (60). From these it follows that

$$V^{(2p)} J_{\lambda_0}^{(pp)} V^{(2p)} = -J_{\lambda_0}^{(pp)\mathsf{T}}.$$

$$K_{\lambda_0}^{(pp)} = \frac{1}{2}\left[\begin{pmatrix} H - H^{\mathsf T} & O \\ O & H^{\mathsf T} - H \end{pmatrix} + i \begin{pmatrix} \lambda_0 E + H & O \\ O & -\lambda_0 E - H \end{pmatrix}\begin{pmatrix} O & V \\ V & O \end{pmatrix} \right.$$
$$\left. - i \begin{pmatrix} O & V \\ V & O \end{pmatrix}\begin{pmatrix} \lambda_0 E + H & O \\ O & -\lambda_0 E - H \end{pmatrix}\right]$$
$$= \frac{1}{2}\begin{pmatrix} H - H^{\mathsf T} & i\,(2\lambda_0 V + HV + VH) \\ -i\,(2\lambda_0 V + HV + VH) & H^{\mathsf T} - H \end{pmatrix}, \tag{64}$$

i.e.,

$$K_{\lambda_0}^{(pp)} = \frac{1}{2}\left\|\begin{array}{ccccc:ccccc} 0 & 1 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & i & 2\lambda_0 \\ -1 & 0 & \ddots & & \vdots & \vdots & & \iddots & 2\lambda_0 & i \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & i & \iddots & \iddots & & \vdots \\ 0 & \cdots & \cdots & -1 & 0 & 2\lambda_0 & i & \cdots & \cdots & 0 \\ \hdashline 0 & \cdots & \cdots & -i & -2\lambda_0 & 0 & -1 & \cdots & \cdots & 0 \\ & & \iddots & -2\lambda_0 & -i & 1 & 0 & \ddots & & \vdots \\ \vdots & \iddots & \iddots & \iddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \iddots & \iddots & & \vdots & \vdots & & \ddots & \ddots & -1 \\ -i & \iddots & & \iddots & & & & & \ddots & \\ -2\lambda_0 & \iddots & -i & \cdots & 0 & 0 & \cdots & \cdots & 1 & 0 \end{array}\right\|. \tag{65}$$

We shall now construct a skew-symmetric matrix $K^{(q)}$ of order q having one elementary divisor λ^q, where q is odd. Obviously, the required skew-symmetric matrix will be similar to the matrix

$$J^{(q)} = \begin{Vmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & \ddots & -1 & 0 \\ & & & & 0 & -1 \\ 0 & & \cdots & & 0 & 0 \end{Vmatrix}. \tag{66}$$

In this matrix all the elements outside the first superdiagonal are equal to zero, and along the first superdiagonal there are at first $(q-1)/2$ elements 1 and then $(q-1)/2$ elements -1. Setting

$$K^{(q)} = TJ^{(q)}T^{-1}, \tag{67}$$

we find from the condition of skew-symmetry:

$$W_1 J^{(q)} + J^{(q)\mathsf T} W_1 = 0, \tag{68}$$

where

$$T^{\mathsf T} T = -2iW_1. \tag{69}$$

By direct verification we can convince ourselves that the matrix

$$W_1 = V^{(q)} = \begin{Vmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdots & 0 & 0 \end{Vmatrix}$$

satisfies the condition (68). Taking this value for W_1 we find from (69), as before:

$$T = E^{(q)} - iV^{(q)}, \quad T^{-1} = \frac{1}{2}\,[E^{(q)} + iV^{(q)}], \tag{70}$$

$$K^{(q)} = \frac{1}{2}\,[E^{(q)} - iV^{(q)}]\,J^{(q)}\,[E^{(q)} + iV^{(q)}]$$
$$= \frac{1}{2}\,[J^{(q)} - J^{(q)\mathsf T} + i\,(J^{(q)}V^{(q)} - V^{(q)}J^{(q)})]. \tag{71}$$

When we perform the corresponding computation, we find:

$$2K^{(q)} = \begin{Vmatrix} 0 & 1 & \cdots & \cdots & \cdots & 0 \\ -1 & 0 & \ddots & & & \vdots \\ & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & -1 \\ & & & & \ddots & \\ 0 & & & & 1 & 0 \end{Vmatrix} + i \begin{Vmatrix} 0 & \cdots & \cdots & \cdots & 1 & 0 \\ & & & \iddots & \iddots & \\ & & \iddots & \iddots & & 1 \\ & \iddots & \iddots & \iddots & & \vdots \\ & \iddots & \iddots & & & \vdots \\ -1 & \iddots & & & & \vdots \\ 0 & -1 & \cdots & & & 0 \end{Vmatrix}. \tag{72}$$

Suppose that arbitrary elementary divisors are given, subject to the conditions of Theorem 7:

$$\left.\begin{array}{c}(\lambda-\lambda_j)^{p_j},\ (\lambda+\lambda_j)^{p_j}\ \ (j=1,\ 2,\ \ldots,\ u),\\ \lambda^{q_k}\ (k=1,\ 2,\ \ldots,\ v;\ q_1,\ q_2,\ \ldots,\ q_v \text{ are odd numbers}).^{17}\end{array}\right\} \tag{73}$$

Then the quasi-diagonal skew-symmetric matrix

$$\tilde{K}=\left\{K_{\lambda_1}^{(p_1p_1)},\ \ldots,\ K_{\lambda_u}^{(p_up_u)};\ K^{(q_1)},\ \ldots,\ K^{(q_v)}\right\} \tag{74}$$

has the elementary divisors (73).

This concludes the proof of the theorem.

COROLLARY: *Every complex skew-symmetric matrix K is orthogonally similar to a skew-symmetric matrix having the normal form $\tilde{K}$ determined by* (74), (65), *and* (72); *i.e., there exists a (complex) orthogonal matrix Q such that*

$$K=Q\tilde{K}Q^{-1}. \tag{75}$$

Note. If K is a real skew-symmetric matrix, then it has linear elementary divisors (see Vol. I, Chapter IX, § 13).

$$\lambda-i\varphi_1,\ \lambda+i\varphi_1,\ \ldots,\ \lambda-i\varphi_u,\ \lambda+i\varphi_u,\ \underbrace{\lambda,\ \ldots,\ \lambda}_{v \text{ times}}\ \ (\varphi_j \text{ are real numbers}).$$

In this case, setting all the $p_j=1$ and all the $q_k=1$ in (74), we obtain as the normal form of a real skew-symmetric matrix

$$\tilde{K}=\left\{\left\|\begin{array}{cc}0 & \varphi_1\\ -\varphi_1 & 0\end{array}\right\|,\ \ldots,\ \left\|\begin{array}{cc}0 & \varphi_u\\ -\varphi_u & 0\end{array}\right\|,\ 0,\ \ldots,\ 0\right\}.$$

§ 5. The Normal Form of a Complex Orthogonal Matrix

1. Let us begin by examining what restrictions the orthogonality of a matrix imposes on its elementary divisors.

THEOREM 9: 1. *If λ_0 ($\lambda_0^2\neq 1$) is a characteristic value of an orthogonal matrix Q and if the elementary divisors*

$$(\lambda-\lambda_0)^{f_1},\ (\lambda-\lambda_0)^{f_2},\ \ldots,\ (\lambda-\lambda_0)^{f_t}$$

---

[17] Some of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_u$ may be zero. Moreover, one of the numbers u and v may be zero; i.e., in some cases there may be elementary divisors of only one type.

correspond to this characteristic value, then $1/\lambda_0$ is also a characteristic value of Q and it has the same corresponding elementary divisors:

$$(\lambda-\lambda_0^{-1})^{f_1},\ (\lambda-\lambda_0^{-1})^{f_2},\ \ldots,\ (\lambda-\lambda_0^{-1})^{f_t}.$$

2. *If $\lambda_0=\pm 1$ is a characteristic value of the orthogonal matrix Q, then the elementary divisors of even degree corresponding to λ_0 are repeated an even number of times.*

Proof. 1. For every non-singular matrix Q on passing from Q to Q^{-1} each elementary divisor $(\lambda-\lambda_0)^f$ is replaced by the elementary divisor $(\lambda-\lambda_0^{-1})^f$.[18] On the other hand, the matrices Q and Q^{T} always have the same elementary divisors. Therefore the first part of our theorem follows at once from the orthogonality condition $Q^{\mathsf{T}}=Q^{-1}$.

2. Let us assume that the number 1 is a characteristic value of Q, while -1 is not ($|E-Q|=0$, $|E+Q|\neq 0$). Then we apply Cayley's formulas (see Vol. I, Chapter IX, § 14), which remain valid for complex matrices. We define a matrix K by the equation

$$K=(E-Q)\ (E+Q)^{-1}. \tag{76}$$

Direct verification shows that $K^{\mathsf{T}}=-K$, so that K is skew-symmetric. When we solve the equation (76) for Q, we find:[19]

$$Q=(E-K)\ (E+K)^{-1}.$$

Setting $f(\lambda)=\frac{1-\lambda}{1+\lambda}$, we have $f'(\lambda)=-\frac{2}{(1+\lambda)^2}\neq 0$. Therefore in the transition from K to $Q=f(K)$ the elementary divisors do not split.[20] Hence in the system of elementary divisors of Q those of the form $(\lambda-1)^{2p}$ are repeated an even number of times, because this holds for the elementary divisors of the form λ^{2p} of K (see Theorem 7).

The case where Q has the characteristic value -1, but not $+1$, is reduced to the preceding case by considering the orthogonal matrix $-Q$.

We now proceed to the most complicated case, where Q has both the characteristic value $+1$ and -1. We denote by $\psi(\lambda)$ the minimal polynomial of Q. Using the first part of the theorem, which has already been proved, we can write $\psi(\lambda)$ in the form

---

[18] See Vol. I, Chapter VI, § 7. Setting $f(\lambda)=1/\lambda$, we have $f'(\lambda)=-1/\lambda^2\neq 0$. Hence it follows that in the transition from Q to Q^{-1} the elementary divisors do not split (see Vol. I, p. 158).

[19] Note that (76) implies that $E+K=2(E+Q)^{-1}$ and therefore

$$|E+K|=2^n\,|E+Q|^{-1}\neq 0.$$

[20] See Vol. I, p. 158.

$$\psi(\lambda)=(\lambda-1)^{m_1}(\lambda+1)^{m_2}\prod_{j=1}^{u}(\lambda-\lambda_j)^{p_j}(\lambda-\lambda_j^{-1})^{p_j}\quad(\lambda_j^2\neq 1;\ j=1,2,\ldots,u).$$

We consider the polynomial $g(\lambda)$ of degree less than m (m is the degree of $\psi(\lambda)$) for which $g(1)=1$ and all the remaining $m-1$ values on the spectrum of Q are zero; and we set:[21]

$$P=g(Q). \tag{77}$$

Note that the functions $(g(\lambda))^2$ and $g(1/\lambda)$ assume on the spectrum of Q the same values as $g(\lambda)$. Therefore

$$P^2=P,\qquad P^{\mathsf{T}}=g(Q^{\mathsf{T}})=g(Q^{-1})=P, \tag{78}$$

i.e., P is a symmetric projective matrix.[22]

We define a polynomial $h(\lambda)$ and a matrix N by the equations

$$h(\lambda)=(\lambda-1)\,g(\lambda), \tag{79}$$

$$N=h(Q)=(Q-E)\,P. \tag{80}$$

Since $(h(\lambda))^{m_1}$ vanishes on the spectrum of Q, it is divisible by $\psi(\lambda)$ without remainder. Hence:

$$N^{m_1}=O,$$

i.e., N is a nilpotent matrix with m_1 as index of nilpotency.

From (80) we find:[23]

$$N^{\mathsf{T}}=(Q^{\mathsf{T}}-E)\,P. \tag{81}$$

---

[21] From the fundamental formula (see Vol. I, p. 104)

$$g(A)=\sum_{k=1}^{s}[g(\lambda_k)Z_{k1}+g'(\lambda_k)Z_{k2}+\cdots]$$

it follows that

$$p=Z_{11}.$$

[22] A hermitian operator $\boldsymbol{P}$ is called *projective* if $\boldsymbol{P}^2=\boldsymbol{P}$. In accordance with this, a hermitian matrix P for which $P^2=P$ is called *projective*. An example of a projective operator $\boldsymbol{P}$ in a unitary space $\boldsymbol{R}$ is the operator of the orthogonal projection of a vector $\boldsymbol{x}\in\boldsymbol{R}$ into a subspace $\boldsymbol{S}=\boldsymbol{PR}$, i.e., $\boldsymbol{Px}=\boldsymbol{x}_S$, where $\boldsymbol{x}_S\in\boldsymbol{S}$ and $(\boldsymbol{x}-\boldsymbol{x}_S)\perp\boldsymbol{S}$ (see Vol. I, p. 248).

[23] All the matrices that occur here, P, N, N^{T}, $Q^{\mathsf{T}}=Q^{-1}$, are permutable among each other and with Q, since they are all functions of Q.

Let us consider the matrix

$$R=N(N^{\mathsf{T}}+2E). \tag{82}$$

From (78), (80), and (81) it follows that

$$R=NN^{\mathsf{T}}+2N=(Q-Q^{\mathsf{T}})\,P.$$

From this representation of R it is clear that R is *skew-symmetric*.

On the other hand, from (82)

$$R^k=N^k(N^{\mathsf{T}}+2E)^k\quad(k=1,2,\ldots). \tag{83}$$

But N^{T}, like N, is nilpotent, and therefore

$$|N^{\mathsf{T}}+2E|\neq 0.$$

Hence it follows from (83) that the matrices R^k and N^k have the same rank for every k.

Now for odd k the matrix R^k is skew-symmetric and therefore (see p. 12) has even rank. Therefore each of the matrices

$$N,\ N^3,\ N^5,\ \ldots$$

has odd rank.

By repeating verbatim for N the arguments that were used on p. 13 for K we may therefore state that among the elementary divisors of N those of the form λ^{2p} are repeated an even number of times. But to each elementary divisor λ^{2p} of N there corresponds an elementary divisor $(\lambda-1)^{2p}$ of Q, and vice versa.[24] Hence it follows that among the elementary divisors of Q those of the form $(\lambda-1)^{2p}$ are repeated an even number of times.

We obtain a similar statement for the elementary divisors of the form $(\lambda+1)^{2p}$ by applying what has just been proved to the matrix $-Q$.

Thus, the proof of the theorem is complete.

2. We shall now prove the converse theorem.

---

[24] Since $h(1)=0$, $h'(1)\neq 0$, in passing from Q to $N=h(Q)$ the elementary divisors of the form $(\lambda-1)^{2p}$ of Q do not split and are therefore replaced by elementary divisors λ^{2p} (see Vol. I, Chapter VI, § 7).

THEOREM 10: *Every system of powers of the form*

$$\left.\begin{array}{l}(\lambda-\lambda_j)^{p_j},\ (\lambda-\lambda_j^{-1})^{p_j}\ (\lambda_j\neq 0;\ j=1,\,2,\,\ldots,\,u),\\ (\lambda-1)^{q_1},\ (\lambda-1)^{q_2},\ \ldots,\ (\lambda-1)^{q_v},\\ (\lambda+1)^{t_1},\ (\lambda+1)^{t_2},\ \ldots,\ (\lambda+1)^{t_w}\\ (q_1,\,\ldots,\,q_v,\ t_1,\,\ldots,\,t_w \textit{ are odd numbers})\end{array}\right\}\qquad(84)$$

is the system of elementary divisors of some complex orthogonal matrix Q.[25]

Proof. We denote by μ_j the numbers connected with the numbers λ_j $(j=1,\,2,\,\ldots,\,n)$ by the equations

$$\lambda_j=e^{\mu_j}\quad(j=1,\,2,\,\ldots,\,u)$$

We now introduce the 'canonical' skew-symmetric matrices (see the preceding section)

$$K_{\mu_j}^{(p_jp_j)}\ (j=1,\,2,\,\ldots,\,u);\ K^{(q_1)},\,\ldots,\,K^{(q_v)};\ K^{(t_1)},\,\ldots,\,K^{(t_w)},$$

with the elementary divisors

$$(\lambda-\mu_j)^{p_j},\ (\lambda+\mu_j)^{p_j}\ (j=1,\,2,\,\ldots,\,u)\ \lambda^{q_1},\,\ldots,\,\lambda^{q_v};\ \lambda^{t_1},\,\ldots,\,\lambda^{t_w}.$$

If K is a skew-symmetric matrix, then

$$Q=e^K$$

is orthogonal $(Q^{\mathsf{T}}=e^{K^{\mathsf{T}}}=e^{-K}=Q^{-1})$. Moreover, to each elementary divisor $(\lambda-\mu)^p$ of K there corresponds an elementary divisor $(\lambda-e^{\mu})^p$ of Q.[26]

Therefore the quasi-diagonal matrix

$$\tilde{Q}=\left\{e^{K_{\mu_1}^{(p_1p_1)}},\,\ldots,\,e^{K_{\mu_u}^{(p_up_u)}};\ e^{K^{(q_1)}},\,\ldots,\,e^{K^{(q_v)}};\ -e^{K^{(t_1)}},\,\ldots,\,-e^{K^{(t_w)}}\right\}\quad(85)$$

is orthogonal and has the elementary divisors (84).

This proves the theorem.

From Theorems 4, 9, and 10 we obtain:

---

[25] Some (or even all) of the numbers λ_j may be ± 1. One or two of the numbers u, v, w may be zero. Then the elementary divisors of the corresponding type are absent in Q.

[26] This follows from the fact that for $f(\lambda)=e^{\lambda}$ we have $f'(\lambda)=e^{\lambda}\neq 0$ for every λ.

COROLLARY: *Every (complex) orthogonal matrix Q is orthogonally similar to an orthogonal matrix having the normal form $\tilde{Q}$;* i.e., *there exists an orthogonal matrix Q_1 such that*

$$Q=Q_1\tilde{Q}Q_1^{-1}.\qquad(86)$$

Note. **Just as we have given a concrete form to the diagonal blocks in the skew-symmetric matrix $\tilde{K}$, so we could for the normal form $\tilde{Q}$.**[27]

---

[27] See [378].

CHAPTER XII

SINGULAR PENCILS OF MATRICES

§ 1. Introduction

1. The present chapter deals with the following problem:

Given four matrices A, B, A_1, B_1 all of dimension $m \times n$ with elements from a number field F, *it is required to find under what conditions there exist two square non-singular matrices P and Q of orders m and n, respectively, such that*[1]

$$PAQ = A_1, \quad PBQ = B_1 \tag{1}$$

By introduction of the pencils of matrices $A + \lambda B$ and $A_1 + \lambda B_1$ the two matrix equations (1) can be replaced by the single equation

$$P(A + \lambda B)\,Q = A_1 + \lambda B_1 \tag{2}$$

Definition 1: *Two pencils of rectangular matrices $A + \lambda B$ and $A_1 + \lambda B_1$ of the same dimensions $m \times n$ connected by the equation (2) in which P and Q are constant square non-singular matrices (i.e., matrices independent of λ) of orders m and n, respectively, will be called strictly equivalent.*[2]

According to the general definition of equivalence of λ-matrices (see Vol. I, Chapter VI, p. 132), the pencils $A + \lambda B$ and $A_1 + \lambda B_1$ are equivalent if an equation of the form (2) holds in which P and Q are two square λ-matrices with constant non-vanishing determinants. For strict equivalence it is required in addition that P and Q do not depend on λ.[3]

A criterion for equivalence of the pencils $A + \lambda B$ and $A_1 + \lambda B_1$ follows from the general criterion for equivalence of λ-matrices and consists in the equality of the invariant polynomials or, what is the same, of the elementary divisors of the pencils $A + \lambda B$ and $A_1 + \lambda B_1$ (see Vol. I, Chapter VI, p. 141).

---

[1] If such matrices P and Q exist, then their elements can be taken from the field F. This follows from the fact that the equations (1) can be written in the form $PA = A_1Q^{-1}$, $PB = B_1Q^{-1}$ and are therefore equivalent to a certain system of linear homogeneous equations for the elements of P and Q^{-1} with coefficients in F.

[2] See Vol. I, Chapter VI, p. 145.

[3] We have replaced the term 'equivalent pencils' that occurs in the literature by 'strictly equivalent pencils,' in order to draw a sharp distinction between Definition 1 and the definition of equivalence in Vol. I, Chapter VI.

In this chapter, we shall establish a criterion for strict equivalence of two pencils of matrices and we shall determine for each pencil a strictly equivalent canonical form.

2. The task we have set ourselves has a natural geometrical interpretation. We consider a pencil of linear operators $\boldsymbol{A} + \lambda \boldsymbol{B}$ mapping $\boldsymbol{R}_n$ into $\boldsymbol{R}_m$. For a definite choice of bases in these spaces the pencil of operators $\boldsymbol{A} + \lambda \boldsymbol{B}$ corresponds to a pencil of rectangular matrices $A + \lambda B$ (of dimension $m \times n$); under a change of bases in $\boldsymbol{R}_n$ and $\boldsymbol{R}_m$ the pencil $A + \lambda B$ is replaced by a strictly equivalent pencil $P(A + \lambda B)Q$, where P and Q are square non-singular matrices of order m and n (see Vol. I, Chapter III, §§ 2 and 4). Thus, a criterion for strict equivalence gives a characterization of that class of matrix pencils $A + \lambda B$ (of dimension $m \times n$) which describe one and the same pencil of operators $\boldsymbol{A} + \lambda \boldsymbol{B}$ mapping $\boldsymbol{R}_n$ into $\boldsymbol{R}_m$ for various choices of bases in these spaces.

In order to obtain a canonical form for a pencil it is necessary to find bases for $\boldsymbol{R}_n$ and $\boldsymbol{R}_m$ in which the pencil of operators $\boldsymbol{A} + \lambda \boldsymbol{B}$ is described by matrices of the simplest possible form.

Since a pencil of operators is given by two operators $\boldsymbol{A}$ and $\boldsymbol{B}$, we can also say that: *The present chapter deals with the simultaneous investigation of two operators $\boldsymbol{A}$ and $\boldsymbol{B}$ mapping $\boldsymbol{R}_n$ into $\boldsymbol{R}_m$.*

3. All the pencils of matrices $A + \lambda B$ of dimension $m \times n$ fall into two basic types: *regular* and *singular* pencils.

Definition 2: *A pencil of matrices $A + \lambda B$ is called regular if*

1) *A and B are square matrices of the same order n; and*

2) *The determinant $|A + \lambda B|$ does not vanish identically.*

In all other cases ($m \neq n$, or $m = n$ but $|A + \lambda B| \equiv 0$), *the pencil is called singular.*

A criterion for strict equivalence of regular pencils of matrices and also a canonical form for such pencils were established by Weierstrass in 1867 [377] on the basis of his theory of elementary divisors, which we have expounded in Chapters VI and VII. The analogous problems for singular pencils were solved later, in 1890, by the investigations of Kronecker [249].[4] Kronecker's results form the primary content of this chapter.

§ 2. Regular Pencils of Matrices

1. We consider the special case where the pencils $A + \lambda B$ and $A_1 + \lambda B_1$ consist of square matrices ($m = n$) $|B| \neq 0$, $|B_1| \neq 0$. In this case, as we have shown in Chapter VI (Vol. I, pp. 145-146), the two concepts of 'equiv-

---

[4] Of more recent papers dealing with singular pencils of matrices we mention [234], [369], and [255].

alence' and 'strict equivalence' of pencils coincide. Therefore, by applying to the pencils the general criterion for equivalence of λ-matrices (Vol. I, p. 141) we are led to the following theorem:

THEOREM 1: *Two pencils of square matrices of the same order* $A + \lambda B$ *and* $A_1 + \lambda B_1$ *for which* $|B| \neq 0$ *and* $|B_1| \neq 0$ *are strictly equivalent if and only if the pencils have the same elementary divisors in* F.

A pencil of square matrices $A + \lambda B$ with $|B| \neq 0$ was called regular in Chapter VI, because it represents a special case of a regular matrix polynomial in λ (see Vol. I, Chapter IV, p. 76). In the preceding section of this chapter we have given a wider definition of regularity. According to this definition it is quite possible in a regular pencil to have $|B| = 0$ (and even $|A| = |B| = 0$).

In order to find out whether Theorem 1 remains valid for regular pencils (with the extended Definition 1), we consider the following example:

$$A + \lambda B = \begin{Vmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 3 & 2 & 6 \end{Vmatrix} + \lambda \begin{Vmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{Vmatrix}, \quad A_1 + \lambda B_1 = \begin{Vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{Vmatrix} + \lambda \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{Vmatrix}. \qquad (3)$$

It is easy to see that here each of the pencils $A + \lambda B$ and $A_1 + \lambda B_1$ has only one elementary divisor, $\lambda + 1$. However, the pencils are not strictly equivalent, since the matrices B and B_1 are of ranks 2 and 1, respectively; whereas if an equation (2) were to hold, it would follow from it that the ranks of B and B_1 are equal. Nevertheless, the pencils (3) are regular according to Definition 1, since

$$|A + \lambda B| \equiv |A_1 + \lambda B_1| \equiv \lambda + 1.$$

This example shows that Theorem 1 is not true with the extended definition of regularity of a pencil.

2. In order to preserve Theorem 1, we have to introduce the concept of 'infinite' elementary divisors of a pencil. We shall give the pencil $A + \lambda B$ in terms of 'homogeneous' parameters λ, μ: $\mu A + \lambda B$. Then the determinant $\Delta(\lambda, \mu) \equiv |\mu A + \lambda B|$ is a homogeneous function of λ, μ. By determining the greatest common divisor $D_k(\lambda, \mu)$ of all the minors of order k of the matrix $\mu A + \lambda B$ ($k = 1, 2, \ldots, n$), we obtain the invariant polynomials by the well known formulas

$$i_1(\lambda, \mu) = \frac{D_n(\lambda, \mu)}{D_{n-1}(\lambda, \mu)}, \quad i_2(\lambda, \mu) = \frac{D_{n-1}(\lambda, \mu)}{D_{n-2}(\lambda, \mu)}, \ \ldots ;$$

here all the $D_k(\lambda, \mu)$ and $i_j(\lambda, \mu)$ are homogeneous polynomials in λ and μ.

Splitting the invariant polynomials into powers of homogeneous polynomials irreducible over F, we obtain the elementary divisors $e_\alpha(\lambda, \mu)$ ($\alpha = 1, 2, \ldots$) of the pencil $\mu A + \lambda B$ in F.

It is quite obvious that if we set $\mu = 1$ in $e_\alpha(\lambda, \mu)$ we are back to the elementary divisors $e_\alpha(\lambda)$ of the pencil $A + \lambda B$. Conversely, from each elementary divisor $e_\alpha(\lambda)$ of degree q we obtain the correspondingly elementary divisor $e_\alpha(\lambda, \mu)$ by the formula $e_\alpha(\lambda, \mu) = \mu^q e_\alpha\left(\frac{\lambda}{\mu}\right)$. We can obtain in this way all the elementary divisors of the pencil $\mu A + \lambda B$ apart from those of the form μ^q.

Elementary divisors of the form μ^q exist if and only if $|B| = 0$ and are called 'infinite' elementary divisors of the pencil $A + \lambda B$.

Since strict equivalence of the pencils $A + \lambda B$ and $A_1 + \lambda B_1$ implies strict equivalence of the pencils $\mu A + \lambda B$ and $\mu A_1 + \lambda B_1$, we see that for strictly equivalent pencils $A + \lambda B$ and $A_1 + \lambda B_1$ not only their 'finite,' but also their 'infinite' elementary divisors must coincide.

Suppose now that $A + \lambda B$ and $A_1 + \lambda B_1$ are two regular pencils for which all the elementary divisors coincide (including the infinite ones). We introduce homogeneous parameters: $\mu A + \lambda B$, $\mu A_1 + \lambda B_1$. Let us now transform the parameters

$$\lambda = \alpha_1 \tilde{\lambda} + \alpha_2 \tilde{\mu}, \quad \mu = \beta_1 \tilde{\lambda} + \beta_2 \tilde{\mu} \quad (\alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0).$$

In the new parameters the pencils are written as follows:

$$\tilde{\mu} \tilde{A} + \tilde{\lambda} \tilde{B}, \ \tilde{\mu} \tilde{A}_1 + \tilde{\lambda} \tilde{B}_1, \text{ where } \tilde{B} = \beta_1 A + \alpha_1 B, \ \tilde{B}_1 = \beta_1 A_1 + \alpha_1 B_1.$$

From the regularity of the pencils $\mu A + \lambda B$ and $\mu A_1 + \lambda B_1$ it follows that we can choose the numbers α_1 and β_1 such that $|\tilde{B}| \neq 0$ and $|\tilde{B}_1| \neq 0$.

Therefore by Theorem 1 the pencils $\tilde{\mu} \tilde{A} + \tilde{\lambda} \tilde{B}$ and $\tilde{\mu} \tilde{A}_1 + \tilde{\lambda} \tilde{B}_1$ and consequently the original pencils $\mu A + \lambda B$ and $\mu A_1 + \lambda B_1$ (or, what is the same, $A + \lambda B$ and $A_1 + \lambda B_1$) are strictly equivalent. Thus, we have arrived at the following generalization of Theorem 1:

THEOREM 2: *Two regular pencils* $A + \lambda B$ *and* $A_1 + \lambda B_1$ *are strictly equivalent if and only if they have the same ('finite' and 'infinite') elementary divisors.*

In our example above the pencils (3) had the same 'finite' elementary divisor $\lambda + 1$, but different 'infinite' elementary divisors (the first pencil has one 'infinite' elementary divisor μ^2; the second has two: μ, μ). Therefore these pencils turn out to be not strictly equivalent.

3. Suppose now that $A + \lambda B$ is an arbitrary regular pencil. Then there exists a number c such that $|A + cB| \neq 0$. We represent the given pencil in the form $A_1 + (\lambda - c)B$, where $A_1 = A + cB$, so that $|A_1| \neq 0$. We multiply the pencil on the left by A_1^{-1}: $E + (\lambda - c)A_1^{-1}B$. By a similarity transformation we put the pencil in the form[5]

$$E + (\lambda - c)\{J_0, J_1\} = \{E - cJ_0 + \lambda J_0,\ E - cJ_1 + \lambda J_1\}, \tag{4}$$

where $\{J_0, J_1\}$ is the quasi-diagonal normal form of $A_1^{-1}B$, J_0 is a nilpotent Jordan matrix,[6] and $|J_1| \neq 0$.

We multiply the first diagonal block on the right-hand side of (4) by $(E - cJ_0)^{-1}$ and obtain: $E + \lambda(E - cJ_0)^{-1}J_0$. Here the coefficient of λ is a nilpotent matrix.[7] Therefore by a similarity transformation we can put this pencil into the form[8]

$$E + \lambda \hat{J}_0 = \{N^{(u_1)}, N^{(u_2)}, \ldots, N^{(u_s)}\}\ (N^{(u)} = E^{(u)} + \lambda H^{(u)}). \tag{5}$$

We multiply the second diagonal block on the right-hand side of (4) by J_1^{-1}; it can then be put into the form $J + \lambda E$ by a similarity transformation, where J is a matrix of normal form[9] and E the unit matrix. We have thus arrived at the following theorem:

THEOREM 3: *Every regular pencil $A + \lambda B$ can be reduced to a (strictly equivalent) canonical quasi-diagonal form*

$$\{N^{(u_1)}, N^{(u_2)}, \ldots, N^{(u_s)}, J + \lambda E\}\ (N^{(u)} = E^{(u)} + \lambda H^{(u)}), \tag{6}$$

where the first s diagonal blocks correspond to infinite elementary divisors $\mu^{u_1}, \mu^{u_2}, \ldots, \mu^{u_s}$ of the pencil $A + \lambda B$ and where the normal form of the last diagonal block $J + \lambda E$ is uniquely determined by the finite elementary divisors of the given pencil.

---

[5] The unit matrices E in the diagonal blocks on the right-hand side of (4) have the same order as J_0 and J_1.

[6] I.e., $J_0^l = O$ for some integer $l > 0$.

[7] From $J_0^l = O$ it follows that $[(E - cJ_0)^{-1}J_0]^l = O$.

[8] Here $E^{(u)}$ is a unit matrix of order u and $H^{(u)}$ is a matrix of order u whose elements in the first superdiagonal are 1, while the remaining elements are zero.

[9] Since the matrix J can be replaced here by an arbitrary similar matrix, we may assume that J has one of the normal forms (for example, the natural form of the first or second kind or the Jordan form (see Vol. I, Chapter VI, § 6)).

§ 3. Singular Pencils. The Reduction Theorem

1. We now proceed to consider a singular pencil of matrices $A + \lambda B$ of dimension $m \times n$. We denote by r the *rank of the pencil*, i.e., the largest of the orders of minors that do not vanish identically. From the singularity of the pencil it follows that at least one of the inequalities $r < n$ and $r < m$ holds, say $r < n$. Then the columns of the λ-matrix $A + \lambda B$ are linearly dependent, i.e., the equation

$$(A + \lambda B)x = o, \tag{7}$$

where x is an unknown column matrix, has a non-zero solution. Every non-zero solution of this equation determines some dependence among the columns of $A + \lambda B$. We restrict ourselves to only such solutions $x(\lambda)$ of (7) as are polynomials in λ,[10] and among these solutions we choose one of least possible degree ε:

$$x(\lambda) = x_0 - \lambda x_1 + \lambda^2 x_2 - \cdots + (-1)^\varepsilon \lambda^\varepsilon x_\varepsilon \qquad (x_\varepsilon \neq 0). \tag{8}$$

Substituting this solution in (7) and equating to zero the coefficients of the powers of λ, we obtain:

$$Ax_0 = o,\ Bx_0 - Ax_1 = o,\ Bx_1 - Ax_2 = o,\ \ldots,\ Bx_{\varepsilon-1} - Ax_\varepsilon = o,\ Bx_\varepsilon = o. \tag{9}$$

Considering this as a system of linear homogeneous equations for the elements of the columns $x_0, -x_1, +x_2, \ldots, (-1)^\varepsilon x_\varepsilon$, we deduce that the coefficient matrix of the system

$$M_\varepsilon = M_\varepsilon[A + \lambda B] = \begin{array}{c} \overbrace{\hphantom{A\quad O\quad \dots\quad O}}^{\varepsilon+1} \\ \begin{pmatrix} A & O & \dots & O \\ B & A & & \vdots \\ O & B & \ddots & \vdots \\ \vdots & \vdots & \ddots & A \\ O & O & \dots & B \end{pmatrix} \end{array} \tag{10}$$

is of rank $\varrho_\varepsilon < (\varepsilon + 1)n$. At the same time, by the minimal property of ε, the ranks $\varrho_0, \varrho_1, \ldots, \varrho_{\varepsilon-1}$ of the matrices

---

[10] For the actual determination of the elements of the column x satisfying (7) it is convenient to solve a system of linear homogeneous equations in which the coefficients of the unknown depend linearly on λ. The fundamental linearly independent solutions x can always be chosen such that their elements are polynomials in λ.

$$M_0=\begin{pmatrix}A\\B\end{pmatrix},\quad M_1=\begin{pmatrix}A & O\\ B & A\\ O & B\end{pmatrix},\ \ldots,\ M_{\varepsilon\text{-}1}=\begin{pmatrix}\overbrace{\begin{matrix}A & O & \ldots & O\end{matrix}}^{\varepsilon}\\ \begin{matrix}B & A & & \cdot\\ \cdot & & \ddots & \cdot\\ \cdot & & \ddots & A\\ O & & \ldots & B\end{matrix}\end{pmatrix} \tag{10'}$$

satisfy the equations $\varrho_0=n,\ \varrho_1=2n,\ \ldots,\ \varrho_{\varepsilon-1}=\varepsilon n.$

Thus: *The number ε is the least value of the index k for which the sign $<$ holds in the relation $\varrho_k \leqq (k+1)n$.*

Now we can formulate and prove the following fundamental theorem:

2. THEOREM 4: *If the equation (7) has a solution of minimal degree ε and $\varepsilon>0$, then the given pencil $A+\lambda B$ is strictly equivalent to a pencil of the form*

$$\begin{pmatrix}L_\varepsilon & O\\ O & \hat{A}+\lambda\hat{B}\end{pmatrix}, \tag{11}$$

where

$$L_\varepsilon=\overbrace{\left\|\begin{matrix}\lambda & 1 & 0 & \ldots & 0 & 0\\ 0 & \lambda & 1 & & \cdot & \cdot\\ \cdot & \cdot & \ddots & \ddots & \cdot & \cdot\\ \cdot & \cdot & & \ddots & \ddots & \\ 0 & 0 & & \ldots & \lambda & 1\end{matrix}\right\|}^{\varepsilon+1}\Bigg\}\,\varepsilon, \tag{12}$$

and $\hat{A}+\lambda\hat{B}$ is a pencil of matrices for which the equation analogous to (7) *has no solution of degree less than ε.*

Proof. We shall conduct the proof of the theorem in three stages. First, we shall show that the given pencil $A+\lambda B$ is strictly equivalent to a pencil of the form

$$\begin{pmatrix}L_\varepsilon & D+\lambda F\\ O & \hat{A}+\lambda\hat{B}\end{pmatrix}, \tag{13}$$

where $D,\ F,\ \hat{A},\ \hat{B}$ are constant rectangular matrices of the appropriate dimensions. Then we shall establish that the equation $(\hat{A}+\lambda\hat{B})\hat{x}=O$ has no solution $x(\lambda)$ of degree less than ε. Finally, we shall prove that by further transformations the pencil (13) can be brought into the quasi-diagonal form (11).

1. The first part of the proof will be couched in geometrical terms. Instead of the pencil of matrices $A+\lambda B$ we consider a pencil of operators $\boldsymbol{A}+\lambda\boldsymbol{B}$ mapping $\boldsymbol{R}_n$ into $\boldsymbol{R}_m$ and show that with a suitable choice of bases in the spaces the matrix corresponding to the operator $A+\lambda B$ assumes the form (13).

Instead of (7) we take the vector equation

$$(\boldsymbol{A}+\lambda\boldsymbol{B})\,\boldsymbol{x}=\boldsymbol{o} \tag{14}$$

with the vector solution

$$\boldsymbol{x}(\lambda)=\boldsymbol{x}_0-\lambda\boldsymbol{x}_1+\lambda^2\boldsymbol{x}_2-\cdots+(-1)^\varepsilon\lambda^\varepsilon\boldsymbol{x}_\varepsilon; \tag{15}$$

the equations (9) are replaced by the vector equations

$$\boldsymbol{A}\boldsymbol{x}_0=\boldsymbol{o},\quad \boldsymbol{A}\boldsymbol{x}_1=\boldsymbol{B}\boldsymbol{x}_0,\quad \boldsymbol{A}\boldsymbol{x}_2=\boldsymbol{B}\boldsymbol{x}_1,\ldots,\boldsymbol{A}\boldsymbol{x}_\varepsilon=\boldsymbol{B}\boldsymbol{x}_{\varepsilon-1},\quad \boldsymbol{B}\boldsymbol{x}_\varepsilon=\boldsymbol{o} \tag{16}$$

Below we shall show that the vectors

$$\boldsymbol{A}\boldsymbol{x}_1,\ \boldsymbol{A}\boldsymbol{x}_2,\ldots,\ \boldsymbol{A}\boldsymbol{x}_\varepsilon \tag{17}$$

are linearly independent. Hence it will be easy to deduce the linear independence of the vectors

$$\boldsymbol{x}_0,\ \boldsymbol{x}_1,\ \ldots,\ \boldsymbol{x}_{\varepsilon'} \tag{18}$$

For since $\boldsymbol{A}\boldsymbol{x}_0=\boldsymbol{o}$ we have from $\alpha_0\boldsymbol{x}_0+\alpha_1\boldsymbol{x}_1+\cdots+\alpha_s\boldsymbol{x}_s=\boldsymbol{o}$ that $\alpha_1\boldsymbol{A}\,\boldsymbol{x}_1+\cdots+\alpha_\varepsilon\boldsymbol{A}\,\boldsymbol{x}_\varepsilon=\boldsymbol{o}$, so that by the linear independence of the vectors (17) $\alpha_1=\alpha_2=\ldots=\alpha_\varepsilon=0$. But $\boldsymbol{x}_0\neq 0$, since otherwise $\frac{1}{\lambda}\boldsymbol{x}(\lambda)$ would be a solution of (14) of degree $\varepsilon-1$, which is impossible. Therefore $\alpha_0=0$ also.

Now if we take the vectors (17) and (18) as the first $\varepsilon-1$ vectors for new bases in $\boldsymbol{R}_m$ and $\boldsymbol{R}_n$, respectively, then in these new bases the operators $\boldsymbol{A}$ and $\boldsymbol{B}$, by (16), will correspond to the matrices

$$\tilde{\boldsymbol{A}}=\left\|\begin{array}{cccccccc}\multicolumn{4}{c}{\overbrace{\hphantom{0\quad 1\quad 1\ldots 0}}^{\varepsilon+1}} & & & & \\ 0 & 1 & & \ldots 0 & * & \ldots & * \\ 0 & 0 & 1 & \ldots 0 & * & \ldots & * \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & & \cdot & & & \\ 0 & 0 & & \ldots 1 & * & \ldots & * \\ 0 & 0 & & \ldots 0 & * & \ldots & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ 0 & 0 & & \ldots 0 & * & \ldots & * \end{array}\right\|,\qquad \tilde{B}=\left\|\begin{array}{ccccccc}\multicolumn{4}{c}{\overbrace{\hphantom{1\quad 0\ldots 0\quad 0}}^{\varepsilon+1}} & & & \\ 1 & 0\ldots & 0 & 0 & * & \ldots & * \\ 0 & 1\ldots & 0 & 0 & * & \ldots & * \\ \cdot & \cdot & & \cdot & & & \\ \cdot & \cdot & & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & & \cdot & \cdot & & & \\ 0 & 0\ldots & 1 & 0 & * & \ldots & * \\ 0 & 0\ldots & 0 & 0 & * & \ldots & * \\ & \cdot\ \cdot\ \cdot & 0 & 0 & * & \ldots & * \\ 0 & 0\ldots & 0 & 0 & * & \ldots & * \end{array}\right\|;$$

hence the λ-matrix $\tilde{A} + \lambda\tilde{B}$ is of the form (13). All the preceding arguments will be justified if we can show that the vectors (17) are linearly independent. Assume the contrary and let $\boldsymbol{A}\boldsymbol{x}_h$ ($h \geqq 1$) be the first vector in (17) that is linearly dependent on the preceding ones:

$$\boldsymbol{A}\boldsymbol{x}_h = \alpha_1 \boldsymbol{A}\boldsymbol{x}_{h-1} + \alpha_2 \boldsymbol{A}\boldsymbol{x}_{h-2} + \cdots + \alpha_{h-1}\boldsymbol{A}\boldsymbol{x}_1 .$$

By (16) this equation can be rewritten as follows:

$$\boldsymbol{B}\boldsymbol{x}_{h-1} = \alpha_1 \boldsymbol{B}\boldsymbol{x}_{h-2} + \alpha_2 \boldsymbol{B}\boldsymbol{x}_{h-3} + \cdots + \alpha_{h-1}\boldsymbol{B}\boldsymbol{x}_0 ,$$

i.e.,

$$\boldsymbol{B}\boldsymbol{x}^*_{h-1} = \boldsymbol{o} ,$$

where

$$\boldsymbol{x}^*_{h-1} = \boldsymbol{x}_{h-1} - \alpha_1 \boldsymbol{x}_{h-2} - \alpha_2 \boldsymbol{x}_{h-3} - \cdots - \alpha_{h-1}\boldsymbol{x}_0 .$$

Furthermore, again by (16),

$$\boldsymbol{A}\boldsymbol{x}^*_{h-1} = \boldsymbol{B}(\boldsymbol{x}_{h-2} - \alpha_1 \boldsymbol{x}_{h-3} - \cdots - \alpha_{h-2}\boldsymbol{x}_0) = \boldsymbol{B}\boldsymbol{x}^*_{h-2} ,$$

where

$$\boldsymbol{x}^*_{h-2} = \boldsymbol{x}_{h-2} - \alpha_1 \boldsymbol{x}_{h-3} - \cdots - \alpha_{h-2}\boldsymbol{x}_0 .$$

Continuing the process and introducing the vectors

$$\boldsymbol{x}^*_{h-3} = \boldsymbol{x}_{h-3} - \alpha_1 \boldsymbol{x}_{h-4} - \cdots - \alpha_{h-3}\boldsymbol{x}_0 , \ \ldots , \ \boldsymbol{x}^*_1 = \boldsymbol{x}_1 - \alpha_1 \boldsymbol{x}_0 , \ \boldsymbol{x}^*_0 = \boldsymbol{x}_0 ,$$

we obtain a chain of equations

$$\boldsymbol{B}\boldsymbol{x}^*_{h-1} = \boldsymbol{o} , \quad \boldsymbol{A}\boldsymbol{x}^*_{h-1} = \boldsymbol{B}\boldsymbol{x}^*_{h-2} , \ldots , \boldsymbol{A}\boldsymbol{x}^*_1 = \boldsymbol{B}\boldsymbol{x}^*_0 , \quad \boldsymbol{A}\boldsymbol{x}^*_0 = \boldsymbol{o}. \tag{19}$$

From (19) it follows that

$$\boldsymbol{x}^*(\lambda) = \boldsymbol{x}^*_0 - \lambda \boldsymbol{x}^*_1 + \cdots + (-1)^{h-1}\boldsymbol{x}^*_{h-1} \quad (\boldsymbol{x}^*_0 = \boldsymbol{x}_0 \neq \boldsymbol{o})$$

is a non-zero solution of (14) of degree $\leqq h - 1 < \varepsilon$, which is impossible. Thus, the vectors (17) are linearly independent.

2 We shall now show that the equation $(\hat{A} + \lambda\hat{B})\,\hat{x} = o$ has no solutions of degree less than ε. To begin with, we observe that the equation $L_\varepsilon y = o$, like (7), has a non-zero solution of least degree ε. We can see this immediately, if we replace the matrix equation $L_\varepsilon y = o$ by the system of ordinary equations

$$\lambda y_1 + y_2 = 0 , \quad \lambda y_2 + y_3 = 0, \ \ldots , \ \lambda y_\varepsilon + y_{\varepsilon+1} = 0 \quad (y = (y_1, y_2, \ldots, y_{\varepsilon+1}));$$

$$y_k = (-1)^{k-1} y_1 \lambda^{k-1} \quad (k = 1, 2, \ldots, \varepsilon + 1) .$$

On the other hand, if the pencil has the 'triangular' form (13) then the corresponding matrix pencil M_k ($k = 0, 1, \ldots, \varepsilon$) (see (10) and (10′) on pp. 29 and 30) can also be brought into triangular form, after a suitable permutation of rows and columns:

$$\begin{pmatrix} M_k[L_\varepsilon] & M_k[D + \lambda F] \\ O & M_k[\hat{A} + \lambda\hat{B}] \end{pmatrix}. \tag{20}$$

For $k = \varepsilon - 1$ all the columns of this matrix, like those of $M_{\varepsilon-1}[L_\varepsilon]$, are linearly independent.[11] But $M_{\varepsilon-1}[L_\varepsilon]$ is a square matrix of order $\varepsilon(\varepsilon + 1)$. Therefore in $M_{\varepsilon-1}[\hat{A} + \lambda\hat{B}]$ also, all the columns are linearly independent and, as we have explained at the beginning of the section, this means that the equation $(\hat{A} + \lambda\hat{B})\,\hat{x} = o$ has no solution of degree less than or equal to $\varepsilon - 1$, which is what we had to prove.

3. Let us replace the pencil (13) by the strictly equivalent pencil

$$\begin{pmatrix} E_1 & Y \\ O & E_2 \end{pmatrix} \begin{pmatrix} L_\varepsilon & D + \lambda F \\ O & \hat{A} + \lambda\hat{B} \end{pmatrix} \begin{pmatrix} E_3 & -X \\ O & E_4 \end{pmatrix} = \begin{pmatrix} L_\varepsilon & D + \lambda F + Y(\hat{A} + \lambda\hat{B}) - L_\varepsilon X \\ O & \hat{A} + \lambda\hat{B} \end{pmatrix}, \tag{21}$$

where E_1, E_2, E_3, and E_4 are square unit matrices of orders ε, $m - \varepsilon$, $\varepsilon + 1$, and $n - \varepsilon - 1$, respectively, and X, Y are arbitrary constant rectangular matrices of the appropriate dimensions. Our theorem will be completely proved if we can show that the matrices X and Y can be chosen such that the matrix equation

$$L_\varepsilon X = D + \lambda F + Y(\hat{A} + \lambda\hat{B}) \tag{22}$$

holds.

We introduce a notation for the elements of D, F, X and also for the rows of Y and the columns of $\hat{A}$ and $\hat{B}$:

$$D = \| d_{ik} \| , \quad F = \| f_{ik} \| , \quad X = \| x_{jk} \|$$

$$(i = 1, 2, \ldots, \varepsilon; \quad k = 1, 2, \ldots, n - \varepsilon - 1; \quad j = 1, 2, \ldots, \varepsilon + 1) ,$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_\varepsilon \end{pmatrix}, \quad \hat{A} = (a_1, a_2, \ldots, a_{n-\varepsilon-1}) , \quad \hat{B} = (b_1, b_2, \ldots, b_{n-\varepsilon-1}) .$$

Then the matrix equation (22) can be replaced by a system of scalar equations that expresses the equality of the elements of the k-th column on the right-hand and left-hand sides of (22) ($k = 1, 2, \ldots, n - \varepsilon - 1$):

---

[11] This follows from the fact that the rank of the matrix (20) for $k = \varepsilon - 1$ is equal to εn; a similar equation holds for the rank of the matrix $M_{\varepsilon-1}[L_\varepsilon]$.

$$\begin{aligned}
x_{2k} + \lambda x_{1k} &= d_{1k} + \lambda f_{1k} + y_1 a_k + \lambda y_1 b_k\,,\\
x_{3k} + \lambda x_{2k} &= d_{2k} + \lambda f_{2k} + y_2 a_k + \lambda y_2 b_k\,,\\
x_{4k} + \lambda x_{3k} &= d_{3k} + \lambda f_{3k} + y_3 a_k + \lambda y_3 b_k\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
x_{\varepsilon+1,k} + \lambda x_{\varepsilon k} &= d_{\varepsilon k} + \lambda f_{\varepsilon k} + y_\varepsilon a_k + \lambda y_\varepsilon b_k\\
&(k = 1, 2, \ldots, n - \varepsilon - 1).
\end{aligned} \tag{23}$$

The left-hand sides of these equations are linear binomials in λ. The free term of each of the first $\varepsilon - 1$ of these binomials is equal to the coefficient of λ in the next binomial. But then the right-hand sides must also satisfy this condition. Therefore

$$\begin{aligned}
y_1 a_k - y_2 b_k &= f_{2k} - d_{1k}\,,\\
y_2 a_k - y_3 b_k &= f_{3k} - d_{2k}\,,\\
&\ldots\ldots\ldots\\
y_{\varepsilon-1} a_k - y_\varepsilon b_k &= f_{\varepsilon k} - d_{\varepsilon-1,k}\\
&(k = 1, 2, \ldots, n - \varepsilon - 1).
\end{aligned} \tag{24}$$

If (24) holds, then the required elements of X can obviously be determined from (23).

It now remains to show that the system of equations (24) for the elements of Y always has a solution for arbitrary d_{ik} and f_{ik} ($i = 1, 2, \ldots, \varepsilon$; $k = 1, 2, \ldots, n - \varepsilon - 1$). Indeed, the matrix formed from the coefficients of the unknown elements of the rows $y_1, -y_2, y_3, -y_4, \ldots$, can be written, after transposition, in the form

$$\begin{pmatrix} \overbrace{\hat{A} & O & \ldots & O}^{\varepsilon-1} \\ \hat{B} & \hat{A} & & \vdots \\ O & \hat{B} & \ddots & \vdots \\ \vdots & & \ddots & \hat{A} \\ O & O & \ldots & \hat{B} \end{pmatrix}.$$

But this is the matrix $M_{\varepsilon-2}$ for the pencil of rectangular matrices $\hat{A} + \lambda\hat{B}$ (see (10′) on p. 30). The rank of the matrix is $(\varepsilon - 1)(n - \varepsilon - 1)$, because the equation $(\hat{A} + \lambda\hat{B})\,\hat{x} = o$, by what we have shown, has no solutions of degree less than ε. Thus, the rank of the system of equations (24) is equal to the number of equations and such a system is consistent (non-contradictory) for arbitrary free terms.

This completes the proof of the theorem.

§ 4. The Canonical Form of a Singular Pencil of Matrices

1. Let $A + \lambda B$ be an arbitrary singular pencil of matrices of dimension $m \times n$. To begin with, we shall assume that neither among the columns nor among the rows of the pencil is there a linear dependence with constant coefficients.

Let $r < n$, where r is the rank of the pencil, so that the columns of $A + \lambda B$ are linearly dependent. In this case the equation $(A + \lambda B)x = o$ has a non-zero solution of minimal degree ε_1. From the restriction made at the beginning of this section it follows that $\varepsilon_1 > 0$. Therefore by Theorem 4 the given pencil can be transformed into the form

$$\begin{pmatrix} L_{\varepsilon_1} & O \\ O & A_1 + \lambda B_1 \end{pmatrix},$$

where the equation $(A_1 + \lambda B_1)\,x^{(1)} = o$ has no solution $x^{(1)}$ of degree less than ε_1.

If this equation has a non-zero solution of minimal degree ε_2 (where, necessarily, $\varepsilon_2 \geqq \varepsilon_1$), then by applying Theorem 4 to the pencil $A_1 + \lambda B_1$ we can transform the given pencil into the form

$$\begin{pmatrix} L_{\varepsilon_1} & O & O \\ O & L_{\varepsilon_2} & O \\ O & O & A_2 + \lambda B_2 \end{pmatrix}.$$

Continuing this process, we can put the given pencil into the quasi-diagonal form

$$\begin{pmatrix} L_{\varepsilon_1} & & & & O \\ & L_{\varepsilon_2} & & & \\ & & \ddots & & \\ & & & L_{\varepsilon_p} & \\ O & & & & A_p + \lambda B_p \end{pmatrix}, \tag{25}$$

where $0 < \varepsilon_1 \leqq \varepsilon_2 \leqq \ldots \leqq \varepsilon_p$ and the equation $(A_p + \lambda B_p)\,x^{(p)} = o$ has no non-zero solution, so that the columns of $A_p + \lambda B_p$ are linearly independent.[12]

If the rows of $A_p + \lambda B_p$ are linearly dependent, then the transposed pencil $A_p^{\mathsf{T}} + \lambda B_p^{\mathsf{T}}$ can be put into the form (25), where instead of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ there occur the numbers $(0 <)\, \eta_1 \leq \eta_2 \leqq \cdots \leq \eta_q$.[13] But then the given pencil $A + \lambda B$ turns out to be transformable into the quasi-diagonal form

---

[12] In the special case where $\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_p = m$ the block $A_p + \lambda B_p$ is absent.

[13] Since no linear dependence with constant coefficients exists among the rows of the pencil $A + \lambda B$ and consequently of $A_p + \lambda B_p$, we have $\eta_1 > 0$.

$$\begin{pmatrix} L_{\varepsilon_1} & & & & & & & & O \\ & L_{\varepsilon_2} & & & & & & & \\ & & \ddots & & & & & & \\ & & & L_{\varepsilon_p} & & & & & \\ & & & & L^{\top}_{\eta_1} & & & & \\ & & & & & L^{\top}_{\eta_2} & & & \\ & & & & & & \ddots & & \\ & & & & & & & L^{\top}_{\eta_q} & \\ O & & & & & & & & A_0 + \lambda B_0 \end{pmatrix}, \tag{26}$$

$$(0 < \varepsilon_1 \leqq \varepsilon_2 \leqq \cdots \leqq \varepsilon_p, \quad 0 < \eta_1 \leqq \eta_2 \leqq \cdots \leqq \eta_q)$$

where both the columns and the rows of $A_0 + \lambda B_0$ are linearly independent, i.e., $A_0 + \lambda B_0$ is a regular pencil.[16]

2. We now consider the general case where the rows and the columns of the given pencil may be connected by linear relations with constant coefficients. We denote the maximal number of constant independent solutions of the equations

$$(A + \lambda B)x = o \quad \text{and} \quad (A^{\top} + \lambda B^{\top}) = o$$

by g and h, respectively. Instead of the first of these equations we consider, just as in the proof of Theorem 4, the corresponding vector equation $(\boldsymbol{A} + \lambda \boldsymbol{B})\boldsymbol{x} = \boldsymbol{o}$ ($\boldsymbol{A}$ and $\boldsymbol{B}$ are operators mapping $\boldsymbol{R}_n$ into $\boldsymbol{R}_m$). We denote linearly independent constant solutions of this equation by $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_g$ and take them as the first g basis vectors in $\boldsymbol{R}_n$. Then the first g columns of the corresponding matrix $\tilde{A} + \lambda B$ consist of zeros

$$\tilde{A} + \lambda \tilde{B} = (\overset{g}{\overline{O}}, \tilde{A}_1 + \lambda \tilde{B}_1). \tag{27}$$

Similarly, the first h rows of the pencil $\tilde{A}_1 + \lambda \tilde{B}_1$ can be made into zeros. The given pencil then assumes the form

$$\begin{pmatrix} {}^{h}[\overset{g}{\overline{O}} & O \\ O & A^0 + \lambda B^0 \end{pmatrix}, \tag{28}$$

---

[16] If in the given pencil $r = n$, i.e., if the columns of the pencil are linearly independent, then the first p diagonal blocks in (26) of the form L_{ε} are absent ($p = 0$). In the same way, if $r = m$, i.e., if the rows of $A + \lambda B$ are linearly independent, then in (26) the diagonal blocks of the form $L^{\top}_{\eta}$ are absent ($q = 0$).

where there is no longer any linear dependence with constant coefficients among the rows or the columns of the pencil $A^0 + \lambda B^0$. The pencil $A^0 + \lambda B^0$ can now be represented in the form (26). Thus, in the general case, the pencil $A + \lambda B$ can always be put into the canonical quasi-diagonal form

$$\{{}^{h}[\overset{g}{\overline{O}}, \quad L_{\varepsilon_{g+1}}, \dots, L_{\varepsilon_p}, \quad L^{\top}_{\eta_{h+1}}, \dots, L^{\top}_{\eta_g}, \quad A_0 + \lambda B_0\}. \tag{29}$$

The choice of indices for ε and η is due to the fact that it is convenient here to take $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_g = 0$ and $\eta_1 = \eta_2 = \cdots = \eta_h = 0$.

When we replace the regular pencil $A_0 + \lambda B_0$ in (29) by its canonical form (6) (see § 2, p. 28), we finally obtain the following quasi-diagonal matrix

$$\{{}^{h}[\overset{g}{\overline{O}}; \quad L_{\varepsilon_{g+1}}, \dots, L_{\varepsilon_p}; \quad L^{\top}_{\eta_{h+1}}, \dots, L^{\top}_{\eta_g}; \quad N^{(u_1)}, \dots, N^{(u_s)}; \quad J + \lambda E\}, \tag{30}$$

where the matrix J is of Jordan normal form or of natural normal form and $N^{(u)} = E^{(u)} + \lambda H^{(u)}$.

The matrix (30) is the canonical form of the pencil $A + \lambda B$ in the most general case.

In order to determine the canonical form (30) of a given pencil immediately, without carrying out the successive reduction processes, we shall, following Kronecker, introduce in the next section the concept of minimal indices of a pencil.

§ 5. The Minimal Indices of a Pencil. Criterion for Strong Equivalence of Pencils

1. Let $A + \lambda B$ be an arbitrary singular pencil of rectangular matrices. Then the k polynomial columns $x_1(\lambda), x_2(\lambda), \dots, x_k(\lambda)$ that are solutions of the equation

$$(A + \lambda B)x = o \tag{31}$$

are linearly dependent if the rank of the polynomial matrix formed from these columns $X = [x_1(\lambda), x_2(\lambda), \dots, x_k(\lambda)]$ is less than k. In that case there exist k polynomials $p_1(\lambda), p_2(\lambda), \dots, p_k(\lambda)$, not all identically zero, such that

$$p_1(\lambda)\, x_1(\lambda) + p_2(\lambda)\, x_2(\lambda) + \cdots + p_k(\lambda)\, x_k(\lambda) \equiv O.$$

But if the rank of X is k, then such a dependence does not exist and the solutions $x_1(\lambda), x_2(\lambda), \dots, x_k(\lambda)$ are linearly independent.

Among all the solutions of (31) we choose a non-zero solution $x_1(\lambda)$ of least degree ε_1. Among all the solutions of the same equation that are linearly independent of $x_1(\lambda)$ we take a solution $x_2(\lambda)$ of least degree ε_2. Obviously, $\varepsilon_1 \leqq \varepsilon_2$. We continue the process, choosing among the solutions that are linearly independent of $x_1(\lambda)$ and $x_2(\lambda)$ a solution $x_3(\lambda)$ of minimal degree ε_3, etc. Since the number of linearly independent solutions of (31) is always at most n, the process must come to an end. We obtain a *fundamental series of solutions* of (31)

$$x_1(\lambda),\ x_2(\lambda),\ \ldots,\ x_p(\lambda) \tag{32}$$

having the degrees

$$\varepsilon_1 \leqq \varepsilon_2 \leqq \cdots \leqq \varepsilon_p. \tag{33}$$

In general, a fundamental series of solutions is not uniquely determined (to within scalar factors) by the pencil $A + \lambda B$. However, *two distinct fundamental series of solutions always have one and the same series of degrees* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$. For let us consider in addition to (32) another fundamental series of solutions $\tilde{x}_1(\lambda)$, $\tilde{x}_2(\lambda)$, ... with the degrees $\tilde{\varepsilon}_1$, $\tilde{\varepsilon}_2$, ... Suppose that in (33)

$$\varepsilon_1 = \cdots = \varepsilon_{n_1} < \varepsilon_{n_1+1} = \cdots = \varepsilon_{n_2} < \cdots$$

and similarly, in the series $\tilde{\varepsilon}_1, \tilde{\varepsilon}_2, \ldots,$

$$\tilde{\varepsilon}_1 = \cdots = \tilde{\varepsilon}_{\tilde{n}_1} < \tilde{\varepsilon}_{\tilde{n}_1+1} = \cdots = \tilde{\varepsilon}_{\tilde{n}_2} < \cdots.$$

Obviously, $\varepsilon_1 = \tilde{\varepsilon}_1$. Every column $\tilde{x}_i(\lambda)$ $(i = 1, 2, \ldots, \tilde{n}_1)$ is a linear combination of the columns $x_1(\lambda), x_2(\lambda), \ldots, x_{n_1}(\lambda)$, since otherwise the solution $x_{n_1+1}(\lambda)$ in (32) could be replaced by $x_i(\lambda)$, which is of smaller degree. It is obvious that, conversely, every column $x_i(\lambda)$ $(i = 1, 2, \ldots, n_1)$ is a linear combination of the columns $\tilde{x}_1(\lambda), \tilde{x}_2(\lambda), \ldots, \tilde{x}_{n_1+1}(\lambda)$. Therefore $n_1 = \tilde{n}_1$ and $\varepsilon_{n_1+1} = \tilde{\varepsilon}_{\tilde{n}_1+1}$. Now by a similar argument we obtain that $n_2 = \tilde{n}_2$ and $\varepsilon_{n_2+1} = \tilde{\varepsilon}_{\tilde{n}_1+1}$, etc.

2. Every solution $x_k(\lambda)$ of the fundamental series (32) yields a linear dependence of degree ε_k among the columns of $A + \lambda B$ $(k = 1, 2, \ldots, p)$. Therefore the numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p$ are called the *minimal indices for the columns* of the pencil $A + \lambda B$.

The *minimal indices* $\eta_1, \eta_2, \ldots, \eta_q$ *for the rows* of the pencil $A + \lambda B$ are introduced similarly. Here the equation $(A + \lambda B)x = o$ is replaced by $(A^{\top} + \lambda B^{\top})y = o$, and $\eta_1, \eta_2, \ldots, \eta_q$ are defined as minimal indices for the columns of the transposed pencil $A^{\top} + \lambda B^{\top}$.

Strictly equivalent pencils have the same minimal indices. For let $A + \lambda B$ and $P(A + \lambda B)Q$ be two such pencils (P and Q are non-singular square matrices). Then the equation (31) for the first pencil can be written, after multiplication on the left by P, as follows:

$$P(A + \lambda B)Q \cdot Q^{-1}x = o.$$

Hence it is clear that all the solutions of (31), after multiplication on the left by Q^{-1}, give rise to a complete system of solutions of the equation

$$P(A + \lambda B)Qz = o.$$

Therefore the pencils $A + \lambda B$ and $P(A + \lambda B)Q$ have the same minimal indices for the columns. That the minimal indices for the rows also coincide can be established by going over to the transposed pencils.

Let us compute the minimal indices for the canonical quasi-diagonal matrix

$$\left\{ {}^{h}[\overset{g}{\overline{O}},\quad L_{\varepsilon_{g+1}}, \ldots, L_{\varepsilon_p};\quad L^{\top}_{\eta_{h+1}}, \ldots, L^{\top}_{\eta_q},\quad A_0 \div \lambda B_0 \right\} \tag{34}$$

($A_0 + \lambda B_0$ is a regular pencil having the normal form (6)).

We note first of all that: *The complete system of indices for the columns (rows) of a quasi-diagonal matrix is obtained as the union of the corresponding systems of minimal indices of the individual diagonal blocks.* The matrix L_ε has only one index ε for the columns, and its rows are linearly independent. Similarly, the matrix $L^{\top}_\eta$ has only one index η for the rows, and its columns are linearly independent. Therefore the matrix (34) has as its minimal indices for the columns

$$\varepsilon_1 = \cdots = \varepsilon_g = 0, \quad \varepsilon_{g+1}, \ldots, \varepsilon_p$$

and for the rows

$$\eta_1 = \cdots = \eta_h = 0, \quad \eta_{h+1}, \ldots, \eta_q.$$

We note further that L_ε has no elementary divisors, since among its minors of maximal order ε there is one equal to 1 and one equal to λ^ε. The same statement is, of course, true for the transposed matrix $L^{\top}_\varepsilon$. Since the elementary divisors of a quasi-diagonal matrix are obtained by combining those of the individual diagonal blocks (see Volume I, Chapter VI, p. 141), *the elementary divisors of the λ-matrix* (34) *coincide with those of its regular 'kernel'* $A_0 + \lambda B_0$.

The canonical form of the pencil (34) *is completely determined by the minimal indices* $\varepsilon_1, \ldots, \varepsilon_p, \eta_1, \ldots, \eta_q$ *and the elementary divisors of the pencil or, what is the same, of the strictly equivalent pencil* $A + \lambda B$. Since two pencils having one and the same canonical form are strictly equivalent, we have proved the following theorem:

THEOREM 5 (Kronecker): *Two arbitrary pencils $A + \lambda B$ and $A_1 + \lambda B_1$ of rectangular matrices of the same dimension $m \times n$ are strictly equivalent if and only if they have the same minimal indices and the same (finite and infinite) elementary divisors.*

In conclusion, we write down, for purposes of illustration, the canonical form of a pencil $A + \lambda B$ with the minimal indices $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $\varepsilon_3 = 2$, $\eta_1 = 0$, $\eta_2 = 0$, $\eta_3 = 2$ and the elementary divisors λ^2, $(\lambda + 2)^2$, μ^3:[15]

$$\left|\begin{array}{ccccccccccccccc}
0 \\
0 \\
& \lambda & 1 \\
&&& \lambda & 1 & 0 \\
&&& 0 & \lambda & 1 \\
&&&&&& \lambda & 0 \\
&&&&&& 1 & \lambda \\
&&&&&& 0 & 1 \\
&&&&&&&& 1 & \lambda & 0 \\
&&&&&&&& 0 & 1 & \lambda \\
&&&&&&&& 0 & 0 & 1 \\
&&&&&&&&&&& \lambda & 1 \\
&&&&&&&&&&& 0 & \lambda \\
&&&&&&&&&&&&& \lambda+2 & 1 \\
&&&&&&&&&&&&& 0 & \lambda+2
\end{array}\right| . \qquad (35)$$

§ 6. Singular Pencils of Quadratic Forms

1. Suppose given two complex quadratic forms:

$$A(x, x) = \sum_{i,k=1}^{n} a_{ik} x_i x_k, \qquad B(x, x) = \sum_{i,k=1}^{n} b_{ik} x_i x_k; \qquad (36)$$

they generate a pencil of quadratic forms $A(x, x) + \lambda B(x, x)$. This pencil of forms corresponds to a pencil of symmetric matrices $A + \lambda B$ ($A^T = A$, $B^T = B$). If we subject the variables in the pencil of forms $A(x, x) + \lambda B(x, x)$ to a non-singular linear transformation $x = Tz$ ($|T| \neq 0$), then the transformed pencil of forms $\tilde{A}(z, z) + \lambda \tilde{B}(z, z)$ corresponds to the pencil of matrices

$$\tilde{A} + \lambda \tilde{B} = T^T (A + \lambda B) T; \qquad (37)$$

here T is a constant (i.e., independent of λ) non-singular square matrix of order n.

Two pencils of matrices $A + \lambda B$ and $\tilde{A} + \lambda \tilde{B}$ that are connected by a relation (36) are called *congruent* (see Definition 1 of Chapter X; Vol. I, p. 296).

Obviously, congruence is a special case of equivalence of pencils of matrices. However, if congruence of two pencils of symmetric (or skew-symmetric) matrices is under consideration, then the concept of congruence coincides with that of equivalence. This is the content of the following theorem.

THEOREM 6: *Two strictly equivalent pencils of complex symmetric (or skew-symmetric) matrices are always congruent.*

Proof. Let $\mathcal{A} \equiv A + \lambda B$ and $\tilde{\mathcal{A}} \equiv \tilde{A} + \lambda \tilde{B}$ be two strictly equivalent pencils of symmetric (skew-symmetric) matrices:

$$\tilde{\mathcal{A}} = P \mathcal{A} Q \quad (\mathcal{A}^T = \pm \mathcal{A},\ \tilde{\mathcal{A}}^T = \pm \tilde{\mathcal{A}};\ |P \neq 0, \quad Q| \neq 0). \qquad (38)$$

By going over to the transposed matrices we obtain:

$$\tilde{\mathcal{A}} = Q^T \mathcal{A} P^T. \qquad (39)$$

From (38) and (39), we have

$$\mathcal{A} Q P^{T-1} = P^{-1} Q^T \mathcal{A}. \qquad (40)$$

Setting

$$U = Q P^{T-1}, \qquad (41)$$

we rewrite (40) as follows:

$$\mathcal{A} U = U^T \mathcal{A}. \qquad (42)$$

From (42) it follows easily that

$$\mathcal{A} U^k = U^{Tk} \mathcal{A} \quad (k = 0, 1, 2, \ldots)$$

and, in general,

$$\mathcal{A} S = S^T \mathcal{A}, \qquad (43)$$

where

$$S = f(U), \qquad (44)$$

and $f(\lambda)$ is an arbitrary polynomial in λ. Let us assume that this polynomial is chosen such that $|S| \neq 0$. Then we have from (43):

$$\mathcal{A} = S^T \mathcal{A} S^{-1}. \qquad (45)$$

---

[15] All the elements of the matrix that are not mentioned expressly are zero.

Substituting this expression for Λ in (38), we have:

$$\tilde{\Lambda} = PS^{\mathsf{T}}\Lambda S^{-1}Q. \tag{46}$$

If this relation is to be a congruence transformation, the following equation must be satisfied:

$$(PS^{\mathsf{T}})^{\mathsf{T}} = S^{-1}Q\,,$$

which can be rewritten as

$$S^2 = QP^{\mathsf{T}-1} = U.$$

Now the matrix $S = f(U)$ satisfies this equation if we take as $f(\lambda)$ the interpolation polynomial for $\sqrt{\lambda}$ on the spectrum of U. This can be done, because the many-valued function $\sqrt{\lambda}$ has a single-valued branch determined on the spectrum of U, since $|U| \neq 0$.

The equation (46) now becomes the condition for congruence

$$\tilde{\Lambda} = T^{\mathsf{T}}\Lambda T \quad \left(T = SQ = \sqrt{QP^{\mathsf{T}-1}}\,Q\right). \tag{47}$$

From this theorem and Theorem 5 we deduce:

COROLLARY: *Two pencils of quadratic forms*

$$A(x, x) + \lambda B(x, x) \quad \textit{and} \quad \tilde{A}(z, z) + \lambda \tilde{B}(z, z)$$

can be carried into one another by a transformation $x = Tz$ $(|T| \neq 0)$ *if and only if the pencils of symmetric matrices* $A + \lambda B$ *and* $\tilde{A} + \lambda \tilde{B}$ *have the same elementary divisors (finite and infinite) and the same minimal indices.*

Note. For pencils of symmetric matrices the rows and columns have the same minimal indices:

$$p = q; \quad \varepsilon_1 = \eta_1, \ldots, \varepsilon_p = \eta_p. \tag{48}$$

2. Let us raise the following question: *Given two arbitrary complex quadratic forms*

$$A(x, x) = \sum_{i,k=1}^{n} a_{ik}x_ix_k\,, \qquad B(x, x) = \sum_{i,k=1}^{n} b_{ik}x_ix_k\,.$$

Under what conditions can the two forms be reduced simultaneously to sums of squares

$$\sum_{i=1}^{n} a_iz_i^2 \quad \textit{and} \quad \sum_{i=1}^{n} b_iz_i^2 \tag{49}$$

by a non-singular transformation of the variables $x = Tz$ $(|T| \neq 0)$?

Let us assume that the quadratic forms $A(x, x)$ and $B(x, x)$ have this property. Then the pencil of matrices $A + \lambda B$ is congruent to the pencil of diagonal matrices

$$\{a_1 + \lambda b_1,\ a_2 + \lambda b_2,\ \ldots,\ a_n + \lambda b_n\}. \tag{50}$$

Suppose that among the diagonal binomials $a_i + \lambda b_i$ there are precisely r $(r \leqq n)$ that are not identically zero. Without loss of generality we can assume that

$$a_1 = b_1 = 0, \ldots, a_{n-r} = b_{n-r} = 0,\ a_i + \lambda b_i \not\equiv 0 \quad (i = n - r + 1, \ldots, n). \tag{51}$$

Setting

$$A_0 + \lambda B_0 = \{a_{n-r+1} + \lambda b_{n-r+1}, \ldots, a_n + \lambda b_n\}, \tag{52}$$

we represent the matrix (51) in the form

$$\{\overbrace{O}^{n-r},\quad A_0 + \lambda B_0\}. \tag{53}$$

Comparing (52) with (34) (p.39), we see that in this case all the minimal indices are zero. Moreover, all the elementary divisors are linear. Thus we have obtained the following theorem:

THEOREM 7: *Two quadratic forms* $A(x, x)$ *and* $B(x, x)$ *can be reduced simultaneously to sums of squares* (49) *by a transformation of the variables if and only if in the pencil of matrices* $A + \lambda B$ *all the elementary divisors (finite and infinite) are linear and all the minimal indices are zero.*

In order to reduce two quadratic forms $A(x, x)$ and $B(x, x)$ *simultaneously* to some canonical form in the general case, we have to replace the pencil of matrices $A + \lambda B$ by a strictly equivalent 'canonical' pencil of symmetric matrices.

Suppose the pencil of symmetric matrices $A + \lambda B$ has the minimal indices $\varepsilon_1 = \ldots = \varepsilon_g = 0$, $\varepsilon_{g+1} \neq 0$, $\ldots$, $\varepsilon_p \neq 0$, the infinite elementary divisors $\mu^{u_1}, \mu^{u_2}, \ldots, \mu^{u_s}$ and the finite ones $(\lambda + \lambda_1)^{c_1}$, $(\lambda + \lambda_2)^{c_2}, \ldots, (\lambda + \lambda_t)^{c_t}$ Then, in the canonical form (30), $g = h$, $p = q$ and $\varepsilon_{g+1} = \eta_{g+1}, \ldots, \varepsilon_p = \eta_p$. We replace in (30) every two diagonal blocks of the form L_ε and $L_\varepsilon^{\mathsf{T}}$ by a single diagonal block $\begin{pmatrix} 0 & L_\varepsilon^{\mathsf{T}} \\ L_\varepsilon & 0 \end{pmatrix}$ and each block of the form $N^{(u)} = E^{(u)} + \lambda H^{(u)}$ by the strictly equivalent symmetric block

$$\tilde{N}^{(u)} = V^{(u)}N^{(u)} = \left\| \begin{matrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & \lambda \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \lambda & \ldots & 0 & 0 \end{matrix} \right\| \quad \text{with} \quad V^{(u)} = \left\| \begin{matrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ \cdot & 1 & & & \cdot \\ 1 & 0 & \ldots & 0 & 0 \end{matrix} \right\| . \tag{54}$$

Moreover, instead of the regular diagonal block $J + \lambda E$ in (30) (J is a Jordan matrix)

$$J + \lambda E = \{(\lambda + \lambda_1)\, E^{(e_1)} + H^{(e_1)}, \ldots, (\lambda + \lambda_t)\, E^{(e_t)} + H^{(e_t)}\},$$

we take the strictly equivalent block

$$\{Z_{\lambda_1}^{(e_1)}, \ldots, Z_{\lambda_t}^{(e_t)}\}, \tag{55}$$

where

$$Z_{\lambda_i}^{(e_i)} = V^{(e_i)}\,[(\lambda + \lambda_i)\, E^{(e_i)} + H^{(e_i)}]$$

$$= \left\| \begin{matrix} 0 & \ldots & 0 & \lambda + \lambda_i \\ 0 & \ldots & \lambda + \lambda_i & 1 \\ \vdots & \iddots & \iddots & \vdots \\ \lambda + \lambda_i & 1 & \ldots & 0 \end{matrix} \right\| \quad (i = 1, 2, \ldots, t). \tag{56}$$

The pencil $A + \lambda B$ is strictly equivalent to the symmetric pencil

$$\tilde{A} + \lambda\tilde{B}$$
$$= \left\{ O, \begin{pmatrix} O & L_{\varepsilon_{g+1}}^{\mathsf{T}} \\ L_{\varepsilon_{g+1}} & O \end{pmatrix}, \ldots, \begin{pmatrix} O & L_{\varepsilon_p}^{\mathsf{T}} \\ L_{\varepsilon_p} & O \end{pmatrix};\ \tilde{N}^{(u_1)}, \ldots, \tilde{N}^{(u_s)};\ Z_{\lambda_1}^{(e_1)}, \ldots, Z_{\lambda_t}^{(e_t)} \right\}. \tag{57}$$

Two quadratic forms with complex coefficients $A(x, x)$ *and* $B(x, x)$ *can be simultaneously reduced to the canonical forms* $\tilde{A}(z, z)$ *and* $\tilde{B}(z, z)$ *defined in* (57) *by a transformation of the variables* $x = Tz$ ($|T| \neq 0$).[17]

---

[17] In the Russian edition the author stated that propositions analogous to Theorems 6 and 7 hold for hermitian forms. A. I. Mal'cev has pointed out to the author that this is not the case. As regards singular pencils of hermitian forms, see [197 1J].

§ 7. Application to Differential Equations

1. The results obtained will now be applied to a system of m linear differential equations of the first order in n unknown functions with constant coefficients:[18]

$$\sum_{k=1}^{n} a_{ik}x_k + \sum_{k=1}^{n} b_{ik}\frac{dx_k}{dt} = f_i(t) \quad (i = 1, 2, \ldots, m), \tag{58}$$

or in matrix notation:

$$Ax + B\frac{dx}{dt} = f(t); \tag{59}$$

here[19]

$$A = \|a_{ik}\|, \quad B = \|b_{ik}\| \quad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n),$$
$$x = (x_1, x_2, \ldots, x_n), \quad f = (f_1, f_2, \ldots, f_m).$$

We introduce new unknown functions $z_1, z_2, \ldots, z_n$ that are connected with the old $x_1, x_2, \ldots, x_n$ by a linear non-singular transformation with constant coefficients:

$$x = Qz \quad (z = (z_1, z_2, \ldots, z_n);\ |Q| \neq 0). \tag{60}$$

Moreover, instead of the equations (58) we can take m arbitrary independent combinations of these equations, which is equivalent to multiplying the matrices A, B, f on the left by a square non-singular matrix P of order m. Substituting Qz for x in (59) and multiplying (59) on the left by P, we obtain:

$$\tilde{A}z + \tilde{B}\frac{dz}{dt} = \tilde{f}(t), \tag{61}$$

where

$$\tilde{A} = PAQ, \quad \tilde{B} = PBQ, \quad \tilde{f} = Pf = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n). \tag{62}$$

The matrix pencils $A + \lambda B$ and $\tilde{A} + \lambda\tilde{B}$ are strictly equivalent:

$$\tilde{A} + \lambda\tilde{B} = P(A + \lambda B)Q. \tag{63}$$

We choose the matrices P and Q such that the pencil $\tilde{A} + \lambda\tilde{B}$ has the canonical quasi-diagonal form

---

[18] The particular case where $m = n$ and the system (58) is solved with respect to the derivatives has been treated in detail in Vol. 1, Chapter V, § 5.

It is well known that a system of linear differential equations with constant coefficients of arbitrary order s can be reduced to the form (58) if all the derivatives of the unknown functions up to and including the order $s - 1$ are included as additional unknown functions.

[19] We recall that parentheses denote column matrices. Thus, $x = (x_1, x_2, \ldots, x_n)$ is the column with the elements $x_1, x_2, \ldots, x_n$.

$$\tilde{A} + \lambda\tilde{B} = \{O,\ L_{\varepsilon_{g+1}}, \ldots, L_{\varepsilon_p},\ L^{\mathsf{T}}_{\eta_{h+1}}, \ldots, L^{\mathsf{T}}_{\eta_q},\ N^{(u_1)}, \ldots, N^{(u_s)},\ J + \lambda E\}. \tag{64}$$

In accordance with the diagonal blocks in (64) the system of differential equations splits into $\nu = p - g + q - h + s + 2$ separate systems of the form

$$O \cdot \overset{1}{z} = \overset{1}{\tilde{f}}, \tag{65}$$

$$L_{\varepsilon_{g+i}}\left(\frac{d}{dt}\right)\overset{1+i}{z} = \overset{1+i}{\tilde{f}} \qquad (i = 1, 2, \ldots, p-g), \tag{66}$$

$$L^{\mathsf{T}}_{\eta_{h+j}}\left(\frac{d}{dt}\right)\overset{p-g+1+j}{z} = \overset{p-g+1+j}{\tilde{f}} \qquad (j = 1, 2, \ldots, q-h), \tag{67}$$

$$N^{(u_k)}\left(\frac{d}{dt}\right)\overset{p-g+q-h+1+k}{z} = \overset{p-g+q-h+1+k}{\tilde{f}} \qquad (k = 1, 2, \ldots, s), \tag{68}$$

$$\left(J + \frac{d}{dt}\right)\overset{\nu}{z} = \overset{\nu}{\tilde{f}}, \tag{69}$$

where

$$z = \begin{pmatrix} \overset{1}{z} \\ \overset{2}{z} \\ \cdot \\ \cdot \\ \cdot \\ \overset{\nu}{z} \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} \overset{1}{\tilde{f}} \\ \overset{2}{\tilde{f}} \\ \cdot \\ \cdot \\ \cdot \\ \overset{\nu}{\tilde{f}} \end{pmatrix}, \tag{70}$$

$$\overset{1}{z} = (z_1, \ldots, z_g),\ \overset{1}{\tilde{f}} = (\tilde{f}_1, \ldots, \tilde{f}_h),\ \overset{2}{z} = (z_{g+1}, \ldots),\ \overset{2}{\tilde{f}} = (\tilde{f}_{h+1}, \ldots) \quad \text{etc.}, \tag{71}$$

$$\varLambda\left(\frac{d}{dt}\right) = A + B\frac{d}{dt}, \qquad \text{if} \quad \varLambda(\lambda) \equiv A + \lambda B. \tag{72}$$

Thus, the integration of the system (59) in the most general case is reduced to the integration of the special systems (65)-(69) of the same type. In these systems the matrix pencil $A + \lambda B$ has the form O, L_ε, L^{T}_η, $N^{(u)}$, and $J + \lambda E$, respectively.

1) The system (65) is not inconsistent if and only if

$$\overset{1}{\tilde{f}} \equiv o,$$

i.e.,

$$\tilde{f}_1 \equiv 0, \ldots, \tilde{f}_h \equiv 0. \tag{73}$$

In that case we can take arbitrary functions of t as the unknown functions $z_1, z_2, \ldots, z_g$ that form the columns $\overset{1}{z}$.

2) The system (66) is of the form

$$L_\varepsilon\left(\frac{d}{dt}\right)z = \tilde{f} \tag{74}$$

or, more explicitly,[20]

$$\frac{dz_1}{dt} + z_2 = \tilde{f}_1(t),\ \frac{dz_2}{dt} + z_3 = \tilde{f}_2(t), \ldots,\ \frac{dz_\varepsilon}{dt} + z_{\varepsilon+1} = \tilde{f}_\varepsilon(t). \tag{75}$$

Such a system is always consistent. If we take for $z_{\varepsilon+1}(t)$ an arbitrary function of t, then all the remaining unknown functions $z_\varepsilon, z_{\varepsilon-1}, \ldots, z_1$ can be determined from (75) by successive quadratures.

3) The system (67) is of the form

$$L^{\mathsf{T}}_\eta\left(\frac{d}{dt}\right)z = \tilde{f} \tag{76}$$

or, more explicitly,[21]

$$\frac{dz_1}{dt} = \tilde{f}_1(t),\ \frac{dz_2}{dt} + z_1 = \tilde{f}_2(t), \ldots,\ \frac{dz_\eta}{dt} + z_{\eta-1} = \tilde{f}_\eta(t),\ \ \tilde{z}_\eta = \tilde{f}_{\eta+1}(t). \tag{77}$$

From all the equations (77) except the first we determine $z_\eta, z_{\eta-1}, \ldots, z_1$ uniquely:

$$\begin{aligned} z_\eta &= \tilde{f}_{\eta+1}, \\ z_{\eta-1} &= \tilde{f}_\eta - \frac{d\tilde{f}_{\eta+1}}{dt}, \\ &\cdots\cdots\cdots\cdots\cdots \\ z_1 &= \tilde{f}_2 - \frac{d\tilde{f}_3}{dt} + \cdots + (-1)^{\eta-1}\frac{d^{\eta-1}\tilde{f}_{\eta+1}}{dt^{\eta-1}}. \end{aligned} \tag{78}$$

Substituting this expression for z_1 into the first equation, we obtain the condition for consistency:

$$\tilde{f}_1 - \frac{d\tilde{f}_2}{dt} + \frac{d^2\tilde{f}_3}{dt^2} - \cdots + (-1)^\eta \frac{d^\eta \tilde{f}_{\eta+1}}{dt^\eta} = 0. \tag{79}$$

---

[20] We have changed the indices of z and $\tilde{f}$ to simplify the notation. In order to return from (75) to (66) we have to replace ε by ε_i and add to each index of z the number $g + \varepsilon_{g+1} + \cdots + \varepsilon_{g+i-1} + i - 1$, to each index of $\tilde{f}$ the number $h + \varepsilon_{g+1} + \cdots + \varepsilon_{g+i-1}$.

[21] Here, as in the preceding case, we have changed the notation. See the preceding footnote.

4) The system (68) is of the form

$$N^{(u)}\left(\frac{d}{dt}\right)z=\tilde{f} \tag{80}$$

or, more explicitly,

$$\frac{dz_2}{dt}+z_1=\tilde{f}_1,\ \frac{dz_3}{dt}+z_2=\tilde{f}_2,\ \ldots,\ \frac{dz_u}{dt}+z_{u-1}=\tilde{f}_{u-1},\ z_u=\tilde{f}_u. \tag{81}$$

Hence we determine successively the unique solutions

$$\begin{aligned} z_u &= \tilde{f}_u, \\ z_{u-1} &= \tilde{f}_{u-1}-\frac{d\tilde{f}_u}{dt}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ z_1 &= \tilde{f}_1-\frac{d\tilde{f}_2}{dt}+\frac{d^2\tilde{f}_3}{dt_2}-\cdots+(-1)^{u-1}\frac{d^{u-1}\tilde{f}_u}{dt^{u-1}}. \end{aligned} \tag{82}$$

5) The system (69) is of the form

$$Jz+\frac{dz}{dt}=\tilde{f}. \tag{83}$$

As we have proved in Vol. I, Chapter V, § 5, the general solution of such a system has the form

$$z=e^{-Jt}z_0+\int_0^t e^{-J(t-\tau)}f(\tau)\,d\tau; \tag{84}$$

here z_0 is a column matrix with arbitrary elements (the initial values of the unknown functions for $t=0$).

The inverse transition from the system (61) to (59) is effected by the formulas (60) and (62), according to which each of the functions $x_1, \ldots, x_n$ is a linear combination of the functions $z_1, \ldots, z_n$ and each of the functions $\tilde{f}_1(t), \ldots, \tilde{f}_m(t)$ is expressed linearly (with constant coefficients) in terms of the functions $f_1(t), \ldots, f_m(t)$.

2. The preceding analysis shows that: *In general, for the consistency of the system* (58) *certain well-defined linear dependence relations* (*with constant coefficients*) *must hold among the right-hand sides of the equations and the derivatives of these right-hand sides.*

If these relations are satisfied, then the general solution of the system contains both arbitrary constants and arbitrary functions linearly.

The character of the consistency conditions and the character of the

solutions (*in particular, the number of arbitrary constants and arbitrary functions*) *are determined by the minimal indices and the elementary divisors of the pencil $A+\lambda B$, because the canonical form* (65)-(69) *of the system of differential equations depends on these minimal indices and elementary divisors.*

CHAPTER XIII

MATRICES WITH NON-NEGATIVE ELEMENTS

In this chapter we shall study properties of real matrices with non-negative elements. Such matrices have important applications in the theory of probability, where they are used for the investigation of Markov chains ('stochastic matrices,' see [46]), and in the theory of small oscillations of elastic systems ('oscillation matrices,' see [17]).

§ 1. General Properties

1. We begin with some definitions.

Definition 1: *A rectangular matrix A with real elements*

$$A = \| a_{ik} \| \qquad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n)$$

is called non-negative (notation: $A \geqq O$) *or positive* (notation: $A > O$) *if all the elements of A are non-negative* ($a_{ik} \geqq 0$) *or positive* ($a_{ik} > 0$).

Definition 2: *A square matrix* $A = \| a_{ik} \|_1^n$ *is called reducible if the index set* 1, 2, . . . , *n can be split into two complementary sets* (*without common indices*) $i_1, i_2, \ldots, i_\mu;\ k_1, k_2, \ldots, k_\nu$ $(\mu + \nu = n)$ *such that*

$$a_{i_\alpha k_\beta} = 0 \quad (\alpha = 1, 2, \ldots, \mu;\ \beta = 1, 2, \ldots, \nu).$$

Otherwise the matrix is called *irreducible.*

By a *permutation* of a square matrix $A = \| a_{ik} \|_1^n$ we mean a permutation of the rows of A combined with the *same* permutation of the columns.

The definition of a reducible matrix and an irreducible matrix can also be formulated as follows:

Definition 2′: *A matrix* $A = \| a_{ik} \|_1^n$ *is called reducible if there is a permutation that puts it into the form*

$$\tilde{A} = \begin{pmatrix} B & O \\ C & D \end{pmatrix},$$

where B and D are square matrices. Otherwise A is called irreducible.

Suppose that $A = \| a_{ik} \|_1^n$ corresponds to a linear operator $\boldsymbol{A}$ in an n-dimensional vector space $\boldsymbol{R}$ with the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. To a permutation of A there corresponds a renumbering of the basis vectors, i.e., a transition from the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ to a new basis $\boldsymbol{e}'_1 = \boldsymbol{e}_{j_1}, \boldsymbol{e}'_2 = \boldsymbol{e}_{j_2}, \ldots, \boldsymbol{e}'_n = \boldsymbol{e}_{j_n}$, where $(j_1, j_2, \ldots, j_n)$ is a permutation of the indices 1, 2, . . . , n. The matrix A then goes over into a similar matrix $\tilde{A} = T^{-1}AT$. (Each row and each column of the transforming matrix T contains a single element 1, and the remaining elements are zero.)

2. By a ν-dimensional *coordinate subspace* of $\boldsymbol{R}$ we mean a subspace of $\boldsymbol{R}$ with a basis $\boldsymbol{e}_{k_1}, \boldsymbol{e}_{k_2}, \ldots, \boldsymbol{e}_{k_\nu}$ $(1 \leqq k_1 < k_2 < \ldots < k_\nu \leqq n)$. There are $\binom{n}{\nu}$ ν-dimensional coordinate subspaces of $\boldsymbol{R}$ connected with a given basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. The definition of a reducible matrix can also be given in the following form:

Definition 2″: *A matrix* $A = \| a_{ik} \|_1^n$ *is called reducible if and only if the corresponding operator* $\boldsymbol{A}$ *has a* ν*-dimensional invariant coordinate subspace with* $\nu < n$.

We shall now prove the following lemma:

Lemma 1: *If* $A \geqq O$ *is an irreducible matrix of order n, then*

$$(E + A)^{n-1} > O. \tag{1}$$

Proof. For the proof of the lemma it is sufficient to show that for every vector[1] (i.e., column) $y \geqq o$ $(y \neq o)$ the inequality

$$(E + A)^{n-1} y > o$$

holds.

This inequality will be established if we can only show that *under the conditions* $y \geqq o$ *and* $y \neq o$ *the vector* $z = (E + A)y$ *always has fewer zero coordinates than y does.* Let us assume the contrary. Then y and z have the same zero coordinates.[2] Without loss of generality we may assume that the columns y and z have the form[3]

---

[1] Here and throughout this chapter we mean by a vector a column of n numbers. In this way we identify, as it were, a vector with the column of its coordinates in that basis in which the given matrix $A = \| a_{ik} \|_1^n$ determines a certain linear operator.

[2] Here we start from the fact that $z = y + Ay$ and $Ay \geqq o$; therefore to positive coordinates of y there correspond positive coordinates of z.

[3] The columns y and z can be brought into this form by means of a suitable renumbering of the coordinates (the same for y and z).

$$y=\begin{pmatrix} u \\ o \end{pmatrix},\quad z=\begin{pmatrix} v \\ o \end{pmatrix}\quad (u>o,\ v>o),$$

where the columns u and v are of the same dimension.

Setting

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

we have

$$\begin{pmatrix} u \\ o \end{pmatrix}+\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} u \\ o \end{pmatrix}=\begin{pmatrix} v \\ o \end{pmatrix};$$

and hence

$$A_{21}u=o\,.$$

Since $u>o$, it follows that

$$A_{21}=O\,.$$

This equation contradicts the irreducibility of A.

Thus the lemma is proved.

We introduce the powers of A:

$$A^q=\|\, a_{ik}^{(q)} \|_1^n \quad (q=1,2,\ldots).$$

Then the lemma has the following corollary:

COROLLARY: *If $A \geqq O$ is an irreducible matrix, then for every index pair i, k $(1 \leqq i, k \leqq n)$ there exists a positive integer q such that*

$$a_{ik}^{(q)}>0. \tag{2}$$

Moreover, q can always be chosen within the bounds

$$\left.\begin{array}{ll} q \leqq m-1 & \text{if } i \neq k, \\ q \leqq m & \text{if } i=k, \end{array}\right\} \tag{3}$$

where m is the degree of the minimal polynomial $\psi(\lambda)$ of A.

For let $r(\lambda)$ denote the remainder on dividing $(\lambda+1)^{n-1}$ by $\psi(\lambda)$. Then by (1) we have $r(A)>O$. Since the degree of $r(\lambda)$ is less than m, it follows from this inequality that for arbitrary i, k $(1 \leqq i, k \leqq n)$ at least one of the non-negative numbers

$$\delta_{ik},\ a_{ik},\ a_{ik}^{(2)},\ \ldots,\ a_{ik}^{(m-1)}$$

is not zero. Since $\delta_{ik}=0$ for $i \neq k$, the first of the relations (3) follows.

The other relation (for $i=k$) is obtained similarly if the inequality $r(A)>O$ is replaced by $Ar(A)>O$.[4]

Note. This corollary of the lemma shows that in (1) the number $n-1$ can be replaced by $m-1$.

§ 2. Spectral Properties of Irreducible Non-negative Matrices

1. In 1907 Perron found a remarkable property of the spectra (i.e., the characteristic values and characteristic vectors) of positive matrices.[5]

THEOREM 1 (Perron): *A positive matrix $A=\|\, a_{ik} \|_1^n$ always has a real and positive characteristic value r which is a simple root of the characteristic equation and exceeds the moduli of all the other characteristic values. To this 'maximal' characteristic value r there corresponds a characteristic vector $z=(z_1, z_2, \ldots, z_n)$ of A with positive coordinates $z_i>0$ $(i=1,2,\ldots,n)$.*[6]

A positive matrix is a special case of an irreducible non-negative matrix. Frobenius[7] has generalized Perron's theorem by investigating the spectral properties of irreducible non-negative matrices.

THEOREM 2 (Frobenius): *An irreducible non-negative matrix $A=\|\, a_{ik} \|_1^n$ always has a positive characteristic value r that is a simple root of the characteristic equation. The moduli of all the other characteristic values do not exceed r. To the 'maximal' characteristic value r there corresponds a characteristic vector with positive coordinates.*

Moreover, if A has h characteristic values $\lambda_0=r, \lambda_1, \ldots, \lambda_{h-1}$ of modulus r, then these numbers are all distinct and are roots of the equation

$$\lambda^h-r^h=0\,. \tag{4}$$

More generally: The whole spectrum $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ of A, regarded as a system of points in the complex λ-plane, goes over into itself under a rotation

---

[4] The product of an irreducible non-negative matrix and a positive matrix is itself positive.

[5] See [316], [317], and [17], p. 100.

[6] Since r is a simple characteristic value, the characteristic vector z belonging to it is determined to within a scalar factor. By Perron's theorem all the coordinates of z are real, different from zero, and of like sign. By multiplying z by -1, if necessary, we can make all its coordinates positive. In the latter case the vector (column) $z=(z_1, z_2, z_3, \ldots, z_n)$ is called *positive* (as in Definition 1).

[7] See [165] and [166].

of the plane by the angle $2\pi/h$. If $h > 1$, then A can be put by means of a permutation into the following 'cyclic' form:

$$A = \begin{pmatrix} O & A_{12} & O & \ldots & O \\ O & O & A_{23} & \ldots & O \\ . & . & . & . & . \\ O & O & O & \ldots & A_{h-1,h} \\ A_{h1} & O & O & \ldots & O \end{pmatrix}, \tag{5}$$

where there are square blocks along the main diagonal.

Since Perron's theorem follows as a special case from Frobenius' theorem, we shall only prove the latter.[8] To begin with, we shall agree on some notation.

We write

$$C \leqq D \text{ or } D \geqq C,$$

where C and D are real rectangular matrices of the same dimensions $m \times n$

$$C = \| c_{ik} \|, \; D = \| d_{ik} \| \quad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n),$$

if and only if

$$c_{ik} \leqq d_{ik} \quad (i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n). \tag{6}$$

If the equality sign can be omitted in *all* the inequalities (6), then we shall write

$$C < D \text{ or } D > C.$$

In particular, $C \geqq O$ ($> O$) means that all the elements of C are non-negative (positive).

Furthermore, we denote by C^+ the matrix mod C which arises from C when all the elements are replaced by their moduli.

2. *Proof of Frobenius' Theorem:*[9] Let $x = (x_1, x_2, \ldots, x_n)$ $(x \neq o)$ be a fixed real vector. We set:

$$r_x = \min_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i} \quad \left((Ax)_i = \sum_{k=1}^{n} a_{ik} x_k;\ i = 1, 2, \ldots, n\right).$$

In the definition of the minimum we exclude here the values of i for which $x_i = 0$. Obviously $r_x \geqq 0$, and r_x is the largest real number ϱ for which

$$\varrho x \leqq Ax.$$

---

[8] For a direct proof of Perron's theorem see [17], p. 100 ff.

[9] This proof is due to Wielandt [384].

We shall show that the function r_x assumes a maximum value r for some vector $z \geqq o$:

$$r = r_z = \max_{(x \geqq o)} r_x = \max_{(x \geqq o)} \min_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i}. \tag{7}$$

From the definition of r_x it follows that on multiplication of a vector $x \geqq o$ $(x \neq o)$ by a number $\lambda > 0$ the value of r_x does not change. Therefore, in the computation of the maximum of r_x we can restrict ourselves to the closed set M of vectors x for which

$$x \geqq o \quad \text{and} \quad (xx) = \sum_{i=1}^{n} x_i^2 = 1.$$

If the function r_x were continuous on M, then the existence of a maximum would be guaranteed. However, though continuous at every 'point' $x > o$, r_x may have discontinuities at the boundary points of M at which one of the coordinates vanishes. Therefore, we introduce in place of M the set N of all the vectors y of the form

$$y = (E + A)^{n-1} x \quad (x \in M).$$

The set N, like M, is bounded and closed and by Lemma 1 consists of *positive* vectors only.

Moreover, when we multiply both sides of the inequality

$$r_x x \leqq Ax,$$

by $(E + A)^{n-1} > O$, we obtain:

$$r_x y \leqq Ay \qquad (y = (E + A)^{n-1} x).$$

Hence, from the definition of r_y we have

$$r_x \leqq r_y.$$

Therefore in the computation of the maximum of r_x we can replace M by the set N which consists of positive vectors only. On the bounded and closed set N the function r_x is continuous and therefore assumes a largest value for some vector $z > o$.

Every vector $z \geqq o$ for which

$$r_z = r \tag{8}$$

will be called *extremal*.

We shall now show that: 1) *The number r defined by* (7) *is positive and is a characteristic value of A*; 2) *Every extremal vector z is positive and is a characteristic vector of A for the characteristic value r, i.e.,*

$$r > 0,\ z > o,\ Az = rz. \tag{9}$$

For if $u = (\underbrace{1, 1, \ldots, 1}_{n})$, then $r_u = \min_{1 \leqq i \leqq n} \sum_{k=1}^{n} a_{ik}$. But then $r_u > 0$, because no row of an irreducible matrix can consist of zeros only. Therefore $r > 0$, since $r \geqq r_u$. Now let

$$x = (E + A)^{n-1} z. \tag{10}$$

Then, by Lemma 1, $x > o$. Suppose that $Az - rz \neq o$. Then by (1), (8), and (10) we obtain successively:

$$Az - rz \geqq o,\ (E + A)^{n-1}(Az - rz) > o,\ Ax - rx > o.$$

The last inequality contradicts the definition of r, because it would imply that $Ax - (r + \varepsilon)x > o$ for sufficiently small $\varepsilon > 0$, i.e., $r_x \geqq r + \varepsilon > r$. Therefore $Az = rz$. But then

$$o < x = (E + A)^{n-1} z = (1 + r)^{n-1} z,$$

so that $z > o$.

We shall now show that *the moduli of all the characteristic values do not exceed r*. Let

$$Ay = \alpha y\ (y \neq o). \tag{11}$$

Taking the moduli of both sides in (11), we obtain:[10]

$$|\alpha| y^+ \leqq Ay^+. \tag{12}$$

Hence

$$|\alpha| \leqq r_{y^+} \leqq r.$$

Let y be some characteristic vector corresponding to r:

$$Ay = ry \quad (y \neq o).$$

Then setting $\alpha = r$ in (11) and (12) we conclude that y^+ is an extremal vector, so that $y^+ > o$, i.e., $y = (y_1, y_2, \ldots, y_n)$, where $y_i \neq 0$ $(i = 1, 2, \ldots, n)$. Hence it follows that *only one characteristic direction corresponds to the characteristic value r*; for if there were two linearly independent characteristic vectors z and z_1, we could chose numbers c and d such that the characteristic vector $y = cz + dz_1$ has a zero coordinate, and by what we have shown this is impossible.

---

[10] Regarding the notation y^+, see p. 54.

We now consider the adjoint matrix of the characteristic matrix $\lambda E - A$:

$$B(\lambda) = \| B_{ik}(\lambda) \|_1^n = \Delta(\lambda)\,(\lambda E - A)^{-1},$$

where $\Delta(\lambda)$ is the characteristic polynomial of A and $B_{ik}(\lambda)$ the algebraic complement of the element $\lambda\delta_{ki} - a_{ki}$ in the determinant $\Delta(\lambda)$. From the fact that only one characteristic vector $z = (z_1, z_2, \ldots, z_n)$ with $z_1 > 0$, $z_2 > 0, \ldots, z_n > 0$ corresponds to the characteristic value r (apart from a factor) it follows that $B(r) \neq O$ and that in every non-zero column of $B(r)$ all the elements are different from zero and are of the same sign. The same is true for the rows of $B(r)$, since in the preceding argument A can be replaced by the transposed matrix A^{T}. From these properties of the rows and columns of A it follows that *all* the $B_{ik}(r)$ $(i, k = 1, 2, \ldots, n)$ are different from zero and are of the same sign σ. Therefore

$$\sigma \Delta'(r) = \sigma \sum_{i=1}^{n} B_{ii}(r) > 0,$$

i.e., $\Delta'(r) \neq 0$ and *r is a simple root of the characteristic equation $\Delta(\lambda) = 0$.*

Since r is the maximal root of $\Delta(\lambda) = \lambda^n + \ldots$, $\Delta(\lambda)$ increases for $\lambda \geqq r$. Therefore $\Delta'(r) > 0$ and $\sigma = 1$, i.e.,

$$B_{ik}(r) > 0 \quad (i, k = 1, 2, \ldots, n). \tag{13}$$

3. Proceeding now to the proof of the second part of Frobenius' theorem, we shall make use of the following interesting lemma:[11]

LEMMA 2: *If $A = \| a_{ik} \|_1^n$ and $C = \| c_{ik} \|_1^n$ are two square matrices of the same order n, where A is irreducible and*[12]

$$C^+ \leqq A, \tag{14}$$

then for every characteristic value γ of C and the maximal characteristic value r of A we have the inequality

$$|\gamma| \leqq r. \tag{15}$$

In the relation (15) *the equality sign holds if and only if*

$$C = e^{i\varphi} DAD^{-1}, \tag{16}$$

where $e^{i\varphi} = \gamma / r$ and D is a diagonal matrix whose diagonal elements are of unit modulus $(D^+ = E)$.

---

[11] See [384].

[12] C is a complex matrix and $A \geqq O$.

Proof. We denote by y a characteristic vector of C corresponding to the characteristic value γ:

$$Cy = \gamma y \quad (\gamma \neq 0). \tag{17}$$

From (14) and (17) we find

$$|\gamma|\, y^+ \leqq C^+ y^+ \leqq A y^+. \tag{18}$$

Therefore

$$|\gamma| \leqq r_{y^+} \leqq r.$$

Let us now examine the case $|\gamma| = r$ in detail. Here it follows from (18) that y^+ is an extremal vector for A, so that $y^+ > o$ and that y^+ is a characteristic vector of A for the characteristic value r. Therefore the relation (18) assumes the form

$$Ay^+ = C^+ y^+ = r y^+, \quad y^+ > o. \tag{19}$$

Hence by (14)

$$C^+ = A. \tag{20}$$

Let $y = (y_1, y_2, \dots, y_n)$, where

$$y_j = |y_j|\, e^{i\varphi_j} \qquad (j = 1, 2, \dots, n).$$

We define a diagonal matrix D by the equation

$$D = \{ e^{i\varphi_1}, e^{i\varphi_2}, \dots, e^{i\varphi_n} \}.$$

Then

$$y = D y^+.$$

Substituting this expression for y in (17) and then setting $\gamma = re^{i\varphi}$, we find easily:

$$F y^+ = r y^+, \tag{21}$$

where

$$F = e^{-i\varphi} D^{-1} C D. \tag{22}$$

Comparing (19) with (21), we obtain

$$F y^+ = C^+ y^+ = A y^+. \tag{23}$$

But by (22) and (20)

$$F^+ = C^+ = A.$$

Therefore we find from (23)

$$F y^+ = F^+ y^+.$$

Since $y^+ > o$, this equation can hold only if

$$F = F^+,$$

i.e.,

$$e^{-i\varphi} D^{-1} C D = A.$$

Hence

$$C = e^{i\varphi} D A D^{-1},$$

and the Lemma is proved.

4. We return to Frobenius' theorem and apply the lemma to an irreducible matrix $A \geqq O$ that has precisely h characteristic values of maximal modulus r:

$$\lambda_0 = re^{i\varphi_0},\ \lambda_1 = re^{i\varphi_1},\ \dots,\ \lambda_{h-1} = re^{i\varphi_{h-1}}$$
$$(0 = \varphi_0 < \varphi_1 < \varphi_2 < \cdots < \varphi_{h-1} < 2\pi).$$

Then, setting $C = A$ and $\gamma = \lambda_k$ in the lemma, we have, for every $k = 0, 1, \dots, h-1$,

$$A = e^{i\varphi_k} D_k A D_k^{-1}, \tag{24}$$

where D_k is a diagonal matrix with $D_k^+ = E$.

Again, let z be a positive characteristic vector of A corresponding to the maximal characteristic value r:

$$Az = rz \qquad (z > o). \tag{25}$$

Then setting

$$\overset{k}{y} = D_k z \qquad \left(\overset{k}{y}{}^+ = z > o\right), \tag{26}$$

we find from (25) and (26):

$$A\overset{k}{y} = \lambda_k \overset{k}{y} \qquad (\lambda_k = re^{i\varphi_k};\quad k = 0, 1, \dots, h-1). \tag{27}$$

The last equation shows that the vectors $\overset{0}{y}, \overset{1}{y}, \dots, \overset{h-1}{y}$ defined in (26) are characteristic vectors of A for the characteristic values $\lambda_0, \lambda_1, \dots, \lambda_{h-1}$.

From (24) it follows not only that $\lambda_0 = r$, but also that each characteristic value $\lambda_1, \dots, \lambda_{h-1}$ of A is simple. Therefore the characteristic vectors $\overset{k}{y}$ and hence the matrices D_k $(k = 0, 1, \dots, h-1)$ are determined to within scalar factors. To define the matrices $D_0, D_1, \dots, D_{h-1}$ uniquely we shall choose their first diagonal element to be 1. Then $D_0 = E$ and $y = z > o$.

Furthermore, from (24) it follows that

$$A = e^{i(\varphi_j \pm \varphi_k)} D_j D_k^{\pm 1} A D_k^{\pm 1} D_j^{-1} \quad (j, k = 0, 1, \ldots, h-1).$$

Hence we deduce similarly that the vector

$$D_j D_k^{\pm 1} z$$

is a characteristic vector of A corresponding to the characteristic value $re^{i(\varphi_j \pm \varphi_k)}$.

Therefore $e^{i(\varphi_j \pm \varphi_k)}$ coincides with one of the numbers $e^{i\varphi_l}$ and the matrix $D_j D_k^{\pm 1}$ with the corresponding matrix D_l; that is, we have, for some l_1, l_2 $(0 \leqq l_1, l_2 \leqq h-1)$

$$e^{i(\varphi_j + \varphi_k)} = e^{i\varphi_{l_1}}, \quad e^{i(\varphi_j - \varphi_k)} = e^{i\varphi_{l_2}},$$

$$D_j D_k = D_{l_1}, \quad D_j D_k^{-1} = D_{l_2}.$$

Thus: *The numbers $e^{i\varphi_0}, e^{i\varphi_1}, \ldots, e^{i\varphi_{h-1}}$ and the corresponding diagonal matrices $D_0, D_1, \ldots, D_{h-1}$ form two isomorphic multiplicative abelian groups.*

In every finite group consisting of h distinct elements the h-th power of every element is equal to the unit element of the group. Therefore $e^{i\varphi_0}, e^{i\varphi_1}, \ldots, e^{i\varphi_{h-1}}$ are h-th roots of unity. Since there are h such roots of unity and $\varphi_0 = 0 < \varphi_1 < \varphi_2 < \cdots < \varphi_{h-1} < 2\pi$,

$$\varphi_k = \frac{2k\pi}{h} \quad (k = 0, 1, 2, \ldots, h-1)$$

and

$$e^{i\varphi_k} = \varepsilon^k \quad (\varepsilon = e^{i\varphi_1} = e^{i\frac{2\pi}{h}}; \quad k = 0, 1, \ldots, h-1), \tag{28}$$

$$\lambda_k = r\varepsilon^k \quad (k = 0, 1, \ldots, h-1). \tag{29}$$

The numbers $\lambda_0, \lambda_1, \ldots, \lambda_{h-1}$ form a complete system of roots of (4).

In accordance with (28), we have:[14]

$$D_k = D^k \quad (D = D_1; \ k = 0, 1, \ldots, h-1). \tag{30}$$

The equation (24) now gives us (for $k = 1$):

$$A = e^{i\frac{2\pi}{h}} DAD^{-1}. \tag{31}$$

---

[14] Here we use the isomorphism of the multiplicative groups $e^{i\varphi_0}, e^{i\varphi_1}, \ldots, e^{i\varphi_{h-1}}$ and $D_0, D_1, \ldots, D_{h-1}$.

Hence it follows that the matrix A on multiplication by $e^{i\frac{2\pi}{h}}$ goes over into a similar matrix and, therefore, that *the whole system of n characteristic values of A on multiplication by $e^{i\frac{2\pi}{h}}$ goes over into itself.*[15]

Further,

$$D^h = E,$$

so that all the diagonal elements of D are h-th roots of unity. By a permutation of A (and similarly of D) we can arrange that D be of the following quasi-diagonal form:

$$D = \{\eta_0 E_0, \eta_1 E_1, \ldots, \eta_{s-1} E_{s-1}\}, \tag{32}$$

where $E_0, E_1, \ldots, E_{s-1}$ are unit matrices and

$$\eta_p = e^{i\psi_p}, \ \psi_p = n_p \frac{2\pi}{h}$$

$$(n_p \text{ is an integer}; \ p = 0, 1, \ldots, s-1; \ 0 < n_0 < \ldots < n_{s-1} < h).$$

Obviously $s \leqq h$.

Writing A in block form (in accordance with (32))

$$A = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1s} \\ A_{21} & A_{22} & \ldots & A_{2s} \\ \cdot & \cdot & \cdot & \cdot \\ A_{s1} & A_{s2} & \ldots & A_{ss} \end{pmatrix}, \tag{33}$$

we replace (31) by the system of equations

$$\varepsilon A_{pq} = \frac{\eta_{q-1}}{\eta_{p-1}} A_{pq} \quad \left(p, q = 1, 2, \ldots, s; \ \varepsilon = e^{i\frac{2\pi}{h}}\right) \tag{34}$$

Hence for every p and q either $\frac{\eta_{q-1}}{\eta_{p-1}} = \varepsilon$ or $A_{pq} = O$.

Let us take $p = 1$. Since the matrices $A_{12}, A_{13}, \ldots, A_{1s}$ cannot vanish simultaneously, one of the numbers $\frac{\eta_1}{\eta_0}, \frac{\eta_2}{\eta_0}, \ldots, \frac{\eta_{s-1}}{\eta_0}$ $(\eta_0 = 1)$ must be equal to ε. This is only possible for $n_1 = 1$. Then $\frac{\eta_1}{\eta_0} = \varepsilon$ and $A_{11} = A_{13} = \ldots = A_{1s} = O$. Setting $p = 2$ in (34), we find similarly that $n_2 = 2$ and that $A_{21} = A_{22} = A_{24} = \ldots = A_{2s} = O$, etc. Finally, we obtain

---

[15] The number h is the largest integer having these properties, because A has precisely h characteristic values of maximal modulus r. Moreover, it follows from (31) that all the characteristic values of the matrix fall into systems (with h numbers in each) of the form $\mu_0, \mu_0\varepsilon, \ldots, \mu_0\varepsilon^{h-1}$ and that within each such system to any two characteristic values there correspond elementary divisors of equal degree. One such system is formed by the roots of the equation (4) $\lambda_0, \lambda_1, \ldots, \lambda_{h-1}$.

$$A = \begin{pmatrix} O & A_{12} & O & \dots & O \\ O & O & A_{23} & \dots & O \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ O & O & O & \dots & A_{s-1,s} \\ A_{s1} & A_{s2} & A_{s3} & \dots & A_{ss} \end{pmatrix}.$$

Here $n_1 = 1$, $n_2 = 2, \dots, n_{s-1} = s-1$. But then for $p = s$ on the right-hand sides of (34) we have the factors

$$\frac{\eta_{q-1}}{\eta_{s-1}} = e^{(q-s)\frac{2\pi}{h}i} \qquad (q = 1, 2, \dots, s).$$

One of these numbers must be equal to $\varepsilon = e^{i\frac{2\pi}{h}}$. This is only possible when $s = h$ and $q = 1$; consequently, $A_{s2} = \dots = A_{ss} = O$.

Thus,

$$D = \{E_0, \varepsilon E_1, \varepsilon^2 E_2, \dots, \varepsilon^{h-1} E_{h-1}\},$$

and the matrix A has the form (5).

Frobenius' theorem is now completely proved.

5. We now make a few general comments on Frobenius' theorem.

Remark 1. In the proof of Frobenius' theorem we have established incidentally that for an irreducible matrix $A \geqq O$ with the maximal characteristic value r the adjoint matrix $B(\lambda)$ is positive for $\lambda = r$:

$$B(r) > O, \tag{35}$$

i.e.,

$$B_{ik}(r) > 0 \qquad (i, k = 1, 2, \dots, n), \tag{35'}$$

where $B_{ik}(r)$ is the algebraic complement of the element $r\delta_{ki} - a_{ki}$ in the determinant $|rE - A|$.

Let us now consider the reduced adjoint matrix (see Vol. I, Chapter IV, § 6)

$$C(\lambda) = \frac{B(\lambda)}{D_{n-1}(\lambda)},$$

where $D_{n-1}(\lambda)$ is the greatest common divisor of all the polynomials $B_{ik}(\lambda)$ $(i, k = 1, 2, \dots, n)$. It follows from (35') that $D_{n-1}(r) \neq 0$. All the roots of $D_{n-1}(\lambda)$ are characteristic values[16] distinct from r. Therefore all the roots of $D_{n-1}(\lambda)$ either are complex or are real and less than r. Hence $D_{n-1}(r) > 0$ and this, in conjunction with (35), yields:[17]

$$C(r) = \frac{B(r)}{D_{n-1}(r)} > O. \tag{36}$$

Remark 2. The inequality (35') enables us to determine bounds for the maximal characteristic value r.

We introduce the notation

$$s_i = \sum_{k=1}^{n} a_{ik} \quad (i = 1, 2, \dots, n), \quad s = \min_{1 \leq i \leq n} s_i, \quad S = \max_{1 \leq i \leq n} s_i.$$

Then: *For an irreducible matrix $A \geqq O$*

$$s \leqq r \leqq S, \tag{37}$$

and the equality sign on the left or the right of r holds for $s = S$ only; i.e., holds only when all the 'row-sums' $s_1, s_2, \dots, s_n$ are equal.[18]

For if we add to the last column of the characteristic determinant

$$\Delta(r) = \begin{vmatrix} r - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & r - a_{22} & \dots & -a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{n1} & -a_{n2} & \dots & r - a_{nn} \end{vmatrix}$$

all the preceding columns and then expand the determinant with respect to the elements of the last column, we obtain:

$$\sum_{k=1}^{n} (r - s_k) B_{nk}(r) = 0.$$

Hence (37) follows by (35').

Remark 3. *An irreducible matrix $A \geqq O$ cannot have two linearly independent non-negative characteristic vectors.* For suppose that, apart from the positive characteristic vector $z > o$ corresponding to the maximal characteristic value r, the matrix A has another characteristic vector $y \geqq o$ (linearly independent of z) for the characteristic value α:

---

[16] $D_{n-1}(\lambda)$ is a divisor of the characteristic polynomial $D_n(\lambda) \equiv |\lambda E - A|$.

[17] In the following section it will be shown for an irreducible matrix $B(\lambda) > O$, that $C(\lambda) > O$ for every real $\lambda \geqq r$.

[18] Narrower bounds for r than (s, S) are established in the papers [256], [295] and [119, IV].

$$Ay = \alpha y \quad (y \neq o;\ y \geqq o).$$

Since r is a simple root of the characteristic equation $|\lambda E - A| = 0$,

$$\alpha \neq r.$$

We denote by u the positive characteristic vector of the transposed matrix A^{T} for $\lambda = r$:

$$A^{\mathsf{T}} u = r u \quad (u > o).$$

Then[19]

$$r(y, u) = (y, A^{\mathsf{T}} u) = (Ay, u) = \alpha (y, u);$$

hence, as $\alpha \neq r$,

$$(y, u) = 0,$$

and this is impossible for $u > o$, $y \geqq o$, $y \neq o$.

Remark 4. In the proof of Frobenius' Theorem we have established the following characterization of the maximal characteristic value r of an irreducible matrix $A \geqq O$:

$$r = \max_{(x \geqq o)} r_x,$$

where r_x is the largest number ϱ for which $\varrho x \leqq Ax$. In other words, since $r_x = \min\limits_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i}$, we have

$$r = \max_{(x \geqq o)} \min_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i}.$$

Similarly, we can define for every vector $x \geqq o$ $(x \neq o)$ a number r^x as the least number σ for which

$$\sigma x \geqq Ax;$$

i.e., we set

$$r^x = \max_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i}.$$

If for some i we have here $x_i = 0$, $(Ax)_i \neq 0$, then we shall take $r^x = +\infty$.

As in the case of the function r_x, it turns out here that the function r^x assumes a least value $\hat{r}$ for some vector $v > o$.

Let us show that the number $\hat{r}$ defined by

$$\hat{r} = \min_{(x \geqq o)} r^x = \min_{(x \geqq o)} \max_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i} \tag{38}$$

---

[19] If $y = (y_1, y_2, \ldots, y_n)$ and $u = (u_1, u_2, \ldots, u_n)$, then we mean by (y, u) the 'scalar product' $y^{\mathsf{T}} u = \sum\limits_{i=1}^{n} y_i u_i$. Then $(y, A^{\mathsf{T}} u) = y^{\mathsf{T}} A^{\mathsf{T}} u$ and $(Ay, u) = (Ay)^{\mathsf{T}} u = y^{\mathsf{T}} A^{\mathsf{T}} u$.

coincides with r and that the vector $v \geqq o$ $(v \neq o)$ for which this minimum is assumed is a characteristic vector of A for $\lambda = r$.

For,

$$\hat{r} v - Av \geqq o \quad (v \geqq o,\ \ v \neq o).$$

Suppose now that the sign $\geqq$ cannot be replaced by the equality sign. Then by Lemma 1

$$(E + A)^{n-1} (\hat{r} v - Av) > o, \quad (E + A)^{n-1} v > o. \tag{39}$$

Setting

$$u = (E + A)^{n-1} v > o,$$

we have

$$\hat{r} u > Au$$

and so for sufficiently small $\varepsilon > 0$

$$(\hat{r} - \varepsilon) u > Au \quad (u > o),$$

which contradicts the definition of $\hat{r}$. Thus

$$Av = \hat{r} v.$$

But then

$$u = (E + A)^{n-1} v = (1 + \hat{r})^{n-1} v.$$

Therefore $u > o$ implies that $v > o$.

Hence, by the Remark 3,

$$\hat{r} = r.$$

Thus we have for r the double characterization:

$$r = \max_{(x \geqq o)} \min_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i} = \min_{(x \geqq o)} \max_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i}. \tag{40}$$

Moreover we have shown that $\max\limits_{(x \geqq o)}$ or $\min\limits_{(x \geqq o)}$ is only assumed for a positive characteristic vector for $\lambda = r$.

From this characterization of r we obtain the inequality[20]

$$\min_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i} \leqq r \leqq \max_{1 \leqq i \leqq n} \frac{(Ax)_i}{x_i} \quad (x \geqq o,\ x \neq o). \tag{41}$$

Remark 5. Since in (40) $\max\limits_{(x \geqq o)}$ and $\min\limits_{(x \geqq o)}$ are only assumed for a positive characteristic vector of the irreducible matrix $A \geqq O$, the inequalities

---

[20] See [128] and also [17], p. 325 ff.

$$rz \leqq Az,\ z \geqq 0,\ z \neq o$$

or

$$rz \geqq Az,\ z \geqq o,\ z \neq o$$

always imply that

$$Az = rz,\ z > o.$$

§ 3. Reducible Matrices

1. The spectral properties of irreducible non-negative matrices that were established in the preceding section are not preserved when we go over to reducible matrices. However, since every non-negative matrix $A \geqq O$ can be represented as the limit of a sequence of irreducible positive matrices A_m

$$A = \lim_{m \to \infty} A_m \quad (A_m > O,\ m = 1, 2, \ldots), \tag{42}$$

some of the spectral properties of irreducible matrices hold in a weaker form for reducible matrices.

For an arbitrary non-negative matrix $A = \| a_{ik} \|_1^n$ we can prove the following theorem:

THEOREM 3: *A non-negative matrix $A = \| a_{ik} \|_1^n$ always has a non-negative characteristic value r such that the moduli of all the characteristic values of A do not exceed r. To this 'maximal' characteristic value r there corresponds a non-negative characteristic vector*

$$Ay = ry \quad (y \geqq o,\ y \neq o).$$

The adjoint matrix $B(\lambda) = \| B_{ik}(\lambda) \|_1^n = (\lambda E - A)^{-1} \Delta(\lambda)$ satisfies the inequalities

$$B(\lambda) \geqq O, \quad \frac{d}{d\lambda} B(\lambda) \geqq O \quad \textit{for} \quad \lambda \geqq r. \tag{43}$$

Proof. Let A be represented as in (42). We denote by $r^{(m)}$ and $y^{(m)}$ the maximal characteristic value of the positive matrix A_m and the corresponding normalized[21] positive characteristic vector:

$$A_m y^{(m)} = r^{(m)} y^{(m)} \quad ((y^{(m)}, y^{(m)}) = 1,\ y^{(m)} > o;\ m = 1, 2, \ldots). \tag{44}$$

Then it follows from (42) that the limit

$$\lim r^{(m)} = r$$

---

[21] By a normalized vector we mean a column $y = (y_1, y_2, \ldots, y_n)$ for which $(y, y) \equiv \sum_{i=1}^{n} y_i^2 = 1$.

exists, where r is a characteristic value of A. From the fact that $r^{(m)} > 0$ and $r^{(m)} > | \lambda_0^{(m)} |$, where $\lambda_0^{(m)}$ is an arbitrary characteristic value of A_m $(m = 1, 2, \ldots)$, we obtain by proceeding to the limit:

$$r \geqq 0,\ r \geqq | \lambda_0 |,$$

where λ_0 is an arbitrary characteristic value of A. This passage to the limit gives us in place of (35)

$$B(r) \geqq O. \tag{45}$$

Furthermore, from the sequence of normalized characteristic vectors $y^{(m)}$ $(m = 1, 2, \ldots)$ we can select a subsequence $y^{(m_p)}$ $(p = 1, 2, \ldots)$ that converges to some normalized (and therefore non-zero) vector y. When we go to the limit on both sides of (44) by giving to m the values m_p $(p = 1, 2, \ldots)$ successively, we obtain:

$$Ay = ry \quad (y \geqq o,\ y \neq o).$$

The inequalities (43) will be established by induction on the order n. For $n = 1$, they are obvious.[22] Let us establish them for a matrix $A = \| a_{ik} \|_1^n$ of order n on the assumption that they are true for matrices of order less than n.

Expanding the characteristic determinant $\Delta(\lambda) = | \lambda E - A |$ with respect to the elements of the last row and the last column, we obtain:

$$\Delta(\lambda) = (\lambda - a_{nn}) B_{nn}(\lambda) - \sum_{i,k=1}^{n-1} B_{ki}^{(n)}(\lambda)\, a_{in} a_{nk}\,. \tag{46}$$

Here $B_{nn}(\lambda) = | \lambda\delta_{ik} - a_{ik} |_1^{n-1}$ is the characteristic determinant of a 'truncated' non-negative matrix of order $n-1$, and $B_{ki}^{(n)}(\lambda)$ is the algebraic complement of the element $\lambda\delta_{ik} - a_{ik}$ in $B_{nn}(\lambda)$ $(i, k = 1, 2, \ldots, n-1)$. The maximal non-negative root of $B_{nn}(\lambda)$ will be denoted by r_n. Then setting $\lambda = r_n$ in (46) and observing that by the induction hypothesis

$$B_{ki}^{(n)}(r_n) \geqq O \qquad (i, k = 1, 2, \ldots, n-1),$$

we obtain from (46):

$$\Delta(r_n) \leqq 0.$$

On the other hand $\Delta(\lambda) = \lambda^n + \ldots$, so that $\Delta(+\infty) = +\infty$. Therefore r_n either is a root of $\Delta(\lambda)$ or is less than some real root of $\Delta(\lambda)$. In both cases,

---

[22] For since $B(\lambda) \equiv (\lambda E - A)^{-1}\Delta(\lambda)$, we have $B(\lambda) \equiv E$, $\frac{d}{d\lambda} B(\lambda) \equiv O$ for $n = 1$.

$$r_n \leqq r.$$

Since every principal minor $B_{jj}(\lambda)$ of order $n-1$ can be brought into the position of $B_{nn}(\lambda)$ by a permutation of A, we have

$$r_j \leqq r \qquad (j = 1, 2, \ldots, n), \tag{47}$$

where r_j denotes the maximal root of the polynomial $B_{jj}(\lambda)$ $(j = 1, 2, \ldots, n)$.

Furthermore, $B_{ik}(\lambda)$ may be represented as a minor of order $n-1$ of the characteristic matrix $\lambda E - A$, multiplied by $(-1)^{i+k}$. When we differentiate this determinant with respect to λ, we obtain:

$$\frac{d}{d\lambda} B_{ik}(\lambda) = \sum B_{ik}^{(j)}(\lambda) \qquad (i, k = 1, 2, \ldots, n-1), \tag{48}$$

where $B^{(j)}(\lambda) = \| B_{ik}^{(j)} \|$ $(i \neq j,\ k \neq j;\ j = 1, 2, \ldots, n)$ is the adjoint matrix of the matrix $\| a_{ik} \|$ $(i, k = 1, 2, \ldots, j-1, j+1, \ldots, n)$ of order $n-1$. But, by the induction hypothesis,

$$B^{(j)}(\lambda) \geqq O \quad \text{for} \quad \lambda \geqq r_j \quad (j = 1, 2, \ldots, n);$$

and so, by (47) and (48),

$$\frac{d}{d\lambda} B(\lambda) \geqq O \quad \text{for} \quad \lambda \geqq r. \tag{49}$$

From (45) and (49) it follows that

$$B(\lambda) \geqq O \quad \text{for} \quad \lambda \geqq r.$$

The proof of the theorem is now complete.

Note. In the passage to the limit (42) the inequalities (37) are preserved. They hold, therefore, for an arbitrary non-negative matrix. However, the conditions under which the equality sign holds in (37) are not valid for a reducible matrix.

2. A number of important propositions follow from Theorem 3:

1. *If $A = \| a_{ik} \|_1^n$ is a non-negative matrix with maximal characteristic value r and $C(\lambda)$ is its reduced adjoint matrix, then*

$$C(\lambda) \geqq O \quad \textit{for} \quad \lambda \geqq r. \tag{50}$$

For

$$C(\lambda) = \frac{B(\lambda)}{D_{n-1}(\lambda)}, \tag{51}$$

where $D_{n-1}(\lambda)$ is the greatest common divisor of the elements of $B(\lambda)$. Since $D_{n-1}(\lambda)$ divides the characteristic polynomial $\Delta(\lambda)$ and $D_{n-1}(\lambda) = \lambda^{n-1} + \ldots$,

$$D_{n-1}(\lambda) > 0 \quad \text{for} \quad \lambda > r. \tag{52}$$

Now (43), (51), and (52) imply (50).

2. *If $A \geqq O$ is an irreducible matrix with maximal characteristic value r, then*

$$B(\lambda) > O,\ C(\lambda) > O \quad \textit{for} \quad \lambda \geqq r. \tag{53}$$

Indeed, by (35) $B(r) > O$. But also (see (43)) $\frac{d}{d\lambda} B(\lambda) \geqq O$ for $\lambda \geqq r$. Therefore

$$B(\lambda) > O \quad \text{for} \quad \lambda \geqq r. \tag{54}$$

The other of the inequalities (53) follows from (51), (52), and (54).

3. *If $A \geqq O$ is an irreducible matrix with maximal characteristic value r, then*

$$(\lambda E - A)^{-1} > O \quad \textit{for} \quad \lambda > r. \tag{55}$$

This inequality follows from the formula

$$(\lambda E - A)^{-1} = \frac{B(\lambda)}{\Delta(\lambda)},$$

since $B(\lambda) > O$ and $\Delta(\lambda) > 0$ for $\lambda > r$.

4. *The maximal characteristic value r' of every principal minor*[23] *(of order less than n) of a non-negative matrix $A = \| a_{ik} \|_1^n$ does not exceed the maximal characteristic value r of A:*

$$r' \leqq r. \tag{56}$$

If A is irreducible, then the equality sign in (56) cannot occur.

If A is reducible, then the equality sign in (56) holds for at least one principal minor.

---

[23] We mean here by a principal minor the matrix formed from the elements of a principal minor.

For the inequality (56) is true for every principal minor of order $n-1$ (see (47)). If A is irreducible, then by (35′) $B_{jj}(r) > 0$ $(j=1, 2, \dots, n)$ and therefore $r' \neq r$.

By descent from $n-1$ to $n-2$, from $n-2$ to $n-3$, etc., we show the truth of (56) for the principal minors of every order.

If A is a reducible matrix, then by means of a permutation it can be put into the form

$$A=\begin{pmatrix} B & O \\ C & D \end{pmatrix}.$$

Then r must be a characteristic value of one of the two principal minors B and D. This proves Proposition 4.

From 4. we deduce:

5. *If $A \geqq O$ and if in the characteristic determinant*

$$\Delta(r)=\begin{vmatrix} r-a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & r-a_{22} & \dots & -a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{n1} & -a_{n2} & \dots & r-a_{nn} \end{vmatrix}$$

any principal minor vanishes (A is reducible!), then every 'augmented' principal minor also vanishes; in particular, so does one of the principal minors of order $n-1$

$$B_{11}(\lambda),\ B_{22}(\lambda),\ \dots,\ B_{nn}(\lambda).$$

From 4. and 5. we deduce:

6. *A matrix $A \geqq O$ is reducible if and only if in one of the relations*

$$B_{ii}(r) \geqq 0 \qquad (i=1, 2, \dots, n)$$

the equality sign holds.

From 4. we also deduce:

7. *If r is the maximal characteristic value of a matrix $A \geqq O$, then for every $\lambda > r$ all the principal minors of the characteristic matrix $A_\lambda \equiv \lambda E - A$ are positive:*

$$A_\lambda\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix}>0 \quad (\lambda > r;\ 1 \leqq i_1 < i_2 < \dots < i_p \leqq n;\ p=1, 2, \dots, n). \quad (57)$$

It is easy to see that, conversely, (57) implies that $\lambda > r$. For

$$\Delta(\lambda+\mu)=|(\lambda+\mu)E-A|=|A_\lambda+\mu E|=\sum_{k=0}^{n} S_k\mu^{n-k},$$

where S_k is the sum of all the principal minors of order k of the characteristic matrix $A_\lambda \equiv \lambda E - A$ $(k=1, 2, \dots, n)$.[24] Therefore, if for some real λ all the principal minors of A_λ are positive, then for $\mu \geqq 0$

$$\Delta(\lambda+\mu) \neq 0,$$

i.e., no number greater than λ is a characteristic value of A. Therefore

$$r < \lambda.$$

Thus, (57) is a necessary and sufficient condition for λ to be an upper bound for the moduli of the characteristic values of A.[25] However, the inequalities (57) are not all independent.

The matrix $\lambda E - A$ is a matrix with non-positive elements outside the main diagonal.[26] D. M. Kotelyanskiĭ has proved that for such matrices, just as for symmetric matrices, all the principal minors are positive, provided the successive principal minors are positive.[27]

LEMMA 3 (Kotelyanskiĭ): *If in a real matrix $G=\| g_{ik} \|_1^n$ all the non-diagonal elements are negative or zero*

$$g_{ik} \leqq 0 \qquad (i \neq k;\ i, k=1, 2, \dots, n) \qquad (58)$$

and the successive principal minors are positive

$$g_{11}=G\begin{pmatrix} 1 \\ 1 \end{pmatrix}>0,\ G\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}>0,\ \dots,\ G\begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}>0, \qquad (59)$$

then all the principal minors are positive:

$$G\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ i_1 & i_2 & \dots & i_p \end{pmatrix}>0 \qquad (1 \leqq i_1 < i_2 < \dots < i_p \leqq n;\ p=1, 2, \dots, n).$$

---

[24] See Vol. I, p. 70.

[25] See [344].

[26] It is easy to see that, conversely, every matrix with negative or zero non-diagonal elements can be represented in the form $\lambda E - A$, where A is a non-negative matrix and λ is a real number.

[27] See [215]. This paper contains a number of results about matrices in which all the non-diagonal elements are of like sign.

Proof. We shall prove the lemma by induction on the order n of the matrix. For $n = 2$ the lemma holds, since it follows from

$$g_{12} \leqq 0 \quad g_{21} \leqq 0, \quad g_{11} > 0, \quad g_{11}g_{22} - g_{12}g_{21} > 0$$

that $g_{22} > 0$. Let us assume now that the lemma is true for matrices of order less than n; we shall then prove it for $G = \| g_{ik} \|_1^n$. We consider the bordered determinants

$$t_{ik} = G\begin{pmatrix}1 & i\\ 1 & k\end{pmatrix} = g_{11}g_{ik} - g_{1k}g_{i1} \qquad (i, k = 2, \ldots, n).$$

From (58) and (59) it follows that

$$t_{ik} \leqq 0 \qquad (i \neq k;\ i, k = 2, \ldots, n).$$

On the other hand, by applying Sylvester's identity (Vol. I, Chapter II, (30), p. 33) to the matrix $T = \| t_{ik} \|_2^n$, we obtain:

$$T\begin{pmatrix}i_1 & i_2 & \ldots & i_p\\ i_1 & i_2 & \ldots & i_p\end{pmatrix} = (g_{11})^{p-1} G\begin{pmatrix}1 & i_1 & i_2 & \ldots & i_p\\ 1 & i_1 & i_2 & \ldots & i_p\end{pmatrix} \qquad \begin{pmatrix}2 \leqq i_1 < i_2 < \cdots < i_p \leqq n,\\ p = 1, 2, \ldots, n-1\end{pmatrix}. \tag{60}$$

Hence it follows by (59) that the successive principal minors of the matrix $T = \| t_{ik} \|_2^n$ are positive:

$$t_{22} = T\begin{pmatrix}2\\ 2\end{pmatrix} > 0, \; T\begin{pmatrix}2 & 3\\ 2 & 3\end{pmatrix} > 0, \ldots, \; T\begin{pmatrix}2 & 3 & \ldots & n\\ 2 & 3 & \ldots & n\end{pmatrix} > 0.$$

Thus, the matrix $T = \| t_{ik} \|_2^n$ of order $n - 1$ satisfies the condition of the lemma. Therefore by the induction hypothesis all the principal minors are positive:

$$T\begin{pmatrix}i_1 & i_2 & \ldots & i_p\\ i_1 & i_2 & \ldots & i_p\end{pmatrix} > 0 \quad (2 \leqq i_1 < i_2 < \cdots < i_p \leqq n;\ p = 1, 2, \ldots, n-1).$$

But then it follows from (60) that all the principal minors of G containing the first row are positive:

$$G\begin{pmatrix}1 & i_1 & i_2 & \ldots & i_p\\ 1 & i_1 & i_2 & \ldots & i_p\end{pmatrix} > 0 \quad (2 \leqq i_1 < i_2 < \cdots < i_p \leqq n;\ p = 1, 2, \ldots, n-1). \tag{61}$$

Let us choose fixed indices $i_1, i_2, \ldots, i_{n-2}$ (where $1 < i_1 < i_2 < \cdots < i_{n-2} \leqq n$) and form the matrix of order $n - 1$:

$$\| g_{\alpha\beta} \| \qquad (\alpha, \beta = 1, i_1, i_2, \ldots, i_{n-2}). \tag{62}$$

The successive principal minors of this matrix are positive, by (61):

$$g_{11} > 0, \; G\begin{pmatrix}1 & i_1\\ 1 & i_1\end{pmatrix} > 0, \; \ldots, \; G\begin{pmatrix}1 & i_1 & i_2 & \ldots & i_{n-2}\\ 1 & i_1 & i_2 & \ldots & i_{n-2}\end{pmatrix} > 0;$$

and the non-diagonal elements are non-positive:

$$g_{\alpha\beta} \leqq 0 \qquad (\alpha \neq \beta;\ \alpha, \beta = 1, i_1, i_2, \ldots, i_{n-2}).$$

But the order of (62) is $n - 1$. Therefore, by the induction hypothesis, all the principal minors of this matrix are positive; in particular,

$$G\begin{pmatrix}i_1 & i_2 & \ldots & i_p\\ i_1 & i_2 & \ldots & i_p\end{pmatrix} > 0 \tag{63}$$
$$(2 \leqq i_1 < i_2 < \cdots < i_p \leqq n;\ p = 1, 2, \ldots, n-2).$$

Thus, *all* the minors of G of order not exceeding $n - 2$ are positive.

Since by (63) $g_{22} > 0$, we may now consider the determinants of order two bordering the element g_{22} (and not g_{11} as before):

$$t_{ik}^{*} = G\begin{pmatrix}2 & i\\ 2 & k\end{pmatrix} \qquad (i, k = 1, 3, \ldots, n).$$

By operating with the matrix $T^{*} = \| t_{ik}^{*} \|$, as we have done above with T, we obtain inequalities analogous to (61):

$$G\begin{pmatrix}2 & i_1 & \ldots & i_p\\ 2 & i_1 & \ldots & i_p\end{pmatrix} > 0 \tag{64}$$
$$(i_1 < i_2 < \cdots < i_p;\quad i_1, \ldots, i_p = 1, 3, \ldots, n;\quad p = 1, 2, \ldots, n-1).$$

Since every principal minor of $G = \| g_{ik} \|_1^n$ contains either the first or the second row or is of order not exceeding $n - 2$, it follows from (61), (63), and (64) that all the principal minors of A are positive. This completes the proof of the lemma.

This lemma allows us to retain only the successive principal minors in the condition (57) and to formulate the following theorem:

---

[25] See [344] and [215]. Since $C = A - \lambda E$ and $A \geqq O$, λ_n is real (this follows from $\lambda_n + \lambda = r$) and the corresponding characteristic vector of C is non-negative: $Cy = \lambda_n y$ $(y \geqq o,\ y \neq o)$.

THEOREM 4: *A real number λ is greater than the maximal characteristic value r of the matrix $A = \| a_{ik} \|_1^n \geqq O$*

$$r < \lambda$$

if and only if for this value λ all the successive principal minors of the characteristic matrix $A_\lambda \equiv \lambda E - A$ are positive:

$$\begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} > 0, \ \ldots, \ \begin{vmatrix} \lambda - a_{11} & -a_{12} & \ldots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \ldots & -a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{n1} & -a_{n2} & \ldots & \lambda - a_{nn} \end{vmatrix} > 0. \qquad (65)$$

Let us consider one application of Theorem 4. Suppose that in the matrix $C = \| c_{ik} \|_1^n$ all the non-diagonal elements are non-negative. Then for some $\lambda > 0$ we have $A = C + \lambda E \geqq O$. We arrange the characteristic values λ_i $(i = 1, 2, \ldots, n)$ of C with their real parts in ascending order:

$$\operatorname{Re} \lambda_1 \leqq \operatorname{Re} \lambda_2 \leqq \ldots \leqq \operatorname{Re} \lambda_n.$$

We denote by r the maximal characteristic value of A. Since the characteristic values of A are the sums $\lambda_i + \lambda$ $(i = 1, 2, \ldots, n)$, we have

$$\lambda_n + \lambda = r.$$

In this case the inequality $r < \lambda$ holds for $\lambda_n < 0$ only, and signifies that all the characteristic values of C have negative real parts. When we write down the inequality (65) for the matrix $-C = \lambda E - A$, we obtain the following theorem:

THEOREM 5: *The real parts of all the characteristic values of a real matrix $C = \| c_{ik} \|_1^n$ with non-negative non-diagonal elements*

$$c_{ik} \geqq 0 \ \ (i \neq k;\ i, k = 1, 2, \ldots, n)$$

are negative if and only if

$$c_{11} < 0, \ \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} > 0, \ \ldots, \ (-1)^n \begin{vmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{vmatrix} > 0. \qquad (66)$$

§ 4. The Normal Form of a Reducible Matrix

1. We consider an arbitrary reducible matrix $A = \| a_{ik} \|_1^n$. By means of a permutation we can put it into the form

$$A = \begin{pmatrix} B & O \\ C & D \end{pmatrix}, \qquad (67)$$

where B and D are square matrices.

If one of the matrices B or D is reducible, then it can also be represented in a form similar to (67), so that A then assumes the form

$$A = \begin{pmatrix} K & O & O \\ H & L & O \\ F & G & M \end{pmatrix}.$$

If one of the matrices K, L, M is reducible, then the process can be continued. Finally, by a suitable permutation we can reduce A to triangular block form

$$A = \begin{pmatrix} A_{11} & O & \ldots & O \\ A_{21} & A_{22} & \ldots & O \\ \cdot & \cdot & \cdot & \cdot \\ A_{s1} & A_{s2} & \ldots & A_{ss} \end{pmatrix}, \qquad (68)$$

where the diagonal blocks are square irreducible matrices.

A diagonal block A_{ii} $(1 \leqq i \leqq s)$ is called *isolated* if

$$A_{ik} = O \quad (k = 1, 2, \ldots, i-1, i+1, \ldots, s).$$

By a permutation of the blocks (see p. 50) in (68) we can put all the isolated blocks in the first places along the main diagonal, so that A then assumes the form

$$A = \begin{pmatrix} A_1 & O & \ldots O & O & \ldots & O \\ O & A_2 & \ldots O & O & \ldots & O \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ O & O & \ldots A_g & O & \ldots & O \\ A_{g+1,1} & A_{g+1,2} & \ldots A_{g+1,g} & A_{g+1} & \ldots & O \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{s1} & A_{s2} & \ldots A_{sg} & A_{s,g+1} & \ldots & A_s \end{pmatrix}; \qquad (69)$$

here $A_1, A_2, \ldots, A_s$ are irreducible matrices, and in each row

$$A_{f1}, A_{f2}, \ldots, A_{f,f-1} \qquad (f = g+1, \ldots, s)$$

at least one matrix is different from zero.

We shall call the matrix (69) the *normal form* of the reducible matrix A.

Let us show that *the normal form of a matrix A is uniquely determined to within a permutation of the blocks and permutations within the diagonal blocks* (the same for rows and columns).[29] For this purpose we consider the operator $\boldsymbol{A}$ corresponding to A in an n-dimensional vector space $\boldsymbol{R}$. To the representation of A in the form (69) there corresponds a decomposition of $\boldsymbol{R}$ into coordinate subspaces

$$\boldsymbol{R}=\boldsymbol{R}_1+\boldsymbol{R}_2+\dots+\boldsymbol{R}_g+\boldsymbol{R}_{g+1}+\dots+\boldsymbol{R}_s; \tag{70}$$

here $\boldsymbol{R}_s, \boldsymbol{R}_{s-1}+\boldsymbol{R}_s, \boldsymbol{R}_{s-2}+\boldsymbol{R}_{s-1}+\boldsymbol{R}_s, \dots$ are invariant coordinate subspaces for $\boldsymbol{A}$, and there is no intermediate invariant subspace between any two adjacent ones in this sequence.

Suppose then that apart from the normal form (69) of the given matrix there is another normal form corresponding to another decomposition of $\boldsymbol{R}$ into coordinate subspaces:

$$\boldsymbol{R}=\hat{\boldsymbol{R}}_1+\hat{\boldsymbol{R}}_2+\dots+\hat{\boldsymbol{R}}_g+\hat{\boldsymbol{R}}_{g+1}+\dots+\hat{\boldsymbol{R}}_t. \tag{71}$$

The uniqueness of the normal form will be proved if we can show that the decompositions (70) and (71) coincide apart from the order of the terms.

Suppose that the invariant subspace $\hat{\boldsymbol{R}}_t$ has coordinate vectors in common with $\boldsymbol{R}_k$, but not with $\boldsymbol{R}_{k+1}, \dots, \boldsymbol{R}_s$. Then $\hat{\boldsymbol{R}}_t$ must be entirely contained in $\boldsymbol{R}_k$, since otherwise $\hat{\boldsymbol{R}}_t$ would contain a 'smaller' invariant subspace, the intersection of $\hat{\boldsymbol{R}}_t$ with $\boldsymbol{R}_k+\boldsymbol{R}_{k+1}+\dots+\boldsymbol{R}_s$. Moreover, $\hat{\boldsymbol{R}}_t$ must coincide with $\boldsymbol{R}_k$, since otherwise the invariant subspace $\hat{\boldsymbol{R}}_t+\boldsymbol{R}_{k+1}+\dots+\boldsymbol{R}_s$ would be intermediate between $\boldsymbol{R}_k+\boldsymbol{R}_{k+1}+\dots+\boldsymbol{R}_s$ and $\boldsymbol{R}_{k+1}+\dots+\boldsymbol{R}_s$. Since $\boldsymbol{R}_k$ coincides with $\hat{\boldsymbol{R}}_t$, $\boldsymbol{R}_k$ is an invariant subspace. Therefore, without infringing the normal form of the matrix, $\boldsymbol{R}_k$ can be put in the place of $\boldsymbol{R}_s$. Thus, we may assume that in (70) and (71) $\boldsymbol{R}_s\equiv\hat{\boldsymbol{R}}_t$.

Let us now consider the coordinate subspace $\hat{\boldsymbol{R}}_{t-1}$. Suppose that it has coordinate vectors in common with $\boldsymbol{R}_l$ $(l<s)$, but not with $\boldsymbol{R}_{l-1}, \dots, \boldsymbol{R}_s$. Then the invariant subspace $\hat{\boldsymbol{R}}_{t-1}+\hat{\boldsymbol{R}}_t$ must be entirely contained in $\boldsymbol{R}_l+\boldsymbol{R}_{l+1}+\dots+\boldsymbol{R}_s$, since otherwise there would be an invariant coordinate subspace intermediate between $\hat{\boldsymbol{R}}_t$ and $\hat{\boldsymbol{R}}_{t-1}+\hat{\boldsymbol{R}}_t$. Therefore $\hat{\boldsymbol{R}}_{t-1}\subset\boldsymbol{R}_l$. Moreover $\hat{\boldsymbol{R}}_{t-1}\equiv\boldsymbol{R}_l$, since otherwise $\hat{\boldsymbol{R}}_{t-1}+\boldsymbol{R}_{l+1}+\dots+\boldsymbol{R}_s$ would be an invariant subspace intermediate between $\boldsymbol{R}_l+\boldsymbol{R}_{l+1}+\dots+\boldsymbol{R}_s$ and $\boldsymbol{R}_{l+1}+\dots+\boldsymbol{R}_s$. From $\hat{\boldsymbol{R}}_{t-1}\equiv\boldsymbol{R}_l$ it follows that $\boldsymbol{R}_l+\boldsymbol{R}_s$ is an invariant subspace. Therefore $\boldsymbol{R}_l$ may be put in the place of $\boldsymbol{R}_{s-1}$ and then we have

$$\hat{\boldsymbol{R}}_{t-1}\equiv\boldsymbol{R}_{s-1},\quad \hat{\boldsymbol{R}}_t\equiv\boldsymbol{R}_s.$$

Continuing this process, we finally reach the conclusion that $s=t$ and that the decompositions (70) and (71) coincide apart from the order of the terms. The corresponding normal forms then coincide to within a permutation of the blocks.

From the uniqueness of the normal form it follows that the numbers g and s are invariants of the non-negative matrix A.[30]

2. Making use of the normal form, we shall now prove the following theorem:

THEOREM 6: *To the maximal characteristic value r of the matrix $A\geqq O$ there belongs a positive characteristic vector if and only if in the normal form* (69) *of A:* 1) *each of the matrices $A_1, A_2, \dots, A_g$ has r as a characteristic value; and* (*in case $g<s$*) 2) *none of the matrices $A_{g+1}, \dots, A_s$ has this property.*

Proof. 1. Let $z>o$ be a positive characteristic vector belonging to the maximal characteristic value r. In accordance with the dissection into blocks in (69) we dissect the column z into parts z^k $(k=1, 2, \dots, s)$. Then the equation

$$Az=rz \qquad (z>o) \tag{72}$$

is replaced by two systems of equations

$$A_iz^i=rz^i \qquad (i=1, 2, \dots, g), \tag{72'}$$

$$\sum_{h=1}^{j-1}A_{jh}z^h+A_jz^j=rz^j \qquad (j=g+1, \dots, s). \tag{72''}$$

From (72′) it follows that r is a characteristic value of each of the matrices $A_1, A_2, \dots, A_g$. From (72″) we find:

$$A_jz^j\leqq rz^j,\quad A_jz^j\neq rz^j \qquad (j=g+1, \dots, s). \tag{73}$$

We denote by r_j the maximal characteristic value of A_j $(j=g+1, \dots s)$. Then (see (41) on p. 65) we find from (73):

$$r_j\leqq\max_i\frac{(A_jz^j)_i}{z_i^j}\leqq r \qquad (j=g+1, \dots, s).$$

---

[29] Without violating the normal form we can permute the first g blocks arbitrarily among each other. Moreover, sometimes certain permutations among the last $s-g$ blocks are possible with preservation of the normal form.

[30] For an irreducible matrix, $g=s=1$.

On the other hand, the equation $r_j = r$ would contradict the second of the relations (73) (see Note 5 on p. 65). Therefore

$$r_j < r \qquad (j = g+1, \ldots, s). \tag{74}$$

2. Suppose now, conversely, that the maximal characteristic values of the matrices A_i $(i = 1, 2, \ldots, g)$ are equal to r and that (74) holds for the matrices A_j $(j = g+1, \ldots, s)$. Then by replacing the required equation (72) by the systems (72′), (72″) we can define positive characteristic columns z^i of the matrices A_i $(i = 1, 2, \ldots, g)$ by means of (72′). Next we find columns z^j $(j = g+1, \ldots, s)$ from (72″):

$$z^j = (rE_j - A_j)^{-1} \sum_{h=1}^{j-1} A_{jh} z^h \qquad (j = g+1, \ldots, s), \tag{75}$$

where E_j is the unit matrix of the same order as A_j $(j = g+1, \ldots, s)$.

Since $r_j < r$ $(j = g+1, \ldots, s)$, we have (see (55) on p. 69)

$$(rE_j - A_j)^{-1} > O \qquad (j = g+1, \ldots, s). \tag{76}$$

Let us prove by induction that the columns $z^{g+1}, \ldots, z^s$ defined by (75) are positive. We shall show that for every j $(g+1 \leqq j \leqq s)$ the fact that $z^1, z^2, \ldots, z^{j-1}$ are positive implies that $z^j > o$. Indeed, in this case,

$$\sum_{h=1}^{j-1} A_{jh} z^h \geqq o, \quad \sum_{h=1}^{j-1} A_{jh} z^h \neq o,$$

which in conjunction with (76) yields, by (75):

$$z^j > o.$$

Thus, the positive column $z = (z^1, \ldots, z^s)$ is a characteristic vector of A for the characteristic value r. This completes the proof of the theorem.

3. The following theorem gives a characterization of a matrix $A \geqq O$ which together with its transpose A^{T} has the property that a positive characteristic vector belongs to the maximal characteristic value.

THEOREM 7:[31] *To the maximal characteristic value r of a matrix $A \geqq O$ there belongs a positive characteristic vector both of A and of A^{T} if and only if A can be represented by a permutation in quasi-diagonal form*

$$A = \{A_1, A_2, \ldots, A_s\}, \tag{77}$$

where $A_1, A_2, \ldots, A_s$ are irreducible matrices each of which has r as its maximal characteristic value.

---

[31] See [166].

Proof. Suppose that A and A^{T} have positive characteristic vectors for $\lambda = r$. Then, by Theorem 6, A is representable in the normal form (69), where $A_1, A_2, \ldots, A_g$ have r as maximal characteristic value and (for $g < s$) the maximal characteristic values of $A_{g+1}, \ldots, A_s$ are less than r. Then

$$A^{\mathsf{T}} = \begin{pmatrix} A_1^{\mathsf{T}} & \ldots & O & A_{g+1,1}^{\mathsf{T}} & \ldots & A_{s1}^{\mathsf{T}} \\ \vdots & \ddots & & & & \\ O & \ldots & A_g^{\mathsf{T}} & A_{g+1,g}^{\mathsf{T}} & \ldots & A_{sg}^{\mathsf{T}} \\ O & \ldots & O & A_{g+1}^{\mathsf{T}} & & \\ \vdots & & & \vdots & \ddots & \\ O & \ldots & O & O & \ldots & A_s^{\mathsf{T}} \end{pmatrix}.$$

Let us reverse here the order of the blocks in this matrix:

$$\begin{pmatrix} A_s^{\mathsf{T}} & O & O \ldots O \\ A_{s,s-1}^{\mathsf{T}} & A_{s-1}^{\mathsf{T}} & O \ldots O \\ \vdots & \ddots & \vdots \\ A_{s1}^{\mathsf{T}} & A_{s-1,1}^{\mathsf{T}} & \ldots A_1^{\mathsf{T}} \end{pmatrix}. \tag{78}$$

Since $A_s^{\mathsf{T}}, A_{s-1}^{\mathsf{T}}, \ldots, A_1^{\mathsf{T}}$ are irreducible, we obtain a normal form for (78) by a permutation of the blocks, placing the isolated blocks first along the main diagonal. One of these isolated blocks is A_s^{T}. Since the normal form of A^{T} must satisfy the conditions of the preceding theorem, the maximal characteristic value of A_s^{T} must be equal to r. This is only possible when $g = s$. But then the normal form (69) goes over into (77).

If, conversely, a representation (77) of A is given, then

$$A^{\mathsf{T}} = \{A_1^{\mathsf{T}}, A_2^{\mathsf{T}}, \ldots, A_s^{\mathsf{T}}\}. \tag{79}$$

We then deduce from (77) and (79), by the preceding theorem, that A and A^{T} have positive characteristic vectors for the maximal characteristic value r.

This proves the theorem.

COROLLARY. *If the maximal characteristic value r of a matrix $A \geqq O$ is simple and if positive characteristic vectors belong to r both in A and A^{T}, then A is irreducible.*

Since, conversely, every irreducible matrix has the properties of this corollary, these properties provide a spectral characterization of an irreducible non-negative matrix.

§ 5. Primitive and Imprimitive Matrices

1. We begin with a classification of irreducible matrices.

DEFINITION 3: *If an irreducible matrix $A \geqq O$ has h characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_h$ of maximal modulus r $(\lambda_1 = |\lambda_2| = \ldots = |\lambda_h| = r)$, then A is called primitive if $h = 1$ and imprimitive if $h > 1$. h is called the index of imprimitivity of A.*

The index of imprimitivity h is easily determined if the coefficients of the characteristic equation of the matrix are known

$$\Delta(\lambda) \equiv \lambda^n + a_1 \lambda^{n_1} + a_2 \lambda^{n_2} + \cdots + a_t \lambda^{n_t} = 0$$
$$(n > n_1 > \cdots > n_t;\ a_1 \neq 0,\ a_2 \neq 0, \ldots, a_t \neq 0)\ ;$$

namely: *h is the greatest common divisor of the differences*

$$n - n_1,\ n_1 - n_2, \ldots, n_{t-1} - n_t. \tag{80}$$

For by Frobenius' theorem the spectrum of A in the complex λ-plane goes over into itself under a rotation through $2\pi/h$ around the point $\lambda = 0$. Therefore the polynomial $\Delta(\lambda)$ must be obtained from some polynomial $g(\mu)$ by the formula

$$\Delta(\lambda) = g(\lambda^h)\, \lambda^{n'}.$$

Hence it follows that h is a common divisor of the differences (80). But then h is the greatest common divisor d of these differences, since the spectrum does not change under a rotation by $2\pi/d$, which is impossible for $h < d$.

The following theorem establishes an important property of a primitive matrix:

THEOREM 8: *A matrix $A \geqq O$ is primitive if and only if some power of A is positive:*

$$A^p > O \qquad (p \geqq 1). \tag{81}$$

Proof. If $A^p > O$, then A is irreducible, since the reducibility of A would imply that of A^p. Moreover, for A we have $h = 1$, since otherwise the positive matrix A^p would have h (> 1) characteristic values

$$\lambda_1^p, \lambda_2^p, \ldots, \lambda_h^p$$

of maximal modulus r^p, and this contradicts Perron's theorem.

Suppose now, conversely, that A is primitive. We apply the formula (23) of Chapter V (Vol. I, p. 107) to A^p

$$A^p = \sum_{k=1}^{s} \frac{1}{(m_k - 1)!} \left[\frac{C(\lambda)\lambda^p}{\overset{k}{\psi}(\lambda)} \right]^{(m_k-1)}_{\lambda=\lambda_k}, \tag{82}$$

where

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_s)^{m_s} \quad (\lambda_j \neq \lambda_f \text{ for } j \neq f)$$

is the minimal polynomial of A, $\overset{k}{\psi}(\lambda) = \frac{\psi(\lambda)}{(\lambda - \lambda_k)^{m_k}}$ $(k = 1, 2, \ldots, s)$ and $C(\lambda) = (\lambda E - A)^{-1}\psi(\lambda)$ is the reduced adjoint matrix.

In this case, we can set:

$$\lambda_1 = r > |\lambda_2| \geqq \cdots \geqq |\lambda_s| \quad \text{and} \quad m_1 = 1. \tag{83}$$

Then (82) assumes the form

$$A^p = \frac{C(r)}{\psi'(r)} r^p + \sum_{k=2}^{s} \frac{1}{(m_k - 1)!} \left[\frac{C(\lambda)\lambda^p}{\overset{k}{\psi}(\lambda)} \right]^{(m_k-1)}_{\lambda=\lambda_k}.$$

Hence it is easy to deduce by (83) that

$$\lim_{p \to \infty} \frac{A^p}{r^p} = \frac{C(r)}{\psi'(r)}. \tag{84}$$

On the other hand, $C(r) > O$ (see (53)) and $\psi'(r) > 0$ by (83). Therefore

$$\lim_{p \to \infty} \frac{A^p}{r^p} > O,$$

and so (73) must hold from some p onwards.[32] This completes the proof.

We shall now prove the following theorem:

THEOREM 9: *If $A \geqq O$ is an irreducible matrix and some power A^q of A is reducible, then A^q is completely reducible,* i.e., *A^q can be represented by means of a permutation in the form*

$$A^q = \{A_1, A_2, \ldots, A_d\}, \tag{85}$$

where $A_1, A_2, \ldots, A_d$ are irreducible matrices having one and the same maximal characteristic value. Here d is the greatest common divisor of q and h, where h is the index of imprimitivity of A.

---

[32] As regards a lower bound for the exponent p in (81), see [384].

Proof. Since A is irreducible, we know by Frobenius' theorem that positive characteristic vectors belong to the maximal characteristic value r, both in A and in A^{T}. But then these positive vectors are also characteristic vectors of the non-negative matrices A^q and $(A^q)^{\mathsf{T}}$ for the characteristic value $\lambda = r^q$. Therefore by applying Theorem 7 to A^q, we represent this matrix (after a suitable permutation) in the form (85), where $A_1, A_2, \ldots, A_d$ are irreducible matrices with the same maximal characteristic value r^q. But A has h characteristic values of maximal modulus r:

$$r, r\varepsilon, \ldots, r\varepsilon^{h-1} \quad \left(\varepsilon = e^{\frac{2\pi i}{h}}\right).$$

Therefore A^q also has h characteristic values of maximal modulus

$$r^q, r^q\varepsilon^q, \ldots, r^q\varepsilon^{q(h-1)},$$

among which d are equal to r^q. This is only possible when d is the greatest common divisor of q and h. This proves the theorem.

For $h = 1$, we obtain:

COROLLARY 1: *A power of a primitive matrix is irreducible and primitive.*

If we set $q = h$ in the theorem, then we obtain:

COROLLARY 2: *If A is an imprimitive matrix with index of imprimitivity h, then A^h splits into h primitive matrices with the same maximal characteristic value.*

§ 6. Stochastic Matrices

1. We consider n possible states of a certain system

$$S_1, S_2, \ldots, S_n \tag{86}$$

and a sequence of instants

$$t_0, t_1, t_2, \ldots.$$

Suppose that at each of these instants the system is in one and only one of the states (86) and that p_{ij} denotes the probability of finding the system in the state S_j at the instant t_k if it is known that at the preceding instant t_{k-1} the system is in the state S_i $(i, j = 1, 2, \ldots, n; k = 1, 2, \ldots)$. We shall assume that the *transition probability* p_{ij} $(i, j = 1, 2, \ldots, n)$ does not depend on the index k (of the instant t_k).

If the matrix of transition probabilities is given,

$$P = \| p_{ij} \|_1^n,$$

then we say that we have a *homogeneous Markov chain with a finite number of states.*[33] It is obvious that

$$p_{ij} \geqq 0, \quad \sum_{j=1}^{n} p_{ij} = 1 \qquad (i, j = 1, 2, \ldots, n). \tag{87}$$

DEFINITION 4: *A square matrix $P = \| p_{ij} \|_1^n$ is called stochastic if P is non-negative and if the sum of the elements of each row of P is* 1, i.e., *if the relations* (87) *hold.*[34]

Thus, for every homogeneous Markov chain the matrix of transition probabilities is stochastic and, conversely, every stochastic matrix can be regarded as the matrix of transition probabilities of some homogeneous Markov chain. This is the basis of the matrix method of investigating homogeneous Markov chains.[35]

A stochastic matrix is a special form of a non-negative matrix. Therefore all the concepts and propositions of the preceding sections are applicable to it.

We mention some specific properties of a stochastic matrix. From the definition of a stochastic matrix it follows that it has the characteristic value 1 with the positive characteristic vector $z = (1, 1, \ldots, 1)$. It is easy to see that, conversely, every matrix $P \geqq O$ having the characteristic vector $(1, 1, \ldots, 1)$ for the characteristic value 1 is stochastic. Moreover, 1 is the maximal characteristic value of a stochastic matrix, since the maximal characteristic value is always included between the largest and the smallest of the row sums[36] and in a stochastic matrix all the row sums are 1. Thus, we have proved the proposition:

1. *A non-negative matrix $P \geqq O$ is stochastic if and only if it has the characteristic vector* $(1, 1, \ldots, 1)$ *for the characteristic value* 1. *For a stochastic matrix the maximal characteristic value is* 1.

Now let $A = \| a_{ik} \|_1^n$ be a non-negative matrix with a positive maximal characteristic value $r > 0$ and a corresponding positive characteristic vector $z = (z_1, z_2, \ldots, z_n) > o$:

---

[33] See [212] and [46}, pp. 9-12.

[34] Sometimes the additional condition $\sum_{i=1}^{n} p_{ij} \neq 0$ $(j = 1, 2, \ldots, n)$ is included in the definition of a stochastic matrix. See [46], p. 13.

[35] The theory of homogeneous Markov chains with a finite (and a countable) number of states was introduced by Kolmogorov (see [212]). The reader can find an account of the later introduction and development of the matrix method with applications to homogeneous Markov chains in the memoir [329] and in the monograph [46] by V. I. Romanovskiĭ (see also [4], Appendix 5).

[36] See (37) and the note on p. 68.

$$\sum_{j=1}^{n} a_{ij} z_j = r z_i \qquad (i = 1, 2, \ldots, n). \tag{88}$$

We introduce the diagonal matrix $Z = \{z_1, z_2, \ldots, z_n\}$ and the matrix $P = \| p_{ij} \|_1^n$

$$P = \frac{1}{r} Z^{-1} A Z.$$

Then

$$p_{ij} = \frac{1}{r} z_i^{-1} a_{ij} z_j \geqq 0 \qquad (i, j = 1, 2, \ldots, n),$$

and by (88)

$$\sum_{j=1}^{n} p_{ij} = 1 \qquad (i = 1, 2, \ldots, n).$$

Thus:

2. *A non-negative matrix $A \geqq O$ with the maximal positive characteristic value $r > 0$ and with a corresponding positive characteristic vector $z = (z_1, z_2, \ldots z_n) > o$ is similar to the product of r and a stochastic matrix:*[37]

$$A = Z r P Z^{-1} \qquad (Z = \{z_1, z_2, \ldots, z_n\} > O). \tag{89}$$

In a preceding section we have given (see Theorem 6, § 4) a characterization of the class of non-negative matrices having a positive characteristic vector for $\lambda = r$. The formula (89) establishes a close connection between this class and the class of stochastic matrices.

2. We shall now prove the following theorem:

THEOREM 10: *To the characteristic value* 1 *of a stochastic matrix there always correspond only elementary divisors of the first degree.*

Proof. We apply the decomposition (69), § 4, to the stochastic matrix $P = \| p_{ij} \|_1^n$.

$$P = \begin{pmatrix} A_1 & O & \cdot & \cdot & \cdot & \cdot & \cdot & O \\ O & A_2 & \cdot & \cdot & \cdot & \cdot & \cdot & O \\ \vdots & & & & & & & \vdots \\ O & \cdot & \cdot & \cdot & A_g & O & \cdot & O \\ A_{g+1,1} & \cdot & \cdot & \cdot & A_{g+1,g} & A_{g+1} & \cdot & O \\ \vdots & & & & & & \ddots & \vdots \\ A_{s1} & \cdot & \cdot & \cdot & A_{sg} & \cdot & \cdot & A_s \end{pmatrix},$$

where $A_1, A_2, \ldots, A_s$ are irreducible and

---

[37] Proposition 2. also holds for $r = 0$, since $A \geqq O$, $z > o$ implies that $A = O$.

$$A_{f1} + A_{f2} + \cdots + A_{f,f-1} \neq O \qquad (f = g + 1, \ldots, s).$$

Here $A_1, A_2, \ldots, A_g$ are *stochastic* matrices, so that each has the simple characteristic value 1. As regards the remaining irreducible matrices $A_{g+1}, \ldots, A_s$, by the Remark 2 on p. 63 their maximal characteristic values are less than 1, since in each of these matrices at least one row sum is less than 1.[38]

Thus, the matrix P is representable in the form

$$P = \begin{pmatrix} Q_1 & O \\ S & Q_2 \end{pmatrix},$$

where in Q_1 to the value 1 there correspond elementary divisors of the first degree, and where 1 is not a characteristic value of Q_2. The theorem now follows immediately from the following lemma:

LEMMA 4: *If a matrix A has the form*

$$A = \begin{pmatrix} Q_1 & O \\ S & Q_2 \end{pmatrix}, \tag{90}$$

where Q_1 and Q_2 are square matrices, and if the characteristic value λ_0 of A is also a characteristic value of Q_1, but not of Q_2,

$$|Q_1 - \lambda_0 E = 0, \quad |Q_2 - \lambda_0 E| \neq 0,$$

then the elementary divisors of A and Q_1 corresponding to the characteristic value λ_0 are the same.

Proof. 1. To begin with, we consider the case where Q_1 and Q_2 do not have characteristic values in common. Let us show that in this case the elementary divisors of Q_1 and Q_2 together form the system of elementary divisors of A, i.e., for some matrix T ($|T| \neq 0$)

$$T A T^{-1} = \begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix}. \tag{91}$$

We shall look for the matrix T in the form

$$T = \begin{pmatrix} E_1 & O \\ U & E_2 \end{pmatrix}$$

---

[38] These properties of the matrices $A_1, \ldots, A_s$ also follow from Theorem 6.

(the dissection of T into blocks corresponds to that of A; E_1 and E_2 are unit matrices). Then

$$TAT^{-1}=\begin{pmatrix}E_1 & O\\ U & E_2\end{pmatrix}\begin{pmatrix}Q_1 & O\\ S & Q_2\end{pmatrix}\begin{pmatrix}E_1 & O\\ -U & E_2\end{pmatrix}=\begin{pmatrix}Q_1 & O\\ UQ_1-Q_2U+S & Q_2\end{pmatrix}. \quad (91')$$

The equation (91′) reduces to (91) if we choose the rectangular matrix U so that it satisfies the matrix equation

$$Q_2U-UQ_1=S.$$

If Q_1 and Q_2 have no characteristic values in common, then this equation always has a unique solution for every right-hand side S (see Vol. I, Chapter VIII, § 3).

2. In the case where Q_1 and Q_2 have characteristic values in common, we replace Q_1 in (90) by its Jordan form J (as a result, A is replaced by a similar matrix). Let $J=\{J_1\,J_2\}$, where all the Jordan blocks with the characteristic value λ_0 are combined in J_1. Then

$$A=\begin{pmatrix}J_1 & O & O & O\\ O & J_2 & O & O\\ S_{11} & S_{12} & \multicolumn{2}{c}{\ldots}\\ S_{21} & S_{22} & \multicolumn{2}{c}{Q_2}\end{pmatrix}=\begin{pmatrix}J_1 & O & O & O\\ O & & \ldots & \\ S_{11} & & \hat{Q}_2 & \\ S_{21} & & & \end{pmatrix}.$$

This matrix falls under the preceding case, since the matrices J_1 and $\hat{Q}_2$ have no characteristic values in common. Hence it follows that the elementary divisors of the form $(\lambda-\lambda_0)^p$ are the same for A and J_1 and therefore also for A and Q_1. This proves the lemma.

If an irreducible stochastic matrix P has a complex characteristic value λ_0 with $|\lambda_0|=1$, then λ_0P is similar to P (see (16)) and so it follows from Theorem 10 that to λ_0 there correspond only elementary divisors of the first degree. With the help of the normal form and of Lemma 4 it is easy to extend this statement to reducible stochastic matrices. Thus we obtain:

COROLLARY 1. *If λ_0 is a characteristic value of a stochastic matrix P and $|\lambda_0|=1$, then the elementary divisors corresponding to λ_0 are of the first degree.*

From Theorem 10 we also deduce by 2. (p. 84):

COROLLARY 2. *If a positive characteristic vector belongs to the maximal characteristic value r of a non-negative matrix A, then all the elementary divisors of A that belong to a characteristic value λ_0 with $|\lambda_0|=r$ are of the first degree.*

We shall now mention some papers that deal with the distribution of the characteristic values of stochastic matrices.

A characteristic value of a stochastic matrix P always lies in the disc $|\lambda|\leqq 1$ of the λ-plane. The set of all points of this disc that are characteristic values of any stochastic matrices of order n will be denoted by M_n.

3. In 1938, in connection with investigation on Markov chains A. N. Kolmogorov raised the problem of determining the structure of the domain M_n. This problem was partially solved in 1945 by N. A. Dmitriev and E. B. Dynkin [133], [133a] and completely in 1951 in a paper by F. I. Karpelevich [209]. It turned out that the boundary of M_n consists of a finite number of points on the circle $|\lambda|=1$ and certain curvilinear arcs joining these points in cyclic order.

We note that by Proposition 2. (p. 84) the characteristic values of the matrices $A=\|a_{ik}\|_1^n\geqq O$ having a positive characteristic vector for $\lambda=r$ with a fixed r form the set $r\cdot M_n$.[39] Since every matrix $A=\|a_{ik}\|_1^n\geqq O$ can be regarded as the limit of a sequence of non-negative matrices of that type and the set $r\cdot M_n$ is closed, the characteristic values of arbitrary matrices $A=\|a_{ik}\|_1^n\geqq O$ with a given maximal characteristic value r fill out the set $r\cdot M_n$.[40]

A paper by H. R. Suleimanova [359] is relevant in this context; it contains sufficiency criteria for n given real numbers $\lambda_1,\lambda_2,\ldots,\lambda_n$ to be the characteristic values of a stochastic matrix $P=\|p_{ij}\|_1^n$.[41]

§ 7. Limiting Probabilities for a Homogeneous Markov Chain with a Finite Number of States

1. Let

$$S_1,\,S_2,\,\ldots,\,S_n$$

be all the possible states of a system in a homogeneous Markov chain and let $P=\|p_{ij}\|_1^n$ be the stochastic matrix determined by this chain that is formed from the transition probabilities p_{ij} $(i,j=1,2,\ldots,n)$ (see p. 82).

We denote by $p_{ij}^{(q)}$ the probability of finding the system in the state S_j at the instant t_k if it is known that at the instant t_{k-q} it is in the state S_i $(i,j=1,2,\ldots,n;\ q=1,2,\ldots)$. Clearly, $p_{ij}^{(1)}=p_{ij}$ $(i,j=1,2,\ldots,n)$.

---

[39] $r\cdot M_n$ is the set of points in the λ-plane of the form $r\mu$, where $\mu\in M_n$.

[40] Kolmogorov has shown (see [133a (1946)], Appendix) that this problem for an arbitrary matrix $A\geqq O$ can be reduced to the analogous problem for a stochastic matrix.

[41] See also [312].

Making use of the theorems on the addition and multiplication of probabilities, we find easily:

$$p_{ij}^{(q+1)} = \sum_{h=1}^{n} p_{ih}^{(q)} p_{hj} \quad (i, j = 1, 2, \ldots, n)$$

or, in matrix notation,

$$\| p_{ij}^{(q+1)} \| = \| p_{ij}^{(q)} \|_1^n \, \| p_{ij} \|_1^n .$$

Hence, by giving to q in succession the values 1, 2, ..., we obtain the important formula[42]

$$\| p_{ij}^{(q)} \| = P^q \quad (q = 1, 2, \ldots).$$

If the limits

$$\lim_{q \to \infty} p_{ij}^{(q)} = p_{ij}^{\infty} \quad (i, j = 1, 2, \ldots, n)$$

or, in matrix notation,

$$\lim_{q \to \infty} P^q = P^{\infty} = \| p_{ij}^{\infty} \|_1^n ,$$

exist, then the values p_{ij}^{∞} $(i, j = 1, 2, \ldots, n)$ are called the *limiting* or *final transition probabilities*.[43]

In order to investigate under what conditions limiting transition probabilities exist and to derive the corresponding formulas, we introduce the following terminology.

We shall call a stochastic matrix P and the corresponding homogeneous Markov chain *regular* if P has no characteristic values of modulus 1 other than 1 itself and *fully regular* if, in addition, 1 is a simple root of the characteristic equation of P.

A regular matrix P is characterized by the fact that in its normal form (69) (p. 75) the matrices $A_1, A_2, \ldots, A_g$ are primitive. For a fully regular matrix we have, in addition, $g = 1$.

Furthermore, a homogeneous Markov chain is *irreducible*, *reducible*, *acyclic* or *cyclic* if the stochastic matrix P of the chain is irreducible, reducible, primitive, or imprimitive, respectively. Just as a primitive stochastic matrix is a special form of a regular matrix, so an acyclic Markov chain is a special form of a regular chain.

We shall prove that: *Limiting transition probabilities exist for regular homogeneous Markov chains only.*

---

[42] It follows from this formula that the probabilities $p_{ij}^{(q)}$ as well as p_{ij} $(i, j = 1, 2, 3, \ldots, n;\ q = 1, 2, \ldots)$ do not depend on the index k of the original instant t_k.

[43] The matrix P^{∞}, as the limit of stochastic matrices, is itself stochastic.

For let $\psi(\lambda)$ be the minimal polynomial of the regular matrix $P = \| p_{ij} \|_1^n$. Then

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_u)^{m_u} \quad (\lambda_i \neq \lambda_k;\ i, k = 1, 2, \ldots, u). \tag{92}$$

By Theorem 10 we may assume that

$$\lambda_1 = 1, \quad m_1 = 1. \tag{93}$$

By the formula (23) of Chapter V (Vol. I, p. 107),

$$P^q = \frac{C(1)}{\overset{1}{\psi}(1)} + \sum_{k=2}^{u} \frac{1}{(m_k - 1)!} \left[\frac{C(\lambda)}{\overset{k}{\psi}(\lambda)} \lambda^q \right]_{\lambda = \lambda_k}^{(m_k - 1)}, \tag{94}$$

where $C(\lambda) = (\lambda E - P)^{-1} \psi(\lambda)$ is the reduced adjoint matrix and

$$\overset{k}{\psi}(\lambda) = \frac{\psi(\lambda)}{(\lambda - \lambda_k)^{m_k}} \quad (k = 1, 2, \ldots, u);$$

moreover

$$\overset{1}{\psi}(\lambda) = \frac{\psi(\lambda)}{\lambda - 1} \quad \text{and} \quad \overset{1}{\psi}(1) = \psi'(1).$$

If P is a regular matrix, then

$$|\lambda_k| < 1 \quad (k = 2, 3, \ldots, u),$$

and therefore all the terms on the right-hand side of (94), except the first, tend to zero for $q \to \infty$. Therefore, for a regular matrix P the matrix P^{∞} formed from the limiting transition probabilities exists, and

$$P^{\infty} = \frac{C(1)}{\psi'(1)}. \tag{95}$$

The converse proposition is obvious. If the limit

$$P^{\infty} = \lim_{q \to \infty} P^q \tag{96}$$

exists, then the matrix P cannot have any characteristic value λ_k for which $\lambda_k \neq 1$ and $|\lambda_k| = 1$, since then the limit $\lim_{q \to \infty} \lambda_k^q$ would not exist. (This limit must exist, since the limit (96) exists.)

We have proved that the matrix P^{∞} exists for a regular homogeneous Markov chain (and for such a regular chain only). This matrix is determined by (95).

We shall now show that P^∞ can be expressed by the characteristic polynomial

$$\Delta(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_u)^{n_u} \tag{97}$$

and the adjoint matrix $B(\lambda) = (\lambda E - P)^{-1}\Delta(\lambda)$.

From the identity

$$\frac{B(\lambda)}{\Delta(\lambda)} = \frac{C(\lambda)}{\psi(\lambda)}$$

it follows by (92), (93), and (97) that

$$\frac{n_1 B^{(n_1-1)}(1)}{\Delta^{(n_1)}(1)} = \frac{C(1)}{\psi'(1)}.$$

Therefore (95) may be replaced by the formula

$$P^\infty = \frac{n_1 B^{(n_1-1)}(1)}{\Delta^{(n_1)}(1)}. \tag{98}$$

For a fully regular Markov chain, inasmuch as it is a special form of a regular chain, the matrix P^∞ exists and is determined by (95) or (98). In this case $n_1 = 1$, and (98) assumes the form

$$P^\infty = \frac{B(1)}{\Delta'(1)}. \tag{99}$$

2. Let us consider a regular chain of general type (not fully regular). We write the corresponding matrix P in the normal form

$$P = \left(\begin{array}{cccccc}
Q_1 & \cdots & O & O & \cdots & O \\
\vdots & \ddots & \vdots & & & \vdots \\
O & & Q_g & O & & O \\
U_{g+1,1} & \cdots & U_{g+1,g} & Q_{g+1} & & \vdots \\
\vdots & & \vdots & & \ddots & \vdots \\
U_{s1} & \cdots & U_{sg} & \cdots & U_{s,s-1} & Q_s
\end{array}\right), \tag{100}$$

where $Q_1, \ldots, Q_g$ are primitive stochastic matrices and the maximal values of the irreducible matrices $Q_{g+1}, \ldots, Q_s$ are less than 1. Setting

$$U = \begin{pmatrix} U_{g+1,1} & \cdots & U_{g+1\cdot g} \\ \cdots & \cdots & \cdots \\ U_{s1} & \cdots & U_{sg} \end{pmatrix}, \quad W = \begin{pmatrix} Q_{g+1} & \cdots & O \\ \cdots & \cdots & \cdots \\ U_{s,g+1} & \cdots & Q_s \end{pmatrix},$$

we write P in the form

$$P = \left(\begin{array}{ccc|c}
Q_1 & \cdots & O & O \\
\vdots & \ddots & \vdots & \vdots \\
O & \cdots & Q_g & O \\
 & U & & W
\end{array}\right).$$

Then

$$P^q = \left(\begin{array}{ccc|c}
Q_1^q & \cdots & O & O \\
\vdots & \ddots & \vdots & \vdots \\
O & \cdots & Q_g^q & O \\
 & U^q & & W^q
\end{array}\right) \tag{101}$$

and

$$P^\infty = \lim_{q\to\infty} P^q = \left(\begin{array}{ccc|c}
Q_1^\infty & \cdots & O & O \\
\vdots & \ddots & \vdots & \vdots \\
O & \cdots & Q_g^\infty & O \\
 & U_\infty & & W^\infty
\end{array}\right).$$

But $W^\infty = \lim_{q\to\infty} W^q = O$, because all the characteristic values of W are of modulus less than 1. Therefore

$$P^\infty = \left(\begin{array}{ccc|c}
Q_1^\infty & \cdots & O & O \\
\vdots & \ddots & \vdots & \vdots \\
O & \cdots & Q_g^\infty & O \\
 & U_\infty & & O
\end{array}\right). \tag{102}$$

Since $Q_1, \ldots, Q_g$ are primitive stochastic matrices, the matrices $Q_1^\infty, \ldots, Q_g^\infty$ are positive, by (99) and (35) (p. 62)

$$Q_1^{\infty} > O, \ \dots, \ Q_g^{\infty} > O,$$

and in each of these matrices all the elements belonging to any one column are equal:

$$Q_h^{\infty} = \| q_{*j}^{(h)} \|_{i,j=1}^{n} \qquad (h = 1, 2, \dots, g).$$

We note that the states $S_1, S_2, \dots, S_n$ of the system fall into groups corresponding to the normal form (100) of P:

$$\Sigma_1, \Sigma_2, \dots, \Sigma_g, \Sigma_{g+1}, \dots, \Sigma_s. \tag{103}$$

To each group Σ in (103) there corresponds a group of rows in (100). In the terminology of Kolmogorov the states of the system that occur in $\Sigma_1, \Sigma_2, \dots, \Sigma_g$ are called *essential* and the states that occur in the remaining groups $\Sigma_{g+1}, \dots, \Sigma_s$ *non-essential*.

From the form (101) of P^q it follows that in any finite number q of steps (from the instant t_{k-q} to t_k) only the following transitions of the system are possible: a) from an essential state to an essential state of the same group; b) from a non-essential state to an essential state; and c) from a non-essential state to a non-essential state of the same or a preceding group.

From the form (102) of P^{∞} it follows that: *A limiting transition can only lead from an arbitrary state to an essential state*, i.e., the probability of transition to any non-essential state tends to zero when the number of steps q tends to infinity. The essential states are therefore sometimes also called *limiting states*.

3. From (95) it follows that[45]

$$(E - P)\, P^{\infty} = O.$$

Hence it is clear that: *Every column of P^{∞} is a characteristic vector of the stochastic matrix P for the characteristic value $\lambda = 1$.*

For a fully regular matrix P, 1 is a simple root of the characteristic equation and (apart from scalar factors) only one characteristic vector $(1, 1, \dots, 1)$ of P belongs to it. Therefore all the elements of the j-th column of P^{∞} are equal to one and the same non-negative number p_{*j}^{∞}:

$$p_{ij}^{\infty} = p_{*j}^{\infty} \geqq 0 \quad (j = 1, 2, \dots, n;\ \sum_{j=1}^{n} p_{*j} = 1). \tag{104}$$

---

[44] See [212] and [46], pp. 37-39.

[45] This formula holds for an arbitrary regular chain and can be obtained from the obvious equation $P^q - P \cdot P^{q-1} = O$ by passing to the limit $q \to \infty$.

Thus, in a fully regular chain the limiting transition probabilities do not depend on the initial state.

Conversely, if in a regular homogeneous Markov chain the limiting transition probabilities do not depend on the initial state, i.e., if (104) holds, then obviously in the scheme (102) for P^{∞} we have $g = 1$. But then $n_1 = 1$ and the chain is fully regular.

For an acyclic chain, which is a special case of a fully regular chain, P is a primitive matrix. Therefore $P^q > O$ (see Theorem 8 on p. 80) for some $q > 0$. But then also $P^{\infty} = P^{\infty} P^q > O$.[46]

Conversely, it follows from $P^{\infty} > O$ that $P^q > O$ for some $q > 0$, and this means by Theorem 8 that P is primitive and hence that the given homogeneous Markov chain is acyclic.

We formulate these results in the following theorem:

THEOREM 11: 1. *In a homogeneous Markov chain all the limiting transition probabilities exist if and only if the chain is regular. In that case the matrix P^{∞} formed from the limiting transition probabilities is determined by* (95) *or* (98).

2. *In a regular homogeneous Markov chain the limiting transition probabilities are independent of the initial state if and only if the chain is fully regular. In that case the matrix P^{∞} is determined by* (99).

3. *In a regular homogeneous Markov chain all the limiting transition probabilities are different from zero if and only if the chain is acyclic.*[47]

4. We now consider the columns of *absolute probabilities*

$$\overset{k}{p} = (\overset{k}{p}_1, \overset{k}{p}_2, \dots, \overset{k}{p}_n) \qquad (k = 0, 1, 2, \dots), \tag{105}$$

where $\overset{k}{p}_i$ is the probability of finding the system in the state S_i ($i = 1, 2, \dots, n$; $k = 0, 1, 2, \dots$) at the instant t_k. Making use of the theorems on the addition and multiplication of probabilities, we find:

$$\overset{k}{p}_i = \sum_{h=1}^{n} \overset{0}{p}_h p_{hi}^{(k)} \qquad (i = 1, 2, \dots, n;\ k = 1, 2, \dots)$$

or, in matrix notation,

---

[46] This matrix equation is obtained by passing to the limit $m \to \infty$ from the equation $P^m = P^{m-q} \cdot P^q$ $(m > q)$. P^{∞} is a stochastic matrix; therefore $P^{\infty} \geqq O$ and there are non-zero elements in every row of P^{∞}. Hence $P^{\infty} P^q > O$. Instead of Theorem 8 we can use here the formula (99) and the inequality (35) (p. 62).

[47] Note that $P^{\infty} > O$ implies that the chain is acyclic and therefore regular. Hence it follows automatically from $P^{\infty} > O$ that the limiting transition probabilities do not depend on the initial state, i.e., that the formulas (104) hold.

$$p=(P^{\mathsf{T}})^k\overset{0}{p} \qquad (k=1, 2, \ldots), \tag{106}$$

where P^{T} is the transpose of P.

All the absolute probabilities (105) can be determined from (106) if the initial probabilities $\overset{0}{p}_1, \overset{0}{p}_2, \ldots, \overset{0}{p}_n$ and the matrix of transition probabilities $P=\| p_{ij} \|_1^n$ are known.

We introduce the *limiting absolute probabilities*

$$\overset{\infty}{p}_i=\lim_{k\to\infty}\overset{k}{p}_i \qquad (i=1, 2, \ldots, n)$$

or

$$\overset{\infty}{p}=(\overset{\infty}{p}_1, \overset{\infty}{p}_2, \ldots, \overset{\infty}{p}_n)=\lim_{k\to\infty}\overset{k}{p}.$$

When we take the limit $k \to \infty$ on both sides of (106), we obtain:

$$\overset{\infty}{p}=(P^{\infty})^{\mathsf{T}}\overset{0}{p}. \tag{107}$$

Note that the existence of the matrix of limiting transition probabilities P^{∞} implies the existence of the limiting absolute probabilities

$$\overset{\infty}{p}=(\overset{\infty}{p}_1, \overset{\infty}{p}_2, \ldots, \overset{\infty}{p}_n)$$

for arbitrary initial probabilities $\overset{0}{p}=(\overset{0}{p}_1, \overset{0}{p}_2, \ldots, \overset{0}{p}_n)$, and vice versa.

From the formula (107) and the form (102) of P^{∞} it follows that: *The limiting absolute probabilities corresponding to non-essential states are zero.*

Multiplying both sides of the matrix equation

$$P^{\mathsf{T}}\cdot(P^{\infty})^{\mathsf{T}}=(P^{\infty})^{\mathsf{T}}$$

by $\overset{0}{p}$ on the right, we obtain by (107):

$$P^{\mathsf{T}}\overset{\infty}{p}=\overset{\infty}{p}, \tag{108}$$

i.e.: *The column of limiting absolute probabilities $\overset{\infty}{p}$ is a characteristic vector of P^{T} for the characteristic value $\lambda=1$.*

If a fully regular Markov chain is given, then $\lambda=1$ is a simple root of the characteristic equation of P^{T}. In this case, the column of limiting absolute probabilities is uniquely determined by (108) (because $\overset{\infty}{p}_i \geqq 0$ ($j=1, 2, \ldots, n$) and $\sum_{j=1}^{n}\overset{\infty}{p}_j=1$).

Suppose that a fully regular Markov chain is given. Then it follows from (104) and (107) that:

$$\overset{\infty}{p}-\sum_{h=1}^{n}\overset{0}{p}_h\overset{\infty}{p}_{hj}=\overset{\infty}{p}_{*j}\sum_{h=1}^{n}\overset{0}{p}_h=\overset{\infty}{p}_{*j} \qquad (j=1, 2, \ldots, n). \tag{109}$$

In this case the limiting absolute probabilities $\overset{\infty}{p}_1, \overset{\infty}{p}_2, \ldots, \overset{\infty}{p}_n$ do not depend on the initial probabilities $\overset{0}{p}_1, \overset{0}{p}_2, \ldots, \overset{0}{p}_n$.

Conversely, $\overset{\infty}{p}$ is independent of $\overset{0}{p}$ on account of (107) if and only if all the rows of P^{∞} are equal, i.e.,

$$\overset{\infty}{p}_{hj}=\overset{\infty}{p}_{*j} \qquad (h, j=1, 2, \ldots, n)$$

so that (by Theorem 11) P is a fully regular matrix.

If P is primitive, then $P^{\infty} > O$ and hence, by (109),

$$\overset{\infty}{p}_j>0 \qquad (j=1, 2, \ldots, n).$$

Conversely, if all the $\overset{\infty}{p}_j$ ($j=1, 2, \ldots, n$) are positive and do not depend on the initial probabilities, then all the elements in every column of P^{∞} are equal and by (109) $P^{\infty} > O$, and this means by Theorem 11 that P is primitive, i.e., that the given chain is acyclic.

From these remarks it follows that Theorem 11 can also be formulated as follows:

THEOREM 11′: 1. *In a homogeneous Markov chain all the limiting absolute probabilities exist for arbitrary initial probabilities if and only if the chain is regular.*

2. *In a homogeneous Markov chain the limiting absolute probabilities exist for arbitrary initial probabilities and are independent of them if and only if the chain is fully regular.*

3. *In a homogeneous Markov chain positive limiting absolute probabilities exist for arbitrary initial probabilities and are independent of them if and only if the chain is acyclic.*[48]

5. We now consider a homogeneous Markov chain of general type with a matrix P of transition probabilities.

---

[48] The second part of Theorem 11′ is sometimes called the *ergodic theorem* and the first part the *general quasi-ergodic theorem* for homogeneous Markov chains (see [4], pp. 473 and 476).

We choose the normal form (69) for P and denote by $h_1, h_2, \ldots, h_g$ the indices of imprimitivity of the matrices $A_1, A_2, \ldots, A_g$ in (69). Let h be the least common multiple of the integers $h_1, h_2, \ldots, h_g$. Then the matrix P^h has no characteristic values, other than 1, of modulus 1, i.e., P^h is regular; here h is the least exponent for which P^h is regular. We shall call h the *period* of the given homogeneous Markov chain.

Since P^h is regular, the limit

$$\lim_{q\to\infty} P^{hq} = (P^h)^\infty$$

exists and hence the limits

$$P_r = \lim_{q\to\infty} P^{r+qh} = P^r (P^h)^\infty \qquad (r = 0, 1, \ldots, h-1)$$

also exist.

Thus, in general, the sequence of matrices

$$P, P^2, P^3, \ldots$$

splits into h subsequences with the limits $P_r = P^r (P^h)^\infty$ $(r = 0, 1, \ldots, h-1)$.

When we go from the transition probabilities to the absolute probabilities by means of (106), we find that the sequence

$$\overset{1}{p}, \overset{2}{p}, \overset{3}{p}, \ldots$$

splits into h subsequences with the limits

$$\lim_{q\to\infty} \overset{r+qh}{p} = (P^{\mathsf{T}h})^\infty \overset{r}{p} \quad (r = 0, 1, 2, \ldots, h-1).$$

For an arbitrary homogeneous Markov chain with a finite number of states the limits of the arithmetic means always exist:

$$\tilde{P} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} P^k = \frac{1}{h}(E + P + \cdots + P^{h-1})(P^h)^\infty \tag{110}$$

and

$$\tilde{p} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \overset{k}{p} = P^{\mathsf{T}} \overset{0}{p}. \tag{110'}$$

Here $\tilde{P} = \| \tilde{p}_{ij} \|_1^n$ and $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)$. The values $\tilde{p}_{ij}$ $(i, j = 1, 2, 3, \ldots, n)$ and $\tilde{p}_j$ $(j = 1, 2, \ldots, n)$ are called the *mean limiting transition probabilities* and *mean limiting absolute probabilities*, respectively.

Since

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=2}^{N+1} P^k = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \tilde{P}^k,$$

we have

$$\tilde{P} P = \tilde{P}$$

and therefore, by (110′),

$$P^{\mathsf{T}} \tilde{p} = \tilde{p}; \tag{111}$$

i.e., $\tilde{p}$ is a characteristic vector of P^{T} for $\lambda = 1$.

Note that by (69) and (110) we may represent $\tilde{P}$ in the form

$$\tilde{P} = \begin{pmatrix} \tilde{A}_1 & O & \cdot & \cdot & \cdot & O & \\ O & \tilde{A}_2 & \cdot & \cdot & \cdot & O & O \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ O & O & \cdot & \cdot & \cdot & \tilde{A}_g & \\ & & & \tilde{U} & & & \tilde{W} \end{pmatrix},$$

where

$$\tilde{A}_i = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} A_i^k \; (i = 1, 2, \ldots, g) \quad \tilde{W} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} W^k,$$

$$W = \begin{pmatrix} A_{g+1} & O & \cdot & \cdot & O \\ * & A_{g+2} & \cdot & \cdot & O \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & \cdot & \cdot & A_s \end{pmatrix}.$$

Since all the characteristic values of W are of modulus less than 1, we have

$$\lim_{k\to\infty} W^k = O,$$

and therefore $\tilde{W} = O$.

Hence

$$\tilde{P} = \begin{pmatrix} \tilde{A}_1 & O & \cdot & \cdot & \cdot & O & \\ O & \tilde{A}_2 & \cdot & \cdot & \cdot & O & O \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ O & O & \cdot & \cdot & \cdot & \tilde{A}_g & \\ & & & \tilde{U} & & & O \end{pmatrix}. \tag{112}$$

Since $\tilde{P}$ is a stochastic matrix, the matrices $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_g$ are also stochastic.

From this representation of $\tilde{P}$ and from (107) is follows that: *The mean limiting absolute probabilities corresponding to non-essential states are always zero.*

If $g=1$ in the normal form of P, then $\lambda=1$ is a simple characteristic value of P^{T}.

In this case $\tilde{p}$ is uniquely determined by (111). and the mean limiting probabilities $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n$ do not depend on the initial probabilities $\overset{0}{p}_1, \overset{0}{p}_2, \dots, \overset{0}{p}_n$. Conversely, if $\tilde{p}$ does not depend on $\overset{0}{p}$. then P is of rank 1 by (110′). But the rank of (112) can be 1 only if $g=1$.

We formulate these results in the following theorem:[49]

THEOREM 12: *For an arbitrary homogeneous Markov chain with period h the probability matrices P^k and $\overset{k}{p}$ tend to a periodic repetition with period h for $k \to \infty$; moreover, the mean limiting transition probabilities and the absolute probabilities $\tilde{P} = \| \tilde{p}_{ij} \|_1^n$ and $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n)$ defined by (110) and (110′) always exist.*

The mean absolute probabilities corresponding to non-essential states are always zero.

If $g=1$ in the normal form of P (and only in this case), the mean limiting absolute probabilities $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n$ are independent of the initial probabilities $\overset{0}{p}_1, \overset{0}{p}_2, \dots, \overset{0}{p}_n$ and are uniquely determined by (111).

§ 8. Totally Non-negative Matrices

In this and the following sections we consider real matrices in which not only the elements, but also all the minors of every order are non-negative. Such matrices have important applications in the theory of small oscillations of elastic systems. The reader will find a detailed study of these matrices and their applications in the book [17]. Here we shall only deal with some of their basic properties.

1. We begin with a definition:

DEFINITION 5: *A rectangular matrix*

$$A = \| a_{ik} \| \qquad (i=1, 2, \dots, m;\ k=1, 2, \dots, n)$$

is called totally non-negative (totally positive) if all its minors of any order are non-negative (positive):

$$A\begin{pmatrix} i_1 & i_2 & \dots & i_p \\ k_1 & k_2 & \dots & k_p \end{pmatrix} \geqq 0 \qquad (>0)$$

$$\left(1 \leq \begin{matrix} i_1 < i_2 < \cdots < i_p \\ k_1 < k_2 < \cdots < k_p \end{matrix} \leq n;\ p = 1, 2, \dots, \min(m, n)\right).$$

In what follows we shall only consider square totally non-negative and totally positive matrices.

Example 1. The *generalized Vandermonde matrix*

$$V = \| a_i^{\alpha_k} \|_1^n \qquad (0 < a_1 < a_2 < \dots < a_n;\ \alpha_1 < \alpha_2 < \dots < \alpha_n)$$

is totally positive. Let us show first that $|V| \neq 0$. Indeed, from $|V| = 0$ it would follow that we could determine real numbers $c_1, c_2, \dots, c_n$, not all equal to zero, such that the function

$$f(x) = \sum_{k=1}^{n} c_k x^{\alpha_k} \qquad (\alpha_i \neq \alpha_j \text{ for } i \neq j)$$

has the n zeros $x_i = a_i$ $(i = 1, 2, \dots, n)$, where n is the number of terms in the above summand. For $n=1$ this is impossible. Let us make the induction hypothesis that it is impossible for a sum of n_1 terms, where $n_1 < n$, and show that it is then also impossible for the given function $f(x)$. Assume the contrary. Then by Rolle's Theorem the function $f_1(x) = [x^{-\alpha_1} f(x)]'$ consisting of $n-1$ terms would have $n-1$ positive zeros, and this contradicts the induction hypothesis.

Thus, $|V| \neq 0$. But for $\alpha_1 = 0, \alpha_2 = 1, \dots, \alpha_n = n-1$ the determinant $|V|$ goes over into the ordinary Vandermonde determinant $| a_i^{k-1} |_1^n$, which is positive. Since the transition from this to the generalized Vandermonde determinant can be carried out by means of a continuous change of the exponents $\alpha_1, \alpha_2, \dots, \alpha_n$ with preservation of the inequalities $\alpha_1 < \alpha_2 < \dots < \alpha_n$, and since, by what we have shown, the determinant does not vanish in this process, we have $|V| > 0$ for arbitrary $0 < a_1 < a_2 < \dots < a_n$.

Since every minor of V can be regarded as the determinant of some generalized Vandermonde matrix, all the minors of V are positive.

Example 2. We consider a *Jacobi matrix*

$$J = \left\| \begin{matrix} a_1 & b_1 & 0 & \dots & 0 & 0 \\ c_1 & a_2 & b_2 & \dots & 0 & 0 \\ 0 & c_2 & a_3 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & c_{n-1} & a_n \end{matrix} \right\|, \tag{113}$$

---

[49] This theorem is sometimes called the *asymptotic theorem* for homogeneous Markov chains. See [4], pp. 479-82.

in which all the elements are zero outside the main diagonal and the first super-diagonal and sub-diagonal. Let us set up a formula that expresses an arbitrary minor of the matrix in terms of principal minors and the elements b, c. Suppose that

$$1 \leqq \begin{matrix} i_1 < i_2 < \cdots < i_p \\ k_1 < k_2 < \cdots < k_p \end{matrix} \leqq n$$

and

$$i_1 = k_1,\ i_2 = k_2, \ldots, i_{\nu_1} = k_{\nu_1}\ ;\ i_{\nu_1+1} \neq k_{\nu_1+1}, \ldots, i_{\nu_2} \neq k_{\nu_2};\ i_{\nu_2+1} = k_{\nu_2+1}, \ldots, i_{\nu_3} = k_{\nu_3}; \ldots ;$$

then

$$J\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ k_1 & k_2 & \ldots & k_p \end{pmatrix} = J\begin{pmatrix} i_1 & \ldots & i_{\nu_1} \\ k_1 & \ldots & k_{\nu_1} \end{pmatrix} J\begin{pmatrix} i_{\nu_1+1} \\ k_{\nu_1+1} \end{pmatrix} \cdots J\begin{pmatrix} i_{\nu_2} \\ k_{\nu_2} \end{pmatrix} J\begin{pmatrix} i_{\nu_2+1} & \ldots & i_{\nu_3} \\ k_{\nu_2+1} & \ldots & k_{\nu_3} \end{pmatrix} \cdots. \qquad (114)$$

This formula is a consequence of the easily verifiable equation:

$$J\begin{pmatrix} i_1 & \ldots & i_p \\ k_1 & \ldots & k_p \end{pmatrix} = J\begin{pmatrix} i_1 & \ldots & i_{\nu-1} \\ k_1 & \ldots & k_{\nu-1} \end{pmatrix} J\begin{pmatrix} i_\nu \\ k_\nu \end{pmatrix} J\begin{pmatrix} i_{\nu+1} & \ldots & i_p \\ k_{\nu+1} & \ldots & k_p \end{pmatrix} \quad (\text{for } i_\nu \neq k_\nu). \qquad (115)$$

From (114) it follows that every minor is the product of certain principal minors and certain elements of J. Thus: *For J to be totally non-negative it is necessary and sufficient that all the principal minors and the elements b, c should be non-negative.*

2. A totally non-negative matrix $A = \| a_{ik} \|_1^n$ always satisfies the following important determinantal inequality:[50]

$$A\begin{pmatrix} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \end{pmatrix} \leqq A\begin{pmatrix} 1 & 2 & \ldots & p \\ 1 & 2 & \ldots & p \end{pmatrix} A\begin{pmatrix} p+1 & \ldots & n \\ p+1 & \ldots & n \end{pmatrix} \quad (p < n). \qquad (116)$$

Before deriving this inequality, we prove the following lemma:

LEMMA 5: *If in a totally non-negative matrix $A = \| a_{ik} \|_1^n$ any principal minor vanishes, then every principal minor 'bordering' it also vanishes.*

Proof. The lemma will be proved if we can show that for a totally non-negative matrix $A = \| a_{ik} \|_1^n$ it follows from

$$A\begin{pmatrix} 1 & 2 & \ldots & q \\ 1 & 2 & \ldots & q \end{pmatrix} = 0 \quad (q < n) \qquad (117)$$

that

$$A\begin{pmatrix} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \end{pmatrix} = 0. \qquad (118)$$

---

[50] See [172] and [17], pp. 111ff, where it is also shown that the equality sign in (116) can only hold in the following obvious cases:

1) One of the factors on the right-hand side of (116) is zero;

2) All the elements a_{ik} $(i = 1, 2, \ldots, p;\ k = p+1, \ldots, n)$ or a_{ik} $(i = p+1, \ldots, n;\ k = 1, 2, \ldots, p)$ are zero.

The inequality (116) has the same outward form as the generalized Hadamard inequality (see (33), Vol. I, p. 255) for a positive-definite hermitian or quadratic form.

For this purpose we consider two cases:

1) $a_{11} = 0$. Since $\begin{vmatrix} a_{11} & a_{1k} \\ a_{i1} & a_{ik} \end{vmatrix} = -a_{i1}a_{1k} \geqq 0$, $a_{i1} \geqq 0$, $a_{1k} \geqq 0$ $(i, k = 2, \ldots, n)$, either all the $a_{i1} = 0$ $(i = 2, \ldots, n)$ or all the $a_{1k} = 0$ $(k = 2, \ldots, n)$. These equations and $a_{11} = 0$ imply (118).

2) $a_{11} \neq 0$. Then for some p $(1 \leqq p \leqq q)$

$$A\begin{pmatrix} 1 & 2 & \ldots & p-1 \\ 1 & 2 & \ldots & p-1 \end{pmatrix} \neq 0, \quad A\begin{pmatrix} 1 & 2 & \ldots & p-1 & p \\ 1 & 2 & \ldots & p-1 & p \end{pmatrix} = 0. \qquad (119)$$

We introduce bordered determinants

$$d_{ik} = A\begin{pmatrix} 1 & 2 & \ldots & p-1 & i \\ 1 & 2 & \ldots & p-1 & k \end{pmatrix} \quad (i, k = p, p+1, \ldots, n) \qquad (120)$$

and form from them a matrix $D = \| d_{ik} \|_p^n$.

By Sylvester's identity (Vol. I, Chapter II, § 3),

$$D\begin{pmatrix} i_1 & i_2 & \ldots & i_g \\ k_1 & k_2 & \ldots & k_g \end{pmatrix}$$
$$= \left[A\begin{pmatrix} 1 & 2 & \ldots & p-1 \\ 1 & 2 & \ldots & p-1 \end{pmatrix} \right]^{g-1} A\begin{pmatrix} 1 & 2 & \ldots & p-1 & i_1 & i_2 & \ldots & i_g \\ 1 & 2 & \ldots & p-1 & k_1 & k_2 & \ldots & k_g \end{pmatrix} \geqq 0 \qquad (121)$$
$$\left(p \leqq \begin{matrix} i_1 < i_2 < \cdots < i_g \\ k_1 < k_2 < \cdots < k_g \end{matrix} \leqq n; \quad g = 1, 2, \ldots, n-p+1 \right),$$

so that D is a totally non-negative matrix.

Since by (119)

$$d_{pp} = A\begin{pmatrix} 1 & 2 & \ldots & p \\ 1 & 2 & \ldots & p \end{pmatrix} = 0,$$

the matrix D falls under the case 1) and

$$D\begin{pmatrix} p & p+1 & \ldots & n \\ p & p+1 & \ldots & n \end{pmatrix} = \left[A\begin{pmatrix} 1 & 2 & \ldots & p-1 \\ 1 & 2 & \ldots & p-1 \end{pmatrix} \right]^{n-p} A\begin{pmatrix} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \end{pmatrix} = 0.$$

Since $A\begin{pmatrix}1 & 2 & \dots & p-1\\ 1 & 2 & \dots & p-1\end{pmatrix} \neq 0$, (118) follows, and the lemma is proved.

3. We may now assume in the derivation of the inequality (116) that all the principal minors of A are different from zero, since by Lemma 5 one of the principal minors can only be zero when $|A| = 0$, and in this case the inequality (116) is obvious.

For $n = 2$, (116) can be verified immediately:

$$A\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \leqq a_{11}a_{22},$$

since $a_{12} \geqq 0$, $a_{21} \geqq 0$. We shall establish (116) for $n > 2$ under the assumption that it is true for matrices of order less than n. Moreover, without loss of generality, we may assume that $p > 1$, since otherwise by reversing the numbering of the rows and columns we could interchange the roles of p and $n - p$.

We now consider again the matrix $D = \| d_{ik} \|_p^n$, where the d_{ik} $(i, k = p, p+1, \dots, n)$ are defined by (120); we use Sylvester's identity twice as well as the basic inequality (116) for matrices of order less than n and obtain:

$$\begin{aligned} A\begin{pmatrix}1 & 2 & \dots & n\\ 1 & 2 & \dots & n\end{pmatrix} &= \frac{D\begin{pmatrix}p & p+1 & \dots & n\\ p & p+1 & \dots & n\end{pmatrix}}{\left[A\begin{pmatrix}1 & 2 & \dots & p-1\\ 1 & 2 & \dots & p-1\end{pmatrix}\right]^{n-p}} \leqq \frac{d_{pp}D\begin{pmatrix}p+1 & \dots & n\\ p+1 & \dots & n\end{pmatrix}}{\left[A\begin{pmatrix}1 & 2 & \dots & p-1\\ 1 & 2 & \dots & p-1\end{pmatrix}\right]^{n-p}}\\ &= \frac{A\begin{pmatrix}1 & 2 & \dots & p\\ 1 & 2 & \dots & p\end{pmatrix} A\begin{pmatrix}1 & 2 & \dots & p-1 & p+1 & \dots & n\\ 1 & 2 & \dots & p-1 & p+1 & \dots & n\end{pmatrix}}{A\begin{pmatrix}1 & 2 & \dots & p-1\\ 1 & 2 & \dots & p-1\end{pmatrix}}\\ &\leqq A\begin{pmatrix}1 & 2 & \dots & p\\ 1 & 2 & \dots & p\end{pmatrix} A\begin{pmatrix}p+1 & \dots & n\\ p+1 & \dots & n\end{pmatrix}. \qquad (122)\end{aligned}$$

Thus, the inequality (116) has been established.

Let us make the following definition:

DEFINITION 6. *A minor*

$$A\begin{pmatrix}i_1 & i_2 & \dots & i_p\\ k_1 & k_2 & \dots & k_p\end{pmatrix} \quad \left(1 \leqq \begin{matrix}i_1 < i_2 < \cdots < i_p\\ k_1 < k_2 < \cdots < k_p\end{matrix} \leqq n\right) \qquad (123)$$

of the matrix $A = \| a_{ik} \|_1^n$ *will be called almost principal if of the differences* $i_1 - k_1, i_2 - k_2, \dots, i_p - k_p$ *only one is not zero.*

We can then point out that the whole derivation of (116) (and the proof of the auxiliary lemma) remain valid if the condition 'A is totally non-negative' is replaced by the weaker condition 'all the principal and almost principal minors of A are non-negative.'[51]

§ 9. Oscillatory Matrices

1. The characteristic values and characteristic vectors of totally positive matrices have a number of remarkable properties. However, the class of totally positive matrices is not wide enough from the point of view of applications to small oscillations of elastic systems. In this respect, the class of totally non-negative matrices is suffiently extensive. But the spectral properties we need do not hold for all totally non-negative matrices. Now there exists an intermediate class (between that of totally positive and that of totally non-negative matrices) in which the spectral properties of totally positive matrices are preserved and which is of sufficiently wide scope for the applications. The matrices of this intermediate class have been called 'oscillatory.' The name is due to the fact that oscillatory matrices form the mathematical apparatus for the study of oscillatory properties of small vibrations of elastic systems.[52]

DEFINITION 7. *A matrix* $A = \| a_{ik} \|_1^n$ *is called oscillatory if* A *is totally non-negative and if there exists an integer* $q > 0$ *such that* A^q *is totally positive.*

Example. A Jacobi matrix J (see (113)) is oscillatory if and only if 1. all the numbers b, c are positive and 2. the successive principal minors are positive:

---

[51] See [214]. We take this opportunity of mentioning that in the second edition of the book [17] by F. R. Gantmacher and M. G. Kreĭn a mistake crept in which was first pointed out to the authors by D. M. Kotelyanskiĭ. On p. 111 of that book an almost principal minor (123) was defined by the equation

$$\sum_{\nu=1}^{p} |i_\nu - k_\nu| = 1.$$

With this definition, the inequality (116) does not follow from the fact that the principal and the almost principal minors are non-negative. However, all the statements and proofs of § 6, Chapter II in [17] that refer to the fundamental inequality remain valid if an almost principal minor is defined as above and as we have done in the paper [214].

[52] See [17], Introduction, Chapter III, and Chapter IV.

$$a_1 > 0, \quad \begin{vmatrix} a_1 & b_1 \\ c_1 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_3 \end{vmatrix} > 0, \ \ldots, \ \begin{vmatrix} a_1 & b_1 & 0 & \ldots & 0 & 0 \\ c_1 & a_2 & b_2 & \ldots & 0 & 0 \\ 0 & c_2 & a_3 & \ldots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \ldots & c_{n-1} & a_n \end{vmatrix} > 0. \qquad (124)$$

Necessity of 1., 2. The numbers b, c are non-negative, because $J \geqq O$. But none of the numbers b, c may be zero, since otherwise the matrix would be reducible and then the inequality $J^q > O$ could not hold for any $q > 0$. Hence, all the numbers b, c are positive. All the principal minors of (124) are positive, by Lemma 5, since it follows from $|J| \geqq 0$ and $|J^q| > 0$ that $|J| > 0$.

Sufficiency of 1., 2. When we expand $|J|$ we easily see that the numbers b, c occur in $|J|$ only as products $b_1c_1, b_2c_2, \ldots, b_{n-1}c_{n-1}$. The same applies to every principal minor of 'zero density,' i.e., a minor formed from successive rows and columns (without gaps). But every principal minor of J is a product of principal minors of zero density. Therefore: *In every principal minor of J the numbers b and c occur only as products $b_1c_1, b_2c_2, \ldots, b_{n-1}c_{n-1}$.*

We now form the symmetrical Jacobi matrix

$$\tilde{J} = \begin{vmatrix} a_1 & \tilde{b}_1 & & & & 0 \\ \tilde{b}_1 & a_2 & \tilde{b}_2 & & & \\ & \tilde{b}_2 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \tilde{b}_{n-1} \\ 0 & & & & \tilde{b}_{n-1} & a_n \end{vmatrix}, \quad \tilde{b}_i = \sqrt{b_i c_i} > 0 \qquad (i = 1, 2, \ldots, n). \qquad (125)$$

From the above properties of the principal minors of a Jacobi matrix it follows that the corresponding principal minors of J and $\tilde{J}$ are equal. But then (124) means that the quadratic form

$$\tilde{J}(x, x)$$

is positive definite (see Vol. I, Chapter X, Theorem 3, p. 306). But in a positive-definite quadratic form all the principal minors are positive. Therefore in J too all the principal minors are positive. Since by 1. all the numbers b, c are positive, by (114) all the minors of J are non-negative; i.e., J is totally non-negative.

That a totally non-negative matrix J for which 1. and 2. are satisfied is oscillatory follows immediately from the following *criterion for an oscillatory matrix*.

A totally non-negative matrix $A = \| a_{ik} \|_1^n$ is oscillatory if and only if:

1) A is non-singular ($|A| > 0$);

2) All the elements of A in the principal diagonal and the first superdiagonals and sub-diagonals are different from zero ($a_{ik} > 0$ for $|i - k| \leqq 1$).

The reader can find a proof of this proposition in [17], Chapter II, § 7.

2. In order to formulate properties of the characteristic values and characteristic vectors of oscillatory matrices, we introduce some preliminary concepts and notations.

We consider a vector (column)

$$u = (u_1, u_2, \ldots, u_n).$$

Let us count the number of variations of sign in the sequence of coordinates $u_1, u_2, \ldots, u_n$ of u, attributing arbitrary signs to the zero coordinates (if any such exist). Depending on what signs we give to the zero coordinates the number of variations of sign will vary within certain limits. The *maximal* and *minimal* number of variations of sign so obtained will be denoted by S_u^+ and S_u^-, respectively. If $S_u^- = S_u^+$, we shall speak of the *exact* number of sign changes and denote it by S_u. Obviously $S_u^- = S_u^+$ if and only if 1. the extreme coordinates u_1 and u_n of u are different from zero, and 2. $u_i = 0$ $(1 < i < n)$ always implies that $u_{i-1}u_{i+1} < 0$.

We shall now prove the following fundamental theorem:

THEOREM 13: 1. *An oscillatory matrix $A = \| a_{ik} \|_1^n$ always has n distinct positive characteristic values*

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0. \qquad (126)$$

2. *The characteristic vector $\overset{1}{u} = (u_{11}, u_{21}, \ldots, u_{n1})$ of A that belongs to the largest characteristic value λ_1 has only non-zero coordinates of like sign; the characteristic vector $\overset{2}{u} = (u_{12}, u_{22}, \ldots, u_{n2})$ that belongs to the second largest characteristic value λ_2 has exactly one variation of sign in its coordinates; more generally, the characteristic vector $\overset{k}{u} = (u_{1k}, u_{2k}, \ldots, u_{nk})$ that belongs to the characteristic value λ_k has exactly $k - 1$ variations of sign* $(k = 1, 2, \ldots, n)$.

3. *For arbitrary real numbers $c_g, c_{g+1}, \ldots, c_h$* $(1 \leqq g \leqq h \leqq n;$ $\sum_{k=g}^{h} c_k^2 > 0)$ *the number of variations of sign in the coordinates of the vector*

$$u = \sum_{k=g}^{h} c_k \overset{k}{u} \qquad (127)$$

lies between $g - 1$ and $h - 1$:

$$g-1 \leqq S_u^- \leqq S_u^+ \leqq h-1\,. \tag{128}$$

Proof. 1. We number the characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A so that

$$|\lambda_1| \geqq |\lambda_2| \geqq \cdots \geqq |\lambda_n|$$

and consider the p-th compound matrix $\mathfrak{A}_p$ $(p=1, 2, \ldots, n)$ (see Chapter I, § 4). The characteristic values of $\mathfrak{A}_p$ are all the possible products of p characteristic values of A (see Vol. I, p. 75), i.e., the products

$$\lambda_1\lambda_2\cdots\lambda_p, \quad \lambda_1\lambda_2\cdots\lambda_{p-1}\lambda_{p+1}, \quad \ldots.$$

From the conditions of the theorem it follows that for some integer q A^q is totally positive. But then $\mathfrak{A}_p \geqq O$, $\mathfrak{A}_p^q > O$;[53] i.e., $\mathfrak{A}_p$ is irreducible, non-negative, and primitive. Applying Frobenius' theorem (see § 2, p. 40) to the primitive matrix $\mathfrak{A}_p$ $(p=1, 2, \ldots, n)$, we obtain

$$\lambda_1\lambda_2\cdots\lambda_p > 0 \quad (p=1, 2, \ldots, n),$$
$$\lambda_1\lambda_2\cdots\lambda_p > \lambda_1\lambda_2\cdots\lambda_{p-1}\lambda_{p+1} \quad (p=1, 2, \ldots, n-1).$$

Hence (126) follows.

2. From this inequality (126) it follows that $A = \| a_{ik} \|_1^n$ is a matrix of simple structure. Then all the compound matrices $\mathfrak{A}_p$ $(p=1, 2, \ldots, n)$ are also of simple structure (see Vol. I, p. 74).

We consider the fundamental matrix $U = \| u_{ik} \|_1^n$ of A (the k-th column of U contains the coordinates of the k-th characteristic vector $\overset{k}{u}$ of A; $k = 1, 2, \ldots, n$). Then (see Vol. I, Chapter III, p. 74), the characteristic vector of $\mathfrak{A}_p$ belonging to the characteristic value $\lambda_1\lambda_2\ldots\lambda_p$ has the coordinates

$$U\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ 1 & 2 & \ldots & p \end{pmatrix} \quad (1 \leqq i_1 < i_2 < \cdots < i_p \leqq n) \tag{129}$$

By Frobenius' theorem all the numbers (129) are different from zero and are of like sign. Multiplying the vectors $\overset{1}{u}, \overset{2}{u}, \ldots, \overset{n}{u}$ by ± 1, we can make all the minors of (129) positive:

$$U\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ 1 & 2 & \ldots & p \end{pmatrix} > 0 \quad \begin{pmatrix} 1 \leqq i_1 < i_2 < \cdots < i_p \leqq n \\ p = 1, 2, \ldots, n \end{pmatrix}. \tag{130}$$

---

[53] The matrix $\mathfrak{A}_p^q$ is the p-th compound matrix A^q (see Vol. I, Chapter I, p. 20.)

The fundamental matrix $U = \| u_{ik} \|_1^n$ is connected with A by the equation

$$A = U\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\, U^{-1}. \tag{131}$$

But then

$$A^{\mathsf{T}} = (U^{\mathsf{T}})^{-1}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\, U^{\mathsf{T}}. \tag{132}$$

Comparing (131) with (132), we see that

$$V = (U^{\mathsf{T}})^{-1} \tag{133}$$

is the fundamental matrix of A^{T} with the same characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$. But since A is oscillatory, so is A^{T}. Therefore in V as well for every $p = 1, 2, \ldots, n$ all the minors

$$V\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ 1 & 2 & \ldots & p \end{pmatrix} \quad (1 \leqq i_1 < i_2 < \cdots < i_p \leqq n) \tag{134}$$

are different from zero and are of the same sign.

On the other hand, by (133) U and V are connected by the equation

$$U^{\mathsf{T}} V = E\,.$$

Going over to the p-th compound matrices (see Vol. I, Chapter I, § 4), we have:

$$\mathfrak{U}_p \mathfrak{V}_p = \mathfrak{E}_p\,.$$

Hence, in particular, noting that the diagonal elements of $\mathfrak{E}_p$ are 1, we obtain:

$$\sum_{1 \leqq i_1 < i_2 < \cdots < i_p \leqq n} U\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ 1 & 2 & \ldots & p \end{pmatrix} V\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ 1 & 2 & \ldots & p \end{pmatrix} = 1\,. \tag{135}$$

On the left-hand side of this equation, the first factor in each of the summands is positive and the second factors are different from zero and are of like sign. It is then obvious that the second factors as well are positive; i.e.,

$$V\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ 1 & 2 & \ldots & p \end{pmatrix} > 0 \quad \begin{pmatrix} 1 \leqq i_1 < i_2 < \cdots < i_p \leqq n \\ p = 1, 2, \ldots, n \end{pmatrix}. \tag{136}$$

Thus, the inequalities (130) and (136) hold for $U = \| u_{ik} \|_1^n$ and $V = (U^{\mathsf{T}})^{-1}$ simultaneously.

When we express the minors of V in terms of those of the inverse matrix $V^{-1} = U^{\mathsf{T}}$ by the well-known formulas (see Vol. I, pp. 21-22), we obtain

$$V\begin{pmatrix} j_1 & j_2 & \cdots & j_{n-p} \\ 1 & 2 & \ldots & n-p \end{pmatrix} = \frac{(-1)^{np+\sum_{\nu=1}^{p} i_\nu}}{|U|} U\begin{pmatrix} i_1 & i_2 & & i_p \\ n & n-1 & \ldots & n-p+1 \end{pmatrix}, \qquad (137)$$

where $i_1 < i_2 < \ldots < i_p$ and $j_1 < j_2 < \ldots < j_{n-p}$ together give the complete system of indices $1, 2, \ldots, n$. Since, by (130), $|U| > 0$ it follows from (136) and (137) that

$$(-1)^{np+\sum_{\nu=1}^{p} i_\nu} U\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ 1 & 2 & \ldots & p \end{pmatrix} > 0 \quad \begin{pmatrix} 1 \leqq i_1 < i_2 < \cdots < i_p \leqq n \\ p = 1, 2, \ldots, n \end{pmatrix}. \qquad (138)$$

Now let $u = \sum_{k=g}^{h} c_k \overset{k}{u}$ $\left(\sum_{k=g}^{h} c_k^2 > 0 \right)$. We shall show that the inequalities (130) imply the second part of (128):

$$S_u^+ \leqq h - 1, \qquad (139)$$

and the inequalities (138), the first part:

$$S_u^- \geqq g - 1. \qquad (140)$$

Suppose that $S_u^+ > h - 1$. Then we can find $h + 1$ coordinates of u

$$u_{i_1}, u_{i_2}, \ldots, u_{i_{h+1}} \quad (1 \leqq i_1 < i_2 < \cdots < i_{h+1} \leqq n) \qquad (141)$$

such that

$$u_{i_\alpha} u_{i_{\alpha+1}} \leqq 0 \quad (\alpha = 1, 2, \ldots, h).$$

Furthermore, the coordinates (141) cannot all be zero; for then we could equate the corresponding coordinates of the vector $u = \sum_{k=1}^{h} c_k \overset{k}{u}$ $(c_1 = \ldots = c_{g-1} = 0;\ \sum_{k=1}^{h} c_k^2 > 0)$ to zero and thus obtain a system of homogeneous equations

$$\sum_{k=1}^{h} c_k u_{i_\alpha k} = 0 \quad (\alpha = 1, 2, \ldots, h)$$

with the non-zero solution $c_1, c_2, \ldots, c_h$, whereas the determinant of the system

$$U\begin{pmatrix} i_1 & i_2 & \cdots & i_h \\ 1 & 2 & \ldots & h \end{pmatrix}$$

is different from zero, by (130).

We now consider the vanishing determinant

$$\begin{vmatrix} u_{i_1 1} & \ldots & u_{i_1 h} & u_{i_1} \\ u_{i_2 1} & \ldots & u_{i_2 h} & u_{i_2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_{i_{h+1} 1} & \ldots & u_{i_{h+1} h} & u_{i_{h+1}} \end{vmatrix} = 0.$$

We expand it with respect to the elements of the last column:

$$\sum_{\alpha=1}^{h+1} (-1)^{h+\alpha+1} u_{i_\alpha} \; U\begin{pmatrix} i_1 & \cdots & i_{\alpha-1} & i_{\alpha+1} & \cdots & i_{h+1} \\ 1 & . & . & . & . & h \end{pmatrix} = 0.$$

But such an equation cannot hold, since on the left-hand side all the terms are of like sign and at least one term is different from zero. Hence the assumption that $S_u^+ > h - 1$ has led to a contradiction, and (139) can be regarded as proved.

We consider the vector

$$\overset{k*}{u} = (u_{1k}^*, u_{2k}^*, \ldots, u_{nk}^*) \qquad (k = 1, 2, \ldots, n),$$

where

$$u_{ik}^* = (-1)^{n+i+k} u_{ik} \qquad (i, k = 1, 2, \ldots, n);$$

then for the matrix $U^* = \| u_{ik}^* \|_1^n$ we have, by (138):

$$U^*\begin{pmatrix} i_1 & i_2 & \ldots & i_p \\ n & n-1 & \ldots & n-p+1 \end{pmatrix} > 0 \quad \begin{pmatrix} 1 \leqq i_1 < i_2 < \cdots < i_p \leqq n \\ p = 1, 2, \ldots, n \end{pmatrix}. \qquad (142)$$

But the inequalities (142) are analogous to (130). Therefore, by setting

$$\overset{*}{u} = \sum_{k=g}^{h} (-1)^k c_k \overset{k*}{u} \qquad (143)$$

we have the inequality analogous to (139):[54]

$$S_{u^*}^+ \leqq n - g. \qquad (144)$$

Let $u = (u_1, u_2, \ldots, u_n)$ and $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$. It is easy to see that

$$u_i^* = (-1)^i u_i \qquad (i = 1, 2, \ldots, n).$$

Therefore

---

[54] In the inequalities (142), the vectors $\overset{k}{u}$ $(k = 1, 2, \ldots, n)$ occur in the inverse order $\overset{n}{u}, \overset{n-1}{u}, \ldots$. The vector $\overset{g}{u}$ is preceded by $n - g$ vectors of this kind.

$$S_{u^*}^+ + S_u^- = n - 1,$$

and so the relation (140) holds, by (144).

This establishes the inequality (128). Since the second statement of the theorem is obtained from (128) by setting $g = h = k$, the theorem is now completely proved.

3. As an application of this theorem, let us study the small oscillations of n masses $m_1, m_2, \ldots, m_n$ concentrated at n movable points $x_1 < x_2 < \ldots < x_n$ of a segmentary elastic continuum (a string or a rod of finite length), stretched (in a state of equilibrium) along the segment $0 \leqq x \leqq l$ of the x-axis.

We denote by $K(x, s)$ $(0 \leqq x, s \leqq l)$ the function of influence of this continuum ($K(x, s)$ is the displacement at the point x under the action of a unit force applied at the point s) and by k_{ij} the coefficients of influence for the given n masses:

$$k_{ij} = K(x_i, x_j) \quad (i, j = 1, 2, \ldots, n).$$

If at the points $x_1, x_2, \ldots, x_n$ n forces $F_1, F_2, \ldots, F_n$ are applied, then the corresponding static displacement $y(x)$ $(0 \leqq x \leqq l)$, is given, by virtue of the linear superposition of displacements, by the formula

$$y(x) = \sum_{j=1}^{n} K(x, x_j) F_j.$$

When we here replace the forces F_j by the inertial forces $-m_j \frac{\partial^2}{\partial t^2} y(x_j, t)$ $(j = 1, 2, \ldots, n)$, we obtain the equation of free oscillations

$$y(x) = -\sum_{j=1}^{n} m_j K(x, x_j) \frac{\partial^2}{\partial t^2} y(x_j, t). \tag{145}$$

We shall seek harmonic oscillations of the continuum in the form

$$y(x) = u(x) \sin(\omega t + \alpha) \quad (0 \leqq x \leqq l). \tag{146}$$

Here $u(x)$ is the amplitude function, ω the frequency, and α the initial phase. Substituting this expression for $y(x)$ in (145) and cancelling $\sin(\omega t + \alpha)$, we obtain

$$u(x) = \omega^2 \sum_{j=1}^{n} m_j K(x, x_j) u(x_j). \tag{147}$$

Let us introduce a notation for the variable displacements and the displacements in amplitude at the points of distribution of mass:

$$y_i = y(x_i, t), \quad u_i = u(x_i) \quad (i = 1, 2, \ldots, n).$$

Then

$$y_i = u_i \sin(\omega t + \alpha) \quad (i = 1, 2, \ldots, n).$$

We also introduce the *reduced amplitude displacements* and the *reduced coefficients of influence*

$$\tilde{u}_i = \sqrt{m_i} u_i, \quad a_{ij} = \sqrt{m_i m_j} k_{ij} \quad (i, j = 1, 2, \ldots, n). \tag{148}$$

Replacing x in (147) by x_i $(i = 1, 2, \ldots, n)$ successively, we obtain a system of equations for the amplitude displacements:

$$\sum_{j=1}^{n} a_{ij} \tilde{u}_j = \lambda \tilde{u}_i \quad \left(\lambda = \frac{1}{\omega^2}; \quad i = 1, 2, \ldots, n\right). \tag{149}$$

Hence it is clear that the amplitude vector $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n)$ is a characteristic vector of $A = \| a_{ij} \|_1^n = \| \sqrt{m_i m_j} k_{ij} \|_1^n$ for $\lambda = 1/\omega^2$ (see Vol. I, Chapter X, § 8).

It can be established, as the result of a detailed analysis,[55] that *the matrix of the coefficients of influence* $\| k_{ij} \|_1^n$ *of a segmentary continuum is always oscillatory.* But then the matrix $A = \| a_{ij} \|_1^n = \| \sqrt{m_i m_j} k_{ij} \|_1^n$ is also oscillatory! Therefore (by Theorem 13) A has n positive characteristic values

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0;$$

i.e., there exist n harmonic oscillations of the continuum with *distinct* frequencies:

$$(0 <) \, \omega_1 < \omega_2 < \cdots < \omega_n \quad \left(\lambda_i = \frac{1}{\omega_i^2}; \quad i = 1, 2, \ldots, n\right).$$

By the same theorem to the fundamental frequency ω_1 there correspond amplitude displacements different from zero and of like sign. Among the displacements in amplitude corresponding to the first overtone with the frequency ω_2 there is exactly one variation of sign and, in general, among the displacements in amplitude for the overtone with the frequency ω_j there are exactly $j - 1$ variations of sign $(j = 1, 2, \ldots, n)$.

---

[55] See [239], [240], and [17], Chapter III.

From the fact that the matrix of the coefficients of influence $\| k_{ij} \|_1^n$ is oscillatory there follow other oscillatory properties of the continuum: 1) For $\omega = \omega_1$ the amplitude function $u(x)$, which is connected with the amplitude displacements by (147), has no nodes; and, in general, for $\omega = \omega_j$ the function has $j-1$ nodes $(j=1, 2, \dots, n)$; 2) The nodes of two adjacent harmonics alternate, etc.

We cannot dwell here on the justification of these properties.[56]

---

[56] See [17], Chapters III and IV.

CHAPTER XIV

APPLICATIONS OF THE THEORY OF MATRICES TO THE INVESTIGATION OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

§ 1. Systems of Linear Differential Equations with Variable Coefficients. General Concepts

1. Suppose given a system of linear homogeneous differential equations of the first order:

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} p_{ik}(t)\, x_k \qquad (i=1, 2, \dots, n), \tag{1}$$

where $p_{ik}(t)$ $(i, k=1, 2, \dots, n)$ are complex functions of a real argument t, continuous in some interval, finite or infinite, of the variable t.[1]

Setting $P(t) = \| p_{ik}(t) \|_1^n$ and $x = (x_1, x_2, \dots, x_n)$, we write (1) as

$$\frac{dx}{dt} = P(t)\, x\,. \tag{2}$$

An *integral matrix* of the system (1) shall be defined as a square matrix $X(t) = \| x_{ik}(t) \|_1^n$ whose columns are n linearly independent solutions of the system.

Since every column of X satisfies (2), the integral matrix X satisfies the equation

$$\frac{dX}{dt} = P(t)\, X\,. \tag{3}$$

In what follows, we shall consider the matrix equation (3) instead of the system (1).

From the theorem on the existence and uniqueness of the solution of a system of differential equations[2] it follows that the integral matrix $X(t)$ is uniquely determined when the value of the matrix for some ('initial')

---

[1] In this section, all the relations that involve functions of t refer to the given interval.

[2] A proof of this theorem will be given in § 5. See also I. G. Petrowski (Petrovskii), *Vorlesungen über die Theorie der gewöhnlichen Differentialgleichungen,* Leipzig, 1954 (translated from the Russian; Moscow, 1952).

value $t = t_0$ is known,[3] $X(t_0) = X_0$. For X_0 we can take an arbitrary non-singular square matrix of order n. In the particular case where $X(t_0) = E$, the integral matrix $X(t)$ will be called *normalized*.

Let us differentiate the determinant of X by differentiating its rows in succession and let us then use the differential relations

$$\frac{dx_{ij}}{dt} = \sum_{k=1}^{n} p_{ik} x_{kj} \quad (i, j = 1, 2, \ldots, n).$$

We obtain:

$$\frac{d|X|}{dt} = (p_{11} + p_{22} + \cdots + p_{nn})|X|.$$

Hence there follows the well-known *Jacobi identity*

$$|X| = ce^{\int_{t_0}^{t} \operatorname{tr} P\,dt}, \tag{4}$$

where c is a constant and

$$\operatorname{tr} P = p_{11} + p_{22} + \ldots + p_{nn}$$

is the trace of $P(t)$.

Since the determinant $|X|$ cannot vanish identically, we have $c \neq 0$. But then it follows from the Jacobi identity that $|X|$ is different from zero for every value of the argument

$$|X| \neq 0;$$

i.e., *an integral matrix is non-singular for every value of the argument.*

If $\tilde{X}(t)$ is a non-singular ($|\tilde{X}(t)| \neq 0$) particular solution of (3), then the general solution is determined by the formula

$$X = \tilde{X}C, \tag{5}$$

where C is an arbitrary constant matrix.

For, by multiplying both sides of the equation

$$\frac{d\tilde{X}}{dt} = P\tilde{X} \tag{6}$$

by C on the right, we see that the matrix $\tilde{X}C$ also satisfies (3). On the other hand, if X is an arbitrary solution of (3), then (6) implies:

---

[3] It is assumed that t_0 belongs to the given interval of t.

$$\frac{dX}{dt} = \frac{d}{dt}(\tilde{X} \cdot \tilde{X}^{-1}X) = \frac{d\tilde{X}}{dt}\tilde{X}^{-1}X + \tilde{X}\frac{d}{dt}(\tilde{X}^{-1}X) = PX + \tilde{X}\frac{d}{dt}(\tilde{X}^{-1}X),$$

and hence by (3)

$$\frac{d}{dt}(\tilde{X}^{-1}X) = 0$$

and

$$\tilde{X}^{-1}X = \text{const.} = C;$$

i.e., (5) holds.

All the integral matrices X of the system (1) are obtained by the formula (5) with $|C| \neq 0$.

2. Let us consider the special case:

$$\frac{dX}{dt} = AX, \tag{7}$$

where A is a constant matrix. Here $\tilde{X} = e^{At}$ is a particular non-singular solution of (7),[4] so that the general solution is of the form

$$X = e^{At}C \tag{8}$$

where C is an arbitrary constant matrix.

Setting $t = t_0$ in (8) we find: $X_0 = e^{At_0}C$. Hence $C = e^{-At_0}X_0$ and therefore (8) can be represented in the form

$$X = e^{A(t-t_0)}X_0. \tag{9}$$

This formula is equivalent to our earlier formula (46) of Chapter V (Vol. I, p. 118).

Let us now consider the so-called *Cauchy system*:

$$\frac{dX}{dt} = \frac{A}{t-a}X \quad (A \text{ is a constant matrix}). \tag{10}$$

This case reduces to the preceding one by a change of argument:

$$u = \ln(t-a).$$

Therefore the general solution of (10) looks as follows:

$$X = e^{A\ln(t-a)}C = (t-a)^A C. \tag{11}$$

The functions e^{At} and $(t-a)^A$ that occur in (8) and (11) may be represented in the form (Vol. I, p. 117)

---

[4] By term-by-term differentiation of the series $e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k$ we find $\frac{d}{dt}e^{At} = Ae^{At}$.

$$e^{At} = \sum_{k=1}^{s} (Z_{k1} + Z_{k2}t + \cdots + Z_{km_k}t^{m_k-1}) e^{\lambda_k t}, \tag{12}$$

$$(t-a)^A = \sum_{k=1}^{s} (Z_{k1} + Z_{k2}\ln(t-a) + \cdots + Z_{km_k}[\ln(t-a)]^{m_k-1}) (t-a)^{\lambda_k}. \tag{13}$$

Here

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_s)^{m_s}$$
$$(\lambda_i \neq \lambda_k \text{ for } i \neq k;\ i, k = 1, 2, \ldots, s)$$

is the minimal polynomial of A, and Z_{kj} $(j = 1, 2, \ldots, m_k;\ k = 1, 2, \ldots, s)$ are linearly independent constant matrices that are polynomials in A.[5]

Note. Sometimes an integral matrix of the system of differential equations (1) is taken to be a matrix W in which the *rows* are linearly independent solutions of the system. It is obvious that W is the transpose of X:

$$W = X^{\mathsf{T}}.$$

When we go over to the transposed matrices on both sides of (3), we obtain instead of (3) the following equation for W:

$$\frac{dW}{dt} = WP(t). \tag{3'}$$

Here W is the first factor on the right-hand side, not the second, as X was in (3).

§ 2. Lyapunov Transformations

1. Let us now assume that in the system (1) (and in the equation (3)) the coefficient matrix $P(t) = \| p_{ik}(t) \|_1^n$ is a continuous bounded function of t in the interval $[t_0, \infty)$.[6]

In place of the unknown functions $x_1, x_2, \ldots, x_n$ we introduce the new unknown functions $y_1, y_2, \ldots, y_n$ by means of the transformation

$$x_i = \sum_{k=1}^{n} l_{ik}(t)\, y_k \qquad (i = 1, 2, \ldots, n). \tag{14}$$

---

[5] Every term $X_k = (Z_{k1} + Z_{k2}t + \cdots + Z_{km_k}t^{m_k-1}) e^{\lambda_k t}$ $(k = 1, 2, \ldots, s)$ on the right-hand side of (12) is a solution of (7). For the product $g(A)e^{At}$, with an arbitrary function $g(\lambda)$, satisfies this equation. But $X_k = f(A) = g(A)e^{At}$ if $f(\lambda) = g(\lambda)e^{At}$ and $g(\lambda^k) = 1$, and all the remaining $m - 1$ values of $g(\lambda)$ on the spectrum of A are zero (see Vol. I, Chapter V, formula (17), on p. 104).

[6] This means that each function $p_{ik}(t)$ $(i, k = 1, 2, \ldots, n)$ is continuous and bounded in the interval $[t_0, \infty)$, i.e., $t \geqq t_0$.

We impose the following restrictions on the matrix $L(t) = \| l_{ik}(t) \|_1^n$ of the transformation:

1. $L(t)$ has a continuous derivative $\frac{dL}{dt}$ in the interval $[t_0, \infty)$;
2. $L(t)$ and $\frac{dL}{dt}$ are bounded in the interval $[t_0, \infty)$;
3. There exists a constant m such that

$$0 < m < \text{absolute value of } | L(t) | \qquad (t \geqq t_0),$$

i.e., the determinant $| L(t) |$ is bounded in modulus from below by the positive constant m.

A transformation (14) in which the coefficient matrix $L(t) = \| l_{ik}(t) \|_1^n$ satisfies 1.-3. will be called a *Lyapunov transformation* and the corresponding matrix $L(t)$ a *Lyapunov matrix.*

Such transformations were investigated by A. M. Lyapunov in his famous memoir 'The General Problem of Stability of Motion' [32].

Examples. 1. If $L =$ const. and $| L | \neq 0$, then L satisfies the conditions 1.-3. Therefore a non-singular transformation with constant coefficients is always a Lyapunov transformation.

2. If $D = \| d_{ik} \|_1^n$ is a matrix of simple structure with pure imaginary characteristic values, then the matrix

$$L(t) = e^{Dt}$$

satisfies the conditions 1.-3. and is therefore a Lyapunov matrix.[7]

2. It is easy to verify that the conditions 1.-3. of a matrix $L(t)$ imply the existence of the inverse matrix $L^{-1}(t)$ also satisfying the conditions 1.-3.; i.e., the inverse of a Lyapunov transformation is itself a Lyapunov transformation. In the same way it can be verified that two Lyapunov transformations in succession yield a Lyapunov transformation. Thus, the Lyapunov transformations form a group. They have the following important property:

If under the transformation (14) *the system* (1) *goes over into*

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} q_{ik}(t)\, y_k \tag{15}$$

and if the zero solution of this system is stable, asymptotically stable, or unstable in the sense of Lyapunov (see Vol. I, Chapter V, § 6), *then the zero solution of the original system* (1) *has the same property.*

---

[7] Here all the $m_k = 1$ in (12) and $\lambda_k = i\varphi_k$ (φ_k real, $k = 1, 2, \ldots, s$).

In other words, Lyapunov transformations do not alter the character of the zero solution (as regards stability). This is the reason why these transformations can be used in the investigation of stability in order to simplify the original system of equations.

A Lyapunov transformation establishes a one-to-one correspondence between the solutions of the systems (1) and (15); moreover, linearly independent solutions remain so after the transformation. Therefore a Lyapunov transformation carries an integral matrix X of (1) into some integral matrix Y of (15) such that

$$X = L(t)\,Y. \tag{16}$$

In matrix notation, the system (15) has the form

$$\frac{dY}{dt} = Q(t)\,Y, \tag{17}$$

where $Q(t) = \| q_{ik}(t) \|_1^n$ is the coefficient matrix of (15).

Substituting LY for X in (3) and comparing the equation so obtained with (17), we easily find the following formula which expresses Q in terms of P and L:

$$Q = L^{-1}PL - L^{-1}\frac{dL}{dt}. \tag{18}$$

Two systems (1) and (15) or, what is the same, (3) and (17) will be called *equivalent* (in the sense of Lyapunov) if they can be carried into one another by a Lyapunov transformation. The coefficient matrices P and Q of equivalent systems are always connected by the formula (18) in which L satisfies the conditions 1.-3.

§ 3. Reducible Systems

1. Among the systems of linear differential equations of the first order the simplest and best known are those with constant coefficients. It is, therefore, of interest to study systems that can be carried by a Lyapunov transformation into systems with constant coefficients. Lyapunov has called such systems *reducible*.

Suppose given a reducible system

$$\frac{dX}{dt} = PX. \tag{19,}$$

Then some Lyapunov transformation

$$X = L(t)\,Y \tag{20}$$

carries it into a system

$$\frac{dY}{dt} = AY, \tag{21}$$

where A is a constant matrix. Therefore (19) has the particular solution

$$\tilde{X} = L(t)\,e^{At}. \tag{22}$$

It is easy to see that, conversely, every system (19) with a particular solution of the form (22), where $L(t)$ is a Lyapunov matrix and A a constant matrix, is reducible and is reduced to the form (21) by means of the Lyapunov transformation (20).

Following Lyapunov, we shall show that: *Every system* (19) *with periodic coefficients is reducible.*[8]

Let $P(t)$ in (19) be a continuous function in $(-\infty, +\infty)$ with period τ:

$$P(t+\tau) = P(t). \tag{23}$$

Replacing t in (19) by $t+\tau$ and using (23), we obtain:

$$\frac{dX(t+\tau)}{dt} = P(t)\,X(t+\tau).$$

Thus, $X(t+\tau)$ is an integral matrix of (19) if $X(t)$ is. Therefore

$$X(t+\tau) = X(t)\,V,$$

where V is a constant non-singular matrix. Since $|V| \neq 0$, we can determine[9]

$$V^{\frac{t}{\tau}} = e^{\frac{t}{\tau}\ln V}.$$

This matrix function of t, just like $X(t)$, is multiplied on the right by V when the argument is increased by τ. Therefore the 'quotient'

$$L(t) = X(t)\,V^{-\frac{t}{\tau}} = X(t)\,e^{-\frac{t}{\tau}\ln V}$$

is continuous and periodic with period τ:

$$L(t+\tau) = L(t),$$

and with $|L| \neq 0$. The matrix $L(t)$ satisfies the conditions 1.-3. of the preceding section and is therefore a Lyapunov matrix.

---

[8] See [32], § 47.

[9] Here $\ln V = f(V)$, where $f(\lambda)$ is any single-valued branch of $\ln \lambda$ in the simply-connected domain G containing all the characteristic values of V, but not containing 0. See Vol. I, Chapter V.

On the other hand, since the solution X of (19) can be represented in the form

$$X = L(t)\, e^{\frac{\ln V}{\tau} t},$$

the system (19) is reducible.

In this case the Lyapunov transformation

$$X = L(t)\, Y,$$

which carries (19) into the form

$$\frac{dY}{dt} = \frac{1}{\tau} \ln V \cdot Y$$

has periodic coefficients with period τ.

Lyapunov has established[10] a very important criterion for stability and instability of a first linear approximation to a non-linear system of differential equations

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik} x_k + (**) \qquad (i = 1, 2, \ldots, n), \tag{24}$$

where we have convergent power series in $x_1, x_2, \ldots, x_n$ on the right-hand side and where (**) denotes the sum of the terms of second and higher orders in $x_1, x_2, \ldots, x_n$; the coefficients a_{ik} $(i, k = 1, 2, \ldots, n)$ of the linear terms are constant.[11]

LYAPUNOV'S CRITERION: *The zero solution of* (24) *is stable (and even asymptotically stable) if all the characteristic values of the coefficient matrix* $A = \| a_{ik} \|_1^n$ *of the first linear approximation have negative real parts, and unstable if at least one characteristic value has a positive real part.*

2. The arguments used above enable us to apply this criterion to a system whose linear terms have periodic coefficients:

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} p_{ik}(t)\, x_k + (**). \tag{25}$$

For on the basis of the preceding arguments we reduce the system (25) to the form (24) by means of a Lyapunov transformation, where

$$A = \| a_{ik} \|_1^n = \frac{1}{\tau} \ln V$$

and where V is the constant matrix by which an integral matrix of the corresponding linear system (19) is multiplied when the argument is changed by τ. Without loss of generality, we may assume that $\tau > 0$. By the properties of Lyapunov transformations the zero solutions of the original and of the transformed systems are simultaneously stable, asymptotically stable, or unstable. But the characteristic values λ_i and ν_i $(i = 1, 2, \ldots, n)$ of A and V are connected by the formula

$$\lambda_i = \frac{1}{\tau} \ln \nu_i \quad (i = 1, 2, \ldots, n).$$

Therefore, by applying Lyapunov's criterion to the reduced systems we find:[12]

The zero solution of (25) *is asymptotically stable if all the characteristic values* $\nu_1, \nu_2, \ldots, \nu_n$ *of* V *are of modulus less than* 1 *and unstable if at least one characteristic value is of modulus greater than* 1.

Lyapunov has established his criterion for the stability of a linear approximation for a considerably wider class of systems, namely those of the form (24) in which the linear approximation is not necessarily a system with constant coefficients, but belongs to a class of systems that he has called regular.[13]

The class of regular linear systems contains all the reducible systems.

A criterion for instability in the case when the first linear approximation is a regular system was set up by N. G. Chetaev.[14]

§ 4. The Canonical Form of a Reducible System. Erugin's Theorem

1. Suppose that a reducible system (19) and an equivalent system

$$\frac{dY}{dt} = AY$$

(in the sense of Lyapunov) are given, where A is a constant matrix.

We shall be interested in the question: *To what extent is the matrix* A *determined by the given system* (19)? This question can also be formulated as follows:

---

[10] See [32], § 24.

[11] The coefficients in the non-linear terms may depend on t. These functional coefficients are subject to certain restrictions (see [32], § 11).

[12] *Loc. cit.*, § 55.

[13] *Loc. cit.*, § 9.

[14] See [9], p. 181.

When are two systems

$$\frac{dY}{dt} = AY \quad \textit{and} \quad \frac{dZ}{dt} = BZ,$$

where A and B are constant matrices, equivalent in the sense of Lyapunov; i.e., when can they be carried into one another by a Lyapunov transformation?

In order to answer this question we introduce the notion of matrices with one and the same real part of the spectrum.

We shall say that two matrices A and B of order n *have one and the same real part of the spectrum* if and only if the elementary divisors of A and B are of the form

$$(\lambda - \lambda_1)^{m_1}, (\lambda - \lambda_2)^{m_2}, \ldots, (\lambda - \lambda_s)^{m_s}; \ (\lambda - \mu_1)^{m_1}, (\lambda - \mu_2)^{m_2}, \ldots, (\lambda - \mu_s)^{m_s},$$

where

$$\operatorname{Re} \lambda_k = \operatorname{Re} \mu_k \quad (k = 1, 2, \ldots, s).$$

Then the following theorem due to N. P. Erugin holds:[15]

THEOREM 1 (Erugin): *Two systems*

$$\frac{dY}{dt} = AY \quad \textit{and} \quad \frac{dZ}{dt} = BZ \tag{26}$$

(A and B are constant matrices of order n) are equivalent in the sense of Lyapunov if and only if the matrices A and B have one and the same real part of the spectrum.

Proof. Suppose that the systems (26) are given. We reduce A to the normal Jordan form[16] (see Vol. I, Chapter VI, § 7)

$$A = T \{ \lambda_1 E_1 + H_1, \lambda_2 E_2 + H_2, \ldots, \lambda_s E_s + H_s \} T^{-1}, \tag{27}$$

where

$$\lambda_k = \alpha_k + i\beta_k \qquad (\alpha_k, \beta_k \text{ are real numbers}; k = 1, 2, \ldots, s). \tag{28}$$

In accordance with (27) and (28) we set

$$\left.\begin{aligned} A_1 &= T \{ \alpha_1 E_1 + H_1, \alpha_2 E_2 + H_2, \ldots, \alpha_s E_s + H_s \} T^{-1}, \\ A_2 &= T \{ i\beta_1 E_1, i\beta_2 E_2, \ldots, i\beta_s E_s \} T^{-1}. \end{aligned}\right\} \tag{29}$$

---

[15] Our proof of the theorem differs from that of Erugin.

[16] E_k is the unit matrix; in H_k the elements of the first superdiagonal are 1, and the remaining elements are zero; the orders of E_k, H_k are the degrees of the k-th elementary divisor of A, i.e., m_k $(k = 1, 2, \ldots, s)$.

Then

$$A = A_1 + A_2, \quad A_1 A_2 = A_2 A_1. \tag{30}$$

We define a matrix $L(t)$ by the equation

$$L(t) = e^{A_2 t}.$$

$L(t)$ is a Lyapunov matrix (see Example 2 on p. 117).

But by (30) a particular solution of the first of the systems (26) is of the form

$$e^{At} = e^{A_2 t} e^{A_1 t} = L(t) e^{A_1 (t)}.$$

Hence it follows that the first of the systems (26) is equivalent to

$$\frac{dU}{dt} = A_1 U, \tag{31}$$

where, by (29), the matrix A_1 has real characteristic values and its spectrum coincides with the real part of the spectrum of A.

Similarly, we replace the second of the systems (26) by the equivalent system

$$\frac{dV}{dt} = B_1 V, \tag{32}$$

where the matrix B_1 has real characteristic values and its spectrum coincides with the real part of the spectrum of B.

Our theorem will be proved if we can show that *the two systems* (31) *and* (32) *in which* A_1 *and* B_1 *are constant matrices with real characteristic values are equivalent if and only if* A_1 *and* B_1 *are similar.*[17]

Suppose that the Lyapunov transformation

$$U = L_1 V$$

carries (31) into (32). Then the matrix L_1 satisfies the equation

$$\frac{dL_1}{dt} = A_1 L_1 - L_1 B_1. \tag{33}$$

This matrix equation for L_1 is equivalent to a system of n^2 differential equations in the n^2 elements of L_1. The right-hand side of (33) is a linear operation on the 'vector' L_1 in an n^2-dimensional space

---

[17] This proposition implies Theorem 1, since the equivalence of the systems (31) and (32) means that the systems (26) are equivalent, and the similarity of A_1 and B_1 means that these matrices have the same elementary divisors, so that the matrices A and B have one and the same real part of the spectrum.

$$\frac{dL_1}{dt} = \widehat{F}(L_1), \quad [\widehat{F}(L_1) = A_1L_1 - L_1B_1]. \tag{33'}$$

Every characteristic value of the linear operator $\widehat{F}$ (and of the corresponding matrix of order n^2) can be represented in the form of a difference $\gamma - \delta$, where γ is a characteristic value of A_1 and δ a characteristic value of B_1.[18] Hence it follows that the operator $\widehat{F}$ has only real characteristic values.

We denote by

$$\widehat{\psi}(\lambda) = (\lambda - \widehat{\lambda}_1)^{\widehat{m}_1} (\lambda - \widehat{\lambda}_2)^{\widehat{m}_2} \cdots (\lambda - \widehat{\lambda}_u)^{\widehat{m}_u}$$

(the $\widehat{\lambda}_i$ are real; $\widehat{\lambda}_i \neq \widehat{\lambda}_j$ for $i \neq j$; $i, j = 1, 2, \dots, u$) the minimal polynomial of $\widehat{F}$. Then the solution $L_1(t) = e^{\widehat{F}t}L^{(0)}$ of (33′) can, by formula (12) (p. 116), be written as follows:

$$L_1(t) = \sum_{k=1}^{u} \sum_{j=0}^{\widehat{m}_k - 1} L_{kj} t^j e^{\widehat{\lambda}_k t}, \tag{34}$$

where the L_{kj} are constant matrices of order n. Since the matrix $L_1(t)$ is bounded in the interval (t_0, ∞), both for every $\widehat{\lambda}_k > 0$ and for $\widehat{\lambda}_k = 0$ and $j > 0$, the corresponding matrices $L_{kj} = O$. We denote by $L_-(t)$ the sum of all the terms in (34) for which $\widehat{\lambda}_k < 0$. Then

$$L_1(t) = L_-(t) + L_0, \tag{35}$$

where

$$\lim_{t \to +\infty} L_-(t) = O, \quad \lim_{t \to +\infty} \frac{dL_-(t)}{dt} = O, \quad L_0 = \text{const.} \tag{35'}$$

Then, by (35) and (35′),

$$\lim_{t \to +\infty} L_1(t) = L_0,$$

---

[18] For let Λ_0 be any characteristic value of the operator $\widehat{F}$. Then there exists a matrix $L \neq O$ such that $\widehat{F}(L) = \Lambda_0 L$, or

$$(A_1 - \Lambda_0 E)L = LB_1. \tag{*}$$

The matrices $A_1 - \Lambda_0 E$ and B_1 have at least one characteristic value in common, since otherwise there would exist a polynomial $g(\lambda)$ such that

$$g(A_1 - \Lambda_0 E) = O, \; g(B_1) = E,$$

and this is impossible, because it follows from (*) that $g(A_1 - \Lambda_0 E) \cdot L = L \cdot g(B_1)$ and $L \neq O$. But if $A_1 - \Lambda_0 E$ and B_1 have a common characteristic value, then $\Lambda_0 = \gamma - \delta$, where γ and δ are characteristic values of A_1 and B_1, respectively. A detailed study of the operator $\widehat{F}$ can be found in the paper [179] by F. Golubchikov.

from which it follows that

$$|L_0| \neq 0,$$

because the determinant $|L_1(t)|$ is bounded in modulus from below.

When we substitute for $L_1(t)$ in (33) the sum $L_-(t) + L_0$, we obtain:

$$\frac{dL_-(t)}{dt} - A_1L_-(t) + B_1L_-(t) = A_1L_0 - B_1L_0;$$

hence by (35′)

$$A_1L_0 - L_0B_1 = O$$

and therefore

$$B_1 = L_0^{-1}A_1L_0. \tag{36}$$

Conversely, if (36) holds, then the Lyapunov transformation

$$U = L_0V$$

carries (31) into (32). This completes the proof of the theorem.

2. From this theorem it follows that: *Every reducible system* (19) *can be carried by the Lyapunov transformation $X = LY$ into the form*

$$\frac{dY}{dt} = JY,$$

where J is a Jordan matrix with real characteristic values. This canonical form of the system is uniquely determined by the given matrix $P(t)$ to within the order of the diagonal blocks of J.

§ 5. The Matricant

1. We consider a system of differential equations

$$\frac{dX}{dt} = P(t)X, \tag{37}$$

where $P(t) = \| p_{ik}(t) \|_1^n$ is a continuous matrix function of the argument t in some interval (a, b).[19]

---

[19] (a, b) is an arbitrary interval (finite or infinite). All the elements $p_{ik}(t)$ $(i, k = 1, 2, \dots, n)$ of $P(t)$ are complex functions of the real argument t, continuous in (a, b). Everything that follows remains valid if, instead of continuity, we require (in every finite subinterval of (a, b)) only boundedness and Riemann integrability of all the functions $p_{ik}(t)$.

We use the method of successive approximations to determine a normalized solution of (37), i.e., a solution that for $t = t_0$ becomes the unit matrix (t_0 is a fixed number of the interval (a, b)). The successive approximations X_k $(k = 0, 1, 2, \ldots)$ are found from the recurrence relations

$$\frac{dX_k}{dt} = P(t) X_{k-1} \qquad (k = 1, 2, \ldots),$$

when X_0 is taken to be the unit matrix E.

Setting $X_k(t_0) = E$ $(k = 0, 1, 2, \ldots)$ we may represent X_k in the form

$$X_k = E + \int_{t_0}^{t} P(\tau) X_{k-1}\, d\tau .$$

Thus

$$X_0 = E, \quad X_1 = E + \int_{t_0}^{t} P(\tau)\, d\tau, \quad X_2 = E + \int_{t_0}^{t} P(\tau)\, d\tau + \int_{t_0}^{t} P(\tau) \int_{t_0}^{\tau} P(\sigma)\, d\sigma\, d\tau, \ldots,$$

i.e., X_k $(k = 0, 1, 2, \ldots)$ is the sum of the first $k + 1$ terms of the matrix series

$$E + \int_{t_0}^{t} P(\tau)\, d\tau + \int_{t_0}^{t} P(\tau) \int_{t_0}^{\tau} P(\sigma)\, d\sigma\, d\tau + \cdots . \tag{38}$$

In order to prove that this series is absolutely and uniformly convergent in every closed subinterval of the interval (a, b) and determines the required solution of (37), we construct a majorant.

We define non-negative functions $g(t)$ and $h(t)$ in (a, b) by the equations[20]

$$g(t) = \max\, [\, | p_{11}(t) |, \, | p_{12}(t) |, \ldots, | p_{nn}(t) | \,], \quad h(t) = \left| \int_{t_0}^{t} g(\tau)\, d\tau \right| .$$

It is easy to verify that $g(t)$, and consequently $h(t)$ as well, is continuous in (a, b).[21]

Each of the n^2 scalar series into which the matrix series (38) splits is majorized by the series

$$1 + h(t) + \frac{nh^2(t)}{2!} + \frac{n^2h^3(t)}{3!} + \cdots . \tag{39}$$

---

[20] By definition, the value of $g(t)$ for any value of t is the largest of the n^2 moduli of the values of $p_{ik}(t)$ $(i, k = 1, 2, \ldots, n)$ for that value of t.

[21] The continuity of $g(t)$ at any point t_1 of the interval (a, b) follows from the fact that the difference $g(t) - g(t_1)$ for t sufficiently near t_1 always coincides with one of the n^2 differences $| p_{ik}(t) | - | p_{ik}(t_1) |$ $(i, k = 1, 2, \ldots, n)$.

For

$$\left| \left(\int_{t_0}^{t} P(\tau)\, d\tau \right)_{i,k} \right| = \left| \int_{t_0}^{t} p_{ik}(\tau)\, d\tau \right| \leq \left| \int_{t_0}^{t} g(\tau)\, d\tau \right| = h(t),$$

$$\left| \left(\int_{t_0}^{t} P(\tau) \int_{t_0}^{\tau} P(\sigma)\, d\sigma\, d\tau \right)_{i,k} \right| = \left| \sum_{j=1}^{n} \int_{t_0}^{t} p_{ij}(\tau) \int_{t_0}^{\tau} p_{jk}(\sigma)\, d\sigma\, d\tau \right| \leq n \left| \int_{t_0}^{t} g(\tau) \int_{t_0}^{\tau} g(\sigma)\, d\sigma\, d\tau \right| = \frac{n h^2(t)}{2},$$

etc.

The series (39) converges in (a, b) and converges uniformly in every closed part of this interval. Hence it follows that the matrix series (38) also converges in (a, b) and does so absolutely and uniformly in every closed interval contained in (a, b).

By term-by-term differentiation we verify that the sum of (38) is a solution of (37); this solution becomes E for $t = t_0$. The term-by-term differentiation of (38) is permissible, because the series obtained after differentiation differs from (38) by the factor P and therefore, like (38), is uniformly convergent in every closed interval contained in (a, b).

Thus we have proved the theorem on the existence of a normal solution of (37). This solution will be denoted by $\Omega_{t_0}^{t}(P)$ or simply $\Omega_{t_0}^{t}$. Every other solution, as we have shown in § 1, is of the form

$$X = \Omega_{t_0}^{t} C,$$

where C is an arbitrary constant matrix. From this formula it follows that every solution, in particular the normalized one, is uniquely determined by its value for $t = t_0$.

This normalized solution $\Omega_{t_0}^{t}$ of (37) is often called the *matricant*.

We have seen that the matricant can be represented in the form of a series[22]

$$\Omega_{t_0}^{t} = E + \int_{t_0}^{t} P(\tau)\, d\tau + \int_{t_0}^{t} P(\tau) \int_{t_0}^{\tau} P(\sigma)\, d\sigma\, d\tau + \cdots, \tag{40}$$

which converges absolutely and uniformly in every closed interval in which $P(t)$ is continuous.

2. We mention a few formulas involving the matricant.

I. $$\Omega_{t_0}^{t} = \Omega_{t_1}^{t} \Omega_{t_0}^{t_1} \quad (t_0, t_1, t \in (a, b)).$$

For since $\Omega_{t_0}^{t}$ and $\Omega_{t_1}^{t}$ are two solutions of (37), we have

---

[22] The representation of the matricant in the form of such a series was first obtained by Peano [308].

$$\Omega_{t_0}^{t} = \Omega_{t_1}^{t} C \quad (C \text{ is a constant matrix}).$$

Setting $t = t_1$ in this equation, we obtain $C = \Omega_{t_0}^{t_1}$.

2. $\Omega_{t_0}^{t}(P+Q) = \Omega_{t_0}^{t}(P)\,\Omega_{t_0}^{t}(S)$ *with* $S = [\Omega_{t_0}^{t}(P)]^{-1} Q \Omega_{t_0}^{t}(P)$.

To derive this formula we set:

$$X = \Omega_{t_0}^{t}(P), \quad Y = \Omega_{t_0}^{t}(P+Q),$$

and

$$Y = XZ. \tag{41}$$

Differentiating (41) term by term, we find:

$$(P + Q)\,XZ = PXZ + X\frac{dZ}{dt}.$$

Hence

$$\frac{dZ}{dt} = X^{-1}QXZ$$

and since it follows from (41) that $Z(t_0) = E$,

$$Z = \Omega_{t_0}^{t}(X^{-1}QX).$$

When we substitute their respective matricants for X, Y, Z in (41), we obtain the formula 2.

3. $$\ln|\Omega_{t_0}^{t}(P)| = \int_{t_0}^{t} \operatorname{tr} P\, d\tau.$$

This formula follows from the Jacobi identity (4) (p. 114) when we substitute $\Omega_{t_0}^{t}(P)$ for $X(t)$ in that identity.

4. *If* $A = \| a_{ik} \|_1^n = \text{const.}$, *then*

$$\Omega_{t_0}^{t}(A) = e^{A(t-t_0)}.$$

We introduce the following notation. If $P = \| p_{ik} \|_1^n$, then we shall mean by mod P the matrix

$$\operatorname{mod} P = \| \, |p_{ik}| \, \|_1^n.$$

Furthermore, if $A = \| a_{ik} \|_1^n$ and $B = \| b_{ik} \|_1^n$ are two real matrices and

$$a_{ik} \leqq b_{ik} \qquad (i, k = 1, 2, \ldots, n),$$

then we shall write

$$A \leqq B.$$

Then it follows from the representation (40) that:

5. *If* mod $P(t) \leqq$ mod $Q(t)$ $(t \geqq t_0)$, *then the series* (40) *for* $\Omega_{t_0}^{t}(P)$ *is majorized, beginning with the first term, by the same series for* $\Omega_{t_0}^{t}(Q)$, *so that for all* $t \geqq t_0$

$$\operatorname{mod} \Omega_{t_0}^{t}(P) \leqq \Omega_{t_0}^{t}(Q), \quad \operatorname{mod}[\Omega_{t_0}^{t}(P) - E] \leqq \Omega_{t_0}^{t}(Q) - E,$$
$$\operatorname{mod}\Big[\Omega_{t_0}^{t}(P) - E - \int_{t_0}^{t} P\,d\tau\Big] \leqq \Omega_{t_0}^{t}(Q) - E - \int_{t_0}^{t} Q\,d\tau, \quad \text{etc.}$$

In what follows we shall denote the matrix of order n in which all the elements are 1 by I:

$$I = \| 1 \|.$$

We consider the function $g(t)$ defined on p. 126. Then we have

$$\operatorname{mod} P(t) \leqq g(t) I.$$

But $\Omega_{t_0}^{t}(g(t)I)$ is the normalized solution of the equation

$$\frac{dX}{dt} = g(t)\,IX.$$

Therefore, by 4.,[23]

$$\Omega_{t_0}^{t}(g(t)\,I) = e^{h(t)I} = E + \Big(h(t) + \frac{nh^2(t)}{2!} + \frac{n^2h^3(t)}{3!} + \cdots\Big) I, \tag{42}$$

where

$$h(t) = \int_{t_0}^{t} g(\tau)\,d\tau, \quad g(t) = \max_{1 \leqq i,\, k \leqq n} |p_{ik}(t)|.$$

Therefore it follows from 5. and (42) that:

6. $$\operatorname{mod} \Omega_{t_0}^{t}(P) \leqq E + \frac{1}{n}(e^{nh(t)} - 1)\,I,$$
$$\operatorname{mod}[\Omega_{t_0}^{t}(P) - E] \leqq \frac{1}{n}(e^{nh(t)} - 1)\,I,$$
$$\operatorname{mod}\Big[\Omega_{t_0}^{t}(P) - E - \int_{t_0}^{t} P\,d\tau\Big] \leqq \frac{1}{n}(e^{nh(t)} - 1 - nh(t))\,I, \quad \text{etc.}$$

We shall now derive an important formula giving an estimate for the modulus of the difference between two matricants:

---

[23] By replacing the independent variable t by $h = \int_{t_0}^{t} g(t)dt$.

7. $\operatorname{mod}[\Omega_{t_0}^t(P)-\Omega_{t_0}^t(Q)] \leqq \frac{1}{n} e^{nq(t-t_0)}(e^{nd(t-t_0)}-1) I \qquad (t \geqq t_0)$,

if

$$\operatorname{mod} Q \leqq qI, \quad \operatorname{mod}(P-Q) \leqq d \cdot I, \quad I = \| 1 \|$$

(q, d are non-negative numbers; n is the order of P and Q).

We denote the difference $P-Q$ by D. Then

$$P = Q + D, \qquad \operatorname{mod} D \leqq d \cdot I.$$

Using the expansion (40) of the matricant in a series, we find:

$$\Omega_{t_0}^t(Q+D)-\Omega_{t_0}^t(Q)$$
$$=\int_{t_0}^{t} D(\tau)\,d\tau+\int_{t_0}^{t} D(\tau)\int_{t_0}^{\tau} Q(\sigma)\,d\sigma\,d\tau+\int_{t_0}^{t} Q(\tau)\int_{t_0}^{\tau} D(\sigma)\,d\sigma\,d\tau+\int_{t_0}^{t} D(\tau)\int_{t_0}^{\tau} D(\sigma)\,d\sigma\,d\tau+\cdots.$$

From this expression it is clear that, for $t \geqq t_0$,

$$\begin{aligned}\operatorname{mod}[\Omega_{t_0}^t(Q+D)-\Omega_{t_0}^t(Q)] &\leqq \Omega_{t_0}^t(\operatorname{mod} Q+\operatorname{mod} D)-\Omega_{t_0}^t(\operatorname{mod} Q)\\ &\leqq \Omega_{t_0}^t((q+d)I)-\Omega_{t_0}^t(qI)=e^{(q+d)I(t-t_0)}-e^{qI(t-t_0)}\\ &=e^{qI(t-t_0)}(e^{dI(t-t_0)}-E)\\ &=\frac{1}{n}\left[E+\frac{1}{n}(e^{nq(t-t_0)})-1)I\right](e^{nd(t-t_0)}-1)I\\ &=\frac{1}{n}\left[I+\frac{1}{n}(e^{nq(t-t_0)}-1)I^2\right](e^{nd(t-t_0)}-1)\\ &=\frac{1}{n}e^{nq(t-t_0)}(e^{nd(t-t_0)}-1)I.\end{aligned}$$

We shall now show how to express by means of the matricant the general solution of a system of linear differential equations with right-hand sides:

$$\frac{dx_i}{dt}=\sum_{k=1}^{n} p_{ik}(t)\,x_k+f_i(t) \qquad (i=1,2,\ldots,n); \tag{43}$$

$p_{ik}(t)$ and $f_i(t)$ $(i, k=1, 2, \ldots, n)$ are continuous functions of t in some interval.

By introducing the column matrices ('vectors') $x=(x_1, x_2, \ldots, x_n)$ and $f=(f_1, f_2, \ldots, f_n)$ and the square matrix $P=\| p_{ik} \|_1^n$, we write the system as follows:

$$\frac{dx}{dt}=P(t)\,x+f(t). \tag{43'}$$

We shall look for a solution of this equation in the form

$$x=\Omega_{t_0}^t(P)\,z, \tag{44}$$

where z is an unknown column depending on t. We substitute this expression for x in (43') and obtain:

$$P\Omega_{t_0}^t(P)\,z+\Omega_{t_0}^t(P)\frac{dz}{dt}=P\Omega_{t_0}^t(P)\,z+f(t);$$

hence

$$\frac{dz}{dt}=[\Omega_{t_0}^t(P)]^{-1}f(t).$$

Integrating this, we find:

$$z=\int_{t_0}^{t}[\Omega_{t_0}^{\tau}(P)]^{-1}f(\tau)\,d\tau+c,$$

where c is an arbitrary constant vector. Substituting this expression in (44), we obtain:

$$x=\Omega_{t_0}^t(P)\int_{t_0}^{t}[\Omega_{t_0}^{\tau}(P)]^{-1}f(\tau)\,d\tau+\Omega_{t_0}^t(P)\,c. \tag{45}$$

When we give to t the value t_0, we find: $x(t_0)=c$. Therefore (45) assumes the form

$$x=\Omega_{t_0}^t(P)\,x(t_0)+\int_{t_0}^{t}K(t,\tau)\,f(\tau)\,d\tau, \tag{45'}$$

where

$$K(t,\tau)=\Omega_{t_0}^t(P)\,[\Omega_{t_0}^{\tau}(P)]^{-1}$$

is the so-called Cauchy matrix.

§ 6. The Multiplicative Integral. The Infinitesimal Calculus of Volterra

1. Let us consider the matricant $\Omega_{t_0}^t(P)$. We divide the basic interval (t_0, t) into n parts by introducing intermediate points $t_1, t_2, \ldots, t_{n-1}$ and set $\Delta t_k=t_k-t_{k-1}$ $(k=1, 2, \ldots, n;\ t_n=t)$. Then by property 1. of the matricant (see the preceding section),

$$\Omega_{t_0}^t=\Omega_{t_{n-1}}^t\cdots\Omega_{t_1}^{t_2}\Omega_{t_0}^{t_1}. \tag{46}$$

In the interval (t_{k-1}, t_k) we choose an intermediate point τ_k $(k=1, 2, \ldots, n)$. By regarding the Δt_k as small quantities of the first order we can take, for the computation of $\Omega_{t_{k-1}}^{t_k}$ to within small quantities of the second order, $P(t) \approx \text{const.} = P(\tau_k)$. Then

$$\Omega_{t_{k-1}}^{t_k} = e^{P(\tau_k)\Delta t_k} + (**) = E + P(\tau_k)\,\Delta t_k + (**); \tag{47}$$

here we denote by the symbol $(**)$ the sum of terms beginning with terms of the second order.

From (46) and (47) we find:

$$\Omega_{t_0}^{t} = e^{P(\tau_n)\Delta t_n} \cdots e^{P(\tau_2)\Delta t_2} e^{P(\tau_1)\Delta t_1} + (*) \tag{48}$$

and

$$\Omega_{t_0}^{t} = [E + P(\tau_n)\,\Delta t_n] \cdots [E + P(\tau_2)\,\Delta t_2]\ [E + P(\tau_1)\,\Delta t_1] + (*). \tag{49}$$

When we pass to the limit by increasing the number of intervals indefinitely and letting the length of these intervals tend to zero (the small terms $(*)$ disappear in the limit),[24] we obtain the exact limit formulas

$$\Omega_{t_0}^{t}(P) = \lim_{\Delta t_k \to 0} [e^{P(\tau_n)\Delta t_n} \cdots e^{P(\tau_2)\Delta t_2} e^{P(\tau_1)\Delta t_1}] \tag{48'}$$

and

$$\Omega_{t_0}^{t}(P) = \lim_{\Delta t_k \to 0} [E + P(\tau_n)\,\Delta t_n] \cdots [E + P(\tau_2)\,\Delta t_2]\ [E + P(\tau_1)\,\Delta t_1]. \tag{49'}$$

The expression under the limit sign on the right-hand side of the latter equation is the *product integral*.[25] We shall call its limit the *multiplicative integral* and denote it by the symbol

$$\widehat{\int}_{t_0}^{t} [E + P(t)\,dt] = \lim_{\Delta t_k \to 0} [E + P(\tau_n)\,\Delta t_n] \cdots [E + P(\tau_1)\,\Delta t_1]. \tag{50}$$

The formula (49′) gives a representation of the matricant in the form of a multiplicative integral

$$\Omega_{t_0}^{t}(P) = \widehat{\int}_{t_0}^{t} (E + P\,dt), \tag{51}$$

and the formulas (48) and (49) may be used for the approximative computation of the matricant.

---

[24] These arguments can be made more precise by an estimate of the terms we have denoted by $(*)$. For a rigorous deduction of (48′) we have to use formula 7. of § 5 in which the matricant $Q(t)$ must be replaced by a piece-wise constant matrix

$$Q(t) = P(t_k) \qquad (t_{k-1} \leqq t \leqq t_k;\ k = 1, 2, \ldots, n).$$

[25] An analogue to the sum integral for the ordinary integral.

The multiplicative integral was first introduced by Volterra in 1887. On the basis of this concept Volterra developed an original infinitesimal calculus for matrix functions (see [63]).[26]

The whole peculiarity of the multiplicative integral is tied up with the fact that the various values of the matrix function $P(t)$ in subintervals are not permutable. In the very special case when all these values are permutable

$$P(t')\,P(t'') = P(t'')\,P(t') \qquad (t', t'' \in (t_0, t)),$$

the multiplicative integral, as is clear from (48′) and (51), reduces to the matrix

$$e^{\int_{t_0}^{t} P(t)\,dt}$$

2. We now introduce the *multiplicative derivative*

$$D_t X = \frac{dX}{dt} X^{-1}. \tag{52}$$

The operations D_t and $\widehat{\int}_{t_0}^{t}$ are mutually inverse:

If

$$D_t X = P,$$

then[27]

$$X = \widehat{\int}_{t_0}^{t} (E + P\,dt) \cdot C \qquad (C = X(t_0)),$$

and vice versa. The last formula can also be written as follows:[28]

$$\widehat{\int}_{t_0}^{t} (E + P\,dt) = X(t)\,X(t_0)^{-1}. \tag{53}$$

We leave it to the reader to verify the following differential and integral formulas:[29]

---

[26] The multiplicative integral (in German, *Produkt-Integral*) was used by Schlesinger in investigating systems of linear differential equations with analytic coefficients [49] and [50]; see also [321].

The multiplicative integral (50) exists not only for a function $P(t)$ that is continuous in the interval of integration, but also under considerably more general conditions (see [116]).

[27] Here the arbitrary constant matrix C is an analogue to the arbitrary additive constant in the ordinary indefinite integral.

[28] An analogue to the formula $\int_{t_0}^{t} P dt = X(t) - X(t_0)$, where $\frac{dX}{dt} = P$.

[29] These formulas can be deduced immediately from the definitions of the multiplicative derivative and multiplicative integral (see [63]). However, the integral formulas are obtained more quickly and simply if the multiplicative integral is regarded as a matricant and the properties of the matricant that were expounded in the preceding section are used (see [49]).

DIFFERENTIAL FORMULAS

$$\begin{aligned}
\text{I. } & D_t(XY) = D_t(X) + XD_t(Y)X^{-1}, \\
& D_t(XC) = D_t(X), \\
& D_t(CY) = CD_t(Y)C^{-1}. \quad (C \text{ is a constant matrix}) \\
\text{II. } & D_t(X^{\top}) = X^{\top}(D_tX)^{\top}X^{\top -1}. \\
\text{III. } & D_t(X^{-1}) = -X^{-1}D_t(X)X = -(D_t(X^{\top}))^{\top}, \\
& D_t((X^{\top})^{-1}) = -(D_t(X))^{\top}.
\end{aligned}$$

INTEGRAL FORMULAS

$$\begin{aligned}
\text{IV. } & \widehat{\int}_{t_0}^{t}(E + P\,d\tau) = \widehat{\int}_{t_1}^{t}(E + P\,d\tau)\widehat{\int}_{t_0}^{t_1}(E + P\,d\tau). \\
\text{V. } & \widehat{\int}_{t_0}^{t}(E + P\,d\tau) = \Big[\widehat{\int}_{t}^{t_0}(E + P\,d\tau)\Big]^{-1}. \\
\text{VI. } & \widehat{\int}_{t_0}^{t}(E + CPC^{-1}\,d\tau) = C\widehat{\int}_{t_0}^{t}(E + P\,d\tau)C^{-1} \ (C \text{ is a constant matrix}) \\
\text{VII. } & \widehat{\int}_{t_0}^{t}[E + (Q + D_tX)\,d\tau] = X(t)\widehat{\int}_{t_0}^{t}(E + X^{-1}QX\,d\tau)\,X(t_0)^{-1}.^{37} \\
\text{VIII. } & \operatorname{mod}\Big[\widehat{\int}_{t_0}^{t}(E + P\,d\tau) - \widehat{\int}_{t_0}^{t}(E + Q\,d\tau)\Big] \leqq \frac{1}{n}e^{nq(t-t_0)}\left(e^{nd(t-t_0)} - 1\right)I\ (t > t_0),
\end{aligned}$$

if

$$\operatorname{mod} Q \leqq q \cdot I, \qquad \operatorname{mod}(P - Q) \leqq d \cdot I, \qquad I = \| 1 \|$$

(q and d are non-negative numbers; n is the order of P and Q).

Suppose now that the matrices P and Q depend on the same parameter α

$$P = P(\tau, \alpha), \qquad Q = Q(\tau, \alpha)$$

and that

$$\lim_{\alpha \to \alpha_0} P(\tau, \alpha) = \lim_{\alpha \to \alpha_0} Q(\tau, \alpha) = P_0(\tau),$$

where the limit is approached uniformly with respect to τ in the interval (t_0, t) in question. Furthermore, let us assume that for $\alpha \to \alpha_0$ the matrix $Q(\tau, \alpha)$ is bounded in modulus by qI, where q is a positive constant. Then, setting

$$\lim_{\alpha \to \alpha_0} d(\alpha) = 0,$$

we have:

$$d(\alpha) = \max_{\substack{1 \leqq i,k \leqq n \\ t_0 \leqq \tau \leqq t}} |p_{ik}(\tau, \alpha) - q_{ik}(\tau, \alpha)|.$$

---

[31] The formula VII can be regarded in a certain sense as a analogue to the formula for integration by parts in ordinary (non-multiplicative) integrals. VII follows from 2. of § 5).

Therefore it follows from formula VIII that:

$$\lim_{\alpha \to \alpha_0}\Big[\widehat{\int}_{t_0}^{t}(E + P\,dt) - \widehat{\int}_{t_0}^{t}(E + Q\,dt)\Big] = O.$$

In particular, if Q does not depend on α $(Q(t, \alpha) = P_0(t))$, we obtain:

$$\lim_{\alpha \to \alpha_0}\widehat{\int}_{t_0}^{t}[E + P(t, \alpha)\,dt] = \widehat{\int}_{t_0}^{t}[E + P_0(t)\,dt],$$

where

$$P_0(t) = \lim_{\alpha \to \alpha_0} P(t, \alpha).$$

§ 7. Differential Systems in a Complex Domain. General Properties

1. We consider a system of differential equations

$$\frac{dx_i}{dz} = \sum_{k=1}^{n} p_{ik}(z)\,x_k. \tag{54}$$

Here the given function $p_{ik}(z)$ and the unknown functions $x_i(z)$ $(i, k = 1, 2, \ldots, n)$ are supposed to be single-valued analytic functions of a complex argument z, regular in a domain G of the complex z-plane.

Introducing the square matrix $P(z) = \| p_{ik}(z) \|_1^n$ and the column matrix $x = (x_1, x_2, \ldots, x_n)$, we can write the system (54), as in the case of a real argument (§ 1), in the form

$$\frac{dx}{dz} = P(z)\,x \tag{54'}$$

Denoting an integral matrix, i.e., a matrix whose columns are n linearly independent solutions of (54), by X, we can write instead of (54'):

$$\frac{dX}{dz} = P(z)\,X \tag{55}$$

Jacobi's formula holds also for a complex argument z:

$$|X| = ce^{\int_{z_0}^{z} \operatorname{tr} P\,dz} \tag{56}$$

Here it is assumed that z_0 and all the points of the path along which $\int_{z_0}^{z}$ is taken are regular points for the single-valued analytic function $\operatorname{tr} P(z) = p_{11}(z) + p_{22}(z) + \ldots + p_{nn}(z)$.[32]

---

[32] Here, and in what follows, the path of integration is taken as a sectionally smooth curve.

2. A peculiar feature of the case of a complex argument is the fact that for a single-valued function $P(z)$ the integral matrix $X(z)$ may well be a many-valued function of z.

As an example, we consider the Cauchy system

$$\frac{dX}{dz} = \frac{U}{z-a} X \qquad (U \text{ is a constant matrix}). \tag{57}$$

One of the solutions of this system, as in the case of a real argument (see p. 115), is the integral matrix

$$X = e^{U \ln (z-a)} = (z-a)^U. \tag{58}$$

For the domain G we take the whole z-plane except the point $z = a$. All the points of this domain are regular points of the coefficient matrix

$$P(z) = \frac{U}{z-a}.$$

If $U \neq O$, then $z = a$ is a singular point (a pole of the first order) of the matrix function $P(z) = U/(z-a)$.

An element of the integral matrix (58) after going around the point $z = a$ once in the positive direction returns with a new value which is obtained from the old one by multiplication on the right by the constant matrix

$$V = e^{2\pi i U}$$

In the general case of a system (55) we see, by the same reasoning as in the case of a real argument, that two single-valued solutions X and $\tilde{X}$ are always connected in some part of the domain G by the formula

$$X = \tilde{X} C,$$

where C is a constant matrix. This formula remains valid under any analytic continuation of the functions $X(z)$ and $\tilde{X}(z)$ in G.

The proof of the theorem on the existence and (for given initial values) uniqueness of the solution of (54) is similar to that of the real case.

Let us consider a simply-connected star domain G_1 (relative to z_0)[33] forming part of G and let the matrix function $P(z)$ be regular[34] in G_1. We form the series

$$E + \int_{z_0}^{z} P(\zeta)\, d\zeta + \int_{z_0}^{z} P(\zeta) \int_{z_0}^{\zeta} P(\zeta')\, d\zeta'\, d\zeta + \cdots. \tag{59}$$

---

[33] A domain is called a *star domain relative to a point* z_0 if every segment joining z_0 to an arbitrary point z of the domain lies entirely in the given domain.

[34] I.e., all the elements $p_{ik}(z)$ $(i, k = 1, 2, \ldots, n)$ of $P(z)$ are regular functions in G_1.

Since G_1 is simply-connected, it follows that every integral that occurs in (59) is independent of the path of integration and is a regular function in G_1. Since G_1 is a star domain relative to z_0, we may assume for the purpose of an estimate of the moduli of these integrals that they are all taken along the straight-line segment joining z_0 and z.

That the series (59) converges absolutely and uniformly in every closed part of G_1 containing z_0 follows from the convergence of the majorant

$$1 + lM + \frac{n}{2!} l^2 M^2 + \frac{n^2}{3!} l^3 M^3 + \cdots.$$

Here M is an upper bound for the modulus of $P(z)$ and l an upper bound for the distance of z from z_0, and both bounds refer to the closed part of G_1 in question.

By differentiating term by term we verify that the sum of the series (59) is a solution of (55). This solution is normalized, because for $z = z_0$ it reduces to the unit matrix E. The single-valued normalized solution of (55) will be called, as in the real case, a matricant and will be denoted by $\Omega_{z_0}^{z}(P)$. Thus we have obtained a representation of the matricant in G_1 in the form of a series[35]

$$\Omega_{z_0}^{z}(P) = E + \int_{z_0}^{z} P(\zeta)\, d\zeta + \int_{z_0}^{z} P(\zeta) \int_{z_0}^{\zeta} P(\zeta')\, d\zeta'\, d\zeta + \cdots. \tag{60}$$

The properties 1.-4. *of the matricant that were set up in* § 5 *automatically carry over to the case of a complex argument.*

Any solution of (55) that is regular in G and reduces to the matrix X_0 for $z = z_0$ can be represented in the form

$$X = \Omega_{z_0}^{z}(P) \cdot C \qquad (C = X_0). \tag{61}$$

The formula (61) comprises all single-valued solutions that are regular in a neighborhood of z_0 (z_0 is a regular point of the coefficient matrix $P(z)$). These solutions when continued analytically in G give all the solutions of (55); i.e., the equation (55) cannot have any solutions for which z_0 would be a singular point.

For the analytic continuation of the matricant in G it is convenient to use the multiplicative integral.

---

[35] Our proof for the existence of a normalized solution and its representation in G_1 by the series (60) remains valid if instead of the assumption that the domain is a star domain we make a wider assumption, namely, that for every closed part of G_1 there exists a positive number l such that every point z of this closed part can be joined to z_0 by a path of length not exceeding l.

§ 8. The Multiplicative Integral in a Complex Domain

1. The multiplicative integral along a curve in the complex plane is defined in the following way.

Suppose that L is some path and $P(z)$ a matrix function, continuous on L. We divide the path L into n parts $(z_0, z_1), (z_1, z_2), \ldots, (z_{n-1}, z_n)$; here z_0 is the beginning, and $z_n = z$ the end of the path, and $z_1, z_2, \ldots, z_{n-1}$ are intermediate points of division. On the segment $z_{k-1}z_k$ we take an arbitrary point ζ_k and we use the notation $\Delta z_k = z_k - z_{k-1}$ $(k = 1, 2, \ldots, n)$. We then define

$$\widehat{\int_L} [E + P(z)\,dz] = \lim_{\Delta z_k \to 0} (E + P(\zeta_n)\,\Delta z_n] \cdots [E + P(\zeta_1)\,\Delta z_1].$$

When we compare this definition with that on p. 132, we see that they coincide in the special case where L is a segment of the real axis. However, even in the general case, where L is located anywhere in the complex plane, the new definition may be reduced to the old one by a change of the variable of integration.

If

$$z = z(t)$$

is a parametric equation of the path, where $z(t)$ is a continuous function in the interval (t_0, t) with a piece-wise continuous derivative $\frac{dz}{dt}$, then it is easy to see that

$$\widehat{\int_L} [E + P(z)\,dz] = \widehat{\int_{t_0}^{t}} \left\{E + P[z(t)]\,\frac{dz}{dt}\,dt\right\}.$$

This formula shows that the multiplicative integral along an arbitrary path exists if the matrix $P(z)$ under the integral sign is continuous along this path.[36]

2. The multiplicative derivative is defined by the previous formula

$$D_z X = \frac{dX}{dz} X^{-1}.$$

Here it is assumed that $X(z)$ is an analytic function.

All the differential formulas (I-III) of the preceding section carry over without change to the case of a complex argument. As regards the integral formulas IV-VI, their outward form has to be modified somewhat:

---

[36] See footnote 26. Even when $P(z)$ is continuous along L, the function $P[z(t)]\frac{dz}{dt}$ may only be sectionally continuous. In this case we can split the interval (t_0, t) into partial intervals in each of which the derivative $\frac{dz}{dt}$ is continuous and can interpret the integral from t_0 to t as the sum of the integrals along these partial intervals.

$$\text{IV}'.\ \widehat{\int_{(L'+L'')}} (E + P\,dz) = \widehat{\int_{L''}} (E + P\,dz) \widehat{\int_{L'}} (E + P\,dz).$$

$$\text{V}'.\ \widehat{\int_{-L}} (E + P\,dz) = \left[\widehat{\int_{L}} (E + P\,dz)\right]^{-1}.$$

$$\text{VI}'.\ \widehat{\int_{L}} (E + CPC^{-1}\,dz) = C \widehat{\int_{L}} (E + P\,dz)\,C^{-1}\ (C \text{ is a constant matrix}).$$

In IV′ we have denoted by $L' + L''$ the composite path that is obtained by traversing first L' and then L''. In V′, $-L$ denotes the path that differs from L only in direction.

The formula VII now assumes the form

$$\text{VII}'.\ \widehat{\int_{L}} [E + (Q + D_z X)\,dz] = X(z) \widehat{\int_{L}} (E + X^{-1}QX\,dz)\,X(z_0)^{-1}.$$

Here $X(z_0)$ and $X(z)$ on the right-hand side denote the values of $X(z)$ at the beginning and at the end of L, respectively.

Formula VIII is now replaced by the formula

$$\text{VIII}'.\ \text{mod}\ \left[\widehat{\int_{L}} (E + P\,dz) - \widehat{\int_{L}} (E + Q\,dz)\right] \leqq \frac{1}{n} e^{nql} (e^{nd \cdot l} - 1)\,I,$$

where mod $Q \leqq qI$, mod $(P - Q) \leqq d \cdot I$, $I = \| 1 \|$, and l is the length of L.

VIII′ is easily obtained from VIII if we make a change of variable in the latter and take as the new variable of integration the arc-length s along L (with $\left|\frac{dz}{ds}\right| = 1$).

3. As in the case of a real argument, there exists a close connection between the multiplicative integral and the matricant.

Suppose that $P(z)$ is a single-valued analytic matrix function, regular in G, and that G_0 is a simply-connected domain containing z_0 and forming part of G. Then the matricant $\Omega_{z_0}^{z}(P)$ is a regular function of z in G_0.

We join the points z_0 and z by an arbitrary path L lying entirely in G_0 and we choose on L intermediate points $z_1, z_2, \ldots, z_{n-1}$. Then, using the equation

$$\Omega_{z_0}^{z} = \Omega_{z_{n-1}}^{z} \cdots \Omega_{z_1}^{z_2} \Omega_{z_0}^{z_1},$$

and proceeding to the limit exactly as in § 6 (p. 132), we obtain:

$$\Omega_{z_0}^{z}(P)=\overset{\frown}{\int\limits_{L}}(E+P\,dz)=\overset{\frown}{\int}_{z_0}^{z}(E+P\,dz). \tag{62}$$

From this formula it is clear that the multiplicative integral depends not on the form of the path, but only on the initial point and the end point if the whole path of integration lies in the simply-connected domain G_0 within which the integrand $P(z)$ is regular. In particular, for a closed contour L in G_0, we have:

$$\overset{\frown}{\oint}(E+P\,dz)=E. \tag{63}$$

This formula is an analogue to Cauchy's well-known theorem according to which the ordinary (non-multiplicative) integral along a closed contour is zero if the contour lies in a simply-connected domain within which the integrand is regular.

4. The representation of the matricant in the form of the multiplicative integral (62) can be used for the analytic continuation of the matricant along an arbitrary path L in G. In this case the formula

$$X=\overset{\frown}{\int}_{z_0}^{z}(E+P\,dz)\,X_0 \tag{64}$$

gives all those branches of the many-valued integral matrix X of the differential equation $\frac{dX}{dz}=PX$ that for $z=z_0$ reduce to X_0 on one of the branches. The various branches are obtained by taking account of the various paths joining z_0 and z.

By Jacobi's formula (56)

$$|X|=|X_0|\,e^{\int\limits_{z_0}^{z}\operatorname{tr}P\,dz}$$

and, in particular, for $X_0=E$,

$$\left|\overset{\frown}{\int}_{z_0}^{z}(E+P\,dz)\right|=e^{\int\limits_{z_0}^{z}\operatorname{tr}P\,dz} \tag{65}$$

From this formula it follows that the multiplicative integral is always a non-singular matrix provided only that the path of integration lies entirely in a domain in which $P(z)$ is regular.

If L is an arbitrary closed path in G and G is not a simply-connected domain, then (63) cannot hold. Moreover, the value of the integral

$$\overset{\frown}{\oint}(E+P\,dz)$$

is not determined by specification of the integrand and the closed path of integration L but also depends on the choice of the initial point of integration z_0 on L. For let us take on the closed curve L two points z_0 and z_1 and let us denote the portions of the path from z_0 to z_1 and from z_1 to z_0 (in the direction of integration) by L_1 and L_2, respectively. Then, by the formula IV′,[37]

$$\overset{\frown}{\oint\limits_{z_0}}=\overset{\frown}{\int\limits_{L_2}}\cdot\overset{\frown}{\int\limits_{L_1}},\qquad \overset{\frown}{\oint\limits_{z_1}}=\overset{\frown}{\int\limits_{L_1}}\cdot\overset{\frown}{\int\limits_{L_2}}$$

and therefore

$$\overset{\frown}{\oint\limits_{z_1}}=\overset{\frown}{\int\limits_{L_1}}\cdot\overset{\frown}{\oint\limits_{z_0}}\cdot\overset{\frown}{\int\limits_{L_1}}^{-1}. \tag{66}$$

The formula (66) shows that the symbol $\overset{\frown}{\oint}(E+Pdz)$ determines a certain matrix to within a similarity transformation, i.e., determines only the elementary divisors of that matrix.

We consider an element $X(z)$ of the solution (64) in a neighborhood of z_0. Let L be an arbitrary closed path in G beginning and ending at z_0. After analytic continuation along L the element $X(z)$ goes over into an element $\tilde{X}(z)$. But the new element $\tilde{X}(z)$ satisfies the same differential equation (55), since $P(z)$ is a single-valued function in G. Therefore

$$\tilde{X}=XV,$$

where V is a non-singular constant matrix. From (64) it follows that

$$\tilde{X}(z_0)=\overset{\frown}{\oint\limits_{z_0}}(E+P\,dz)\,X_0.$$

Comparing this equation with the preceding one, we find:

$$V=X_0^{-1}\overset{\frown}{\oint\limits_{z_0}}(E+P\,dz)\,X_0. \tag{67}$$

In particular, for the matricant $X=\Omega_{z_0}^{z}$, we have $X_0=E$, and then

$$V=\overset{\frown}{\oint\limits_{z_0}}(E+P\,dz). \tag{68}$$

---

[37] To simplify the notation we have omitted the expression to be integrated, $E+Pdz$, which is the same for all the integrals.

§ 9. Isolated Singular Points

1. We shall now deal with the behavior of a solution (an integral matrix) in a neighborhood of an isolated singular point a.

Let the matrix function $P(z)$ be regular for the values of z satisfying the inequality

$$0 < | z - a | < R.$$

The set of these values forms a doubly-connected domain G. The matrix function $P(z)$ has in G an expansion in a Laurent series

$$P(z) = \sum_{n=-\infty}^{+\infty} P_n (z-a)^n. \tag{69}$$

An element $X(z)$ of the integral matrix, after going once around a in the positive direction along a path L, goes over into an element

$$X^+(z) = X(z)\, V\,,$$

where V is a constant non-singular matrix.

Let U be the constant matrix that is connected with V by the relation

$$V = e^{2\pi i U}. \tag{70}$$

Then the matrix function $(z-a)^U$ after going around a along L goes over into $(z-a)^U V$. Therefore the matrix function

$$F(z) = X(z)\,(z-a)^{-U}\,, \tag{71}$$

which is analytic in G, goes over into itself (remains unchanged) by analytic continuation along L.[38] Therefore the matrix function $F(z)$ is regular in G and can be expanded in G in a Laurent series

$$F(z) = \sum_{n=-\infty}^{+\infty} F_n (z-a)^n. \tag{72}$$

From (71) it follows that:

$$X(z) = F(z)\,(z-a)^U. \tag{73}$$

Thus every integral matrix $X(z)$ can be represented in the form (73), where the single-valued function $F(z)$ and the constant matrix U depend on

the coefficient matrix $P(z)$. However, the algorithmic determination of U and of the coefficients F_n in (72) from the coefficients P_n in (69) is, in general, a complicated task.

A special case of the problem, where

$$P(z) = \sum_{n=-1}^{\infty} P_n (z-a)^n$$

will be analyzed completely in § 10. In this case, the point a is called a *regular singularity* of the system (55).

If the expansion (69) has the form

$$P(z) = \sum_{n=-q}^{\infty} P_n (z-a)^n \qquad (q > 1;\ \ P_{-q} \neq O)$$

then a is called an *irregular singularity of the type of a pole.* Finally, if there is an infinity of non-zero matrix coefficients P_n with negative powers of $z-a$ in (69), then a is called an *essential singularity* of the given differential system.

From (73) it follows that under an arbitrary single circuit in the positive direction (along some closed path L) an integral matrix $X(z)$ is multiplied on the right by one and the same matrix

$$V = e^{2\pi i U}$$

If this circuit begins (and ends) at z_0, then by (67)

$$V = X(z_0)^{-1} \overset{\curvearrowleft}{\oint_{z_0}} (E + P\,dz)\, X(z_0)\,. \tag{74}$$

If instead of $X(z)$ we consider any other integral matrix $\hat{X}(z) = X(z)C$ (C is a constant matrix; $|C| \neq 0$), then, as is clear from (74), V is replaced by the similar matrix

$$\hat{V} = C^{-1} V C$$

Thus, the 'integral substitutions' V of the given system form a class of similar matrices.

From (74) it also follows that the integral

$$\overset{\curvearrowleft}{\oint_{z_0}} (E + P\,dz) \tag{75}$$

is determined by the initial point z_0 and does not depend on the form of the

---

[38] Hence it follows that when z traverses any other closed path in G, the function $F(z)$ returns to its original value.

curved path.[39] If we change the point z_0, then the various values of the integral that are so obtained are similar.[40]

These properties of the integral (75) can also be confirmed directly. For let L and L' be two closed paths in G around $z=a$ with the initial points z_0 and z_0' (see Fig. 6).

The doubly-connected domain between L and L' can be made simply-connected by introducing the cut from z_0 to z_0'. The integral along the cut will be denoted by

$$T=\widehat{\int}_{z_0}^{z_0'}(E+P\,dz).$$

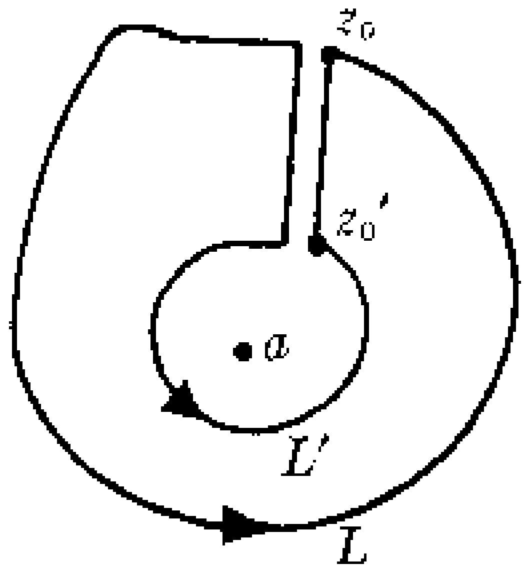

Fig. 6

Since the multiplicative integral along a closed contour of a simply-connected domain is E, we have

$$\widehat{\int_{L'}}\,T\,\widehat{\int_{L}}^{-1}T^{-1}=E;$$

hence

$$\widehat{\int_{L'}}=T\,\widehat{\int_{L}}\,T^{-1}.$$

Thus, the integral $\widehat{\oint}(E+Pdz)$, like V, is determined to within similarity, and we shall occasionally write (74) in the form

$$V\sim\widehat{\oint}(E+P\,dz);$$

meaning that the elementary divisors of the matrices on the left-hand and right-hand sides of the equation coincide.

2. As an example, we consider a system with a regular singularity

$$\frac{dX}{dz}=P(z)\,X$$

where

$$P(z)=\frac{P_{-1}}{z-a}+\sum_{n=0}^{+\infty}P_n(z-a)^n.$$

Let

$$Q(z)=\frac{P_{-1}}{z-a}.$$

---

[39] Under the condition, of course, that the path of integration goes around a once in the positive direction.

[40] This follows from (74), or from (66).

Using the formula VIII′ of the preceding section, we estimate the modulus of the difference

$$D=\widehat{\oint}(E+P\,dz)-\widehat{\oint}(E+Q\,dz)\,, \tag{76}$$

taking as path of integration a circle of radius r $(r<R)$ in the positive direction. Then with

$$\operatorname{mod}P_{-1}\leqq p_{-1}I\,,\qquad \operatorname*{mod}_{|z-a|=r}\sum_{n=0}^{\infty}P_n(z-a)^n\leqq d(r)\,I\,,\qquad I=\|1\|;$$

we set in VIII′:

$$q=\frac{p_{-1}}{r},\quad d=d(r),\quad l=2\pi r$$

and then obtain

$$\operatorname{mod}D\leqq\frac{1}{n}e^{2\pi p_{-1}}(e^{2\pi nrd(r)}-1)\,I\,.$$

Hence it is clear that[41]

$$\lim_{r\to 0}D=O. \tag{77}$$

On the other hand, the system

$$\frac{dY}{dz}=QY$$

is a Cauchy system, and in that case we have for an arbitrary choice of the initial point z_0 and for every $r<R$

$$\widehat{\oint_{z_0}}(E+Q\,dz)=e^{2\pi iP_{-1}}.$$

Therefore it follows from (76) and (77) that:

$$\lim_{r\to 0}\widehat{\oint_{z_0}}(E+P\,dz)=e^{2\pi iP_{-1}}. \tag{78}$$

But the elementary divisors of the integral $\widehat{\oint_{z_0}}(E+Pdz)$ do not depend on z_0 and r and coincide with those of the integral substitution V.

From this Volterra in his well-known memoir (see [374]) and his book [63] (pp. 117-120) deduces that the matrices V and $e^{2\pi iP_{-1}}$ are similar, so that the integral substitution V is determined to within similarity by the 'residue' matrix P_{-1}.

But this assertion of Volterra is incorrect.

---

[41] Here we have used the fact that for a suitable choice of $d(r)$

$$\lim_{r\to 0}d(r)=d_0,$$

where d_0 is the greatest of the moduli of the elements of P_0.

From (74) and (78) we can only deduce that *the characteristic values of the integral substitution V coincide with those of the matrix* $e^{2\pi i P_{-1}}$. However, the elementary divisors of these matrices may be distinct. For example, for every $r \neq 0$ the matrix

$$\begin{Vmatrix} \alpha & r \\ 0 & \alpha \end{Vmatrix}$$

has one elementary divisor $(\lambda - \alpha)^2$, but the limit of the matrix for $r \to 0$, i.e., the matrix $\begin{Vmatrix} \alpha & 0 \\ 0 & \alpha \end{Vmatrix}$, has two elementary divisors $\lambda - \alpha$, $\lambda - \alpha$.

Thus, Volterra's assertion does not follow from (74) and (78). It is not even true in general, as the following example shows.

Let

$$P(z) = \begin{Vmatrix} 0 & 0 \\ 0 & -1 \end{Vmatrix} \frac{1}{z} + \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}.$$

The corresponding system of differential equations has the form:

$$\frac{dx_1}{dz} = x_2, \qquad \frac{dx_2}{dz} = -\frac{x_2}{z}.$$

Integrating the system we find:

$$x_1 = c \ln z + d, \qquad x_2 = \frac{c}{z}.$$

The integral matrix

$$X(z) = \begin{Vmatrix} \ln z & 1 \\ z^{-1} & 0 \end{Vmatrix}$$

when the singular point $z = 0$ is encircled once in the positive direction, is multiplied on the right by the matrix

$$V = \begin{Vmatrix} 1 & 0 \\ 2\pi i & 1 \end{Vmatrix}.$$

This matrix has one elementary divisor $(\lambda - 1)^2$. At the same time the matrix

$$e^{2\pi i P_{-1}} = e^{2\pi i \begin{Vmatrix} 0 & 0 \\ 0 & -1 \end{Vmatrix}} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} = E$$

has two elementary divisors $\lambda - 1$, $\lambda - 1$.

3. We now consider the case where the matrix $P(z)$ has a finite number of negative powers of $z - a$ (a is a regular or irregular singularity of the type of a pole):

$$P(z) = \frac{P_{-q}}{(z-a)^q} + \cdots + \frac{P_{-1}}{z-a} + \sum_{n=0}^{\infty} P_n (z-a)^n \qquad (q \geqq 1;\ P_{-q} \neq O).$$

We transform the given system

$$\frac{dX}{dz} = PX \tag{79}$$

by setting

$$X = A(z)\,Y, \tag{80}$$

where $A(z)$ is a matrix function that is regular at $z = 0$ and assumes there the value E:

$$A(z) = E + A_1(z-a) + A_2(z-a)^2 + \cdots;$$

the power series on the right-hand side converges for $|z - a| < r_1$.

The well-known American mathematician G. D. Birkhoff has published a theorem in 1913 (see [117]) according to which the transformation (80) can always be chosen such that the coefficient matrix of the transformed system

$$\frac{dY}{dz} = P^*(z)\,Y \tag{79'}$$

contains only negative powers of $z - a$:

$$P^*(z) = \frac{P^*_{-q}}{(z-a)^q} + \cdots + \frac{P^*_{-1}}{z-a}.$$

Birkhoff's theorem with its complete proof is reproduced in the book *Ordinary Differential Equations,* by E. L. Ince.[42] Moreover, on the basis of these 'canonical' systems (79′) he investigates the behavior of the solution of an arbitrary system in the neighborhood of a singular point.

Nevertheless, *Birkhoff's proof contains an error, and the theorem is not true.* As a counter-example we can take the same example by which we have above refuted Volterra's claim.[43]

In this example $q = 1$, $a = 0$ and

$$P_{-1} = \begin{Vmatrix} 0 & 0 \\ 0 & -1 \end{Vmatrix}, \quad P_0 = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \quad P_n = O \quad \text{for } n = 1, 2, \ldots.$$

---

[42] See [20], pp. 632-41. Birkhoff and Ince formulate the theorem for the singular point $z = \infty$. This is no restriction, because every singular point $z = a$ can be carried by the transformation $z' = 1/(z-a)$ into $z' = \infty$.

[43] In the case $q = 1$ the erroneous statement of Birkhoff coincides in essence with Volterra's mistake (see p. 145).

Applying Birkhoff's theorem and substituting in (79) the product AY for X in (79), we obtain after replacing $\frac{dY}{dz}$ by $\frac{P^*_{-1}}{z}$ and cancelling Y:

$$A\frac{P^*_{-1}}{z}+\frac{dA}{dz}=PA.$$

Equating the coefficients of $1/z$ and of the free terms we find:

$$P^*_{-1}=P_{-1}, \quad A_1P_{-1}-P_{-1}A_1+A_1=P_0.$$

Setting

$$A_1=\begin{Vmatrix} a & b \\ c & d \end{Vmatrix},$$

we obtain:

$$\begin{Vmatrix} a & 0 \\ c & 0 \end{Vmatrix}-\begin{Vmatrix} 0 & 0 \\ -c & -d \end{Vmatrix}=\begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}.$$

This is a contradictory equation.

In the following section we shall examine, for the case of a regular singularity, what canonical form the system (79) can be transformed into by means of a transformation (80).

§ 10. Regular Singularities

In studying the behavior of a solution in a neighborhood of a singular point we can assume without loss of generality that the singular point is $z=0$.[44]

1. Let the given system be

$$\frac{dX}{dz}=P(z)X, \tag{81}$$

where

$$P(z)=\frac{P_{-1}}{z}+\sum_{m=0}^{\infty}P_mz^m \tag{82}$$

and the series $\sum\limits_{m=0}^{\infty}P_mz^m$ converges in the circle $|z|<r$.

We set

$$X=A(z)Y, \tag{83}$$

where

$$A(z)=E+A_1z+A_2z^2+\cdots. \tag{84}$$

---

[44] By the transformation $z'=z-a$ or $z'=1/z$ every finite point $z=a$ or $z=\infty$ can be carried into $z'=0$.

Leaving aside for the time being the problem of convergence of the series (84), let us try to determine the matrix coefficients A_m such that the transformed system

$$\frac{dY}{dz}=P^*(z)Y, \tag{85}$$

where

$$P^*(z)=\frac{P^*_{-1}}{z}+\sum_{m=0}^{\infty}P^*_mz^m \tag{86}$$

is of the simplest possible ('canonical') form.[45]

When we substitute the product AY for X in (81) and use (85), we obtain:

$$A(z)P^*(z)Y+\frac{dA}{dz}Y=P(z)A(z)Y.$$

Multiplying both sides of the equation by Y^{-1} on the right we find:

$$P(z)A(z)-A(z)P^*(z)=\frac{dA}{dz}.$$

When we replace here $P(z)$, $A(z)$, and $P^*(z)$ by the series (82), (84), and (86) and equate the coefficients of equal powers of z on the two sides, we obtain an infinite system of matrix equations for the unknown coefficients $A_1, A_2, \ldots$:[46]

$$\left.\begin{array}{rl} 1. & P_{-1}=P^*_{-1}, \\ 2. & P_{-1}A_1-A_1(P_{-1}+E)+P_0=P^*_0, \\ 3. & P_{-1}A_2-A_2(P_{-1}+2E)+P_0A_1-A_1P^*_0+P_1=P^*_1, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (m+2). & P_{-1}A_{m+1}-A_{m+1}[P_{-1}+(m+1)E]+ \\ & +P_0A_m-A_mP^*_0+P_1A_{m-1}-A_{m-1}P^*_1+\cdots+P_m=P^*_m. \end{array}\right\} \tag{87}$$

2. We consider several cases separately:

1. *The matrix P_{-1} does not have distinct characteristic values that differ from each other by an integer.*

---

[45] We shall aim at having only a finite number (and indeed the smallest possible number) of non-zero coefficients P^*_m in (86).

[46] In all the equations beginning with the second we replace P^*_{-1} by P_{-1} in accordance with the first equation.

In this case the matrices P_{-1} and $P_{-1}+kE$ do not have characteristic values in common for any $k=1, 2, 3, \ldots$, and therefore (see Vol. I, Chapter VIII, § 3)[47] the matrix equation

$$P_{-1}U - U(P_{-1}+kE) = T$$

has one and only one solution for an arbitrary right-hand side T.

We shall denote this solution by

$$\Phi_k(P_{-1}, T).$$

We can therefore set all the matrices P_m^* $(m=0, 1, 2, \ldots)$ in (87) equal to zero and determine $A_1, A_2, \ldots$ successively by means of the equation

$$A_1 = \Phi_1(P_{-1}, -P_0), \quad A_2 = \Phi_2(P_{-1}, -P_1 - P_0A_1), \ \ldots .$$

The transformed system is then a Cauchy system

$$\frac{dY}{dz} = \frac{P_{-1}}{z}\, Y,$$

and so the solution X of the original system (81) is of the form[48]

$$X = A(z)\, z^{P_{-1}}. \tag{88}$$

2. *Among the distinct characteristic values of P_{-1} there are some whose difference is an integer; furthermore, the matrix P_{-1} is of simple structure.*

We denote the characteristic values of P_{-1} by $\lambda_1, \lambda_2, \ldots, \lambda_n$ and order them in such a way that the inequalities

$$\mathrm{Re}\,(\lambda_1) \geqq \mathrm{Re}\,(\lambda_2) \geqq \ldots \geqq \mathrm{Re}\,(\lambda_n) \tag{89}$$

hold.

---

[47] However, we can also prove this without referring to Chapter VIII. The proposition in which we are interested is equivalent to the statement that the matrix equation

$$P_{-1}U = U(P_{-1}+kE) \tag{*}$$

has only the solution $U=O$. Since the matrices P_{-1} and $P_{-1}+kE$ have no characteristic values in common, there exists a polynomial $f(\lambda)$ for which

$$f(P_{-1}) = O, \; f(P_{-1}+kE) = E.$$

But from (*) it follows that

$$f(P_{-1})U = Uf(P_{-1}+kE).$$

Hence $U=O$.

[48] The formula (88) defines one integral matrix of the system (81). Every integral matrix is obtained from (88) by multiplication on the right by an arbitrary constant non-singular matrix C.

Without loss of generality we can replace P_{-1} by a similar matrix. This follows from the fact that when we multiply both sides of (81) on the left by a non-singular matrix T and on the right by T^{-1}, we in fact replace all the P_m by TP_mT^{-1} $(m=-1, 0, 1, 2, \ldots)$; moreover, X is replaced by TXT^{-1}. Therefore we may assume in this case that P_{-1} is a diagonal matrix:

$$P_{-1} = \| \lambda_i \delta_{ik} \|_1^n. \tag{90}$$

We introduce a notation for the elements of P_m, P_m^* and A_m:

$$P_m = \| p_{ik}^{(m)} \|_1^n, \quad P_m^* = \| p_{ik}^{(m)*} \|_1^n, \quad A_m = \| x_{ik}^{(m)} \|_1^n. \tag{91}$$

In order to determine A_1, we use the second equation in (87). This matrix equation can be replaced by the scalar equations

$$(\lambda_i - \lambda_k - 1)\, x_{ik}^{(1)} + p_{ik}^{(0)} = p_{ik}^{(0)*} \qquad (i, k = 1, 2, \ldots, n) \tag{92}$$

If none of the differences $\lambda_i - \lambda_k$ is 1, we can set $P_0^* = O$. We then have from (87_2) that $A_1 = \Phi_1(P_{-1} - P_0)$.[49]

In that case the elements of A_1 are uniquely determined from (92):

$$x_{ik}^{(1)} = -\frac{p_{ik}^{(0)}}{\lambda_i - \lambda_k - 1} \qquad (i, k = 1, 2, \ldots, n). \tag{93}$$

But if for some[50] i, k

$$\lambda_i - \lambda_k = 1,$$

then the corresponding $p_{ik}^{(0)*}$ is determined from (92):

$$p_{ik}^{(0)*} = p_{ik}^{(0)},$$

and the corresponding $x_{ik}^{(1)}$ can be chosen quite arbitrarily.

For those i and k for which $\lambda_i - \lambda_k \neq 1$ we set:

$$p_{ik}^{(0)*} = 0,$$

and find the corresponding $x_{ik}^{(1)}$ from (93).

Having determined A_1, we next determine A_2 from the third equation of (87). We replace this matrix equation by a system of n^2 scalar equations:

$$(\lambda_i - \lambda_k - 2)\, x_{ik}^{(2)} = p_{ik}^{(1)*} - p_{ik}^{(1)} - (P_0A_1 - A_1P_0^*)_{ik} \tag{94}$$
$$(i, k = 1, 2, \ldots, n).$$

Here we proceed exactly as in the determination of A_1.

---

[49] We use the rotation introduced in dealing with the case 1.

[50] By (89) this is only possible for $i < k$.

If $\lambda_i - \lambda_k \neq 2$, then we set:

$$p_{ik}^{(1)*} = 0;$$

and find from (94):

$$x_{ik}^{(2)} = -\frac{1}{\lambda_1 - \lambda_2 - 2}\,[p_{ik}^{(1)} - (P_0 A_1 - A_1 P_0^*)_{ik}].$$

But if $\lambda_i - \lambda_k = 2$, then it follows from (94) for these i and k that:

$$p_{ik}^{(1)*} = p_{ik}^{(1)} + (P_0 A_1 - A_1 P_0^*)_{ik}.$$

In this case $x_{ik}^{(2)}$ is chosen arbitrarily.

Continuing this process we determine all the matrices $P_{-1}^*, P_0^*, P_1^*, \ldots$ and $A_1, A_2, \ldots$ in succession.

Furthermore, only a finite number of the matrices P_m^* is different from zero and, as is easy to see, $P^*(z)$ is of the form[51]

$$P^*(z) = \left\| \begin{matrix} \frac{\lambda_1}{z} & a_{12} z^{\lambda_1 - \lambda_2 - 1} & \ldots & a_{1n} z^{\lambda_1 - \lambda_n - 1} \\ 0 & \frac{\lambda_2}{z} & \ldots & a_{2n} z^{\lambda_2 - \lambda_n - 1} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \frac{\lambda_n}{z} \end{matrix} \right\|, \tag{95}$$

where $a_{ik} = 0$, when $\lambda_i - \lambda_k$ is not a positive integer, and $a_{ik} = p_{ik}^{(\lambda_i - \lambda_k - 1)*}$, when $\lambda_i - \lambda_k$ is a positive integer.

We denote by m_i the integral part of the numbers Re λ_i:[52]

$$m_i = [\operatorname{Re}(\lambda_i)] \qquad (i = 1, 2, \ldots, n). \tag{96}$$

Then, by (89),

$$m_1 \geqq m_2 \geqq \cdots \geqq m_n.$$

If $\lambda_i - \lambda_k$ is an integer, then

$$\lambda_i - \lambda_k = m_i - m_k.$$

---

[51] P_m^* $(m \geqq 0)$ can be different from zero only when there exist characteristic values λ_i and λ_k of P_{-1} such that $\lambda_i - \lambda_k - 1 = m$ (and, by (89), $i < k$). For a given m there corresponds to each such equation an element $p_{ik}{}^{(m)*} = a_{ik}$ of the matrix P_m^*; this element may be different from zero. All the remaining elements of P_m^* are zero.

[52] I.e., m_i is the largest integer not exceeding Re λ_i $(i = 1, 2, \ldots, n)$.

Therefore in the expression (95) for the canonical matrix $P^*(z)$ we can replace all the differences $\lambda_i - \lambda_k$ by $m_i - m_k$. Furthermore, we set:

$$\tilde{\lambda}_i = \lambda_i - m_i \qquad (i = 1, 2, \ldots, n), \tag{91'}$$

$$M = \| m_i \delta_{ik} \|_1^n, \quad U = \left\| \begin{matrix} \tilde{\lambda}_1 & a_{12} & \ldots & a_{1n} \\ 0 & \tilde{\lambda}_2 & \ldots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \tilde{\lambda}_n \end{matrix} \right\|. \tag{97}$$

Then it follows from (95) (see formula I on p. 134):

$$P^*(z) = z^M \frac{U}{z} z^{-M} + \frac{M}{z} = D_z(z^M z^U).$$

Hence $Y = z^M z^U$ is a solution of (85) and

$$X = A(z)\, z^M z^U \tag{98}$$

is a solution of (81).[53]

3. *The general case.* As we have explained above, we may replace P_{-1} without loss of generality by an arbitrary similar matrix. We shall assume that P_{-1} has the Jordan normal form[54]

$$P_{-1} = \{ \lambda_1 E_1 + H_1, \lambda_2 E_2 + H_2, \ldots, \lambda_u E_u + H_u \}, \tag{99}$$

with

$$\operatorname{Re}(\lambda_1) \geqq \operatorname{Re}(\lambda_2) \geqq \cdots \geqq \operatorname{Re}(\lambda_u). \tag{100}$$

Here E denotes the unit matrix and H the matrix in which the elements of the first superdiagonal are 1 and all the remaining elements zero. The orders of the matrices E_i and H_i in distinct diagonal blocks are, in general, different; their orders coincide with the degrees of the corresponding elementary divisors of P_{-1}.[55]

In accordance with the representation (99) of P_{-1} we split all the matrices P_m, P_m^*, A_m into blocks:

---

[53] The special form of the matrices (97) corresponds to the canonical form of P_{-1}. If P_{-1} does not have the canonical form, then the matrices M and U in (98) are similar to the matrices (97).

[54] See Vol. I, Chapter VI, § 6.

[55] To simplify the notation, the index that indicates the order of the matrices is omitted from E_i and H_i.

$$P_m = (P_{ik}^{(m)})_1^u,\ P_m^* = (P_{ik}^{(m)*})_1^u,\ A_m = (X_{ik}^{(m)})_1^u.$$

Then the second of the equations (87) may be replaced by a system of equations

$$(\lambda_i E_i + H_i) X_{ik}^{(1)} - X_{ik}^{(1)} [(\lambda_k + 1) E_k + H_k] + P_{ik}^{(0)} = P_{ik}^{(0)*}, \tag{101}$$

which can also be written as follows:

$$(\lambda_i - \lambda_k - 1) X_{ik}^{(1)} + H_i X_{ik}^{(1)} - X_{ik}^{(1)} H_k + P_{ik}^{(0)} = P_{ik}^{(0)*} \quad (i, k = 1, 2, \ldots, u). \tag{102}$$

Suppose that[56]

$$X_{ik}^{(1)} = \begin{Vmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \cdots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{Vmatrix} = \| x_{st} \|,\ P_{ik}^{(0)} = \| p_{st}^{(0)} \|,\ P_{ik}^{(0)*} = \| p_{st}^{(0)*} \|.$$

Then the matrix equation (102) (for fixed i and k) can be replaced by a system of scalar equations of the form[57]

$$\begin{gathered} (\lambda_i - \lambda_k - 1) x_{st} + x_{s+1,t} - x_{s,t-1} + p_{st}^{(0)} = p_{st}^{(0)*} \\ (s = 1, 2, \ldots, v;\ t = 1, 2, \ldots, w;\ x_{v+1,t} = x_{s,0} = 0), \end{gathered} \tag{103}$$

where v and w are the orders of the matrices $\lambda_i E_i + H_i$ and $\lambda_k E_k + H_k$ in (99).

If $\lambda_i - \lambda_k \neq 1$, then in (103) we can set all the $p_{st}{}^{(0)*}$ equal to zero and determine all the x_{st} uniquely from the recurrence relations (103). This means that in the matrix equations (102) we set

$$P_{ik}^{(0)*} = O$$

and determine $X_{ik}^{(1)}$ uniquely.

If $\lambda_i - \lambda_k = 1$, then the relations (103) assume the form

$$\begin{gathered} x_{s+1,t} - x_{s,t-1} + p_{st}^{(0)} = p_{st}^{(0)*} \\ (x_{v+1,t} = x_{s,0} = 0;\ s = 1, 2, \ldots, v;\ t = 1, 2, \ldots, w). \end{gathered} \tag{104}$$

---

[56] To simplify the notation, we omit the indices i, k in the elements of the matrices X_{ik}, $P_{ik}^{(0)}$, $P_{ik}^{(0)*}$.

[57] The reader should bear in mind the properties of the matrix H that were developed on pp. 13-15 of Vol. I.

It is not difficult to show that the elements x_{st} of $X_{ik}^{(1)}$ can be determined from (104) so that the matrix $P_{ik}^{(0)*}$ has, depending on its dimensions ($v \times w$), one of the forms

$$\underset{(v = w)}{\begin{Vmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \cdot & \cdot & \ddots & \cdot \\ \cdot & & \ddots & \cdot \\ \cdot & & \ddots & \cdot \\ a_{v-1} & a_{v-2} & \cdots\ a_1 & a_0 \end{Vmatrix}},\quad \underset{(v < w)}{\begin{Vmatrix} a_0 & 0 & \cdots & 0 & 0 & \ldots & 0 \\ a_1 & a_0 & \cdots & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \ddots & & & & \cdot \\ \cdot & & \ddots & & & & \cdot \\ \cdot & & \ddots & & & & \cdot \\ a_{v-1} & & a_1 & a_0 & 0 & \ldots & 0 \end{Vmatrix}},$$

$$\underset{(v > w)}{\begin{Vmatrix} 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \\ a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \cdot & \cdot & \ddots & \cdot \\ \cdot & & \ddots & \cdot \\ \cdot & & \ddots & \cdot \\ a_{w-1} & \cdots & a_1 & a_0 \end{Vmatrix}}. \tag{105}$$

We shall say of the matrices (105) that they have the *regular lower triangular form*.[58]

From the third of the equations (87) we can determine A_2. This equation can be replaced by the system

$$\begin{gathered} (\lambda_i - \lambda_k - 2) X_{ik}^{(2)} + H_i X_{ik}^{(2)} - X_{ik}^{(2)} H_k + \{ P_0 A_1 - A_1 P_0 \}_{ik} + P_{ik}^{(1)} = P_{ik}^{(1)*} \\ (i, k = 1, 2, \ldots, u). \end{gathered} \tag{106}$$

In the same way that we determine A_1, we determine $X_{ik}^{(2)}$ uniquely with $P_{ik}^{(1)*} = O$ from (106) provided $\lambda_i - \lambda_k \neq 2$. But if $\lambda_i - \lambda_k = 2$, then $X_{ik}^{(2)}$ can be determined so that $P_{ik}^{(1)*}$ is of regular lower triangular form.

---

[58] Regular upper triangular matrices are defined similarly. The elements of $X_{ik}^{(1)}$ are not all uniquely determined from (104); there is a certain degree of arbitrariness in the choice of the elements x_{st}. This is immediately clear from (102): for $\lambda_i - \lambda_k = 1$ we may add to $X_{ik}^{(1)}$ an arbitrary matrix permutable with H, i.e., an arbitrary regular upper triangular matrix.

Continuing this process, we determine all the coefficient matrices A_1, $A_2, \ldots$ and P^*_{-1}, P^*_0 . $P^*_1, \ldots$ in succession. Only a finite number of the coefficients P^*_m is different from zero, and the matrix $P^*(z)$ has the following block form:[59]

$$P^*(z) = \begin{pmatrix} \frac{\lambda_1 E_1 + H_1}{z} & B_{12} z^{\lambda_1 - \lambda_2 - 1} & \ldots & B_{1u} z^{\lambda_1 - \lambda_u - 1} \\ O & \frac{\lambda_2 E_2 + H_2}{z} & \ldots & B_{2u} z^{\lambda_2 - \lambda_u - 1} \\ \ldots & \ldots & \ldots & \ldots \\ O & O & \ldots & \frac{\lambda_u E_u + H_u}{z} \end{pmatrix}, \tag{107}$$

where

$$B_{ik} = \begin{cases} O & \text{if } \lambda_i - \lambda_k \text{ is not a positive integer,} \\ P_{ik}^{(\lambda_i - \lambda_k - 1)*} & \text{if } \lambda_i - \lambda_k \text{ is a positive integer.} \end{cases}$$

All the matrices B_{ik} $(i, k = 1, 2, \ldots, u;\ i < k)$ are of regular lower triangular form.

As in the preceding case, we denote by m_i the integral part of Re λ_i

$$m_i = [\mathrm{Re}\,(\lambda_i)] \qquad (i = 1, 2, \ldots, u) \tag{108}$$

and we set

$$\lambda_i = m_i + \tilde{\lambda_i} \qquad (i = 1, 2, \ldots, u). \tag{108'}$$

Then in the expression (107) for $P^*(z)$ we may again replace the difference $\lambda_i - \lambda_k$ everywhere by $m_i - m_k$. If we introduce the diagonal matrix M with integer elements and the upper triangular matrix U by means of the equations[60]

$$M = (m_i E_i \delta_{ik})_1^u, \quad U = \begin{pmatrix} \tilde{\lambda}_1 E_1 + H_1 & B_{12} & \ldots & B_{1u} \\ O & \tilde{\lambda}_2 E_2 + H_2 & \ldots & B_{2u} \\ \ldots & \ldots & \ldots & \ldots \\ O & O & \ldots & \tilde{\lambda}_u E_u + H_u \end{pmatrix}, \tag{109}$$

then we easily obtain, starting from (107), the following representation of $P^*(z)$:

$$P^*(z) = z^M \frac{U}{z} \cdot z^{-M} + \frac{M}{z} = D_z(z^M z^U).$$

---

[59] The dimensions of the square matrices E_i, H_i and the rectangular matrices B_{ik} are determined by the dimensions of the diagonal blocks in the Jordan matrix P_{-1}, i.e., by the degrees of the elementary divisors of P_{-1}.

[60] Here the splitting into blocks corresponds to that of P_{-1} and $P^*(z)$.

Hence it follows that the solution (85) can be given in the form

$$Y = z^M z^U$$

and the solution of (81) can be represented as follows:

$$X = A(z)\, z^M z^U. \tag{110}$$

Here $A(z)$ is the matrix series (84), M is a constant diagonal matrix whose elements are integers, and U is a constant triangular matrix. The matrices M and U are defined by (108), (108′), and (109).[61]

3. We now proceed to prove the convergence of the series

$$A(z) = E + A_1 z + A_2 z^2 + \cdots.$$

We shall use a lemma which is of independent interest.

LEMMA: *If the series*[62]

$$x = a_0 + a_1 z + a_2 z^2 + \cdots \tag{111}$$

formally satisfies the system

$$\frac{dx}{dz} = P(z)\, x \tag{112}$$

for which $z = 0$ *is a regular singularity, then* (111) *converges in every neighborhood of* $z = 0$ *in which the expansion of the coefficient matrix* $P(z)$ *in the series* (82) *converges.*

Proof. Let us suppose that

$$P(z) = \frac{P_{-1}}{z} + \sum_{q=0}^{\infty} P_q z^q,$$

where the series $\sum_{m=0}^{\infty} P^m z_m$ converges for $|z| < r$. Then there exist positive constants p_{-1} and p such that[63]

$$\mathrm{mod}\, P_{-1} \leqq p_{-1} I, \ \ \mathrm{mod}\, P_m \leqq \frac{p}{r^m} I, \quad I = \|1\| \qquad (m = 0, 1, 2, \ldots). \tag{113}$$

Substituting the series (111) for x in (112) and comparing the coefficients of like powers on both sides of (112), we obtain an infinite system of (column) vector equations

---

[61] See footnote 53.

[62] Here $x = (x_1, x_2, \ldots, x_n)$ is a column of unknown functions; $a_0, a_1, a_2, \ldots$ are constant columns; $P(z)$ is a square coefficient matrix.

[63] For the definition of the modulus of a matrix, see p. 128.

$$\left.\begin{aligned} P_{-1}a_0 &= o, \\ (E-P_{-1})\,a_1 &= P_0a_0, \\ (2E-P_{-1})\,a_2 &= P_0a_1+P_1a_0, \\ &\dots\dots\dots\dots\dots\dots \\ (mE-P_{-1})\,a_m &= P_0a_{m-1}+P_1a_{m-2}+\cdots+P_{m-1}a_0, \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \quad (114)$$

It is sufficient to prove that every remainder of the series (111)

$$x^{(k)}=a_kz^k+a_{k+1}z^{k+1}+\cdots \qquad (115)$$

converges in a neighborhood of $z=0$. The number k is subject to the inequality

$$k>np_{-1}.$$

Then k exceeds the moduli of all the characteristic values of P_{-1},[64] so that for $m \geqq k$ we have $|\,mE-P_{-1}\,| \neq 0$ and

$$(mE-P_{-1})^{-1}=\frac{1}{m}\left(E-\frac{P_{-1}}{m}\right)^{-1}=\frac{1}{m}E+\frac{1}{m^2}P_{-1}+\frac{1}{m^3}P^2_{-1}+\cdots \qquad (116)$$
$$(m=k,k+1,\ldots).$$

In the last part of this equation there is a convergent matrix series. With the help of this series and by using (114), we can express all the coefficients of (115) in terms of $a_0, a_1, \ldots, a_{k-1}$ by means of the recurrence relations

$$a_m=\left(\frac{1}{m}E+\frac{1}{m^2}P_{-1}+\frac{1}{m^3}P^2_{-1}+\cdots\right)(f_{m-1}+P_0a_{m-1}+\cdots+P_{m-k-1}a_k),$$
$$(m=k,k+1,\ldots) \qquad (117)$$

where

$$f_{m-1}=P_{m-k}a_{k-1}+\cdots+P_{m-1}a_0 \qquad (m=k,\,k+1,\ldots). \qquad (118)$$

Note that this series (115) formally satisfies the differential equation

---

[64] If λ_0 is a characteristic value of $A=\|a_{ik}\|_n^1$, then $|\,\lambda_0\,| \leqq n \cdot \max\limits_{1 \leqq i,\,k \leqq n} |\,a_{ik}\,|$. For let $Ax=\lambda_0x$, where $x=(x_1, x_2, \ldots, x_n) \neq 0$. Then

$$\lambda_0x_i=\sum_{k=1}^{n}a_{ik}x_k \qquad (i=1,2,\ \ldots\ n).$$

Let $|\,x_j\,|=\max\{|\,x_1\,|, |\,x_2\,|, \ldots, |\,x_n\,|\}$. Then

$$|\,\lambda_0\,|\,|\,x_j\,| \leqq \sum_{k=1}^{n}|\,a_{jk}\,|\,|\,x_k\,| \leqq |\,x_j\,|\,n \max_{1 \leqq i,\,k \leqq n}|\,a_{ik}\,|.$$

Dividing through $|\,x_j\,|$, we obtain the required inequality.

$$\frac{dx^{(k)}}{dz}=P(z)\,x^{(k)}+f(z), \qquad (119)$$

where

$$f(z)=\sum_{m=k-1}^{\infty}f_mz^m=P(z)\,(a_0+a_1z+\cdots+a_{k-1}z^{k-1})-$$
$$-a_1-2\,a_2z-\cdots-(k-1)\,a_{k-1}z^{k-2}. \qquad (120)$$

From (120) it follows that the series

$$\sum_{m=k-1}^{\infty}f_mz^m$$

converges for $|\,z\,|<r$; hence there exists an integer $N>0$ such that[65]

$$\operatorname{mod} f_m \leqq \left\|\frac{N}{r^m}\right\| \qquad (m=k-1,k,\ldots). \qquad (121)$$

From the form of the recurrence relations (117) it follows that when the matrices P_{-1}, P_q, f_{m-1} in these relations are replaced by the majorant matrices $p_{-1}I$, $pr^{-q}I$, $\left\|\frac{N}{r^{m-1}}\right\|$ and the column a_m by $\|\,\alpha_m\,\|$ $(m=k,\,k+1, \ldots;\,q=0,1,2,\ldots)$,[66] then we obtain relations that determine upper bounds $\|\,\alpha_m\,\|$ for $\operatorname{mod} a_m$:

$$\operatorname{mod} a_m \leqq \|\,\alpha_m\,\|. \qquad (122)$$

Therefore the series

$$\xi^{(k)}=\alpha_kz^k+\alpha_{k+1}z^{k+1}+\cdots \qquad (123)$$

after term-by-term multiplication with the column $\|\,1\,\|$ becomes a majorant series for (115).

By replacing in (119) the matrix coefficients P_{-1}, P_q, f_m of the series

$$P(z)=\frac{P_{-1}}{z}+\sum_{q=0}^{\infty}P_qz^q, \qquad f(z)=\sum_{m=k-1}^{\infty}f_mz^m$$

by the corresponding majorant matrices $p_{-1}\,I$, $\frac{p}{r^q}\,I$, $\left\|\frac{N}{r^m}\right\|$, $\|\,\xi^{(k)}\,\|$, we obtain a differential equation for $\xi^{(k)}$:

$$\frac{d\xi^{(k)}}{dz}=n\left(\frac{p_{-1}}{z}+\frac{p}{1-zr^{-1}}\right)\xi^{(k)}+\frac{N\dfrac{z^{k-1}}{r^{k-1}}}{1-\dfrac{z}{r}}. \qquad (124)$$

---

[65] Here $\|\,N/r^m\,\|$ denotes the column in which all the elements are equal to one and the same number, N/r^m.

[66] Here $\|\,\alpha_m\,\|$ denotes the column $(\alpha_m, \alpha_m, \ldots, \alpha_m)$ (α_m is a constant, $m=k$, $k+1, \ldots$).

This linear differential equation has the particular solution

$$\xi^{(k)} = \frac{N}{r^{k-1}} \frac{z^{np_{-1}}}{(1-zr^{-1})^{npr}} \int\limits_0^z z^{k-np_{-1}-1}\left(1-\frac{z}{r}\right)^{npr-1} dz , \tag{125}$$

which is regular for $z=0$ and can be expanded in a neighborhood of this point in the power series (123) which is convergent for $|z| < r$.

From the convergence of the majorant series (123) it follows that the series (115) is convergent for $|z| < r$, and the lemma is proved.

Note 1. This proof enables us to determine all the solutions of the differential system (112) that are regular at the singular point, provided such solutions exist.

For the existence of regular solutions (not identically zero) it is necessary and sufficient that the residue matrix P_{-1} have a non-negative integral characteristic value. If s is the greatest integral characteristic value, then columns $a_0, a_1, \dots, a_s$ that do not all vanish can be determined from the first $s+1$ of the equations (114); for the determinant of the corresponding linear homogeneous equation is zero:

$$\varDelta = |P_{-1}|\,|E-P_{-1}| \cdots |sE-P_{-1}| = 0 .$$

From the remaining equations of (114) the columns $a_{s+1}, a_{s+2}, \dots$ can be expressed uniquely in terms of $a_0, a_1, \dots, a_s$. The series (111) so obtained converges, by the lemma. Thus, the linearly independent solutions of the first $s+1$ equations (114) determine all the linearly independent solutions of the system (112) that are regular at the singular point $z=0$.

If $z=0$ is a singular point, then a regular solution (111) at that point (if such a solution exists) is not uniquely determined when the initial value a_0 is given. However, a solution that is regular at a regular singularity is uniquely determined when $a_0, a_1, \dots, a_s$ are given, i.e., when the initial values at $z=0$ of this solution and the initial values of its first s derivatives are given (s is the largest non-negative integral characteristic value of the residue matrix P_{-1}).

Note 2. The proof of the lemma remains valid for $P_{-1}=O$. In this case an arbitrary positive number can be chosen for p_{-1} in the proof of the lemma. For $P_{-1}=O$ the lemma states the well-known proposition on the existence of a regular solution in a neighborhood of a regular point of the system. In this case the solution is uniquely determined when the initial value a_0 is given.

4. Suppose given the system

$$\frac{dX}{dz} = P(z)\,X , \tag{126}$$

where

$$P(z) = \frac{P_{-1}}{z} + \sum_{m=0}^{\infty} P_m z^m$$

and the series on the right-hand side converges for $|z| < r$.

Suppose, further, that by setting

$$X = A(z)\,Y \tag{127}$$

and substituting for $A(z)$ the series

$$A(z) = A_0 + A_1 z + A_2 z^2 + \cdots , \tag{128}$$

we obtain after formal transformations:

$$\frac{dY}{dz} = P^*(z)\,Y , \tag{129}$$

where

$$P^*(z) = \frac{P^*_{-1}}{z} + \sum_{m=0}^{\infty} P^*_m z^m ,$$

and that here, as in the expression for $P(z)$, the series on the right-hand side converges for $|z| < r$.

We shall show that the series (128) also converges in the neighborhood $|z| < r$ of $z=0$.

Indeed, it follows from (126), (127), and (129) that the series (128) formally satisfies the following matrix differential equation

$$\frac{dA}{dz} = P(z)\,A - AP^*(z). \tag{130}$$

We shall regard A as a vector (column) in the space of all matrices of order n, i.e., a space of dimension n^2. If in this space a linear operator $\widehat{P}(z)$ on A, depending analytically on a parameter z, is defined by the equation

$$\widehat{P}(z)\,[A] = P(z)\,A - AP^*(z) , \tag{131}$$

then the differential equation (130) can be written in the form

$$\frac{dA}{dz} = \widehat{P}(z)\,[A]. \tag{132}$$

The right-hand side of this equation can be considered as the product of the matrix $r\,(z)$ of order n^2 and the column A of n^2 elements. From (131) it is clear that $z=0$ is a regular singularity of the system (132). The series (128) formally satisfies this system. Therefore, by applying the lemma, we conclude that (128) converges in the neighborhood $|z| < r$ of $z=0$.

In particular, the series for $A(z)$ in (110) also converges.

Thus, we have proved the following theorem:

THEOREM 2. *Every system*

$$\frac{dX}{dz} = P(z)\,X, \tag{133}$$

with a regular singularity at $z=0$

$$P(z) = \frac{P_{-1}}{z} + \sum_{m=0}^{\infty} P_m z^m,$$

has a solution of the form

$$X = A(z)\,z^M z^U, \tag{134}$$

where $A(z)$ *is a matrix function that is regular for* $z=0$ *and becomes the unit matrix* E *at that point, and where* M *and* U *are constant matrices,* M *being of simple structure and having integral characteristic values, whereas the difference between any two distinct characteristic values of* U *is not an integer.*

If the matrix P_{-1} *is reduced to the Jordan form by means of a non-singular matrix* T

$$P_{-1} = T\{\lambda_1 E_1 - H_1,\ \lambda_2 E_2 + H_2, \ldots, \lambda_s E_s - H_s\}\,T^{-1} \tag{135}$$
$$(\operatorname{Re}(\lambda_1) \geqq \operatorname{Re}(\lambda_2) \geqq \cdots \geqq \operatorname{Re}(\lambda_s)),$$

then M *and* U *can be chosen in the form*

$$M = T\{m_1 E_1,\ m_2 E_2, \ldots, m_s E_s\}\,T^{-1}, \tag{136}$$

$$U = T\begin{pmatrix} \tilde{\lambda}_1 E_1 + H_1 & B_{12} & \ldots & B_{1s} \\ O & \tilde{\lambda}_2 E_2 + H_2 & \ldots & B_{2s} \\ \cdot & \cdot & \cdot & \cdot \\ O & O & \ldots & \tilde{\lambda}_s E_s + H_s \end{pmatrix} T^{-1}, \tag{137}$$

where

$$m_i = [\lambda_i], \quad \tilde{\lambda}_i = \lambda_i - m_i \qquad (i = 1, 2, \ldots, s). \tag{138}$$

The B_{ik} *are regular lower triangular matrices* $(i, k = 1, 2, \ldots, s)$ *and* $B_{ik} = O$ *if* $\lambda_i - \lambda_k$ *is not a positive integer* $(i, k = 1, 2, \ldots, s)$.

In the particular case where none of the differences $\lambda_i - \lambda_k$ $(i, k = 1, 2, 3, \ldots, s)$ *is a positive integer, we can set in* (134) $M = O$ *and* $U = P_{-1}$; *i.e., in this case the solution can be represented in the form*

$$X = A(z)\,z^{P_{-1}}. \tag{139}$$

Note 1. We wish to point out that in this section we have developed an algorithm to determine the coefficients of the series $A(z) = \sum_{m=0}^{\infty} A_m z^m$ $(A_0 = E)$ in terms of the coefficients P_m of the series for $P(z)$. Moreover, the theorem also determines the integral substitution V by which the solution (134) is multiplied when a circuit is made once in the positive direction around the singular point $z=0$:

$$V = e^{2\pi i U}.$$

Note 2. From the enunciation of the theorem it follows that

$$B_{ik} = O \quad \text{for} \quad \tilde{\lambda}_i \neq \tilde{\lambda}_k \qquad (i, k = 1, 2, \ldots, s).$$

Therefore the matrices

$$\tilde{\Lambda} = T\{\tilde{\lambda}_1 E_1,\ \tilde{\lambda}_2 E_2, \ldots, \tilde{\lambda}_s E_s\}\,T^{-1} \quad \text{and} \quad \tilde{U} = T\begin{pmatrix} O & B_{12} & \ldots & B_{1s} \\ O & O & \ldots & B_{2s} \\ \cdot & \cdot & \cdot & \cdot \\ O & O & \ldots & O \end{pmatrix} T^{-1} \tag{140}$$

are permutable:

$$\tilde{\Lambda}\tilde{U} = \tilde{U}\tilde{\Lambda}.$$

Hence

$$z^M z^U = z^M z^{\tilde{\Lambda} + \tilde{U}} = z^M z^{\tilde{\Lambda}} z^{\tilde{U}} = z^{\Lambda} z^{\tilde{U}}, \tag{141}$$

where

$$\Lambda = M + \tilde{\Lambda} = T\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\,T^{-1} \tag{142}$$

and where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all the characteristic values of P_{-1} arranged in the order $\operatorname{Re}\lambda_1 \geqq \operatorname{Re}\lambda_2 \geqq \ldots \geqq \operatorname{Re}\lambda_n$.

On the other hand,

$$z^{\tilde{U}} = h(\tilde{U}),$$

where $h(\lambda)$ is the Lagrange-Sylvester interpolation polynomial for $f(\lambda) = z^{\lambda}$. Since all the characteristic values of $\tilde{U}$ are zero, $h(\lambda)$ depends linearly on $f(0), f'(0), \ldots, f^{(g-1)}(0)$, i.e., on $1, \ln z, \ldots, (\ln z)^{g-1}$ (g is the least exponent for which $\tilde{U}^g = O$). Therefore

$$h(\lambda) = \sum_{j=0}^{g-1} h_j(\lambda)(\ln z)^j$$

and

$$z^{\tilde{U}} = h(\tilde{U}) = \sum_{j=0}^{g-1} h_j(\tilde{U})(\ln z)^j = T\begin{pmatrix} 1 & q_{12} & \ldots & q_{1n} \\ 0 & 1 & \ldots & q_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 1 \end{pmatrix} T^{-1}, \tag{143}$$

where q_{ij} $(i, j = 1, 2, \ldots, n;\ i < j)$ are polynomials in $\ln z$ of degree less than g.

By (134), (141), (142), and (143) a particular solution of (126) can be chosen in the form

$$X = A(z) \begin{Vmatrix} z^{\lambda_1} & 0 & \ldots & 0 \\ 0 & z^{\lambda_2} & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & z^{\lambda_n} \end{Vmatrix} \begin{Vmatrix} 1 & q_{12} & \cdots & q_{1n} \\ 0 & 1 & \ldots & q_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 1 \end{Vmatrix}. \tag{144}$$

Here $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic values of P_{-1} arranged in the order $\mathrm{Re}\,\lambda_1 \geqq \mathrm{Re}\,\lambda_2 \geqq \ldots \geqq \mathrm{Re}\,\lambda_n$, and q_{ij} $(i, j = 1, 2, \ldots, n;\ i < j)$ are polynomials in $\ln z$ of degree not higher than $g - 1$, where g is the maximal number of characteristic values λ_i that differ from each other by an integer; $A(z)$ is a matrix function, regular at $z = 0$, and $A(0) = T$ $(|T| \neq 0)$. If P_{-1} has the Jordan form, then $T = E$.

§ 11. Reducible Analytic Systems

1. As an application of the theorem of the preceding section we shall investigate in what cases the system

$$\frac{dX}{dt} = Q(t)\,X, \tag{145}$$

where

$$Q(t) = \sum_{m=1}^{\infty} \frac{Q_m}{t^m} \tag{146}$$

is a convergent series for $t > t_0$, is reducible (in the sense of Lyapunov), i.e., in what cases the system has a solution of the form

$$X = L(t)\,e^{Bt}, \tag{147}$$

where $L(t)$ is a Lyapunov matrix (i.e., $L(t)$ satisfies the conditions 1.-3. on p. 117) and B is a constant matrix.[67] Here X and Q are matrices with complex elements and t is a real variable.

We make the transformation

$$z = \frac{1}{t}.$$

---

[67] If the equation (147) holds, then the Lyapunov transformation $X = L(t)Y$ carries the system (145) into the system $\frac{dY}{dt} = BY$.

Then the system (145) assumes the form

$$\frac{dX}{dz} = P(z)\,X, \tag{148}$$

where

$$P(z) = -z^{-2} Q\left(\frac{1}{z}\right) = -\frac{Q_1}{z} - \sum_{m=0}^{\infty} Q_{m+2} z^m. \tag{149}$$

The series on the right-hand side of the expression for $P(z)$ converges for $|z| < 1/t_0$. Two cases can arise:

1) $Q_1 = O$. In that case $z = 0$ is not a singular point of the system (148). The system has a solution that is regular and normalized at $z = 0$. This solution is given by a convergent power series

$$X(z) = E + X_1 z + X_2 z^2 + \cdots \quad \left(|z| < \frac{1}{t_0}\right).$$

Setting

$$L(t) = X\left(\frac{1}{t}\right), \qquad B = O,$$

we obtain the required representation (147). The system is reducible.

2) $Q_1 \neq O$. In that case the system (148) has a regular singularity at $z = 0$.

Without loss of generality we may assume that the residue matrix $P_{-1} = -Q_1$ is reduced to the Jordan form in which the diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$ are arranged in the order $\mathrm{Re}\,\lambda_1 \geqq \mathrm{Re}\,\lambda_2 \geqq \ldots \geqq \mathrm{Re}\,\lambda_n$.

Then in (144) $T = E$, and therefore the system (148) has the solution

$$X = A(z) \begin{Vmatrix} z^{\lambda_1} & 0 & \ldots & 0 \\ 0 & z^{\lambda_2} & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & z^{\lambda_n} \end{Vmatrix} \begin{Vmatrix} 1 & q_{12} & \cdots & q_{1n} \\ 0 & 1 & \ldots & q_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 1 \end{Vmatrix},$$

where the function $A(z)$ is regular for $z = 0$ and assumes at this point the value E, and where q_{ik} $(i, k = 1, 2, \ldots, n;\ i < k)$ are polynomials in $\ln z$. When we replace z by $1/t$, we have:

$$X = A\left(\frac{1}{t}\right) \begin{Vmatrix} \left(\frac{1}{t}\right)^{\lambda_1} & 0 & \ldots & 0 \\ 0 & \left(\frac{1}{t}\right)^{\lambda_2} & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \left(\frac{1}{t}\right)^{\lambda_n} \end{Vmatrix} \begin{Vmatrix} 1 & q_{12}\left(\ln\frac{1}{t}\right) & \ldots & q_{1n}\left(\ln\frac{1}{t}\right) \\ 0 & 1 & \ldots & q_{2n}\left(\ln\frac{1}{t}\right) \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & 1 \end{Vmatrix}. \tag{150}$$

Since $X = A(1/t)Y$ is a Lyapunov transformation, the system (145) is reducible to a system with constant coefficients if and only if the product

$$L_1(t) = \left\| \begin{matrix} t^{-\lambda_1} & 0 & \dots & 0 \\ 0 & t^{-\lambda_2} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & t^{-\lambda_n} \end{matrix} \right\| \left\| \begin{matrix} 1 & q_{12}\left(\ln \frac{1}{t}\right) & \dots & q_{1n}\left(\ln \frac{1}{t}\right) \\ 0 & 1 & \dots & q_{2n}\left(\ln \frac{1}{t}\right) \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \end{matrix} \right\| e^{-Bt}, \qquad (151)$$

where B is a constant matrix, is a Lyapunov matrix, i.e., when the matrices $L_1(t), \frac{dL_1}{dt}$, and $L_1^{-1}(t)$ are bounded. It follows from the theorem of Erugin (§ 4) that the matrix B can be assumed here to have real characteristic values.

Since $L_1(t)$ and $L_1^{-1}(t)$ are bounded for $t > t_0$, all the characteristic values of B must be zero. This follows from the expression for e^{Bt} and e^{-Bt} obtained from (151). Moreover, all the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ must be pure imaginary, because by (151) the fact that the elements of the last row of $L_1(t)$ and of the first column of $L_1^{-1}(t)$ are bounded implies that $\operatorname{Re} \lambda_n \geqq 0$ and $\operatorname{Re} \lambda_1 \leqq 0$.

But if all the characteristic values of P_{-1} are pure imaginary, then the difference between any two distinct characteristic values of P_{-1} cannot be an integer. Therefore the formula (139) holds

$$X = A(z)\, z^{P_{-1}} = A\left(\frac{1}{t}\right) t^{Q_1},$$

and for the reducibility of the system it is necessary and sufficient that the matrix

$$L_2(t) = t^{Q_1} e^{-Bt} \qquad (152)$$

together with its inverse be bounded for $t > t_0$.

Since all the characteristic values of B must be zero, the minimal polynomial of B is of the form λ^d. We denote by

$$\psi(\lambda) = (\lambda - \mu_1)^{c_1} (\lambda - \mu_2)^{c_2} \cdots (\lambda - \mu_u)^{c_u} \qquad (\mu_i \neq \mu_k \text{ for } i \neq k)$$

the minimal polynomial of Q_1. As $Q_1 = -P_{-1}$, the numbers $\mu_1, \mu_2, \dots, \mu_u$ differ only in sign from the corresponding numbers λ_i and are therefore all pure imaginary. Then (see the formulas (12), (13) on p. 116)

$$t^{Q_1} = \sum_{k=1}^{u} [U_{k0} + U_{k1} \ln t + \cdots + U_{k, c_k - 1} (\ln t)^{c_k - 1}]\, t^{\mu_k}, \qquad (153)$$

$$e^{Bt} = V_0 + V_1 t + \cdots + V_{d-1} t^{d-1}. \qquad (154)$$

Substituting these expressions in the equation

$$L_2(t)\, e^{Bt} = t^{Q_1},$$

we obtain

$$[L_2(t)\, V_{d-1} + (*)]\, t^{d-1} = Z_0(t)\, (\ln t)^{c-1}, \qquad (155)$$

where c is the greatest of the numbers $c_1, c_2, \dots, c_n$, $(*)$ denotes a matrix that tends to zero for $t \to \infty$, and $Z_0(t)$ is a bounded matrix for $t > t_0$.

Since the matrices on both sides of (155) must be of equal order of magnitude for $t \to \infty$, we have

$$d = c = 1,$$

i.e.,

$$B = O,$$

and the matrix Q_1 has simple elementary divisors.

Conversely, if Q_1 has simple elementary divisors and pure imaginary characteristic values $\mu_1, \mu_2, \dots, \mu_n$, then

$$X = A(z)\, z^{-Q_1} = A(z) \left\| z^{-\mu_i} \delta_{ik} \right\|_1^n$$

is a solution of (149). Setting $z = 1/t$, we find:

$$X = A\left(\frac{1}{t}\right) \left\| t^{\mu_i} \delta_{ik} \right\|_1^n.$$

The function $X(t)$ as well as $\frac{dX(t)}{dt}$ and the inverse matrix $X^{-1}(t)$ are bounded for $t > t_0$. Therefore the system is reducible $(B = O)$. Thus we have proved the following theorem:[68]

THEOREM 3: *The system*

$$\frac{dX}{dt} = Q(t)\, X,$$

where the matrix $Q(t)$ *can be represented in a series convergent for* $t > t_0$

$$Q(t) = \frac{Q_1}{t} + \frac{Q_2}{t^2} + \cdots,$$

is reducible if and only if all the elementary divisors of the residue matrix Q_1 *are simple and all its characteristic values pure imaginary.*

---

[68] See Erugin [13]. The theorem is proved for the case where Q_1 does not have distinct characteristic values that differ from each other by an integer.

§ 12. Analytic Functions of Several Matrices and their Application to the Investigation of Differential Systems. The Papers of Lappo-Danilevskiĭ

1. An analytic function of m matrices $X_1, X_2, \ldots, X_m$ of order n can be given by a series

$$F(X_1, X_2, \ldots, X_m) = \alpha_0 + \sum_{\nu=1}^{\infty} \sum_{j_1, j_2, \ldots, j_\nu}^{(1, \ldots, m)} \alpha_{j_1 j_2 \ldots j_\nu} X_{j_1} X_{j_2} \cdots X_{j_\nu} \qquad (156)$$

convergent for all matrices X_j of order n that satisfy the inequality

$$\operatorname{mod} X_j < R_j \qquad (j = 1, 2, \ldots, m). \qquad (157)$$

Here the coefficients

$$\alpha_0, \ \alpha_{j_1 j_2 \ldots j_\nu} \qquad (j_1, j_2, \ldots, j_\nu = 1, 2, \ldots, m; \ \nu = 1, 2, 3, \ldots)$$

are complex numbers, R_j $(j = 1, 2, \ldots, m)$ are constant matrices of order n with positive elements, and X_j $(j = 1, 2, \ldots, m)$ are permutable matrices of the same order with complex elements.

The theory of analytic functions of several matrices was developed by I. A. Lappo-Danilevskiĭ. He used this theory as a basis for fundamental investigations on systems of linear differential equations with rational coefficients.

A system with rational coefficients can always be reduced to the form

$$\frac{dX}{dz} = \sum_{j=1}^{m} \left\{ \frac{U_{j0}}{(z-a_j)^{s_j}} + \frac{U_{j1}}{(z-a_j)^{s_j-1}} + \cdots + \frac{U_{j, s_j-1}}{z-a_j} \right\} X \qquad (158)$$

after a suitable transformation of the independent variable, where U_{jk} are constant matrices of order n, a_j are complex numbers, and s_j are positive integers $(k = 0, 1, \ldots, s_{j-1}; \ j = 1, 2, \ldots, m)$.[69]

We shall illustrate some of Lappo-Danilevskiĭ's results in the special case of the so-called *regular* systems. The latter are characterized by the condition $s_1 = s_2 = \ldots = s_m = 1$ and can be written in the form

$$\frac{dX}{dz} = \sum_{j=1}^{m} \frac{U_j}{z-a_j} X. \qquad (159)$$

---

[69] In the system (158) all the coefficients are regular rational fractions in z. Arbitrary rational coefficients can be reduced to this form by carrying a finite point $z = c$ that is regular (for all coefficients) by means of a fractional linear transformation on z into $z = \infty$.

Following Lappo-Danilevskiĭ, we introduce special analytic functions, namely hyperlogarithms, which are defined by the following recurrence relations:

$$l_b(z; a_{j_1}) = \int_b^z \frac{dz}{z - a_{j_1}},$$

$$l_b(z; a_{j_1}, a_{j_2}, \ldots, a_{j_\nu}) = \int_b^z \frac{l_b(z; a_{j_2}, a_{j_3}, \ldots, a_{j_\nu})}{z - a_{j_1}} dz.$$

Regarding $a_1, a_2, \ldots, a_m, \infty$ as branch points of logarithmic type, we construct the corresponding Riemann surface $S(a_1, a_2, \ldots, a_m; \infty)$. Every hyperlogarithm is a single-valued function on this surface. On the other hand, the matricant Ω_b^z of the system (159) (i.e., the solution normalized at $z = b$) after analytic continuation can also be regarded as a single-valued function on $S(a_1, a_2, \ldots, a_m; \infty)$; here b can be chosen as an arbitrary finite point on S other than $a_1, a_2, \ldots, a_m$.

For the normalized solution Ω_b^z Lappo-Danilevskiĭ gives an explicit expression in terms of the defining matrices $U_1, U_2, \ldots, U_m$ of (159) in the form of a series

$$\Omega_b^z = E + \sum_{\nu=1}^{\infty} \sum_{j_1, \ldots, j_\nu}^{(1, \ldots, m)} l_b(z; a_{j_1}, a_{j_2}, \ldots, a_{j_\nu}) U_{j_1} U_{j_2} \cdots U_{j_\nu}. \qquad (160)$$

This expansion converges uniformly in z for arbitrary $U_1, U_2, \ldots, U_m$ and represents Ω_b^z in any finite domain on $S(a_1, a_2, \ldots, a_m; \infty)$ provided only that the domain does not contain $a_1, a_2, \ldots, a_m$ in the interior or on the boundary.

If the series (156) converges for arbitrary matrices $X_1, X_2, \ldots, X_m$, then the corresponding function $F(X_1, X_2, \ldots, X_m)$ is called *entire*. Ω_b^z is an entire function of the matrices $U_1, U_2, \ldots, U_m$.

If in (160) we let the argument z go around the point a_j once in the positive direction along a contour that does not enclose other points a_i (for $i \neq j$), then we obtain the expression for the *integral substitution* V_j corresponding to the point $z = a_j$:

$$V_j = E + \sum_{\nu=1}^{\infty} \sum_{j_1, \ldots, j_\nu}^{(1, \ldots, m)} p_j(b; a_{j_1}, a_{j_2}, \ldots, a_{j_\nu}) \ U_{j_1} U_{j_2} \ldots U_{j_\nu} \qquad (161)$$

$$(j = 1, 2, \ldots, m),$$

where in a readily understandable notation

$$p_j(b; a_{j_1}) = \int\limits_{(a_j)} \frac{dz}{z - a_{j_1}},$$

$$p_j(b; a_{j_1}, a_{j_2}, \ldots, a_{j_\nu}) = \int\limits_{(a_j)} \frac{l_b(z; a_{j_2}, a_{j_3}, \ldots, a_{j_\nu})}{z - a_{j_1}}\, dz$$

$$\begin{pmatrix} j_1, j_2, \ldots, j_\nu, j = 1, 2, \ldots, m \\ \nu = 1, 2, 3, \ldots \end{pmatrix}.$$

The series (161), like (160), is an entire function of $U_1, U_2, \ldots, U_m$.

2. Generalizing the theory of analytic functions to the case[70] of a countably infinite set of matrix arguments $X_1, X_2, X_3, \ldots$, Lappo-Danilevskiĭ has used it to study the behavior of a solution of a system in a neighborhood of an irregular singularity.[71] We quote the basic result.

The normalized solution Ω_b^z of the system

$$\frac{dX}{dz} = \sum_{j=-q}^{\infty} P_j z^j X,$$

where the power series on the right-hand side converges for $|z| < r$ $(r > 1)$,[72] can be represented by a series

$$\begin{aligned} \Omega_b^z = E + \sum_{\nu=1}^{\infty} \sum_{j_1, j_2, \ldots, j_\nu = -q}^{\infty} P_{j_1} \cdots P_{j_\nu} \times \\ \times \sum_{\mu=0}^{\nu} b^{j_{\mu+1} + \cdots + j_\nu + \nu - \mu} z^{j_1 + \cdots + j_\mu + \mu} \sum_{\lambda=0}^{n-\mu} \alpha_{j_{\mu+1}, \ldots, j_\nu}^{*(\lambda)} \ln^\lambda b \sum_{\varkappa=0}^{\mu} \alpha_{j_1, \ldots, j_\mu}^{(\varkappa)} \ln^\varkappa z. \end{aligned} \tag{162}$$

Here $\alpha_{j_{\mu+1}, \ldots, j_\nu}^{*(\lambda)}$ and $\alpha_{j_1, \ldots, j_\mu}^{(\varkappa)}$ are scalar coefficients that are defined by special formulas. The series (162) converges for arbitrary matrices $P_1, P_2, \ldots$ in an annulus

$$\varrho < z < r$$

(ϱ is any positive number less than r). The point b must also lie in this annulus ($\varrho < |b| < r$).

---

[70] See [29], Vol. I, Memoir 1.

[71] See [29], Vol. I, Memoir 3. See also [252], [253], [254], [146], and [147].

[72] The restriction $r > 1$ is not essential, since this condition can always be obtained by replacing z by αz, where α is a suitably chosen positive number.

Since in this book we cannot possibly describe the contents of the papers of Lappo-Danilevskiĭ in sufficient detail, we have had to restrict ourselves to giving above statements of a few basic results and we must refer the reader to the appropriate literature.

All the papers of Lappo-Danilevskiĭ that deal with differential equations have been published posthumously in three volumes ([29]: *Mémoires sur la théorie des systèmes des équations différentielles linéaires* (1934-36)). Moreover, his fundamental results are expounded in the papers [252], [253], [254] and the small book [28]. A concise exposition of some of the results can also be found in the book by V. I. Smirnov [56], Vol. III.

CHAPTER XV

THE PROBLEM OF ROUTH-HURWITZ AND RELATED QUESTIONS

§ 1. Introduction

In Chapter XIV, § 3 we explained that according to Lyapunov's theorem the zero solution of the system of differential equations

$$\frac{dx_i}{dt} = \sum_{k=1}^{n} a_{ik} x_k + (**) \tag{1}$$

(a_{ik} $(i, k = 1, 2, \ldots, n)$ are constant coefficients) with arbitrary terms (**) of the second and higher orders in $x_1, x_2, \ldots, x_n$ is stable if all the characteristic values of the matrix $A = \| a_{ik} \|_1^n$, i.e., all the roots of the secular equation $\Delta(\lambda) \equiv | \lambda E - A | = 0$, have negative real parts.

Therefore the task of establishing necessary and sufficient conditions under which all the roots of a given algebraic equation lie in the left half-plane is of great significance in a number of applied fields in which the stability of mechanical and electrical systems is investigated.

The importance of this algebraic task was clear to the founders of the theory of governors, the British physicist J. C. Maxwell and the Russian scientific research engineer I. A. Vyshnegradskiĭ who, in their papers on governors,[1] established and extensively applied the above-mentioned algebraic conditions for equations of a degree not exceeding three.

In 1868 Maxwell proposed the mathematical problem of discovering corresponding conditions for algebraic equations of arbitrary degree. Actually this problem had already been solved in essence by the French mathematician Hermite in a paper [187] published in 1856. In this paper he had established a close connection between the number of roots of a complex polynomial $f(z)$ in an arbitrary half-plane (and even inside an arbitrary triangle) and the signature of a certain quadratic form. But Hermite's

---

[1] J. C. Maxwell, 'On governors' Proc. Roy. Soc. London, vol. 10 (1868); I. A. Vyshnegradskiĭ, 'On governors with direct action' (1876). These papers were reprinted in the survey 'Theory of automatic governors' (Izd. Akad. Nauk SSSR, 1949). See also the paper by A. A. Andronov and I. N. Voznesenskiĭ, 'On the work of J. C. Maxwell, I. A. Vyshnegradskiĭ, and A. Stodol in the theory of governors of machines.'

results had not been carried to a stage at which they could be used by specialists working in applied fields and therefore his paper did not receive due recognition.

In 1875 the British applied mathematician Routh [47], [48], using Sturm's theorem and the theory of Cauchy indices, set up an algorithm to determine the number k of roots of a real polynomial in the right half-plane ($\mathrm{Re}\, z > 0$). In the particular case $k = 0$ this algorithm then gives a criterion for stability.

At the end of the 19th century, the Austrian research engineer A. Stodola, the founder of the theory of steam and gas turbines, unaware of Routh's paper, again proposed the problem of finding conditions under which all the roots of an algebraic equation have negative real parts, and in 1895 A. Hurwitz [204] on the basis of Hermite's paper gave another solution (independent of Routh's). The determinantal inequalities obtained by Hurwitz are known nowadays as the inequalities of Routh-Hurwitz.

However, even before Hurwitz' paper appeared, the founder of the modern theory of stability, A. M. Lyapunov, had proved in his celebrated dissertation ('The general problem of stability of motion,' Kharkov, 1892)[2] a theorem which yields necessary and sufficient conditions for all the roots of the characteristic equation of a real matrix $A = \| a_{ik} \|_1^n$ to have negative real parts. These conditions are made use of in a number of papers on the theory of governors.[3]

A new criterion of stability was set up in 1914 by the French mathematicians Liénard and Chipart [259]. Using special quadratic forms, these authors obtained a criterion of stability which has a definite advantage over the Routh-Hurwitz criterion (the number of determinantal inequalities in the Liénard-Chipart criterion is roughly half of that in the Routh-Hurwitz criterion).

The famous Russian mathematicians P. L. Chebyshev and A. A. Markov have proved two remarkable theorems on continued-fraction expansions of a special type. These theorems, as will be shown in § 16, have an immediate bearing on the Routh-Hurwitz problem.

The reader will see that in the sphere of problems we have outlined, the theory of quadratic forms (Vol. I, Chapter X) and, in particular, the theory of Hankel forms (Vol. I, Chapter X, § 10) forms an essential tool.

§ 2. Cauchy Indices

1. We begin with a discussion of the so-called Cauchy indices.

---

[2] See [32], § 20.

[3] See, for example, [102].

DEFINITION 1: *The Cauchy index of a real rational function* $R(x)$ *between the limits* a *and* b (notation: $I_a^b R(x)$; *a and b are real numbers or* $\pm\infty$) *is the difference between the numbers of jumps of* $R(x)$ *from* $-\infty$ *to* $+\infty$ *and that of jumps from* $+\infty$ *to* $-\infty$ *as the argument changes from* a *to* b.[5]

According to this definition, if

$$R(x) = \sum_{i=1}^{p} \frac{A_i}{x-\alpha_i} + R_1(x),$$

where A_i, α_i $(i=1, 2, \ldots, p)$ are real numbers and $R_1(x)$ is a rational function[6] without real poles, then[7]

$$I_{-\infty}^{\infty} R(x) = \sum_{i=1}^{p} \operatorname{sgn} A_i \tag{2}$$

and, in general,

$$I_a^b R(x) = \sum_{a<\alpha_i<b} \operatorname{sgn} A_i \quad (a<b). \tag{2'}$$

In particular, if $f(x) = a_0 (x-\alpha_1)^{n_1} \cdots (x-\alpha_m)^{n_m}$ is a real polynomial $(\alpha_i \neq \alpha_k$ for $i \neq k$; $i, k = 1, 2, \ldots, m)$ and if among its roots $\alpha_1, \alpha_2, \ldots, \alpha_m$ only the first p are real, then

$$\frac{f'(x)}{f(x)} = \sum_{j=1}^{m} \frac{n_j}{x-\alpha_j} = \sum_{i=1}^{p} \frac{n_i}{x-\alpha_i} + R_1(x), \tag{2''}$$

where $R_1(x)$ is a real rational function without real poles.

Therefore, by (2′): *The index*

$$I_a^b \frac{f'(x)}{f(x)} \quad (a<b)$$

is equal to the number of distinct real roots of $f(x)$ *in the interval* (a, b).

An arbitrary real rational function $R(x)$ can always be represented in the form

$$R(x) = \sum_{i=1}^{p} \left\{ \frac{A_1^{(i)}}{x-\alpha_i} + \cdots + \frac{A_{n_i}^{(i)}}{(x-\alpha_i)^{n_i}} \right\} + R_1(x),$$

where all the α and A are real numbers $(A^{(i)} \neq 0;\ i=1, 2, \ldots, p)$ and $R_1(x)$ has no real poles.

Then

---

[5] In counting the number of jumps, the extreme values of x—the limits a and b—are not included.

[6] The poles of a rational function are those values of the argument for which the function becomes infinite.

[7] By sign a (a is a real number) we mean $+1$, -1, or 0 according as $a>0$, $a<0$, or $a=0$.

$$I_{-\infty}^{+\infty} R(x) = \sum_{n_i \text{ odd}} \operatorname{sgn} A_{n_i}^{(i)} \tag{3}$$

and, in general,[8]

$$I_a^b R(x) = \sum_{a<\alpha_i<b,\ n_i \text{ odd}} \operatorname{sgn} A_{n_i}^{(i)} \quad (a<b). \tag{3'}$$

2. One of the methods of computing the index $I_a^b R(x)$ is based on the classical *theorem of Sturm.*

We consider a sequence of real polynomials

$$f_1(x),\ f_2(x),\ \ldots,\ f_m(x) \tag{4}$$

that has the two following properties with respect to the interval (a, b):[9]

1. For every value x $(a<x<b)$, if any $f_k(x)$ vanishes, the two adjacent functions $f_{k-1}(x)$ and $f_{k+1}(x)$ have values different from zero and of opposite signs; i.e., for $a<x<b$ it follows from $f_k(x)=0$ that

$$f_{k-1}(x) f_{k+1}(x) < 0.$$

2. The last function $f_m(x)$ in (4) does not vanish in the interval (a, b); i.e., $f_m(x) \neq 0$ for $a<x<b$.

Such a sequence (4) of polynomials is called a *Sturm chain in the interval* (a, b).

We denote by $V(x)$ the number variations of sign in (4) for a fixed value x.[10] Then the value of $V(x)$, as x varies from a to b, can only change when one of the functions in (4) passes through zero. But by 1., when the functions $f_k(x)$ $(k=2, \ldots, m-1)$ pass through zero, the value of $V(x)$ does not change. When $f_1(x)$ passes through zero, then one variation of sign in (4) is lost or gained according as the ratio $f_2(x)/f_1(x)$ goes from $-\infty$ to $+\infty$ or vice versa. Hence we have:

THEOREM 1 (Sturm): *If* $f_1(x), f_2(x), \ldots, f_m(x)$ *is a Sturm chain in* (a, b) *and* $V(x)$ *is the number of variations of sign in the chain, then*

$$I_a^b \frac{f_2(x)}{f_1(x)} = V(a) - V(b). \tag{5}$$

---

[8] In (3) the sum is extended over all the values i for which the corresponding n_i is odd. In (3′) the sum is extended over all the i for which n_i is odd and $a<\alpha_i<b$.

[9] Here a may be $-\infty$ and b may be $+\infty$.

[10] If $a<x<b$ and $f_1(x) \neq 0$, then by 1. in the determination of $V(x)$ a zero value in (4) may be omitted or an arbitrary sign may be attributed to this value. If a is finite, then $V(a)$ must be interpreted as $V(a+\varepsilon)$, where ε is a positive number sufficiently small that in the half-closed interval $(a, a+\varepsilon]$ none of the functions $f_i(x)$ vanishes. In exactly the same way, if b is finite, $V(b)$ is to be interpreted as $V(b-\varepsilon)$, where the number ε is defined similarly.

Note. Let us multiply all the terms of a Sturm chain by one and the same arbitrary polynomial $d(x)$. The chain of polynomials so obtained is called a *generalized Sturm chain.* Since the multiplication of all the terms of (4) by one and the same polynomial alters neither the left-hand nor the right-hand side of (5), Sturm's theorem remains valid for generalized Sturm chains.

Note that if $f(x)$ and $g(x)$ are any two polynomials (where the degree of $f(x)$ is not less than that of $g(x)$), then we can always construct a generalized Sturm chain (4) beginning with $f_1(x) \equiv f(x)$, $f_2(x) \equiv g(x)$ by means of the Euclidean algorithm.

For if we denote by $-f_3(x)$ the remainder on dividing $f_1(x)$ by $f_2(x)$, by $-f_4(x)$ the remainder on dividing $f_2(x)$ by $f_3(x)$, etc., then we have the chain of identities

$$\begin{aligned} f_1(x) &= q_1(x) f_2(z) - f_3(x), \\ &\cdots\cdots\cdots\cdots\cdots \\ f_{k-1}(x) &= q_{k-1}(x) f_k(x) - f_{k+1}(x), \\ &\cdots\cdots\cdots\cdots\cdots \\ f_{m-1}(x) &= q_{m-1}(x) f_m(x), \end{aligned} \tag{6}$$

where the last remainder $f_m(x)$ that is not identically zero is the greatest common divisor of $f(x)$ and $g(x)$ and also of all the functions of the sequence (4) so constructed. If $f_m(x) \neq 0$ $(a < x < b)$ then this sequence (4) satisfies the conditions 1., 2. by (6) and is a Sturm chain. If the polynomial $f_m(x)$ has roots in the interval (a, b), then (4) is a generalized Sturm chain, because it becomes a Sturm chain when all the terms are divided by $f_m(x)$.

From what we have shown it follows that the index of every rational function $R(x)$ can be determined by Sturm's theorem. For this purpose it is sufficient to represent $R(x)$ in the form $Q(x) + \frac{g(x)}{f(x)}$, where $Q(x)$, $f(x)$, $g(x)$ are polynomials and the degree of $g(x)$ does not exceed that of $f(x)$. If we then construct the generalized Sturm chain for $f(x)$, $g(x)$, we have

$$I_a^b R(x) = I_a^b \frac{g(x)}{f(x)} = V(a) - V(b).$$

By means of Sturm's theorem we can determine the number of distinct real roots of a polynomial $f(x)$ in the interval (a, b), since this number, as we have seen, is $I_a^b \frac{f'(x)}{f(x)}$.

§ 3. Routh's Algorithm

1. Routh's problem consists in determining the number k of roots of a real polynomial $f(x)$ in the right half-plane $(\mathrm{Re}\, z > 0)$.

To begin with, we treat the case where $f(z)$ *has no roots on the imaginary axis.* In the right half-plane we construct the semicircle of radius R with its center at the origin and we consider the domain bounded by this semicircle and the segment of the imaginary axis (Fig. 7). For sufficiently large R all the zeros of $f(z)$ with positive real parts lie inside this domain. Therefore $\arg f(z)$ increases by $2k\pi$ on going in the positive direction along the contour of the domain.[11] On the other hand, the increase of $\arg f(z)$ along the semicircle of radius R for $R \to \infty$ is determined by the increase of the argument of the highest term $a_0 z^n$ and is therefore $n\pi$. Hence the increase of $\arg f(z)$ along the imaginary axis $(R \to \infty)$ is given by the expression

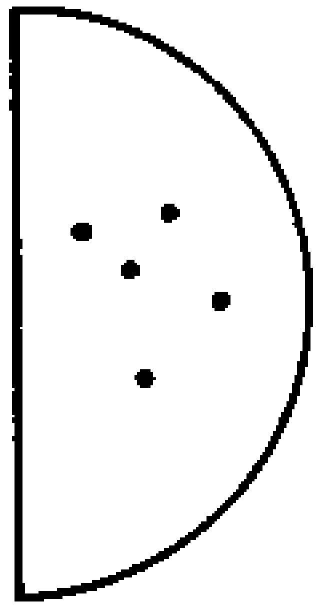

Fig. 7

$$\Delta_{-\infty}^{+\infty} \arg f(i\omega) = (n - 2k)\pi. \tag{7}$$

We introduce a somewhat unusual notation for the coefficients of $f(z)$; namely, we set

$$f(z) = a_0 z^n + b_0 z^{n-1} + a_1 z^{n-2} + b_1 z^{n-3} + \cdots \quad (a_0 \neq 0).$$

Then

$$f(i\omega) = U(\omega) + iV(\omega), \tag{8}$$

where for even n

$$\left. \begin{aligned} U(\omega) &= (-1)^{\frac{n}{2}} (a_0\omega^n - a_1\omega^{n-2} + a_2\omega^{n-4} - \cdots), \\ V(\omega) &= (-1)^{\frac{n}{2}-1} (b_0\omega^{n-1} - b_1\omega^{n-3} + b_2\omega^{n-5} - \cdots) \end{aligned} \right\} \tag{8'}$$

and for odd n

$$\left. \begin{aligned} U(\omega) &= (-1)^{\frac{n-1}{2}} (b_0\omega^{n-1} - b_1\omega^{n-3} + b_2\omega^{n-5} - \cdots), \\ V(\omega) &= (-1)^{\frac{n-1}{2}} (a_0\omega^n - a_1\omega^{n-2} + a_2\omega^{n-4} - \cdots). \end{aligned} \right\} \tag{8''}$$

---

[11] For if $f(z) = a_0 \prod_{i=1}^{n} (z - z_i)$, then $\Delta \arg f(z) = \sum_{i=1}^{n} \Delta \arg (z - z_i)$. If the point z_i lies inside the domain in question, then $\Delta \arg (z - z_i) = 2\pi$; if z_i lies outside the domain, then $\Delta \arg (z - z_i) = 0$.

Following Routh, we make use of the Cauchy index. Then[12]

$$\frac{1}{\pi}\Delta_{-\infty}^{+\infty}\arg f(i\omega)=\begin{cases} I_{-\infty}^{+\infty}\frac{U(\omega)}{V(\omega)} & \text{for } \lim\limits_{\omega\to\infty}\frac{U(\omega)}{V(\omega)}=0,\\ -I_{-\infty}^{+\infty}\frac{V(\omega)}{U(\omega)} & \text{for } \lim\limits_{\omega\to\infty}\frac{V(\omega)}{U(\omega)}=0.\end{cases} \tag{9}$$

The equations (8′) and (8″) show that for even n the lower formula in (9) must be taken and for odd n, the upper. Then we easily obtain from (7), (8′), (8″), and (9) that for every n (even or odd)[13]

$$I_{-\infty}^{+\infty}\frac{b_0\omega^{n-1}-b_1\omega^{n-3}+\cdots}{a_0\omega^n-a_1\omega^{n-2}+\cdots}=n-2k. \tag{10}$$

2. In order to determine the index on the left-hand side of (10) we use Sturm's theorem (see the preceding section). We set

$$f_1(\omega)=a_0\omega^n-a_1\omega^{n-2}+\cdots,\qquad f_2(\omega)=b_0\omega^{n-1}-b_1\omega^{n-3}+\cdots \tag{11}$$

and, following Routh, construct a generalized Sturm chain (see p. 176)

$$f_1(\omega), f_2(\omega), f_3(\omega), \ldots, f_m(\omega). \tag{12}$$

by the Euclidean algorithm.

First we consider the *regular case*: $m=n+1$. In this case the degree of each function in (12) is one less than that of the preceding, and the last function $f_m(\omega)$ is of degree zero.[14]

From Euclid's algorithm (see (6)) it follows that

$$f_3(\omega)=\frac{a_0}{b_0}\omega f_2(\omega)-f_1(\omega)=c_0\omega^{n-2}-c_1\omega^{n-4}+c_2\omega^{n-6}-\cdots,$$

where

$$c_0=a_1-\frac{a_0}{b_0}b_1=\frac{b_0a_1-a_0b_1}{b_0},\quad c_1=a_2-\frac{a_0}{b_0}b_2=\frac{b_0a_2-a_0b_2}{b_0},\ \ldots. \tag{13}$$

Similarly

$$f_4(\omega)=\frac{b_0}{c_0}\omega f_3(\omega)-f_2(\omega)=d_0\omega^{n-3}-d_1\omega^{n-5}+\cdots,$$

where

$$d_0=b_1-\frac{b_0}{c_0}c_1=\frac{c_0b_1-b_0c_1}{c_0},\quad d_1=b_2-\frac{b_0}{c_0}c_2=\frac{c_0b_2-b_0c_2}{c_0},\ \ldots. \tag{13'}$$

The coefficients of the remaining polynomials $f_5(\omega), \ldots, f_{n+1}(\omega)$ are similarly determined.

---

[12] Since $\arg f(i\omega)=\operatorname{arccot}\frac{U(\omega)}{V(\omega)}=\arctan\frac{V(\omega)}{U(\omega)}$.

[13] We recall that the formula (10) was derived under the assumption that $f(z)$ has no roots on the imaginary axis.

[14] In the regular case (12) is the ordinary (not generalized) Sturm chain.

Each polynomial

$$f_1(\omega), f_2(\omega), \ldots, f_{n+1}(\omega) \tag{14}$$

is an even or an odd function and two adjacent polynomials always have opposite parity.

We form the *Routh scheme*

$$\left.\begin{matrix} a_0, & a_1, & a_2, & \ldots, \\ b_0, & b_1, & b_2, & \ldots, \\ c_0, & c_1, & c_2, & \ldots, \\ d_0, & d_1, & d_2, & \ldots, \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \end{matrix}\right\}. \tag{15}$$

The formulas (13), (13′) show that every row in this scheme is determined by the two preceding rows according to the following rule:

From the numbers of the upper row we subtract the corresponding numbers of the lower row multiplied by the number that makes the first difference zero. Omitting this zero difference, we obtain the required row.

The regular case is obviously characterized by the fact that the repeated application of this rule never yields a zero in the sequence

$$b_0, c_0, d_0, \ldots.$$

Figs. 8 and 9 show the skeleton of Routh's scheme for an even n ($n=6$) and an odd n ($n=7$). Here the elements of the scheme are indicated by dots.

In the regular case, the polynomials $f_1(\omega)$ and $f_2(\omega)$ have the greatest common divisor $f_{n+1}(\omega)=\text{const.}\neq 0$. Therefore these polynomials, and hence $U(\omega)$ and $V(\omega)$ (see (8′), (8″), and (11)), do not vanish simultaneously; i.e., $f(i\omega)=U(\omega)+iV(\omega)\neq 0$ for real ω. Therefore: *In the regular case the formula* (10) *holds.*

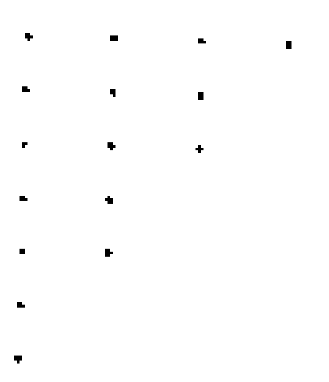

Fig. 8

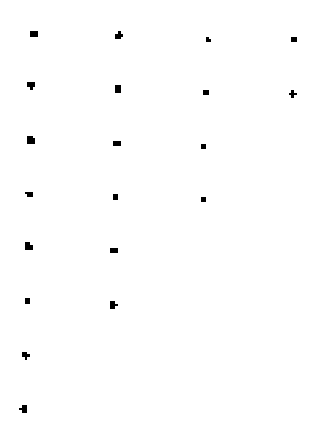

Fig. 9

When we apply Sturm's theorem in the interval $(-\infty, +\infty)$ to the left-hand side of this formula and make use of (14), we obtain by (10):

$$V(-\infty)-V(+\infty)=n-2k. \tag{16}$$

In our case[15]

$$V(+\infty)=V(a_0, b_0, c_0, d_0, \ldots)$$

and

---

[15] The sign of $f_k(\omega)$ for $\omega=+\infty$ coincides with the sign of the highest coefficient and for $\omega=-\infty$ differs from it by the factor $(-1)^{n-k+1}$ ($k=1, 2, \ldots, n+1$).

$$V(-\infty) = V(a_0, -b_0, c_0, -d_0, \ldots).$$

Hence

$$V(-\infty) = n - V(+\infty). \tag{17}$$

From (16) and (17) we find:

$$k = V(a_0, b_0, c_0, d_0, \ldots). \tag{18}$$

Thus we have proved the following theorem:

Theorem 2 (Routh): *The number of roots of the real polynomial* $f(z)$ *in the right half-plane* $\operatorname{Re} z > 0$ *is equal to the number of variations of sign in the first column of Routh's scheme.*

3. We consider the important special case where all the roots of $f(z)$ have negative real parts ('case of stability'). If in this case we construct for the polynomials (11) the generalized Sturm chain (14), then, since $k = 0$, the formula (16) can be written as follows:

$$V(-\infty) - V(+\infty) = n. \tag{19}$$

But $0 \leqq V(-\infty) \leqq m - 1 \leqq n$ and $0 \leqq V(+\infty) \leqq m - 1 \leqq n$. Therefore (19) is possible only when $m = n + 1$ (regular case!) and $V(+\infty) = 0$, $V(-\infty) = m - 1 = n$. The formula (18) then implies:

Routh's Criterion. *All the roots of the real polynomial* $f(z)$ *have negative real parts if and only if in the carrying out of Routh's algorithm all the elements of the first column of Routh's scheme are different from zero and of like sign.*

4. In deriving Routh's theorem we have made use of the formula (10). In what follows we shall have to generalize this formula. The formula (10) was deduced under the assumption that $f(z)$ has no roots on the imaginary axis. We shall now show that in the general case, where the polynomial $f(z) = a_0 z^n + b_0 z^{n-1} + a_1 z^{n-2} + \ldots (a_0 \neq 0)$ has k roots in the right half-plane and s roots on the imaginary axis, the formula (10) is replaced by

$$I_{-\infty}^{+\infty} \frac{b_0 \omega^{n-1} - b_1 \omega^{n-3} + b_2 \omega^{n-5} - \cdots}{a_0 \omega^n - a_1 \omega^{n-2} + a_2 \omega^{n-4} - \cdots} = n - 2k - s. \tag{20}$$

For

$$f(z) = d(z) f^*(z),$$

where the real polynomial $d(z) = z^s + \ldots$ has s roots on the imaginary axis and the polynomial $f^*(z)$ of degree $n^* = n - s$ has no such roots.

For the sake of definiteness, we consider the case where s is even (the case where s is odd is analyzed similarly).

Let

$$f(i\omega) = U(\omega) + iV(\omega) = d(i\omega)\,[U^*(\omega) + iV^*(\omega)].$$

Since in our case $d(i\omega)$ is a real polynomial in ω, we have

$$\frac{U(\omega)}{V(\omega)} = \frac{U^*(\omega)}{V^*(\omega)}.$$

Since n and n^* have equal parity, we find by using (8′), (8″), and the notation (11):

$$\frac{f_2(\omega)}{f_1(\omega)} = \frac{f_2^*(\omega)}{f_1^*(\omega)}.$$

We apply formula (10) to $f^*(z)$. Therefore

$$I_{-\infty}^{+\infty} \frac{f_2(\omega)}{f_1(\omega)} = I_{-\infty}^{+\infty} \frac{f_2^*(\omega)}{f_1^*(\omega)} = n^* - 2k = n - 2k - s,$$

and this is what we had to prove.

§ 4. The Singular Case. Examples

1. In the preceding section we have examined the regular case where in Routh's scheme none of the numbers $b_0, c_0, d_0, \ldots$ vanish.

We now proceed to deal with the singular cases, where among the numbers $b_0, c_0, \ldots$ there occurs a zero, say, $h_0 = 0$. Routh's algorithm stops with the row in which h_0 occurs, because to obtain the numbers of the following row we would have to divide by h_0.

The singular cases can be of two types:

1) *In the row in which* h_0 *occurs there are numbers different from zero.* This means that at some place of (12) the degree drops by more than one.

2) *All the numbers of the row in which* h_0 *occurs vanish simultaneously.* Then this row is the $(m+1)$-th, where m is the number of terms in the generalized Sturm chain (12). In that case, the degrees of the functions in (12) decrease by unity from one function to the next, but the degree of the last function $f_m(\omega)$ is greater than zero. In both cases the number of functions in (12) is $m < n + 1$.

Since the ordinary Routh's algorithm comes to an end in both cases, Routh gives a special rule for continuing the scheme in the cases 1), 2).

2. In case 1), according to Routh, we have to substitute for $h_0 = 0$ a 'small' value ε of definite (but arbitrary) sign and continue to fill in the scheme. Then the subsequent elements of the first column of the scheme are rational functions of ε. The signs of these elements are determined by the 'smallness' and the sign of ε. If any one of these elements vanishes identically in ε, then we replace this element by another small value η and continue the algorithm.

Example:

$$f(z) = z^4 + z^3 + 2z^2 + 2z + 1.$$

Routh's scheme (with a small parameter ε):

$$\begin{array}{lll} 1, & 2, & 1 \\ 1, & 2 & \\ \varepsilon, & 1 & \qquad k = V\left(1, 1, \varepsilon, 2 - \dfrac{1}{\varepsilon}, 1\right) = 2. \\ 2 - \dfrac{1}{\varepsilon} & & \\ 1 & & \end{array}$$

This special method of varying the elements of the scheme is based on the following observation:

Since we assume that there is no singularity of the second type, the functions $f_1(\omega)$ and $f_2(\omega)$ are relatively prime. Hence it follows that the polynomial $f(z)$ has no roots on the imaginary axis.

In Routh's scheme all the elements are expressed rationally in terms of the elements of the first two rows, i.e., the coefficients of the given polynomial. But it is not difficult to observe in the formulas (13), (13′) and the analogous formulas for the subsequent rows that, once we have given arbitrary values to the elements of any two adjacent rows of Routh's scheme and to the first element of the preceding row, we can express all the elements in the first two rows, i.e., the coefficients of the original polynomial, in integral rational form in terms of these elements. Thus, for example, all the numbers a, b can be represented as integral rational functions of

$$a_0, b_0, c_0, \ldots, g_0, g_1, g_2, \ldots, h_0, h_1, h_2, \ldots$$

Therefore, in replacing $g_0 = 0$ by ε we in fact modify our original polynomial. Instead of the scheme for $f(z)$ we have the Routh scheme for a polynomial $F(z, \varepsilon)$, where $F(z, \varepsilon)$ is an integral rational function of z and ε which reduces to $f(z)$ for $\varepsilon = 0$. Since the roots of $F(z, \varepsilon)$ change continuously with a change of the parameter ε and since there are no roots on the imaginary axis for $\varepsilon = 0$, the number k of roots in the right half-plane is the same for $F(z, \varepsilon)$ and $F(z, 0) = f(z)$ for values of ε of small modulus.

3. Let us now proceed to a singularity of the second type. Suppose that in Routh's scheme

$$a_0 \neq 0,\ b_0 \neq 0, \ldots, e_0 \neq 0,\ g_0 = 0,\ g_1 = 0,\ g_2 = 0, \ldots$$

In this case, the last polynomial in the generalized Sturm chain (16) is of the form:

$$f_m(\omega) = e_0\omega^{n-m+1} - e_1\omega^{n-m-1} + \cdots.$$

Routh proposes to replace $f_{m+1}(\omega)$, which is zero, by $f'_m(\omega)$; i.e., he proposes to write instead of $g_0, g_1, \ldots$ the corresponding coefficients

$$(n - m + 1)\, e_0, \quad (n - m - 1)\, e_1, \ldots$$

and to continue the algorithm.

The logical basis for this rule is as follows:

By formula (20)

$$I_{-\infty}^{+\infty} \frac{f_2(\omega)}{f_1(\omega)} = n - 2k - s$$

(the s roots of $f(z)$ on the imaginary axis coincide with the real roots of $f_m(\omega)$). Therefore, if these real roots are simple, then (see p. 174)

$$I_{-\infty}^{+\infty} \frac{f'_m(\omega)}{f_m(\omega)} = s$$

and therefore

$$I_{-\infty}^{+\infty} \frac{f_2(\omega)}{f_1(\omega)} + I_{-\infty}^{+\infty} \frac{f'_m(\omega)}{f_m(\omega)} = n - 2k.$$

This formula shows that the missing part of Routh's scheme must be filled by the Routh scheme for the polynomials $f_m(\omega)$ and $f'_m(\omega)$. The coefficients of $f'_m(\omega)$ are used to replace the elements of the zero row in Routh's scheme.

But if the roots of $f_m(\omega)$ are not simple, then we denote by $d(\omega)$ the greatest common divisor of $f_m(\omega)$ and $f'_m(\omega)$, by $e(\omega)$ the greatest common divisor of $d(\omega)$ and $d'(\omega)$, etc., and we have:

$$I_{-\infty}^{+\infty} \frac{f'_m(\omega)}{f_m(\omega)} + I_{-\infty}^{+\infty} \frac{d'(\omega)}{d(\omega)} + I_{-\infty}^{+\infty} \frac{e'(\omega)}{e(\omega)} + \cdots = s.$$

Thus the required number k can be found if the missing part of Routh's scheme is filled by the Routh scheme for $f_m(\omega)$ and $f'_m(\omega)$, then the scheme for $d(\omega)$ and $d'(\omega)$, then that for $e(\omega)$ and $e'(\omega)$, etc., i.e., Routh's rule has to be applied several times to dispose of a singularity of the second type.

Example. $f(z) = z^{10} + z^9 - z^8 - 2z^7 + z^6 + 3z^5 + z^4 - 2z^3 - z^2 + z + 1$.

Scheme

$$\begin{array}{ll}
\omega^{10} & \begin{array}{rrrrrr} 1 & -1 & 1 & 1 & -1 & 1 \end{array}\\
\omega^{9} & \begin{array}{rrrrr} 1 & -2 & 3 & -2 & 1 \end{array}\\
\omega^{8} & \begin{array}{rrrrr} 1 & -2 & 3 & -2 & 1 \end{array}\\
\omega^{7} & \left\{\begin{array}{rrrr} 8 & -12 & 12 & -4\\ 2 & -3 & 3 & -1 \end{array}\right.\\
\omega^{6} & \begin{array}{rrrr} -1 & 3 & -3 & 2 \end{array}\\
\omega^{5} & \left\{\begin{array}{rrr} 3 & -3 & 3\\ 1 & -1 & 1 \end{array}\right.\\
\omega^{4} & \left\{\begin{array}{rrr} 2 & -2 & 2\\ 1 & -1 & 1 \end{array}\right.\\
\omega^{3} & \left\{\begin{array}{rr} 4 & -2\\ 2 & -1 \end{array}\right.\\
\omega^{2} & \begin{array}{rr} -1 & 2 \end{array}\\
\omega & \begin{array}{r} 1 \end{array}\\
\omega^{0} & \left\{\begin{array}{r} 2\\ 1 \end{array}\right.
\end{array}
\qquad k = V(1, 1, 1, 2, -1, 1, 1, 2, -1, 1, 1) = 4\,.$$

Note. All the elements of any one row may be multiplied by one and the same number without changing the signs of the elements of the first column. This remark has been used in constructing the scheme.

4. However, the application of both rules of Routh does not enable us to determine the number k in all the cases. The application of the first rule (introduction of small parameters $\varepsilon, \ldots$) is justified only when $f(z)$ has no roots on the imaginary axis.

If $f(z)$ has roots on the imaginary axis, then by varying the parameter ε some of these roots may pass over into the right half-plane and change k.

Example. $f(z) = z^6 + z^5 + 3z^4 + 3z^3 + 3z^2 + 2z + 1$.

Scheme

$$\begin{array}{llll}
\omega^{6} & 1 & 3 & 3 \quad 1\\
\omega^{5} & 1 & 3 & 2\\
\omega^{4} & \varepsilon & 1 & 1\\
\omega^{3} & 3 - \dfrac{1}{\varepsilon} & 2 - \dfrac{1}{\varepsilon} & \\
\omega^{2} & 1 - \dfrac{2\varepsilon - 1}{3 - \dfrac{1}{\varepsilon}} & 1 & \\
\omega & u & & \\
\omega^{0} & 1 & &
\end{array}$$

$$\left(u = 2 - \frac{1}{\varepsilon} - \cfrac{3 - \dfrac{1}{\varepsilon}}{1 - \cfrac{2\varepsilon - 1}{3 - \dfrac{1}{\varepsilon}}} = -\varepsilon + \cdots \right)$$

$$V\left(1, 1, \varepsilon, 3 - \frac{1}{\varepsilon}, 1, -\varepsilon, 1\right) = \begin{cases} 4 & \text{for} \quad \varepsilon > 0,\\ 2 & \text{for} \quad \varepsilon < 0. \end{cases}$$

The question of the value of k remains open.

In the general case, where $f(z)$ has roots on the imaginary axis, we have to proceed as follows:

Setting $f(z) = F_1(z) + F_2(z)$, where

$$F_1(z) = a_0 z^n + a_1 z^{n-2} + \ldots, \qquad F_2(z) = b_0 z^{n-1} + b_1 z^{n-2} + \ldots,$$

we must find the greatest common divisor $d(z)$ of $F_1(z)$ and $F_2(z)$. Then $f(z) = d(z) f^*(z)$.

If $f(z)$ has a root z for which $-z$ is also a root (all the roots on the imaginary axis have this property), then it follows from $f(z) = 0$ and $f(-z) = 0$ that $F_1(z) = 0$ and $F_2(z) = 0$, i.e., z is a root of $d(z)$. Therefore $f^*(z)$ has no roots z for which $-z$ is also a root of $f^*(z)$.

Then

$$k = k_1 + k_2,$$

where k_1 and k_2 are the respective numbers of roots of $f^*(z)$ and $d(z)$ in the right half-plane; k_1 is determined by Routh's algorithm and $k_2 = (q - s)/2$, where q is the degree of $d(z)$ and s the number of real roots of $d(i\omega)$.[16]

In the last example,

$$d(z) = z^2 + 1, \quad f^*(z) = z^4 + z^3 + 2z^2 + 2z + 1.$$

Therefore (see example on p. 182), we have $k_2 = 0$, $k_1 = 2$, and hence

$$k = 2.$$

§ 5. Lyapunov's Theorem

1. From the investigations of A. M. Lyapunov published in 1892 in his monograph 'The General Problem of Stability of Motion' there follows a theorem[17] that gives necessary and sufficient conditions for all the roots of the characteristic equation $|\lambda E - A| = 0$ of a real matrix $A = \| a_{ik} \|_1^n$ to have negative real parts. Since every polynomial

$$f(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \ldots + a_n \qquad (a_0 \neq 0)$$

---

[16] $d(i\omega)$ is a real polynomial or becomes one after cancelling i. The number of its real roots can be determined by Sturm's theorem.

[17] See [32], § 20.

can be represented as a characteristic determinant $|\lambda E - A|$.[18] Lyapunov's theorem is of general character and is applicable to an arbitrary algebraic equation $f(\lambda) = 0$.

Suppose given a real matrix $A = \| a_{ik} \|_1^n$ and a homogeneous polynomial of dimension m in the variables $x_1, x_2, \dots, x_n$:

$$V(\underbrace{x, x, \dots, x}_{m}) \qquad (x = (x_1, x_2, \dots, x_n)).$$

Let us find the total derivative with respect to t of $V(x, x, \dots, x)$ under the assumption that x is a solution of the differential system

$$\frac{dx}{dt} = Ax.$$

Then

$$\begin{aligned} \frac{d}{dt} V(x, x, \dots, x) &= V(Ax, x, \dots, x) \\ &\quad + V(x, Ax, \dots, x) + \cdots + V(x, x, \dots, Ax) \\ &= W(x, x, \dots, x), \end{aligned} \tag{21}$$

where $W(x, x, \dots, x)$ is again a homogeneous polynomial of dimension m in $x_1, x_2, \dots, x_n$. The equation (21) defines a linear operator $\widehat{A}$ which associates with every homogeneous polynomial of dimension m $V(x, x, \dots, x)$ a certain homogeneous polynomial $W(x, x, \dots, x)$ of the same dimension m

$$W = \widehat{A}(V).$$

We restrict ourselves to the case $m = 2$.[19] Then $V(x, x)$ and $W(x, x)$ are quadratic forms in the variables $x_1, x_2, \dots, x_n$ connected by the equation

$$\frac{d}{dt} V(x, x) = V(Ax, x) + V(x, Ax) = W(x, x); \tag{22}$$

hence[20]

$$W = \widehat{A}(V) = A^{\top}V + VA. \tag{23}$$

---

[18] For this purpose it is sufficient to set, for example:

$$A = \begin{Vmatrix} 0 & 0 & \dots & 0 & -\frac{a_n}{a_0} \\ 1 & 0 & \dots & 0 & -\frac{a_{n-1}}{a_0} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 & -\frac{a_1}{a_0} \end{Vmatrix}.$$

[19] A. M. Lyapunov has proved his theorem for every positive integer m.

[20] Because $V(x, y) = x^{\top}Vy$.

Here $V = \| v_{ik} \|_1^n$ and $W = \| w_{ik} \|_1^n$ are symmetric matrices formed, respectively, from the coefficients of the forms $V(x, x)$ and $W(x, x)$. The linear operator $\widehat{A}$ in the space of matrices of order n is completely determined by specification of the matrix $A = \| a_{ik} \|_1^n$.

If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the characteristic values of the matrix A, then every characteristic value of the operator $\widehat{A}$ can be represented in the form $\lambda_i + \lambda_k$ $(1 \leqq i, k \leqq n)$.[21]

Therefore, if the matrix $A = \| a_{ik} \|_1^n$ has no zero characteristic value and no two that are opposites, then the operator $\widehat{A}$ is non-singular. In this case the matrix W in (23) determines the matrix V uniquely.

If V is symmetric, then the matrix W defined by (23) is also symmetric. If $\widehat{A}$ is a non-singular operator, then the converse statement also holds: Every symmetric matrix W corresponds by (23) to a symmetric matrix V. For in this case we find, by going over to the transposed matrices on both sides of (23), that the matrix $V^{\top}$, as well as V, satisfies (23). By the uniqueness of the solution, $V^{\top} = V$.

Thus: *If the matrix $A = \| a_{ik} \|_1^n$ has no zero and no two opposite characteristic values, then every quadratic form $W(x, x)$ corresponds to one and only one quadratic form $V(x, x)$ connected with $W(x, x)$ by* (22).

Now we can formulate Lyapunov's theorem.

THEOREM 3 (Lyapunov): *If all the characteristic values of the real matrix $A = \| a_{ik} \|_1^n$ have negative real parts, then to every negative-definite quadratic form $W(x, x)$ there corresponds a positive-definite quadratic form $V(x, x)$ connected with $W(x, x)$—taking*

$$\frac{dx}{dt} = Ax \tag{24}$$

into account—by the equation

$$\frac{d}{dt} V(x, x) = W(x, x). \tag{25}$$

Conversely, if for every negative-definite form $W(x, x)$ there exists a positive-definite form $V(x, x)$ connected with $W(x, x)$ by the equation (25) *—taking* (24) *into account—then all the characteristic values of the matrix $A = \| a_{ik} \|_1^n$ have negative real parts.*

Proof. 1. Suppose that all the characteristic values of A have negative real parts. Then for every solution $x = e^{At}x_0$ of (24) we have $\lim_{t \to +\infty} x = o$.[22] Suppose that the forms $V(x, x)$ and $W(x, x)$ are connected by (25) and that

---

[21] See footnote 18.

[22] See Vol. I, Chapter V, § 6.

$W(x,x) < 0$ $(x \neq o)$.[23]

Let us assume that for some $x_0 \neq o$

$$V_0 = V(x_0, x_0) \leqq 0.$$

But $\frac{d}{dt} V(x,x) = W(x,x) < 0$ $(x = e^{At} x_0)$. Therefore for $t > 0$ the value of $V(x,x)$ is negative and decreases for $t \to \infty$, which results in a contradiction to the equation $\lim\limits_{t\to\infty} V(x,x) = \lim\limits_{x \to o} V(x,x) = 0$. Therefore $V(x,x) > 0$ for $x \neq o$, i.e., $V(x,x)$ is a positive-definite quadratic form.

2. Suppose, conversely, that in (25)

$$W(x,x) < 0, \quad V(x,x) > 0 \qquad (x \neq o).$$

From (25) it follows that

$$V(x, x) = V(x_0, x_0) + \int_0^t W(x, x)\, dt \qquad (x = e^{At} x_0). \tag{25'}$$

We shall show that for every $x_0 \neq o$ the column $x = e^{At} x_0$ comes arbitrarily near to zero for arbitrarily large values of $t > 0$. Assume the contrary. Then there exists a number $\nu > 0$ such that

$$W(x, x) < -\nu < 0 \qquad (x = e^{At} x_0, \quad x_0 \neq o, \quad t > 0).$$

But then from (25′)

$$V(x, x) < V(x_0, x_0) - \nu t,$$

and so for sufficiently large values of t we have $V(x,x) < 0$, which contradicts our assumption.

From what we have shown, it follows that for certain sufficiently large values of t the value of $V(x,x)$ $(x = e^{At} x_0, x_0 \neq 0)$ will be arbitrarily near to zero. But $V(x,x)$ decreases monotonically for $t > 0$, since $\frac{d}{dt} V(x,x) = W(x,x) < 0$. Therefore $\lim\limits_{t\to\infty} V(x,x) = 0$.

Hence it follows that for every $x_0 \neq o$, $\lim\limits_{t\to\infty} e^{At} x_0 = o$, i.e., $\lim\limits_{t\to\infty} e^{At} = O$. This is only possible if all the characteristic values of A have negative real parts (see Vol. I, Chapter V, § 6).

The theorem is now completely proved.

For the form $W(x,x)$ in Lyapunov's theorem we can take any negative-definite form, in particular, the form $-\sum\limits_{i=1}^{n} x_i^2$. In this case the theorem admits of the following matrix formulation:

---

[23] The form $W(x,x)$ is given arbitrarily. The form $V(x,x)$ is uniquely determined by (25), because A has in this case neither the characteristic value zero nor pairs of opposite characteristic values.

THEOREM 3′: *All the characteristic values of the real matrix* $A = \| a_{ik} \|_1^n$ *have negative real parts if and only if the matrix equation*

$$A'V + VA = -E \tag{26}$$

has as its solution V *the coefficient matrix of some positive-definite quadratic form* $V(x,x) > 0$.

2. From this theorem we derive a criterion for determining the stability of a non-linear system from its linear approximation.[24]

Suppose that it is required to prove the asymptotic stability of the zero solution of the non-linear system of differential equations (1) (p. 172) in the case where the coefficients a_{ik} $(i, k = 1, 2, \dots, n)$ in the linear terms on the right-hand side form a matrix $A = \| a_{ik} \|_1^n$ having only characteristic values with negative real parts. Then, if we determine a positive-definite form $V(x,x)$ by the matrix equation (26) and calculate its total derivative with respect to time under the assumption that $x = (x_1, x_2, \dots, x_n)$ is a solution of the given system (1), we have:

$$\frac{d}{dt} V(x, x) = -\sum_{i=1}^{n} x_i^2 + R(x_1, x_2, \dots, x_n),$$

where $R(x_1, x_2, \dots, x_n)$ is a series containing terms of the third and higher total degree in $x_1, x_2, \dots, x_n$. Therefore, in some sufficiently small neighborhood of $(0, 0, \dots, 0)$ we have simultaneously for every $x \neq o$

$$V(x, x) > 0, \quad \frac{d}{dt} V(x, x) < 0.$$

By Lyapunov's general criterion of stability[25] this also indicates the asymptotic stability of the zero solution of the system of differential equations.

If we express the elements of V from the matrix equation (26) in terms of the elements of A and substitute these expressions in the inequalities

$$v_{11} > 0, \quad \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} > 0, \quad \dots, \quad \begin{vmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{vmatrix} > 0,$$

then we obtain the inequalities that the elements of a matrix $A = \| a_{ik} \|_1^n$ must satisfy in order that all the characteristic values of the matrix should

---

[24] See [32], § 26; [9], pp. 113 ff.; [36], pp. 66 ff.

[25] See [32], § 16; [9], pp. 19-21 and 31-33; [36], pp. 32-34.

have negative real parts. However, these inequalities can be obtained in a considerably simpler form from the criterion of Routh-Hurwitz, which will be discussed in the following section.

Note. Lyapunov's theorem (3) or (3′) can be generalized immediately to the case of an arbitrary *complex* matrix $A = \| a_{ik} \|_1^n$. The quadratic forms $V(x, x)$ and $W(x, x)$ are then replaced by Hermitian forms

$$V(x, x) = \sum_{i,k=1}^{n} v_{ik}\bar{x}_i x_k, \quad W(x, x) = \sum_{i,k=1}^{n} w_{ik}\bar{x}_i x_k.$$

Correspondingly, the matrix equation (26) is replaced by the equation

$$A^* V + VA = -E \qquad (A^* = \overline{A'}).$$

§ 6. The Theorem of Routh-Hurwitz

1. In the preceding sections we have explained the method of Routh, unsurpassed in its simplicity, of determining the number k of roots in the right half-plane of a real polynomial whose coefficients are given as explicit numbers. If the coefficients of the polynomial depend on parameters and it is required to determine for what values of the parameters the number k has one value or another—in particular, the value 0 ('domain of stability')[26] —then it is desirable to have explicit expressions for the values of $c_0, d_0, \ldots$ in terms of the coefficients of the given polynomial. In solving this problem, we obtain a method of determining k and, in particular, a stability criterion in a form in which it was established by Hurwitz [204].

We again consider the polynomial

$$f(z) = a_0 z^n + b_0 z^{n-1} + a_1 z^{n-2} + b_1 z^{n-3} + \cdots \quad (a_0 \neq 0).$$

By the *Hurwitz matrix* we mean the square matrix of order n

$$H = \begin{Vmatrix} b_0 & b_1 & b_2 & \ldots & b_{n-1} \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & b_0 & b_1 & \ldots & b_{n-2} \\ 0 & a_0 & a_1 & \ldots & a_{n-2} \\ 0 & 0 & b_0 & \ldots & b_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix} \quad \left(\begin{array}{l} a_k = 0 \quad \text{for} \quad k > \left[\frac{n}{2}\right], \\ b_k = 0 \quad \text{for} \quad k > \left[\frac{n-1}{2}\right] \end{array} \right). \tag{27}$$

---

[26] For this is precisely the situation in planning new mechanical or electrical systems of governors.

We transform the matrix by subtracting from the second, fourth, ... rows the first, third, ..., row, multiplied by a_0/b_0.[27] We obtain the matrix

$$\begin{Vmatrix} b_0 & b_1 & b_2 & \ldots & b_{n-1} \\ 0 & c_0 & c_1 & \ldots & c_{n-2} \\ 0 & b_0 & b_1 & \ldots & b_{n-2} \\ 0 & 0 & c_0 & \ldots & c_{n-3} \\ 0 & 0 & b_0 & \ldots & b_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix}.$$

In this matrix $c_0, c_1, \ldots$ is the third row of Routh's scheme supplemented by zeros ($c_k = 0$ for $k > [n/2] - 1$).

We transform this matrix again by subtracting from the third, fifth, ... rows the second, fourth, ... row, multiplied by b_0/c_0:

$$\begin{Vmatrix} b_0 & b_1 & b_2 & b_3 & \ldots \\ 0 & c_0 & c_1 & c_2 & \ldots \\ 0 & 0 & d_0 & d_1 & \ldots \\ 0 & 0 & c_0 & c_1 & \ldots \\ 0 & 0 & 0 & d_0 & \ldots \\ 0 & 0 & 0 & c_0 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix}.$$

Continuing this process, we ultimately arrive at a triangular matrix of order n

$$R = \begin{Vmatrix} b_0 & b_1 & b_2 & \ldots \\ 0 & c_0 & c_1 & \ldots \\ 0 & 0 & d_0 & \ldots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{Vmatrix}. \tag{28}$$

which we call the *Routh matrix*. It is obtained from Routh's scheme (see (15)) by: 1) deleting the first row; 2) shifting the rows to the right so that their first elements come to lie on the main diagonal; and 3) completing it by zeros to a square matrix of order n.

---

[27] We begin by dealing with the regular case where $b_0 \neq 0,\ c_0 \neq 0,\ d_0 \neq 0, \ldots$.

DEFINITION 2: *Two matrices* $A = \| a_{ik} \|_1^n$ *and* $B = \| b_{ik} \|_1^n$ *will be called equivalent if and only if for every* $p \leqq n$ *the corresponding minors of order* p *in the first* p *rows are equal:*

$$A\begin{pmatrix}1 & 2 & \dots & p\\ i_1 & i_2 & \dots & i_p\end{pmatrix} = B\begin{pmatrix}1 & 2 & \dots & p\\ i_1 & i_2 & \dots & i_p\end{pmatrix} \quad \begin{pmatrix} i_1,\ i_2,\ \dots,\ i_p = 1, 2, \dots, n\\ p = 1, 2, \dots, n\end{pmatrix}.$$

Since we do not change the values of the minors of order p in the first p rows when we subtract from any row of the matrix an arbitrary multiple of any preceding row, the Hurwitz and Routh matrices H and R are equivalent in the sense of Definition 2:

$$H\begin{pmatrix}1 & 2 & \dots & p\\ i_1 & i_2 & \dots & i_p\end{pmatrix} = R\begin{pmatrix}1 & 2 & \dots & p\\ i_1 & i_2 & \dots & i_p\end{pmatrix} \quad \begin{pmatrix} i_1,\ i_2,\ \dots,\ i_p = 1, 2, \dots, n,\\ p = 1, 2, \dots, n\end{pmatrix}. \tag{29}$$

The equivalence of the matrices H and R enables us to express all the elements of R, i.e. of the Routh scheme, in terms of the minors of the Hurwitz matrix H and, therefore, in terms of the coefficients of the given polynomial. For when we give to p in (29) the values 1, 2, 3, ... in succession, we obtain

$$\left.\begin{aligned}
&H\begin{pmatrix}1\\1\end{pmatrix} = b_0, \quad H\begin{pmatrix}1\\2\end{pmatrix} = b_1, \quad H\begin{pmatrix}1\\3\end{pmatrix} = b_2, \ \dots,\\
&H\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix} = b_0c_0, \quad H\begin{pmatrix}1 & 2\\1 & 3\end{pmatrix} = b_0c_1, \quad H\begin{pmatrix}1 & 2\\1 & 4\end{pmatrix} = b_0c_2, \ \dots,\\
&H\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix} = b_0c_0d_0, \quad H\begin{pmatrix}1 & 2 & 3\\1 & 2 & 4\end{pmatrix} = b_0c_0d_1, \quad H\begin{pmatrix}1 & 2 & 3\\1 & 2 & 5\end{pmatrix} = b_0c_0d_2, \ \dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\}, \tag{30}$$

etc.

Hence we find the following expressions for the elements of Routh's scheme:

$$\left.\begin{aligned}
&b_0 = H\begin{pmatrix}1\\1\end{pmatrix}, \quad b_1 = H\begin{pmatrix}1\\2\end{pmatrix}, \quad b_2 = H\begin{pmatrix}1\\3\end{pmatrix}, \ \dots,\\
&c_0 = \frac{H\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}}{H\begin{pmatrix}1\\1\end{pmatrix}}, \quad c_1 = \frac{H\begin{pmatrix}1 & 2\\1 & 3\end{pmatrix}}{H\begin{pmatrix}1\\1\end{pmatrix}}, \quad c_2 = \frac{H\begin{pmatrix}1 & 2\\1 & 4\end{pmatrix}}{H\begin{pmatrix}1\\1\end{pmatrix}}, \ \dots,\\
&d_0 = \frac{H\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}}{H\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}}, \quad d_1 = \frac{H\begin{pmatrix}1 & 2 & 3\\1 & 2 & 4\end{pmatrix}}{H\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}}, \quad d_2 = \frac{H\begin{pmatrix}1 & 2 & 3\\1 & 2 & 5\end{pmatrix}}{H\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}}, \ \dots,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \tag{31}$$

The successive principal minors of H are usually called the *Hurwitz determinants.* We shall denote them by

$$\Delta_1 = H\begin{pmatrix}1\\1\end{pmatrix} = b_0, \quad \Delta_2 = H\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix} = \begin{vmatrix} b_0 & b_1\\ a_0 & a_1\end{vmatrix}, \ \dots$$

$$\dots, \ \Delta_n = H\begin{pmatrix}1 & 2 & \dots & n\\1 & 2 & \dots & n\end{pmatrix} = \begin{vmatrix} b_0 & b_1 & \dots & b_{n-1}\\ a_0 & a_1 & \dots & a_{n-1}\\ 0 & b_0 & \dots & b_{n-2}\\ 0 & a_0 & \dots & a_{n-2}\\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot \end{vmatrix}. \tag{32}$$

Note 1. By the formulas (30),[28]

$$\Delta_1 = b_0, \quad \Delta_2 = b_0c_0, \quad \Delta_3 = b_0c_0d_0, \quad \dots. \tag{33}$$

From $\Delta_1 \neq 0, \dots, \Delta_p \neq 0$ it follows that the first p of the numbers $b_0, c_0, \dots$ are different from zero, and vice versa; in this case the p successive rows of Routh's scheme beginning with the third are completely determined and the formulas (31) hold for them.

Note 2. The regular case (all the $b_0, c_0, \dots$ have a meaning and are different from zero) is characterized by the inequalities

$$\Delta_1 \neq 0, \quad \Delta_2 \neq 0, \quad \dots, \quad \Delta_n \neq 0.$$

Note 3. The definition of the elements of Routh's scheme by means of the formulas (31) is more general than that by means of Routh's algorithm. Thus, for example, if $b_0 = H\begin{pmatrix}1\\1\end{pmatrix} = 0$, then Routh's algorithm does not give us anything except the first two rows formed from the coefficients of the given polynomial. However if for $\Delta_1 = 0$ the remaining determinants $\Delta_2, \Delta_3, \dots$ are different from zero, then by omitting the row of c's we can determine by means of the formulas (31) all the remaining rows of Routh's scheme.

By the formulas (33),

$$b_0 = \Delta_1, \quad c_0 = \frac{\Delta_2}{\Delta_1}, \quad d_0 = \frac{\Delta_3}{\Delta_2}, \ \dots$$

and therefore

---

[28] If the coefficients of $f(z)$ are given numerically, then the formulas (33)—reducing this computation, as they do, to the formation of the Routh scheme—give by far the simplest method for computing the Hurwitz determinants.

$$V(a_0, b_0, c_0, \dots) = V\left(a_0, \Delta_1, \frac{\Delta_2}{\Delta_1}, \dots, \frac{\Delta_n}{\Delta_1}\right) = V(a_0, \Delta_1, \Delta_3, \dots) + V(1, \Delta_2, \Delta_4, \dots).$$

Hence Routh's theorem can be restated as follows:

THEOREM 4 (Routh-Hurwitz): *The number of real roots of the polynomial $f(z) = a_0 z^n + \dots$ in the right half-plane is determined by the formula*

$$k = V\left(a_0, \Delta_1, \frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_2}, \dots, \frac{\Delta_n}{\Delta_{n-1}}\right) \tag{34}$$

or (what is the same) *by*

$$k = V(a_0, \Delta_1, \Delta_3, \dots) + V(1, \Delta_2. \Delta_4, \dots). \tag{34'}$$

Note. This statement of the Routh-Hurwitz theorem assumes that we have the regular case

$$\Delta_1 \neq 0, \quad \Delta_2 \neq 0, \quad \dots, \quad \Delta_n \neq 0.$$

In the following section we shall show how this formula can be used in the singular cases where some of the Hurwitz determinants Δ_i are zero.

2. We now consider the special case where all the roots of $f(z)$ are in the left half-plane $\operatorname{Re} z < 0$. By Routh's criterion, all the $a_0, b_0, c_0, d_0, \dots$ must then be different from zero and of like sign. Since we are concerned here with the regular case, we obtain from (34) for $k = 0$ the following criterion:

CRITERION OF ROUTH-HURWITZ: *All the roots of the real polynomial $f(z) = a_0 z^n + \dots$ ($a_0 \neq 0$) have negative real parts if and only if the inequalities*

$$\left.a_0\Delta_1 > 0, \quad \Delta_2 > 0, \quad a_0\Delta_3 > 0, \quad \Delta_4 > 0, \quad \dots, \quad \begin{matrix} a_0\Delta_n > 0 \ (\textit{for odd } n), \\ \Delta_n > 0 \ (\textit{for even } n) \end{matrix}\right\} \tag{35}$$

hold.

Note. If $a_0 > 0$, these conditions can be written as follows:

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \dots, \quad \Delta_n > 0. \tag{36}$$

If we use the usual notation for the coefficients of the polynomial

$$f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_{n-1} z + a_n,$$

then for $a_0 > 0$ the Routh-Hurwitz conditions (36) can be written in the form of the following determinantal inequalities:

$$|a_1| > 0, \quad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0, \quad \dots, \quad \begin{vmatrix} a_1 & a_3 & a_5 & \dots & 0 \\ a_0 & a_2 & a_4 & \dots & 0 \\ 0 & a_1 & a_3 & \dots & 0 \\ 0 & a_0 & a_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_n \end{vmatrix} > 0. \tag{36'}$$

A real polynomial $f(z) = a_0 z^n + \dots$ whose coefficients satisfy (35), i.e., whose roots have negative real parts, is often called a *Hurwitz polynomial.*

3. In conclusion, we mention a remarkable property of Routh's scheme.

Let $f_0, f_1, \dots$ and $g_0, g_1, \dots$ be the $(m+1)$-th and $(m+2)$-th rows of the scheme ($f_0 = \Delta_m / \Delta_{m-1}$, $g_0 = \Delta_{m+1} / \Delta_m$). Since these two rows together with the subsequent rows form a Routh scheme of their own, the elements of the $(m+p+1)$-th row (of the original scheme) can be expressed in terms of the elements of the $(m+1)$-th and $(m+2)$-th rows $f_0, f_1, \dots$ and $g_0, g_1, \dots$ by the same formulas as the $(p+1)$-th row can in terms of the elements of the first two rows $a_0, a_1, \dots$ and $b_0, b_1, \dots$; that is, if we set

$$\tilde{H} = \begin{matrix} g_0 & g_1 & g_2 & \dots \\ f_0 & f_1 & f_2 & \dots \\ 0 & g_0 & g_1 & \dots \\ 0 & f_0 & f_1 & \dots \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix},$$

then we have

$$\frac{H\begin{pmatrix} 1 \dots m+p-1 & m+p \\ 1 \dots m+p-1 & m+p+k-1 \end{pmatrix}}{H\begin{pmatrix} 1 \dots m+p-1 \\ 1 \dots m+p-1 \end{pmatrix}} = \frac{\tilde{H}\begin{pmatrix} 1 \dots p-1 & p \\ 1 \dots p-1 & p+k-1 \end{pmatrix}}{\tilde{H}\begin{pmatrix} 1 \dots p-1 \\ 1 \dots p-1 \end{pmatrix}}. \tag{37}$$

The Hurwitz determinant Δ_{m+p} is equal to the product of the first $m+p$ numbers in the sequence $b_0, c_0, \dots$:

$$\Delta_{m+p} = b_0 c_0 \dots f_0 g_0 \dots l_0.$$

But

$$\Delta_m = b_0 c_0 \dots f_0, \quad \tilde{\Delta}_p = g_0 \dots l_0.$$

Therefore the following important relation[29] holds:

$$\Delta_{m+p} = \Delta_m \tilde{\Delta}_p. \tag{38}$$

---

[29] Here $\tilde{\Delta}_p$ is the minor of order p in the top left-hand corner of $\tilde{H}$.

The formula (38) holds whenever the numbers $f_0, f_1, \dots$ and $g_0, g_1, \dots$ are well defined, i.e., under the conditions $\Delta_{m-1} \neq 0$, $\Delta_m \neq 0$.

The formula (37) has a meaning if in addition to the conditions $\Delta_{m-1} \neq 0$, $\Delta_m \neq 0$ we also have $\Delta_{m+p-1} \neq 0$. From this condition it follows that the denominator of the fraction on the right-hand side of (37) is also different from zero: $\tilde{\Delta}_{p-1} \neq 0$.

§ 7. Orlando's Formula

1. In the discussion of the cases where some of the Hurwitz determinants are zero we shall have to use the following formula of Orlando [294], which expresses the determinant Δ_{n-1} in terms of the highest coefficient a_0 and the roots $z_1, z_2, \dots, z_n$ of $f(z)$:[30]

$$\Delta_{n-1} = (-1)^{\frac{n(n-1)}{2}} a_0^{n-1} \prod_{i<k}^{1,\dots,n} (z_i + z_k). \tag{39}$$

For $n = 2$ this reduces to the well-known formula for the coefficient b_0 in the quadratic equation $a_0 z^2 + b_0 z + a_1 = 0$:

$$\Delta_1 = b_0 = -a_0 (z_1 + z_2).$$

Let us assume that the formula (39) is true for polynomials of degree n, $f(z) = a_0 z^n + b_0 z^{n-1} + \cdots$ and show that it is then true for polynomials of degree $n+1$

$$\begin{aligned} F(z) &= (z+h) f(z) \\ &= a_0 z^{n+1} + (b_0 + h a_0) z^n + (a_1 + h b_0) z^{n-1} + \cdots \qquad (h = -z_{n+1}). \end{aligned}$$

For this purpose we form the auxiliary determinant of order $n+1$

$$D = \begin{vmatrix} b_0 & b_1 & \dots & b_{n-1} & h^n \\ a_0 & a_1 & \dots & a_{n-1} & -h^{n-1} \\ 0 & b_0 & \dots & b_{n-2} & h^{n-2} \\ 0 & a_0 & \dots & a_{n-2} & -h^{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & \dots & (-1)^n \end{vmatrix} \qquad \left(\begin{array}{lll} a_k = 0 & \text{for} & k > \left[\frac{n}{2}\right], \\ b_k = 0 & \text{for} & k > \left[\frac{n-1}{2}\right] \end{array} \right).$$

---

[30] The coefficients of $f(z)$ may be arbitrary complex numbers.

We multiply the first row of D by a_0 and add to it the second row multiplied by $-b_0$, the third multiplied by a_1, the fourth by $-b_1$, etc. Then in the first row all the elements except the last are zero, and the last element is $f(h)$. Hence we deduce that

$$D = (-1)^n \Delta_{n-1} f(h).$$

On the other hand, when we add to each row of D (except the last) the next multiplied by h we obtain, apart from a factor $(-1)^n$, the Hurwitz determinant Δ_n^* of order n for the polynomial $F(z)$:

$$D = (-1)^n \begin{vmatrix} b_0 + h a_0 & b_1 + h a_1 & \dots \\ a_0 & a_1 + h b_0 & \dots \\ 0 & b_0 + h a_0 & \dots \\ 0 & a_0 & \dots \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = (-1)^n \Delta_n^*.$$

Thus

$$\Delta_n^* = \Delta_{n-1} f(h) = a_0 \Delta_{n-1} \prod_{i=1}^{n} (h - z_i).$$

When we replace Δ_{n-1} by its expression (39) and set $h = -z_{n+1}$, we obtain

$$\Delta_n^* = (-1)^{\frac{(n+1)n}{2}} a_0^n \prod_{i<k}^{1,\dots,n+1} (z_i + z_k).$$

Thus, by mathematical induction Orlando's formula is established for polynomials of every degree.

From Orlando's formula it follows that: $\Delta_{n-1} = 0$ *if and only if the sum of two roots of $f(z)$ is zero.*[31]

Since $\Delta_n = c \Delta_{n-1}$, where c is the constant term of the polynomial $f(z)$ ($c = (-1)^n a_0 z_1 z_2 \dots z_n$), it follows from (39) that:

$$\Delta_n = (-1)^{\frac{n(n-1)}{2}} a_0^n z_1 z_2 \cdots z_n \prod_{i<k}^{1,\dots,n} (z_i + z_k). \tag{40}$$

The last formula shows that: *Δ_n vanishes if and only if $f(z)$ has a pair of opposite roots z and $-z$.*

---

[31] In particular, $\Delta_{n-1} = 0$ when $f(z)$ has at least one pair of conjugate pure imaginary roots or multiple zero roots.

§ 8. Singular Cases in the Routh-Hurwitz Theorem

In discussing the singular cases where some of the Hurwitz determinants are zero, we may assume that $\Delta_n \neq 0$ (and consequently $\Delta_{n-1} \neq 0$).

For if $\Delta_n = 0$, then, as we have seen at the end of the preceding section, the real polynomial $f(z)$ has a root z' for which $-z'$ is also a root. If we set $f(z) = F_1(z) + F_2(z)$, where

$$F_1(z) = a_0 z^n + a_1 z^{n-2} + \cdots, \quad F_2(z) = b_0 z^{n-1} + b_1 z^{n-3} + \cdots,$$

then we can deduce from $f(z') = f(-z') = 0$ that $F_1(z') = F_2(z') = 0$. Therefore z' is a root of the greatest common divisor $d(z)$ of the polynomials $F_1(z)$ and $F_2(z)$. Setting $f(z) = d(z) f^*(z)$, we reduce the Routh-Hurwitz problem for $f(z)$ to that for the polynomial $f^*(z)$ for which the last Hurwitz determinant is different from zero.

1. To begin with, we examine the case where

$$\Delta_1 = \ldots = \Delta_p = 0, \quad \Delta_{p+1} \neq 0, \quad \ldots, \quad \Delta_n \neq 0. \tag{41}$$

From $\Delta_1 = 0$ it follows that $b_0 = 0$; from $\Delta_2 = \begin{vmatrix} 0 & b_1 \\ a_0 & a_1 \end{vmatrix} = -a_0 b_1 = 0$ it follows that $b_1 = 0$. But then we have automatically

$$\Delta_3 = \begin{vmatrix} 0 & b_1 & b_2 \\ a_0 & a_1 & a_2 \\ 0 & 0 & b_1 \end{vmatrix} = -a_0 b_1^2 = 0.$$

From

$$\Delta_4 = \begin{vmatrix} 0 & 0 & b_2 & b_3 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & b_2 \\ 0 & a_0 & a_1 & a_2 \end{vmatrix} = -a_0^2 b_2^2 = 0$$

it follows that $b_2 = 0$ and then $\Delta_5 = -a_0^2 b_2^3 = 0$, etc.

This argument shows that in (41) p is always an *odd* number $p = 2h - 1$. Then $b_0 = b_1 = b_2 = \ldots = b_{h-1} = 0$, $b_h \neq 0$, and[32]

$$\Delta_{p+1} = \Delta_{2h} = (-1)^{\frac{h(h+1)}{2}} a_0^h b_h^h, \quad \Delta_{p+2} = \Delta_{2h+1} = (-1)^{\frac{h(h+1)}{2}} a_0^h b_h^{h+1} = \Delta_{p+1} b_h. \tag{42}$$

Let us vary the coefficients $b_0, b_1, \ldots, b_{h-1}$ in such a way that for the new, slightly altered values $b_0^*, b_1^*, \ldots, b_{h-1}^*$ all the Hurwitz determinants $\Delta_1^*, \Delta_2^*, \ldots, \Delta_n^*$ become different from zero and $\Delta_{p+1}^*, \ldots, \Delta_n^*$ keep their previous signs. We shall take $b_0^*, b_1^*, \ldots, b_{h+1}^*$ as 'small' values of different orders of 'smallness'; indeed, we shall assume that every b_{j-1}^* is in absolute value 'considerably' smaller than b_j^* ($j = 1, 2, \ldots, h$; $b_h^* = b_h$). The latter means that in computing the sign of an integral algebraic expression in the b_i^* we can neglect terms in which some b_i^* have an index less than j in comparison with terms where all the b_i^* have an index at least j. We can then easily find the 'sign-determining' terms of $\Delta_1^*, \Delta_2^*, \ldots, \Delta_p^*$ $(p = 2h - 1)$:[33]

$$\Delta_1^* = b_0^*, \ \Delta_2^* = -a_0 b_1^* + \cdots, \ \Delta_3^* = -a_0 b_1^{*2} + \cdots, \ \Delta_4^* = -a_0^2 b_2^{*2} + \cdots,$$
$$\Delta_5^* = -a_0^2 b_2^{*3} + \cdots, \ \Delta_6^* = a_0^3 b_3^{*3} + \cdots,$$

etc.; in general,

$$\left.\begin{aligned} \Delta_{2j}^* &= (-1)^{\frac{j(j-1)}{2}} a_0^j b_j^{*j} + \cdots & (j = 1, 2, \ldots, h-1), \\ \Delta_{2j+1}^* &= (-1)^{\frac{j(j+1)}{2}} a_0^j b_j^{*j+1} + \cdots & (j = 0, 1, \ldots, h-1). \end{aligned}\right\} \tag{43}$$

We choose $b_0^*, b_1^*, \ldots, b_{2h-1}$ as positive; then the sign of Δ_i^* is determined by the formula

$$\operatorname{sign} \Delta_i^* = (-1)^{\frac{j(j+1)}{2}} \operatorname{sign} a_0^j \qquad \left(j = \left[\frac{i}{2}\right], \ i = 1, 2, \ldots, p\right). \tag{44}$$

In any small variation of the coefficients of the polynomial the number k remains unchanged, because $f(z)$ has no roots on the imaginary axis. Therefore, starting from (44) we determine the number of roots in the right half-plane by the formula

$$k = V\left(a_0, \Delta_1^*, \frac{\Delta_2^*}{\Delta_1^*}, \ldots, \frac{\Delta_{p-1}}{\Delta_p^*}, \frac{\Delta_{p+2}}{\Delta_{p+1}}\right) + V\left(\frac{\Delta_{p+2}}{\Delta_{p+1}}, \ldots, \frac{\Delta_n}{\Delta_{n-1}}\right). \tag{45}$$

An elementary calculation based on (42) and (44) shows that

$$V\left(a_0, \Delta_1^*, \frac{\Delta_2^*}{\Delta_1^*}, \ldots, \frac{\Delta_{p-1}}{\Delta_p^*}, \frac{\Delta_{p+2}}{\Delta_{p+1}}\right) = h + \frac{1 - (-1)^h \varepsilon}{2} \left(\begin{aligned} p &= 2h - 1 \\ \varepsilon &= \operatorname{sign}\left(a_0 \frac{\Delta_{p+2}}{\Delta_{p+1}}\right) \end{aligned}\right) \tag{46}$$

Note that the value on the left-hand side of (46) does not depend on the method of varying the coefficients and retains one and the same sign for arbitrary small variations. This follows from (45), because k does not change its value under small variations of the coefficients.

---

[32] From (42) it follows that for odd h sign $\Delta_{p+2} = (-1)^{\frac{h+1}{2}}$ sign a_0, and for even h sign $\Delta_{p+1} = (-1)^{\frac{h}{2}}$.

[33] Essentially the same terms have already been computed above for $\Delta_1, \Delta_2, \ldots, \Delta_p$.

2. Suppose now that for $s > 0$

$$\Delta_{s+1} = \cdots = \Delta_{s+p} = 0 \tag{47}$$

and that all the remaining Hurwitz determinants are different from zero.

We denote by $\tilde{a}_0, \tilde{a}_1, \ldots$ and $\tilde{b}_0, \tilde{b}_1, \ldots$ the elements of the $(s+1)$-th rows in Routh's scheme ($\tilde{a}_0 = \Delta_s/\Delta_{s-1}$, $\tilde{b}_0 = \Delta_{s+1}/\Delta_s$). We denote the corresponding determinants by $\tilde{\Delta}_1, \tilde{\Delta}_2, \ldots, \tilde{\Delta}_{n-s}$. By formula (38) (p. 195),

$$\Delta_{s+1} = \Delta_s\tilde{\Delta}_1, \ \ldots, \ \Delta_{s+p} = \Delta_s\tilde{\Delta}_p, \ \ \Delta_{s+p+1} = \Delta_s\tilde{\Delta}_{p+1}, \ \ \Delta_{s+p+2} = \Delta_s\tilde{\Delta}_{p+2}. \tag{48}$$

Then by **1.** it follows that p is odd, say $p = 2h - 1$.[34]

Let us vary the coefficients of $f(z)$ in such a way that all the Hurwitz determinants become different from zero and that those that were different from zero before the variation retain their sign. Since the formula (46) is applicable to the determinants $\tilde{\Delta}$, we then obtain, starting from (48):

$$V\left(\frac{\Delta_s}{\Delta_{s-1}}, \frac{\Delta^*_{s+1}}{\Delta_s}, \ldots, \frac{\Delta_{s+p+1}}{\Delta^*_{s+p}}, \frac{\Delta_{s+p+2}}{\Delta_{s+p+1}}\right)$$
$$= h + \frac{1-(-1)^h\varepsilon}{2} \quad \left(\begin{array}{c} p = 2h-1, \\ \varepsilon = \operatorname{sign}\left(\dfrac{\Delta_s}{\Delta_{s-1}}\dfrac{\Delta_{s+p+2}}{\Delta_{s+p+1}}\right)\end{array}\right), \tag{49}$$

$$k = V\left(a_0, \Delta_1, \ldots, \frac{\Delta_s}{\Delta_{s-1}}\right) + V\left(\frac{\Delta_s}{\Delta_{s-1}}, \frac{\Delta^*_{s+1}}{\Delta_s}, \ldots, \frac{\Delta_{s+p+2}}{\Delta_{s+p+1}}\right) + V\left(\frac{\Delta_{s+p+2}}{\Delta_{s+p+1}}, \ldots, \frac{\Delta_n}{\Delta_{n-1}}\right).$$

The value on the left-hand side of (49) again does not depend on the method of variation.

3. Finally, let us assume that among the Hurwitz determinants there are ν groups of zero determinants. We shall show that for every such group (47) the value on the left-hand side of (49) does not depend on the method of variation and is determined by that formula.[35] We have proved this statement for $\nu = 1$. Let us assume that it is true for $\nu - 1$ groups and then show that it is also true for ν groups. Suppose that (47) is the second of the ν groups; we determine $\tilde{\Delta}_1, \tilde{\Delta}_2$ in the same way as was done under **2.**; then for this variation

---

[34] In accordance with footnote 32, for $p = 2h - 1$ and odd h,

$$\operatorname{sign}\Delta_{s+p+2} = (-1)^{\frac{h+1}{2}}\operatorname{sign}\Delta_{s-1};$$

and for even h,

$$\operatorname{sign}\Delta_{s+p+1} = (-1)^{\frac{h}{2}}\operatorname{sign}\Delta_s.$$

[35] From (47) and $\Delta_s \neq 0, \Delta_{s+p+1} \neq 0$ it follows by (48) and (42) that $\Delta_{s-1} \neq 0$, $\Delta_{s+p+2} \neq 0$.

$$V\left(\frac{\Delta^*_s}{\Delta^*_{s-1}}, \ldots, \frac{\Delta^*_n}{\Delta^*_{n-1}}\right) = V\left(\tilde{a}^*_0, \tilde{\Delta}^*_1, \ldots, \frac{\tilde{\Delta}^*_{n-s}}{\tilde{\Delta}^*_{n-s-1}}\right).$$

Since we have only $\nu - 1$ groups of zero determinants on the right-hand side of this equation, our statement holds for the right-hand side and hence for the left-hand side of the equation. In other words, the formula (49) holds for the second, ..., ν-th group of zero Hurwitz determinants. But then it follows from the formula

$$k = V\left(a^*_0, \Delta^*_1, \frac{\Delta^*_2}{\Delta^*_1}, \ldots, \frac{\Delta^*_n}{\Delta^*_{n-1}}\right)$$

that the value of $V\left(\frac{\Delta_s}{\Delta_{s-1}}, \frac{\Delta^*_{s+1}}{\Delta_s}, \frac{\Delta^*_{s+2}}{\Delta^*_{s+1}}, \ldots, \frac{\Delta_{s+p+2}}{\Delta_{s+p+1}}\right)$ does not depend on the method of variation for the first group of zero determinants, and therefore that (49) holds for this group as well.

Thus we have proved the following theorem:

THEOREM 5: *If some of the Hurwitz determinants are zero, but $\Delta_n \neq 0$, then the number of roots of the real polynomial $f(z)$ in the right half-plane is determined by the formula*

$$k = V\left(a_0, \Delta_1, \frac{\Delta_2}{\Delta_1}, \ldots, \frac{\Delta_n}{\Delta_{n-1}}\right)$$

in which for the calculation of the value of V for every group of p successive zero determinants (p is always odd!)

$$(\Delta_s \neq 0) \ \Delta_{s+1} = \cdots = \Delta_{s+p} = 0 \quad (\Delta_{s+p+1} \neq 0)$$

we have to set

$$V\left(\frac{\Delta_s}{\Delta_{s-1}}, \frac{\Delta_{s+1}}{\Delta_s}, \ldots, \frac{\Delta_{s+p+2}}{\Delta_{s+p+1}}\right) = h + \frac{1-(-1)^h\varepsilon}{2} \tag{50}$$

where[36]

$$p = 2h - 1 \quad \textit{and} \quad \varepsilon = \operatorname{sign}\left(\frac{\Delta_s}{\Delta_{s-1}}\frac{\Delta_{s+p+2}}{\Delta_{s+p+1}}\right).$$

§ 9. The Method of Quadratic Forms. Determination of the Number of Distinct Real Roots of a Polynomial

Routh obtained his algorithm by applying Sturm's theorem to the computation of the Cauchy index of a regular rational fraction of special type (see formula (10) on p. 178). Of the two polynomials in this fraction—numera-

---

[36] For $s = 1$ $\frac{\Delta_s}{\Delta_{s-1}}$ is to be replaced by Δ_1; and for $s = 0$, by a_0.

tor and denominator—one contains only even, the other only odd powers of the argument z.

In this and in the following sections we shall explain the deeper and more comprehensive method of quadratic forms, due to Hermite, in its application to the Routh-Hurwitz problem. By means of this method we shall obtain an expression for the index of an arbitrary rational fraction in terms of the coefficients of the numerator and denominator. The method of quadratic forms enables us to apply the results of Frobenius' subtle investigations in the theory of Hankel forms (Vol. I, Chapter X, § 10) to the Routh-Hurwitz problem and to establish a close connection of certain remarkable theorems of Chebyshev and Markov with the problem of stability.

1. We shall acquaint the reader with the method of quadratic forms first in the comparatively simple problem of determining the number of distinct real roots of a polynomial.

In the solution of this problem we may restrict ourselves to the case where $f(z)$ is a real polynomial. For suppose that $f(z) = u(z) + iv(z)$ is a complex polynomial ($u(z)$ and $v(z)$ being real polynomials). Each real root of $f(z)$ makes $u(z)$ and $v(z)$ vanish simultaneously. Therefore the complex polynomial $f(z)$ has the same real roots as the real polynomial $d(z)$, the greatest common divisor of $u(z)$ and $v(z)$.

Thus, let $f(z)$ be a real polynomial with the distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_q$ of the respective multiplicities $n_1, n_2, \ldots, n_q$:

$$f(z) = a_0 (z - \alpha_1)^{n_1} (z - \alpha_2)^{n_2} \cdots (z - \alpha_q)^{n_q}$$

$$(a_0 \neq 0;\ \alpha_i \neq \alpha_k \quad \text{for} \quad i \neq k;\ i, k = 1, 2, \ldots, q).$$

We introduce Newton's sums

$$s_p = \sum_{j=1}^{q} n_j \alpha_j^p \quad (p = 0, 1, 2, \ldots).$$

With these sums we form the Hankel forms

$$S_n(x, x) = \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k,$$

where n is an arbitrary integer, $n \geqq q$.

Then the following theorem holds:

THEOREM 6: *The number of all the distinct roots of $f(z)$ is equal to the rank, and the number of all the distinct real roots to the signature, of the form $S_n(x, x)$.*

Proof. From the definition of the form $S_n(x, x)$ we immediately obtain the following representation:

$$S_n(x, x) = \sum_{j=1}^{q} n_j (x_0 + \alpha_j x_1 + \alpha_j^2 x_2 + \cdots + \alpha_j^{n-1} x_{n-1})^2. \tag{51}$$

Here to each root α_j of $f(z)$ there corresponds the square of a linear form $Z_j = x_0 + \alpha_j x_1 + \ldots + \alpha_j^{n-1} x_{n-1}$ ($j = 1, 2, \ldots, q$). The forms $Z_1, Z_2, \ldots, Z_q$ are linearly independent, since their coefficients form the Vandermonde matrix $\| \alpha_j^k \|$ whose rank is equal to the number of distinct α_j, i.e., to q. Therefore (see Vol. I, p. 297) the rank of the form $S_n(x, x)$ is q.

In the representation (51) to each real root α_j there corresponds a positive square. To each pair of conjugate complex roots α_j and $\overline{\alpha}_j$ there correspond two complex conjugate forms:

$$Z_j = P_j + iQ_j, \quad \overline{Z}_j = P_j - iQ_j;$$

the corresponding terms in (51) together give one positive and one negative square:

$$n_j Z_j^2 + n_j \overline{Z}_j^2 = 2 n_j P_j^2 - 2 n_j Q_j^2.$$

Hence it is easy to see[37] that the signature of $S_n(x, x)$, i.e., the difference between the number of positive and negative squares, is equal to the number of distinct real α_j.

This proves the theorem.

2. Using the rule for determining the signature of a quadratic form that we established in Chapter X (Vol. I, p. 303), we obtain from the theorem the following corollary:

COROLLARY: *The number of distinct real roots of the real polynomial $f(z)$ is equal to the excess of permanences of sign over variations of sign in the sequence*

$$1,\ s_0,\ \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix},\ \ldots,\ \begin{vmatrix} s_0 & s_1 & \ldots & s_{n-1} \\ s_1 & s_2 & \ldots & s_n \\ . & . & . & . \\ s_{n-1} & s_n & \ldots & s_{2n-2} \end{vmatrix}, \tag{52}$$

where the s_p ($p = 0, 1, \ldots$) are Newton's sums for $f(z)$ and n is any integer not less than the number q of distinct roots of $f(z)$ (in particular, n can be chosen as the degree of $f(z)$).

---

[37] The quadratic form $S_n(x, x)$ is representable as an (algebraic) sum of q squares of the real forms Z_j (for real α_j) and P_j and Q_j (for complex α_j). These forms are linearly independent, since the rank of $S_n(x, x)$ is q.

This rule for determining the number of distinct real roots is directly applicable only when all the numbers in (52) are different from zero. However, since we deal here with the computation of the signature of a Hankel form, by the results of Vol. I, Chapter X, § 10, the rule with proper refinements remains valid in the general case (for further details see § 11 of that chapter).

From our theorem it follows that: *All the forms*

$$S_n(x, x) \quad (n = q, q+1, \dots)$$

have the same rank and the same signature.

In applying Theorem 6 (or its corollary) to determine the number of distinct real roots, we may take n to be the degree of $f(z)$.

The number of distinct real roots of the real polynomial $f(z)$ is equal to the index $I_{-\infty}^{+\infty} \frac{f'(z)}{f(z)}$ (see p. 175). Therefore the corollary to Theorem 6 gives the formula

$$I_{-\infty}^{+\infty} \frac{f'(z)}{f(z)} = n - 2V\left(1, s_0, \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}, \dots, \begin{vmatrix} s_0 & s_1 & \dots & s_{n-1} \\ s_1 & s_2 & \dots & s_n \\ . & . & . & . \\ s_{n-1} & s_n & \dots & s_{2n-2} \end{vmatrix}\right),$$

where $s_p = \sum_{j=1}^{q} n_j \alpha_j^p$ $(p = 0, 1, \dots)$ are Newton's sums and n is the degree of $f(z)$.

In § 11 we shall establish a similar formula for the index of an arbitrary rational fraction. The information on infinite Hankel matrices that will be required for this purpose will be given in the next section.

§ 10. Infinite Hankel Matrices of Finite Rank

1. Let

$$s_0, s_1, s_2, \dots$$

be a sequence of complex numbers. This determines an infinite symmetric matrix

$$S = \left\| \begin{matrix} s_0 & s_1 & s_2 & \dots \\ s_1 & s_2 & s_3 & \dots \\ s_2 & s_3 & s_4 & \dots \\ . & . & . & . \end{matrix} \right\|,$$

which is usually called a *Hankel matrix.* Together with the infinite Hankel matrices we shall consider[38] the finite Hankel matrices $S_n = \| s_{i+k} \|_0^{n-1}$ and their associated Hankel forms

$$S_n(x, x) = \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k.$$

The successive principal minors of S will be denoted by $D_1, D_2, D_3, \dots$

$$D_p = | s_{i+k} |_0^{p-1} \qquad (p = 1, 2, \dots).$$

Infinite matrices may be of finite or of infinite rank. In the latter case, the matrices have non-zero minors of arbitrarily large order. The following theorem gives a necessary and sufficient condition for a sequence of numbers $s_0, s_1, s_2, \dots$ to generate an infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$ of finite rank.

THEOREM 7: *The infinite matrix $S = \| s_{i+k} \|_0^\infty$ is of finite rank r if and only if there exist r numbers $\alpha_1, \alpha_2, \dots, \alpha_r$ such that*

$$s_q = \sum_{g=1}^{r} \alpha_g s_{q-g} \qquad (q = r, r+1, \dots) \tag{53}$$

and r is the least number having this property.

Proof. If the matrix $S = \| s_{i+k} \|_0^\infty$ has finite rank r, then its first $r+1$ rows $R_1, R_2, \dots, R_{r+1}$ are linearly dependent. Therefore there exists a number $h \leqq r$ such that $R_1, R_2, \dots, R_h$ are linearly independent and R_{h+1} is a linear combination of them:

$$R_{h+1} = \sum_{g=1}^{h} \alpha_g R_{h-g+1}.$$

We consider the rows $R_{q+1}, R_{q+2}, \dots, R_{q+h+1}$, where q is any non-negative integer. From the structure of S it is immediately clear that the rows $R_{q+1}, R_{q+2}, \dots, R_{q+h+1}$ are obtained from $R_1, R_2, \dots, R_{h+1}$ by a 'shortening' process in which the elements in the first q columns are omitted. Therefore

$$R_{q+h+1} = \sum_{g=1}^{h} \alpha_g R_{q+h-g+1} \qquad (q = 0, 1, 2, \dots).$$

Thus, every row of S beginning with the $(h+1)$-th can be expressed linearly in terms of the h preceding rows and therefore in terms of the linearly

---

[38] See Vol. I, Chapter X, § 10.

independent first h rows. Hence it follows that the rank of S is $r=h$.[39] The linear dependence

$$R_{q+h+1}=\sum_{g=1}^{h}\alpha_g R_{q+h-g+1}$$

after replacement of h by r and written in more convenient notation yields (53).

Conversely, if (53) holds, then every row (column) of S is a linear combination of the first r rows (columns). Therefore all the minors of S whose orders exceed r are zero and S is of rank at most r. But the rank cannot be less than r, since then, as we have already shown, there would be relations of the form (53) with a smaller value than r, and this contradicts the second condition of the theorem. The proof of the theorem is now complete.

COROLLARY: *If the infinite Hankel matrix* $S=\|s_{i+k}\|_0^\infty$ *is of finite rank* r, *then*

$$D_r=|s_{i+k}|_0^{r-1}\neq 0.$$

For it follows from the relations (53) that every row (column) of S is a linear combination of the first r rows (columns). Therefore every minor of S of order r can be represented in the form αD_r, where α is a constant. Hence it follows that $D_r\neq 0$.

Note. For finite Hankel matrices of rank r the inequality $D_r\neq 0$ need not hold. For example $S_2=\begin{Vmatrix}s_0 & s_1\\ s_1 & s_2\end{Vmatrix}$ for $s_0=s_1=0$, $s_2\neq 0$ is of rank 1, whereas $D_1=s_0=0$.

2. We shall now explain certain remarkable connections between infinite Hankel matrices and rational functions.

Let

$$R(z)=\frac{g(z)}{h(z)}$$

be a proper rational fractional function, where

$$h(z)=a_0z^m+\cdots+a_m\quad (a_0\neq 0),\qquad g(z)=b_1z^{m-1}+b_2z^{m-2}+\cdots+b_m.$$

We write the expansion of $R(z)$ in a power series of negative powers of z:

$$R(z)=\frac{g(z)}{h(z)}=\frac{s_0}{z}+\frac{s_1}{z^2}+\frac{s_2}{z^3}+\cdots.$$

---

[39] The statement 'The number of linearly independent rows in a rectangular matrix is equal to its rank' is true not only for finite rows but also for infinite rows.

If all the poles of $R(z)$, i.e., all the values of z for which $R(z)$ becomes infinite, lie in the circle $|z|\leq a$, then the series on the right-hand side of the expansion converges for $|z|>a$. We multiply both sides by the denominator $h(z)$:

$$(a_0z^m+a_1z^{m-1}+\cdots+a_m)\left(\frac{s_0}{z}+\frac{s_1}{z^2}+\frac{s_2}{z^3}+\cdots\right)=b_1z^{m-1}+b_2z^{m-2}+\cdots+b_m.$$

Equating coefficients of equal powers of z on both sides of this identity, we obtain the following system of relations:

$$\left.\begin{aligned}&a_0s_0=b_1,\\&a_0s_1+a_1s_0=b_2,\\&\cdots\cdots\cdots\\&a_0s_{m-1}+a_1s_{m-2}+\cdots+a_{m-1}s_0=b_m,\end{aligned}\right\}\qquad(54)$$

$$a_0s_q+a_1s_{q-1}+\cdots+a_ms_{q-m}=0\qquad (q=m,\ m+1,\ldots).\qquad (54')$$

Setting

$$\alpha_g=-\frac{a_g}{a_0}\qquad (g=1,2,\ldots,m),$$

we can write the relations (54′) in the form (53) (for $r=m$). Therefore, by Theorem 7, the infinite Hankel matrix

$$S=\|s_{i+k}\|_0^\infty$$

formed from the coefficients $s_0, s_1, s_2, \ldots$ is of finite rank ($\leq m$).

Conversely, if the matrix $S=\|s_{i+k}\|_0^\infty$ is of finite rank r, then the relations (53) hold, which can be written in the form (54′) (for $m=r$). Then, when we define the numbers $b_1, b_2, \ldots, b_m$ by the equations (54) we have the expansion

$$\frac{b_1z^{m-1}+\cdots+b_m}{a_0z^m+a_1z^{m-1}+\cdots+a_m}=\frac{s_0}{z}+\frac{s_1}{z^2}+\cdots.$$

The least degree of the denominator m for which this expansion holds is the same as the least integer m for which the relations (53) hold. By Theorem 7, this least value of m is the rank of $S=\|s_{i+k}\|_0^\infty$.

Thus we have proved the following theorem:

THEOREM 8: *The matrix* $S=\|s_{i+k}\|_0^\infty$ *is of finite rank if and only if the sum of the series*

$$R(z)=\frac{s_0}{z}+\frac{s_1}{z^2}+\frac{s_2}{z^3}+\cdots$$

is a rational function of z. *In this case the rank of* S *is the same as the number of poles of* $R(z)$, *counting each pole with its proper multiplicity.*

§ 11. Determination of the Index of an Arbitrary Rational Fraction by the Coefficients of Numerator and Denominator

1. Suppose given a rational function. We write its expansion in a series of descending powers of z:[40]

$$R(z) = s_{-u-1}z_u + \cdots + s_{-2}z + s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots. \tag{55}$$

The sequence of coefficients of the negative powers of z

$$s_0, s_1, s_2, \ldots$$

determines an infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$.

We have thus established a correspondence

$$R(z) \sim S.$$

Obviously two rational functions whose difference is an integral function correspond to one and the same matrix S. However, not every matrix $S = \| s_{i+k} \|_0^\infty$ corresponds to some rational function. In the preceding section we have seen that an infinite matrix S corresponds to a rational function if and only if it is of finite rank. This rank is equal to the number of poles of $R(z)$ (multiplicities taken into account), i.e., to the degree of the denominator $f(z)$ in the reduced fraction $g(z)/f(z) = R(z)$. By means of the expansion (55) we have a one-to-one corespondence between proper rational functions $R(z)$ and Hankel matrices $S = \| s_{i+k} \|_0^\infty$ of finite rank.

We mention some properties of the correspondence:

1. If $R_1(z) \sim S_1$, $R_2(z) \sim S_2$, then for arbitrary numbers c_1, c_2

$$c_1R_1(z) + c_2R_2(z) \sim c_1S_1 + c_2S_2.$$

In what follows we shall have to deal with the case where the coefficients of the numerator and the denominator of $R(z)$ are integral rational functions of a parameter α; R is then a rational function of z and α. From the expansion (54) it follows that in this case the numbers $s_0, s_1, s_2, \ldots$, i.e., the elements of S, depend rationally on α. Differentiating (55) term by term with respect to α, we obtain:

2. If $R(z, \alpha) \sim S(\alpha)$, then $\frac{\partial R}{\partial \alpha} \sim \frac{\partial S}{\partial \alpha}$.[41]

---

[40] The series (55) converges outside every circle (with center at $z = 0$) containing all the poles of $R(z)$.

[41] If $S = \| s_{i+k} \|_0^\infty$, then $\frac{\partial s}{\partial \alpha} = \| \frac{\partial s_{i+k}}{\partial \alpha} \|_0^\infty$.

2. Let us write down the expansion of $R(z)$ in partial fractions:

$$R(z) = Q(z) + \sum_{j=1}^{q} \left\{ \frac{A_1^{(j)}}{z - \alpha_j} + \frac{A_2^{(j)}}{(z - \alpha_j)^2} + \cdots + \frac{A_{\nu_j}^{(j)}}{(z - \alpha_j)^{\nu_j}} \right\}, \tag{56}$$

where $Q(z)$ is a polynomial; we shall show how to construct the matrix S corresponding to $R(z)$ from the numbers α and A.

For this purpose we consider first the simple rational function

$$\frac{1}{z - \alpha} = \sum_{p=0}^{\infty} \frac{\alpha^p}{z^{p+1}}.$$

It corresponds to the matrix

$$S_\alpha = \| \alpha^{i+k} \|_0^\infty.$$

The form $S_{\alpha n}(x, x)$ associated with this matrix is

$$S_{\alpha n}(x, x) = \sum_{i,k=0}^{n-1} \alpha^{i+k} x_i x_k = (x_0 + \alpha x_1 + \cdots + \alpha^{n-1} x_{n-1})^2.$$

If

$$R(z) = Q(z) + \sum_{j=1}^{q} \frac{A^{(j)}}{z - \alpha_j},$$

then by 1. the corresponding matrix S is determined by the formula

$$S = \sum_{j=1}^{q} A^{(j)} S_{\alpha_j} = \| \sum_{j=1}^{q} A^{(j)} \alpha_j^{i+k} \|_0^\infty$$

and the corresponding quadratic form is

$$S_n(x, x) = \sum_{j=1}^{q} A^{(j)} (x_0 + \alpha_j x_1 + \cdots + \alpha_j^{n-1} x_{n-1})^2.$$

In order to proceed to the general case (56), we first differentiate the relation

$$\frac{1}{z - \alpha} \sim S_\alpha = \| \alpha^{i+k} \|_0^\infty$$

$h - 1$ times term by term. By 1. and 2., we obtain:

$$\frac{1}{(z - \alpha)^h} \sim \frac{1}{(h-1)!} \frac{\partial^{h-1} S_\alpha}{\partial \alpha^{h-1}} = \left\| \binom{i+k}{h-1} \alpha^{i+k-h+1} \right\|_0^\infty, \binom{i+k}{h-1} = 0 \text{ for } i + k < h - 1.$$

Therefore, by using rule **1**. again we find in the general case, where $R(z)$ has the expansion (56):

$$R(z) \sim S = \sum_{j=1}^{q}\left(A_1^{(j)} + A_2^{(j)}\frac{\partial}{\partial\alpha_j} + \cdots + \frac{1}{(\nu_j-1)!}A_{\nu_j}^{(j)}\frac{\partial^{\nu_j-1}}{\partial\alpha_j^{\nu_j-1}}\right)S_{\alpha_j}. \tag{57}$$

By carrying out the differentiation, we obtain:

$$S = \left\| \sum_{j=1}^{q} A_1^{(j)}\alpha_j^{i+k} + A_2^{(j)}\binom{i+k}{1}\alpha_j^{i+k-1} + \cdots + A_{\nu_j}^{(j)}\binom{i+k}{\nu_j-1}\alpha_j^{i+k-\nu_j+1} \right\|_0^\infty .$$

The corresponding Hankel form $S_n(x, x) = \sum_{i,k=0}^{n-1} s_{i+k}x_ix_k$ is

$$S_n(x,x) = \sum_{j=1}^{q}\left(A_1^{(j)} + A_2^{(j)}\frac{\partial}{\partial\alpha_j} + \cdots + \frac{1}{(\nu_j-1)!}A_{\nu_j}^{(j)}\frac{\partial^{\nu_j-1}}{\partial\alpha_j^{\nu_j-1}}\right)(x_0 + \alpha_jx_1 + \cdots + \alpha_j^{n-1}x_{n-1})^2. \tag{57''}$$

3. Now we are in a position to enunciate and prove the fundamental theorem:[42]

THEOREM 9: *If*

$$R(z) \sim S$$

and m is the rank of S,[43] *then the Cauchy index* $I_{-\infty}^{+\infty} R(z)$ *is equal to the signature*[44] *of the form* $S_n(x, x)$ *for any* $n \geqq m$:

$$I_{-\infty}^{+\infty} R(z) = \sigma\,[S_n(x,x)].$$

Proof. Suppose that the expansion (56) holds. Then, by (57),

$$S = \sum_{j=1}^{q} T_{\alpha_j},$$

where each term is of the form

$$T_\alpha = \left(A_1 + A_2\frac{\partial}{\partial\alpha} + \cdots + \frac{1}{(\nu-1)!}A_\nu\frac{\partial^{\nu-1}}{\partial\alpha^{\nu-1}}\right)S_\alpha, \quad S_\alpha = \left\|\alpha^{i+k}\right\|_0^\infty \tag{58}$$

and

$$S_n(x,x) = \sum_{j=1}^{q} T_{\alpha_j}(x,x) = \sum_{\alpha_j \text{ real}} T_{\alpha_j}(x,x) + \sum_{\alpha_j \text{ complex}} [T_{\alpha_j}(x,x) + T_{\overline{\alpha}_j}(x,x)]$$

---

[42] This theorem was proved by Hermite in 1856 for the simplest case where $R(z)$ has no multiple poles [187]. In the general case it was proved by Hurwitz [204] (see also [25], pp. 17-19). The proof in the text differs from Hurwitz' proof.

[43] As we have already mentioned, m is the degree of the denominator in the reduced representation of the rational fraction $R(z)$ (see Theorem 8 on p. 207).

[44] We denote the signature of $S_n(x, x)$ by $\sigma[S_n(x, x)]$.

By Theorem 8, the rank of the matrix T_{α_j}, and hence of the form $T_{\alpha_j}(x, x)$, is ν_j $(j = 1, 2, \ldots, q)$ and the rank of $S_n(x, x)$ is $m = \sum_{j=1}^{q} \nu_j$. But if the rank of the sum of certain real quadratic forms is equal to the sum of the ranks of the constituent forms, then the same relation holds for the signatures:

$$\sigma\,[S_n(x,x)] = \sum_{\alpha_j \text{ real}} \sigma\,[T_{\alpha_j}(x,x)] + \sum_{\alpha_j \text{ complex}} \sigma\,[T_{\alpha_j}(x,x) + T_{\overline{\alpha}_j}(x,x)]. \tag{59}$$

We consider two cases separately:

1) α is real. Under any variation of the parameters $A_1, A_2, \ldots, A_{\nu-1}$ and α in

$$\frac{A_1}{z-\alpha} + \frac{A_2}{(z-\alpha)^2} - \cdots + \frac{A_\nu}{(z-\alpha)^\nu} \tag{60}$$

the rank of the corresponding matrix T_α remains unchanged $(= \nu)$; therefore the signature of $T_\alpha(x, x)$ also remains unchanged (see Vol. I, p. 309). Therefore $\sigma[T_\alpha(x, x)]$ does not change if we set in (59) and (60): $A_1 = \ldots = A_{\nu-1} = 0$ and $\alpha = 0$, i.e., if for T_α we take the matrix

$$\frac{1}{(\nu-1)!}\frac{\partial^{\nu-1}S_\alpha}{\partial\alpha^{\nu-1}} = \left\| \begin{array}{lllll} \overbrace{0 \quad 0 \ldots 0}^{\nu-1} & A_\nu & 0 & 0 & \ldots \\ 0 & & & \cdot^{\cdot^{\cdot}} & \\ \vdots & & \cdot^{\cdot^{\cdot}} & & \\ 0 & \cdot^{\cdot^{\cdot}} & & & \\ A_\nu & & & & \\ 0 & & & & \\ 0 & & & & \\ \vdots & & & & \end{array} \right\|$$

The corresponding quadratic form is equal to

$$\left.\begin{array}{l} 2A_\nu\,(x_0x_{\nu-1} + x_1x_{\nu-2} + \cdots + x_{s-1}x_s) \text{ for } \nu = 2s, \\ A_\nu\,[2\,(x_0x_{\nu-1} + \cdots - x_{s-2}x_s) - x_{s-1}^2] \text{ for } \nu = 2s-1, \end{array}\right\} \quad (s = 1, 2, 3, \ldots).$$

But the signature of the upper form is always zero and that of the lower form is sign A_ν. Thus, if α is real, then

$$\sigma[T_\alpha(x,x)] = \begin{cases} 0, \text{ for even } \nu \\ \text{sign } A_\nu, \text{ for odd } \nu \end{cases} \tag{61}$$

2) α is complex.

$$T_\alpha(x,x) = \sum_{k=1}^{\nu} (P_k + iQ_k)^2, \quad T_{\bar{\alpha}_j}(x,x) = \sum_{k=1}^{\nu} (P_k - iQ_k)^2,$$

where P_k, Q_k $(k = 1, 2, \dots, \nu)$ are real linear forms in the variables $x_0, x_1, x_2, \dots, x_{n-1}$. Then

$$T_\alpha(x,x) + T_{\bar{\alpha}}(x,x) = 2\sum_{k=1}^{\nu} P_k^2 - 2\sum_{k=1}^{\nu} Q_k^2. \tag{62}$$

Since the rank of this quadratic form is 2ν, the P_k, Q_k $(k = 1, 2, \dots, \nu)$ are linearly independent, so that by (62) for a complex α

$$\sigma[T_\alpha(x,x) + T_{\bar{\alpha}}(x,x)] = 0. \tag{63}$$

From (59), (61), and (63) it follows that

$$\sigma[S_n(x,x)] = \sum_{\left(\substack{\alpha_j \text{ real} \\ \nu \text{ odd}}\right)} \text{sign } A_\nu^{(j)}.$$

But on p. 175 we saw that the sum on the right-hand side of this equation is $I_{-\infty}^{+\infty} R(z)$. This completes the proof.

From this theorem we deduce:

COROLLARY 1: *If* $R(z) \sim S = \| s_{i+k} \|_0^\infty$ *and* m *is the rank of* S, *then all the quadratic forms* $S_n(x,x) = \sum_{i,k=0}^{n-1} s_{i+k} x_i x_k$ $(n = m, m+1, \dots)$ *have one and the same signature.*

In Chapter X, § 10 (Vol. I, pp. 343-44) we established a rule for computing the signature of a Hankel form; moreover, Frobenius' investigations enabled us to formulate a rule that embraces all singular cases. By the

---

[45] Each of the products $x_0 x_{\nu-1}, x_1 x_{\nu-2}, \dots$ can be replaced by a difference of squares

$$\left(\frac{x_0 + x_{\nu-1}}{2}\right)^2 - \left(\frac{x_0 - x_{\nu-1}}{2}\right)^2, \left(\frac{x_1 + x_{\nu-2}}{2}\right)^2 - \left(\frac{x_1 - x_{\nu-2}}{2}\right)^2, \dots.$$

All the squares so obtained are linearly independent.

theorem above we can apply this rule to compute the Cauchy index. Thus we obtain:

COROLLARY 2: *The index of an arbitrary rational function* $R(z)$ *whose corresponding matrix* $S = \| s_{i+k} \|_0^\infty$ *is of rank* m, *is determined by the formula*

$$I_{-\infty}^{+\infty} R(z) = m - 2V(1, D_1, D_2, \dots, D_m), \tag{64}$$

where

$$D_f = |s_{i+k}|_0^{f-1} = \begin{vmatrix} s_0 & s_1 & \dots & s_{f-1} \\ s_1 & s_2 & \dots & s_f \\ \cdot & \cdot & \cdot & \cdot \\ s_{f-1} & s_f & \dots & s_{2f-2} \end{vmatrix} \quad (f = 1, 2, \dots, m); \tag{65}$$

if among $D_1, D_2, \dots, D_m$ *there is a group of vanishing determinants*[46]

$$(D_h \neq 0) \quad D_{h+1} = \cdots = D_{h+p} = 0 \quad (D_{h+p+1} \neq 0),$$

then in the computation of $V(D_h, D_{h+1}, \dots, D_{h+p+1})$ *we can take*

$$\text{sign } D_{h+j} = (-1)^{\frac{j(j-1)}{2}} \text{sign } D_h \quad (j = 1, 2, \dots, p)$$

and this gives

$$V(D_h, D_{h+1}, \dots, D_{h+p+1}) = \begin{cases} \dfrac{p+1}{2} & \text{for odd } p, \\ \dfrac{p+1-\varepsilon}{2} & \text{for even } p \text{ and } \varepsilon = (-1)^{\frac{p}{2}} \text{sign } \dfrac{D_{h+p+1}}{D_h}. \end{cases} \tag{66}$$

In order to express the index of a rational function in terms of the coefficients of the numerator and denominator we shall require some additional relations.

First of all, we can always represent $R(z)$ in the form[47]

$$R(z) = Q(z) + \frac{g(z)}{h(z)},$$

where $Q(z)$, $g(z)$, $h(z)$ are polynomials and

$$h(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m \ (a_0 \neq 0), \ g(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_m.$$

Obviously,

$$I_{-\infty}^{+\infty} R(z) = I_{-\infty}^{+\infty} \frac{g(z)}{h(z)}.$$

---

[46] Here we always have $D_m \neq 0$ (p. 206).

[47] It is not necessary to replace $R(z)$ by a proper fraction. For what follows it is sufficient that the degree of $g(z)$ does not exceed that of $h(z)$.

Let

$$\frac{g(z)}{h(z)} = s_{-1} + \frac{s_0}{z} + \frac{s_1}{z^2} + \cdots .$$

If we now get rid of the denominator and then equate equal powers of z on the two sides of the equation, we obtain:

$$\begin{aligned} &a_0 s_{-1} = b_0 , \\ &a_0 s_0 + a_1 s_{-1} = b_1 , \\ &\cdots\cdots\cdots \\ &a_0 s_{m-1} + a_1 s_{m-2} + \cdots + a_m s_{-1} = b_m , \\ &a_0 s_t + a_1 s_{t-1} + \cdots + a_m s_{t-m} = 0 \quad (t = m, m+1, \ldots) . \end{aligned} \tag{67}$$

Using (67), we find an expression for the following determinant of order $2p$ in which we put $a_j = 0$, $b_j = 0$ for $j > m$:

$$\begin{vmatrix} a_0 & a_1 & a_2 \ldots a_{2p-1} \\ b_0 & b_1 & b_2 \ldots b_{2p-1} \\ 0 & a_0 & a_1 \ldots a_{2p-2} \\ 0 & b_0 & b_1 \ldots b_{2p-2} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \ldots 0 \\ s_{-1} & s_0 & s_1 \ldots s_{2p-2} \\ 0 & 1 & 0 \ldots 0 \\ 0 & s_{-1} & s_0 \ldots s_{2p-3} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} \cdot \begin{vmatrix} a_0 & a_1 & a_2 \ldots a_{2p-1} \\ 0 & a_0 & a_1 \ldots a_{2p-2} \\ 0 & 0 & a_0 \ldots a_{2p-3} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots a_0 \end{vmatrix}$$

$$= (-1)^{\frac{p(p-1)}{2}} a_0^{2p} \begin{vmatrix} s_{p-1} & s_p & \ldots s_{2p-2} \\ s_{p-2} & s_{p-1} & \ldots s_{2p-3} \\ \cdots & \cdots & \cdots \\ s_0 & s_1 & \ldots s_{p-1} \end{vmatrix} = a_0^{2p} \begin{vmatrix} s_0 & s_1 & \ldots s_{p-1} \\ s_1 & s_2 & \ldots s_p \\ \cdots & \cdots & \cdots \\ s_{p-1} & s_p & \ldots s_{2p-2} \end{vmatrix} = a_0^{2p} D_p . \tag{68}$$

We introduce the abbreviation

$$\nabla_{2p} = \begin{vmatrix} a_0 & a_1 \ldots a_{2p-1} \\ b_0 & b_1 \ldots b_{2p-1} \\ 0 & a_0 \ldots a_{2p-2} \\ 0 & b_0 \ldots b_{2p-2} \\ \cdots & \cdots \end{vmatrix} \qquad (p = 1, 2, \ldots; \ a_j = b_j = 0 \text{ for } j > m). \tag{69}$$

Then (68) can be written as follows:

$$\nabla_{2p} = a_0^{2p} D_p \quad (p = 1, 2, \ldots) . \tag{68'}$$

By this formula, Corollary 2 above leads to the following theorem:

THEOREM 10: *If* $\nabla_{2m} \neq 0$,[48] *then*

$$I_{-\infty}^{+\infty} \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{a_0 z^m + a_1 z^{m-1} + \cdots + a_m} = m - 2V(1, \nabla_2, \nabla_4, \ldots, \nabla_{2m}) \quad (a_0 \neq 0) , \tag{70}$$

where ∇_{2p} $(p = 1, 2, \ldots, m)$ *is determined by* (69); *if there is a group of zero determinants*

$$(\nabla_{2h} \neq 0) \quad \nabla_{2h+2} = \cdots = \nabla_{2h+2p} = 0 \quad (\nabla_{2h+2p+2} \neq 0) ,$$

then in computing $V(\nabla_{2h}, \nabla_{2h+2}, \ldots, \nabla_{2h+2p+2})$ *we have to set:*

$$\operatorname{sign} \nabla_{2h+2j} = (-1)^{\frac{j(j-1)}{2}} \operatorname{sign} \nabla_{2h} \quad (j = 1, 2, \ldots, p)$$

or, what is the same,

$$V(\nabla_{2h}, \ldots, \nabla_{2h+2p+2}) = \begin{cases} \dfrac{p+1}{2} & \textit{for odd } p \\ \dfrac{p+1-\varepsilon}{2} & \textit{for even } p \textit{ and } \varepsilon = (-1)^{\frac{p}{2}} \operatorname{sign} \dfrac{\nabla_{2h+2p+2}}{\nabla_{2h}} . \end{cases}$$

Note. If $\nabla_{2m} \neq 0$, i.e., if the fraction under the index sign in (70) is reducible, then (70) must be replaced by another formula

$$I_{-\infty}^{+\infty} \frac{b_0 z^m + b_1 z^{m-1} - \cdots \pm b_m}{a_0 z^m - a_1 z^{m-1} + \cdots + a_m} = r - 2V(1, \nabla_2, \nabla_4, \ldots, \nabla_{2r}) , \tag{70'}$$

where r is the number of poles (including multiplicities) of the rational fraction under the index sign (i.e., r is the degree of the denominator in the reduced fraction).

For in this case the index we are interested in is

$$r - 2V(1, D_1, D_2, \ldots, D_r) ,$$

since r is the rank of the corresponding matrix $S = \| s_{i+k} \|_0^\infty$. But the equation (68′) is of a formal character and also holds for reduced fractions. Therefore

$$V(1, D_1, D_2, \ldots, D_r) = V(1, \nabla_2, \nabla_4, \ldots, \nabla_{2r}) ,$$

and we have reached (70′).

Formula (70′) enables us to express the index of every rational fraction in which the degree of the numerator does not exceed that of the denominator in terms of the coefficients of numerator and denominator.

---

[48] The condition $\nabla_{2m} \neq 0$ means that $D_m \neq 0$, so that the fraction under the index sign in (70) is reduced.

§ 12. Another Proof of the Routh-Hurwitz Theorem

1. In § 6 we proved the Routh-Hurwitz theorem with the help of Sturm's theorem and the Routh algorithm. In this section we shall give an alternative proof based on Theorem 10 of § 11 and on properties of the Cauchy indices.

We mention a few properties of the Cauchy indices that will be required in what follows.

1. $I_a^b R(x) = -I_b^a R(x)$.[49]

2. $I_a^b R_1(x) R(x) = \operatorname{sign} R_1(x)\, I_a^b R(x)$ *if* $R_1(x) \neq 0, \infty$ *within the interval* (a, b).

3. *If* $a < c < b$, *then* $I_a^b R(x) = I_a^c R(x) + I_c^b R(x) + \eta_c$, *where* $\eta_c = 0$ *if* $R(c)$ *is finite and* $\eta_c = \pm 1$ *if* $R(x)$ *becomes infinite at* c: *here* $\eta_c = +1$ *corresponds to a jump from* $-\infty$ *to* $+\infty$ *at* c *(for increasing* x*), and* $\eta_c = -1$ *to a jump from* $+\infty$ *to* $-\infty$.

4. *If* $R(-x) = -R(x)$, *then* $I_{-a}^0 R(x) = I_0^a R(x)$. *If* $R(-x) = R(x)$, *then* $I_{-a}^0 R(x) = -I_0^a R(x)$.

5. $I_a^b R(x) + I_a^b (1/R(x)) = \frac{\varepsilon_a - \varepsilon_b}{2}$, *where* ε_a *is the sign of* $R(x)$ *within* (a, b) *near* a *and* ε_b *is the sign of* $R(x)$ *within* (a, b) *near* b.

The first four properties follow immediately from the definition of the Cauchy index (see § 2). Property 5. follows from the fact that the sum of the indices $I_a^b R(x)$ and $I_a^b \frac{1}{R(x)}$ is equal to the difference $n_1 - n_2$, where n_1 is the number of times $R(x)$ changes from negative to positive when x changes from a to b, and n_2 the number of times $R(x)$ changes from positive to negative.

We consider a real polynomial[50]

$$f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n \quad (a_0 > 0),$$

We can represent it in the form

$$f(z) = h(z^2) + z g(z^2),$$

where

$$h(u) = a_n + a_{n-2} u + \cdots, \quad g(u) = a_{n-1} + a_{n-3} u + \cdots.$$

---

[49] Here and in what follows the lower limit of the index may be $-\infty$ and the upper limit may be $+\infty$.

[50] We have here reverted to the usual notation for the coefficients of a polynomial.

We shall use the notation

$$\varrho = I_{-\infty}^{+\infty} \frac{a_1 z^{n-1} - a_3 z^{n-3} + \cdots}{a_0 z^n - a_2 z^{n-2} + \cdots}. \tag{71}$$

In § 3 we proved (see (20) on p. 180) that

$$\varrho = n - 2k - s, \tag{72}$$

where k is the number of roots of $f(z)$ with positive real parts and s the number of roots of $f(z)$ on the imaginary axis.

We shall transform the expression (71) for ϱ.

To begin with, we deal with the case where n is even. Let $n = 2m$. Then

$$h(u) = a_0 u^m + a_2 u^{m-1} + \cdots + a_n, \quad g(u) = a_1 u^{m-1} + a_3 u^{m-2} + \cdots + a_{n-1}.$$

Using the properties 1.-4. and setting $\eta = \pm 1$ if $\lim_{u \to 0-} \frac{g(u)}{h(u)} = \pm \infty$, respectively, and $\eta = 0$ otherwise, we have:

$$\varrho = -I_{-\infty}^{+\infty} \frac{z g(-z^2)}{h(-z^2)} = -(I_{-\infty}^0 + I_0^{+\infty} + \eta) = -2 I_{-\infty}^0 \frac{z g(-z^2)}{h(-z^2)} - \eta$$

$$= 2 I_{-\infty}^0 \frac{g(-z^2)}{h(-z^2)} - \eta = 2 I_{-\infty}^0 \frac{g(u)}{h(u)} - \eta = I_{-\infty}^0 \frac{g(u)}{h(u)} - I_{-\infty}^0 \frac{u g(u)}{h(u)} - \eta$$

$$= I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} - I_{-\infty}^{+\infty} \frac{u g(u)}{h(u)}.$$

Similarly we have for odd n, $n = 2m + 1$:

$$h(u) = a_1 u^m + a_3 u^{m-1} + \cdots + a_n, \quad g(u) = a_0 u^m + a_2 u^{m-1} + \cdots + a_{n-1}.$$

Setting[51] $\zeta = \operatorname{sign} \left[\frac{g(u)}{h(u)}\right]_{u=0-}$ if $\lim_{u \to 0-} \frac{g(u)}{h(u)} = 0$ and $\zeta = 0$ otherwise, we find:

$$\varrho = I_{-\infty}^{+\infty} \frac{h(-z^2)}{z g(-z^2)} = I_{-\infty}^0 + I_0^{+\infty} + \zeta = 2 I_{-\infty}^0 \frac{h(-z^2)}{z g(-z^2)} + \zeta = 2 I_{-\infty}^0 \frac{h(u)}{u g(u)} + \zeta$$

$$= I_{-\infty}^0 \frac{h(u)}{u g(u)} - I_{-\infty}^0 \frac{h(u)}{g(u)} + \zeta = I_{-\infty}^{+\infty} \frac{h(u)}{u g(u)} - I_{-\infty}^{+\infty} \frac{h(u)}{g(u)}.$$

Thus[52]

---

[51] Here we mean by sign $[g(u)/h(u)]_{u=0-}$ the sign of $g(u)/h(u)$ for negative values of u of sufficiently small modulus.

[52] If $a_1 \neq 0$, then the two formulas (73′) and (73″) may be combined into the single formula

$$\varrho = I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} + I_{-\infty}^{+\infty} \frac{h(u)}{u g(u)}. \tag{73‴}$$

$$\varrho = I_{-\infty}^{+\infty}\frac{g(u)}{h(u)} - I_{-\infty}^{+\infty}\frac{ug(u)}{h(u)} \quad (n = 2m), \tag{73'}$$

$$\varrho = I_{-\infty}^{+\infty}\frac{h(u)}{ug(u)} - I_{-\infty}^{+\infty}\frac{h(u)}{g(u)} \quad (n = 2m+1). \tag{73''}$$

As before, we denote by $\Delta_1, \Delta_2, \ldots, \Delta_n$ the Hurwitz determinants of $f(z)$. We assume that $\Delta_n \neq 0$.[53]

1) $n = 2m$. By (70),[54]

$$I_{-\infty}^{+\infty}\frac{g(u)}{h(u)} = m - 2V(1, \Delta_1, \Delta_3, \ldots, \Delta_{n-1}), \tag{74}$$

$$\begin{aligned} I_{-\infty}^{+\infty}\frac{ug(u)}{h(u)} &= m - 2V(1, -\Delta_2, +\Delta_4, -\Delta_6, \ldots) \\ &= -m + 2V(1, \Delta_2, \Delta_4, \ldots, \Delta_n). \end{aligned} \tag{75}$$

But then, by (73'),

$$\varrho = n - 2V(1, \Delta_1, \Delta_3, \ldots, \Delta_{n-1}) - 2V(1, \Delta_2, \Delta_4, \ldots, \Delta_n),$$

which in conjunction with $\varrho = n - 2k$ gives

$$k = V(1, \Delta_1, \Delta_3, \ldots, \Delta_{n-1}) + V(1, \Delta_2, \Delta_4, \ldots, \Delta_n). \tag{76}$$

2) $n = 2m + 1$. By (70),[55]

$$I_{-\infty}^{+\infty}\frac{h(u)}{ug(u)} = m + 1 - 2V(1, \Delta_1, \Delta_3, \ldots, \Delta_n), \tag{77}$$

$$\begin{aligned} I_{-\infty}^{+\infty}\frac{h(u)}{g(u)} &= m - 2V(1, -\Delta_2, +\Delta_4, -\cdots) \\ &= -m + 2V(1, \Delta_2, \Delta_4, \ldots, \Delta_{n-1}). \end{aligned} \tag{78}$$

The equation $\varrho = 2m + 1 - 2k$ together with (73''), (77), and (78) again gives (76).

This proves the Routh-Hurwitz theorem (see p. 194).

---

[53] In this case $s = 0$, so that $\varrho = n - 2k$. Moreover, $\Delta_n \neq 0$ means that the fractions under the index signs in (73') and (73'') are reduced.

[54] In computing $\nabla_2, \nabla_4, \ldots, \nabla_{2m}$ the values $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_m$ must be replaced by $a_0, a_2, \ldots, a_{2m}$ and $0, a_1, a_3, \ldots, a_{2m-1}$ respectively in computing the first index and by $a_0, a_2, \ldots, a_{2m}$ and $a_1, a_3, \ldots, a_{2m-1}, 0$ respectively in computing the second index.

[55] In computing the first index in (70) we take $a_0, a_2, \ldots, a_{2m}, 0$ and $0, a_1, a_3, \ldots, a_{2m-1}$, respectively, instead of $a_0, a_1, \ldots, a_{m+1}$ and $b_0, b_1, \ldots, b_{m+1}$; and in computing the second index we take $a_1, a_3, \ldots, a_{2m+1}$ and $a_0, a_2, \ldots, a_{2m}$, respectively, instead of $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_m$.

2. *Note* 1. If in the formula

$$k = V(1, \Delta_1, \Delta_3, \ldots) + V(1, \Delta_2, \Delta_4, \ldots)$$

some intermediate Hurwitz determinants are zero, then the formula remains valid, only in each group of successive zero determinants

$$(\Delta_l \neq 0) \quad \Delta_{l+2} = \Delta_{l+4} = \cdots = \Delta_{l+2p} = 0 \quad (\Delta_{l+2p+2} \neq 0)$$

the following signs must be attributed to these determinants (in accordance with Theorem 7)

$$\operatorname{sign} \Delta_{l+2j} = (-1)^{\frac{j(j-1)}{2}} \operatorname{sign} \Delta_l \quad (j = 1, 2, \ldots, p),$$

which yields:

$$V(\Delta_l, \Delta_{l+2}, \ldots, \Delta_{l+2p+2}) = \begin{cases} \dfrac{p+1}{2}, & \text{for odd } p, \\[2ex] \dfrac{p+1-\varepsilon}{2}, & \text{for even } p \text{ and } \varepsilon = (-1)^{\frac{p}{2}} \operatorname{sign} \dfrac{\Delta_{l+2p+2}}{\Delta_l}. \end{cases} \tag{79}$$

A careful comparison of this rule for computing k in the presence of vanishing Hurwitz determinants with the rule given in Theorem 5 (p. 201) shows that the two rules coincide.[56]

Note 2. If $\Delta_n = 0$, then the polynomials $ug(u)$ and $h(u)$ are not co-prime. We denote by $d(u)$ the greatest common divisor of $g(u)$ and $h(u)$ and by $u^\gamma d(u)$ that of $ug(u)$ and $h(u)$ ($\gamma = 0$ or 1). We denote the degree of $d(u)$ by δ and we set $h(u) = d(u)h_1(u)$ and $g(u) = d(u)g_1(u)$.

The irreducible rational fraction $g_1(u)/h_1(u)$ always corresponds to an infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$ of rank r, where r is the degree of $h_1(u)$. The corresponding determinant $D_r \neq 0$ and $D_{r+1} = D_{r+2} = \ldots = 0$. By (68') $\nabla_{2r} \neq 0, \nabla_{2r+2} = \nabla_{2r+4} = \ldots = 0$. Moreover,

$$I_{-\infty}^{+\infty}\frac{g_1(u)}{h_1(u)} = r - 2V(1, \nabla_2, \ldots, \nabla_{2r}).$$

When we apply all this to the fractions under the index sign in (74), (75), (77), and (78) we easily find that for every n (even or odd) and $\varkappa = 2\delta + \gamma$

$$\Delta_{n-\varkappa-1} \neq 0, \; \Delta_{n-\varkappa} \neq 0, \quad \overbrace{\Delta_{n-\varkappa+1} = \cdots = \Delta_n}^{\varkappa} = 0$$

and that the formulas (74), (75), (77), and (78) all remain valid in this case, provided we omit all the Δ_i with $i > n - \varkappa$ on the right-hand sides and replace the number m (in (77), $m + 1$) by the degree of the corresponding

---

[56] We have to take account here of the remark made in footnote 36 (p. 201).

denominator of the fraction under the index, after reduction. We then obtain by taking (73′) and (73″) into account:

$$\varrho = n - \varkappa - 2V(1, \Delta_1, \Delta_3, \ldots) - 2V(1, \Delta_2, \Delta_4, \ldots).$$

Together with the formula $\varrho = n - 2k - s$ this gives:

$$k_1 = V(1, \Delta_1, \Delta_3, \ldots) + V(1, \Delta_2, \Delta_4, \ldots), \tag{80}$$

where $k_1 = k + s/2 - \varkappa/2$ is the number of all the roots of $f(z)$ in the right half-plane, excluding those that are also roots of $f(-z)$.[57]

§ 13. Some Supplements to the Routh-Hurwitz Theorem. Stability Criterion of Liénard and Chipart

1. Suppose given a polynomial with real coefficients

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad (a_0 > 0).$$

Then the Routh-Hurwitz conditions that are necessary and sufficient for all the roots of $f(z)$ to have negative real parts can be written in the form of the inequalities

$$\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0, \tag{81}$$

where

$$\Delta_i = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & \\ a_0 & a_2 & a_4 & \cdots & \\ 0 & a_1 & a_3 & \cdots & \\ 0 & a_0 & a_2 & a_4 & \\ & & & \ddots & \\ & & & & a_i \end{vmatrix} \quad (a_k = 0 \text{ for } k > n)$$

is the Hurwitz determinant of order i $(i = 1, 2, \ldots, n)$.

If (81) is satisfied, then $f(z)$ can be represented in the form of a product of a_0 with factors of the form $z + u$, $z^2 + vz + w$ $(u > 0,\ v > 0,\ w > 0)$, so that all the coefficients of $f(z)$ are positive:[58]

---

[57] This follows from the fact that $\varkappa$ is the degree of the greatest common divisor of $h(u)$ and $ug(u)$; $\varkappa$ is the number of 'special' roots of $f(z)$, i.e., those roots z^* for which $-z^*$ is also a root of $f(z)$. The number of these special roots is equal to the number of determinants in the last uninterrupted sequence of vanishing Hurwitz determinants (including Δ_n): $\Delta_{n-\varkappa+1} = \cdots = \Delta_n = 0$.

[58] $a_0 > 0$, by assumption.

$$a_1 > 0,\ a_2 > 0,\ \ldots,\ a_n > 0. \tag{82}$$

Unlike (81), the conditions (82) are necessary but by no means sufficient for all the roots of $f(z)$ to lie in the left half-plane $\operatorname{Re} z < 0$.

However, when the conditions (82) hold, then the inequalities (81) are not independent. For example: For $n = 4$ the Routh-Hurwitz conditions reduce to the single inequality $\Delta_3 > 0$; for $n = 5$, to the two: $\Delta_2 > 0$, $\Delta_4 > 0$; for $n = 6$ to the two: $\Delta_3 > 0$, $\Delta_5 > 0$.[59]

This circumstance was investigated by the French mathematicians Liénard and Chipart[60] in 1914 and enabled them to set up a stability criterion different from the Routh-Hurwitz criterion.

THEOREM 11 (Stability Criterion of Liénard and Chipart): *Necessary and sufficient conditions for all the roots of the real polynomial* $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ $(a_0 > 0)$ *to have negative real parts can be given in any one of the following four forms:*[61]

1) $a_n > 0,\ a_{n-2} > 0, \ldots;\ \Delta_1 > 0,\ \Delta_3 > 0, \ldots,$
2) $a_n > 0,\ a_{n-2} > 0, \ldots;\ \Delta_2 > 0,\ \Delta_4 > 0, \ldots,$
3) $a_n > 0;\ a_{n-1} > 0,\ a_{n-3} > 0, \ldots;\ \Delta_1 > 0,\ \Delta_3 > 0, \ldots,$
4) $a_n > 0;\ a_{n-1} > 0,\ a_{n-3} > 0, \ldots;\ \Delta_2 > 0,\ \Delta_4 > 0, \ldots.$

From Theorem 11 it follows that Hurwitz's determinant inequalities (81) are not independent for a real polynomial $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ $(a_0 > 0)$ in which all the coefficients (or even only part of them: $a_n, a_{n-2}, \ldots$ or $a_n, a_{n-1}, a_{n-3}, \ldots$) are positive. In fact: *If the Hurwitz determinants of odd order are positive, then those of even order are also positive, and vice versa.*

Liénard and Chipart obtained the condition 1) in the paper [259] by means of special quadratic forms. We shall give a simpler derivation of the condition 1) (and also of 2), 3), 4)) based on Theorem 10 of § 11 and the theory of Cauchy indices and we shall obtain these conditions as a special case of a much more general theorem which we are now about to expound.

We again consider the polynomials $h(u)$ and $g(u)$ that are connected with $f(z)$ by the identity

---

[59] This fact has been established for the first few values of n in a number of papers on the theory of governors, independently of the general criterion of Liénard and Chipart, with which the authors of these papers were obviously not acquainted.

[60] See [259]. An account of some of the basic results of Liénard and Chipart can be found in the fundamental survey by M. G. Kreĭn and M. A. Naĭmark [25].

[61] Conditions 1), 2), 3), and 4) have a decided advantage over Hurwitz' conditions, because they involve only about half the number of determinantal inequalities.

$$f(z) = h(z^2) + z\,g(z^2)\,.$$

If n is even, $n = 2m$, then

$$h(u) = a_0u^m + a_2u^{m-1} + \cdots + a_n\,, \quad g(u) = a_1u^{m-1} + a_3u^{m-2} + \cdots + a_{n-1};$$

if n is odd, $n = 2m + 1$, then

$$h(u) = a_1u^m + a_3u^{m-1} + \cdots + a_n\,,\ g(u) = a_0u^m + a_2u^{m-1} + \cdots + a_{n-1}\,.$$

The conditions $a_n > 0$, $a_{n-2} > 0, \ldots$ (or $a_{n-1} > 0$, $a_{n-3} > 0, \ldots$) can therefore be replaced by the more general condition: $h(u)$ (or $g(u)$) does not change sign for $u > 0$.[62]

Under these conditions we can deduce a formula for the number of roots of $f(z)$ in the right half-plane, using only Hurwitz determinants of odd order or of even order.

THEOREM 12: *If for the real polynomial*

$$f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n = h(z^2) + zg(z^2) \quad (a_0 > 0)$$

$h(u)$ (or $g(u)$) does not change sign for $u > 0$ and the last Hurwitz determinant $\Delta_n \neq 0$, then the number k of roots of $f(z)$ in the right half-plane is determined by the formulas

	$n = 2m$	$n = 2m + 1$
$h(u)$ does not change sign for $u > 0$	$k = 2V(1, \Delta_1, \Delta_3, \ldots, \Delta_{n-1})$ $= 2V(1, \Delta_2, \Delta_4, \ldots, \Delta_n)$	$k = 2V(1, \Delta_1, \Delta_3, \ldots, \Delta_n) - \frac{1-\varepsilon_\infty}{2}$ $= 2V(1, \Delta_2, \Delta_4, \ldots, \Delta_{n-1}) + \frac{1-\varepsilon_\infty}{2}$
$g(u)$ does not change sign for $u > 0$	$k = 2V(1, \Delta_1, \Delta_3, \ldots, \Delta_{n-1}) + \frac{\varepsilon_\infty - \varepsilon_0}{2}$ $= 2V(1, \Delta_2, \Delta_4, \ldots, \Delta_n) - \frac{\varepsilon_\infty - \varepsilon_0}{2}$	$k = 2V(1, \Delta_1, \Delta_3, \ldots, \Delta_n) - \frac{1-\varepsilon_0}{2}$ $= 2V(1, \Delta_2, \Delta_4, \ldots, \Delta_{n-1}) + \frac{1-\varepsilon_0}{2}$

(83)

where[63]

$$\varepsilon_\infty = \operatorname{sign}\left[\frac{g(u)}{h(u)}\right]_{u=+\infty},\ \varepsilon_0 = \operatorname{sign}\left[\frac{g(u)}{h(u)}\right]_{u\to 0}. \tag{84}$$

---

[62] I.e., $h(u) \geqq 0$ or $h(u) \leqq 0$ for $u > 0$ ($g(u) \geqq 0$ or $g(u) \leqq 0$ for $u > 0$).

[63] If $a_1 \neq 0$, then $\varepsilon_\infty = \operatorname{sign} a_1$; and, more generally, if $a_1 = a_3 = \ldots = a_{2\mu-1} = 0$, $a_{2\mu+1} \neq 0$, then $\varepsilon_\infty = \operatorname{sign} a_{2\mu+1}$. If $a_{n-1} \neq 0$, then $\varepsilon_0 = \operatorname{sign} a_{n-1}/a_n$; and, more generally, if $a_{n-1} = a_{n-3} = \ldots = a_{n-2\mu-1} = 0$ and $a_{n-2\mu-1} \neq 0$, then $\varepsilon_0 = \operatorname{sign} a_{n-2\mu-1}/a_n$.

Proof. Again we use the notation

$$\varrho = I_{-\infty}^{+\infty} \frac{a_1z^{n-1} - a_3z^{n-3} + \cdots}{a_0z^n - a_2z^{n-2} + \cdots}\,.$$

Corresponding to the table (83) we consider four cases:

1) $n = 2m$; $h(u)$ does not change sign for $u > 0$. Then[64]

$$I_0^{+\infty} \frac{g(u)}{h(u)} = I_0^{+\infty} \frac{ug(u)}{h(u)} = 0\,,$$

and so the obvious equation

$$I_{-\infty}^{0} \frac{g(u)}{h(u)} = -I_{-\infty}^{0} \frac{ug(u)}{h(u)}$$

implies that:[65]

$$I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} = -I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)}\,.$$

But then we have from (74) and (75):

$$V(1, \Delta_1, \Delta_3, \ldots) = V(1, \Delta_2, \Delta_4, \ldots),$$

and therefore the Routh-Hurwitz formula (76) gives:

$$k = 2V(1, \Delta_1, \Delta_3, \ldots, \Delta_{n-1}) = 2V(1, \Delta_2, \Delta_4, \ldots, \Delta_n)\,.$$

2) $n = 2m$; $g(u)$ does not change sign for $u > 0$. In this case,

$$I_0^{+\infty} \frac{h(u)}{g(u)} = I_0^{+\infty} \frac{h(u)}{ug(u)} = 0\,,$$

$$I_{-\infty}^{0} \frac{h(u)}{g(u)} + I_{-\infty}^{0} \frac{h(u)}{ug(u)} = 0\,,$$

so that with the notation (84) we have:

$$I_{-\infty}^{+\infty} \frac{h(u)}{g(u)} + I_{-\infty}^{+\infty} \frac{h(u)}{ug(u)} - \varepsilon_0 = 0\,. \tag{85}$$

When we replace the functions under the index sign by their reciprocals, then we obtain by 5. (see p. 216):

$$I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} + I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)} = \varepsilon_\infty - \varepsilon_0\,.$$

---

[64] If $h(u_1) = 0$ $(u_1 > 0)$, then $g(u_1) \neq 0$, because $\Delta_n \neq 0$. Therefore $h(u) \geqq 0$ $(u > 0)$ implies that $g(u)/h(u)$ does not change sign in passing through $u = u_1$.

[65] From $\Delta_n = a_n \Delta_{n-1} \neq 0$ it follows that $h(0) = a_n \neq 0$.

But this by (74) and (75) gives:

$$V(1, \Delta_2, \Delta_4, \dots) - V(1, \Delta_1, \Delta_3, \dots) = \frac{\varepsilon_\infty - \varepsilon_0}{2}.$$

Hence, in conjunction with the Routh-Hurwitz formula (76), we obtain:

$$k = 2V(1, \Delta_1, \Delta_3, \dots) + \frac{\varepsilon_\infty - \varepsilon_0}{2} = 2V(1, \Delta_2, \Delta_4, \dots) - \frac{\varepsilon_\infty - \varepsilon_0}{2}.$$

3) $n = 2m + 1$, $g(u)$ does not change sign for $u > 0$.

In this case, as in the preceding one, (85) holds. When we substitute the expressions for the indices from (77) and (78) into (85), we obtain:

$$V(1, \Delta_1, \Delta_3, \dots) - V(1, \Delta_2, \Delta_4, \dots) = \frac{1 - \varepsilon_0}{2}.$$

In conjunction with the Routh-Hurwitz formula this gives:

$$k = 2V(1, \Delta_1, \Delta_3, \dots) - \frac{1 - \varepsilon_0}{2} = 2V(1, \Delta_2, \Delta_4, \dots) + \frac{1 - \varepsilon_0}{2}.$$

4) $n = 2m + 1$, $h(u)$ does not change sign for $u > 0$.

From the equations

$$I_0^\infty \frac{g(u)}{h(u)} = I_0^\infty \frac{u\,g(u)}{h(u)} = 0 \text{ and } I_{-\infty}^{0} \frac{g(u)}{h(u)} + I_{-\infty}^{0} \frac{u\,g(u)}{h(u)} = 0$$

we deduce:

$$I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} + I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)} = 0.$$

Taking the reciprocals of the functions under the index sign, we obtain:

$$I_{-\infty}^{+\infty} \frac{u(u)}{g(u)} + I_{-\infty}^{+\infty} \frac{h(u)}{ug(u)} = \varepsilon_\infty.$$

Again, when we substitute the expressions for the indices from (77) and (78), we have:

$$V(1, \Delta_1, \Delta_3, \dots) - V(1, \Delta_2, \Delta_4, \dots) = \frac{1 - \varepsilon_\infty}{2}.$$

From this and the Routh-Hurwitz formula it follows that:

$$k = 2V(1, \Delta_1, \Delta_3, \dots) - \frac{1 - \varepsilon_\infty}{2} = 2V(1, \Delta_2, \Delta_4, \dots) + \frac{1 - \varepsilon_\infty}{2}.$$

This completes the proof of Theorem 12.

From Theorem 12 we obtain Theorem 11 as a special case.

2. COROLLARY TO THEOREM 12: *If the real polynomial*

$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n \quad (a_0 > 0)$$

has positive coefficients

$$a_0 > 0, \quad a_1 > 0, \quad a_2 > 0, \dots, a_n > 0,$$

and $\Delta_n \neq 0$, *then the number* k *of its roots in the right half-plane* $\operatorname{Re} z > 0$ *is determined by the formula*

$$k = 2V(1, \Delta_1, \Delta_3, \dots) = 2V(1, \Delta_2, \Delta_4, \dots).$$

Note. If in the last formula, or in (83), some of the intermediate Hurwitz determinants are zero, then in the computation of $V(1, \Delta_1, \Delta_3, \dots)$ and $V(1, \Delta_2, \Delta_4, \dots)$ the rule given in Note 1 on p. 219 must be followed.

But if $\Delta_n = \Delta_{n-1} = \dots = \Delta_{n-\varkappa+1} = 0$, $\Delta_{n-\varkappa} \neq 0$, then we disregard the determinants $\Delta_{n-\varkappa+1}, \dots, \Delta_n$ in (83)[66] and determine from these formulas the number k_1 of the 'non-singular' roots of $f(z)$ in the right half-plane, provided only that $h(u) \neq 0$ for $u > 0$ or $g(u) \neq 0$ for $u > 0$.[67]

§ 14. Some Properties of Hurwitz Polynomials. Stieltjes' Theorem. Representation of Hurwitz Polynomials by Continued Fractions

1. Let

$$f(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n \qquad (a_0 \neq 0)$$

be a real polynomial. We represent it in the form

$$f(z) = h(z^2) + zg(z^2).$$

We shall investigate what conditions have to be imposed on $h(u)$ and $g(u)$ in order that $f(z)$ be a Hurwitz polynomial.

Setting $k = s = 0$ in (20) (p. 180), we obtain a necessary and sufficient condition for $f(z)$ to be a Hurwitz polynomial, in the form

$$\varrho = n,$$

where, as in the preceding sections,

$$\varrho = I_{-\infty}^{+\infty} \frac{a_1 z^{n-1} - a_3 z^{n-3} + \dots}{a_0 z^n - a_2 z^{n-2} + \dots}.$$

---

[66] See p. 220.

[67] In this case the polynomials $h_1(u)$ and $g_1(u)$ obtained from $h(u)$ and $g(u)$ by dividing them by their greatest common divisor $d(u)$ satisfy the conditions of Theorem 12.

Let $n = 2m$. By (73′) (p. 218), this condition can be written as follows:

$$n = 2m = I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} - I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)} . \tag{86}$$

Since the absolute value of the index of a rational fraction cannot exceed the degree of the denominator (in this case, m), the equation (86) can hold if and only if

$$I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} = m \text{ and } I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)} = -m \tag{87}$$

hold simultaneously.

For $n = 2m + 1$ the equation (73″) gives (on account of $\varrho = n$):

$$n = I_{-\infty}^{+\infty} \frac{h(u)}{ug(u)} - I_{-\infty}^{+\infty} \frac{h(u)}{g(u)} .$$

When we replace the fractions under the index signs by their reciprocals (see 5. on p. 216) and observe that $h(u)$ and $g(u)$ are of the same degree m, we obtain:[68]

$$n = 2m + 1 = I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} - I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)} + \varepsilon_{\infty} . \tag{88}$$

Starting again from the fact that the absolute value of the index of a fraction cannot exceed the degree of the denominator we conclude that (88) holds if and only if

$$I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} = m, \quad I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)} = -m \text{ and } \varepsilon_{\infty} = 1 \tag{89}$$

hold simultaneously.

If $n = 2m$, the first of equations (87) indicates that $h(u)$ has m distinct real roots $u_1 < u_2 < \ldots < u_m$ and that the proper fractions $g(u)/h(u)$ can be represented in the form

$$\frac{g(u)}{h(u)} = \sum_{i=1}^{m} \frac{R_i}{u - u_i} , \tag{90}$$

where

$$R_i = \frac{g(u_i)}{h'(u_i)} > 0 \qquad (i = 1, 2, \ldots, m). \tag{90′}$$

From this representation of $g(u)/h(u)$ it follows that between any two roots u_i, u_{i+1} of $h(u)$ there is a real root u_i' of $g(u)$ $(i = 1, 2, \ldots, m-1)$ and that the highest coefficients of $h(u)$ and $g(u)$ are of like sign, i.e.,

---

[68] As in the preceding section, $\varepsilon_{\infty} = \text{sign} \left[\frac{g(u)}{h(u)}\right]_{u=+\infty}$.

$$h(u) = a_0 (u - u_1) \cdots (u - u_m), \qquad g(u) = a_1 (u - u_1') \cdots (u - u_{m-1}'),$$
$$u_1 < u_1' < u_2 < u_2' < \cdots < u_{m-1} < u_{m-1}' < u_m; \qquad a_0 a_1 > 0 .$$

The second of equations (87) adds only one condition

$$u_m < 0 .$$

By this condition all the roots of $h(u)$ and $g(u)$ must be negative.

If $n = 2m + 1$, then it follows from the first of equations (89) that $h(u)$ has m distinct real roots $u_1 < u_2 < \ldots < u_m$ and that

$$\frac{g(u)}{h(u)} = s_{-1} + \sum_{i=1}^{m} \frac{R_i}{u - u_i} \qquad (s_{-1} \neq 0), \tag{91}$$

where

$$R_i = \frac{g(u_i)}{h'(u_i)} > 0 \qquad (i = 1, 2, \ldots, m). \tag{91′}$$

The third of equations (89) implies that

$$s_{-1} > 0 , \tag{92}$$

i.e., that the highest coefficients a_0 and a_1 are of like sign. Moreover, it follows from (91), (91′), and (92) that $g(u)$ has m real roots $u'_1 < u'_2 < \ldots < u'_m$ in the intervals $(-\infty, u_1), (u_1, u_2), \ldots, (u_{m-1}, u_m)$. In other words,

$$h(u) = a_1 (u - u_1) \cdots (u - u_m), \qquad g(u) = a_0 (u - u_1') \cdots (u - u_m'),$$
$$u_1' < u_1 < u_2' < u_2 < \cdots < u_m' < u_m; \qquad a_0 a_1 > 0 .$$

The second of equations (89), as in the case $n = 2m$, only adds one further inequality

$$u_m < 0 .$$

DEFINITION 3. *We shall say that two polynomials $h(u)$ and $g(u)$ of degree m (or the first of degree m and the second of degree $m - 1$) form a positive pair*[69] *if the roots $u_1, u_2, \ldots, u_m$ and $u_1', u_2', \ldots, u_m'$ (or $u_1', u_2', \ldots, u_{m-1}'$) are all distinct, real, and negative and they alternate as follows:*

$$u_1' < u_1 < u_2' < u_2 < \cdots < u_m' < u_m < 0$$
$$(\text{or } u_1 < u_1' < u_2 < \cdots < u_{m-1}' < u_m < 0)$$

and their highest coefficients are of like sign.[70]

---

[69] See [17], p. 333. The definition of a positive pair of polynomials given here differs slightly from that given in the book [17].

[70] If we omit the condition that the roots be negative, we obtain a real pair of polynomials. For the application of this concept to the Routh-Hurwitz problem, see [36].

When we introduce the positive numbers $v_i = -u_i$ and $v_i' = -u_i'$ and multiply $h(u)$ and $g(u)$ by $+1$ or -1 so that their highest coefficients are positive, then we can write the polynomials of this positive pair in the form

$$h(u) = a_1 \prod_{i=1}^{m} (u + v_i), \quad g(u) = a_0 \prod_{i=1}^{m} (u + v_i'), \tag{93}$$

where

$$a_1 > 0,\ a_0 > 0, \quad 0 < v_m < v_m' < v_{m-1} < v_{m-1}' < \cdots < v_1 < v_1',$$

in case both $h(u)$ and $g(u)$ are of degree m, and in the form

$$h(u) = a_0 \prod_{i=1}^{m} (u + v_i), \qquad g(u) = a_1 \prod_{i=1}^{m-1} (u + v_i'), \tag{93'}$$

where

$$a_0 > 0,\ a_1 > 0, \quad 0 < v_m < v_{m-1}' < v_{m-1} < \cdots < v_1' < v_1,$$

in case $h(u)$ is of degree m and $g(u)$ of degree $m-1$.

By our earlier arguments we have proved the following two theorems:

THEOREM 13: *The polynomial $f(z) = h(z^2) + zg(z^2)$ is a Hurwitz polynomial if and only if $h(u)$ and $g(u)$ form a positive pair.*[71]

THEOREM 14: *Two polynomials $h(u)$ and $g(u)$ the first of which is of degree m and the second of degree m or $m-1$ form a positive pair if and only if the equations*

$$I_{-\infty}^{-\infty} \frac{g(u)}{h(u)} = m, \quad I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)} = -m \tag{94}$$

hold and, when $h(u)$ and $g(u)$ are of equal degree, the additional condition

$$\varepsilon_\infty = \operatorname{sign}\left[\frac{g(u)}{h(u)}\right]_{+\infty} = 1 \tag{95}$$

holds.

2. Using properties of the Cauchy indices we can easily deduce from the last theorem a theorem of Stieltjes on the representation of a fraction $g(u)/h(u)$ as a continued fraction of a special type, provided $h(u)$ and $g(u)$ form a positive pair of polynomials.

The proof of Stieltjes' theorem will be based on the following lemma:

---

[71] This theorem is a special case of the so-called Hermite-Biehler theorem (see [7], p. 21).

LEMMA. *If the polynomials $h(u)$ and $g(u)$ ($h(u)$ of degree m) form a positive pair and*

$$\frac{g(u)}{h(u)} = c + \cfrac{1}{du + \cfrac{h_1(u)}{g_1(u)}}, \tag{96}$$

where c, d are constants and $h_1(u)$, $g_1(u)$ are polynomials of degree not exceeding $m-1$, then

1. *$c \geqq 0$, $d > 0$;*
2. *$h_1(u)$, $g_1(u)$ are of degree $m-1$;*
3. *$h_1(u)$ and $g_1(u)$ form a positive pair.*

Given $h(u)$ and $g(u)$, the polynomials $h_1(u)$ and $g_1(u)$ are uniquely determined (to within a common constant factor) and so are c and d.

Conversely, from (96) and 1., 2., 3. it follows that $h(u)$ and $g(u)$ form a positive pair, that $h(u)$ is of degree m, and $g(u)$ is of degree m or $m-1$ according as $c > 0$ or $c = 0$.

Proof. Let $h(u)$, $g(u)$ be a positive pair. Then it follows from (94) and (96) that:

$$m = I_{-\infty}^{+\infty} \frac{g(u)}{h(u)} = I_{-\infty}^{+\infty} \cfrac{1}{du + \cfrac{h_1(u)}{g_1(u)}}. \tag{97}$$

This equation implies that $g_1(u)$ is of degree $m-1$ and that $d \neq 0$.

Further, from (97) we find:

$$m = -I_{-\infty}^{+\infty}\left[du + \frac{h_1(u)}{g_1(u)}\right] + \operatorname{sign} d = -I_{-\infty}^{+\infty} \frac{h_1(u)}{g_1(u)} + \operatorname{sign} d.$$

Hence it follows that $d > 0$ and that

$$I_{-\infty}^{+\infty} \frac{h_1(u)}{g_1(u)} = -(m-1). \tag{98}$$

The second of equations (94) now gives:

$$\begin{aligned} -m = I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)} &= I_{-\infty}^{+\infty}\left[cu + \cfrac{1}{d + \cfrac{h_1(u)}{ug_1(u)}}\right] \\ &= I_{-\infty}^{+\infty} \cfrac{1}{d + \cfrac{h_1(u)}{ug_1(u)}} = -I_{-\infty}^{+\infty}\left[d + \frac{h_1(u)}{ug_1(u)}\right] = -I_{-\infty}^{+\infty} \frac{h_1(u)}{ug_1(u)}. \end{aligned} \tag{99}$$

Hence it follows that $h_1(u)$ is of degree $m-1$.

Condition (95) yields, by (96): $c > 0$. But if $g(u)$ is of smaller degree than $h(u)$, then it follows from (96) that $c = 0$.

(98) and (99) imply:

$$I_{-\infty}^{+\infty}\frac{g_1(u)}{h_1(u)} = m-1, \quad I_{-\infty}^{+\infty}\frac{ug_1(u)}{h_1(u)} = -m+\varepsilon_{\infty}^{(1)}, \tag{100}$$

where

$$\varepsilon_{\infty}^{(1)} = \operatorname{sign}\left[\frac{g_1(u)}{h_1(u)}\right]_{u=+\infty}.$$

Since the second of the indices (100) is in absolute value less than $m-1$, we have

$$\varepsilon_{\infty}^{(1)} = 1, \tag{101}$$

and then we conclude from (100) and (101), by Theorem 12, that the polynomials $h_1(u)$ and $g_1(u)$ form a positive pair.

From (96) it follows that

$$c = \lim_{u\to\infty}\frac{g(u)}{h(u)}, \quad \lim_{u\to\infty}\left[\frac{g(u)}{h(u)} - c\right]u = \frac{1}{d}.$$

After c and d have been found, the ratio $\frac{h_1(u)}{g_1(u)}$ is determined by (96).

The relations (97), (98), (99), (100), and (101) applied in the reverse order, establish the second part of the lemma. Thus the proof of the lemma is complete.

Suppose given a positive pair of polynomials $h(u)$, $g(u)$, with $h(u)$ of degree m. Then when we divide $g(u)$ by $h(u)$ and denote the quotient by c_0 and the remainder by $g_1(u)$, we obtain:

$$\frac{g(u)}{h(u)} = c_0 + \frac{g_1(u)}{h(u)} = c_0 + \cfrac{1}{\cfrac{h(u)}{g_1(u)}}.$$

$\frac{h(u)}{g_1(u)}$ can be represented in the form $d_0u + \frac{h_1(u)}{g_1(u)}$, where $h_1(u)$, like $g_1(u)$, is of degree less than m. Hence

$$\frac{g(u)}{h(u)} = c_0 + \cfrac{1}{d_0u + \cfrac{h_1(u)}{g_1(u)}}. \tag{102}$$

Thus, the representation (96) always holds for a positive pair $h(u)$ and $g(u)$. By the lemma

$$c_0 \geqq 0,\ d_0 > 0,$$

and the polynomials $h_1(u)$ and $g_1(u)$ are of degree $m-1$ and form a positive pair.

When we apply the same arguments to the positive pair $h_1(u)$, $g_1(u)$, we obtain

$$\frac{g_1(u)}{h_1(u)} = c_1 + \cfrac{1}{d_1u + \cfrac{h_2(u)}{g_2(u)}}, \tag{102'}$$

where

$$c_1 > 0,\ d_1 > 0,$$

and the polynomials $h_2(u)$ and $g_2(u)$ are of degree $m-2$ and form a positive pair. Continuing the process, we finally end up with a positive pair h_m and g_m, where h_m and g_m are constants of like sign. We set:

$$\frac{g_m}{h_m} = c_m. \tag{102$^{(m)}$}$$

Then it follows from (102), (102′), ..., (102$^{(m)}$) that:

$$\frac{g(u)}{h(u)} = c_0 + \cfrac{1}{d_0u + \cfrac{1}{c_1 + \cfrac{1}{d_1u + \cfrac{1}{c_2 + \cfrac{}{\ddots + \cfrac{1}{d_{m-1}u + \cfrac{1}{c_m}}}}}}}.$$

Using the second part of the lemma, we show similarly that for arbitrary $c_0 \geqq 0,\ c_1 > 0, \ldots, c_m > 0,\ d_0 > 0,\ d_1 > 0, \ldots, d_{m-1} > 0$ the above continued fraction determines uniquely (to within a common constant factor) a positive pair of polynomials $h(u)$ and $g(u)$, where $h(u)$ is of degree m and $g(u)$ is of degree m when $c_0 > 0$ and of degree $m-1$ when $c_0 = 0$.

Thus we have proved the following theorem.[72]

---

[72] A proof of Stieltjes' theorem that is not based on the theory of Cauchy indices can be found in the book [17], pp. 333-37.

THEOREM 15 (Stieltjes): *If* $h(u)$, $g(u)$ *is a positive pair of polynomials and* $h(u)$ *is of degree* m, *then*

$$\frac{g(u)}{h(u)} = c_0 + \cfrac{1}{d_0u + \cfrac{1}{c_1 + \cfrac{1}{d_1u + \cfrac{1}{c_2 + \ddots + \cfrac{1}{d_{m-1}u + \cfrac{1}{c_m}}}}}}, \tag{103}$$

where

$$c_0 \geqq 0,\ c_1 > 0, \ldots, c_m > 0, \quad d_0 > 0, \ldots, d_{m-1} > 0 .$$

Here $c_0 = 0$ *if* $g(u)$ *is of degree* $m-1$ *and* $c_0 > 0$ *if* $g(u)$ *is of degree* m. *The constants* c_i, d_k *are uniquely determined by* $h(u)$, $g(u)$.

Conversely, for arbitrary $c_0 \geqq 0$ *and arbitrary positive* $c_1, \ldots, c_m$, $d_0, \ldots, d_{m-1}$, *the continued fraction* (103) *determines a positive pair of polynomials* $h(u)$, $g(u)$, *where* $h(u)$ *is of degree* m.

From Theorem 13 and Stieltjes' Theorem we deduce:

THEOREM 16: *A real polynomial of degree* n $f(z) = h(z^2) + zg(z^2)$ *is a Hurwitz polynomial if and only if the formula* (103) *holds with non-negative* c_0 *and positive* $c_1, \ldots, c_m, d_0, \ldots, d_{m-1}$. *Here* $c_0 > 0$ *when* n *is odd and* $c_0 = 0$ *when* n *is even.*

§ 15. Domain of Stability. Markov Parameters

1. With every real polynomial of degree n we can associate a point of an n-dimensional space whose coordinates are the quotients of the coefficients divided by the highest coefficient. In this 'coefficient space' all the Hurwitz polynomials form a certain n-dimensional domain which is determined[73] by the Hurwitz inequalities $\Delta_1 > 0$, $\Delta_2 > 0, \ldots, \Delta_n > 0$, or, for example, by the Liénard-Chipart inequalities $a_n > 0$, $a_{n-2} > 0, \ldots, \Delta_1 > 0$, $\Delta_3 > 0, \ldots$. We shall call it the *domain of stability*. If the coefficients are given as functions of p parameters, then the domain of stability is constructed in the space of these parameters.

---

[73] For $a_0 = 1$.

The study of the domain of stability is of great practical interest; for example, it is essential in the design of new systems of governors.[74]

In § 17 we shall show that two remarkable theorems which were found by Markov and Chebyshev in connection with the expansion of continued fractions in power series with negative powers of the argument are closely connected with the investigation of the domain of stability. In formulating and proving these theorems it is convenient to give the polynomial not by its coefficients, but by special parameters, which we shall call *Markov parameters.*

Suppose that

$$f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n \qquad (a_0 \neq 0)$$

is a real polynomial. We represent it in the form

$$f(z) = h(z^2) + zg(z^2).$$

We may assume that $h(u)$ and $g(u)$ are co-prime ($\Delta_n \neq 0$). We expand the irreducible rational fraction $\frac{g(u)}{h(u)}$ in a series of decreasing powers of u:[75]

$$\frac{g(u)}{h(u)} = s_{-1} + \frac{s_0}{u} - \frac{s_1}{u^2} + \frac{s_2}{u^3} - \frac{s_3}{u^4} + \cdots. \tag{104}$$

The sequence $s_0, s_1, s_2, \ldots$ determines an infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$. We define a rational function $R(v)$ by

$$R(v) = -\frac{g(-v)}{h(-v)}. \tag{105}$$

Then

$$R(v) = -s_{-1} + \frac{s_0}{v} + \frac{s_1}{v^2} + \frac{s_2}{v^3} + \cdots, \tag{106}$$

so that we have the relation (see p. 208)

$$R(v) \sim S. \tag{107}$$

Hence it follows that the matrix S is of rank $m = [n/2]$, since m, being the degree of $h(u)$, is equal to the number of poles of $R(v)$.[76]

For $n = 2m$ (in this case, $s_{-1} = 0$), the matrix S determines the irreducible fraction $\frac{g(u)}{h(u)}$ uniquely and therefore determines $f(z)$ to within a

---

[74] A number of papers by Y. I. Naĭmark deal with the investigation of the domain of stability and also of the domains corresponding to various values of k (k is the number of roots in the right half-plane). (See the monograph [41].)

[75] In what follows it is convenient to denote the coefficients of the even negative powers of u by $-s_1$, $-s_3$, etc.

[76] See Theorem 8 (p. 207).

constant factor. For $n = 2m + 1$, in order to give $f(z)$ by means of S it is necessary also to know the coefficient s_{-1}.

On the other hand, in order to give the infinite Hankel matrix S of rank m it is sufficient to know the first $2m$ numbers $s_0, s_1, \ldots, s_{2m-1}$. These numbers may be chosen arbitrarily subject to only one restriction

$$D_m = | s_{i+k} |_0^m \neq 0 ; \tag{108}$$

all the subsequent coefficients $s_{2m}, s_{2m+1}, \ldots$ of (104) are uniquely (and rationally) expressible in terms of the first $2m$: $s_0, s_1, \ldots, s_{2m-1}$. For in the infinite Hankel matrix S of rank m the elements are connected by a recurrence relation (see Theorem 7 on p. 205)

$$s_q = \sum_{g=1}^{m} \alpha_g s_{q-g} \qquad (q = m, m+1, \ldots). \tag{109}$$

If the numbers $s_0, s_1, \ldots, s_{m-1}$ satisfy (108), then the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_m$ in (109) are uniquely determined by the first m relations; the subsequent relations then determine $s_{2m}, s_{2m+1}, \ldots$.

Thus, a real polynomial $f(z)$ of degree $n = 2m$ with $\Delta_n \neq 0$ can be given uniquely[77] by $2m$ numbers $s_0, s_1, \ldots, s_{2m-1}$ satisfying (108). When $n = 2m + 1$, we have to add s_{-1} to these numbers.

We shall call the n values $s_0, s_1, \ldots, s_{2m-1}$ (for $n = 2m$) or $s_{-1}, s_0, \ldots, s_{2m-1}$ (for $n = 2m + 1$) the *Markov parameters* of the polynomial $f(z)$. These parameters may be regarded as the coordinates in an n-dimensional space of a point that represents the given polynomial $f(z)$.

We shall find out what conditions must be imposed on the Markov parameters in order that the corresponding polynomial be a Hurwitz polynomial. In this way we shall determine the domain of stability in the space of Markov parameters.

A Hurwitz polynomial is characterized by the conditions (94) and the additional condition (95) for $n = 2m + 1$. Introducing the function $R(v)$ (see (105)), we write (94) as follows:

$$I_{-\infty}^{+\infty} R(v) = m, \qquad I_{-\infty}^{+\infty} vR(v) = m. \tag{110}$$

The additional condition (95) for $n = 2m + 1$ gives:

$$s_{-1} > 0.$$

Apart from the matrix $S = \| s_{i+k} \|_0^\infty$ we introduce the infinite Hankel matrix $S^{(1)} = \| s_{i+k+1} \|_0^\infty$. Then, since by (106)

$$vR(v) = -s_{-1}v + s_0 + \frac{s_1}{v} + \frac{s_2}{v^2} + \cdots ,$$

the following relation holds:

$$vR(v) \sim S^{(1)}. \tag{111}$$

The matrix $S^{(1)}$, like S, is of finite rank m, since the function $vR(v)$, like $R(v)$, has m poles. Therefore the forms

$$S_m(x, x) = \sum_{i,k=0}^{m-1} s_{i+k} x_i x_k , \qquad S_m^{(1)}(x, x) = \sum_{i,k=0}^{m-1} s_{i+k+1} x_i x_k$$

are of rank m. But by Theorem 9 (p. 190) the signatures of these forms, in virtue of (107) and (111), are equal to the indices (110) and hence also to m. Thus, the conditions (110) mean that the quadratic forms $S_m(x, x)$ and $S_m^{(1)}(x, x)$ are positive definite. Hence:

THEOREM 17: *A real polynomial $f(z) = h(z^2) + zg(z^2)$ of degree $n = 2m$ or $n = 2m + 1$ is a Hurwitz polynomial if and only if:*[78]

1. *The quadratic forms*

$$S_m(x, x) = \sum_{i,k=0}^{m-1} s_{i+k} x_i x_k , \qquad S_m^{(1)}(x, x) = \sum_{i,k=0}^{m-1} s_{i+k+1} x_i x_k \tag{112}$$

are positive definite; and

2. *(For $n = 2m + 1$)*

$$s_{-1} > 0. \tag{113}$$

Here $s_{-1}, s_0, s_1, \ldots, s_{2m-1}$ are the coefficients of the expansion

$$\frac{g(u)}{h(u)} = s_{-1} + \frac{s_0}{u} - \frac{s_2}{u^2} + \frac{s_2}{u^3} - \frac{s_3}{u^4} + \cdots .$$

---

[77] To within a constant factor.

[78] We do not mention the inequality $\Delta_n \neq 0$ expressly, because it follows automatically from the conditions of the theorem. For if $f(z)$ is a Hurwitz polynomial, then it is known that $\Delta_n \neq 0$. But if the conditions 1., 2. are given, then the fact that the form $S_m^{(1)}(x, x)$ is positive definite implies that

$$-I_{-\infty}^{+\infty} \frac{ug(u)}{h(u)} = I_{-\infty}^{+\infty} vR(v) = m,$$

and from this it follows that the fraction $ug(u)/h(u)$ is reduced, which can be expressed by the inequality $\Delta_n \neq 0$.

In exactly the same way, it follows automatically from the conditions of the theorem that $D_m = | s_{i+k} |_0^{m-1} \neq 0$, i.e., that the numbers $s_0, s_1, \ldots, s_{2m-1}$, and (for $n = 2m + 1$) s_{-1} are the Markov parameters of $f(z)$.

We introduce a notation for the determinants

$$D_p = \left| s_{i+k} \right|_0^{p-1}, \quad D_p^{(1)} = \left| s_{i+k+1} \right|_0^{p-1} \quad (p=1,\, 2,\, \dots,\, m). \tag{114}$$

Then condition 1. is equivalent to the system of determinantal inequalities

$$\left.\begin{array}{l} D_1 = s_0 > 0,\ D_2 = \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} > 0,\ \dots,\ D_m = \begin{vmatrix} s_0 & s_1 & \dots & s_{m-1} \\ s_1 & s_2 & \dots & s_m \\ \cdot & \cdot & \cdot & \cdot \\ s_{m-1} & s_m & \dots & s_{2m-2} \end{vmatrix} > 0, \\ D_1^{(1)} = s_1 > 0, D_2^{(1)} = \begin{vmatrix} s_1 & s_2 \\ s_2 & s_3 \end{vmatrix} > 0, \dots, D_m^{(1)} = \begin{vmatrix} s_1 & s_2 & \dots & s_m \\ s_2 & s_3 & \dots & s_{m+1} \\ \cdot & \cdot & \cdot & \cdot \\ s_m & s_{m+1} & & s_{2m-1} \end{vmatrix} > 0. \end{array}\right\} \tag{115}$$

If $n = 2m$, the inequalities (115) determine the domain of stability in the space of Markov parameters. If $n = 2m + 1$, we have to add the further inequality:

$$s_{-1} > 0. \tag{116}$$

In the next section we shall find out what properties of S follow from the inequalities (115) and, in so doing, shall single out the special class of infinite Hankel matrices S that correspond to Hurwitz polynomials.

§ 16. Connection with the Problem of Moments

1. We begin by stating the following problem:

PROBLEM OF MOMENTS FOR THE POSITIVE AXIS $0 < v < \infty$:[79] *Given a sequence s_0, s_1, … of real numbers, it is required to determine positive numbers*

$$\mu_1 > 0, \quad \mu_2 > 0, \ \dots, \ \mu_m > 0, \quad 0 < v_1 < v_2 < \cdots < v_m \tag{117}$$

such that the following equations hold:

$$s_p = \sum_{j=1}^{m} \mu_j v_j^p \qquad (p=0,1,2,\dots). \tag{118}$$

It is not difficult to see that the system (118) of equations is equivalent to the following expansion in a series of negative powers of u:

---

[79] This problem of moments ought to be called discrete in contrast to the general exponential problem of moments, in which the sums $\sum_{j=1}^{m} \mu_j v_j^p$ are replaced by Stieltjes integrals $\int_0^\infty v^p \, d\mu(v)$ (see [55]).

$$\sum_{j=1}^{m} \frac{\mu_j}{u + v_j} = \frac{s_0}{u} - \frac{s_1}{u^2} + \frac{s_2}{u^3} - \cdots. \tag{119}$$

In this case the infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$ is of finite rank m and by (117) in the irreducible proper fraction

$$\frac{g(u)}{h(u)} = \sum_{j=1}^{m} \frac{\mu_j}{u + v_j} \tag{120}$$

(we choose the highest coefficients of $h(u)$ and $g(u)$ to be positive) the polynomials $h(u)$ and $g(u)$ form a positive pair (see (91) and (91′)).

Therefore (see Theorem 14), our problem of moments has a solution if and only if the sequence s_0, s_1, s_2, … determines by means of (119) and (120) a Hurwitz polynomial $f(z) = h(z^2) + zg(z^2)$ of degree $2m$.

The solution of the problem of moments is unique, because the positive numbers v_j and μ_j ($j = 1, 2, \dots, m$) are uniquely determined from the expansion (119).

Apart from the 'infinite' problem of moments (118) we also consider the 'finite' problem of moments given by the first $2m$ equations (118)

$$s_p = \sum_{j=1}^{m} \mu_j v_j^p \qquad (p=0,\, 1,\, \dots,\, 2m-1). \tag{121}$$

These relations already determine the following expressions for the Hankel quadratic forms:

$$\left.\begin{array}{l} \sum_{i,k=0}^{m-1} s_{i+k} x_i x_k = \sum_{j=1}^{m} \mu_j (x_0 + x_1 v_j + \cdots + x_{m-1} v_j^{m-1})^2, \\ \sum_{i,k=0}^{m-1} s_{i+k+1} x_i x_k = \sum_{j=1}^{m} \mu_j v_j (x_0 + x_1 v_j + \cdots + x_{m-1} v_j^{m-1})^2. \end{array}\right\} \tag{122}$$

Since the linear forms in the variables x_0, x_1, …, x_{m-1}

$$x_0 + x_1 v_j + \cdots + x_{m-1} v_j^{m-1} \qquad (j=1,2,\,\dots,\,m)$$

are independent (their coefficients form a non-vanishing Vandermonde determinant), the quadratic forms (122) are positive definite. But then by Theorem 17 the numbers s_0, s_1, …, s_{2m-1} are the Markov parameters of a certain Hurwitz polynomial $f(z)$. They are the first $2m$ coefficients of the expansion (119). Together with the remaining coefficients s_{2m}, s_{2m+1}, … they determine the infinite solvable problem of moments (118), which has the same solution as the finite problem (121).

Thus we have proved the following theorem:

THEOREM 18: 1) *The finite problem of moments*

$$s_p = \sum_{j=1}^{m} \mu_j v_j^p \tag{123}$$

$(p = 0, 1, \ldots, 2m-1; \mu_1 > 0, \ldots, \mu_m > 0; 0 < v_1 < v_2 < \ldots < v_m)$, *where s_p are given real numbers and v_j and μ_j are unknown real numbers* $(p = 0, 1, \ldots, 2m-1;\ j = 1, 2, \ldots, m)$ *has a solution if and only if the quadratic forms*

$$\sum_{i,k=0}^{m-1} s_{i+k} x_i x_k, \quad \sum_{i,k=0}^{m-1} s_{i+k+1} x_i x_k \tag{124}$$

are positive definite, i.e., if the numbers $s_0, s_1, \ldots, s_{2m-1}$ are the Markov parameters of some Hurwitz polynomial of degree $2m$.

2) *The infinite problem of moments*

$$s_p = \sum_{j=1}^{m} \mu_j v_j^p \tag{125}$$

$(p = 0, 1, 2, \ldots; \mu_1 > 0, \ldots, \mu_m > 0; 0 < v_1 < v_2 < \ldots < v_m)$, *where s_p are given real numbers and v_j and μ_j are unknown real numbers* $(p = 0, 1, \ldots;\ j = 1, 2, \ldots, m)$ *has a solution if and only if* 1. *the quadratic forms* (124) *are positive definite and* 2. *the infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$ is of rank m, i.e., if the series*

$$\frac{s_0}{u} - \frac{s_1}{u^2} + \frac{s_2}{u^3} - \cdots = \frac{g(u)}{h(u)} \tag{126}$$

determines a Hurwitz polynomial $f(z) = h(z^2) = zg(z^2)$ of degree $2m$.

3) *The solution of the problem of moments, both the finite* (123) *and the infinite* (124) *problem, is always unique.*

2. We shall use this theorem in investigating the minors of an infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$ of rank m corresponding to some Hurwitz polynomial, i.e., one for which the quadratic form (124) is positive definite. In this case the generating numbers $s_0, s_1, s_2, \ldots$ of S can be represented in the form (123), so that for an arbitrary minor of S of order $h \leqq m$ we have:

$$\begin{vmatrix} s_{i_1+k_1} & \cdots & s_{i_1-k_h} \\ \cdots & \cdots & \cdots \\ s_{i_h+k_1} & \cdots & s_{i_h+k_h} \end{vmatrix} = \begin{vmatrix} \mu_1 v_1^{i_1} & \mu_2 v_2^{i_1} & \ldots & \mu_m v_m^{i_1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_1 v_1^{i_h} & \mu_2 v_2^{i_h} & \ldots & \mu_m v_m^{i_h} \end{vmatrix} \cdot \begin{vmatrix} v_1^{k_1} & \ldots & v_1^{k_h} \\ v_2^{k_1} & \ldots & v_2^{k_h} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ v_m^{k_1} & \ldots & v_m^{k_h} \end{vmatrix}$$

and therefore

$$S\begin{pmatrix} i_1 & i_2 & \ldots & i_h \\ k_1 & k_2 & \ldots & k_h \end{pmatrix}$$

$$= \sum_{1 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_h \leqq m} \mu_{\alpha_1} \mu_{\alpha_2} \ldots \mu_{\alpha_h} \begin{vmatrix} v_{\alpha_1}^{i_1} & v_{\alpha_2}^{i_1} & \ldots & v_{\alpha_h}^{i_1} \\ v_{\alpha_1}^{i_2} & v_{\alpha_2}^{i_2} & \ldots & v_{\alpha_h}^{i_2} \\ \cdots & \cdots & \cdots & \cdots \\ v_{\alpha_1}^{i_h} & v_{\alpha_1}^{i_h} & \ldots & v_{\alpha_h}^{i_h} \end{vmatrix} \begin{vmatrix} v_{\alpha_1}^{k_1} & v_{\alpha_1}^{k_2} & \ldots & v_{\alpha_1}^{k_h} \\ v_{\alpha_2}^{k_1} & v_{\alpha_2}^{k_2} & \ldots & v_{\alpha_2}^{k_h} \\ \cdots & \cdots & \cdots & \cdots \\ v_{\alpha_h}^{k_1} & v_{\alpha_h}^{k_2} & \ldots & v_{\alpha_h}^{k_h} \end{vmatrix}. \tag{127}$$

But from the inequalities

$$0 < v_1 < v_2 < \cdots < v_m, \quad i_1 < i_2 < \cdots < i_h, \quad k_1 < k_2 < \cdots < k_h$$

it follows that the generalized Vandermonde determinants[80]

$$\begin{vmatrix} v_{\alpha_1}^{i_1} & v_{\alpha_2}^{i_1} & \ldots & v_{\alpha_h}^{i_1} \\ v_{\alpha_1}^{i_2} & v_{\alpha_2}^{i_2} & \ldots & v_{\alpha_h}^{i_2} \\ \cdots & \cdots & \cdots & \cdots \\ v_{\alpha_1}^{i_h} & v_{\alpha_2}^{i_h} & \ldots & v_{\alpha_h}^{i_h} \end{vmatrix} > 0, \qquad \begin{vmatrix} v_{\alpha_1}^{k_1} & v_{\alpha_1}^{k_2} & \ldots & v_{\alpha_1}^{k_h} \\ v_{\alpha_2}^{k_1} & v_{\alpha_2}^{k_2} & \ldots & v_{\alpha_2}^{k_h} \\ \cdots & \cdots & \cdots & \cdots \\ v_{\alpha_h}^{k_1} & v_{\alpha_h}^{k_2} & \ldots & v_{\alpha_h}^{k_h} \end{vmatrix} > 0$$

are positive.

Since the numbers μ_j are positive $(j = 1, 2, \ldots, m)$, it therefore follows from (127) that

$$S\begin{pmatrix} i_1 & i_2 & \ldots & i_h \\ k_1 & k_2 & \ldots & k_h \end{pmatrix} > 0 \quad \left(0 \leqq \begin{matrix} i_1 < i_2 < \cdots < i_h \\ k_1 < k_2 < \cdots < k_h, \end{matrix} \quad h = 1, 2, \ldots, m\right). \tag{128}$$

Conversely, if in an infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$ of rank m all the minors of every order $h \leqq m$ are positive, then the quadratic forms (124) are positive definite.

DEFINITION 4: *An infinite matrix $A = \| a_{ik} \|_1^\infty$ will be called totally positive of rank m if and only if all the minors of A of order $h \leqq m$ are positive and all the minors of order $h > m$ are zero.*

The property of S that we have found can now be expressed in the following theorem:[81]

THEOREM 19: *An infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$ is totally positive of rank m if and only if* 1) *S is of rank m and* 2) *the quadratic forms*

$$\sum_{i,k=0}^{m-1} s_{i+k} x_i x_k, \quad \sum_{i,k=0}^{m-1} s_{i+k+1} x_i x_k$$

are positive definite.

---

[80] See p. 99, Example 1.

[81] See [173].

From this theorem and Theorem 17 we obtain:

THEOREM 20: *A real polynomial $f(z)$ of degree n is a Hurwitz polynomial if and only if the corresponding infinite Hankel matrix $S = \| s_{i+k} \|_0^\infty$ is totally positive of degree $m = [n/2]$ and if, in addition, $s_{-1} > 0$ when n is odd.*

Here the elements $s_0, s_1, s_2, \ldots$ of S and s_{-1} are determined by the expansion

$$\frac{g(u)}{h(u)} = s_{-1} + \frac{s_0}{u} - \frac{s_1}{u^2} + \frac{s_2}{u^3} - \cdots, \tag{129}$$

where

$$f(z) = h(z^2) + zg(z^2).$$

§ 17. Theorems of Markov and Chebyshev

1. In a notable memoir 'On functions obtained by converting series into continued fractions'[82] Markov proved two theorems, the second of which had been established in 1892 by Chebyshev by other methods, and not in the same generality.[83]

In this section we shall show that these theorems have an immediate bearing on the study of the domain of stability in the Markov parameters and shall give a comparatively simple proof (without reference to continued fractions) which is based on Theorem 19 of the preceding section.

In proceeding to state the first theorem, we quote the corresponding passage from the above-mentioned memoir of Markov:[84]

> On the basis of what has preceded it is not difficult to prove two remarkable theorems with which we conclude our paper.
>
> One is concerned with the determinants[85]
>
> $$\Delta_1, \Delta_2, \ldots, \Delta_m, \quad \Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(m)}$$
>
> and the other with the roots of the equation[86]
>
> $$\psi_m(x) = 0.$$

---

[82] Zap. Petersburg Akad. Nauk, Petersburg, 1894 [in Russian]; see also [38], pp. 78-105.

[83] This theorem was first published in Chebyshev's paper 'On the expansion in continued fractions of series in descending powers of the variable' [in Russian]. See [8], pp. 307-62.

[84] [38], p. 95, beginning with line 3 from below.

[85] In our notation, $D_1, D_2, \ldots, D_m, D_1^{(1)}, D_2^{(1)}, \ldots, D_m^{(1)}$. (See p. 236.)

[86] In our notation, $h(-x) = 0$.

THEOREM ON DETERMINANTS: *If we have for the numbers*

$$s_0, s_1, s_2, \ldots, s_{2m-2}, s_{2m-1}$$

two sets of values

1. $s_0 = a_0, s_1 = a_1, s_2 = a_2, \ldots, s_{2m-2} = a_{2m-2}, s_{2m-1} = a_{2m-1}$,
2. $s_0 = b_0, s_1 = b_1, s_2 = b_2, \ldots, s_{2m-2} = b_{2m-2}, s_{2m-1} = b_{2m-1}$

for which all the determinants

$$\Delta_1 = s_0, \Delta_2 = \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}, \ldots, \Delta_m = \begin{vmatrix} s_0 & s_1 & \ldots & s_{m-1} \\ s_1 & s_2 & \ldots & s_m \\ \cdot & \cdot & \cdot & \cdot \\ s_{m-1} & s_m & & s_{2m-2} \end{vmatrix},$$

$$\Delta^{(1)} = s_1, \Delta^{(2)} = \begin{vmatrix} s_1 & s_2 \\ s_2 & s_3 \end{vmatrix}, \ldots, \Delta^{(m)} = \begin{vmatrix} s_1 & s_2 & \ldots & s_m \\ s_2 & s_3 & \ldots & s_{m+1} \\ \cdot & \cdot & \cdot & \cdot \\ s_m & s_{m+1} & & s_{2m-1} \end{vmatrix}$$

turn out to be positive numbers satisfying the inequalities

$$a_0 \geqq b_0, b_1 \geqq a_1, a_2 \geqq b_2, b_3 \geqq a_3, \ldots, a_{2m-2} \geqq b_{2m-2}, b_{2m-1} \geqq a_{2m-1},$$

then our determinant

$$\Delta_1, \Delta_2, \ldots, \Delta_m; \; \Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(m)}$$

must be positive for all values

$$s_0, s_1, s_2, \ldots, s_{2m-1}$$

satisfying the inequalities

$$a_0 \geqq s_0 \geqq b_0, \; b_1 \geqq s_1 \geqq a_1, \; a_2 \geqq s_2 \geqq b_2, \ldots,$$
$$a_{2m-2} \geqq s_{2m-2} \geqq b_{2m-2}, \; b_{2m-1} \geqq s_{2m-1} \geqq a_{2m-1}.$$

Under these conditions we have

$$\begin{vmatrix} a_0 & a_1 & \ldots & a_{k-1} \\ a_1 & a_2 & \ldots & a_k \\ \cdot & \cdot & \cdot & \cdot \\ a_{k-1} & a_k & \ldots & a_{2k-2} \end{vmatrix} \geqq \begin{vmatrix} s_0 & s_1 & \ldots & s_{k-1} \\ s_1 & s_2 & \ldots & s_k \\ \cdot & \cdot & \cdot & \cdot \\ s_{k-1} & s_k & \ldots & s_{2k-2} \end{vmatrix} \geqq \begin{vmatrix} b_0 & b_1 & \ldots & b_{k-1} \\ b_1 & b_2 & \ldots & b_k \\ \cdot & \cdot & \cdot & \cdot \\ b_{k-1} & b_k & \ldots & b_{2k-2} \end{vmatrix}$$

and

$$\begin{vmatrix} b_1 & b_2 & \dots & b_k \\ b_2 & b_3 & \dots & b_{k+1} \\ \dots & \dots & \dots & \dots \\ b_k & b_{k+1} & \dots & b_{2k-1} \end{vmatrix} \geqq \begin{vmatrix} s_1 & s_2 & \dots & s_k \\ s_2 & s_3 & \dots & s_{k+1} \\ \dots & \dots & \dots & \dots \\ s_k & s_{k+1} & \dots & s_{2k-1} \end{vmatrix} \geqq \begin{vmatrix} a_1 & a_2 & \dots & a_k \\ a_2 & a_3 & \dots & a_{k+1} \\ \dots & \dots & \dots & \dots \\ a_k & a_{k+1} & \dots & a_{2k-1} \end{vmatrix}$$

for $k = 1, 2, \dots, m$.

In order to give another statement of this theorem in connection with the problem of stability, we introduce some concepts and notations.

The Markov parameters $s_0, s_1, \dots, s_{2m-1}$ (for $n = 2m$) or $s_{-1}, s_0, s_1, \dots, s_{2m-1}$ (for $n = 2m + 1$) will be regarded as the coordinates of some point $\boldsymbol{P}$ in an n-dimensional space. The domain of stability in this space will be denoted by $\boldsymbol{G}$. The domain $\boldsymbol{G}$ is characterized by the inequalities (115) and (116) (p. 236).

We shall say that a point $\boldsymbol{P} = \{s_i\}$ 'precedes' a point $\boldsymbol{P}^* = \{s_i^*\}$ and shall write $\boldsymbol{P} \prec \boldsymbol{P}^*$ if

$$\left.\begin{array}{l} s_0 \leqq s_0^*, \; s_1^* \leqq s_1, \; s_2 \leqq s_2^*, \; s_3^* \leqq s_3, \; \dots, \; s_{2m-1}^* \leqq s_{2m-1} \\ \text{and (for } n = 2m + 1) \qquad s_{-1} \leqq s_{-1}^* \end{array}\right\} \tag{130}$$

and the sign $<$ holds in at least one of these relations.

If only the relations (130) hold, without the last clause, then we shall write:

$$\mathfrak{P} \preceq \mathfrak{P}^*.$$

We shall say that a point $\boldsymbol{Q}$ lies 'between' $\boldsymbol{P}$ and $\boldsymbol{R}$ if $\boldsymbol{P} \prec \boldsymbol{Q} \prec \boldsymbol{R}$.

To every point $\boldsymbol{P}$ there corresponds an infinite Hankel matrix of rank m: $S = \| s_{i+k} \|_0^\infty$. We shall denote this matrix by $S_{\mathfrak{P}}$.

Now we can state Markov's theorem in the following way:

THEOREM 21 (Markov): *If two points* $\boldsymbol{P}$ *and* $\boldsymbol{R}$ *belong to the domain of stability* $\boldsymbol{G}$ *and if* $\boldsymbol{P}$ *precedes* $\boldsymbol{R}$, *then every point* $\boldsymbol{Q}$ *between* $\boldsymbol{P}$ *and* $\boldsymbol{R}$ *also belongs to* $\boldsymbol{G}$, *i.e.,*

from $\boldsymbol{P}, \boldsymbol{R} \in \boldsymbol{G}, \boldsymbol{P} \preceq \boldsymbol{Q} \preceq \boldsymbol{R}$ *it follows that* $\boldsymbol{Q} \in \boldsymbol{G}$.

Proof. From $\boldsymbol{P} \preceq \boldsymbol{Q} \preceq \boldsymbol{R}$ it follows that $\boldsymbol{P}$ and $\boldsymbol{Q}$ can be connected by an arc of a curve

$$s_i = (-1)^i \varphi_i(t) \; [\alpha \leqq t \leqq \gamma; i = 0, 1, \dots, 2m-1 \text{ and (for } n = 2m+1) \; i = -1] \tag{131}$$

passing through $\boldsymbol{Q}$ such that: 1) the functions $\varphi_i(t)$ are continuous, monotonic increasing, and differentiable when t varies from $t = \alpha$ to $t = \gamma$; and 2) the values α, β, γ ($\alpha < \beta < \gamma$) of t correspond to the points $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{R}$ on the curve.

From the values (131) we form the infinite Hankel matrix $S = S(t) = \| s_{i+k}(t) \|_0^\infty$ of rank m. We consider part of this matrix, namely the rectangular matrix

$$\begin{Vmatrix} s_0 & s_1 & \dots & s_{m-1} & s_m \\ s_1 & s_2 & \dots & s_m & s_{m+1} \\ \dots & \dots & \dots & \dots & \dots \\ s_{m-1} & s_m & \dots & s_{2m-2} & s_{2m-1} \end{Vmatrix}. \tag{132}$$

By the conditions of the theorem, the matrix $S(t)$ is totally positive of rank m for $t = \alpha$ and $t = \gamma$, so that all the minors of (132) of order $p = 1, 2, 3, \dots, m$ are positive.

We shall now show that this property also holds for every intermediate value of t ($\alpha < t < \gamma$).

For $p = 1$, this is obvious. Let us prove the statement for the minors of order p, on the assumption that it is true for those of order $p - 1$. We consider an arbitrary minor of order p formed from successive rows and columns of (132):

$$D_p^{(q)} = \begin{vmatrix} s_q & s_{q+1} & \dots & s_{q+p-1} \\ s_{q+1} & s_{q+2} & \dots & s_{q+p} \\ \dots & \dots & \dots & \dots \\ s_{q+p-1} & s_{q+p} & \dots & s_{q+2p-2} \end{vmatrix} \qquad [q = 0, 1, \dots, 2(m-p) \div 1]. \tag{133}$$

We compute the derivative of this minor

$$\frac{d}{dt} D_p^{(q)} = \sum_{i,k=0}^{p-1} \frac{\partial D_p^{(q)}}{\partial s_{q+i+k}} \frac{ds_{q+i+k}}{dt}. \tag{134}$$

$\dfrac{\partial D_p^{(q)}}{\partial s_{q+i+k}}$ ($i, k = 0, 1, \dots, p-1$) are the algebraic complements (adjoints) of the elements of (133). Since by assumption all the minors of this determinant are positive, we have

$$(-1)^{i+k} \frac{\partial D_p^{(q)}}{\partial s_{q+i+k}} > 0 \qquad (i, k = 0, 1, \dots, p-1). \tag{135}$$

On the other hand, we find from (131):

$$(-1)^{q+i+k}\frac{ds_{q+i+k}}{dt}=\frac{d\varphi_{q+i+k}}{dt}\geqq 0 \qquad (i,\ k=0,\ 1,\ \ldots\ ,\ p-1). \quad (136)$$

From (134), (135), and (136) it follows that

$$(-1)^{q}\frac{d}{dt}D_p^{(q)}\geqq 0 \qquad \begin{pmatrix} q=0,\ 1,\ \ldots\ ,\ 2(m-p)+1, \\ p=1,\ 2,\ \ldots\ ,\ m, \\ \alpha\leqq t\leqq\gamma \end{pmatrix}. \qquad (137)$$

Thus, when the argument increases from $t=\alpha$, to $t=\gamma$, then every minor (133) with even q is a monotone non-decreasing function and with odd q is a monotone non-increasing function; but since the minor is positive for $t=\alpha$ and $t=\gamma$, it is also positive for every intermediate value of t $(\alpha<t<\gamma)$.

From the fact that the minors of (132) of order $p-1$ and those of order p that are formed from successive rows and columns are positive, it now follows that *all* the minors of (131) of order p are positive.[87]

What we have proved implies that for every t $(\alpha\leqq t\leqq\gamma)$ the values $s_0, s_1, \ldots, s_{2m-1}$ and (for $n=2m+1$) s_{-1} satisfy the inequalities (115) and (116), i.e., that for every t these values are the Markov parameters of a certain Hurwitz polynomial. In other words, the whole arc (131) and, in particular, the point $\boldsymbol{Q}$ lies in the domain of stability $\boldsymbol{G}$.

This completes the proof of Markov's Theorem.

Note. Since we have proved that every point of the arc (131) belongs to $\boldsymbol{G}$, the values of (131) for every t $(\alpha\leqq t\leqq\gamma)$ determine a totally positive matrix $S(t)=\| s_{i+k}(t)\|_0^m$ of rank m. Therefore the inequalities (135) and consequently (137) as well hold for every t $(\alpha\leqq t\leqq\gamma)$, i.e., with increasing t every $D_p^{(q)}$ increases for even q and decreases for odd q $(q=0, 1, 2, \ldots, 2(m-p)+1;\ p=1, \ldots, m)$. In other words, from $\boldsymbol{P}\prec\boldsymbol{Q}\preceq\boldsymbol{R}$ it follows that

$$(-1)^qD_p^{(q)}(\mathfrak{P})\leqq(-1)^qD_p^{(q)}(\mathfrak{Q})\leqq(-1)^qD_p^{(q)}(\mathfrak{R})$$
$$(q=0,\ 1,\ \ldots,\ 2(m-p)+1;\ p=1,\ \ldots,\ m).$$

These inequalities for $q=0$, 1 give Markov's inequalities (pp. 241).

We now come to the Chebyshev-Markov theorem mentioned at the beginning of this section. Again we quote from Markov's memoir:[88]

---

[87] This follows from Fekete's determinant indentity (see [17], pp. 306-7).

[88] See [38], p. 103, beginning with line 5.

THEOREM ON ROOTS: *If the numbers*

$$a_0,\ a_1,\ a_2,\ \ldots,\ a_{2m-2},\ a_{2m-1},$$
$$s_0,\ s_1,\ s_2,\ \ldots,\ s_{2m-2},\ s_{2m-1},$$
$$b_0,\ b_1,\ b_2,\ \ldots,\ b_{2m-2},\ b_{2m-1}$$

satisfy all the conditions of the preceding theorem,[89] *then the equations*

$$\begin{vmatrix} a_0 & a_1 & \ldots & a_{m-1} & 1 \\ a_1 & a_2 & \ldots & a_m & x \\ a_2 & a_3 & \ldots & a_{m+1} & x^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_m & a_{m+1} & \ldots & a_{2m-1} & x^m \end{vmatrix}=0,$$

$$\begin{vmatrix} s_0 & s_1 & \ldots & s_{m-1} & 1 \\ s_1 & s_2 & \ldots & s_m & x \\ s_2 & s_3 & \ldots & s_{m+1} & x^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_m & s_{m+1} & \ldots & s_{2m-1} & x^m \end{vmatrix}=0,$$

$$\begin{vmatrix} b_0 & b_1 & \ldots & b_{m-1} & 1 \\ b_1 & b_2 & \ldots & b_m & x \\ b_2 & b_3 & \ldots & b_{m+1} & x^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_m & b_{m+1} & \ldots & b_{2m-1} & x^m \end{vmatrix}=0$$

of degree m in the unknown x do not have multiple or imaginary or negative roots.

And the roots of the second equation are larger than the corresponding roots of the first equation and smaller than the corresponding roots of the last equation.

Let us find out the connection of this theorem with the domain of stability in the space of the Markov parameters. Setting $f(z)=h(z^2)+zg(z^2)$ and

$$h(-v)=c_0v^m+c_1v^{m-1}+\cdots+c_m \qquad (c_0\neq 0),$$

we obtain from the expansion (105)

$$R(v)=-\frac{g(-v)}{h(-v)}=-s_{-1}+\frac{s_0}{v}+\frac{s_1}{v^2}+\cdots$$

the identity

---

[89] He refers to the preceding theorem, Markov's theorem on determinants (pp. 241).

$$-g(-v)=\left(-s_{-1}+\frac{s_0}{v}+\frac{s_1}{v^2}+\cdots\right)(c_0v^m+c_1v^{m-1}+\cdots+c_m).$$

Equating to zero the coefficients of the powers $v^{-1}, v^{-2}, \ldots, v^{-m}$, we find:

$$\left.\begin{array}{l} s_0c_m \quad +s_1c_{m-1}+\cdots+s_mc_0 \quad =0,\\ s_1c_m \quad +s_2c_{m-1}+\cdots+s_{m+1}\,c_0=0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ s_{m-1}c_m+s_mc_{m-1}+\cdots+s_{2m-1}c_0-0; \end{array}\right\} ; \tag{138}$$

to these relations we add the equation

$$h(-v)=0, \tag{139}$$

written as

$$c_m+vc_{m-1}+\cdots+v^mc_0=0. \tag{139'}$$

Eliminating from (138) and (139′) the coefficients $c_0, c_1, \ldots, c_m$, we represent the equation (139) in the form

$$\left|\begin{array}{ccccc} s_0 & s_1 & \ldots & s_{m-1} & 1\\ s_1 & s_2 & \ldots & s_m & v\\ s_2 & s_3 & \ldots & s_{m+1} & v^2\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ s_m & s_{m+1} & \ldots & s_{2m-1} & v^m \end{array}\right|=0. \tag{139''}$$

Thus, the algebraic equation in the Chebyshev-Markov theorem coincides with (139) and the inequalities imposed on $s_0, s_1, \ldots, s_{2m-1}$ coincide with the inequalities (115) that determine the domain of stability in the space of the Markov parameters.

The Chebyshev-Markov theorem shows how the roots $u_1=-v_1$, $u_2=-v_2, \ldots, u_m=-v_m$ of $h(u)$ change when the corresponding Markov parameters $s_0, s_1, \ldots, s_{2m-1}$ vary in the domain of stability.

The first part of the theorem states something we already know: When the inequalities (115) are satisfied, then all the roots $u_1, u_2, \ldots, u_m$ of $h(u)$ are simple, real, and negative.[90] We denote them as follows:

$$u_1(\boldsymbol{P}),\ u_2(\boldsymbol{P}), \ldots, u_m(\boldsymbol{P}),$$

where $\boldsymbol{P}$ is the corresponding point of $\boldsymbol{G}$.

The second (fundamental) part of the Chebyshev-Markov theorem can be stated as follows:

---

[90] See Theorem 13, on p. 228.

THEOREM 22 (Chebyshev-Markov): *If* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *are two points of* $\boldsymbol{G}$ *and* $\boldsymbol{P}$ *'precedes'* $\boldsymbol{Q}$,

$$\boldsymbol{P} \prec \boldsymbol{Q}, \tag{140}$$

then[91]

$$u_1(\boldsymbol{P})<u_1(\boldsymbol{Q}),\ u_2(\boldsymbol{P})<u_2(\boldsymbol{Q}), \ldots, u_m(\boldsymbol{P})<u_m(\boldsymbol{Q}). \tag{141}$$

Proof. The coefficients of $h(u)$ can be expressed rationally in terms of the parameters $s_0, s_1, \ldots, s_{2m-1}$.[92] Then

$$h(u_i)=0 \qquad (i=1,\ 2,\ \ldots,\ m)$$

implies that:[93]

$$\frac{\partial h(u_i)}{\partial s_l}+h'(u_i)\frac{du_i}{ds_l}=0 \qquad (i=1,\ 2,\ \ldots,\ m; \quad l=0,\ 1,\ \ldots,\ 2m-1). \tag{142}$$

On the other hand, when we differentiate the expansion

$$\frac{g(u)}{h(u)}=s_{-1}+\frac{s_0}{u}-\frac{s_1}{u^2}+\frac{s_2}{u^3}-\cdots$$

term by term with respect to s, we find:

$$\frac{h(u)\dfrac{\partial g(u)}{\partial s_l}-g(u)\dfrac{\partial h(u)}{\partial s_l}}{h^2(u)}=\frac{(-1)^l}{u^{l+1}}+\frac{1}{u^{2m+1}}(*). \tag{143}$$

Multiplying both sides of this equation by $\dfrac{h^2(u)}{u-u_i}$ and denoting the coefficient of u^l in this polynomial by C_{il}, we obtain:

$$\frac{h(u)}{u-u_i}\frac{\partial g(u)}{\partial s_l}-\frac{g(u)\dfrac{\partial h(u)}{\partial s_l}}{u-u_i}=\frac{(-1)^lC_{il}}{u}+\cdots. \tag{144}$$

Comparing the coefficients of $1/u$ (the residues) on the two sides of (144), we find:

$$(-1)^{l-1}g(u_i)\frac{\partial h(u_i)}{\partial s_l}=C_{il}, \tag{145}$$

which gives in conjunction with (142):

$$\frac{du_i}{ds_l}=\frac{(-1)^lC_{il}}{g(u_i)h'(u_i)}.$$

---

[91] In other words, the roots $u_1, u_2, \ldots, u_m$ increase with increasing $s_0, s_2, \ldots, s_{2m-2}$ and with decreasing $s_1, s_3, \ldots, s_{2m-1}$,

[92] For example, by the equations (138) if, for simplicity, we set $c_0=1$ in these equations

[93] Here $\dfrac{\partial h(u_i)}{\partial s_l}=\left[\dfrac{\partial h(u)}{\partial s_l}\right]_{u=u_i}$.

Introducing the values

$$R_i = \frac{g\,(u_i)}{h'\,(u_i)} \quad (l = 1,\ 2,\ \ldots,\ m), \tag{146}$$

we obtain the *formula of Chebyshev-Markov*:

$$\frac{du_i}{ds_l} = \frac{(-1)_l C_{il}}{R_i[h'\,(u_i)]^2} \qquad (i = 1,\ 2,\ \ldots,\ m; \quad l = 0,\ 1,\ \ldots,\ 2m-1). \tag{147}$$

But in the domain of stability the values R_i $(i = 1, 2, \ldots, m)$ are positive (see (90′) on p. 226). The same can be said of the coefficients C_{il}. For

$$\frac{h^2\,(u)}{u - u_i} = c_0^2\,(u+v_1)^2 \cdots (u+v_{i-1})^2\,(u+v_i)\,(u+v_{i-1})^2 \cdots (u+v_m)^2, \tag{148}$$

where

$$v_i = -\,u_i > 0 \quad (i = 1,\ 2,\ \ldots,\ m),$$

From (148) it is clear that all the coefficients C_{il} in the expansion of $\frac{h^2\,(u)}{u-u_i}$ in powers of u are positive. Thus, we obtain from the Chebyshev-Markov formula:

$$(-1)^l\,\frac{du_i}{ds_l} > 0. \tag{149}$$

In the proof of Markov's theorem we have shown that any two points $\boldsymbol{P} \prec \boldsymbol{Q}$ of $\boldsymbol{G}$ can be joined by an arc $s_l = (-1)^l \varphi_l\,(t)$ $(l = 0, 1, \ldots, 2m-1)$, where $\varphi_l(t)$ is a monotonic increasing differentiable function of t (t varies within the limits α and β $(\alpha < \beta)$ and $t = \alpha$ corresponds to $\boldsymbol{P}$, $t = \beta$ to $\boldsymbol{Q}$). Then along this arc we have, by (149):[94]

$$\frac{du_i}{dt} = \sum_{l=0}^{2m-1} \frac{du_i}{ds_l}\,\frac{ds_l}{dt} \geqq 0, \quad \frac{du_i}{dt} \not\equiv 0 \quad (\alpha \leqq t \leqq \beta). \tag{150}$$

Hence by integrating we obtain:

$$u_{i(t=\alpha)} = u_i\,(\boldsymbol{P}) < u_{i(t=\beta)} = u_i\,(\boldsymbol{Q}) \qquad (i = 1,\ 2,\ \ldots,\ m).$$

This completes the proof of the Chebyshev-Markov theorem.

§ 18. The Generalized Routh-Hurwitz Problem

1. In this section we shall give a rule to determine the number of roots in the right half-plane of a polynomial $f(z)$ with complex coefficients.

---

[94] Since $(-1)^l \frac{ds_l}{dt} = \frac{d\varphi_l}{dt} \geqq 0$ $(\alpha \leqq t \leqq \beta)$ and for at least one l there exist values of t for which $(-1)^l \frac{ds_l}{dt} > 0$.

Suppose that

$$f\,(iz) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n + i\,(a_0 z^n + a_1 z^{n-1} + \cdots + a_n), \tag{151}$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$ are real numbers. If the degree of $f(z)$ is n, then $b_0 + ia_0 \neq 0$. Without loss of generality we may assume that $a_0 \neq 0$ (otherwise we could replace $f(z)$ by $if(z)$).

We shall assume that the real polynomials

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_n \quad \text{and} \quad b_0 z^n + b_1 z^{n-1} + \cdots + b_n \tag{152}$$

are co-prime, i.e., that their resultant does not vanish:[95]

$$\nabla_{2n} + \begin{vmatrix} a_0 & a_1 \ldots a_n & 0 & \ldots 0 \\ b_0 & b_1 \ldots b_n & 0 & \ldots 0 \\ 0 & a_0 \ldots a_{n-1} & a_n & \ldots 0 \\ 0 & b_0 \ldots b_{n-1} & b_n & \ldots 0 \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{vmatrix} \neq 0. \tag{153}$$

Hence it follows, in particular, that the polynomials (152) have no roots in common and that, therefore, $f(z)$ has no roots on the imaginary axis.

We denote by k the number of roots of $f(z)$ with positive real parts. By considering the domain in the right half-plane bounded by the imaginary axis and the semi-circle of radius R $(R \to \infty)$ and by repeating verbatim the arguments used on p. 177 for the real polynomial $f(z)$, we obtain the formula for the increment of $\arg f(z)$ along the imaginary axis

$$\Delta_{-\infty}^{-\infty} \arg f\,(z) = (n - 2k)\,\pi. \tag{154}$$

Hence we obtain, by (151), in view of $a_0 \neq 0$:

$$I_{-\infty}^{+\infty} \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{a_0 z^n + a_1 z^{n-1} + \cdots + z_n} = n - 2k. \tag{155}$$

Using Theorem 10 of § 11 (p. 215), we now obtain:

$$k = V\,(1,\ \nabla_2,\ \nabla_4,\ \ldots,\ \nabla_{2n}), \tag{156}$$

where

---

[95] ∇_{2n} is a determinant of order $2n$.

$$\nabla_{2p} = \begin{vmatrix} a_0 & a_1 & \cdots & a_{2p-1} \\ b_0 & b_1 & \cdots & b_{2p-1} \\ 0 & a_0 & \cdots & a_{2p-2} \\ 0 & b_0 & \cdots & a_{2p-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \quad (p=1,\ 2,\ \ldots,\ n;\ a_k=b_k=0 \text{ for } k>n). \qquad (157)$$

We have thus arrived at the following theorem.

Theorem 23: *If a complex polynomial $f(z)$ is given for which*

$$f(iz) = b_0z^n + b_1z^{n-1} + \cdots + b_n + i\,(a_0z^n + a_1z^{n-1} + \cdots + a_n) \qquad (a_0 \neq 0)$$

and if the polynomials $a_0z^n + \ldots + a_n$ and $b_0z^n + \ldots + b_n$ are co-prime ($\nabla_{2n} \neq 0$), then the number of roots of $f(z)$ in the right half-plane is determined by the formulas (156) *and* (157).

Moreover, if some of the determinants (157) *vanish, then for each group of successive zeros*

$$(\nabla_{2h} \neq 0) \ \nabla_{2h+2} = \cdots = \nabla_{2h+2p} = 0 \ (\nabla_{2h+2p+2} \neq 0) \qquad (158)$$

in the calculation of $V(1, \nabla_2, \nabla_4, \ldots, \nabla_{2n})$ we must set:

$$\operatorname{sign} \nabla_{2h+2j} = (-1)^{\frac{j(j-1)}{2}} \operatorname{sign} \nabla_{2h} \qquad (j=1,\ 2,\ \ldots,\ p) \qquad (159)$$

or, what is the same,

$$V(\nabla_{2h},\ \nabla_{2h+2},\ \ldots,\ \nabla_{2h+2p},\ \nabla_{2h+2p+2}) = \begin{cases} \dfrac{p+1}{2} & \text{for odd } p\,, \\ \dfrac{p+1-\varepsilon}{2} & \text{for even } p \text{ and } \varepsilon = (-1)^{\frac{p}{2}} \operatorname{sign} \dfrac{\nabla_{2h+2p+2}}{\nabla_{2h}}\,. \end{cases} \qquad (160)$$

We leave it to the reader to verify that in the special case where $f(z)$ is a real polynomial we can obtain the Routh-Hurwitz theorem (see § 6) from Theorem 23.[96]

In conclusion, we mention that in this chapter we have dealt with the application of quadratic forms (in particular, Hankel forms) to one problem of the disposition of the roots of a polynomial in the complex plane. Quadratic and hermitian forms also have interesting applications to other problems of the disposition of roots. We refer the reader who is interested in these questions to the survey, already quoted, of M. G. Kreĭn and M. A. Naĭmark 'The method of symmetric and hermitian forms in the theory of separation of roots of algebraic equations,' (Kharkov, 1936).

---

[96] Suitable algorithms for the solution of the generalized Routh-Hurwitz problem can be found in the monograph [41] and in the paper [39]. See also [7] and [37].

BIBLIOGRAPHY

BIBLIOGRAPHY

Items in the Russian language are indicated by *

Part A. Textbooks, Monographs, and Surveys

[1] Achieser (Akhieser), N. J., *Theory of Approximation.* New York: Ungar, 1956. [Translated from the Russian.]

[2] Aitken, A. C., *Determinants and matrices.* 9th ed., Edinburgh: Oliver and Boyd, 1956.

[3] Bellman, R., *Stability Theory of Differential Equations.* New York: McGraw-Hill, 1953.

*[4] Bernstein, S. N., *Theory of Probability.* 4th ed., Moscow: Gostekhizdat, 1946.

[5] Bodewig, E., *Matrix Calculus.* 2nd ed., Amsterdam: North Holland, 1959.

[6] Cahen, G., *Éléments du calcul matriciel.* Paris: Dunod, 1955.

*[7] Chebotarëv, N. G., and Meĭman, N. N., *The problem of Routh-Hurwitz for polynomials and integral functions.* Trudy Mat. Inst. Steklov., vol. 26 (1949).

*[8] Chebyshev, P. L., *Complete collected works.* vol. III. Moscow: Izd. AN SSSR, 1948.

*[9] Chetaev, N. G., *Stability of motion.* Moscow: Gostekhizdat, 1946.

[10] Collatz, L., *Eigenwertaufgaben mit technischen Anwendungen.* Leipzig: Akad. Velags., 1949.

[11] ——— *Eigenwertprobleme und ihre numerische Behandlung.* New York: Chelsea, 1948.

[12] Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, vol. I. Trans. and revised from the German original. New York: Interscience, 1953.

*[13] Erugin, N. R., *The method of Lappo-Danilevskiĭ in the theory of linear differential equations.* Leningrad: Leningrad University, 1956.

*[14] Faddeev, D. K. and Sominskiĭ, I. S., *Problems in higher algebra.* 2nd ed., Moscow, 1949; 5th ed. Moscow: Gostekhizdat, 1954.

[15] Faddeeva, V. N., *Computational methods of linear algebra.* New York: Dover Publications, 1959. [Translated from the Russian.]

[16] Frazer, R. A., Duncan, W. J., and Collar, A., *Elementary Matrices and Some Applications to Dynamics and Differential Equations.* Cambridge: Cambridge University Press, 1938.

*[17] Gantmacher (Gantmakher), F. R. and Kreĭn, M. G., *Oscillation matrices and kernels and small vibrations of dynamical systems.* 2nd ed., Moscow: Gostekhizdat, 1950. [A German translation is in preparation.]

[18] Gröbner, W., *Matrizenrechnung.* Munich: Oldenburg, 1956.

[19] Hahn, W., *Theorie und Anwendung der direkten Methode von Lyapunov* (Ergebnisse der Mathematik, Neue Folge, Heft 22). Berlin: Springer, 1959. [Contains an extensive bibliography.]

[20] Ince, E. L., *Ordinary Differential Equations.* New York: Dover, 1948.

[21] Jung, H., *Matrizen und Determinanten. Eine Einführung.* Leipzig, 1953.

[22] Klein, F., *Vorlesungen über höhere Geometrie.* 3rd ed., New York: Chelsea, 1949.

[23] Kowalewski, G., *Einführung in die Determinantentheorie.* 3rd ed., New York: Chelsea, 1949.

*[24] Kreĭn, M. G., *Fundamental propositions in the theory of λ-zone stability of a canonical system of linear differential equations with periodic coefficients.* Moscow: Moscow Academy, 1955.

*[25] Kreĭn, M. G. and Naĭmark, M. A., *The method of symmetric and hermitian forms in the theory of separation of roots of algebraic equations.* Kharkov: GNTI, 1936.

*[26] Kreĭn, M. G. and Rutman, M. A., *Linear operators leaving a cone in a Banach space invariant.* Uspehi Mat. Nauk, vol. 3 no. 1, (1948).

*[27] Kudryavchev, L. D., *On some mathematical problems in the theory of electrical networks.* Uspehi Mat. Nauk, vol. 3 no. 4 (1948).

*[28] Lappo-Danilevskiĭ, I. A., *Theory of functions of matrices and systems of linear differential equations.* Moscow, 1934.

[29] ——— *Mémoires sur la théorie des systèmes des équations différentielles linéaires.* 3 vols., Trudy Mat. Inst. Steklov, vols. 6-8 (1934-1936). New York: Chelsea, 1953.

[30] Lefschetz, S., *Differential Equations: Geometric Theory.* New York: Interscience, 1957.

[31] Lichnerowicz, A., *Algèbre et analyse linéaires.* 2nd ed., Paris: Masson, 1956.

[32] Lyapunov (Liapounoff), A. M., *Le Problème général de la stabilité du mouvement* (Annals of Mathematics Studies, No. 17). Princeton: Princeton Univ. Press, 1949.

[33] MacDuffee, C. C., *The Theory of Matrices.* New York: Chelsea, 1946.

[34] ——— *Vectors and matrices.* La Salle: Open Court, 1943.

*[35] Malkin, I. G., *The method of Lyapunov and Poincaré in the theory of non-linear oscillations.* Moscow: Gostekhizdat, 1949.

[36] ——— *Theory of stability of motion.* Moscow: Gostekhizdat, 1952. [A German translation is in preparation.]

[37] Marden, M., *The geometry of the zeros of a polynomial in a complex variable* (Mathematical Surveys, No. 3). New York: Amer. Math. Society, 1949.

*[38] Markov, A. A., *Collected works.* Moscow, 1948.

*[39] Meĭman, N. N., *Some problems in the disposition of roots of polynomials.* Uspehi Mat. Nauk, vol. 4 (1949).

[40] Mirsky, L., *An Introduction to Linear Algebra.* Oxford: Oxford University Press, 1955.

*[41] Naĭmark, Y. I., *Stability of linearized systems.* Leningrad: Leningrad Aeronautical Engineering Academy, 1949.

[42] Parodi, M., *Sur quelques propriétés des valeurs caractéristiques des matrices carrées* (Mémorial des Sciences Mathématiques, vol. 118), Paris: Gauthiers-Villars, 1952.

[43] Perlis, S., *Theory of Matrices.* Cambridge. (Mass.): Addison-Wesley, 1952.

[44] Pickert, G., *Normalformen von Matrizen* (Enz. Math. Wiss., Band I, Teil B. Heft 3, Teil I). Leipzig: Teubner, 1953.

*[45] Potapov, V. P., *The multiplicative structure of J-inextensible matrix functions.* Trudy Moscow Mat. Soc., vol. 4 (1955).

*[46] Romanovskiĭ, V. I., *Discrete Markov chains.* Moscow: Gostekhizdat, 1948.

[47] Routh, E. J., *A treatise on the stability of a given state of motion.* London: Macmillan, 1877.

[48] ——— *The advanced part of a Treatise on the Dynamics of a Rigid Body.* 6th ed., London: Macmillan, 1905; repr., New York: Dover, 1959.

[49] Schlesinger, L., *Vorlesungen über lineare Differentialgleichungen.* Berlin, 1908.

[50] ——— *Einführung in die Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage.* Berlin, 1922.

[51] Schmeidler, W., *Vortrage über Determinanten und Matrizen mit Anwendungen in Physik und Technik.* Berlin: Akademie-Verlag, 1949.

[52] Schreier, O. and Sperner, E., *Vorlesungen über Matrizen.* Leipzig: Teubner, 1932. [A slightly revised version of this book appears as Chapter V of [53].]

[53] ——— *Introduction to Modern Algebra and Matrix Theory.* New York: Chelsea, 1958.

[54] Schwerdtfeger, H., *Introduction to Linear Algebra and the Theory of Matrices.* Groningen: Noordhoff, 1950.

[55] Shohat, J. A. and Tamarkin, J. D., *The problem of moments* (Mathematical Surveys, No. 1). New York: Amer. Math. Society, 1943.

[56] Smirnow, W. I. (Smirnov, V. I.), *Lehrgang der höheren Mathematik,* Vol. III. Berlin, 1956. [This is a translation of the 13th Russian edition.]

[57] Specht, W., *Algebraische Gleichungen mit reellen oder komplexen Koeffizienten* (Enz. Math. Wiss., Band I, Teil B, Heft 3, Teil II). Stuttgart: Teubner, 1958.

[58] Stieltjes, T. J., *Oeuvres Complètes.* 2 vols., Groningen: Noordhoff.

[59] Stoll, R. R., *Linear Algebra and Matrix Theory.* New York: McGraw-Hill, 1952.

[60] Thrall, R. M. and Tornheim, L., *Vector spaces and matrices.* New York: Wiley, 1957.

[61] Turnbull, H. W., *The Theory of Determinants, Matrices and Invariants.* London: Blackie, 1950.

[62] Turnbull, H. W. and Aitken, A. C., *An Introduction to the Theory of Canonical Matrices.* London: Blackie, 1932.

[63] Volterra, V. et Hostinsky, B., *Opérations infinitésimales linéaires.* Paris: Gauthiers-Villars, 1938.

[64] Wedderburn, J. H. M., *Lectures on matrices.* New York: Amer. Math. Society, 1934.

[65] Weyl, H., *Mathematische Analyse des Raumproblems.* Berlin, 1923. [A reprint is in preparation: Chelsea, 1960.]

[66] Wintner, A., *Spektraltheorie der unendlichen Matrizen.* Leipzig, 1929.

[67] Zurmühl, R., *Matrizen.* Berlin, 1950.

PART B. Papers

[101] Afriat, S., *Composite matrices,* Quart. J. Math. vol. 5, pp. 81-89 (1954).

*[102] Aizerman (Aisermann), M. A., *On the computation of non-linear functions of several variables in the investigation of the stability of an automatic regulating system,* Avtomat. i Telemeh. vol. 8 (1947).

[103] Aisermann, M. A. and F. R. Gantmacher, *Determination of stability by linear approximation of a periodic solution of a system of differential equations with discontinuous right-hand sides,* Quart. J. Mech. Appl. Math. vol. 11, pp. 385-98 (1958).

[104] Aitken, A. C., *Studies in practical mathematics. The evaluation, with applications, of a certain triple product matrix.* Proc. Roy. Soc. Edinburgh vol. 57, (1936-37).

[105] Amir Moéz Ali, R., *Extreme properties of eigenvalues of a hermitian transformation and singular values of the sum and product of linear transformations,* Duke Math. J. vol. 23, pp. 463-76 (1956).

*[106] Artashenkov, P. V., *Determination of the arbitrariness in the choice of a matrix reducing a system of linear differential equations to a system with constant coefficients.* Vestnik Leningrad. Univ., Ser. Mat., Phys. i Chim., vol. 2, pp. 17-29 (1953).

*[107] Arzhanych, I. S., *Extension of Krylov's method to polynomial matrices,* Dokl. Akad. Nauk SSSR, Vol. 81, pp. 749-52 (1951).

*[108] Azbelev, N. and R. Vinograd, *The process of successive approximations for the computation of eigenvalues and eigenvectors,* Dokl. Akad. Nauk., vol. 83, pp. 173-74 (1952).

[109] Baker, H. F., *On the integration of linear differential equations,* Proc. London Math. Soc., vol. 35, pp. 333-78 (1903).

[110] Barankin, E. W., *Bounds for characteristic roots of a matrix,* Bull. Amer. Math. Soc., vol. 51, pp. 767-70 (1945).

[111] Bartsch, H., *Abschätzungen für die Kleinste charakteristische Zahl einer positiv-definiten hermitschen Matrix,* Z. Angew. Math. Mech., vol. 34, pp. 72-74 (1954).

[112] Bellman, R., *Notes on matrix theory,* Amer. Math. Monthly, vol. 60, pp. 173-75, (1953); vol. 62, pp. 172-73, 571-72, 647-48 (1955); vol. 64, pp. 189-91 (1957).

[113] Bellman, R. and A. Hoffman, *On a theorem of Ostrowski,* Arch. Math., vol. 5, pp. 123-27 (1954).

[114] Bendat, J. and S. Silverman, *Monotone and convex operator functions,* Trans. Amer. Math. Soc., vol. 79, pp. 58-71 (1955).

[115] Berge, C., *Sur une propriété des matrices doublement stochastiques,* C. R. Acad. Sci. Paris, vol. 241, pp. 269-71 (1955).

[116] Birkhoff, G., *On product integration,* J. Math. Phys., vol. 16, pp. 104-32 (1937).

[117] Birkhoff, G. D., *Equivalent singular points of ordinary linear differential equations,* Math. Ann., vol. 74, pp. 134-39 (1913).

[118] Bott, R. and R. Duffin, *On the algebra of networks,* Trans. Amer. Math. Soc., vol. 74, pp. 99-109 (1953).

[119] Brauer, A., *Limits for the characteristic roots of a matrix,* Duke Math. J., vol. 13, pp. 387-95 (1946); vol. 14, pp. 21-26 (1947); vol. 15, pp. 871-77 (1948); vol. 19, pp. 73-91, 553-62 (1952); vol. 22, pp. 387-95 (1955).

[120] ——— *Über die Lage der charakteristischen Wurzeln einer Matrix,* J. Reine Angew. Math., vol. 192, pp. 113-16 (1953).

[121] ——— *Bounds for the ratios of the coordinates of the characteristic vectors of a matrix,* Proc. Nat. Acad. Sci. U.S.A., vol. 41, pp. 162-64 (1955).

[122] ——— *The theorems of Ledermann and Ostrowski on positive matrices,* Duke Math. J., vol. 24, pp. 265-74 (1957).

[123] Brenner, J., *Bounds for determinants,* Proc. Nat. Acad. Sci. U.S.A., vol. 40, pp. 452-54 (1954); Proc. Amer. Math. Soc., vol. 5, pp. 631-34 (1954); vol. 8, pp. 532-34 (1957); C. R. Acad. Sci. Paris, vol. 238, pp. 555-56 (1954).

[124] Bruijn, N., *Inequalities concerning minors and eigenvalues,* Nieuw Arch. Wisk., vol. 4, pp. 18-35 (1956).

[125] Bruijn, N. and G. Szekeres, *On some exponential and polar representatives of matrices,* Nieuw Arch. Wisk., vol. 3, pp. 20-32 (1955).

*[126] Bulgakov, B. V., *The splitting of rectangular matrices,* Dokl. Akad. Nauk SSSR, vol. 85, pp. 21-24 (1952).

[127] Cayley, A., *A memoir on the theory of matrices,* Phil. Trans. London, vol. 148, pp. 17-37 (1857); Coll. Works, vol. 2, pp. 475-96.

[128] Collatz, L., *Einschliessungssatz für die charakteristischen Zahlen von Matrizen,* Math. Z., vol. 48, pp. 221-26 (1942).

[129] ——— *Über monotone systeme linearen Ungleichungen,* J. Reine Angew. Math., vol. 194, pp. 193-94 (1955).

[130] Cremer, L., *Die Verringerung der Zahl der Stabilitätskriterien bei Voraussetzung positiven koeffizienten der charakteristischen Gleichung,* Z. Angew. Math. Mech., vol. 33, pp. 222-27 (1953).

*[131] Danilevskiĭ, A. M., *On the numerical solution of the secular equation,* Mat. Sb., vol. 2, pp. 169-72 (1937).

[132] Diliberto, S., *On systems of ordinary differential operations.* In: *Contributions to the Theory of Non-linear Oscillations,* vol. I, edited by S. Lefschetz (Annals of Mathematics Studies, No. 20). Princeton: Princeton Univ. Press (1950), pp. 1-38.

*[133] Dmitriev, N. A. and E. B. Dynkin, *On the characteristic roots of stochastic matrices,* Dokl. Akad. Nauk SSSR, vol. 49, pp. 159-62 (1945).

*[133a] ——— *Characteristic roots of Stochastic Matrices,* Izv. Akad. Nauk, Ser. Fiz-Mat., vol. 10, pp. 167-94 (1946).

[134] Dobsch, O., *Matrixfunktionen beschränkter Schwankung,* Math. Z., vol. 43, pp. 353-88 (1937).

*[135] Donskaya, L. I., *Construction of the solution of a linear system in the neighborhood of a regular singularity in special cases,* Vestnik Leningrad. Univ., vol. 6 (1952).

*[136] ——— *On the structure of the solution of a system of linear differential equations in the neighbourhood of a regular singularity,* Vestnik Leningrad. Univ., vol. 8, pp. 55-64 (1954).

*[137] Dubnov, Y. S., *On simultaneous invariants of a system of affinors,* Trans. Math. Congress in Moscow 1927, pp. 236-37.

*[138] ——— *On doubly symmetric orthogonal matrices,* Bull. Ass. Inst. Univ. Moscow, pp. 33-35 (1927).

*[139] ——— *On Dirac's matrices,* Uč. zap. Univ. Moscow, vol. 2, pp. 2, 43-48 (1934).

*[140] Dubnov, Y. S. and V. K. Ivanov, On the reduction of the degree of affinor *polynomials,* Dokl. Akad. Nauk SSSR, vol. 41, pp. 99-102 (1943).

[141] Duncan, W., *Reciprocation of triply-partitioned matrices,* J. Roy. Aero. Soc., vol. 60, pp. 131-32 (1956).

[142] Egerváry, E., *On a lemma of Stieltjes on matrices,* Acta. Sci. Math., vol. 15, pp. 99-103 (1954).

[143] ——— *On hypermatrices whose blocks are commutable in pairs and their application in lattice-dynamics,* Acta Sci. Math., vol. 15, pp. 211-22 (1954).

[144] Epstein, M. and H. Flanders, *On the reduction of a matrix to diagonal form,* Amer. Math. Monthly, vol. 62, pp. 168-71 (1955).

*[145] Ershov, A. P., *On a method of inverting matrices,* Dokl. Akad. Nauk SSSR, vol. 100, pp. 209-11 (1955).

[146] Erugin, N. P., *Sur la substitution exposante pour quelques systèmes irreguliers,* Mat. Sb., vol. 42, pp. 745-53 (1935).

*[147] ——— *Exponential substitutions of an irregular system of linear differential equations,* Dokl. Akad. Nauk SSSR, vol. 17, pp. 235-36 (1935).

*[148] ——— *On Riemann's problem for a Gaussian system*, Uč. Zap. Ped. Inst., vol. 28, pp. 293-304 (1939).

*[149] Faddeev, D. K., *On the transformation of the secular equation of a matrix*, Trans. Inst. Eng. Constr., vol. 4, pp. 78-86 (1937).

[150] Faedo, S., *Un nuove problema di stabilità per le equationi algebriche a coefficienti reali*, Ann. Scuola Norm. Sup. Pisa, vol. 7, pp. 53-63 (1953).

*[151] Fage, M. K., *Generalization of Hadamard's determinant inequality*, Dokl. Akad. Nauk SSSR, vol. 54, pp. 765-68 (1946).

*[152] ——— *On symmetrizable matrices*, Uspehi Mat. Nauk, vol. 6, no. 3, pp. 153-56 (1951).

[153] Fan, K., *On a theorem of Weyl concerning eigenvalues of linear transformations*, Proc. Nat. Acad. Sci. U.S.A., vol. 35, pp. 652-55 (1949); vol. 36, pp. 31-35 (1950).

[154] ——— *Maximum properties and inequalities for the eigenvalues of completely continuous operators*, Proc. Nat. Acad. Sci. U.S.A., vol. 37, pp. 760-66 (1951).

[155] ——— *A comparison theorem for eigenvalues of normal matrices*, Pacific J. Math., vol. 5, pp. 911-13 (1955).

[156] ——— *Some inequalities concerning positive-definite Hermitian matrices*, Proc. Cambridge Philos. Soc., vol. 51, pp. 414-21 (1955).

[157] ——— *Topological proofs for certain theorems on matrices with non-negative elements*, Monatsh. Math., vol. 62, pp. 219-37 (1958).

[158] Fan, K. and A. Hoffman, *Some metric inequalities in the space of matrices*, Proc. Amer. Math. Soc., vol. 6, pp. 111-16 (1958).

[159] Fan, K. and G. Pall, *Imbedding conditions for Hermitian and normal matrices*, Canad. J. Math., vol. 9, pp. 298-304 (1957).

[160] Fan, K. and J. Todd, *A determinantal inequality*, J. London Math. Soc., vol. 30, pp. 58-64 (1955).

[161] Frobenius, G., *Über lineare substitutionen und bilineare Formen*, J. Reine Angew. Math., vol. 84, pp. 1-63 (1877).

[162] ——— *Über das Trägheitsgesetz der quadratischen Formen*, S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl., 1894, pp. 241-56, 407-31.

[163] ——— *Über die cogredienten transformationen der bilinearer Formen*, S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl., 1896, pp. 7-16.

[164] ——— *Über die vertauschbaren Matrizen*, S.-B. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl., 1896, pp. 601-614.

[165] ——— *Über Matrizen aus positiven Elementen*, S.-B. Deutsch. Akad. Wiss. Berlin. Math-Nat. Kl. 1908, pp. 471-76; 1909, pp. 514-18.

[166] ——— *Über Matrizen aus nicht negativen Elementen*, S.-B. Deutsch. Akad. Wiss. Berlin Math.-Nat. Kl., 1912, pp. 456-77.

*[167] Gantmacher, F. R., *Geometric theory of elementary divisors after Krull*, Trudy Odessa Gos. Univ. Mat., vol. 1, pp. 89-108 (1935).

*[168] ——— *On the algebraic analysis of Krylov's method of transforming the secular equation*, Trans. Second Math. Congress, vol. II, pp. 45-48 (1934).

[169] ——— *On the classification of real simple Lie groups*, Mat. Sb., vol. 5, pp. 217-50 (1939).

*[170] Gantmacher, F. R. and M. G. Kreĭn, *On the structure of an orthogonal matrix*, Trans. Ukrain, Acad. Sci. Phys.-Mat. Kiev (Trudy fiz.-mat. otdela VUAN, Kiev), 1929, pp. 1-8.

*[171] ——— *Normal operators in a hermitian space*, Bull. Phys-Mat. Soc. Univ. Kasan (Izvestiya fiz.-mat. ob-va pri Kazanskom universitete), IV, vol. 1, ser. 3, pp. 71-84 (1929-30).

*[172] ——— *On a special class of determinants connected with Kellogg's integral kernels*, Mat. Sb., vol. 42, pp. 501-8 (1935).

[173] ——— *Sur les matrices oscillatoires et completement non-négatives*, Compositio Math., vol. 4, pp. 445-76 (1937).

[174] Gantschi, W., *Bounds of matrices with regard to an hermitian metric*, Compositio Math., vol. 12, pp. 1-16 (1954).

*[175] Gelfand, I. M. and V. B. Lidskiĭ, *On the structure of the domains of stability of linear canonical systems of differential equations with periodic coefficients*, Uspehi. Mat. Nauk, vol. 10, no. 1, pp. 3-40 (1955).

[176] Gershgorin, S. A., *Über die Abgrenzung der Eigenwerte einer Matrix*, Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 6, pp. 749-54 (1931).

[177] Goddard, L., *An extension of a matrix theorem of A. Brauer*, Proc. Int. Cong. Math. Amsterdam, 1954, vol. 2, pp. 22-23.

[178] Goheen, H. E., *On a lemma of Stieltjes on matrices*, Amer. Math. Monthly, vol. 56, pp. 328-29 (1949).

*[179] Golubchikov, A. F., *On the structure of the automorphisms of the complex simple Lie groups*, Dokl. Akad. Nauk SSSR, vol. 27, pp. 7-9 (1951).

*[180] Grave, D. A., *Small oscillations and some propositions in algebra*, Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 2, pp. 563-70 (1929).

*[181] Grossman, D. P., *On the problem of a numerical solution of systems of simultaneous linear algebraic equations*, Uspehi Mat. Nauk, vol. 5, no. 3, pp. 87-103 (1950).

[182] Hahn, W., *Eine Bemerkung zur zweiten Methode von Lyapunov*, Math. Nachr., vol. 14, pp. 349-54 (1956).

[183] ——— *Über die Anwendung der Methode von Lyapunov auf Differenzengleichungen*, Math. Ann., vol. 136, pp. 430-41 (1958).

[184] Haynsworth, E., *Bounds for determinants with dominant main diagonal*, Duke Math. J., vol. 20, pp. 199-209 (1953).

[185] ——— *Note on bounds for certain determinants*, Duke Math. J., vol. 24, pp. 313-19 (1957).

[186] Hellmann, O., *Die Anwendung der Matrizanten bei Eigenwertaufgaben*, Z. Angew. Math. Mech., vol. 35, pp. 300-15 (1955).

[187] Hermite, C., *Sur le nombre des racines d'une équation algébrique comprise entre des limites données*, J. Reine Angew. Math., vol. 52, pp. 39-51 (1856).

[188] Hjelmsler, J., *Introduction à la théorie des suites monotones*, Kgl. Danske Vid. Selbsk. Forh. 1914, pp. 1-74.

[189] Hoffman, A. and O. Taussky, *A characterization of normal matrices*, J. Res. Nat. Bur. Standards, vol. 52, pp. 17-19 (1954).

[190] Hoffman, A. and H. Wielandt, *The variation of the spectrum of a normal matrix*, Duke Math. J., vol. 20, pp. 37-39 (1953).

[191] Horn, A., *On the eigenvalues of a matrix with prescribed singular values*, Proc. Amer. Math. Soc., vol. 5, pp. 4-7 (1954).

[192] Hotelling, H., *Some new methods in matrix calculation*, Ann. Math. Statist., vol. 14, pp. 1-34 (1943).

[193] Householder, A. S., *On matrices with non-negative elements*, Monatsh. Math., vol. 62, pp. 238-49 (1958).

[194] Householder, A. S. and F. L. Bauer, *On certain methods for expanding the characteristic polynomial*, Numer. Math., vol. 1, pp. 29-35 (1959).

[195] Hsu, P. L., *On symmetric, orthogonal, and skew-symmetric matrices*, Proc. Edinburgh Math. Soc., vol. 10, pp. 37-44 (1953).

[196] ——— *On a kind of transformation of matrices*, Acta Math. Sinica, vol. 5, pp. 333-47 (1955).

[197] Hua, L.-K., *On the theory of automorphic functions of a matrix variable*, Amer. J. Math., vol. 66, pp. 470-88; 531-63 (1944).

[198] ——— *Geometries of matrices*, Trans. Amer. Math. Soc., vol. 57, pp. 441-90 (1945).

[199] ——— *Orthogonal classification of Hermitian matrices*, Trans. Amer. Math. Soc., vol. 59, pp. 508-23 (1946).

*[200] ——— *Geometries of symmetric matrices over the real field*, Dokl. Akad. Nauk SSSR, vol. 53, pp. 95-98; 195-96 (1946).

*[201] ——— *Automorphisms of the real symplectic group*, Dokl. Akad. Nauk SSSR, vol. 53, pp. 303-306 (1946).

[202] ——— *Inequalities involving determinants*, Acta Math. Sinica, vol. 5, pp. 463-70 (1955).

*[203] Hua, L.-K. and B. A. Rosenfeld, *The geometry of rectangular matrices and their application to the real projective and non-euclidean geometries*, Izv. Higher Ed. SSSR, Matematika, vol. 1, pp. 233-46 (1957).

[204] Hurwitz, A., *Über die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt*, Math. Ann., vol. 46, pp. 273-84 (1895).

[205] Ingraham, M. H., *On the reduction of a matrix to its rational canonical form*, Bull. Amer. Math. Soc., vol. 39, pp. 379-82 (1933).

[206] Ionescu, D., *O identitate importantá si descompunere a unei forme bilineare into sumá de produse*, Gaz. Mat. Ser. Fiz. A. 7, vol. 7, pp. 303-312 (1955).

[207] Ishak, M., *Sur les spectres des matrices*, Sém. P. Dubreil et Ch. Pisot, Fac. Sci. Paris, vol. 9, pp. 1-14 (1955/56).

*[208] Kagan, V. F., *On some number systems arising from Lorentz transformations*, Izv. Ass. Inst. Moscow Univ. 1927, pp. 3-31.

*[209] Karpelevich, F. I., *On the eigenvalues of a matrix with non-negative elements*, Izv. Akad. Nauk SSSR Ser. Mat., vol. 15, pp. 361-83 (1951).

[210] Khan, N. A., *The characteristic roots of a product of matrices*, Quart. J. Math., vol. 7, pp. 138-43 (1956).

*[211] Khlodovskiĭ, I. N., *On the theory of the general case of Krylov's transformation of the secular equation*, Izv. Akad. Nauk, Ser. Fiz.-Mat., vol. 7, pp. 1076-1102 (1933).

*[212] Kolmogorov, A. N., *Markov chains with countably many possible states*, Bull. Univ. Moscow (A), vol. 1:3 (1937).

*[213] Kotelyanskiĭ, D. M., *On monotonic matrix functions of order n*, Trans. Univ. Odessa, vol. 3, pp. 103-114 (1941).

*[214] ——— *On the theory of non-negative and oscillatory matrices*, Ukrain. Mat. Z., vol. 2, pp. 94-101 (1950).

*[215] ——— *On some properties of matrices with positive elements*, Mat. Sb., vol. 31, pp. 497-506 (1952).

*[216] ——— *On a property of matrices of symmetric signs*, Uspehi Mat. Nauk, vol. 8, no. 4, pp. 163-67 (1953).

*[217] ——— *On some sufficient conditions for the spectrum of a matrix to be real and simple*, Mat. Sb., vol. 36, pp. 163-68 (1955).

*[218] ——— *On the influence of Gauss' transformation on the spectra of matrices*, Uspehi Mat. Nauk, vol. 9, no. 3, pp. 117-21 (1954).

*[219] ——— *On the distribution of points on a matrix spectrum*, Ukrain. Mat. Z., vol. 7, pp. 131-33 (1955).

*[220] ——— *Estimates for determinants of matrices with dominant main diagonal*, Izv. Akad. Nauk SSSR, Ser. Mat., vol. 20, pp. 137-44 (1956).

*[221] Kovalenko, K. R. and M. G. Kreĭn, *On some investigations of Lyapunov concerning differential equations with periodic coefficients*, Dokl. Akad. Nauk SSSR, vol. 75, pp. 495-99 (1950).

[222] Kowalewski, G., *Natürliche Normalformen linearer Transformationen*, Leipz. Ber., vol. 69, pp. 325-35 (1917).

*[223] Krasovskiĭ, N. N., *On the stability after the first approximation*, Prikl. Mat. Meh., vol. 19, pp. 516-30 (1955).

*[224] Krasnosel'skiĭ, M. A. and M. G. Kreĭn, *An iteration process with minimal deviations*, Mat. Sb., vol. 31, pp. 315-34 (1952).

[225] Kraus, F., *Über konvexe Matrixfunktionen*, Math. Z., vol. 41, pp. 18-42 (1936).

*[226] Kravchuk, M. F., *On the general theory of bilinear forms*, Izv. Polyt. Inst. Kiev, vol. 19, pp. 17-18 (1924).

*[227] ——— *On the theory of permutable matrices*, Zap. Akad. Nauk Kiev, Ser. Fiz.-Mat., vol. 1:2, pp. 28-33 (1924).

*[228] ——— *On a transformation of quadratic forms*, Zap. Akad. Nauk Kiev, Ser. Fiz.-Mat., vol. 1:2, pp. 87-90 (1924).

*[229] ——— *On quadratic forms and linear transformations*, Zap. Akad. Nauk Kiev, Ser. Fiz.-Mat., vol. 1:3, pp. 1-89 (1924).

*[230] ——— *Permutable sets of linear transformations*, Zap. Agr. Inst. Kiev, vol. 1, pp. 25-58 (1926).

[231] ——— *Über vertauschbare Matrizen*, Rend. Circ. Mat. Palermo, vol. 51, pp. 126-30 (1927).

*[232] ——— *On the structure of permutable groups of matrices*, Trans. Second. Mat. Congress 1934, vol. 2, pp. 11-12.

*[233] Kravchuk, M. F. and Y. S. Gol'dbaum, *On groups of commuting matrices*, Trans. Av. Inst. Kiev, 1929, pp. 73-98; 1936, pp. 12-23.

*[234] ——— *On the equivalence of singular pencils of matrices*, Trans. Av. Inst. Kiev, 1936, pp. 5-27.

*[235] Kreĭn, M. G., *Addendum to the paper 'On the structure of an orthogonal matrix,'* Trans. Fiz.-Mat. Class. Akad. Nauk Kiev, 1931, pp. 103-7.

*[236] ——— *On the spectrum of a Jacobian form in connection with the theory of torsion oscillations of drums*, Mat. Sb., vol. 40, pp. 455-66 (1933).

*[237] ——— *On a new class of hermitian forms*, Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 9, pp. 1259-75 (1933).

*[238] ——— *On the nodes of harmonic oscillations of mechanical systems of a special type*, Mat. Sb., vol. 41, pp. 339-48 (1934).

[239] ——— *Sur quelques applications des noyaux de Kellog aux problèmes d'oscillations*, Proc. Charkov Mat. Soc. (4), vol. 11, pp. 3-19 (1935).

[240] ——— *Sur les vibrations propres des tiges dont l'une des extrémités est encastrée et l'autre libre*, Proc. Charkov. Mat. Soc. (4), vol. 12, pp. 3-11 (1935).

*[241] ——— *Generalization of some results of Lyapunov on linear differential equations with periodic coefficients*, Dokl. Akad. Nauk SSSR, vol. 73, pp. 445-48 (1950).

*[242] ——— *On an application of the fixed-point principle in the theory of linear transformations of spaces with indefinite metric*, Uspehi Mat. Nauk, vol. 5, no. 2, pp. 180-90 (1950).

*[243] ——— *On an application of an algebraic proposition in the theory of monodromy matrices*, Uspehi Mat. Nauk, vol. 6, no. 1, pp. 171-77 (1951).

*[244] ——— *On some problems concerning Lyapunov's ideas in the theory of stability*, Uspehi Mat. Nauk, vol. 3, no. 3, pp. 166-69 (1948).

*[245] ——— *On the theory of integral matrix functions of exponential type*, Ukrain. Mat. Z., vol. 3, pp. 164-73 (1951).

*[246] ——— *On some problems in the theory of oscillations of Sturm systems*, Prikl. Mat. Meh., vol. 16, pp. 555-68 (1952).

*[247] Kreĭn, M. G. and M. A. Naĭmark (Neumark), *On a transformation of the Bézoutian leading to Sturm's theorem*, Proc. Charkov Mat. Soc., (4), vol. 10, pp. 33-40 (1933).

*[248] ——— *On the application of the Bézoutian to problems of the separation of roots of algebraic equations*, Trudy Odessa Gos. Univ. Mat., vol. 1, pp. 51-69 (1935).

[249] Kronecker, L., *Algebraische Reduction der Schaaren bilinearer Formen*, S.-B. Akad. Berlin 1890, pp. 763-76.

[250] Krull, W., *Theorie und Anwendung der verallgemeinerten Abelschen Gruppen*, S.-B. Akad. Heidelberg 1926, p. 1.

*[251] Krylov, A. N., *On the numerical solution of the equation by which the frequency of small oscillations is determined in technical problems*, Izv. Akad. Nauk SSSR Ser. Fiz.-Mat., vol. 4, pp. 491-539 (1931).

[252] Lappo-Danilevskiĭ, I. A., *Résolution algorithmique des problèmes réguliers de Poincaré et de Riemann*, J. Phys. Mat. Soc. Leningrad, vols. 2:1, pp. 94-120; 121-54 (1928).

[253] ——— *Théorie des matrices satisfaisantes à des systèmes des équations differentielles linéaires a coefficients rationnels arbitraires*, J. Phys. Mat. Soc. Leningrad, vols. 2:2, pp. 41-80 (1928).

*[254] ——— *Fundamental problems in the theory of systems of linear differential equations with arbitrary rational coefficients*, Trans. First Math. Congr., ONTI, 1936, pp. 254-62.

[255] Ledermann, W., *Reduction of singular pencils of matrices*, Proc. Edinburgh Math. Soc., vol. 4, pp. 92-105 (1935).

[256] ——— *Bounds for the greatest latent root of a positive matrix*, J. London Math. Soc., vol. 25, pp. 265-68 (1950).

*[257] Lidskiĭ, V. B., *On the characteristic roots of a sum and a product of symmetric matrices*, Dokl. Akad. Nauk SSSR, vol. 75, pp. 769-72 (1950).

*[258] ——— *Oscillation theorems for canonical systems of differential equations*, Dokl. Akad. Nauk SSSR, vol. 102, pp. 111-17 (1955).

[259] Liénard, and Chipart, *Sur la signe de la partie réelle des racines d'une équation algébrique*, J. Math. Pures Appl. (6), vol. 10, pp. 291-346 (1914).

*[260] Lipin, N. V., *On regular matrices*, Trans. Inst. Eng 8. Transport, vol. 9, p. 105 (1934).

*[261] Livshitz, M. S. and V. P. Potapov, *The multiplication theorem for characteristic matrix functions*, Dokl. Akad. Nauk SSSR, vol. 72, pp. 164-73 (1950).

*[262] Lopshitz, A. M., *Vector solution of a problem on doubly symmetric matrices*, Trans. Math. Congress Moscow, 1927, pp. 186-87.

*[263] ——— *The characteristic equation of lowest degree for affinors and its application to the integration of differential equations*, Trans. Sem. Vectors and Tensors, vols. 2/3 (1935).

*[264] ——— *A numerical method of determining the characteristic roots and characteristic planes of a linear operator*, Trans. Sem. Vectors and Tensors, vol. 7, pp. 233-59 (1947).

*[265] ——— *An extremal theorem for a hyper-ellipsoid and its application to the solution of a system of linear algebraic equations*, Trans. Sem. Vectors and Tensors, vol. 9, pp. 183-97 (1952).

[266] Löwner, K., *Über monotone Matrixfunktionen*, Math. Z., vol. 38, pp. 177-216 (1933); vol. 41, pp. 18-42 (1936).

[267] ——— *Some classes of functions defined by difference on differential inequalities*, Bull. Amer. Math. Soc., vol. 56, pp. 308-19 (1950).

*[268] Lusin, N. N., *On Krylov's method of forming the secular equation*, Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 7, pp. 903-958 (1931).

*[269] ——— *On some properties of the displacement factor in Krylov's method*, Izv. Akad. Nauk SSSR, Ser. Fiz.-Mat., vol. 8, pp. 596-638; 735-62; 1065-1102 (1932).

*[270] ——— *On the matrix theory of differential equations*, Avtomat. i Telemeh, vol. 5, pp. 3-66 (1940).

*[271] Lyusternik, L. A., *The determination of eigenvalues of functions by an electric scheme*, Electričestvo, vol. 11, pp. 67-8 (1946).

*[272] ——— *On electric models of symmetric matrices*, Uspehi Mat. Nauk, vol. 4, no. 2, pp. 198-200 (1949).

*[273] Lyusternik, L. A. and A. M. Prokhorov, *Determination of eigenvalues and functions of certain operators by means of an electrical network*, Dokl. Akad Nauk SSSR, vol. 55, pp. 579-82; Izv. Akad. Nauk SSSR, Ser. Mat., vol. 11, pp. 141-45 (1947).

[274] Marcus, M., *A remark on a norm inequality for square matrices*, Proc. Amer. Math. Soc., vol. 6, pp. 117-19 (1955).

[275] ——— *An eigenvalue inequality for the product of normal matrices*, Amer. Math. Monthly, vol. 63, pp. 173-74 (1956).

[276] ——— *A determinantal inequality of H. P. Robertson*, II, J. Washington Acad. Sci., vol. 47, pp. 264-66 (1957).

[277] ——— *Convex functions of quadratic forms*, Duke Math. J., vol. 24, pp. 321-26 (1957).

[278] Marcus, M. and J. L. McGregor, *Extremal properties of Hermitian matrices*, Canad. J. Math., vol. 8, pp. 524-31 (1956).

[279] Marcus, M. and B. N. Moyls, *On the maximum principle of Ky Fan*, Canad. J. Math., vol. 9, pp. 313-20 (1957).

[280] ——— *Maximum and minimum values for the elementary symmetric functions of Hermitian forms*, J. London Math. Soc., vol. 32, pp. 374-77 (1957).

*[281] Mayants, L. S., *A method for the exact determination of the roots of secular equations of high degree and a numerical analysis of their dependence on the parameters of the corresponding matrices*, Dokl. Akad. Nauk SSSR, vol. 50, pp. 121-24 (1945).

[282] Mirsky, L., *An inequality for positive-definite matrices*, Amer. Math. Monthly, vol. 62, pp. 428-30 (1955).

[283] ——— *The norm of adjugate and inverse matrices*, Arch. Math., vol. 7, pp. 276-77 (1956).

[284] ——— *The spread of a matrix*, Mathematika, vol. 3, pp. 127-30 (1956).

[285] ——— *Inequalities for normal and Hermitian matrices*, Duke Math. J., vol. 24, pp. 591-99 (1957).

[286] Mitrović, D., *Conditions graphiques pour que toutes les racines d'une équation algébrique soient à parties réelles négatives*, C. R. Acad. Sci. Paris, vol. 240, pp. 1177-79 (1955).

[287] Morgenstern, D., *Eine Verschärfung der Ostrowskischen Determinantenabschätzung*, Math. Z., vol. 66, pp. 143-46 (1956).

[288] Motzkin, T. and O. Taussky, *Pairs of matrices with property L.*, Trans. Amer. Math. Soc., vol. 73, pp. 108-14 (1952); vol. 80, pp. 387-401 (1954).

*[289] Neĭgaus (Neuhaus), M. G. and V. B. Lidskiĭ, *On the boundedness of the solutions of linear systems of differential equations with periodic coefficients*, Dokl. Akad. Nauk SSSR, vol. 77, pp. 183-93 (1951).

[290] Neumann, J., *Approximative of matrices of high order*, Portugal. Math., vol. 3, pp. 1-62 (1942).

*[291] Nudel'man, A. A. and P. A. Shvartsman, *On the spectrum of the product of unitary matrices*, Uspehi Mat. Nauk, vol. 13, no. 6, pp. 111-17 (1958).

[292] Okamoto, M., *On a certain type of matrices with an application to experimental design*, Osaka Math. J., vol. 6, pp. 73-82 (1954).

[293] Oppenheim, A., *Inequalities connected with definite Hermitian forms*, Amer. Math. Monthly, vol. 61, pp. 463-66 (1954).

[294] Orlando, L., *Sul problema di Hurwitz relàtivo alle parti reali delle radici di un' equatione algebrica*, Math. Ann., vol. 71, pp. 233-45 (1911).

[295] Ostrowski, A., *Bounds for the greatest latent root of a positive matrix*, J. London Math. Soc., vol. 27, pp. 253-56 (1952).

[296] ——— *Sur quelques applications des fonctions convexes et concaves au sens de I. Schur*, J. Math. Pures Appl., vol. 31, pp. 253-92 (1952).

[297] ——— *On nearly triangular matrices*, J. Res. Nat. Bur. Standards, vol. 52, pp. 344-45 (1954).

[298] ——— *On the spectrum of a one-parametric family of matrices*, J. Reine Angew. Math., vol. 193, pp. 143-60 (1954).

[299] ——— *Sur les déterminants à diagonale dominante*, Bul. Soc. Math. Belg., vol. 7, pp. 46-51 (1955).

[300] ——— *Note on bounds for some determinants*, Duke Math. J., vol. 22, pp. 95-102 (1955).

[301] ——— *Über Normen von Matrizen*, Math. Z., vol. 63, pp. 2-18 (1955).

[302] ——— *Über die Stetigkeit von charakteristischen Wurzeln in Abhängigkeit von den Matrizenelementen*, Jber. Deutsch. Math. Verein., vol. 60, pp. 40-42 (1957).

*[303] Papkovich, P. F., *On a method of computing the roots of a characteristic determinant*, Prikl. Mat. Meh., vol. 1, pp. 314-18 (1933).

[304] Papulis, A., *Limits on the zeros of a network determinant*, Quart. Appl. Math., vol. 15, pp. 193-94 (1957).

[305] Parodi, M., *Remarques sur la stabilité*, C. R. Acad. Sci. Paris, vol. 228, pp. 51-2; 807-8; 1198-1200 (1949).

[306] ——— *Sur une propriété des racines d'une équation qui intervient en mécanique*, C. R. Acad. Sci. Paris, vol. 241, pp. 1019-21 (1955).

[307] ——— *Sur la localisation des valeurs caractéristiques des matrices dans le plan complexe*, C. R. Acad. Sci. Paris, vol. 242, pp. 2617-18 (1956).

[308] Peano, G., *Intégration par series des équations différentielles linéaires*, Math. Ann., vol. 32, pp. 450-56 (1888).

[309] Penrose, R., *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc., vol. 51, pp. 406-13 (1955).

[310] ——— *On best approximate solutions of linear matrix equations*, Proc. Cambridge Philos. Soc., vol. 52, pp. 17-19 (1956).

[311] Perfect, H., *On matrices with positive elements*, Quart. J. Math., vol. 2, pp. 286-90 (1951).

[312] ——— *On positive stochastic matrices with real characteristic roots*, Proc. Cambridge Philos. Soc., vol. 48, pp. 271-76 (1952).

[313] ——— *Methods of constructing certain stochastic matrices*, Duke Math. J., vol. 20, pp. 395-404 (1953); vol. 22, pp. 305-11 (1955).

[314] ——— *A lower bound for the diagonal elements of a non-negative matrix*, J. London Math. Soc., vol. 31, pp. 491-93 (1956).

[315] Perron, O., *Jacobischer Kettenbruchalgorithmus*, Math. Ann., vol. 64, pp. 1-76 (1907).

[316] ——— *Über Matrizen*, Math. Ann., vol. 64, pp. 248-63 (1907).

[317] ——— *Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzengleichungen*, J. Reine Angew. Math., vol. 161, pp. 41-64 (1929).

[318] Phillips, H. B., *Functions of matrices*, Amer. J. Math., vol. 41, pp. 266-78 (1919).

*[319] Pontryagin, L. S., *Hermitian operators in a space with indefinite metric*, Izv. Akad. Nauk SSSR, Ser. Mat., vol. 8, pp. 243-80 (1944).

*[320] Potapov, V. P., *On holomorphic matrix functions bounded in the unit circle*, Dokl. Akad. Nauk SSSR, vol. 72, pp. 849-53 (1950).

[321] Rasch, G., *Zur Theorie und Anwendung der Produktintegrals*, J. Reine Angew. Math., vol. 171, pp. 65-119 (19534).

[322] Reichardt, H., *Einfache Herleitung der Jordanschen Normalform*, Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe, vol. 6, pp. 445-47 (1953/54).

*[323] Rechtman-Ol'shanskaya, P. G., *On a theorem of Markov*, Uspehi Mat. Nauk, vol. 12, no. 3, pp. 181-87 (1957).

[324] Rham, G. de, *Sur un théorème de Stieltjes relatif à certains matrices*, Acad. Serbe Sci. Publ. Inst. Math., vol. 4, pp. 133-54 (1952).

[325] Richter, H., *Über Matrixfunktionen*, Math. Ann., vol. 122, pp. 16-34 (1950).

[326] ——— *Bemerkung zur Norm der Inversen einer Matrix*, Arch. Math., vol. 5, pp. 447-48 (1954).

[327] ——— *Zur Abschätzung von Matrizennormen*, Math. Nachr., vol. 18, pp. 178-87 (1958).

[328] Romanovskiĭ, V. I., *Un théorème sur les zéros des matrices non-négatives*, Bull. Soc. Math. France, vol. 61, pp. 213-19 (1933).

[329] ——— *Recherches sur les chaines de Markoff*, Acta Math., vol. 66, pp. 147-251 (1935).

[330] Roth, W., *On the characteristic polynomial of the product of two matrices*, Proc. Amer. Math. Soc., vol. 5, pp. 1-3 (1954).

[331] ——— *On the characteristic polynomial of the product of several matrices*, Proc. Amer. Math. Soc., vol. 7, pp. 578-82 (1956).

[332] Roy, S., *A useful theorem in matrix theory*, Proc. Amer. Math. Soc., vol. 5, pp. 635-38 (1954).

*[333] Sakhnovich, L. A., *On the limits of multiplicative integrals*, Uspehi Mat. Nauk, vol. 12 no. 3, pp. 205-11 (1957).

*[334] Sarymsakov, T. A., *On sequences of stochastic matrices*, Dokl. Akad. Nauk, vol. 47, pp. 331-33 (1945).

[335] Schneider, H., *An inequality for latent roots applied to determinants with dominant principal diagonal*, J. London Math. Soc., vol. 28, pp. 8-20 (1953).

[336] ——— *A pair of matrices with property P*, J. Amer. Math. Monthly, vol. 62, pp. 247-49 (1955).

[337] ——— *A matrix problem concerning projections*, Proc. Edinburgh Math. Soc., vol. 10, pp. 129-30 (1956).

[338] ——— *The elementary divisors, associated with 0, of a singular M-matrix*, Proc. Edinburgh Math. Soc., vol. 10, pp. 108-22 (1956).

[339] Schoenberg, J., *Über variationsvermindernde lineare Transformationen*, Math. Z., vol. 32, pp. 321-28 (1930).

[340] ——— *Zur Abzählung der reellen Wurzeln algebraischer Gleichungen*, Math. Z., vol. 38, p. 546 (1933).

[341] Schoenberg, I. J., and A. Whitney, *A theorem on polygons in n dimensions with application to variation diminishing linear transformations*, Compositio Math., vol. 9, pp. 141-60 (1951).

[342] Schur,.I., *Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die theorie der Integralgleichungen*, Math. Ann., vol. 66, pp. 488-510 (1909).

*[343] Semendyaev, K. A., *On the determination of the eigenvalues and invariant manifolds of matrices by means of iteration*, Prikl. Matem. Meh., vol. 3, pp. 193-221 (1943).

*[344] Sevast'yanov, B. A., *The theory of branching random processes*, Uspehi Mat. Nauk, vol. 6, no. 6, pp. 46-99 (1951).

*[345] Shiffner, L. M., *The development of the integral of a system of differential equations with regular singularities in series of powers of the elements of the differential substitution*, Trudy Mat. Inst. Steklov. vol. 9, pp. 235-66 (1935).

*[346] ——— *On the powers of matrices*, Mat. Sb., vol. 42, pp. 385-94 (1935).

[347] Shoda, K., *Über mit einer Matrix vertauschbare Matrizen*, Math. Z., vol. 29, pp. 696-712 (1929).

*[348] Shostak, P. Y., *On a criterion for the conditional definiteness of quadratic forms in n linearly independent variables and on a sufficient condition for a conditional extremum of a function of n variables*, Uspehi Mat. Nauk, vol. 8, no. 4, pp. 199-206 (1954).

*[349] Shreĭder, Y. A., *A solution of systems of linear algebraic equations*, Dokl. Akad. Nauk, vol. 76, pp. 651-55 (1950).

*[350] Shtaerman (Steiermann), I. Y., *A new method for the solution of certain algebraic equations which have application to mathematical physics*, Z. Mat., Kiev, vol. 1, pp. 83-89 (1934); vol. 4, pp. 9-20 (1934).

*[351] Shtaerman (Steiermann), I. Y. and N. I. Akhieser (Achieser), *On the theory of quadratic forms*, Izv. Polyteh., Kiev, vol. 19, pp. 116-23 (1934).

*[352] Shura-Bura, M. R., *An estimate of error in the numerical computation of matrices of high order*, Uspehi Mat. Nauk, vol. 6, no. 4, pp. 121-50 (1951).

*[353] Shvartsman (Schwarzmann), A. P., *On Green's matrices of self-adjoint differential operators*, Proc. Odessa Univ. Matematika, vol. 3, pp. 35-77 (1941).

[354] Siegel, C. L., *Symplectic Geometry*, Amer. J. Math., vol. 65, pp. 1-86 (1943).

*[355] Skal'kina, M. A., *On the preservation of asymptotic stability on transition from differential equations to the corresponding difference equations*, Dokl. Akad. Nauk SSSR, vol. 104, pp. 505-8, (1955).

*[356] Smogorzhevskiĭ, A. S., *Sur les matrices unitaires du type de circulants*, J. Mat. Circle Akad. Nauk Kiev, vol. 1, pp. 89-91 (1932).

*[356a] Smogorzhevskiĭ, A. S. and M. F. Kravchuk, *On orthogonal transformations*, Zap. Inst. Ed. Kiev, vol. 2, pp. 151-56 (1927).

[357] Stenzel, H., *Über die Darstellbarkeit einer Matrix als Produkt von zwei symmetrischen Matrizen*, Math. Z., vol. 15, pp. 1-25 (1922).

[358] Stöhr, A., *Oszillationstheoreme für die Eigenvektoren spezieller Matrizen*, J. Reine Angew. Math., vol. 185, pp. 129-43 (1943).

*[359] Suleĭmanova, K. R., *Stochastic matrices with real characteristic values*, Dokl. Akad. Nauk SSSR, vol. 66, pp. 343-45 (1949).

*[360] ——— *On the characteristic values of stochastic matrices*, Uč. Zap. Moscow Ped. Inst., Ser. 71, Math., vol. 1, pp. 167-97 (1953).

*[361] Sultanov, R. M., *Some properties of matrices with elements in a non-commutative ring*, Trudy Mat. Sectora Akad. Nauk Baku, vol. 2, pp. 11-17 (1946).

*[362] Sushkevich, A. K., *On matrices of a special type*, Uč. Zap. Univ. Charkov, vol. 10, pp. 1-16 (1937).

[363] Sz-Nagy, B., *Remark on S. N. Roy's paper 'A useful theorem in matrix theory,'* Proc. Amer. Math. Soc., vol. 7, p. 1 (1956).

[364] Ta Li, *Die Stabilitätsfrage bei Differenzengleichungen*, Acta Math., vol. 63, pp. 99-141 (1934).

[365] Taussky, O., *Bounds for characteristic roots of matrices*, Duke Math. J., vol. 15, pp. 1043-44 (1948).

[366] ——— *A determinantal inequality of H. P. Robertson*, I, J. Washington Acad. Sci., vol. 47, pp. 263-64 (1957).

[367] ——— *Commutativity in finite matrices*, Amer. Math. Monthly, vol. 64, pp. 229-35 (1957).

[368] Toeplitz, O., *Das algebraische Analogon zu einem Satz von Fejér*, Math. Z., vol. 2, pp. 187-97 (1918).

[369] Turnbull, H. W., *On the reduction of singular matrix pencils*, Proc. Edinburgh Math. Soc., vol. 4, pp. 67-76 (1935).

*[370] Turchaninov, A. S., *On some applications of matrix calculus to linear differential equations*, Uč. Zap. Univ. Odessa, vol. 1, pp. 41-48 (1921).

*[371] Verzhbitskiĭ, B. D., *Some problems in the theory of series compounded from several matrices*, Mat. Sb., vol. 5, pp. 505-12 (1939).

*[372] Vilenkin, N. Y., *On an estimate for the maximal eigenvalue of a matrix*, Uč. Zap. Moscow Ped. Inst., vol. 108, pp. 55-57 (1957).

[373] Vivier, M., *Note sur les structures unitaires et paraunitaires*, C. R. Acad. Sci. Paris, vol. 240, pp. 1039-41 (1955).

[374] Volterra, V., *Sui fondamenti della teoria delle equazioni differenziali lineari*, Mem. Soc. Ital. Sci. (3), vol. 6, pp. 1-104 (1887); vol. 12, pp. 3-68 (1902).

[375] Walker, A. and J. Weston, *Inclusion theorems for the eigenvalues of a normal matrix*, J. London Math. Soc., vol. 24, pp. 28-31 (1949).

[376] Wayland, H., *Expansions of determinantal equations into polynomial form*, Quart. Appl. Math., vol. 2, pp. 277-306 (1945).

[377] Weierstrass, K., *Zur theorie der bilinearen und quadratischen Formen*, Monatsh. Akad. Wiss. Berlin, 1867, pp. 310-38.

[378] Wellstein, J., *Über symmetrische, alternierende und orthogonale Normalformen von Matrizen*, J. Reine Angew. Math., vol. 163, pp. 166-82 (1930).

[379] Weyl, H., *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Nat. Acad. Sci., vol. 35, pp. 408-11 (1949).

[380] Weyr, E., *Zur Theorie der bilinearen Formen*, Monatsh. f. Math. und Physik, vol. 1, pp. 163-236 (1890).

[381] Whitney, A., *A reduction theorem for totally positive matrices*, J. Analyse Math., vol. 2, pp. 88-92 (1952).

[382] Wielandt, H., *Ein Einschliessungssatz für charakteristische Wurzeln normaler Matrizen*, Arch. Math., vol. 1, pp. 348-52 (1948/49).

[383] ——— *Die Einschliessung von Eigenwerten normaler Matrizen*, Math. Ann. vol. 121, pp. 234-41 (1949).

[384] ——— *Unzerlegbare nicht-negative Matrizen*, Math. Z., vol. 52, pp. 642-48 (1950).

[385] ——— *Lineare Scharen von Matrizen mit reellen Eigenwerten*, Math. Z., vol. 53, pp. 219-25 (1950).

[386] ——— *Pairs of normal matrices with property L*, J. Res. Nat. Bur. Standards, vol. 51, pp. 89-90 (1953).

[387] ——— *Inclusion theorems for eigenvalues*, Nat. Bur. Standards, Appl. Math. Sci., vol. 29, pp. 75-78 (1953).

[388] ——— *An extremum property of sums of eigenvalues*, Proc. Amer. Math. Soc., vol. 6, pp. 106-110 (1955).

[389] ——— *On eigenvalues of sums of normal matrices*, Pacific J. Math., vol. 5, pp. 633-38 (1955).

[390] WINTNER, A., *On criteria for linear stability*, J. Math. Mech., vol. 6, pp. 301-9 (1957).

[391] WONG, Y., *An inequality for Minkowski matrices*, Proc. Amer. Math. Soc., vol. 4, pp. 137-41 (1953).

[392] ——— *On non-negative valued matrices*, Proc. Nat. Acad. Sci. U.S.A., vol. 40, pp. 121-24 (1954).

*[393] YAGLOM, I. M., *Quadratic and skew-symmetric bilinear forms in a real symplectic space*, Trudy Sem. Vect. Tens. Anal. Moscow, vol. 8, pp. 364-81 (1950).

*[394] YAKUBOVICH, V. A., *Some criteria for the reducibility of a system of differential equations*, Dokl. Akad. Nauk SSSR, vol. 66, pp. 577-80 (1949).

*[395] ZEITLIN (Tseĭtlin), M. L., *Application of the matrix calculus to the synthesis of relay-contact schemes*, Dokl. Akad. Nauk SSSR, vol. 86, pp. 525-28 (1952).

*[396] ZIMMERMANN (Tsimmerman), G. K., *Decomposition of the norm of a matrix into products of norms of its rows*, Nauč. Zap. Ped. Inst. Nikolaevsk, vol. 4, pp. 130-35 (1953).

INDEX

INDEX

[Numbers in italics refer to Volume Two]