Covering Spaces with Singularities†

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THE familiar concept of *unbranched covering* (Unverzweigte Ueberlagerung) is a topological concept (cf. [14, 23]) abstracted from the analytical concept *Riemann surface*, or rather that part of the Riemann surface remaining after the branch points have been deleted. Hitherto the concept of *branched covering* (Verzweigte Ueberlagerung) has apparently been formulated only in combinatorial terms. For example, Heegard [12], Tietze [21], Alexander [1, 2, 3] and Reidemeister [16] considered combinatorially defined branched coverings of spherical n-dimensional space. In fact, Tietze conjectured and Alexander [1] proved that every orientable n-dimensional manifold can be represented by such a covering. Later Tucker [22] gave a combinatorial definition of a more general type of covering in which there is allowed not only 'branching' but 'folding' as well. Seifert [17] gave a combinatorial definition of a covering of a 3-dimensional manifold branched over a (single or multiple) knot, and [18, 19, 20] derived important knot-invariants therefrom.

The principal object of this note is to formulate as a topological concept the idea of a *branched covering space*. This topological concept encompasses the above-mentioned combinatorial concept used by Heegard, Tietze, Alexander, Reidemeister and Seifert. This has as a consequence that the knot-invariants defined by Seifert (the linking invariants of the cyclic coverings) are invariants of the topological type of the knot (i.e. are unaltered by an orientation-preserving autohomeomorphism of 3-space). Without the developments of this note I am unable to see any simple proof that these invariants are invariants of anything more than the combinatorial type of the knot.

It appears that the best way to look at branched covering is as a 'completion' of unbranched côvering. This completion process appears

† This paper is based mainly on a course of lectures that I gave at the Instituto de Matemáticas de la Universidad Nacional Autónoma de México in the summer of 1951. I also had the opportunity of developlng my ideas by lecturing on this topic to the American Mathematical Soceity (1949), to the Summer Seminar of the Canadian Mathematical Society (1953), and at the Universities of Delft and Stockholm, while on a Fulbright grant (1952).

in its simplest form if it is applied to a somewhat wider class of objects. It is for this reason that I introduce the concept of a *spread* (a concept that encompasses, in particular, the 'branched and folded coverings' of Tucker). The basic theory of spreads is developed in §§ 1–3 for locally connected T_1-spaces. In § 4 it is shown how Freudenthal's compactification process [8] can be evolved out of the new process. In § 5 the branched covering concept is given a precise meaning. Conditions are found in § 6 ensuring that a branched covering of a complex (or manifold) be a complex (or manifold). The fundamental group of a branched covering is calculated in § 7, and a possible further line of development is indicated in § 8.

1. Spreads and their completions

A mapping g of a locally connected T_1-space Y into a locally connected T_1-space Z will be called a *spread* if the components of the inverse images of the open sets of Z form a basis of Y. The *antecedent* is Y and the *space over which the antecedent is spread* is the subset $g(Y)$ of Z. A point z of Z will be called an *ordinary point* if it has a neighborhood W in Z that is evenly covered [4] by g, i.e. if $g^{-1}(W)$ is non-vacuous and each component of $g^{-1}(W)$ is mapped topologically upon W by g. The points of Z that are not ordinary will be called *singular points*. In order that a map $g: Y \to Z$ be a spread it is necessary that $g^{-1}(z)$ be 0-dimensional for each point z of $g(Y)$. This may be expressed by saying that the antecedent of a spread must lie over the image space in thin sheets.

If $g: Y \to Z$ is a spread and Z is regular, then Y must also be regular. Let y be a point of Y and V a basic open set containing y. There is an open set W of Z containing $z = g(y)$ such that V is a component of $g^{-1}(W)$. Since Z is regular, there is an open neighborhood W_1 of z such that $\overline{W}_1 \subset W$. Let V_1 be the component of $g^{-1}(W_1)$ that contains y. Then $V_1 \subset V \cap g^{-1}(W_1)$, and hence $\overline{V}_1 \subset \overline{V} \cap g^{-1}(\overline{W}_1) \subset \overline{V} \cap g^{-1}(W) = V$.

A spread $g: Y \to Z$ will be said to be *complete* if for every point z of Z the following condition is satisfied: If to every open neighborhood W of z there is selected a component V of $g^{-1}(W)$ in such a way that $V_1 \subset V_2$ whenever $W_1 \subset W_2$, then $\bigcap_W V$ is non-vacuous (and is therefore a point).

Any locally connected subset X of the antecedent Y of a spread g is itself the antecedent of a spread; the spread with which it is associated is $f = g \mid X$, and the space over which it is spread is $f(X)$. In this circumstance the spread g is an *extension* of f. A more precise definition would be the following: an *extension* of a spread $f: X \to Z$ is a spread

$g\colon Y \to Z$ together with a homeomorphism i of X into Y that satisfies $gi = f$. However, I shall use the more informal definition, as this is unlikely to cause any real confusion. Two extensions $g_1\colon Y_1 \to Z$ and $g_2\colon Y_2 \to Z$ are *equivalent* if there is a homeomorphism ϕ of Y_1 upon Y_2 satisfying $g_2\phi = g_1$ and $\phi_2 | X = 1$. An extension $g\colon Y \to Z$ of a spread $f\colon X \to Z$ will be called a *completion* of f if g is complete and X is dense and locally connected† in Y.

2. The existence theorem

EXISTENCE THEOREM. *Every spread has a completion.*

Given a spread $f\colon X \to Z$, we are going to construct a space Y in which X is contained and a mapping g of Y into Z in such a way that g is a completion of f.

(a) *The points of Y and the function g.* Let z be any point of Z. A point y of the subset $g^{-1}(z)$ of Y is a function that associates to each open neighborhood W of z a component yW of $f^{-1}(W)$ in such a way that yW_1 is contained in yW_2 whenever W_1 is contained in W_2. This defines simultaneously the set Y and the function g. (Of course there may be points z for which $g^{-1}(z)$ is vacuous.)

(b) *The topology of Y.* Given any open set W of Z and any component U of $f^{-1}(W)$ define U/W to be the set of those points of Y for which $yW = U$. For any union $\bigcup_\alpha U_\alpha$ of components U_α of $f^{-1}(W)$ define $(\bigcup_\alpha U_\alpha)/W = \bigcup_\alpha (U_\alpha/W)$. Consider components U_1 and U_2 of $f^{-1}(W_1)$ and $f^{-1}(W_2)$ respectively. It is obvious that

$$U_1/W_1 \cap U_2/W_2 \subset U_1 \cap U_2/W_1 \cap W_2.$$

If conversely, $y \in U_1 \cap U_2/W_1 \cap W_2$ then $y(W_1 \cap W_2) \subset U_j$ $(j = 1, 2)$. But yW_j is the component of $f^{-1}(W_j)$ that contains the component $y(W_1 \cap W_2)$ of $f^{-1}(W_1 \cap W_2)$. Since U_j is a component of $f^{-1}(W_j)$ that contains $y(W_1 \cap W_2)$, it follows that $U_j = yW_j$, i.e. that $y \in U_j/W_j$. Thus it has been shown that

$$U_1/W_1 \cap U_2/W_2 = U_1 \cap U_2/W_1 \cap W_2.$$

This formula justifies the use of the collection of sets U/W, W ranging over the open sets of Z and U over the components of $f^{-1}(W)$, as a basis of Y; a topology is thereby defined in Y. It is easily verified that Y is a T_1-space.

† A space X is *locally connected* in a space Y if there is a basis of Y such that $V \cap X$ is connected for every basic open set V. An example of a space X not locally connected in a space Y is the following: Y is the Cartesian plane, $Y - X$ is the origin and the positive half of the real axis. Here X fails to be locally connected at any of the points of $Y - X$ except the origin. If $Z = Y$ the identity map of Y into Z is not the completion of the identity map of X into Z; in the completion of $i\colon X \to Z$ each point of $Y - X$ other than the origin gets covered by two points corresponding to the two sides of the real axis.

(c) *The imbedding of X in Y.* For any point x of X and open neighborhood W of $f(x)$ define xW to be that component of $f^{-1}(W)$ in which x is contained. It is clear that $xW_1 \subset xW_2$ whenever $W_1 \subset W_2$, so that x determines a point of Y. Since X and Z are T_1-spaces, distinct points of X determine distinct points of Y. We shall identify each point of X with the point of Y that it determines; X is then a subset of Y. It is obvious that, for any basic open set U of X,

$$U/W \cap X = U,$$

so that the topology of X is identical with the relativization topology induced in X by Y. Since the intersection of X with any basic open set of Y is non-vacuous and connected it follows that X is dense and locally connected in Y. Furthermore $f = g \mid X$ (and hence

$$f(X) \subset g(Y) \subset \overline{f(X)}).$$

(d) *The continuity of g.* This is an immediate consequence of the fact that, for any open set W of Z,

$$g^{-1}(W) = f^{-1}(W)/W.$$

(e) *The spread property of g.* For any open set W of Z and component U of $f^{-1}(W)$, we have

$$U \subset U/W \subset \overline{U},$$

so that each U/W is seen to be connected. (This shows that Y is locally connected.) On the other hand,

$$g^{-1}(W) = \cup_U U/W,$$

U ranging over the components of $f^{-1}(W)$, and

$$U_1/W \cap U_2/W = 0 \quad \text{if} \quad U_1 \neq U_2,$$

so that each U/W is clopen (closed and open) in $g^{-1}(W)$. Thus the components of $g^{-1}(W)$ are the sets U/W, U ranging over the components of $f^{-1}(W)$.

(f) *The completeness of g.* It was shown in (e) that a component V of $g^{-1}(W)$ is of the form U/W, where U is a component of $f^{-1}(W)$. The condition '$U_1/W_1 \subset U_2/W_2$ whenever $W_1 \subset W_2$' is equivalent to the condition '$U_1 \subset U_2$ whenever $W_1 \subset W_2$'. Thus $\cap_W V = \cap_W U/W$ contains the point y, where $yW = U$.

LEMMA. *If X and Z are separable then Y is also separable.*

If X is separable the components U of $f^{-1}(W)$ are enumerable, and if Z is also separable a countable basis of Y is made up of the sets U/W, W ranging over a countable basis of Z, and U ranging over the components of $f^{-1}(W)$.

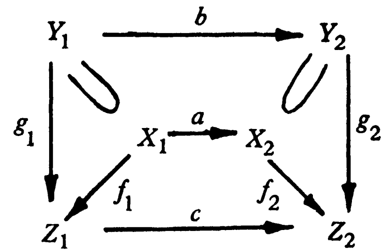
3. The uniqueness theorem and the extension theorem

LEMMA. *If X is dense and locally connected in Y then the intersection of X with any connected open set of Y is connected.*

Let V be a connected open subset of Y and suppose that the set $U = V \cap X$ (which is not vacuous, because X is dense in Y) is not connected. Then $U = A_1 \cup A_2$, where A_1 and A_2 are disjoint non-vacuous open subsets of X. Since X is locally connected in Y, any point y of V has an open neighborhood $N(y)$ contained in V whose intersection $M(y)$ with X is connected. Clearly either $M(y) \subset A_1$ or $M(y) \subset A_2$. Let $B_j = \{y \mid M(y) \subset A_j\}$ $(j = 1, 2)$. Then $V = B_1 \cup B_2$ and $B_1 \cap B_2 = 0$; B_j is open because $B_j = \bigcup_{y \in B_j} N(y)$; B_j is non-vacuous because $B_j \supset A_j$. Hence V cannot be connected. This contradiction shows that U must be connected.

Let f_1: $X_1 \to Z_1$ and f_2: $X_2 \to Z_2$ be spreads. A mapping a of X_1 into X_2 *covers* a mapping c of Z_1 into Z_2 if $f_2 a = c f_1$. Let g_1: $Y_1 \to Z_1$ and g_2: $Y_2 \to Z_2$ be completions of f_1 and f_2 respectively.

EXTENSION THEOREM. *The mapping a: $X_1 \to X_2$ can be extended to a mapping b: $Y_1 \to Y_2$ that also covers the mapping c: $Z_1 \to Z_2$.*



Let y_1 be any point of Y_1 and consider any open neighborhood W_2 of $z_2 = c(g_1(y_1))$. Then $W_1 = c^{-1}(W_2)$ is an open neighborhood of $z_1 = g_1(y_1)$. Let V_1 be the component of $g_1^{-1}(W_1)$ that contains y_1. By the lemma, $U_1 = V_1 \cap X_1$ is a component of $f_1^{-1}(W_1)$. Since $a(U_1)$ is a connected subset of $f_2^{-1}(W_2)$, it is contained in a component V_2 of $g_2^{-1}(W_2)$. Clearly $V_2 \subset V_2'$ whenever $W_2 \subset W_2'$. Since g_2 is complete, $\cap V_2$ is a point y_2. Define $b(y_1) = y_2$. It is obvious that $b \mid X_1 = a$ and that $g_2 b = c g_1$.

To prove that b is continuous, consider a basic open set of Y_2, i.e. a component V_2 of $g_2^{-1}(W_2)$ for some open set W_2 of Z_2. Let $U_2 = V_2 \cap X_2$ and $W_1 = c^{-1}(W_2)$. It is easily seen that $b^{-1}(V_2)$ is the union of those components of $g_1^{-1}(W_1)$ that intersect $a^{-1}(U_2)$. Hence $b^{-1}(V_2)$ is an open set.

UNIQUENESS THEOREM. *Any two completions of a spread are equivalent.*

Let $g_1: Y_1 \to Z$ and $g_2: Y_2 \to Z$ be completions of a spread $f: X \to Z$. By the extension theorem, there exist mappings $\phi: Y_1 \to Y_2$ and $\psi: Y_2 \to Y_1$ such that $\phi \mid X = \psi \mid X = 1$, $g_2\phi = g_1$, and $g_1\psi = g_2$. The map $\psi\phi: Y_1 \to Y_1$ is an extension of the identity map $1: X \to X$; since X is dense in Y_1, $\psi\phi = 1$. Similarly $\phi\psi = 1$; therefore ϕ is a homeomorphism of Y_1 upon Y_2 and $\psi = \phi^{-1}$.

By virtue of the uniqueness theorem we may speak of *the* completion of a spread.

COROLLARY OF THE EXTENSION THEOREM. *Let $f: X \to Z$ be a spread and $g: Y \to Z$ its completion. Let Q be any locally connected T_1-space, let $c: Q \times [0, 1] \to Z$ be a homotopy of Q in Z and let the 'open' homotopy $c \mid Q \times [0, 1)$ be covered by an 'open' homotopy*

$$a: \; Q \times [0, 1) \to X.$$

Then a can be extended to a homotopy $b: Q \times [0, 1] \to Y$ that covers c.

The identity mapping of $Q \times [0, 1)$ into $Q \times [0, 1]$ is clearly a spread, and its completion is the identity mapping of $Q \times [0, 1]$ upon itself. The corollary follows immediately. Of particular interest is the special case where Q is a point. The corollary then says that *an open path a in X that covers the interior of a path c in Z can be extended to a path b in Y that covers the path c.*

4. The ideal compactification

LEMMA. *Let $f: X \to Z$ be a spread and $g: Y \to Z$ its completion. Suppose that X and Z are separable, that $\overline{f(X)}$ is compact, and that Z has a basis such that, for each basic open set W, the number of components of $f^{-1}(W)$ is finite. Then Y is compact.*

Since Y is separable and X is dense in Y, it suffices to show that any sequence of points $x_1, x_2, \ldots$ of X has a subsequence converging in Y. Let $z_j = f(x_j)$; since $\overline{f(X)}$ is compact it is no loss of generality to assume that the sequence $z_1, z_2, \ldots$ converges to a point z_0. Let $W_1 \supset W_2 \supset \ldots$ be a local basis of Z at z_0 such that the number of components of each set $f^{-1}(W_n)$ is finite. Let U_1 be a component of $f^{-1}(W_1)$ that contains an infinite subsequence of $\{x_j\}$. Select, inductively, for each $n > 1$, a component U_n of $f^{-1}(W_n)$ in such a way that $U_n \subset U_{n-1}$ and U_n contains an infinite subsequence of $\{x_j\}$. This can be done because $f^{-1}(W_n) \cap U_{n-1}$ contains an infinite subsequence of $\{x_j\}$ and has only a finite number of components. Define $yW_n = U_n$. Any neighborhood W of z_0 contains W_n for some index n; define yW to be that component of $f^{-1}(W)$ that contains U_n. Thus a point y of Y is defined. It is obvious from the construction that some subsequence of $\{x_j\}$ converges to y.

It is well known that any locally compact Hausdorff space X that is not already compact can be compactified by the adjunction of one point, i.e. there is a compact Hausdorff space Z containing X such that $Z-X$ is a point. Furthermore, if X is connected, locally connected, separable, and regular, then so is Z. On the other hand Freudenthal [8] has shown that any connected, locally connected, locally compact, separable, regular space X has an *ideal compactification*, i.e. X is contained in a connected, locally connected, compact, separable, regular space Y in such a way that X is dense, open and locally connected in Y, and the set $Y-X$ is 0-dimensional, hence discrete. The concept of completion of a spread allows us to establish a relation between these two kinds of compactification.

COMPACTIFICATION THEOREM. *Let X be connected, locally connected, locally compact, separable, regular, and not already compact, and let $Z=X\cup z_0$ be its one-point compactification. The identity mapping $1\colon X\to Z$ is a spread; let $g\colon Y\to Z$ be its completion. Then Y is the ideal compactification of X.*

Since the ideal compactification is determined by the properties listed above, it suffices to check that Y has them. Compactness of Y is the only one of these properties that is not clear from the preceding sections. According to the lemma it suffices to show that Z has a local basis at z_0 such that, for each open set W of this local basis, the number of components of $f^{-1}(W)$ is finite.

Consider any neighborhood W_1 of z_0. There is an open neighborhood W_2 of z_0 such that $\overline{W_2}\subset W_1$. Since the boundary B of W_2 is a compact subset of the locally connected space X, it can meet only a finite number of the components of W_1-z_0, say $U_1, \ldots, U_n$. Define

$$W=z_0\cup U_1\cup \ldots \cup U_n.$$

Since X is connected, no component of W_1-z_0 lies within W_2. Hence W is an open neighborhood of z_0. Obviously $W\subset W_1$, and

$$f^{-1}(W)=W-z_0$$

has only a finite number of components.

5. Covering spaces

A spread f: $X \to Z$ over a connected† set Z is an *unbranched* (or *non-singular*) *covering* if the antecedent space X is connected and there are no singular points; the antecedent X is an *unbranched* (or *non-singular*) *covering space*, the map f is onto, and the space $f(X) = Z$ over which the antecedent is spread is the *base space*. If z and z' are any two points of Z, the number of points in $f^{-1}(z)$ and $f^{-1}(z')$ is the same (the set of points z' for which the number of points in $f^{-1}(z')$ is the same as the number of points in $f^{-1}(z)$ for some fixed z is easily seen to be non-vacuous and clopen in Z); it is the *index* j_f (Blaetterzahl) of f.

If g: $Y \to Z$ is any spread, the set Z_0 of ordinary points is obviously an open subset of Z. Hence $X = g^{-1}(Z_0)$ is an open subset of Y, and therefore locally connected. Thus $f = g \mid X$ is a spread. If Z_0 is non-vacuous and connected and its inverse image X is connected, the spread f: $X \to Z_0$ is an unbranched covering; I shall call it *the unbranched covering associated with* g.

I shall call a spread g: $Y \to Z$ a *branched covering*, or simply a *covering*,‡ if (1) Z_0 is connected, dense and locally connected in Z, (2) $g^{-1}(Z_0)$ is connected (so that g has an associated unbranched covering), and (3) g is the completion of its associated unbranched covering. The space Y is a *covering space* (or a *branched covering space*); Z is the *base space* of g. The set $Z_s = Z - Z_0$ is the *singular set*. An unbranched covering is a covering whose singular set is vacuous. Riemann surfaces [23] and Riemann spreads [1, 2] are covering spaces.

If y is any point of the covering space Y, W any connected open neighborhood of $z = g(y)$ such that $W_0 = W \cap Z_0$ is also connected,

† Here I have adopted the customary requirement that an unbranched covering space is connected. Although this is convenient, it is not really essential. It could be weakened to the requirement that the inverse image of each component of Z be connected, without causing any other than verbal difficulties. Of course this last condition is absolutely indispensable if one has any hopes of defining a universal covering space.

‡ Condition (3) excludes 'adhesions' of all sorts, in particular the 'folded coverings' of [22] are excluded. (An example of a spread with an 'adhesion' is the projection onto the plane $z = 0$ of the double cone $x^2 + y^2 = z^2$.) Condition (1) excludes 'slits' (exemplified in footnote, p. 245) and certain undesirable pathological singularities (such as isolated points).

A puzzling kind of spread is given by Fox and Kershner [7]. Here an open 2-dimensional manifold (of infinite genus) is mapped onto the plane. The branch points lie over a dense subset of the plane, so that every point is singular. Nevertheless, the branch points are isolated and the projection is a local homeomorphism at all other points. According to the present definition this is not a covering space, although its exclusion might be debatable.

V the component of $g^{-1}(W)$ that contains y, $U = V \cap X$, $q = g \mid V$ and $p = q \mid U$, then q: $V \to W$ is a covering with p: $U \to W_0 = W \cap Z_0$ its associated unbranched covering. Denote by $j(y, W)$ the index of p (over W_0). Obviously $j(y, W) \leqq j(y, W')$ whenever $W \subset W'$. Denote by $j(y)$ the minimum of the numbers $j(y, W)$; this is the *index of branching* of the point y. The number $\mu(y) = j(y) - 1$ is the classical *order of branching* [23] of y. Clearly z is a singular point if $j(y) > 1$; the converse need not be true. It is not clear from the literature what a *branch point* is, but it seems most probable that it is a point y for which $j(y) > 1$.

I shall call a covering g: $Y \to Z$ *finitely branched* if the index of branching $j(y)$ is finite for each point y of Y. I shall call a covering *regular* if its associated unbranched covering is regular.

6. Covering complexes

It is obvious that *a simplicial mapping g of a locally finite simplicial complex Y into a locally finite simplicial complex Z is a spread if and only if no simplex is mapped degenerately.* Such a mapping may be called a *simplicial spread.* Its singular set $Z_s = Z - Z_0$ is a subcomplex of Z. Furthermore, a principal open simplex of Z (i.e. one which is not on the boundary of any other simplex of Z) belongs to Z_0 or not according as it does or does not belong to $g(Y)$. Thus Z_0 is dense in Z if and only if g maps Y onto Z. The condition that Z_0 be dense and locally connected in Z is equivalent to the condition that, for each simplex τ of the subcomplex Z_s, the intersection $S(\tau)$ of Z_0 with the open star $\mathrm{st}_Z \tau$ of τ be non-vacuous and connected. Thus we are led to the following statement:

If the locally finite simplicial complex Z is connected, a simplicial spread g: $Y \to Z$ is a covering if and only if (1) *$S(\tau) = Z_0 \cap \mathrm{st}_Z \tau$ is non-vacuous and connected for every simplex τ of Z_s,* (2) *$X = g^{-1}(Z_0)$ is connected and,* (3) *$S(\sigma) = X \cap \mathrm{st}_Y \sigma$ is non-vacuous and connected for every simplex σ of $Y - X$.* Such a mapping may be called a *simplicial covering*, and the antecedent may be called a *covering complex.*

THEOREM. *Let Z be a barycentrically subdivided, connected, locally finite simplicial complex and let g: $Y \to Z$ be any (not necessarily simplicial) covering whose singular set Z_s is a subcomplex such that $S(\tau) = Z_0 \cap \mathrm{st}_Z \tau$ is non-vacuous and connected for every simplex τ of Z_s. If the index of branching $j(y)$ is finite for each point y of Y, then Y is a locally finite simplicial complex and g is a simplicial covering.*

Let $X = g^{-1}(Z_0)$ and $f = g \mid X$, so that f: $X \to Z_0$ is an unbranched covering. We are going to define a locally finite simplicial complex Y'

containing X and a simplicial mapping $g'\colon Y' \to Z$ such that $g' \mid X = f$; it will be clear that g' is a completion of f and hence equivalent to g.

If τ is any open n-dimensional simplex of Z_o, the components of $f^{-1}(\tau) = g'^{-1}(\tau)$ are open n-dimensional simplexes σ_i, i ranging over the cosets of the subgroup H of $\pi_1(Z_o)$ to which f belongs. If τ and τ^* are simplexes of Z_o such that $\tau < \tau^*$ (i.e. τ is on the boundary of τ^*) then there is a permutation $\rho = \rho_{\tau,\tau^*}$ of the cosets of H such that $\sigma_i < \sigma_j^*$ if and only if $i = \rho(j)$.

If τ is any open n-dimensional simplex of Z_s, the components of $g'^{-1}(\tau)$ are to be open n-dimensional simplexes σ_i, i ranging over the components $S_i(\tau)$ of $f^{-1}(S(\tau)) = f^{-1}(\mathrm{st}\,\tau)$. Since the index of branching of g is finite at each point of $g^{-1}(\tau)$, the number of simplexes comprised in any $S_i(\tau)$ is finite.

If τ and τ^* are simplexes of Z_s such that $\tau < \tau^*$, then $S(\tau) \supset S(\tau^*)$, so that each $S_j(\tau^*)$ is contained in some $S_i(\tau)$. The incidence relations in Y' that are to cover the incidence relation $\tau < \tau^*$ are: $\sigma_i < \sigma_j^*$ if and only if $S_i(\tau) \supset S_j(\tau^*)$.

If τ is a simplex of Z_s and τ^* a simplex of Z_o such that $\tau < \tau^*$ then the incidence relations in Y' covering this are to be: $\sigma_i < \sigma_j^*$ if and only if $S_i(\tau) \supset \sigma_j^*$.

It is easy to verify that the simplexes σ of Y', with the incidence relations described above, form a locally finite simplicial complex. (In order to prove that no two simplexes of Y' have the same vertices it is necessary to use the fact that Z has been barycentrically subdivided. For example, a simplicial subdivision of the 2-sphere Z might not be covered by a simplicial subdivision of a given Riemann surface if there were branching over both end-points of some 1-dimensional simplex of Z.) It is also easy to verify that g' is a completion of f and hence equivalent to g. The homeomorphism of Y' on Y induces the triangulation of Y.

Of special interest are the finitely branched coverings of a connected n-dimensional (combinatorial) manifold Z whose singular sets are pure $(n-2)$-dimensional simplicial complexes tamely imbedded in the interior of Z. (An $(n-2)$-dimensional simplicial complex is *pure* if every principal simplex is $(n-2)$-dimensional.) If we assume, as we may, that Z is triangulated in such a way that Z_s is a subcomplex, and then barycentrically subdivided, then, by the preceding theorem, such a covering is simplicial and the antecedent Y is a locally finite simplicial complex. Under what conditions is the covering complex Y also an n-dimensional manifold?

It is well known that, for $n = 2$, Y is always a manifold. For $n > 2$

the situation is more complicated. In any particular case it can be decided (in principle) by the following method. Let z be any vertex of the singular set Z_s, assumed to be in the interior of Z; then the boundary B of the star of z is a triangulated $(n-1)$-dimensional sphere and $B \cap Z_s$ is a pure $(n-3)$-dimensional subcomplex. The components of $g^{-1}(B)$ are finitely branched covering complexes of B whose singular sets are subcomplexes of $B \cap Z_s$. One has only to examine these components and decide whether or not they are $(n-1)$-dimensional spheres. The answer is affirmative in one general case (which includes all the examples that have been considered in the literature):

THEOREM. *Let Z be a connected, barycentrically subdivided, combinatorial n-dimensional manifold and let L be a polyhedrally imbedded combinatorial $(n-2)$-dimensional manifold such that the star of any vertex in L is flat*[11] *in Z. Then any finitely branched covering complex of Z whose singular set is a subcomplex of L is a combinatorial n-dimensional manifold.*

For simplicity let it be assumed that L is in the interior of Z. If L intersects the boundary of Z the proof following has to be modified.

Let g be the finitely branched covering, Y the covering complex, $K = g^{-1}(L)$, so that $Y - K$ is an unbranched covering space, with associated covering $e = g \mid Y - K$. If τ is any q-dimensional open simplex of L and τ_i, a q-dimensional open simplex of K, one of the components of $g^{-1}(\tau)$, then the closed star $\operatorname{St}\tau_i$ is a covering space of the closed star $\operatorname{St}\tau$ (the associated covering being the restriction h of g to $\operatorname{St}\tau_i$) whose singular set is a subcomplex of $\operatorname{St}_L \tau = L \cap \operatorname{St}\tau$. Since there is a homeomorphism that maps $\operatorname{St}\tau$ onto the Cartesian product $C \times E^{n-2}$ of the plane disc C: $x_1^2 + x_2^2 \leqq 1$ and the $(n-2)$-cell E^{n-2}: $0 \leqq x_i \leqq 1$ $(i = 3, \ldots, n)$ in such a way that $\operatorname{St}_L \tau$ is mapped onto $p \times E^{n-2}$ (where p denotes the point $x_1 = x_2 = 0$), the covering h must be of the form $d_m \times i$, where i maps E^{n-2} identically upon itself and d_m is the cyclic covering of C with branching index m at p exemplified in the branch point that the Riemann surface of the function $w = \sqrt[m]{z}$ has at the origin. Since all of these maps are simplicial it is clear that $\operatorname{St}\tau_i$ is a combinatorial n-cell.

7. The fundamental group of a branched covering

If S is an open subset of a space Y, an element of $\pi_1(Y)$ will be said to be *represented in* S if it is represented by a loop of the form $\alpha\gamma\alpha^{-1}$, where γ is a loop in S and α is a path in Y from the base point of $\pi_1(Y)$ to the base point of γ. Note that, if $S_1, S_2, \ldots$ are the components of S, an element of $\pi_1(Y)$ is represented in S if and only if it is represented in some S_i.

LEMMA. *Let Y be a connected, barycentrically subdivided, locally finite complex and let K be a subcomplex such that, for each vertex u of K, the intersection $S(u)$ of $Y-K$ with the open star* st u *of u is non-vacuous and connected. Then the injection homomorphism:*

$$\phi\colon\ \pi_1(Y-K)\to\pi_1(Y)$$

is onto, and its kernel is the consequence† of those elements of $\pi_1(Y-K)$ that are represented in $\bigcup_u S(u)$.

Since Y has been barycentrically subdivided, the stars st u are the components of $\bigcup_u \operatorname{st} u$ (and the sets $S(u)$ are the components of $\bigcup_u S(u)$). Let T be a simplicial tree in $Y-K$ rooted at the base point of $\pi_1(Y-K)$ and meeting each St u at exactly one point. The set $T\cup\bigcup_u S(u)$ is connected, and the image of the injection homomorphism $\pi_1(T\cup\bigcup_u S(u))\to\pi_1(Y-K)$ is the consequence of the elements of $\pi_1(Y-K)$ that are represented in $\bigcup_u S(u)$. The image of the injection homomorphism $\pi_1(T\cup\bigcup_u S(u))\to\pi_1(Y)$ is clearly 1. The theorem follows from an application of van Kampen's theorem [13], regarding Y as the union of $Y-K$ and $T\cup\bigcup_u \operatorname{st} u$.

THEOREM. *Let Z be a barycentrically subdivided, connected, locally finite complex and let L be a subcomplex such that, for each vertex v of L, the intersection $S(v)$ of $Z-L$ with the open star* st v *of v is non-vacuous and connected. Let Y be a finitely branched covering of Z whose singular set Z_s is a subcomplex of L. Let H be the subgroup of $G=\pi_1(Z-L)$ to which the associated unbranched covering of $Z-L$ belongs. Then $\pi_1(Y)\approx H/N$, where N is the consequence of those elements of H that are represented in $\bigcup_v S(v)$.*

By a preceding theorem, Y is a locally finite complex, mapped simplicially onto Z. After another barycentric subdivision, Y and K, the inverse image of L, satisfy the conditions of the lemma. The theorem follows from the observations that an element of G is covered by an element of $\pi_1(Y-K)$ if and only if it belongs to H and that an element of $\pi_1(Y-K)$ is represented in $\bigcup_u S(u)$ if and only if the element of G that it covers is represented in $\bigcup_v S(v)$.

The following application of this theorem may be of some interest. In [5], I proved that the group $F=(S_1, S_2, \dots, S_d\colon \prod_{i=1}^{d} S_i=1, S_i^{n_i}=1\ (i=1,\dots,d))$, where each n_i is a positive integer greater than 1, has a normal subgroup N with finite index in F and contains no element

† By the *consequence* of a set of elements in a group is meant the smallest normal subgroup that contains all these elements.

of finite order other than the identity. Let Z be the 2-sphere and select $d > 1$ points $s_1, s_2, \ldots, s_d$ of Z. The fundamental group of $Z - (s_1 \cup s_2 \cup \ldots \cup s_d)$ is $(x_1, x_2, \ldots, x_d : \prod_{i=1}^d x_i = 1)$, where x_i is represented by a small loop around s_i. Denote by ϕ the homomorphism $x_i \to S_i$ of this group upon the group F. Since N has no elements of finite order, $x_i^m \in W = \phi^{-1}(N)$ if and only if $m \equiv 0 \pmod{n_i}$. Let X be the unbranched covering space of $Z - (s_1 \cup s_2 \cup \ldots \cup s_d)$ determined by W. Thus X is a regular covering and $W \approx \pi_1(X)$. The branched covering space Y of Z to which X is associated has fundamental group N. Thus we have proved the following theorem:

If $s_1, s_2, \ldots, s_d$ $(d > 1)$ are points of the 2-sphere Z and $n_1, n_2, \ldots, n_d$ any positive integers greater than 1, *there exists a regular covering Y of Z of finite index for which the index of branching is equal to n_i at each point over s_i.*

Naturally Y is an orientable surface of genus

$$p = 1 - n + (n/2) \sum_{i=1}^d (1 - (1/n_i)),$$

where n is the index of N in F.

8. Generalizations

In § 1, I defined a spread $f: X \to Z$ only when X and Z are locally connected. If X and Z are arbitrary T_1-spaces, which are not necessarily locally connected, a mapping $f: X \to Z$ may be defined to be a spread† if the clopen subsets of the sets $f^{-1}(W)$, W ranging over the open sets of Z, form a basis of X. To such a spread a 'completion' $g: Y \to Z$ may be constructed, by a generalization of the process of § 2. A point y of $g^{-1}(z)$ is a function that associates to each open neighborhood W of z a quasi-component yW of $f^{-1}(W)$ in such a way that $yW_1 \subset yW_2$ whenever $W_1 \subset W_2$. Basic open sets U/W are defined as in § 2 for any clopen subset U of $f^{-1}(W)$. However, there are difficulties with this generalization in connection with the uniqueness theorem. Furthermore, its relation to Freudenthal's generalized ideal

† If Z is separable, regular, and X is compact, Hausdorff, a mapping f of X into Z is a spread if and only if $f^{-1}(z)$ is totally disconnected for every z, i.e. if and only if f is a so-called *light* mapping. (Let x be any point in any open set G of X and let $W_1 \supset W_2 \supset \ldots$ be a basic sequence of neighborhoods of $z = f(x)$. Let F_n be the component of $f^{-1}(\overline{W}_n)$ that contains x. Since $f^{-1}(z)$ is totally disconnected, and $\cap_n F_n$ is connected, $\cap_n F_n = x$. Hence, for some index n, $F_n \subset G$. Thus f is a spread.) That the compactness of X is essential here is shown by the following example constructed by John Milnor: Let X be the plane set consisting of all straight lines $y = ax + b$, a and b rational; let Z be the x-axis and let f map X upon Z by orthogonal projection. This is a light mapping (and X is locally connected), but f is not a spread. In fact, for any open interval W of Z the set $f^{-1}(W)$ is connected.

compactification is unclear. For these reasons I am not certain that it is the proper generalization, and have accordingly restricted myself to the locally connected case.

It would be interesting to generalize our theory of covering spaces with singularities to a theory of fibre spaces with singularities. A satisfactory definition of 'fibre space with singularities' should encompass at least the types considered by Seifert [17] and probably also the type considered by Montgomery and Samelson [15]. In an attempt at such a generalization, I replaced the set of components of quasi-components of $f^{-1}(W)$ by a decomposition of $f^{-1}(W)$ subject to suitable conditions. However, the resulting theory turned out to be rather unsatisfactory, in that the associated non-singular fibre space has to have a 'totally disconnected group'. Such a restriction is obviously much too severe. The example of the lens spaces, which are singular fibre spaces in the sense of Seifert [17], shows that a singular fibre space cannot be uniquely recovered from its associated non-singular fibre space, at least unless some additional structure is posited. In the case of the Seifert singular fibre spaces the additional structure is roughly the type of torus knot determined by a non-singular fibre in the neighborhood of a singular fibre, and is given by the numbers α, β in the 'symbol' (cf. [17]).

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