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FREE DIFFERENTIAL CALCULUS, V. THE ALEXANDER MATRICES RE-EXAMINED

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In FDC II, I defined *the Alexander polynomial of a group* G (having a finite presentation in which there are more generators than relations) only in the case where the commutator quotient group H of G is torsion free. Furthermore, I remarked (p. 209) that the situation when H is not torsion free is complicated, and that its treatment would be left open for future consideration. Unfortunately it is just this case that is involved in my method of classifying the lens spaces. Of course a polynomial can be defined by mapping the Alexander matrix into the Betti group, but the fact that inclusion of a space X into a containing space Y does not induce a homomorphism of the Betti group of X into the homology group of Y, but only into the homology group of Y modulo the image of the torsion group of X generates complications that threaten to become unbearable. Although Brody succeeded [44] in surmounting these difficulties and giving a complete proof that my classification of the lens spaces is, in fact, topological, the retreat to the Betti group seemed to me unaesthetic, and I was always searching for an alternative procedure.

Some years ago Blanchfield remarked that the Alexander polynomial is really a derivative, and it is this fact, which I recently rediscovered, that is the key to the resolution of the difficulty.

1. Derivatives modulo the order ideal

In order to utilize fully the remark of Blanchfield it is necessary to generalize slightly, for an abelian group, the notion of a *derivative* to that of a *derivative modulo the order ideal*. In general let H be any group, $\mathcal{J}$ an ideal in JH and $\natural$ the canonical homomorphism of JH upon $JH/\mathcal{J}$. By a *derivative modulo* $\mathcal{J}$ in H will be meant any linear mapping D of JH into $JH/\mathcal{J}$ that satisfies the product rule $D(uv) = Du \cdot v^0 + u^\natural \cdot Dv$. The case of interest here is: H abelian and $\mathcal{J}$ the order ideal $\mathfrak{E}_0(H)$. I shall denote by $\mathfrak{D}_0(H)$ the right $JH/\mathfrak{E}_0(H)$ module of derivatives modulo $\mathfrak{E}_0(H)$ in H. [Cf., FDC I p. 549].

It can be shown that $\mathfrak{D}_0(H)$ is the cyclic module generated by the basic inner derivative

$$I : v \longrightarrow (v^\natural - v^0)$$

unless either H is finite cyclic or the direct product of the infinite cyclic group with a finite abelian group T. In the first case $H = (x : x^q = 1)_\varphi$, $\mathfrak{E}_0(H) = ((1 + x + \cdots + x^{q-1})^\varphi)$ and $\mathfrak{D}_0(H)$ is the cyclic module generated by the derivative

$$D_x : u^\varphi \to \left(\frac{\partial u}{\partial x}\right)^{\natural\varphi} .$$

In the second case $H = (x :)_\varphi \times T$, $\mathfrak{E}_0(H) = (0)$, and $\mathfrak{D}_0(H)$ is generated by I and the derivative

$$K : u^\varphi \to \left(\frac{\partial u}{\partial x}\right)^{\varphi} \cdot \sum_{t \in T} t .$$

The derivatives I and K are not independent but satisfy the identity

$$I \cdot \sum_{t \in T} t = K \cdot (x^\varphi - 1) .$$

If D is any derivative mod $\mathfrak{E}_0(H)$ in H then the fact that any two elements u, v of JH commute implies the identity

$$(u^\natural - u^0)Dv = (v^\natural - v^0)Du. \tag{1.1}$$

Comparison of this formula with formula (4.4) of [1], or with § 6 of FDC II, was actually the clue that led to the writing of this installment.

Let θ be a homomorphism of an abelian group H onto an abelian group K. It is easily seen that $\theta : JH \to JK$ maps $\mathfrak{E}_0(H)$ into $\mathfrak{E}_0(K)$, so that a homomorphism of $JH/\mathfrak{E}_0(H)$ onto $JK/\mathfrak{E}_0(K)$ is induced; I shall denote it by the same symbol θ. Now consider a derivative $D \in \mathfrak{D}_0(H)$. For any element $v \in JH$, $Dv \in JH/\mathfrak{E}_0(H)$, hence $(Dv)^\theta \in JK/\mathfrak{E}_0(K)$. Suppose $w \in JH$ such that $w^\theta = 0$ and let v be an arbitrary element of JH. Then, by (1.1),

$$(v^{\natural\theta} - v^0)(Dw)^\theta = (w^{\natural\theta} - w^0)(Dv)^\theta = 0 .$$

Since θ is onto, this means that $(Dw)^\theta$ is annihilated by every element of the fundamental ideal of JK. It is easily seen [cf., FDC II § 5] that this implies that $(Dw)^\theta = 0$.

Define $D^\theta(w^\theta) = (D(w))^\theta$. It follows that D^θ induces a map of $JH/(\text{kernel of } \theta) \approx JK$ into $JK/\mathfrak{E}_0(K)$. Denote this map also by D^θ. It is easily verified that D^θ is a derivative mod $\mathfrak{E}_0(K)$ in K and that $(D_1 + D_2)^\theta = D_1^\theta + D_2^\theta$ and $(u^\theta \cdot D^\theta)(w^\theta) = u^\theta(D^\theta w^\theta)$.

(1.2) *In this sense, θ induces a homomorphism $D \to D^\theta$ of the module $\mathfrak{D}_0(H)$ into the module $\mathfrak{D}_0(K)$.*

2. The module of elementary derivatives

Let G be any finitely generated group, denote its commutator quotient group by H and its abelianizer by ψ. Let $(\mathbf{x} : \mathbf{r})$ be any finitely generated presentation of G, denote by φ the canonical homomorphism onto G of the free group $F(\mathbf{x})$ whose free basis is $\mathbf{x}$ and by R the consequence in $F(\mathbf{x})$ of $\mathbf{r}$.

Let $x_1, \cdots, x_n$ and $r_1, r_2, \cdots$ be the elements of $\mathbf{x}$ and $\mathbf{r}$ respectively, written down in some order. Let h be an arbitrary element of the group ring JH and let f be any element of $JF(\mathbf{x})$ such that $f^{\psi\varphi} = h$. Write f_j and r_{ij} for $(\partial f/\partial x_j)^{\natural\psi\varphi}$ and $(\partial r_i/\partial x_j)^{\natural\psi\varphi}$, and define

$$\omega h = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ r_{11} & r_{12} & \cdots & r_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ r_{n-1\,1} & r_{n-1\,2} & \cdots & r_{n-1\,n} \end{vmatrix} \in JH/\mathfrak{E}_0(H)$$

I claim that ωh does not depend on the choice of the element f. For if $f'^{\psi\varphi} = h$, then $f' - f$ belongs to the kernel of $\psi\varphi$, and hence $\omega h' - \omega h = 0$. Thus we have defined a mapping ω of JH into $JH/\mathfrak{E}_0(H)$. It is easily verified that $\omega \in \mathfrak{D}_0(H)$.

Clearly the derivative ω depends on the choice of the finitely generated presentation $(\mathbf{x} : \mathbf{r})$ of G and on the ordering $x_1, x_2, \cdots, x_n$ of $\mathbf{x}$ and $r_1, r_2, \cdots$ of $\mathbf{r}$ as well as on G. Let us take a fixed finitely generated presentation $\mathfrak{P} = (\mathbf{x} : \mathbf{r})$ of G and look at the derivatives ω that we get by reordering $\mathbf{x}$ and $\mathbf{r}$ in all possible ways. Denote by $El(\mathfrak{P})$ the submodule of $\mathfrak{D}_0(H)$ that is generated by these derivatives ω.

I claim that $El(\mathfrak{P})$ depends only on G, and not on the selected presentation $\mathfrak{P}$. To see this we need only examine the effect on $El(\mathfrak{P})$ of applying Tietze transformations to $\mathfrak{P}$. That adjunction to $\mathbf{r}$ of any set of its consequences does not alter $El(\mathfrak{P})$ is entirely obvious. Suppose then that $\mathfrak{P}'$ is obtained from $\mathfrak{P}$ by adjoining a new generator x_0 and a new relator $x_0 u^{-1}$ where $u \in F(x_1, \cdots, x_n)$. We have the following diagram:

$$\begin{array}{ccc} F(x_0, x_1, \cdots, x_n) & \searrow^{\varphi} & \\ \rho \big\downarrow & & G \xrightarrow{\ \psi\ } H\,, \\ F(x_1, \cdots, x_n) & \nearrow_{\varphi} & \end{array}$$

where ρ is the retraction $x_0^\rho = u$, $x_1^\rho = x_1, \cdots, x_n^\rho = x_n$. Let f' be any element of $F(x_0, x_1, \cdots, x_n)$ such that $f'\rho = f$. By FDC I (2.6) we have

$$f_j = f'_j + f'_0 u_j \qquad (j = 1, \cdots, n)\,,$$

where f'_j and u_j denote $(\partial f'/\partial x_j)^{\natural\psi\varphi}$ and $(\partial u/\partial x_j)^{\natural\psi\varphi}$ respectively. The new submodule $El(\mathfrak{P}')$ is generated by the derivatives of the following two types:

(i)

$$\omega' h = \begin{vmatrix} f_0' & f_1' & \cdots & f_n' \\ -1 & u_1 & \cdots & u_n \\ 0 & r_{11} & \cdots & r_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & r_{n-1\,1} & \cdots & r_{n-1\,n} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & f_1' + f_0' u_1 & \cdots & f_n' + f_0' u_n \\ -1 & u_1 & \cdots & u_n \\ 0 & r_{11} & \cdots & r_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & r_{n-1\,1} & \cdots & r_{n-1\,n} \end{vmatrix}$$

$$= \begin{vmatrix} f_1 & \cdots & f_n \\ r_{11} & \cdots & r_{1n} \\ \cdot & \cdots & \cdot \\ r_{n-1\,1} & \cdots & r_{n-1\,n} \end{vmatrix} = \omega h \ ,$$

(ii)

$$\omega' h = \begin{vmatrix} f_0' & f_1' & \cdots & f_n' \\ 0 & r_{11} & \cdots & r_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ 0 & r_{n1} & \cdots & r_{nn} \end{vmatrix} = 0 \ .$$

Thus application of Tietze transformations has no effect on $El(\mathfrak{P})$. As we have now shown that $El(\mathfrak{P})$ does not depend on $\mathfrak{P}$ but only on G we may simply write $El = El(G)$ instead of $El(\mathfrak{P})$. I call the elements of this submodule $El(G)$ of the module $\mathfrak{D}_0(H)$ the *elementary derivatives* induced by G in H.

3. The Alexander derivatives

By the *deficiency* of a finite presentation $(x_1, \cdots, x_n : r_1, \cdots, r_m)$ I mean the number $n - m$. A group has *deficiency* d if it has a presentation of deficiency d but none of deficiency $d + 1$. Since the Betti number of a group is an upper bound for the deficiencies of its finite presentations, as is easily shown, it follows that any finitely presented group G has a finite deficiency $d(G)$.

The groups of positive deficiency are of special interest. If G is such a group its commutator quotient group H is necessarily infinite, so that $\mathfrak{E}_0(H) = (0)$ and the elements of $\mathfrak{D}_0(H)$ are derivatives in JH. Furthermore such a group has at least one presentation of deficiency 1, and therefore the module $El(G)$ is necessarily cyclic; a generating element $\nabla : JH \rightarrow JH$ of $El(G)$ is determined up to multiplication by certain elements[1] of JH, and I shall call such a generating element an *Alexander derivative* of G. Note however that $\nabla = 0$ if $d(G) > 1$.

It is easy to verify that

(3.1) *If $d(G) = 1$ and H is the infinite cyclic group generated by t then, for every $v \in JH$,*

$$\nabla v \doteq \frac{v - v^0}{t - 1} \cdot \Delta(t)$$

where $\Delta(t)$ denotes the Alexander polynomial of G.

(3.2) *If $d(G) = 1$ and H is free abelian of rank $\mu > 1$ generated by $t_1, \cdots, t_\mu$ then, for every $v \in JH$,*

$$\nabla v \doteq (v - v^0) \cdot \Delta(t_1, \cdots, t_\mu) ,$$

where $\Delta(t_1, \cdots, t_\mu)$ denotes the Alexander polynomial of G.

Thus the Alexander derivative is determined by the Alexander polynomial (and conversely) whenever the latter is defined; i.e., whenever H is torsion free [cf. FDC II § 6]. To see what the derivative looks like when the polynomial is not defined, consider for example the fundamental group $G = (x, y : yxy = x)$ of the Klein bottle. Here $H = (x :) \times (y : y^2 = 1)$. An Alexander matrix is $\| 1 - yx + y \|^{\psi\varphi}$. Thus an Alexander derivative is

$$\nabla v = \left((x + y)\frac{\partial u}{\partial x} + (y - 1)\frac{\partial u}{\partial y} \right)^{\psi\varphi} , \qquad u^{\psi\varphi} = v \in JH .$$

It is determined, of course, by its values

$$\nabla x^{\psi\varphi} = x^{\psi\varphi} + y^{\psi\varphi} \quad \text{and} \quad \nabla y^{\psi\varphi} = y^{\psi\varphi} - 1 .$$

In terms of the generators I and K of $\mathfrak{D}_0(H)$, $\nabla = I + K$. Consider on the other hand the group $G' = (x, y : y^2 = 1)$. Its commutator quotient group is isomorphic to H and it may be verified that an Alexander derivative is $\nabla = K$.

4. Presentations of homomorphisms

By a *presentation of homomorphism* θ: $G_0 \rightarrow G_1$ will be meant a symbol $(\mathbf{x}; \mathbf{y} : \mathbf{r}; \mathbf{s})$ such that

[1] If (w) is a cyclic module over a commutative ring R then rw, $r \in R$, is a generator of (w) if and only if there exists an element $r^* \in R$ such that $rr^* - 1$ annihilates w. Such an element may be termed *a unit of R modulo annihilators of w*. I shall write $\doteq$ for equations relating Alexander polynomials $\nabla_1, \nabla_2, \cdots$ to indicate that equality holds if $\nabla_1, \nabla_2, \cdots$ are replaced by $\varepsilon_1\nabla_1, \varepsilon_2\nabla_2, \cdots$ where ε_i denotes some properly chosen unit of R modulo annihilators of ∇_i.

(i) $(\mathbf{x}, \mathbf{y} : \mathbf{r}, \mathbf{s})_{\varphi_1}$ is a presentation of G_1;

(ii) $(\mathbf{x} : \mathbf{r})_{\varphi_0}$ is a presentation of G_0;

(iii) the injection ι of $F(\mathbf{x})$ into $F(\mathbf{x}, \mathbf{y})$ is such that the diagram

$$\begin{array}{ccc} F(\mathbf{x}) & \xrightarrow{\iota} & F(\mathbf{x}, \mathbf{y}) \\ {\scriptstyle \varphi_0}\downarrow & & \downarrow{\scriptstyle \varphi_1} \\ G_0 & \xrightarrow{\theta} & G_1 \end{array}$$

is consistent.

This generalizes the notion of *presentation of a pair* that I defined in FDC II p. 197. In fact, if $G_0 \subset G_1$, θ is the injection of G_0 into G_1, and $(\mathbf{x}; \mathbf{y} : \mathbf{r}; \mathbf{s})$ is a presentation of θ then $(\mathbf{y}; \mathbf{x} : \mathbf{r} \cup \mathbf{s})$ is a presentation of the pair (G_1, G_0). In particular, a presentation $(\mathbf{y} : \mathbf{s})$ of a group G_1 may be considered as a presentation $(; \mathbf{y} : ; \mathbf{s})$ of the homomorphism $1 \to G_1$.

The cardinality of the set $\mathbf{y}$ is the *rank of the presentation* $(\mathbf{x}; \mathbf{y} : \mathbf{r}; \mathbf{s})$. The *rank of the homomorphism* θ is the minimum of the ranks of its presentations; thus the rank of θ is the smallest n such that G_1 may be generated by $\theta(G_0)$ and n elements of G_1.

A presentation $(\mathbf{x}; \mathbf{y} : \mathbf{r}; \mathbf{s})$ is *finite* if the sets $\mathbf{x}, \mathbf{y}, \mathbf{r}$ and $\mathbf{s}$ are finite. Clearly a homomorphism $\theta : G_0 \to G_1$ has a finite presentation if and only if G_0 and G_1 are each finitely presentable. By the *deficiency of a finite presentation* $(x_1, \cdots, x_\nu; y_1, \cdots, y_n : r_1, \cdots, r_\mu; s_1, \cdots, s_m)$, I mean the number $n - m$. A homomorphism θ has deficiency d if it has a presentation of deficiency d but none of deficiency $d+1$. Since the Betti number of $G_1/$(consequence of $\theta(G_0)$ in G_1) is an upper bound for the deficiencies of the finite presentations of θ, it follows that any finitely presentable homomorphism has a finite deficiency.

The following Tietze transformations of a presentation $(\mathbf{x}; \mathbf{y} : \mathbf{r}; \mathbf{s})$ do not alter the isomorphism type of the homomorphism $\theta : G_0 \to G_1$:

(I) Adjoin to $\mathbf{r}$ any of its consequences;

(I′) Adjoin to $\mathbf{s}$ any consequences of $\mathbf{r} \cup \mathbf{s}$;

(II) Adjoin x_0 to $\mathbf{x}$ and $x_0 u^{-1}$, where $u \in F(\mathbf{x})$, to $\mathbf{r}$;

(II′) Adjoin y_0 to $\mathbf{y}$ and $y_0 v^{-1}$, where $v \in F(\mathbf{x}, \mathbf{y})$, to $\mathbf{s}$.

By a proof very similar to the proof of FDC II (1.1) p. 198 one can prove the following:

(4.1) TIETZE THEOREM. *If $\mathfrak{P}$ and $\mathfrak{Q}$ are finitely generated presentations of a homomorphism $\theta : G_0 \to G_1$ then it is possible to pass from the one to the other by a finite sequence of Tietze transformations* $(\mathrm{I})^{\pm 1}$, $(\mathrm{I}')^{\pm 1}$, $(\mathrm{II})^{\pm 1}$, $(\mathrm{II}')^{\pm 1}$.

Using Tietze transformations it is easy to prove

(4.2) *If α is a homomorphism of A into B and β is a homomorphism of B into C then $d(\beta\alpha) \geqq d(\alpha) + d(\beta)$.*

PROOF. Let $(\mathbf{x}; \mathbf{y} : \mathbf{r}; \mathbf{s})$ be a presentation of α of deficiency $d(\beta)$ and let $(\mathbf{u}; \mathbf{v} : \mathbf{p}; \mathbf{q})$ be a presentation of β of deficiency $d(\alpha)$. Since $(\mathbf{x}, \mathbf{y} : \mathbf{r}, \mathbf{s})$ and $(\mathbf{u} : \mathbf{p})$ are both presentations of B there exist homomorphisms γ of $F(\mu)$ onto $F(\mathbf{x}, \mathbf{y})$ and δ of $F(\mathbf{x}, \mathbf{y})$ onto $F(\mathbf{u})$ such that $\mathbf{p}^\gamma$ is contained in the consequence of $\mathbf{r} \cup \mathbf{s}$ and $\mathbf{r}^\delta \cup \mathbf{s}^\delta$ is contained in the consequence of $\mathbf{p}$. Extend γ to a homomorphism of $F(\mathbf{u}, \mathbf{v})$ onto $F(\mathbf{x}, \mathbf{y}, \mathbf{v})$ by defining $v^\gamma = v$ for every $v \in \mathbf{v}$. Then $(\mathbf{x}, \mathbf{y}; \mathbf{v} : \mathbf{r}, \mathbf{s}; \mathbf{q}^\gamma)$ is a presentation of β, and $(\mathbf{x}; \mathbf{y}, \mathbf{v} : \mathbf{r}; \mathbf{s}, \mathbf{q}^\gamma)$ is a presentation of $\beta\alpha$. The deficiency of this last presentation is obviously $d(\alpha) + d(\beta)$.

(4.3) COROLLARY. *If θ is a homomorphism of G_0 into G_1 then $d(\theta) \leqq d(G_1) - d(G_0)$.*

PROOF. $A = 1$, $B = G_0$, $C = G_1$, $\beta = \theta$.

In (4.2) and (4.3) equality need not hold, as is shown by the following examples:

(1) G_1 the group of a non-trivial knot, G_0 one of the maximal peripheral subgroups, θ the injection. Here $d(G_0) = d(G_1) = 1$ but $d(\theta) = -1$.

(2) G_0 the group of a non-trivial knot, G_1 the infinite cyclic group, θ the abelianizer. Again $d(G_0) = d(G_1) = 1$ but $d(\theta) = -1$.

By *the Jacobian of the presentation* $(\mathbf{x}; \mathbf{y} : \mathbf{r}; \mathbf{s})$ of $\theta : G_0 \to G_1$ will be meant the matrix $(\partial\mathbf{s}/\partial\mathbf{y})^{\varphi_1}$ of elements of JG_1. It is clearly unaltered by the Tietze transformations I and II. The effect of Tietze I′ is to adjoin new rows that are linear combinations of the old ones (since $\partial r_i/\partial y_j = 0$), and the effect of Tietze II′ is to replace $\mathbf{M} = (\delta\mathbf{s}/\partial\mathbf{y})^{\varphi_1}$ by $\begin{pmatrix}\mathbf{M} & 0\\ * & 1\end{pmatrix}$. Hence

(4.4) *The Jacobians of the finitely generated presentations of θ are equivalent over JG_1.*

Thus we can associate to any finitely generated homomorphism a chain of elementary ideals $\mathfrak{E}_0(\theta) \subset \mathfrak{E}_1(\theta) \subset \cdots$ of JH_1 exactly as in FDC II, etc. Here I am especially interested in the order ideal $\mathfrak{E}_0(\theta)$.

(4.5) *Let G_0 be a group of positive deficiency and $\theta : G_0 \to G_1$ a homomorphism of deficiency $\geqq 0$. Then the order ideal $\mathfrak{E}_0(\theta)$ is a principal ideal, the Alexander derivatives ∇_0 of G_0 and ∇_1 of G_1 exist and, for any $v \in JH_0$*

$$(\nabla_0 v)^\theta \cdot \sigma \doteq \nabla_1 v^\theta ,$$

where σ denotes a generator of $\mathfrak{E}_0(\theta)$.

PROOF. Since $d(G_0) \geqq 1$ the existence of ∇_0 is assured; by (4.3) $d(G_1) \geqq 1$, so that ∇_1 also exists. Since $d(\theta) \geqq 0$ it is obvious that $\mathfrak{E}_0(\theta)$ is a principal ideal. Let $(\mathbf{x}; \mathbf{y} : \mathbf{r}; \mathbf{s})$ be a presentation of θ of deficiency 0 such that the presentation $(\mathbf{x} : \mathbf{r})$ of G_0 has deficiency 1. We have the consistent diagram

$$\begin{array}{ccc} F(\mathbf{x}) & \subset & F(\mathbf{x}, \mathbf{y}) \\ {\scriptstyle\varphi_0}\big\downarrow & & {\scriptstyle\varphi_1}\big\downarrow \\ G_0 & \xrightarrow{\theta} & G_1 \\ {\scriptstyle\psi_0}\big\downarrow & & {\scriptstyle\psi_1}\big\downarrow \\ H_0 & \xrightarrow{\theta} & H_1 \end{array}$$

Let u be an element of $JF(\mathbf{x})$ such that $u^{\psi_0\varphi_0} = v \in JH_0$. Then

$$\begin{aligned} \nabla_1 v^\theta &\doteq \begin{vmatrix} \mathbf{u_x} & 0 \\ \mathbf{r_x} & 0 \\ \mathbf{s_x} & \mathbf{s_y} \end{vmatrix}^{\psi_1\varphi_1} \\ &\doteq \begin{vmatrix} u_{\mathbf{x}}^{\psi_0\varphi_0} \\ \mathbf{r}_{\mathbf{x}}^{\psi_0\varphi_0} \end{vmatrix}^{\theta} \cdot \begin{vmatrix} \mathbf{s_y} \end{vmatrix}^{\psi_1\varphi_1} \\ &\doteq (\nabla_0 v)^\theta \cdot \sigma\ . \end{aligned}$$

Note that if $d(\theta) > 0$ we must have $d(G_1) > 1$ so that both σ and $\nabla_1 v^\theta$ must vanish.

5. The Alexander derivative of a certain direct limit

Let $\alpha_1 : G_0 \to G_1$ and $\alpha_2 : G_0 \to G_2$ be homomorphisms and denote by V the smallest normal subgroup of the free product $G_1 * G_2$ that contains all the elements $g^{\alpha_1} g^{-\alpha_2}$, $g \in G_0$. The group $G = G_1 * G_2 / V$ is called the *direct limit* of the system $G_0, G_1, G_2, \alpha_1, \alpha_2$ and the homomorphism β_i compounded of the inclusion of G_i into $G_1 * G_2$ and the natural homomorphism $G_1 * G_2 \to G$ is called the *projection* of G_i into G $(i = 1, 2)$. If G_0, G_1, G_2 have presentations $(\mathbf{x} : \mathbf{r})$, $(\mathbf{x}, \mathbf{y} : \mathbf{r}, \mathbf{s})$, $(\mathbf{x}, \mathbf{z} : \mathbf{r}, \mathbf{t})$ respectively then $(\mathbf{x}, \mathbf{y}, \mathbf{z} : \mathbf{r}, \mathbf{s}, \mathbf{t})$ is a presentation of G.

$$\begin{array}{ccccc} & & G_0 & & \\ & {\scriptstyle\alpha_1}\swarrow & & \searrow{\scriptstyle\alpha_2} & \\ G_1 & & & & G_2 \\ & {\scriptstyle\beta_1}\searrow & & \swarrow{\scriptstyle\beta_2} & \\ & & G & & \end{array} \qquad \beta_1\alpha_1 = \beta_2\alpha_2 = \beta\alpha\ .$$

Since **x**, **y**, **r**, **s** can be so chosen that the deficiency of the presentation (**x**; **y** : **r**; **s**) of α_1 is equal to $d(\alpha_1)$, and since (**x**, **z**; **y** : **r**, **t**; **s**) is a presentation of β_2, it follows that $d(\alpha_1) \leqq d(\beta_2)$. Similarly $d(\alpha_2) \leqq d(\beta_1)$.

Suppose that G_0, G_1 and G_2 are all of positive deficiency. In order that the direct limit G also be of positive deficiency it is sufficient, because of (4.3) and the above remark, that $d(\alpha_1) \geqq 0$ or $d(\alpha_2) \geqq 0$. These conditions ensure the existence of Alexander derivatives ∇_0, ∇_1, ∇_2, ∇ in G_0, G_1, G_2 G respectively. Furthermore it is not hard to construct examples to show that $d(G)$ need not be positive if $d(\alpha_1) < 0$ and $d(\alpha_2) < 0$.

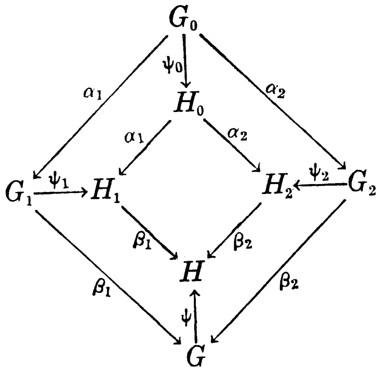
(5.1) *If* $d(G_0) > 0$, $d(G_1) > 0$, $d(G_2) > 0$, $d(\alpha_1) \geqq 0$, *so that* $d(G) > 0$, *then, for any* $u \in JH_0$ *and* $w \in JH_2$,

$$(\nabla_0 u)^{\beta\alpha}(\nabla w^{\beta_2}) \doteq (\nabla_1 u^{\alpha_1})^{\beta_1}(\nabla_2 w)^{\beta_2} .$$

(5.2) *If, in addition,* $\alpha_2(G_0) = G_2$, *then, for any* $u \in JH_0$ *and* $v \in JH_1$,

$$(\nabla_0 u)^{\beta\alpha}(\nabla v^{\beta_1}) \doteq (\nabla_1 v)^{\beta_1}(\nabla_2 u^{\alpha_2})^{\beta_2} .$$

PROOF. We have the consistent diagram



The order $\mathfrak{E}_0(\alpha_1)$ is generated by $\sigma = \det(\partial \mathbf{s}/\partial \mathbf{y})^{\psi_1\varphi_2}$, where (**x**; **y** : **r**; **s**) is a presentation of α_1 of deficiency 0. Then β_2 has a presentation (**x**, **z**; **y** : **r**, **t**; **s**), so that the order ideal $\mathfrak{E}_0(\beta_2)$ is generated by σ^{β_1}. Then, by (4.5), $u \in JH_0$ and $w \in JH_2$,

$$(\nabla_0 u)^{\alpha_1} \cdot \sigma \doteq \nabla_1 u^{\alpha_2}$$
$$(\nabla_2 w)^{\beta_2} \cdot \sigma^{\beta_1} \doteq \nabla w^{\beta_2} .$$

Hence

$$(\nabla_0 u)^{\beta\alpha}(\nabla w^{\beta_2}) \doteq (\nabla_0 u)^{\beta_1\alpha_1}(\nabla_2 w)^{\beta_2} \cdot \sigma^{\beta_1}$$
$$\doteq (\nabla_1 u^{\alpha_1})^{\beta_1}(\nabla_2 w)^{\beta_2} .$$

Now suppose that $\alpha_2(G_0) = G_2$. Let $(\mathbf{x} : \mathbf{r})_{\varphi_0}$ be a presentation of G_0 of deficiency 1 and let $(\mathbf{x} : \mathbf{r}^*)$ be a presentation of G_2 of deficiency 1. As shown in the proof of (4.2) there is a presentation $(\mathbf{x}; \mathbf{y} : \mathbf{r}; \mathbf{s})$ of α_1 of deficiency 0; clearly $(\mathbf{x}, \mathbf{y} : \mathbf{r}, \mathbf{s})$ is a presentation of G_1 of deficiency 1 and $(\mathbf{x}; \mathbf{y} : \mathbf{r}^*, \mathbf{s})$ is a presentation of G of deficiency 1. Now the consequence R of $\mathbf{r}$ in $F(\mathbf{x})$ is contained in the consequence R^* of $\mathbf{r}^*$ in $F(\mathbf{x})$. Thus each $r_i \in \mathbf{r}$ is a consequence of $\mathbf{r}^*$. Let $\mathbf{r} = r_1, \cdots, r_{n-1}$ and $\mathbf{r}^* = r_1^*, \cdots, r_{n-1}^*$. Then there exist elements $a_{ik} \in JH_2$ such that

$$r_{ij}^{\alpha_2} = \sum_{k=1}^{n-1} a_{ik} r_{kj}^* \qquad (i = 1, \cdots, n-1, j = 1, \cdots, n);$$

hence

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ r_{11} & r_{12} & \cdots & r_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ r_{n-1\,1} & r_{n-1\,2} & \cdots & r_{n-1\,n} \end{vmatrix}^{\alpha_2} = \det(a_{ik}) \cdot \begin{vmatrix} u_1^{\alpha_2} & u_2^{\alpha_2} & \cdots & u_n^{\alpha_2} \\ r_{11}^* & r_{12}^* & \cdots & r_{1n}^* \\ \cdot & \cdot & \cdots & \cdot \\ r_{n-1\,1}^* & r_{n-1\,2}^* & \cdots & r_{n-1\,n}^* \end{vmatrix},$$

where

$$u_j = \left(\frac{\partial u}{\partial x_j}\right)^{\psi_0\varphi_0}, \qquad r_{ij} = \left(\frac{\partial r_i}{\partial x_j}\right)^{\psi_0\varphi_0}, \qquad r_{ij}^* = \left(\frac{\partial r_i^*}{\partial x_j}\right)^{\alpha_2\psi_0\varphi_0}.$$

Thus $(\nabla_0 u)^{\alpha_2} \doteq \det(a_{ik}) \cdot (\nabla_2 u^{\alpha_2})$. Similarly $(\nabla_1 v)^{\beta_1} \doteq \det(a_{ik})^{\beta_2}(\nabla v^{\beta_1})$ so that

$$\begin{aligned} (\nabla_0 u)^{\beta_2\alpha_2}(\nabla v^{\beta_1}) &\doteq \det(a_{ik})^{\beta_2}(\nabla_1 v^{\beta_1})(\nabla_2 u^{\alpha_2})^{\beta_2} \\ &\doteq (\nabla_1 v)^{\beta_1}(\nabla_2 u^{\alpha_2})^{\beta_2}. \end{aligned}$$

6. An application to topology

(6.1) *If S is a closed surface then* $d(\pi(S)) \geqq 1 - \chi(S)$;

(6.2) *If M is a compact 3-manifold with boundary N then*

$$\begin{aligned} d(\pi(M)) &\geqq 0 && \textit{if } N \textit{ is vacuous,} \\ &\geqq 1 - \tfrac{1}{2}\chi(N) && \textit{if } N \textit{ is non-vacuous;} \end{aligned}$$

(6.3) *If S is one of the components of the boundary N of M, and if θ is the inclusion homomorphism $\pi(S) \to \pi(M)$, then*

$$\begin{aligned} d(\theta) &\geqq \tfrac{1}{2}\chi(S) - 1 && \textit{if } S = N \\ &\geqq \chi(S) - \tfrac{1}{2}\chi(N) && \textit{if } S \neq N. \end{aligned}$$

PROOF. We prove (6.2) and (6.3); the proof of (6.1) is rather trivial and occurs only incidentally. Let M be given a fixed triangulation, denote by $\alpha_n(M)$ the number of n-cells in this triangulation ($n = 0, 1, 2, 3$) and by $\alpha_n(S)$ the number of these that belong to S. Let T' be a maximal tree in S and let T be a maximal tree in M such that $T \cap S = T'$. The number of edges in T is $\alpha_0(M) - 1$, and the number of edges in T' is $\alpha_0(S) - 1$.

There is a presentation $\mathfrak{P}'$ of $\pi(S)$ having a generator corresponding to each edge of S not in T' and a relator corresponding to each 2-cell of S. Thus $d(\mathfrak{P}') = (\alpha_1(S) - (\alpha_0(S) - 1)) - \alpha_2(S) = 1 - \chi(S)$. This proves (6.1).

Construct in M a maximal cave C, i.e., a maximal tree in the dual triangulation; it consists of all the (open) 3-cells of M and $\alpha_3(M) - 1$ of the interior 2-cells of M together with any one of the 2-cells of N if N is non-vacuous. There is a presentation $\mathfrak{P}$ of $\pi(M)$ having a generator corresponding to each edge of M not on T and a relator corresponding to each 2-cell of M not in C. Thus, if N is vacuous, $d(\mathfrak{P}) = (\alpha_1(M) - (\alpha_0(M) - 1)) - (\alpha_2(M) - (\alpha_3(M) - 1)) = 0$, and, if N is non-vacuous, $d(\mathfrak{P}) = (\alpha_1(M) - (\alpha_0(M) - 1)) - (\alpha_2(M) - \alpha_3(M)) = 1 - \chi(M) = 1 - \frac{1}{2}\chi(N)$. (cf. [45, p. 223]). This proves (6.2).

If $S \neq N$, the maximal cave C can be chosen in $M - S$, so that $\mathfrak{P}'$ is contained in $\mathfrak{P}$. Consequently there is a presentation $\mathfrak{P}^*$ of θ such that $d(\mathfrak{P}^*) = d(\mathfrak{P}) - d(\mathfrak{P}') = \chi(S) - \frac{1}{2}\chi(N)$. On the other hand if $S = N$, the maximal cave C must contain a 2-cell of S so that $\mathfrak{P}'$ has a single relator that is not in $\mathfrak{P}$. Consequently, in this case, $d(\mathfrak{P}^*) = (d(\mathfrak{P}) - 1) - d(\mathfrak{P}') = \chi(S) - \frac{1}{2}\chi(N) - 1 = \frac{1}{2}\chi(S) - 1$. This proves (6.3).

Note that $\pi(S)$ has positive deficiency for every closed surface S except the 2-sphere and the projective plane; $\pi(M)$ has positive deficiency whenever N is non-vacuous and contains no 2-spheres or projective planes; $d(\theta)$ is non-negative if S is a torus and $N \neq S$ and contains no 2-spheres or projective planes. From this and §§ 4, 5 several conclusions can be drawn:

(6.4) *Let M be a compact 3-manifold whose boundary N is not connected and does not contain any 2-spheres or projective planes. Let S be a torus that is a component of N. Then $\pi(S)$ and $\pi(M)$ have Alexander derivatives ∇_S and ∇_M, and the order ideal of the inclusion homomorphism $\theta : \pi(S) \to \pi(M)$ is principal with generator denoted by σ. For any $v \in JH(S)$,*

$$\nabla_M v^\theta \doteq \sigma \cdot (v^\theta - v^0) .$$

PROOF. Apply (4.5), noting that $\nabla_S v \doteq v - v^0$.

Let M be a compact 3-manifold whose boundary N is non-vacuous and contains no 2-spheres or projective planes. Let M_0 be a torus semilinearly imbedded in the interior of M, that separates M into two components, whose closures are compact 3-manifolds M_1 and M_2. Denote the Alexander derivatives of $\pi(M_0)$, $\pi(M_1)$, $\pi(M_2)$, $\pi(M)$ by ∇_0, ∇_1, ∇_2, ∇ respectively.

(6.5) *If* $M_1 \cap N$ *is non-vacuous then, for any* $u \in JH(M_0)$ *and* $w \in JH(M_2)$,

$$(u^{\beta\alpha} - u^0)(\nabla w^{\beta_2}) \doteq (\nabla_1 u^{\alpha_1})^{\beta_1}(\nabla_2 w)^{\beta_2} ,$$

where α_i *and* β_i *denote the respective inclusions*

$$\pi(M_0) \to \pi(M_i) \quad \textit{and} \quad \pi(M_i) \to \pi(M) ;$$

(6.6) *If* M_2 *is a solid torus then, for any* $v \in JH(M_1)$,

$$(m^{\beta_2} - 1)(\nabla v^{\beta_1}) \doteq (\nabla_1 v)^{\beta_1} ,$$

where m *is a generating element of* $H(M_2)$.

PROOF. Apply (5.1) and (5.2), noting that $\nabla_0 u \doteq u - u^0$ and that, if M_2 is a solid torus, $\nabla_2 m \doteq 1$.

The second part, (6.6), can be restated in the following way:

(6.7) THEOREM. *Let* M *be a compact 3-manifold whose boundary is non-vacuous and contains no 2-spheres or projective planes. Let* k *be an element of* $H(M)$ *and let* K *be a simple closed polygon in the interior of* M *representing* k. *Then, for any* $v \in JH(M - K)$,

$$(\nabla_{M-K} v)^\theta \doteq (k - 1)\nabla_M v^\theta ,$$

where ∇_M *and* ∇_{M-K} *denote the Alexander derivatives of* M *and* $M - K$ *respectively and* θ *denotes the inclusion homomorphism* $H(M-K) \to H(M)$.

PROOF. Replace K by its closed neighborhood M_2 in the second barycentric subdivision of M. Then M_2 is a solid torus and $\pi(K) \to \pi(M_2)$ is an isomorphism onto.

(6.8) COROLLARY (Torres [12]). *Let* $L = L_1 \cup \cdots \cup L_\mu$ *be a tame link in spherical 3-space* S. *Let* $\Delta(t_1, \cdots, t_\mu)$ *be the Alexander polynomial of* L *and let* $\Delta(t_1, \cdots, t_{\mu-1})$ *be the Alexander polynomial of* $L_1 \cup \cdots \cup L_{\mu-1}$. *Then, denoting by* l_{ij} *the linking number of* L_i *and* L_j,

$$\Delta(t_1, \cdots, t_{\mu-1}, 1) \doteq (t_1^{l_{1\mu}} \cdots t_{\mu-1}^{l_{1\mu-1}} - 1)\Delta(t_1, \cdots, t_{\mu-1}) \qquad \text{if } \mu > 2 ,$$

$$\Delta(t_1, 1) \doteq \frac{t_1^{l_{12}} - 1}{t_1 - 1}\Delta(t_1)$$

PROOF. Apply (6.7) to the polygon $K = L_\mu$ in the complement M of the union of the neighborhoods of $L_1, L_2, \cdots, L_{\mu-1}$ in the second barycentric subdivision of a triangulation of the 3-sphere in which $L_1, L_2, \cdots, L_\mu$ are polygons. By (3.1) and (3.2)

$$\nabla_M v^\theta \doteq \frac{v^\theta - v^0}{t_1 - 1}\Delta(t_1) \qquad \text{if } \mu - 1 = 1 ,$$

$$\doteq (v^\theta - v^0)\Delta(t_1, \cdots, t_{\mu-1}) \qquad \text{if } \mu - 1 > 1 ,$$

$$\nabla_{M-K} v \doteq (v - v^0)\Delta(t_1, \cdots, t_\mu) .$$

Clearly $k = t_1^{l_{1\mu}} \cdots t_{\mu-1}^{l_{1\mu-1}}$. Thus, if $\mu > 2$,

$$(v^\theta - v^0)\Delta(t_1, \cdots, t_{\mu-1}, 1) \doteq (\nabla_{M-K} v)^\theta \doteq (k-1)\nabla_M v^\theta$$
$$\doteq (t_1^{l_{1\mu}} \cdots t_{\mu-1}^{l_{1\mu-1}} - 1)(v^\theta - v^0)\Delta(t_1, \cdots, t_{\mu-1}) ,$$

and, if $\mu = 2$,

$$(v^\theta - v^0)\Delta(t_1, 1) \doteq (\nabla_{M-K} v)^\theta \doteq (k-1)\nabla_M v^\theta$$
$$\doteq (t_1^{l_{12}} - 1)\frac{v^\theta - v^0}{t_1 - 1}\Delta(t_1) .$$

The result follows by choosing the element $v \in H(M-K)$ so that $v^\theta \neq 1$.

On the boundary $\dot{V}$ of a solid torus V in spherical 3-space S there is a simple closed curve uniquely determined up to homotopy that bounds in V but not on $\dot{V}$. Such a curve is called a *meridian* of V. A simple closed curve (also uniquely determined up to homotopy) that bounds in $\overline{S - V}$ but not on $\dot{V}$ is called a *longitude* of V. If V and V' are two solid tori in S a homeomorphism of V on V' is called *faithful* if it preserves the orientations induced by the orientation of S in V and V' and if it transforms longitudes into longitudes. (Such a homeomorphism necessarily transforms meridians into meridians).

(6.9) COROLLARY. *Let V and V' be polyhedral solid tori in spherical 3-space S. Let $L = L_1 \cup \cdots \cup L_\mu$ be a polygonal link of multiplicity μ contained in V and let f be a faithful simplical homeomorphism of V on V', so that the link $L' = f(L)$ is also polyhedral. Let $\Delta(t_1, \cdots, t_\mu)$ and $\Delta'(t_1, \cdots, t_\mu)$ denote the Alexander polynomials of the links L and L' and let $\Delta(t)$ and $\Delta'(t)$ denote the Alexander polynomials of the knots V and V'. The linking number of L_i and the meridian of V is equal to the linking number of L'_i and the meridian of V' (suitable orientations having been assigned) and is denoted by l_i. Then*[2]

$$\frac{\Delta(t_1, \cdots, t_\mu)}{\Delta(t_1^{l_1} \cdots t_\mu^{l_\mu})} \doteq \frac{\Delta'(t_1, \cdots, t_\mu)}{\Delta'(t_1^{l_1} \cdots t_\mu^{l_\mu})} ,$$

PROOF. Case I. Not all of $l_1, \cdots, l_\mu$ are equal to zero. Apply (6.5) with $M = S - L$, $M_1 = V - L$ and $M_2 = S - V$, $M_0 = \dot{V}$, $u \in H(\dot{V})$ and

[2] Here, and in the following, an equation of the form $a/b \doteq c/d$ is meant to be read as a proportion, i.e., $ad \doteq cb$.

$w \in H(S - V)$ meridians of V. Let t_i denote the element of $H(S - L)$ represented by a meridian of L_i and let t denote the element of $H(S - L)$ represented by a meridian of V. Then $t_1^{l_1} \cdots t_\mu^{l_\mu} = t \neq 1$, and $w^{\beta_2} = t$, $u^{\beta\alpha} = t$. By (3.1) $(\nabla_2 w)^{\beta_2} \doteq \Delta(t_1^{l_1} \cdots t_\mu^{l_\mu})$, and, by (3.2), $\nabla w^{\beta_2} \doteq (t-1)\Delta(t_1, \cdots, t_\mu)$. Hence by (5.1)

$$(t-1)^2\Delta(t_1, \cdots, t_\mu) \doteq \Delta(t)(\nabla_{V-L}(u^{\alpha_1}))^{\beta_1} .$$

Similarly

$$(t-1)^2\Delta'(t_1, \cdots, t_\mu) \doteq \Delta'(t)(\nabla_{V-L'}(u'^{\alpha'_1}))^{\beta'_1} ,$$

where I have written $t, t_1, \cdots, t_\mu$ for $t^f, t_1^f, \cdots, t_\mu^f$, $u' = u^f \in H(\dot{V}')$, $\alpha'_1 : H(\dot{V}') \to H(S - V')$, $\beta'_1 : H(S - V') \to H(S - L')$. Since $t - 1$ and $\Delta(t)$ are not zero divisors it follows that

$$\frac{\Delta(t_1, \cdots, t_\mu)}{\Delta(t)} \doteq \frac{\nabla_{V-L}(u^{\alpha_1})^{\beta_1}}{(t-1)^2} \doteq \frac{\Delta'(t_1, \cdots, t_\mu)}{\Delta'(t)} .$$

Case II. $l_1 = \cdots = l_\mu = 0$. Construct in the interior of $V - L$ a polygonal knot $L_{\mu+1}$ such that

(1) its linking (in S) number $l_{\mu+1}$ with a meridian of V is different from zero, and

(2) its linking (in S) number $l_{1\mu+1}$ with L_1 is different from zero.

Let $L'_{\mu+1} = f(L_{\mu+1})$, so that the linking numbers of $L'_{\mu+1}$ with a meridian of V' and L_1 are $l_{\mu+1}$ and $l_{1\mu+1}$ respectively. Then, denoting by $\Delta(t_1, \cdots, t_{\mu+1})$ and $\Delta'(t_1, \cdots, t_{\mu+1})$ the polynomials of $S - L \cup L_{\mu+1}$ and $S - L' \cup L'_{\mu+1}$ respectively, we have, by Case I,

$$\frac{\Delta(t_1, \cdots, t_{\mu+1})}{\Delta(t_{\mu+1}^{l_{\mu+1}})} \doteq \frac{\Delta'(t_1, \cdots, t_{\mu+1})}{\Delta'(t_{\mu+1}^{l_{\mu+1}})} .$$

But, by (6.8), $\Delta(t_1, \cdots, t_\mu, 1) \doteq (t_1^{l_{1\mu+1}} \cdots t_\mu^{l_{\mu\mu+1}} - 1)\Delta(t_1, \cdots, t_\mu)$, $\Delta'(t_1, \cdots, t_\mu, 1) \doteq (t_1^{l_{1\mu+1}} \cdots t_\mu^{l_{\mu\mu+1}} - 1)\Delta'(t_1, \cdots, t_\mu)$. Since, furthermore, $\Delta(1) \doteq 1 \doteq \Delta'(1)$, and since $t_1^{l_{1\mu+1}} \cdots t_\mu^{l_{\mu\mu+1}} - 1 \neq 0$, it follows that $\Delta(t_1, \cdots, t_\mu) \doteq \Delta'(t_1, \cdots, t_\mu)$ as required.

This result was proved for the case $\mu = 1$ by Seifert [46]. The case treated by Seifert contains as special cases theorems by Alexander [1] on composite knots, by Burau [47] on cable knots (Schlauchknoten), and by Whitehead [48] on doubled knots.

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