The Nonfiniteness of Nil

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THE NONFINITENESS OF Nil

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ABSTRACT. We show that \(\text{Nil } R\) is finitely generated only when it vanishes.

Bass and Murthy [2] gave the first examples of "nice" groups \(G\) such that \(\text{Wh } G\) is not finitely generated. Their examples result from calculating \(\text{Nil } R\) for certain rings \(R\). (See \([1, \text{ Chapter 12}]\) for the basic facts about \(\text{Nil } R\).) We show this is a general "pathology" for \(\text{Nil } R\). Let \(R\) be any ring with unit 1.

THEOREM. If \(\text{Nil } R \neq 0\), then \(\text{Nil } R\) is not finitely generated.

Our proof is based on three lemmas which we now discuss. Let \(n\) be a positive integer, \(t\) an indeterminate, \(R[\{t\}]\) and \(R[\{t^n\}]\) polynomial rings, and \(\sigma: R[\{t^n\}] \to R[\{t\}]\) the canonical inclusion. Recall that \(\sigma\) induces induction and restriction (transfer) maps

\[
\begin{align*}
\sigma_*: K_1R[\{t^n\}] &\to K_1R[\{t\}], \\
\sigma^*: K_1R[\{t\}] &\to K_1R[\{t^n\}],
\end{align*}
\]

respectively. The following is immediate.

LEMMA 1. The composite \(\sigma^* \sigma_*\) is multiplication by \(n\) on \(K_1R[\{t^n\}]\).

Next, we recall how \(\text{Nil } R\) embeds (as a direct summand) in \(K_1R[\{t\}]\) and in \(K_1R[\{t^n\}]\). If the nilpotent matrix \(N\) represents an element of \(\text{Nil } R\) then \(I + Nt\) represents the corresponding element of \(K_1R[\{t\}]\) where \(I\) is the identity matrix. Denote this map by \(\alpha\) and use \(\alpha'\) for the map \(\text{Nil } R \to K_1R[\{t^n\}]\) induced by \(N \to I + Nt^n\).

LEMMA 2. The image of \(\alpha'\) is mapped into the image of \(\alpha\) by \(\sigma_*\).

PROOF. Let \(N\) represent some \(x \in \text{Nil } R\), then \(I + Nt^n\) represents \(\sigma_* \alpha'(x) \in K_1R[\{t\}]\). It is well known that the image of \(\alpha\) is precisely the kernel of \(\epsilon_*: K_1R[\{t\}] \to K_1R\) where \(\epsilon: R[\{t\}] \to R\) is the evaluation homomorphism \(p(t) \to p(0)\) for \(p(t) \in R[\{t\}]\); clearly, the class of \(I + Nt^n\) is in the kernel of \(\epsilon_\ast\).

LEMMA 3. For each \(x \in \text{Nil } R\), there exists an integer \(K(x)\) such that \(\sigma^* \alpha(x) = 0\) for all integers \(n \geq K(x)\).
PROOF. Represent $x$ by an $s \times s$ nilpotent matrix $N$, then the $ns \times ns$ matrix $M$ (described below in blocked form) represents $\sigma^* \alpha(x)$

$$M = \begin{bmatrix}
I & & & Nt^n \\
N & I & & \\
& \ddots & \ddots & \\
& & N & I
\end{bmatrix};$$

namely, $M$ has $I$ down the diagonal, $N$ down the first subdiagonal, $Nt^n$ in the upper right corner, and the 0 matrix elsewhere. Let $K(x)$ be the degree of nilpotency of $N$; i.e., $N^n = 0$ for all $n > K(x)$. Let $A$ be the $ns \times ns$ matrix whose blocked description is the same as $M$ except $Nt^n$ is replaced by 0; i.e.,

$$A = \begin{bmatrix}
I & & & \\
N & I & & \\
& \ddots & \ddots & \\
& & N & I
\end{bmatrix}.$$

One sees (by an easy calculation) that $A^{-1} M = I + T$ where $T$ is a strictly upper triangular matrix provided $n > K(x)$. Since $A$ and $I + T$ clearly represent zero in $K_1 R[t^n]$, so does $M$.

PROOF OF THEOREM. We proceed by contradiction; i.e., assume Nil $R$ is nonzero but finitely generated. Hence there is a prime $p$ such that multiplication by $p$ is a monomorphism of Nil $R$. In particular, Lemma 1 together with Lemma 2 imply that $\sigma^* \alpha$: Nil $R \to K_1 R[t^n]$ is nonzero when $n = p^i$ for all $i > 0$. But this contradicts Lemma 3 since we assumed Nil $R$ is finitely generated.

REMARKS. There are other Nil-type groups in algebraic $K$ and $L$-theory. They arise geometrically when studying codimension-1 submanifolds. It seems plausible to conjecture the analogue of our Theorem for all these groups. All examples known to me of nonfinite generation for $K$ and $L$ functors applied to "nice" rings arise from these Nil-type groups. For other particular instances of this, see the following articles: [3] and [4] for UNil, [6] for Nil$_2$, and [5] for Wh$_1(\_\_\_)$). This phenomena is a major stumbling block to understanding nonsimply connected manifolds.

REFERENCES


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