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CLASSIFICATION OF CERTAIN HIGHER-DIMENSIONAL KNOTS OF CODIMENSION TWO

M. Sh. Farber

An n-dimensional knot is a pair (S^{n+2}, k^n) consisting of an oriented sphere S^{n+2} and a smooth closed oriented submanifold k that is homotopy-equivalent to an n-dimensional sphere. Two n-dimensional knots (S^{n+2}, k_ν) ($\nu = 1$ or 2) are equivalent (or of the same isotopy type) if there is an orientation-preserving isotopy of S^{n+2} taking k_1 to k_2. In this lecture we consider the problem of describing the set of isotopy types of n-dimensional knots. We use terminology of differential topology.

1. **Homotopy Seifert pairings.** Let V be a connected compact oriented $(n+1)$-dimensional submanifold of the sphere S^{n+2} with a non-empty boundary ∂V. Let Y be the closure of the complement of an open tubular neighbourhood of V in S^{n+2}. We denote by $u\colon V \wedge Y \to S^{n+1}$ the canonical pairing of Spanier–Whitehead duality. Let $i_+\colon V \to Y$ be the map given by a small shift along the field of positive normals to V in S^{n+2}. A *homotopy Seifert pairing* of the manifold V is the composition

$$\theta\colon V \wedge V \xrightarrow{1\wedge i_+} V \wedge Y \xrightarrow{u} S^{n+1}.$$

It is clear that θ defines a unique embedding $V \subset S^{n+2}$ up to homotopy.

If n is odd, then θ induces the classical Seifert pairing on the middle-dimensional homology [1].

A homotopy pairing $\theta\colon K \wedge K \to S^{n+1}$, where K is a finite complex,is *spherical* if K has the homotopy type of a complex of dimension $\leqslant n$, and the pairing $\theta + (-1)^{n+1}\theta'\colon K \wedge K \to S^{n+1}$ is a Spanier–Whitehead duality. Here θ' is the composition of the map $K \wedge K \to K \wedge K$ interchanging the factors and the map θ, and the plus or minus sign is understood as operating in the cohomotopy group $\pi^{n+1}(K \wedge K)$. Two homotopy pairings $\theta_\nu\colon K_\nu \wedge K_\nu \to S^{n+1}$ ($\nu = 1$ or 2) are *stably congruent* if there is an S-equivalence $f\colon K_1 \to K_2$ for which $\theta_2 \circ (f \wedge f)$ is stably homotopic to θ_1.

THEOREM 1. *A homotopy Seifert pairing of an $(n+1)$-dimensional submanifold $V \subset S^{n+2}$ is spherical if and only if ∂V is a homology sphere.*

THEOREM 2. *If* $3r \geqslant n+1 \geqslant 6$, *then the association of a submanifold with its homotopy Seifert pairing realizes a bijection of the set of isotopy classes of embeddings of r-connected* $(n+1)$*-dimensional oriented submanifolds of* S^{n+2} *that are bounded by homotopy spheres into the set of classes of stably congruent spherical homotopy pairings* $\theta: K \wedge K \to S^{n+1}$ *given on finite r-connected complexes.*

2. **The classification of knots.** For each knot (S^{n+2}, k^n) there is a connected orientable $(n+1)$-dimensional submanifold $V \subset S^{n+2}$ with $\partial V = k$. It is called a *Seifert manifold* of the knot. The orientation of a Seifert manifold can be chosen canonically, using the orientation of k. The Seifert manifold defined by the knot is not unique. Later we shall explain how the homotopy pairings corresponding to the various Seifert manifolds of a knot are related.

We say that two homotopy pairings $\theta_\nu: K_\nu \wedge K_\nu \to S^{n+1}$ $(\nu = 1$ or $2)$ *abut* if there exist connected complexes L and M and pairings $\alpha: K_1 \wedge K_2 \to S^{n+1}$ and $u: L \wedge M \to S^{n+1}$, the latter being a Spanier–Whitehead duality, and an S-equivalence $h: K_1 \vee K_2 \to L \vee M$ such that $\eta \circ (h \wedge h)$ is stably homotopic to ξ, where the homotopy pairings $\eta: (L \vee M) \wedge (L \vee M) \to S^{n+1}$ and $\xi: (K_1 \vee K_2) \wedge (K_1 \vee K_2) \to S^{n+1}$ are given, respectively, by the matrices

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \text{and} \begin{pmatrix} \theta_1 & \alpha \\ (-1)^n \alpha' & (-1)^n \theta_2' \end{pmatrix}.$$

Here, as above, a dash denotes transposition, that is, the composition of the map with the interchange of the factors.

A pairing that abuts a spherical pairing is itself spherical. The relation of abutting is symmetric and reflexive on the set of spherical pairings. The equivalence relation generated by abutting is called an *R-equivalence.* More precisely, two homotopy pairings $\theta_\nu: K_\nu \wedge K_\nu \to S^{n+1}$ $(\nu = 1$ or $2)$ are *R-equivalent* if there is a sequence $\eta_i: N_i \wedge N_i \to S^{n+1}$ $(i = 1, \ldots, s)$ of homotopy pairings such that $\eta_1 = \theta_1$, $\eta_s = \theta_2$, and for each i the pairings η_i and η_{i+1} abut.

THEOREM 3. *The homotopy pairings of any two Seifert manifolds of a knot are R-equivalent.*

THEOREM 4. *Let* V_ν *be an r-connected Seifert manifold of a knot* (S^{n+2}, k_ν) $(\nu = 1$ *or* $2)$, *and let* $\theta_\nu: V_\nu \wedge V_\nu \to S^{n+1}$ *be the corresponding homotopy Seifert pairing. If* θ_1 *and* θ_2 *are R-equivalent and* $3r \geqslant n+1 \geqslant 6$, *then the knots* (S^{n+2}, k_ν) $(\nu = 1, 2)$ *are equivalent.*

Levine [2] has proved that the knot (S^{n+2}, k^n) has an r-connected Seifert manifold if and only if $\pi_i(S^{n+2} - k) \approx \pi_i(S^1)$ for $i \leqslant r$. Let us denote by $K_{r,n}$ the set of isotopy types of such knots. This is a semigroup under the operation of forming the connected sum. Moreover, $K_{0,n}$ is the semigroup of isotopy types of all n-dimensional knots. The semigroups $K_{r,n}$ determine a decreasing filtration $K_{0,n} \supset K_{1,n} \supset K_{2,n} \supset \ldots$ If $2r \geqslant n \geqslant 5$, then $K_{r,n}$ consists of a single element, the type of the trivial knot.

Let $\Sigma_{r,n}$ denote the set of R-equivalence classes of spherical homotopy pairings $\theta\colon K \wedge K \to S^{n+1}$ on finite r-connected complexes K. The theorems stated above imply the following classification theorem.

CLASSIFICATION THEOREM. *The map that associates with a knot the R-equivalence class of homotopy Seifert pairings of a certain Seifert manifold spanned by this knot defines a bijection* $K_{r,n} \to \Sigma_{r,n}$ *if* $3r \geqslant n + 1 \geqslant 6$.

Note also the following useful fact: if two spherical homotopy pairings $\theta_\nu\colon K_\nu \wedge K_\nu \to S^{n+1}$ ($\nu = 1$ or 2) are R-equivalent and the complexes K_1 and K_2 are r-connected, then the sequence $\eta_i\colon N_i \wedge N_i \to S^{n+1}$ in the definition of the R-equivalence can be selected so that the complexes N_i are also r-connected.

3. **Periodic knots.** In [3] Bredon suggested a suspension construction for embeddings of codimension 2: if (S^{n+2}, l^n) is a pair consisting of an oriented sphere S^{n+2} and a smooth closed oriented submanifold l^n, then the suspension of this pair is $(S^{n+4}, \omega(S^{n+2}, l))$, where $\omega(S^{n+2}, l)$ is the double covering of S^{n+2} branching over l and canonically embedded in S^{n+4}. The manifold $\omega(S^{n+2}, l)$ need not be a homotopy sphere, even if l is one. However, the twice iterated suspension ω^2 sends knots into knots and defines a homomorphism of the semigroup of isotopy types of n-dimensional knots into the same semigroup of $(n + 4)$-dimensional knots [3]. In addition, if (S^{n+2}, k^n) bounds an r-connected manifold, then $\omega^2(S^{n+2}, k^n)$ bounds an $(r + 2)$-connected manifold. Consequently, ω^2 can be regarded as a homomorphism $K_{r,n} \to K_{r+2,n+4}$.

THEOREM 5. *The homomorphism* $\omega^2\colon K_{r,n} \to K_{r+2,n+4}$ *is an isomorphism if* $3r \geqslant n + 1 \geqslant 6$.

The knots of $K_{r,n}$ are naturally called *stable* when $3r \geqslant n + 1 \geqslant 6$. The homomorphism ω^2 sends stable knots to stable knots and for each n-dimensional knot K the knot $\omega^{2N}K$ is stable if $2N \geqslant n + 1$.

Theorem 5 asserts that the set $K_{r,n}$ of stable knots depends only on the residue of n modulo 4 and on $n - 2r$.

This theorem is fairly easily deduced from the classification theorem of §2. If $\theta\colon K \wedge K \to S^{n+1}$ is a certain homotopy pairing, then we define $\sigma(\theta)$ to be the composition

$$SK \wedge SK = (S^1 \wedge K) \wedge (S^1 \wedge K) \to$$
$$\to S^1 \wedge S^1 \wedge (K \wedge K) \xrightarrow{1\wedge 1\wedge\theta} S^1 \wedge S^1 \wedge S^{n+1} = S^{n+3},$$

where S is the suspension as above and the unnamed map is the interchange of the second and third factors. If θ is a spherical pairing, then $\sigma^2(\theta) = \sigma(\sigma(\theta))$ is also spherical. If θ_1 and θ_2 are R-equivalent, then so are $\sigma^2(\theta_1)$ and $\sigma^2(\theta_2)$. Hence, σ^2 defines a map $\Sigma_{r,n} \to \Sigma_{r+2,n+4}$. Further the diagram

$$\begin{array}{ccc} K_{r,n} & \xrightarrow{\omega^2} & K_{r+2,\,n+4} \\ \downarrow & & \downarrow \\ \Sigma_{r,n} & \xrightarrow{\sigma^2} & \Sigma_{r+2,\,n+4} \end{array},$$

in which the vertical arrows denote maps analogous to those in the classification theorem, is commutative.

If $3r \geqslant n + 1 \geqslant 6$, then these maps are bijective. The map σ^2 is also bijective in this case; this is a consequence of a theorem on suspensions [4]. So we deduce that ω^2 is bijective.

Theorem 5 was proved by Bredon [3] when $n - 2r = 1, r \geqslant 2$.

4. Knot complements. The question as to what extent the complement of a knot defines its type has been much studied. It was proved in papers by Gluck [5], Browder [6], Lashoff and Shaneson [7] that for $n \geqslant 2$ there are at most two distinct knots having diffeomorphic complements. It is known that the complement of a knot defines its type uniquely in Levine's class of simple odd-dimensional knots [8] and in the class of knots obtained by superspinning [5], [9]. Examples of non-equivalent knots with diffeomorphic complements were constructed recently in [10] and [11].

THEOREM 6. *Stable knots are equivalent if and only if their complements are diffeomorphic.*

5. The classification of fibred knots. The results of §2 simplify considerably for fibred knots; here one can avoid using R-equivalence and obtain an immediate classification in terms of the invariants of an infinite cyclic covering. These results can be used to study isolated singularities of polynomial maps $\mathbf{R}^m \to \mathbf{R}^2$.

A knot $K = (S^{n+2}, k^n)$ is said to be *fibred* if there is a map $b: S^{n+2} \to D^2$ such that $0 \in D^2$ is a regular value, $b^{-1}(0) = k$, and the map $\bar{b}: S^{n+2} - k \to S^1$, $\bar{b}(x) = b(x)/\|b(x)\|$ is a smooth fibration. Let $\alpha \in S^1$ and $[0, \alpha]$ be a radial segment joining $0 \in D^2$ and α in D^2. Then $V = b^{-1}([0, \alpha])$ is a Seifert manifold of K, which we call the *fibre of the knot.*

THEOREM 7. *A homotopy Seifert pairing of the fibre of any fibred knot is a duality. Conversely, if a homotopy pairing of a certain r-connected Seifert manifold of an n-dimensional knot is a duality and* $r \geqslant 1, n \geqslant 4$, *then the knot is fibred.*

THEOREM 8. *If two spherical homotopy pairings* $\theta_\nu: K_\nu \wedge K_\nu \to S^{n+1}$ ($\nu = 1$ *or* 2) *are Spanier–Whitehead dualities, then they are R-equivalent if and only if they are stably congruent.*

This generalizes a theorem due to Trotter [12] about S-equivalent unimodular Seifert matrices.

Let (S^{n+2}, k^n) be a certain fibred knot and V its fibre. We denote by X the complement $S^{n+2} - k$ and let $p: \widetilde{X} \to X$ be an infinite cyclic covering. We choose a generator $t: \widetilde{X} \to \widetilde{X}$ of the group of covering transformations of p by the following condition: if $x_i \in \widetilde{X}$ and ω is a path beginning at x_0 and ending at $t(x_0)$ in $\widetilde{X}$, then the intersection index of the loop $p \circ \omega$ with V in S^{n+2} is 1.

The embedding i: int $V \to X$ can be lifted to a covering $\tilde{i}$: int $V \to \widetilde{X}$, where $\tilde{i}$ is a homotopy equivalence. Let ψ: $\widetilde{X} \to V$ be a homotopy equivalence that is the composition of a homotopy equivalence inverse to $\tilde{i}$ and the embedding int $V \to V$. We consider the pairing

$$u\colon \widetilde{X} \wedge \widetilde{X} \to S^{n+1},$$

given by

$$u = [\theta + (-1)^{n+1}\theta']\cdot(\psi \wedge \psi),$$

where $\theta\colon\ V \wedge V \to S^{n+1}$ is a homotopy Seifert pairing. Theorem 8 implies that the *pairing u is, up to stable congruence, an invariant of the fibred knot* (S^{n+2}, k^n). Furthermore, a) u is a duality; b) $u' \sim (-1)^{n+1}u$; c) $u \circ (t \wedge t) \sim u$; d) t is an S-equivalence; e) $t - 1$ is an S-equivalence, where 1 denotes the identity map.

An *n-isometry* is a triple (L, u, t), where L is a finite polyhedron, $u\colon L \wedge L \to S^{n+1}$ is a continuous map, and $t\colon L \to L$ is an S-map satisfying a)–e). An n-isometry is said to be r-connected if L is r-connected. Two n-isometries (L_ν, u_ν, t_ν) ($\nu = 1$ or 2) are equivalent if there is an S-equivalence $f\colon L_1 \to L_2$ such that $u_1 \sim u_2 \circ (f \wedge f)$ and $t_2 \circ f$ is stably homotopic to $f \circ t_1$. The set of equivalence classes of r-connected n-isometries is denoted by $I_{r,n}$.

We saw above that each n-dimensional fibred knot defines an n-isometry $(\widetilde{X}, u, t)$. This is r-connected if the original knot belongs to $K_{r,n}$. Thus, denoting by the symbol $FK_{r,n}$ the set of equivalence classes of fibred knots in $K_{r,n}$, we obtain a map $FK_{r,n} \to I_{r,n}$.

THEOREM 9. *If* $3r \geqslant n + 1 \geqslant 6$, *this map is bijective.*

The proof uses the classification theorem, Theorems 7 and 8, and the following commutative diagram

$$\begin{array}{ccc} FK_{r,n} & \longrightarrow & K_{r,n} \\ \downarrow & \cdot & \downarrow \\ I_{r,n} & \longrightarrow & \Sigma_{r,n} \end{array},$$

in which the lower horizontal map sends the class of the n-isometry (L, u, t) to the R-equivalence class of the spherical pairing $\theta\colon\ L \wedge L \to S^{n+1}$, where $\theta = u \circ (1 \wedge (1 - t)^{-1})$. Here $(1 - t)^{-1}$ is a certain S-map inverse to $1 - t\colon L \to L$.

6. The algebraic classification of knots. The results set out above reduce the differential-topological problem of describing the isotopy types of stable knots to homotopy problems such as the problems of classifying spherical homotopy pairings with respect to R-equivalence and the classification of n-isometries. The difficulty of these homotopy problems increases sharply with $n - 2r$.

The simplest case is when $n - 2r = 1$. This corresponds to the knots studied by Levine [8]. Applying the classification theorem to this class of knots leads automatically to an algebraic classification in terms of Seifert matrices, which coincides essentially with Levine's classification [8]. The only difference is that we arrive at a slightly different (but equivalent) form of the equivalence relation between Seifert matrices. For fibred knots results are obtained similar to [13] (in the latter the concept of a "knot" is taken in a broader sense than here).

Let us explain the algebraic classification of fibred knots in $FK_{r,n}$ when $n - 2r = 2$. The stability condition is satisfied if $r \geqslant 3$. By Theorem 9, the isotopy types of such knots are in one-to-one correspondence with the equivalence classes of n-isometries (L, u, t), where L is an r-connected complex. Since u is a Spanier–Whitehead duality and $n = 2r + 2$, L can have only two non-zero homology groups, $H_{r+1}L$ and $H_{r+2}L$, and the latter must be free Abelian. Hence it follows that the complex L has the homotopy type of a one-point union of Moore spaces $M(H_{r+1}L, r+1) \vee M(H_{r+2}L, r+2)$. In particular, these two groups determine completely the homotopy type of L. The pairing u and the S-map t give a well-defined algebraic structure on these groups. Omitting the intermediate calculations we arrive at the resulting invariants.

The S-equivalence $t: L \to L$ defines the structure of a $\mathbf{Z}[t, t^{-1}]$-module on the group $A = \pi_{r+1}L$. Since $t - 1$ is an S-equivalence (this is part of the definition of an n-isometry), A can be regarded as a module over the ring $\Lambda = \mathbf{Z}[t, t^{-1}, (1-t)^{-1}]$.

The homotopy pairing u defines a bilinear form $l: T(A) \otimes_{\mathbf{Z}} T(A) \to \mathbf{Q}/\mathbf{Z}$, where $T(A) = \mathrm{Tor}_{\mathbf{Z}}A$, in the following way. Let $x, y \in T(A)$. Since L is r-connected, we can treat x and y as elements of $H_{r+1}L$. Let $z \in H_{r+2}(L; \mathbf{Q}/\mathbf{Z})$ be a certain class that goes into x under the Bockstein homomorphism corresponding to the extension $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$. We put $l(x, y) = (u^*s, z \wedge y) \in \mathbf{Q}/\mathbf{Z}$, where $s \in H^{2r+3}(S^{2r+3}; \mathbf{Z})$ is a fundamental class. This gives a well-defined form l.

Let us consider the group $B = \pi_{r+3}L$. The S-map t determines the structure of a Λ-module on it, and the homotopy pairing u gives a bilinear form $\psi: B \otimes_{\mathbf{Z}} B \to \sigma_3$, where σ_m denotes the m-th stable homotopy group of the spheres. The form ψ is defined as follows: if $b_\nu: S^{r+3} \to L$ ($\nu = 1$ or 2) are maps, then $\psi([b_1], [b_2])$ is the homotopy class of the composition

$$S^{2r+6} = S^{r+3} \wedge S^{r+3} \xrightarrow{b_1 \wedge b_2} L \wedge L \xrightarrow{u} S^{2r+3}.$$

There are Λ-homomorphisms

$$\alpha: A \otimes_{\mathbf{Z}} \sigma_2 \to B, \quad \beta: B \to \mathrm{Hom}_{\mathbf{Z}}(A, \sigma_1) = \bar{A},$$

of which the first is given by composition with a non-trivial element of $\sigma_2 = \pi_{r+3} S^{r+1}$ and the second as follows. Let b: $S^{r+3} \to L$ and a: $S^{r+1} \to L$ be continuous maps. Then $\beta([b])([a])$ is the homotopy class of the composition

$$S^{2r+4} = S^{r+3} \wedge S^{r+1} \xrightarrow{b \wedge a} L \wedge L \xrightarrow{u} S^{2r+3}.$$

Then β is a Λ-homomorphism if we introduce the following Λ-module structure in $\bar{A}$: $(tf)(x) = f(t^{-1}x)$, where $f \in \bar{A}$, $x \in A$.

It is not difficult to show that the sequence

$$0 \to A \otimes \sigma_2 \xrightarrow{\alpha} B \xrightarrow{\beta} \bar{A} \to 0$$

is exact. Hence, in particular, the exponent of B divides 4 and ψ takes values in $\mathbf{Z}_4 \subset \sigma_2$.

It can be proved that *the invariants we have constructed form a complete system. Thus, an r-connected* $(2r + 2)$*-isometry* (L, u, t) *is determined by the following algebraic objects*:

(1) *the* Λ*-module* A;

(2) *the* $\mathbf{Z}$*-homomorphism* l: $T(A) \otimes_{\mathbf{Z}} T(A) \to \mathbf{Q}/\mathbf{Z}$;

(3) *the* Λ*-extension*

$$E: 0 \to A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B \xrightarrow{\beta} \bar{A} \to 0,$$

where $\bar{A} = \mathrm{Hom}_{\mathbf{Z}}(A; \mathbf{Z}_2)$;

(4) *the* $\mathbf{Z}$*-homomorphism* ψ: $B \otimes_{\mathbf{Z}} B \to \mathbf{Z}_4$.

Here the following conditions are satisfied:

(a) *A is a finitely generated Abelian group*;

(b) *the form l is non-degenerate*;

(c) *the forms l and ψ are* $(-1)^r$*-symmetric*;

(d) *multiplication by $t \in \Lambda$ defines an isometry between l and ψ*;

(e) $\psi(\alpha(a_1), \alpha(a_2)) = 0$, $a_1, a_2 \in A \otimes \mathbf{Z}_2$;

(f) [1]) $\psi(\alpha(a), b) = \beta(b)(a)$, $a \in A \otimes \mathbf{Z}_2$, $b \in B$;

(g) *let $b \in B$; by* (b), *there is a unique element $a \in T(A)$ such that*[2] $\beta(b)(x) = l(a, x)$ *for any* $x \in T(A)$.

Then $2b = \alpha(a)$.

Every collection A, l, E, ψ satisfying (a)–(g) *can be realized by a certain r-connected* $(2r + 2)$*-isometry.* By Theorem 9, this implies the following result:

THEOREM 10. *The association of the fibred knot* (S^{2r+4}, k^{2r+2}) *with the* Λ*-module* $A = \pi_{r+1}(S^{2r+4} - k^{2r+2})$, *the form* l: $T(A) \otimes T(A) \to \mathbf{Q}/\mathbf{Z}$, *the* Λ*-extension* $0 \to A \otimes \mathbf{Z}_2 \to B \to \bar{A} \to 0$, *where* $B = \pi_{r+3}(S^{2r+4} - k)$, *and the form* ψ: $B \otimes B \to \mathbf{Z}_4$ *defines for* $r \geq 3$ *a bijection between the set of isotopy types of the knots* $FK_{r,2r+2}$ *and the set of isomorphism classes of objects* (1) *to* (4) *satisfying* (a)–(g).

A construction of the form l suitable for knots that are not fibred can be found in [14].

[1] It is understood that $\mathbf{Z}_2$ is embedded in $\mathbf{Z}_4$.

[2] It is understood that $\mathbf{Z}_2$ is embedded in $\mathbf{Q}/\mathbf{Z}$.

Knots of $K_{r,2r+2}$ having a module A without 2-torsion were studied by Kearton [15]. See also [18], where it is assumed that $T(A) = 0$.

The results of §§1 and 2 are given in greater detail in [16].

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