ON PRODUCTS OF COMPLEXES.*

By SAMUEL EILENBERG and J. A. ZILBER.

The objective of this note is to establish a theorem (stated in § 1) concerning the equivalence, from the point of view of homology, of two kinds of products that may be defined for complete semi-simplicial complexes (see below for a definition). The proof (§ 2) uses the method of acyclic models established in the paper [1] just preceding. Some applications are listed in § 3.

1. The theorem. We write $[m]$ for the set $(0, 1, \cdots, m)$ where m is an integer $\geqq 0$. By a map $\alpha : [m] \to [n]$ will be meant a function satisfying $\alpha(i) \leqq \alpha(j)$ for $0 \leqq i \leqq j \leqq m$.

A complete semi-simplicial (abbreviated: c. s. s.) complex K is a collection of "simplexes" σ, to each of which is attached a dimension $q \geqq 0$, such that for each q-simplex σ and each map $\alpha : [m] \to [q]$ $(m \geqq 0)$ there is defined an m-simplex $\sigma\alpha$ of K, subject to the conditions

(1) If $\epsilon_q : [q] \to [q]$ is the identity then $\sigma\epsilon_q = \sigma$,

(2) If $\beta : [n] \to [m]$ then $(\sigma\alpha)\beta = \sigma(\alpha\beta)$.

Let $\epsilon_q{}^i : [q-1] \to [q]$ be the map which covers all of $[q]$ except i $(= 0, \cdots, q)$. Then $\sigma\epsilon_q{}^i$ is called the i-th *face* of σ, and the boundary of σ is defined as the chain

$$\partial\sigma = \sum_{i=0}^{q} (-1)^i \sigma\epsilon_q{}^i.$$

If K and L are c. s. s. complexes, a function $f : K \to L$ mapping q-simplexes into q-simplexes and such that $f(\sigma\alpha) = (f\sigma)\alpha$ is called a c. s. s. map. For further details see [3, § 8].

Let K and L be two c. s. s. complexes. The *cartesian product* $K \times L$ is a c. s. s. complex whose q-simplexes are pairs (σ, τ) where σ and τ are q-simplexes of K and L respectively. For each map $\alpha : [m] \to [q]$ we define $(\sigma, \tau)\alpha = (\sigma\alpha, \tau\alpha)$.

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The *tensor* product $K \otimes L$ is an abstract cell complex with r-cells $\sigma \otimes \tau$ where σ is a p-simplex of K, τ a q-simplex of L with $p+q=r$ and

$$\partial(\sigma \otimes \tau) = \partial\sigma \otimes \tau + (-1)^p \sigma \otimes \partial\tau.$$

Both $K \times L$ and $K \otimes L$ may be regarded as chain complexes and may be compared by means of chain transformations and chain homotopies.

THEOREM. *For any two complete semi-simplicial complexes K and L, there exist chain transformations*

$$f : K \times L \to K \otimes L, \qquad g : K \otimes L \to K \times L$$

and chain homotopies

$$D : gf \cong \textit{identity}, \qquad E : fg \cong \textit{identity}$$

such that for 0-simplexes $\sigma \in K$, $\tau \in L$,

$$f(\sigma, \tau) = \sigma \otimes \tau, \quad g(\sigma \otimes \tau) = (\sigma, \tau), \quad D(\sigma, \tau) = 0, \quad E(\sigma \otimes \tau) = 0.$$

Moreover, f, g, D and E are natural in the following sense. Let $\phi : K \to K'$, $\psi : L \to L'$ be c. s. s. maps. We consider the induced maps

$$\phi \times \psi : K \times L \to K' \times L', \qquad (\phi \times \psi)(\sigma, \tau) = (\phi\sigma, \psi\tau),$$

$$\phi \otimes \psi : K \otimes L \to K' \otimes L', \qquad (\phi \otimes \psi)(\sigma \otimes \tau) = \phi\sigma \otimes \psi\tau.$$

Then these maps commute properly with f, g, D, E. For example, the diagram

$$\begin{array}{ccc} K \times L & \xrightarrow{\ \phi \times \psi\ } & K' \times L' \\ \downarrow f & & \downarrow f \\ K \otimes L & \xrightarrow[\ \phi \otimes \psi\ ]{} & K' \otimes L' \end{array}$$

is commutative.

2. Proof of the theorem. For each integer $m \geqq 0$ we define a c. s. s. complex $K[m]$ as follows. A q-simplex of $K[m]$ is any map $\sigma : [q] \to [m]$. For each map $\alpha : [n] \to [q]$, $\sigma\alpha$ is defined as the composite map.

Let $\mathcal{A}$ be the category whose objects are pairs (K, L), where K and L are c. s. s. complexes. A map $(\phi, \psi) : (K, L) \to (K', L')$ *in* $\mathcal{A}$ is a pair of c. s. s. maps $\phi : K \to K'$, $\psi : L \to L'$. Composition is defined by $(\phi', \psi')(\phi, \psi) = (\phi'\phi, \psi'\psi)$ whenever $\phi'\phi$ and $\psi'\psi$ are defined.

In $\mathcal{A}$ we consider the set $\mathfrak{M}$ of models consisting of all pairs $(K[m], K[n])$.

On $\mathcal{A}$ we define two covariant functors P and Q with values in the category of chain complexes as follows. $P(K, L)$ (resp. $Q(K, L)$) is the chain complex obtained from $K \times L$ (resp. $K \otimes L$) by adjoining the group of integers as group of chains in dimension -1 with $\partial(\sigma \times \tau) = 1$ (resp. $\partial(\sigma \otimes \tau) = 1$) for 0-simplexes $\sigma \varepsilon K$, $\tau \varepsilon L$. The maps $P(\phi, \psi)$ (resp. $Q(\phi, \psi)$) are defined as extensions of $\phi \times \psi$ (resp. $\phi \otimes \psi$) obtained by keeping the chains of dimension -1 (i. e. the integers) pointwise fixed.

We first show that for each dimension $r \geqq 0$ the functors P_r and Q_r are representable. If σ is an n-simplex in a c. s. s. complex K, then we denote by ϕ_σ the map $\phi_\sigma : K[n] \to K$ defined for each α in $K[n]$ as $\phi_\sigma \alpha = \sigma\alpha$. In particular $\phi_\sigma \epsilon_n = \sigma$. With these definitions it is clear that the maps

$$\sigma \times \tau \to ((\phi_\sigma, \phi_\tau), \epsilon_r \times \epsilon_r),\ \dim \sigma = \dim \tau = r$$

$$\sigma \otimes \tau \to ((\phi_\sigma, \phi_\tau), \epsilon_p \otimes \epsilon_q),\ \dim \sigma = p, \dim \tau = q, p + q = r$$

yield representations of the functors P_r and Q_r.

Next we prove that the homology groups of the complexes $P(K[m], K[n])$ and $Q(K[m], K[n])$ are all trivial.

For any map $\alpha : [q] \to [r]$ we define a map $F(\alpha) : [q+1] \to [r]$ by setting

$$F(\alpha)(0) = 0, \qquad F(\alpha)(i) = \alpha(i-1) \text{ for } i = 1, \cdots, q+1.$$

Further, we define $\theta_r : [0] \to [r]$ by $\theta_r(0) = 0$.

Then

$$F(\alpha)\epsilon_{q+1}{}^0 = \alpha$$

$$F(\alpha)\epsilon_{q+1}{}^i = F(\alpha\epsilon_q{}^{i-1}) \qquad q > 0, i = 1, \cdots, q+1$$

$$F(\alpha)\epsilon_1{}^1 = \theta_r \qquad q = 0.$$

Next, we define in $P(K[m], K[n])$ and $Q(K[m], K[n])$ homotopy operators G and H as follows:

$$G(\sigma, \tau) = (F(\sigma), F(\tau)), \qquad G(1) = (\theta_m, \theta_n),$$

$$H(\sigma \otimes \tau) = F(\sigma) \otimes \tau \text{ if } \dim \sigma > 0,$$

$$H(\sigma \otimes \tau) = F(\sigma) \otimes \tau + \theta_m \otimes F(\tau) \text{ if } \dim \sigma = 0,$$

$$H(1) = \theta_m \otimes \theta_n.$$

A simple calculation, using the face formulae for $F(\alpha)$ shows that $\partial G + G\partial$

and $\partial H + H\partial$ are identity operators. This proves the assertion concerning the triviality of the homology groups.

The remainder of the proof is now a direct application of Theorem II of [1]. We define the maps $f: P \to Q$ and $g: Q \to P$ in dimension -1 by $f(1) = 1 = g(1)$ and in dimension zero by

$$f(\sigma, \tau) = \sigma \otimes \tau, \qquad g(\sigma \otimes \tau) = (\sigma, \tau).$$

Then f and g can be extended to maps defined in all dimensions. Since gf and fg coincide with the identity maps in dimensions < 1, the homotopies D and E required by the theorem, also exist in virtue of Theorem II of [1].

Although the proof given here appears to be purely existential, using the representations given for the functors P_r and Q_r and using the homotopies G and H above, explicit formulae for f, g, D and E may be readily found. Such formulae will be found in [2].

3. Applications. Let $X \times Y$ be the cartesian product of two topological spaces X and Y. A q-dimensional singular simplex in $X \times Y$ defines by projection a singular q-simplex in X and one in Y. Conversely a pair of singular q-simplexes one in X and one in Y determine a singular q-simplex in $X \times Y$. It follows that the total singular complex $S(X \times Y)$ (which is a c. s. s. complex; see [3, § 8]), may be identified with the product $S(X) \times S(Y)$. Thus the theorem allows us to assert that from the point of view of homology $S(X \times Y)$ is equivalent with $S(X) \otimes S(Y)$.

Let A and B be subspaces of X and Y respectively. We write $S(X, A)$ for the quotient of $S(X)$ by its subcomplex $S(A)$. Since the maps and homotopies asserted in the theorem are natural, it follows that the relative homology groups

(1) $$H_q(S(X \times Y)/S(A \times Y) \cup S(X \times B))$$

and

(2) $$H_q(S(X, A) \otimes S(Y, B))$$

are isomorphic. We consider the triple of complexes

$$(S(X \times Y), S(A \times Y \cup X \times B), S(A \times Y) \cup S(X \times B)).$$

If all the homology groups

(3) $$H_q(S(A \times Y \cup X \times B)/S(A \times Y) \cup S(X \times B))$$

are trivial, then it follows from the exactness of the homology sequence of the triple above that the groups (1) are isomorphic with

$$H_q(X \times Y, A \times Y \cup X \times B). \tag{4}$$

Thus in this case (2) and (4) are isomorphic.

Our second application concerns the simplicial product of simplicial complexes. Let K and L be simplicial complexes. The *simplicial product* $K \triangle L$ has as vertices pairs (A, B) of vertices $A \varepsilon L$, $B \varepsilon L$. A set $(A^0, B^0), \cdots, (A^n, B^n)$ of vertices of $K \triangle L$ forms a simplex of $K \triangle L$ if and only if $A^0, \cdots, A^n$ are in a simplex of K and $B^0, \cdots, B^n$ are in a simplex of L.

With each simplicial complex K we associate a c. s. s. complex $O(K)$ as follows. The q-simplexes of $O(K)$ are sequences $A^0 \cdots A^q$ of vertices of K contained in a simplex of K. For each map $\alpha: [m] \to [q]$ we define $(A^0 \cdots A^q)\alpha = A^{\alpha(0)} \cdots A^{\alpha(m)}$. The homology theories of K and $O(K)$ are equivalent.

With these definitions it is easy to see that $O(K \triangle L) = O(K) \times O(L)$. Thus the theorem of this paper asserts that $O(K \triangle L)$ and $O(K) \otimes O(L)$ are homologically equivalent. It follows that the homology theories of $K \triangle L$ and of $K \otimes L$ (regarding K and L as chain complexes) are equivalent.

This result may be applied in the following situation. Let U and V be coverings of spaces X and Y respectively and let $U \times V$ be the "product" covering of $X \times Y$. Then it is easy to verify the following relation between the nerves of these coverings: $N(U \times V) = N(U) \triangle N(V)$. It follows that that $N(U \times V)$ is homologically equivalent with $N(U) \otimes N(V)$.

COLUMBIA UNIVERSITY,
THE JOHNS HOPKINS UNIVERSITY.

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