Journal of Knot Theory and Its Ramifications, Vol. 9, No. 5 (2000) 611–615


# ON WHITE'S FORMULA

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Received 1 September 1999

**Abstract**

A smooth knot is non-trivial precisely if it cannot be unlinked from every small deformation of itself. White's formula expresses the linking number of a representative of a knot and a deformation of it as a sum of two, not necessarily integral, terms – a writhe and a twist. An elementary proof is given of this formula. A byproduct of the proof is an easy derivation of the expression for the writhe of the knot in terms of standard invariants of the curve and the writhe of the knot diagram obtained by projection onto a plane.

## 1 Statement of White's Formula

Let $K$ be a smooth knot in $\mathbb{R}^3$ and $k : S \to \mathbb{R}^3$, where $S$ is a circle, be the smooth, injective, arc length preserving map that parametrises $K$. Let $v$ be a smooth non-zero vector field defined on a neighbourhood of $K$. In White's formula one considers the case where at each point $k(s)$ of $K$ the vector $v(s)$ has unit length and is perpendicular to the unit tangent $k'(s)$ of $K$. One further supposes that the scale has been chosen sufficiently small so that the intervals in $\mathbb{R}^3$ that join $k(s)$ to $k(s)+v(s)$ are disjoint for distinct $s \in S$. These intervals form a ribbon, as $s$ varies, with edges $K$ and $K_v$ (say).

The **linking number** of $K$ and $K_v$ is given by the well-known formula of Gauss

$$Lk(K, K_v) = -\frac{1}{4\pi} \iint\limits_{S\times S} g_1^*(w) ,$$

where $g_1 : S \times S \to \mathbb{R}^3 \setminus \{0\}$ is given by $g_1(s,t) = k(s) - k(t) - v(t)$ and where $w$ is the 2-form $(x_1^2+x_2^2+x_3^2)^{-\frac{3}{2}}(x_1 dx_2{\wedge}dx_3 - x_2 dx_1{\wedge}dx_3 + x_3 dx_1{\wedge}dx_2)$ on $\mathbb{R}^3 \setminus \{0\}$. The **twisting number** $Tw(K, v)$ is defined to be

$$Tw(K,v) = \frac{1}{2\pi} \int\limits_S (k'(s) \times v(s)).v'(s) ds .$$

Using the Serret-Frenet equations one can rewrite this definition as the formula, which first appeared in [3],