3. A. Grothendieck (notes by R. Hartshorne), "Local cohomology, Lecture Notes in Math.," Vol. *41*, Springer-Verlag, Berlin, 1967.
4. E. Hamann, On the R-invariance of $R[x]$, Thesis, University of Minnesota, 1973.
5. M. Hochster, Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, *Ann. of Math.* **96** (1972), 318–337.
6. M. Hochster and J. L. Roberts, Actions of reductive groups on regular rings and Cohen–Macaulay rings, *Bull. Amer. Math. Soc.* **80** (1974), 281–284.
7. M. Hochster and J. L. Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen–Macaulay, *Advances in Math.* **13** (1974), 115–175.
8. E. Kunz, Characterization of regular local rings of characteristic p, *Amer. J. Math.* **91** (1969), 772–784.
9. M. Nagata, "Local rings," Interscience, New York, 1962.
10. G. Reisner, Cohen–Macaulay quotients of polynomial rings, Thesis, University of Minnesota, 1974.
11. I. Reiten, The converse to a theorem of Sharp on Gorenstein modules, *Proc. Amer. Math. Soc.* **32** (1972), 417–420.
12. J. Roberts, Generic projections of algebraic varieties, *Amer. J. Math.* **93** (1971), 193–214.
13. J. P. Serre, Faisceaux algébriques cohérents, *Ann. of Math.* **61** (1955), 197–278.
14. R. Stanley, Cohen–Macaulay rings and constructible polytopes, MIT Preprint.
15. R. Stanley, The upper bound conjecture and Cohen–Macaulay rings, MIT Preprint.
16. D. Taylor, Ideals generated by monomials in an R-sequence, Theis, University of Chicago, 1966.
17. C. Traverso, Seminormality and Picard group, *Ann. Scuola Norm. Sup. Pisa* **24**, No. 3 (1970), 585–595.
18. R. B. Warfield, Jr., Purity and algebraic compactness for modules, *Pacific J. Math.* **28** (1969), 699–719.

ADVANCES IN MATHEMATICS **21**, 173–195 (1976)

On the Morse Index in Variational Calculus

J. J. Duistermaat

Mathematisch Instituut der Rijksuniversiteit, Budapestlaan 6, De Uithof, Utrecht, The Netherlands

Communicated by M. Atiyah

Introduction

The main purpose of this paper is to show that the intersection theory of curves of Lagrange spaces is a very flexible tool in the study of the Morse index in variational calculus.

In Section 1 the stage is set with a brief review of the classical translation of the Morse index into the number of negative eigenvalues of a Sturm–Liouville problem. After a translation into Hamilton systems using the Legendre transformation, this can then be read as the intersection number of a curve of Lagrange spaces $\mu \mapsto \operatorname{graph} \Phi(\mu, T)$, μ running from -1 to 0, with a fixed one ρ. Here $\Phi(\mu, t)\colon T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is the solution matrix of the linear Hamilton system with eigenvalue parameter μ, T is the final time, and ρ is a Lagrange space in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ determined by the (arbitrary) boundary condition in the variational problem.

In Section 2 the main properties of the intersection number of curves of Lagrange spaces $\omega(t)$ with a fixed one α are collected. Firstly, it follows from the results of Arnol'd [1] that it is invariant under a homotopy of ω, keeping the initial and end-point of ω (which do not intersect α) fixed. Secondly, replacing α by another Lagrange space α' changes the intersection number by an integer, computed by Hörmander [10] and given explicitly in terms of the signatures of some quadratic forms defined by α, α', and only the initial and end-point of ω. This allows for the definition of an index of ω as the intersection number with α plus a correction term making it independent of α.

In Section 3 it is shown that for any curve of symplectic transformations $\Phi(t)$ and any Lagrange space V in $T^*\mathbb{R}^n$, the index of the curve $t \mapsto \operatorname{graph} \Phi(t)$ of Lagrange spaces in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ is equal to the index of the curve $t \mapsto \Phi(t)^{-1}(V)$ of Lagrange spaces in $T^*\mathbb{R}^n$, plus an integer $j(\Phi(T), V)$ depending only on $\Phi(T)$ and V, generalizing the "order of concavity" of Morse [12].

Applying a homotopy argument, the Morse index is expressed in Section 4 as the index of the curve $t \mapsto \text{graph } \Phi(0, t)$, t running from 0 to T, plus a correction term which can be computed from $\Phi(0, T)$ and the boundary condition of the original variational problem. Because $\Phi(0, t)$ is related to the differential of the Euler–Lagrange flow by means of the Legendre transformation, this can be regarded as a geometric formula for the index.

The intersection number of $t \mapsto \Phi(0, t)^{-1}(V)$ with a fixed Lagrange space U, related to the index of $t \mapsto \text{graph } \Phi(0, t)$ according to the formula of Section 3, can be interpreted as a number of focal points along the stationary curve if V = vertical space. It is equal to the number of conjugate points if in addition $U = V$.

Section 4 is concluded with a short discussion of the fixed end-point and the periodic boundary condition, thus recovering some classical formulas of Morse [12]. For the periodic boundary condition a special choice of U adapted to $\Phi(T)$ leads to the formula of Klingenberg [11] (obtained in the Riemannian case and with a much more direct method).

This paper originated in an attempt to identify the exponent in a power of i occurring in the asymptotic expansions of [8] for an elliptic operator on a compact manifold, with a Morse index for periodic geodesics. This relation was suggested by similar expansions of Colin de Verdière [5] for the Laplace operator on a Riemannian manifold, where the exponent of i was equal to this Morse index almost by construction.

If $\Phi(t)$ is the solution matrix of any periodic linear Hamilton system with period T, then the index of graph $\Phi(t)$, t running from 0 to $k \cdot T$ (k an integer) can be expressed in terms of the index of graph $\Phi(t)$, t running from 0 to T, the number k and the normal form of the real symplectic linear transformation $\Phi(T)$. This application of the intersection theory will be worked out in a subsequent paper [6] with R. Cushman. It generalizes the formulas obtained by Bott [3] and Klingenberg [11] in the sense that no positivity assumptions are made for the Hamilton system. However, their results are formulated in the framework of hermitian forms of signature 0, for which an intersection number has been introduced by Edwards [13, Sect. 4] in the same fashion as here, but without using the analog of Hörmander's signature number. Some comments on the relation between the real symplectic theory and the hermitian one will be given in the final section of [6].

I finally would like to thank A. Weinstein, W. Klingenberg, and R. Cushman for stimulating discussions on this subject.

1. Some Classical Translations of the Morse Index

Let X be a smooth manifold of dimension n, points in its tangent bundle TX will be denoted by (x, v), with $x \in X$, $v \in T_xX$. Let f be a real-valued smooth function on an open subset Z of $\mathbb{R} \times TX$. Then

$$E(c) = \int_0^T f\left(t, c(t), \frac{dc}{dt}(t)\right) dt \tag{1.1}$$

defines a real-valued smooth function E on the space of curves

$$\mathscr{C} = \left\{c \in C^1([0, T], X);\ \left(t, c(t), \frac{dc}{dt}(t)\right) \in Z \text{ for all } t \in [0, T]\right\}. \tag{1.2}$$

$\mathscr{C}$ is a smooth Banach manifold modeled on the Banach space $C^1([0, T], \mathbb{R}^n)$ with its usual topology of uniform convergence of the curves and their derivatives.

Boundary conditions will be introduced by restricting E to the set of curves

$$\mathscr{C}_R = \{c \in \mathscr{C};\ (c(0), c(T)) \in R\}, \tag{1.3}$$

here R is a given smooth submanifold of $X \times X$. The most familiar examples are $R = \{(x_0, x_T)\}$ and $R = \{(x, y) \in X \times X;\ x = y\}$. In the first case $\mathscr{C}_R$ is the space of curves with prescribed initial- and end-point, in the second case $\mathscr{C}_R$ is the space of closed curves. In the general case $\mathscr{C}_R$ is a smooth submanifold of $\mathscr{C}$ of codimension equal to the codimension of R in $X \times X$, with tangent space equal to

$$T_c\mathscr{C}_R = \{\delta c \in C^1([0, T], c^*TX);\ (\delta c(0), \delta c(T)) \in T_{(c(0), c(T))}R\}. \tag{1.4}$$

$c \in \mathscr{C}_R$ is called a stationary curve for the boundary relation R if the restriction of E to $\mathscr{C}_R$ has a stationary point at c, that is, if $DE(c)(\delta c) = 0$ for all $\delta c \in T_c\mathscr{C}_R$. For such a curve the second-order differential $D^2E(c)$ of E at c is an invariantly defined symmetric bilinear form on

$T_c\mathscr{C}_R$. Now the *Morse index of the stationary curve c for the boundary relation R* is defined as

$$i_R(c) = \sup\{\dim L;\, L \text{ is a linear subspace of } T_c\mathscr{C}_R \text{ on which } D^2E(c) \text{ is negative definite}\}. \tag{1.5}$$

It is a classical result that $i_R(c)$ can only be finite if $D_v^2 f(\tilde{c}(t))$ is positive semidefinite for all $t \in [0, T]$, and that conversely

$$D_v^2 f(\tilde{c}(t)) \text{ is positive definite for all } t \in [0, T] \tag{1.6}$$

implies that $i_R(c)$ is finite. Here D_v denotes differentiation of functions on Ω with respect to $v \in T_xX$, keeping t and x fixed, and we have used the abbreviation

$$\tilde{c}(t) = \left(t, c(t), \frac{dc}{dt}(t)\right). \tag{1.7}$$

(1.6) is called the *sufficient condition of Legendre.*

We repeat the proof of (1.6) $\Rightarrow i_R(c) < \infty$ briefly here, because it contains a translation of the Morse index which will be used in the sequel. For convenience we also reduce the computations of the differentials of E to the case that X is an open subset of $\mathbb{R}^n$. This can be done for instance by introducing a smooth map $\Gamma\colon [0, T] \times Y \to X$, Y an open neighborhood of 0 in $\mathbb{R}^n$, such that $y \mapsto \Gamma(t, y)$ is a diffeomorphism from Y to an open neighborhood of $c(t)$ in X, mapping 0 to $c(t)$, for each $t \in [0, T]$. The t-dependent substitution of variables $x = \Gamma(t, y)$ then induces a diffeomorphism between a neighborhood of c in $\mathscr{C}$ and a neighborhood of the zero curve in $\mathbb{R}^n$.

Let $q(t)$ be a positive definite symmetric bilinear form on $\mathbb{R}^n$, depending continuously on $t \in [0, T]$, such that

$$\begin{pmatrix} q(t) + D_x^2 f(\tilde{c}(t)) & D_v D_x f(\tilde{c}(t)) \\ D_x D_v f(\tilde{c}(t)) & D_v^2 f(\tilde{c}(t)) \end{pmatrix} \text{ is positive definite for all } t \in [0, T]. \tag{1.8}$$

(The possibility of finding such $q(t)$ is equivalent to (1.6).)

Writing

$$Q(\delta c, \delta c') = \int_0^T q(t)(\delta c(t), \delta c'(t))\, dt \tag{1.9}$$

it follows that Q is an inner product on $T_c\mathscr{C}_R$ inducing the L^2-topology, and that $Q + D^2E(c)$ is an inner product on $T_c\mathscr{C}_R$ inducing the $H^{(1)}$-

topology of L^2-convergence of the curves and their derivatives. Let $H_R{}^0$, resp. $H_R{}^1$ be the completion of $T_c\mathscr{C}_R$ with respect to Q, resp. $Q + D^2E(c)$.

Now define the linear operator in $H_R{}^1$ by

$$D^2E(c) = (Q + D^2E(c)) \circ \mathscr{E}; \tag{1.10}$$

here bilinear forms are regarded as mappings from the vector space to its dual. Then $\mathscr{E} = I - K$, with $(Q + D^2E(c)) \circ K = Q$, so K is continuous: $H_R{}^0 \to H_R{}^1$ and therefore compact as an operator in $H_R{}^0$ in view of Ascoli's theorem. It follows that K is a symmetric positive operator in $H_R{}^1$ with a discrete spectrum converging to 0, so $\mathscr{E}$ is a symmetric operator in $H_R{}^1$ with a discrete spectrum $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ such that $\lambda_j \nearrow 1$ as $j \to \infty$. In particular the sum E^- of the eigenspaces of $\mathscr{E}$ for the negative eigenvalues is finite-dimensional and $\mathscr{E} \geqslant 0$ on the orthogonal complement E^+ of E^- in $H_R{}^1$.

We conclude that $H_R{}^1 = E^- \oplus E^+$, $D^2E(c) < 0$ on E^- and $D^2E(c) \geqslant 0$ on E^+. If L is another linear subspace of $H_R{}^1$ on which $D^2E(c) < 0$ then the linear projection to E^- along E^+ is injective on L, so $\dim L \leqslant \dim E^-$. Because $T_c\mathscr{C}_R$ is dense in $H_R{}^1$ one can find a linear subspace L of $T_c\mathscr{C}_R$ such that $\dim L = \dim E^-$ and $D^2E(c) < 0$ on L (we will see below that in fact $E^- \subset T_c\mathscr{C}_R$), so we have proved:

LEMMA 1.1. *If (1.6) holds then $i_R(c)$ is equal to the number of negative eigenvalues of $\mathscr{E}$, counted with multiplicity, and this number is finite.*

Straightforward calculations, involving a partial integration with respect to t, show that $c \in \mathscr{C}_R$ is a stationary curve for the boundary relation R if and only if

$$\frac{d}{dt}(D_v f(\tilde{c}(t))) = D_x f(\tilde{c}(t)) \qquad (\textit{Euler–Lagrange}) \tag{1.11}$$

and

$$(D_v f(\tilde{c}(0)), -D_v f(\tilde{c}(T))) \in (T_{(c(0), c(T))}R)^\perp. \tag{1.12}$$

Moreover, for such a curve c, and $\delta c \in T_c\mathscr{C}_R$, we have $\mathscr{E}(\delta c) = \lambda \cdot \delta c$ if and only if

$$\frac{d}{dt}\left[D_x D_v f(\tilde{c}(t)) \cdot \delta c(t) + D_v^2 f(\tilde{c}(t)) \cdot \frac{d(\delta c)}{dt}(t)\right] = [D_x^2 f(\tilde{c}(t)) - \mu \cdot q(t)] \cdot \delta c(t) + D_v D_x f(\tilde{c}(t)) \cdot \frac{d(\delta c)}{dt}(t), \tag{1.13}$$

with $\mu = \lambda/(1-\lambda)$, and

$$\left(\begin{pmatrix}\delta c(0)\\ \frac{d(\delta c)}{dt}(0)\end{pmatrix}, \begin{pmatrix}\delta c(T)\\ \frac{d(\delta c)}{dt}(T)\end{pmatrix}\right) \in T_{\left(\begin{pmatrix}c(0)\\ \frac{dc}{dt}(0)\end{pmatrix}, \begin{pmatrix}c(T)\\ \frac{dc}{dt}(T)\end{pmatrix}\right)} \tilde{R}, \tag{1.14}$$

with

$$\tilde{R} = \left\{\left(\begin{pmatrix}x\\ v\end{pmatrix}, \begin{pmatrix}y\\ w\end{pmatrix}\right); (x, y) \in R, (D_v f(0, x, v), -D_v f(T, y, w)) \in (T_{(x,y)}R)^{\perp}\right\}.$$

The differential equations (1.11), (1.13) hold in distribution sense, but their solutions are automatically smooth and satisfy the equations in the classical sense because $D_v^2 f$ is nondegenerate. The equations (1.13), (1.14) for δc are known in the literature as a *Sturm–Liouville problem.* Note that (1.6) implies that it has only nonzero solutions δc for $\mu > -1$ ($\Leftrightarrow \lambda < 1$) and that the sum of the dimensions of the solution spaces for $-1 < \mu < 0$ ($\Leftrightarrow \lambda < 0$) is finite. For $\mu = 0$ the equations (1.13), (1.14) are just the variational equations of (1.11), (1.12).

In many cases the choice (1.2) of the space of curves on which the function E is studied is not the most appropriate one. For instance if $E(t, x, v) = \| v \|_x^2$ for a Riemannian structure on X, then it is more natural to define E on the Hilbert manifold of $H^{(1)}$-curves in X, as in Flaschel and Klingenberg [9]. However, if $\mathscr{H}_R^1$ is the subset of the $H^{(1)}$-curves satisfying the boundary condition R, then $c \in \mathscr{H}_R^1$ is a stationary point for $E: \mathscr{H}_R^1 \to \mathbb{R}$ if and only if (1.11), (1.12) hold, that is, $c \in \mathscr{C}_R$ and c is a stationary point for $E: \mathscr{C}_R \to \mathbb{R}$. Moreover, $T_c\mathscr{H}_R^1 = H_R^1$ and the definition of the Morse index does not change if we replace $\mathscr{C}_R$ by $\mathscr{H}_R^1$. One can also replace $\mathscr{C}_R$ by the finite-dimensional manifold $\mathscr{G}_R^{(k)}$ of broken geodesics with k corners satisfying the boundary relation R, and conclude that the Morse index of $E: \mathscr{G}_R^{(k)} \to \mathbb{R}$ is the same as the one defined above if k is sufficiently large. This follows from the observation that the spaces $T_c\mathscr{G}_R^{(k)}$, $k = 1, 2, \ldots$ form an increasing sequence of linear subspaces of H_R^1, the union of which is dense in H_R^1, so one of these subspaces contains a linear subspace L with $\dim L = \dim E^-$ and $D^2E(c) < 0$ on L.

We conclude this section by another classical translation, called the *Legendre transformation.* Consider the mapping

$$\mathscr{L}: (t, x, v) \mapsto (t, x, D_v f(t, x, v)). \tag{1.15}$$

The condition that $D_v^2 f$ is nondegenerate means that $\mathscr{L}$ is a smooth covering map from Ω to an open subset of $\mathbb{R} \times T^*X$, here T^*X denotes the cotangent bundle of X. Because coverings induce local diffomorphisms between the corresponding spaces of curves, we will make the abuse of notation of treating $\mathscr{L}$ as if it were a global diffeomorphism.

Let $v = v(t, x, \xi)$ be the solution of $\xi = D_v f(t, x, v)$. Define

$$p(t, x, \xi) = \langle v(t, x, \xi), \xi\rangle - f(t, x, v(t, x, \xi)). \tag{1.16}$$

Writing $\xi(t) = D_v f(t, x(t), v(t))$, $x(t) = c(t)$, the equations $c \in \mathscr{C}_R$, $dx/dt(t) = v(t)$, (1.11) and (1.12) are equivalent to

$$\frac{dx}{dt}(t) = D_\xi p(t, x(t), \xi(t)), \qquad \frac{d\xi}{dt}(t) = -D_x p(t, x(t), \xi(t)), \qquad (\textit{Hamilton equations}) \tag{1.17}$$

with boundary conditions

$$(x(0), x(T)) \in R, (\xi(0), -\xi(T)) \in (T_{(x(0),x(T))}R)^{\perp}. \tag{1.18}$$

Writing $\delta\xi(t) = D_x D_v f(\tilde{c}(t)) \cdot \delta x(t) + D_v^2 f(c(t)) \cdot \delta v(t)$, $x(t) = c(t)$, $\delta x(t) = \delta c(t)$, the equations $d(\delta x)/dt = \delta v$, $\delta c \in H_R^1$, (1.13) and (1.14) are equivalent to

$$\frac{d}{dt}\begin{pmatrix}\delta x(t)\\ \delta\xi(t)\end{pmatrix} = A(\mu, t)\begin{pmatrix}\delta x(t)\\ \delta\xi(t)\end{pmatrix}, \tag{1.19}$$

with

$$A(\mu, t) = \begin{pmatrix} D_x D_\xi p(t, x(t), \xi(t)) & D_\xi^2 p(t, x(t), \xi(t)) \\ -\mu q(t) - D_x^2 p(t, x(t), \xi(t)) & -D_\xi D_x p(t, x(t), \xi(t))\end{pmatrix} \tag{1.20}$$

and with boundary conditions

$$\left(\begin{pmatrix}\delta x(0)\\ \delta\xi(0)\end{pmatrix}, \begin{pmatrix}\delta x(T)\\ \delta\xi(T)\end{pmatrix}\right) \in \rho,$$

with

$$\rho = \text{tangent space at } \left(\begin{pmatrix}x(0)\\ \xi(0)\end{pmatrix}, \begin{pmatrix}x(T)\\ \xi(T)\end{pmatrix}\right)$$

$$\text{of the manifold } \left\{\left(\begin{pmatrix}x\\ \xi\end{pmatrix}, \begin{pmatrix}y\\ \eta\end{pmatrix}\right); (x, y) \in R, (\xi, -\eta) \in (T_{(x,y)}R)^{\perp}\right\}. \tag{1.21}$$

607/21/2-5

Let $\Phi(\mu, t)$ be the fundamental solution of (1.19), that is,

$$\frac{\partial \Phi}{\partial t}(\mu, t) = A(\mu, t) \circ \Phi(\mu, t), \qquad \Phi(\mu, 0) = I = \text{identity in } T^*\mathbb{R}^n. \tag{1.22}$$

Then we have translated Lemma 1.1 into

$$i_R(c) = \sum_{-1<\mu<0} \dim(\text{graph } \Phi(\mu, T) \cap \rho). \tag{1.23}$$

The mapping $A(\mu, t)$ is an infinitesimal symplectic transformation in $T^*\mathbb{R}^n$ for the canonical symplectic form $\sigma = \sigma_{T^*\mathbb{R}^n}$ in $T^*\mathbb{R}^n$ defined by

$$\sigma\left(\begin{pmatrix}\delta x\\ \delta\xi\end{pmatrix}, \begin{pmatrix}\delta x'\\ \delta\xi'\end{pmatrix}\right) = \langle \delta\xi, \delta x'\rangle - \langle \delta x, \delta\xi'\rangle. \tag{1.24}$$

This means that $\sigma(A(\mu, t) \cdot u, v) + \sigma(u, A(\mu, t) \cdot v) = 0$ for all $u, v \in T^*\mathbb{R}^n$. It follows that $\Phi(\mu, t)$ is a symplectic transformation in $T^*\mathbb{R}^n$ for all t, that is $\sigma(\Phi(\mu, t) \cdot u, \Phi(\mu, t) \cdot v) = \sigma(u, v)$ for all $u, v \in T^*\mathbb{R}^n$. This can also be expressed by saying that the symplectic form

$$\sigma = \sigma_{\text{first factor}} \oplus -\sigma_{\text{second factor}} \tag{1.25}$$

in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ vanishes on the graph of $\Phi(\mu, t)$. A $2n$-dimensional linear subspace of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ on which σ vanishes is called a *Lagrange subspace* of the symplectic vector space $(T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \sigma)$. It is easily verified that ρ is also a Lagrange subspace of $(T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \sigma)$. Indeed, the normal bundle in $T^*X \times T^*X \cong T^*(X \times X)$ of $R \subset X \times X$ is a Lagrange manifold for $\sigma_{\text{first factor}} \oplus \sigma_{\text{second factor}}$ (cf. [7, Sect. 3.7]), and ρ is obtained from its tangent space by a flip of sign in the fiber of the second factor T^*X. In this way the Morse index is reduced to a number of intersections of a curve of Lagrange spaces with a fixed one.

2. Intersection Theory for Curves of Lagrange Spaces

Let E be a real vector space and let σ be a symplectic form on E, that is, σ is a nondegenerate antisymmetric bilinear form on E. For a linear subspace α of E write

$$\alpha^\sigma = \{v \in E; \sigma(u, v) = 0 \text{ for all } u \in \alpha\} \tag{2.1}$$

for the orthogonal complement of α with respect to σ. α is called *isotropic*

if $\alpha \subset \alpha^\sigma$ and α is called a *Lagrange subspace* of the symplectic vector space (E, σ) if it is maximal with this property. The set of Lagrange subspaces of (E, σ) will be denoted by $\Lambda = \Lambda(E, \sigma)$. It is easily seen that $\alpha \in \Lambda$ if and only if $\alpha = \alpha^\sigma$, so $\dim E = 2 \dim \alpha$ and all $\alpha \in \Lambda$ have the same dimension d, $\dim E = 2d$. If $\alpha \in \Lambda$, write $\Lambda^k(\alpha) = \{\beta \in \Lambda; \dim \beta \cap \alpha = k\}$.

If $\alpha, \beta \in \Lambda$, $\alpha \cap \beta = 0$, then any d-dimensional linear subspace γ of E with $\gamma \cap \beta = 0$ can be written as

$$\gamma = \{u + Cu; u \in \alpha\} \qquad \text{for a linear mapping } C\colon \alpha \to \beta. \tag{2.2}$$

Then the bilinear form

$$Q(\alpha, \beta; \gamma)\colon (u, v) \mapsto \sigma(Cu, v) \tag{2.3}$$

on α is symmetric if and only if $\gamma \in \Lambda$, so

$$Q(\alpha, \beta)\colon \gamma \mapsto Q(\alpha, \beta; \gamma) \tag{2.4}$$

is a bijection from $\Lambda^0(\beta)$ to the space $S^2\alpha$ of symmetric bilinear forms on α.

Proposition 2.1. *Λ is a regular algebraic variety of dimension $\frac{1}{2} \cdot d \cdot (d + 1)$ in the Grassmann-variety of all d-dimensional linear subspaces of E. The mappings $Q(\alpha, \beta)$ with $\alpha, \beta \in \Lambda$, $\alpha \cap \beta = 0$ form an atlas of Λ. The differential of $Q(\alpha, \beta)$ at α does not depend on the choice of $\beta \in \Lambda^0(\alpha)$ and therefore defines a canonical identification of $T_\alpha\Lambda$ with $S^2\alpha$.*

For the straightforward proof, see for instance [7, Sect. 3.4]. Now let $\gamma \in \Lambda^k(\alpha)$, choose $\beta \in \Lambda^0(\alpha) \cap \Lambda^0(\gamma)$. On a suitable basis of α we can write

$$Q(\alpha, \beta; \gamma) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

with A a nonsingular $(n - k) \times (n - k)$ matrix. If $\gamma' \in \Lambda$ is close to γ, then $\gamma' \in \Lambda^0(\beta)$ and

$$Q(\alpha, \beta; \gamma') = \begin{pmatrix} B & C \\ {}^tC & D \end{pmatrix},$$

with C, D small and B close to A, in particular B is invertible too. Now

$$\begin{pmatrix} 0 & 0 \\ -{}^tCB^{-1} & I \end{pmatrix}\begin{pmatrix} B & C \\ {}^tC & D \end{pmatrix}\begin{pmatrix} I & -B^{-1}C \\ 0 & I \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & D - {}^tCB^{-1}C \end{pmatrix}, \tag{2.5}$$

so taking $R(\gamma') = D - {}^tCB^{-1}C \in S^2(\operatorname{Ker} Q(\alpha, \beta; \gamma)) = S^2(\alpha \cap \gamma)$ we have proved:

LEMMA 2.2. *For every $\gamma \in \Lambda^k(\alpha)$ there exists a submersion R from an open neighborhood $\mathcal{O}$ of γ in Λ to $S^2(\alpha \cap \gamma)$ such that*

(i) $\gamma' \in \Lambda^{k'}(\alpha) \cap \mathcal{O} \Leftrightarrow \dim \operatorname{Ker} R(\gamma') = k'$, *and*

(ii) $DR(\gamma) = \rho_{\alpha\cap\gamma} \circ q_\gamma := q_{\alpha\cap\gamma}$.

Here $\rho_{\alpha\cap\gamma}$ denotes the restriction mapping: $S^2\gamma \to S^2(\alpha \cap \gamma)$. If $t \mapsto \omega(t)$ is a differentiable curve in Λ and $q_{\alpha\cap\omega(t_0)}(d\omega/dt(t_0))$ is nondegenerate, then

$$\operatorname{sgn} Q(\alpha, \beta; \omega(t)) = \operatorname{sgn} Q(\alpha, \beta; \omega(t_0)) + \operatorname{sgn} q_{\alpha\cap\omega(t_0)}\left(\frac{d\omega}{dt}(t_0)\right) \tag{2.6}$$

for $\beta \in \Lambda^0(\alpha) \cap \Lambda^0(\omega(t_0))$, $t > t_0$, $t - t_0$ sufficiently small.

It follows that $\bigcup_{k'\geqslant k} \Lambda^{k'}(\alpha)$ is an algebraic variety in Λ with regular part equal to $\Lambda^k(\alpha)$. For every $\gamma \in \Lambda^k(\alpha)$, $T_\gamma\Lambda^k(\alpha) = \operatorname{Ker} q_{\alpha\cap\gamma}$, so $\Lambda^k(\alpha)$ has codimension $\frac{1}{2} \cdot k \cdot (k+1)$ in Λ. In particular $\Sigma(\alpha) := \Lambda \backslash \Lambda^0(\alpha)$ is an algebraic variety of codimension 1 in Λ, with regular part equal to $\Lambda^1(\alpha)$ and singular part $\bigcup_{k'\geqslant 2} \Lambda^{k'}(\alpha)$ of codimension $\frac{1}{2} \cdot 2 \cdot (2+1) = 3$ in Λ. Moreover, $\Lambda^1(\alpha)$ is oriented in Λ, because for each $\gamma \in \Lambda^1(\alpha)$, $q_{\alpha\cap\gamma}$ induces an isomorphism: $T_\gamma\Lambda/T_\gamma\Lambda^1(\alpha) \to S^2(\alpha \cap \gamma)$, and the 1-dimensional space $S^2(\alpha \cap \gamma)$ is oriented by calling an element positive if it is a positive definite bilinear form on $\alpha \cap \gamma$.

If ω: $S^1 \to \Lambda$ is a differentiable loop in Λ intersecting $\Sigma(\alpha)$ only in $\Lambda^1(\alpha)$ and transversally, then the intersection number of ω with $\Sigma(\alpha)$ is given by

$$[\omega] := \sum_{\omega(t)\in\Sigma(\alpha)} \operatorname{sgn} q_{\alpha\cap\omega(t)}\left(\frac{d\omega}{dt}(t)\right). \tag{2.7}$$

Because $\Sigma(\alpha)\backslash\Lambda^1(\alpha)$ is a finite union of smooth manifolds of codimension $\geqslant 3$, a smooth homotopy of loops generically avoids $\Sigma(\alpha)\backslash\Lambda^1(\alpha)$ and is transversal to $\Lambda^1(\alpha)$, showing that $\omega \mapsto [\omega]$ is homotopy invariant, so it defines a homomorphism: $\pi_1(\Lambda) \to \mathbb{Z}$. Using that $\Lambda^0(\alpha)$ is simply connected and $\Lambda^1(\alpha)$ is connected it follows that ω is contractible in Λ if $[\omega] = 0$. On the other hand it is easy to find a loop ω with $[\omega] = 1$, so the map $\omega \mapsto [\omega]$ induces an isomorphism between $\pi_1(\Lambda)$ and $\mathbb{Z}$. $[\omega]$ does not depend on the choice of $\alpha \in \Lambda$ because Λ is connected. In other words, $\Sigma(\alpha)$ defines an oriented cycle of codimension 1 in Λ not depending on $\alpha \in \Lambda$, which is dual by Poincaré duality to a generator of $H^1(\Lambda, \mathbb{Z}) \cong \mathbb{Z}$.

For more details see [7, Sect. 3.4]. The results are due to Arnol'd [1] who gives a somewhat different proof. $[\omega]$ will be called the *Maslov–Arnol'd index* of the loop ω.

If $\omega \in C^0([a, b], \Lambda)$, $\omega(a)$, $\omega(b) \in \Lambda^0(\alpha)$, then the *intersection number of ω with α* will be defined as

$$[\omega : \alpha] = [\tilde{\omega}], \text{ where } \tilde{\omega} \text{ is the loop consisting of } \omega \text{ followed by a curve } \omega' \text{ in } \Lambda^0(\alpha) \text{ from } \omega(b) \text{ to } \omega(a). \tag{2.8}$$

Because $\Lambda^0(\alpha)$ is simply connected, this definition does not depend on the choice of ω'. Of course, $[\omega : \alpha] = \sum_{i=1}^{j} [\omega_i : \alpha]$ if $a = t_0 < t_1 < \cdots < t_j = b$, $\omega(t_i) \in \Lambda^0(\alpha)$ for all $i = 0, 1,..., j$, and ω_i is equal to the restriction of ω to $[t_{i-1}, t_i]$. Also $[\omega : \alpha] = [\omega]$ if ω is a loop.

If α' is another Lagrange space and $\omega(a)$, $\omega(b) \in \Lambda^0(\alpha')$, then

$$[\omega : \alpha'] = [\omega : \alpha] + s(\alpha, \alpha'; \omega(a), \omega(b)). \tag{2.9}$$

Here $s(\alpha, \alpha'; \beta, \beta')$ is the Maslov–Arnol'd index of the loop consisting of a curve in $\Lambda^0(\alpha)$ from β to β' followed by a curve in $\Lambda^0(\alpha')$ from β' back to β. This number has been introduced by Hörmander [10, Sect. 3.3], who also gave the explicit formula

$$s(\alpha, \alpha'; \beta, \beta') = \tfrac{1}{2}\{\operatorname{sgn} Q(\alpha, \alpha'; \beta) - \operatorname{sgn} Q(\alpha, \alpha'; \beta')\}. \tag{2.10}$$

Here the right-hand side has the following interpretation when $\alpha \cap \alpha' \neq 0$. If ϵ is an isotropic subspace of (E, σ), then σ defines a symplectic form on ϵ^σ/ϵ. Moreover, for each $\delta \in \Lambda(E, \sigma)$, the image $\pi_\epsilon(\delta)$ of $\delta \cap \epsilon^\sigma$ under the canonical homomorphism: $\epsilon^\sigma \to \epsilon^\sigma/\epsilon$ is a Lagrange subspace of $(\epsilon^\sigma/\epsilon, \sigma)$. This allows us to define

$$Q(\alpha, \alpha'; \gamma) = Q(\pi\alpha, \pi\alpha'; \pi\gamma) \quad \text{if} \quad \gamma \in \Lambda, \quad \gamma \cap \alpha' = 0, \quad \text{here} \quad \pi = \pi_{\alpha\cap\alpha'}. \tag{2.11}$$

Note that if $\gamma \in \Lambda^0(\alpha) \cap \Lambda^0(\alpha')$ then $Q(\alpha, \alpha'; \gamma)$ is similar to $Q(\alpha', \gamma; \alpha)$, read as a bilinear form on $\pi\alpha'$ by dividing out its null space $\alpha \cap \alpha'$. So (2.10) also can be read as

$$s(\alpha, \alpha'; \beta, \beta') = \tfrac{1}{2}\{\operatorname{sgn} Q(\alpha', \beta; \alpha) - \operatorname{sgn} Q(\alpha', \beta'; \alpha)\}. \tag{2.10'}$$

In order to prove (2.10), note that $s(\alpha, \alpha'; \beta, \beta') = [\hat{\omega} : \alpha]$ for any curve $\hat{\omega}$ from β' to β in $\Lambda^0(\alpha')$. $\hat{\omega}$ can be chosen such that it intersects $\Sigma(\alpha)$

only at $\Lambda^1(\alpha)$ and transversally. The bilinear form $Q(\pi\hat{\omega}(t_0), \pi\alpha'; \pi\hat{\omega}(t))$ is similar to the restriction of $Q(\hat{\omega}(t_0), \alpha'; \hat{\omega}(t))$ to $\hat{\omega}(t_0) \cap (\alpha + \alpha')$, and $\alpha + \alpha' = (\alpha \cap \alpha')^\sigma$, so

$$\operatorname{sgn} q_{\pi\alpha \cap \pi\hat{\omega}(t_0)} \frac{d\pi\hat{\omega}}{dt}(t_0) = \operatorname{sgn} q_{\alpha \cap \hat{\omega}(t_0)} \frac{d\hat{\omega}}{dt}(t_0). \tag{2.12}$$

In view of (2.5) it follows that $\operatorname{sgn} Q(\pi\alpha, \pi\alpha'; \pi\hat{\omega}(t))$ jumps by $+2$, resp. -2 at each positive, resp. negative crossing of $\Lambda^1(\alpha)$ by B and remains constant elsewhere, thus proving (2.10).

We will also use the formula

$$s(\alpha, \alpha'; \beta, \beta') = -s(\beta, \beta'; \alpha, \alpha'); \tag{2.13}$$

see [10, (3.3.7)]. Combining (2.9), (2.13), and (2.10′) it follows that the number $[\omega : \alpha] - \frac{1}{2} \operatorname{sgn} Q(\omega(b), \alpha; \omega(a))$ does not depend on the choice of $\alpha \in \Lambda^0(\omega(a)) \cap \Lambda^0(\omega(b))$. In order to obtain an integer we propose the following

Definition 2.3. The *index* of a continuous curve $\omega \in C^0([a, b], \Lambda)$, not necessarily closed, is given by

$$\operatorname{ind}(\omega) = [\omega : \alpha] + \operatorname{ind} Q(\omega(b), \alpha; \omega(a)). \tag{2.14}$$

Here $Q(\omega(b), \alpha; \omega(a))$ is the symmetric bilinear form on $\omega(b)$ describing $\omega(a)$ as in (2.2), (2.3). Because the index of a symmetric bilinear form Q on a vector space F is equal to $\frac{1}{2}(\dim F - \dim \operatorname{Ker} Q - \operatorname{sgn} Q)$, the right-hand side in (2.14) does not depend on the choice of $\alpha \in \Lambda^0(\omega(a)) \cap \Lambda^0(\omega(b))$.

If $\omega \in C^0([a, c], \Lambda)$, $b \in [a, c]$, and ω_1, resp. ω_2 is the restriction of ω to $[a, b]$, resp. $[b, c]$, then for any $\delta \in \Lambda^0(\omega(a)) \cap \Lambda^0(\omega(b)) \cap \Lambda^0(\omega(c))$ the formula $[\omega : \delta] = [\omega_1 : \delta] + [\omega_2 : \delta]$ implies that

$$\begin{aligned}\operatorname{ind}(\omega_1) &+ \operatorname{ind}(\omega_2) - \operatorname{ind}(\omega) \\ &= \operatorname{ind} Q(\omega(b), \delta; \omega(a)) + \operatorname{ind} Q(\omega(c), \delta; \omega(b)) - \operatorname{ind} Q(\omega(c), \delta; \omega(a)).\end{aligned} \tag{2.15}$$

Choosing a curve ω such that $\omega(a) = \alpha$, $\omega(b) = \beta$, $\omega(c) = \gamma$, it therefore follows that for any $\alpha, \beta, \gamma \in \Lambda$ the number

$$i(\alpha, \beta, \gamma) = \operatorname{ind} Q(\beta, \delta; \alpha) + \operatorname{ind} Q(\gamma, \delta; \beta) - \operatorname{ind} Q(\gamma, \delta; \alpha) \tag{2.16}$$

does not depend on the choice of $\delta \in \Lambda^0(\alpha) \cap \Lambda^0(\beta) \cap \Lambda^0(\gamma)$.

Lemma 2.4. *Writing* $\pi = \pi_{(\alpha\cap\beta)+(\beta\cap\gamma)}$,

$$i(\alpha, \beta, \gamma) = (\textit{index} + \textit{nullity})\, Q(\pi\alpha, \pi\beta; \pi\gamma). \tag{2.17}$$

Proof. By letting the curve ω run in $\alpha + \beta + \gamma$ one can reduce the computation to the symplectic space $(\alpha + \beta + \gamma)/(\alpha \cap \beta \cap \gamma)$, so we may assume that $\alpha \cap \beta \cap \gamma = 0$.

Let $t \mapsto \alpha(t)$ be a differentiable curve in Λ such that $\alpha(0) = \alpha$ and $q_\alpha(d\alpha/dt)(0)$ is positive definite. Taking $\delta = \alpha(t)$, $t > 0$, t sufficiently small, we have $\operatorname{ind} Q(\beta, \delta; \alpha) = \operatorname{ind} Q(\alpha, \beta; \delta) = 0$ and similarly $\operatorname{ind} Q(\gamma, \delta; \alpha) = 0$, so $i(\alpha, \beta, \gamma) = \operatorname{ind} Q(\gamma, \delta; \beta)$. Because $\alpha \cap (\beta \cap \gamma) = 0$, the curve $t \mapsto \pi_{\beta\cap\gamma}\alpha(t)$ is differentiable in $\Lambda((\beta + \gamma)/(\beta \cap \gamma), \sigma)$ and has positive definite derivative at $t = 0$, so reducing to $(\beta + \gamma)/(\beta \cap \gamma)$ we may now assume that $\beta \cap \gamma = 0$.

Write $\alpha = (\alpha \cap \beta) \oplus \alpha_1$ for a linear subspace $\alpha_1 \supset \alpha \cap \gamma$, and let $t \mapsto \alpha'(t)$ be a differentiable curve in Λ such that $\alpha'(0) = \alpha$, $\alpha \cap \beta \subset \alpha'(t)$ for all t and $q_{\alpha_1}(d\alpha'/dt)(0)$ is positive definite. Taking $\delta' = \alpha'(t')$, $t' > 0$, t' sufficiently small, it can be arranged that δ is attained from δ' by traveling into the positive definite direction, so $\operatorname{ind} Q(\gamma, \delta; \beta) = \operatorname{ind} Q(\beta, \gamma; \delta) = \operatorname{ind} Q(\beta, \gamma; \delta')$. Because $\alpha \cap \beta \subset \alpha'(t)$ for all t the curve $t \mapsto \pi_{\alpha\cap\beta}\alpha'(t)$ is smooth in $\Lambda((\alpha + \beta)/(\alpha \cap \beta), \sigma)$ and has positive definite derivative at $t = 0$, so reducing to $(\alpha + \beta)/(\alpha \cap \beta)$ we may now assume that $\alpha \cap \beta = 0$.

But then $\operatorname{ind} Q(\beta, \gamma; \delta) = \operatorname{ind} -Q(\gamma, \beta; \delta) = \operatorname{ind} -Q(\gamma, \beta; \alpha) + \dim(\gamma \cap \alpha) = (\text{index} + \text{nullity})\, Q(\alpha, \beta; \gamma)$. Because first reducing to $(\beta + \gamma)/(\beta \cap \gamma)$ and then to $(\pi_{\beta\cap\gamma}\alpha + \pi_{\beta\cap\gamma}\beta)/(\pi_{\beta\cap\gamma}\alpha \cap \pi_{\beta\cap\gamma}\beta)$ is the same as reducing to ϵ^σ/ϵ with $\epsilon = (\alpha \cap \beta) + (\beta \cap \gamma)$, (2.17) is proved.

We conclude this section with the computation of some intersection numbers.

Lemma 2.5. *Let ω intersect $\sum(\alpha)$ at time t and have a right, resp. left derivative at time t equal to θ^+, resp. θ^-. Assume that $Q^\pm = q_{\alpha\cap\omega(t)}\theta^\pm$ is nondegenerate. Then, restricting ω to a sufficiently small neighborhood of t, this is the only intersection of ω with $\sum(\alpha)$ and*

$$[\omega : \alpha] = \tfrac{1}{2}\{\operatorname{sgn} Q^- + \operatorname{sgn} Q^+\}. \tag{2.18}$$

Proof. Take $\beta \in \Lambda^0(\alpha)$. Then, applying (2.6),

$$\begin{aligned}[\omega : \alpha] &= [\omega : \beta] - s(\alpha, \beta; \omega(a), \omega(b)) = -s(\alpha, \beta; \omega(a), \omega(b)) \\ &= \tfrac{1}{2}\{\operatorname{sgn} Q(\alpha, \beta; \omega(t)) + \operatorname{sgn} Q^+ - (\operatorname{sgn} Q(\alpha, \beta; \omega(t)) - \operatorname{sgn} Q^-)\} \\ &= \tfrac{1}{2}\{\operatorname{sgn} Q^+ + \operatorname{sgn} Q^-\}, \text{ if } a < t < b,\ a, b \text{ sufficiently close to } t.\end{aligned}$$

A curve $\omega \in C^1([a, b], \Lambda)$ is called a *plus-curve* if $q_{\omega(t)}(d\omega/dt)(t)$ is positive definite for every $t \in [a, b]$.

COROLLARY 2.6. *If ω is a plus-curve, $\omega(a)$, $\omega(b) \in \Lambda^0(\alpha)$, then ω intersects $\sum(\alpha)$ in finitely many points and*

$$[\omega : \alpha] = \sum_{\omega(t)\cap\alpha\neq 0} \dim(\omega(t) \cap \alpha). \tag{2.19}$$

3. CURVES OF SYMPLECTIC TRANSFORMATIONS

Let (F, σ) be a real symplectic vector space and $E = F \times F$ be provided with the symplectic form $\sigma = \sigma_{\text{first factor}} \oplus -\sigma_{\text{second factor}}$ as in (1.25). Then the mapping $\Phi \mapsto \text{graph } \Phi$ is a diffeomorphism from the group $\text{Sp}(F, \sigma)$ of symplectic transformations in (F, σ) to a dense open subset of $\Lambda = \Lambda(F \times F, \sigma)$. In order to describe its differential we identify $T_{\text{graph }\Phi}\Lambda$ with $S^2(\text{graph } \Phi)$ by means of the mapping $q_{\text{graph }\Phi}$ in Proposition 2.1, and in turn $S^2(\text{graph } \Phi)$ with $S^2(F)$ using $u \mapsto (u, \Phi u)$ as a similarity transformation.

LEMMA 3.1. *With these identifications, $\delta\Phi \in T_\Phi(\text{Sp}(F, \sigma))$ corresponds to the symmetric bilinear form*

$$(v, \tilde{v}) \mapsto \sigma(-\Phi^{-1} \circ \delta\Phi v, \tilde{v}) \text{ on } F. \tag{3.1}$$

Proof. Take $\Psi \in \text{Sp}(F, \sigma)$ such that its graph is transversal to graph Φ, that is, $\Psi - \Phi$ is invertible. For Φ' close to Φ write $(u, \Phi' u) = (v, \Phi v) + (w, \Psi w)$, implying that $w = (\Psi - \Phi')^{-1}(\Phi' - \Phi)v$. So $Q(\text{graph } \Phi, \text{graph } \Psi; \text{graph } \Phi')$ is given by

$$\begin{aligned}(v, \tilde{v}) &\mapsto \sigma((w, \Psi w), (\tilde{v}, \Phi v))\\ &= \sigma(w, \tilde{v}) - \sigma(\Psi w, \Phi\tilde{v}) = \sigma((I - \Phi^{-1}\Psi)w, \tilde{v})\\ &= \sigma((I - \Phi^{-1}\Psi)(\Psi - \Phi')^{-1}(\Phi' - \Phi)v, \tilde{v}).\end{aligned}$$

Differentiation with respect to Φ' at $\Phi' = \Phi$ now gives (3.1).

LEMMA 3.2. *For each $V \in \Lambda(F, \sigma)$, $\Psi \mapsto \Psi(V)$ is a smooth mapping from $\text{Sp}(F, \sigma)$ to $\Lambda(F, \sigma)$. The image of $\delta\Psi \in T_\Psi(\text{Sp}(F, \sigma))$ under its differential corresponds to the symmetric bilinear form*

$$(v, \tilde{v}) \mapsto \sigma(\delta\Psi \circ \Psi^{-1} v, \tilde{v}) \text{ on } \Psi(V). \tag{3.2}$$

Proof. Choose $U \in \Lambda^0(V)$. For Ψ close to I, write $v + u = \Psi v'$, with $v, v' \in V$, $u \in U$. Writing π_V for the linear projection onto V along U, this leads to $\sigma(u, \tilde{v}) = \sigma(\Psi v', \tilde{v}) = \sigma(\Psi(\pi_V\Psi)^{-1}v, \tilde{v})$ if $v \in V$. Differentiating this with respect to Ψ at $\Psi = I$ (treating $(\pi_V\Psi)^{-1}$ as a map: $V \to V$) leads to (3.2) for $\Psi = I$. The general case follows by replacing $\Psi(V)$ by V, and remarking that

$$(\Psi + \delta\Psi)(V) = (I + \delta\Psi \circ \Psi^{-1})(\Psi(V)).$$

COROLLARY 3.3. *Let $t \mapsto \Phi(t)$, t running from 0 to T, be a curve in $\text{Sp}(F, \sigma)$ with $\Phi(0) = I$. Write φ for the curve $t \mapsto \text{graph } \Phi(t)$ in $\Lambda(F \times F, \sigma)$, and φ_V for the curve $t \mapsto \Phi(t)^{-1}(V)$ in $\Lambda(F, \sigma)$ for any $V \in \Lambda(F, \sigma)$. Then*

$$[\varphi_V : U] = [\varphi : U \times V] \tag{3.3}$$

if $U \in \Lambda^0(V) \cap \Lambda^0(\Phi(T)^{-1}(V))$.

Proof. By a homotopy we can make φ differentiable, intersecting $\sum(U \times V)$ only in its regular part and transversally. In view of Lemma 3.1, $q_{(U\times V)\cap\varphi(t)}(d\varphi/dt)(t)$ is similar to the symmetric bilinear form $(u, \tilde{u}) \mapsto \sigma(-\Phi(t)^{-1}(d\Phi/dt)(t)u, \tilde{u})$, restricted to $U \cap \Phi(t)^{-1}(V)$. In view of Lemma 3.2 and using that

$$\frac{d}{dt}\Phi(t)^{-1} = -\Phi(t)^{-1}\frac{d\Phi}{dt}(t)\,\Phi(t)^{-1}$$

this is in turn similar to $q_{U\cap\Phi(t)^{-1}(V)}(d\varphi_V/dt)(t)$. So φ_V intersects $\sum(U)$ only in its regular part and transversally, and (3.3) follows now from the definition of the intersection number.

In view of Definition 2.3, (3.3) also can be read as

$$\begin{aligned}&\text{ind}(\varphi) - \text{ind}(\varphi_V)\\ &\quad= \text{ind}\, Q(\text{graph } \Phi(T), U \times V; \Delta) - \text{ind}\, Q(\Phi(T)^{-1}(V), U; V).\end{aligned} \tag{3.4}$$

Here $\Delta = \text{graph } I = \text{diagonal in } F \times F$. Choosing a curve in $\text{Sp}(F, \sigma)$ from I to Φ, it follows that for any $\Phi \in \text{Sp}(F, \sigma)$, $V \in \Lambda(F, \sigma)$ the number

$$j(\Phi, V) = \text{ind}\, Q(\text{graph } \Phi, U \times V, \Delta) - \text{ind}\, Q(\Phi^{-1}(V), U; V) \tag{3.5}$$

does not depend on the choice of $U \in \Lambda^0(V) \cap \Lambda^0(\Phi^{-1}(V))$.