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A Short Proof that Compact 2-Manifolds Can Be Triangulated

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The result mentioned in the title of this paper was first proved by RADO [1]; a proof can also be found in [2]. The idea for the present proof is that of the first-named author, who discovered it while investigating the possibilities of engulfing in low dimensions. It is shorter than previous proofs, and is presented in the interests of economy.

We commence by listing a few familiar facts from geometric topology:

The Jordan-Schoenflies Theorem. A simple closed curve J in E^2 separates E^2 into two regions. There exists a self-homeomorphism of E^2 under which J is mapped onto a circle.

Thickening an Arc. Each arc in the interior of a 2-manifold lies in the interior of a 2-cell. This 2-cell can be chosen to be disjoint from any preassigned compact set in the complement of the arc.

Cellularity and Quotients. For our purposes, a *cellular* set K is one that can be written as the intersection of a sequence of 2-cells

$$K=\bigcap_{i=1}^{\infty} E_i, \quad \text{where} \quad E_i \subset \operatorname{Int} E_{i-1} \qquad (i=2,3,\ldots).$$

If K is a cellular subset of a 2-manifold M, then M/K is homeomorphic to M. (The corresponding statement also holds in n dimensions; see [3], for example.)

We shall also have need for the following

Lemma. *Let M be a closed 2-manifold, and let C be a connected subset of M which is the union of n simple closed curves,*

$$C=\bigcup_{i=1}^{n} C_i.$$

Let A be a compact, totally-disconnected subset of C. Then A lies in the interior of a closed 2-cell in M. (A *totally-disconnected* set is characterized by the property that each of its connected subsets consists of at most one point.)

Proof. Since A is compact and totally-disconnected, some subarc S of C_1 contains no points of A. Thicken $C_1 - S$ to obtain a closed 2-cell D_1 whose interior contains $A \cap C_1$.

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Next, suppose that a 2-cell D_k has been constructed to satisfy

$$A \cap \bigcup_{i=1}^{k} C_i \subset \operatorname{Int} D_k.$$

$C_{k+1} - D_k$ is the union of a collection of countably many mutually disjoint open arcs X_μ whose endpoints lie on $\operatorname{Bd} D_k$. Each X_μ contains some subarc S_μ for which $A \cap S_\mu = \emptyset$, so that the two arcs comprising $X_\mu - S_\mu$ can be thickened to form closed 2-cells $E_{\mu 1}$ and $E_{\mu 2}$ which (a) avoid each other, (b) avoid X_λ for $\lambda \neq \mu$, and (c) meet D_k in closed 2-cells. Then

$$D_{k+1} = D_k \cup \bigcup_{\mu} (E_{\mu 1} \cup E_{\mu 2})$$

is a closed 2-cell whose interior contains

$$A \cap \bigcup_{i=1}^{k+1} C_i. \quad \square$$

Theorem. *Any closed 2-manifold M can be triangulated.*

Proof. Cover M irreducibly by a finite collection of closed disks $\{B_1, B_2, \ldots, B_n\}$, and put $C_i = \operatorname{Bd} B_i$. Let A be the singular set of

$$C = \bigcup_{i=1}^{n} C_i,$$

i.e. A consists of those points of C which do not have 1-dimensional euclidean neighborhoods in C. A is compact and totally-disconnected, and, since the cover $\{B_i\}$ was irreducible, no proper subset of $\{B_i\}$ covers M; thus C is connected, so by the lemma, A lies in the interior of a closed 2-cell D in M. Note that $M - C$ is the union of a countable collection of mutually disjoint open 2-cells, each of whose closures in M is a closed 2-cell; this is an easy consequence of the Jordan-Schoenflies Theorem. Note also that $C - D$ consists of countably many mutually disjoint open arcs whose endpoints all lie on $\operatorname{Bd} D$.

Now $D \subset U \subset M$, where U is an open 2-cell, as can be seen by thickening $\operatorname{Bd} D$ slightly; hence D is cellular (again by the Jordan-Schoenflies Theorem), and so $M_1 = M/D$ is topologically equivalent to M. M_1 is the copy of M with which we shall work from this point on.

Let $R \subset M_1$ be the image of $\overline{C - D}$ under the quotient map; thus R is the one-point-union of a countable collection of simple closed curves, and is locally euclidean except at one point p (the image of D under the quotient map). Moreover, $M_1 - R$ is topologically equivalent to $M - (C \cup D)$. Furthermore, any 2-cell neighborhood V of p will contain all but a finite number of the simple closed curves which comprise R; this follows from the compactness of $\overline{C - D}$, hence of R. If each of the simple closed curves lying within V is spanned by the disk it bounds, a cellular set T, consisting of a one-point-union of closed disks, is obtained. (To see that T is cellular, let N be any neighborhood of T. Then N contains a 2-cell neighborhood V' of p, which must in turn contain all but finitely many of the simple closed curves comprising R. Thickening the arcs of R which lie outside V' and appending the resulting cells to $T \cup V'$ yields a cell Q, where $T \subset \operatorname{Int} Q \subset Q \subset N$.) Therefore, passing to M_1/T if necessary, we lose no generality in assuming that R is an r-leafed rose whose complement in M_1 is composed of finitely many components that are open 2-cells.

Finally, enclose p in a small closed 2-cell E meeting each simple closed curve in R in two points of its boundary, $\operatorname{Bd} E$. $E \cup R$ is then a disk with a finite number of mutually disjoint closed arcs $A_1, A_2, \ldots, A_r$ meeting $\operatorname{Bd} E$ in their endpoints. Each A_i may be enclosed (except for its end-points) in the interior of a closed disk meeting E in a pair of arcs on its boundary. By selecting these disks to be disjoint in pairs one obtains a triangulable 2-manifold N with boundary. The triangulation of N is now extended to the closure of each component of $M_1 - N$. □

Corollary 1. *Any compact 2-manifold can be triangulated.*

Proof. Include the boundary curves in the set C of the proof of the theorem; minor modifications of the above argument will convert it into a proof of the corollary. □

Corollary 2. *Any compact 2-manifold with non-void boundary is embeddable in* E^3.

Proof. Given such a manifold M, M can be embedded in a closed manifold M_1. Let $M_1 = E^2 \cup R$, a standard decomposition [4], where R is an r-leafed rose disjoint from the open 2-cell E^2. Then a copy of M lies in every neighborhood of R, for any embedding of R in E^3. □

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