

The geometric Hopf invariant and double points

Michael Crabb and Andrew Ranicki

Abstract. The geometric Hopf invariant of a stable map F is a stable $\mathbb{Z}/2$ -equivariant map $h(F)$ such that the stable $\mathbb{Z}/2$ -equivariant homotopy class of $h(F)$ is the primary obstruction to F being homotopic to an unstable map. In this paper, we express the geometric Hopf invariant of the Umkehr map F of an immersion $f : M^m \looparrowright N^n$ in terms of the double point set of f . We interpret the Smale–Hirsch–Haefliger regular homotopy classification of immersions f in the metastable dimension range $3m < 2n - 1$ (when a generic f has no triple points) in terms of the geometric Hopf invariant.

Mathematics Subject Classification (2010). Primary 55Q25; Secondary 57R42.

Keywords. Geometric Hopf invariant, immersion, double point.

Introduction

The original Hopf invariant $H(F) \in \mathbb{Z}$ of a map $F : S^3 \rightarrow S^2$ was interpreted by Steenrod as the evaluation of the cup product in the mapping cone $C(F)$. The mod 2 Hopf invariant $H_2(F) \in \mathbb{Z}/2$ of a map $F : S^j \rightarrow S^k$ was then defined using the functional Steenrod squares of F . The geometric Hopf invariant of a stable map $F : \Sigma^\infty X \rightarrow \Sigma^\infty Y$ is the stable $\mathbb{Z}/2$ -equivariant map

$$h(F) = (F \wedge F) \Delta_X - \Delta_Y F : \Sigma^\infty X \rightarrow \Sigma^\infty (Y \wedge Y)$$

measuring the failure of F to preserve the diagonal maps of X and Y , with $\mathbb{Z}/2$ acting by the identity on X and by the transposition $T : (y_1, y_2) \mapsto (y_2, y_1)$ on $Y \wedge Y$. Thus $h(F)$ is a homotopy-theoretic generalization of the functional Steenrod squares. The stable homotopy class of $h(F)$ is the primary obstruction to F being homotopic to an unstable map.

Given an immersion $f : M^m \looparrowright N^n$ with normal bundle $\nu(f)$, we express the geometric Hopf invariant $h(F)$ of the Umkehr map $F : \Sigma^\infty N^+ \rightarrow \Sigma^\infty M^{\nu(f)}$ in terms of the double point set of f , where $M^{\nu(f)}$ denotes the

Thom space. There are many antecedents for this expression! The stable homotopy class of $h(F)$ depends only on the regular homotopy class of f . If f is regular homotopic to an embedding, then $h(F)$ is stably null-homotopic. We interpret the Smale–Hirsch–Haefliger regular homotopy classification of immersions f in the metastable dimension range $3m < 2n - 1$ (when a generic f has no triple points) in terms of the geometric Hopf invariant.

In [6] we shall provide a considerably more detailed exposition of the geometric Hopf invariant $h(F)$ and its applications to manifolds. This will include the $\pi_1(N)$ -equivariant geometric Hopf invariant $\tilde{h}(F)$ needed for a homotopy-theoretic treatment of the double point invariant $\mu(f)$ of Wall [23] for a generic immersion $f : M^m \looparrowright N^{2m}$ which plays such an important role in non-simply-connected surgery theory, with $M = S^m$. When both M and N are connected and oriented and f induces the trivial map $\pi_1(M) \rightarrow \pi_1(N)$, $\mu(f)$ is an element of the group

$$\mathbb{Z}[\pi_1(N)] / \langle g - (-)^m g^{-1} \mid g \in \pi_1(N) \rangle$$

and $\mu(f) = 0$ if (and, for $m > 2$, only if) f is regular homotopic to an embedding, by the Whitney trick for removing double points. In [6], $\tilde{h}(F)$ will be shown to induce the quadratic construction ψ_F of [19] on the chain level.

The present paper is set out as follows. Section 1 describes briefly the construction of the geometric Hopf invariant and its fibrewise generalization. The double point theorem is stated and proved in Section 2. In Section 3, building on the work of Dax [7], Hatcher and Quinn [10], Salomonsen [20] and Li, Liu and Zhang [18], we relate the geometric Hopf invariant in a stable range to Haefliger's obstruction to the existence of a regular homotopy from an immersion to an embedding. The papers of Boardman and Steer [2] and Koschorke and Sanderson [17] are also relevant. The variation of the geometric Hopf invariant of an immersion under a (not necessarily regular) homotopy is computed in Section 4 in terms of the Smale–Hirsch–Haefliger classification. In Section 5 we use Whitney's figure-of-eight immersion [24] to construct, in a metastable range, immersions close to a given embedding. Prerequisites for that section, on the differential topological classification of vector bundle monomorphisms, are given in the appendix.

We shall write the one-point compactification of a locally compact Hausdorff topological space X as X^+ . A subscript “+” will be used for the adjunction of a disjoint basepoint to a space. If X is compact $X^+ = X_+$. For a Euclidean vector bundle ξ over a general space X , we write $D(\xi)$ for the closed unit disc bundle, $S(\xi)$ for the sphere bundle and $B(\xi)$ for open unit disc bundle. The Thom space of ξ is written as X^ξ . To simplify notation, we shall sometimes write Y^ξ , rather than $Y^{p^*\xi}$, for the Thom space of the pullback $p^*\xi$ by a map $p : Y \rightarrow X$, if the map p is clear from the context. Similarly, we sometimes write the trivial vector bundle over X with fibre over the vector space V as just V , instead of $X \times V$.

Methods of fibrewise homotopy theory will be used extensively. Fibre-wise constructions, such as the one-point compactification, the Thom space, or the smash product, over a base B will be indicated by attaching a subscript “ B ” to the relevant symbol. We follow the notation for (fibrewise) stable homotopy adopted in [4]. Consider fibrewise pointed spaces $X \rightarrow B$ and $Y \rightarrow B$ over a Euclidean neighbourhood retract (ENR) base B . If B is compact and A is a closed sub-ENR, we write

$$\omega_B^0\{X; Y\} \quad \text{and} \quad \omega_{(B,A)}^0\{X; Y\},$$

respectively, for the abelian group of stable fibrewise maps $X \rightarrow Y$ over B and the relative group defined in terms of homotopy classes of maps that are zero over the subspace A . (See, for example, [5, Part II, Section 3].) We also need to consider fibrewise maps with compact supports. When B is not necessarily compact, we write

$${}_c\omega_B^0\{X; Y\}$$

for the group of fibrewise stable maps that are zero outside a compact subspace of B . The ω^0 -theories are extended, using the fibrewise suspension Σ_B over B , to ω^i -cohomology theories indexed by $i \in \mathbb{Z}$. When $Y \rightarrow B$ is a trivial bundle $B \times S^i \rightarrow B$, there are natural identifications of the fibrewise groups with the reduced stable cohomotopy of an appropriate pointed space:

$$\omega_B^0\{X; B \times S^i\} = \tilde{\omega}^i(X/B), \quad \text{and} \quad \omega_{(B,A)}^0\{X; B \times S^i\} = \tilde{\omega}^i(X/(X_A \cup B)),$$

where $X_A \rightarrow A$ denotes the restriction of $X \rightarrow B$.

We shall also need $\mathbb{Z}/2$ -equivariant stable homotopy theory, which we indicate by a prefix as ${}^{\mathbb{Z}/2}\omega^*$. So, for example, the equivariant stable cohomotopy of a point, ${}^{\mathbb{Z}/2}\omega^0(*)$, is the direct limit over all finite-dimensional \mathbb{R} -vector spaces V and W of the homotopy classes of pointed $\mathbb{Z}/2$ -maps $(V \oplus LW)^+ \rightarrow (V \oplus LW)^+$. Here, and throughout the paper, we write L for the nontrivial 1-dimensional representation \mathbb{R} of $\mathbb{Z}/2$ with the involution -1 , and, for a finite-dimensional real vector space W , abbreviate the tensor product $L \otimes W$ to LW .

1. A review of the geometric Hopf invariant

Let X and Y be pointed topological spaces. It is convenient to assume that X is a compact ENR and that Y is an absolute neighbourhood retract (ANR).

We shall be working both with the unreduced suspension of an unpointed space A ,

$$sA = [0, 1] \times A / \sim, \quad (0, a) \sim (0, a'), \quad (1, a) \sim (1, a') \quad (a, a' \in A),$$

and with the reduced suspension of a pointed space $(B, * \in B)$,

$$\Sigma B = [0, 1] \times B / \sim, \quad (0, b) \sim (1, b') \sim (t, *) \quad (b, b' \in B, t \in [0, 1]).$$

Let V be a finite-dimensional Euclidean space, and let

$$S(V) = \{u \in V \mid \|u\| = 1\}$$

be the unit sphere in V with respect to an inner product $\| \cdot \|$ on V . For $t \in [0, 1]$, $u \in S(V)$ we write

$$[t, u] = \frac{tu}{1-t} \in V^+$$

with $[0, u] = 0$, $[1, u] = + \in V^+$. The maps

$$\begin{aligned} sS(V) &\rightarrow V^+; (t, u) \mapsto [t, u], \\ \Sigma S(V)_+ &\rightarrow V^+/\{0, +\}; (t, u) \mapsto [t, u] \end{aligned}$$

are homeomorphisms.

Let the generator $T \in \mathbb{Z}/2$ act on $X \wedge X$ by transposition

$$T : X \wedge X \rightarrow X \wedge X; (x, y) \mapsto (y, x).$$

The diagonal map

$$\Delta_X : X \rightarrow X \wedge X; x \mapsto (x, x)$$

is $\mathbb{Z}/2$ -equivariant. The diagonal map Δ_{V^+} extends to a $\mathbb{Z}/2$ -equivariant homeomorphism

$$\kappa_V : LV^+ \wedge V^+ \rightarrow V^+ \wedge V^+; (u, v) \mapsto (u + v, -u + v).$$

The $\mathbb{Z}/2$ -action $u \mapsto -u$ on LV has fixed point $\{0\}$; the $\mathbb{Z}/2$ -action on the unit sphere $S(LV)$ is free.

The geometric Hopf invariant of a map $F : V^+ \wedge X \rightarrow V^+ \wedge Y$ measures the difference $(F \wedge F)\Delta_X - \Delta_Y F$, given that $(F \wedge F)\Delta_{V^+ \wedge X} = \Delta_{V^+ \wedge Y} F$. The diagram of $\mathbb{Z}/2$ -equivariant maps

$$\begin{array}{ccc} LV^+ \wedge V^+ \wedge X & \xrightarrow{1 \wedge \Delta_X} & LV^+ \wedge V^+ \wedge X \wedge X \\ \downarrow 1 \wedge F & & \downarrow G \\ LV^+ \wedge V^+ \wedge Y & \xrightarrow{1 \wedge \Delta_Y} & LV^+ \wedge V^+ \wedge Y \wedge Y \end{array}$$

does not commute in general, with G defined by

$$\begin{aligned} G &= (\kappa_V^{-1} \wedge 1)(F \wedge F)(\kappa_V \wedge 1) : \\ &LV^+ \wedge V^+ \wedge X \wedge X \rightarrow LV^+ \wedge V^+ \wedge Y \wedge Y; \\ &(u, v, x_1, x_2) \mapsto ((w_1 - w_2)/2, (w_1 + w_2)/2, y_1, y_2) \\ &(F(u + v, x_1) = (w_1, y_1), F(-u + v, x_2) = (w_2, y_2)). \end{aligned}$$

However, the $\mathbb{Z}/2$ -equivariant maps defined by

$$\begin{aligned} p &= G(1 \wedge \Delta_X) : LV^+ \wedge V^+ \wedge X \rightarrow LV^+ \wedge V^+ \wedge Y \wedge Y; \\ &(u, v, x) \mapsto ((w_1 - w_2)/2, (w_1 + w_2)/2, y_1, y_2) \\ &(F(u + v, x) = (w_1, y_1), F(-u + v, x) = (w_2, y_2)), \\ q &= (1 \wedge \Delta_Y)(1 \wedge F) : LV^+ \wedge V^+ \wedge X \rightarrow LV^+ \wedge V^+ \wedge Y \wedge Y; \\ &(u, v, x) \mapsto (u, w, y, y) \quad (F(v, x) = (w, y)) \end{aligned}$$

agree on $0^+ \wedge V^+ \wedge X = V^+ \wedge X \subseteq LV^+ \wedge V^+ \wedge X$, with

$$p| = q| = (\kappa_V^{-1} \wedge 1) \Delta_{V^+ \wedge Y} F = (\kappa_V^{-1} \wedge 1) (F \wedge F) (\kappa_V \wedge 1) \Delta_{V^+ \wedge X} : \\ V^+ \wedge X \rightarrow LV^+ \wedge V^+ \wedge Y \wedge Y.$$

Definition 1.1 (see [5, pp. 306–308]). The *geometric Hopf invariant* of a pointed map $F : V^+ \wedge X \rightarrow V^+ \wedge Y$ is the $\mathbb{Z}/2$ -equivariant map given by the relative difference of the $\mathbb{Z}/2$ -equivariant maps p, q

$$h_V(F) = \delta(p, q) : \Sigma S(LV)_+ \wedge V^+ \wedge X \rightarrow LV^+ \wedge V^+ \wedge Y \wedge Y; \\ (t, u, v, x) \mapsto \begin{cases} q([1 - 2t, u], v, x) & \text{if } 0 \leq t \leq 1/2, \\ p([2t - 1, u], v, x) & \text{if } 1/2 \leq t \leq 1 \end{cases} \\ (t \in [0, 1], u \in S(LV), v \in V, x \in X).$$

We are primarily interested in the stable $\mathbb{Z}/2$ -equivariant class of $h_V(F)$, which depends only on the homotopy class of F , and which we write simply as

$$h_V(F) \in {}^{\mathbb{Z}/2}\omega^0\{\Sigma S(LV)_+ \wedge X; LV^+ \wedge (Y \wedge Y)\}.$$

Using duality, we can rewrite this stable homotopy group in different ways and thus obtain two other versions of the geometric Hopf invariant as follows. Smashing $h_V(F)$ with the identity on the sphere $S(LV)_+$ and composing with the duality map

$$LV^+ \rightarrow \Sigma S(LV)_+ \rightarrow S(LV)_+ \wedge \Sigma S(LV)_+,$$

we get a map

$$V^+ \wedge LV^+ \wedge X \rightarrow V^+ \wedge LV^+ \wedge S(LV)_+ \wedge (Y \wedge Y)$$

and a second version of the geometric Hopf invariant as an element

$$h'_V(F) \in {}^{\mathbb{Z}/2}\omega^0\{X; S(LV)_+ \wedge (Y \wedge Y)\}.$$

Remark 1.2. The $\mathbb{Z}/2$ -equivariant cofibration sequence

$$S(LV)_+ \rightarrow S^0 = \{0\}^+ \rightarrow LV^+$$

induces an exact sequence of stable $\mathbb{Z}/2$ -equivariant homotopy groups

$${}^{\mathbb{Z}/2}\omega^0\{X; S(LV)_+ \wedge (Y \wedge Y)\} \rightarrow {}^{\mathbb{Z}/2}\omega^0\{X; Y \wedge Y\} \\ \rightarrow {}^{\mathbb{Z}/2}\omega^0\{X; LV^+ \wedge Y \wedge Y\}.$$

The stable class $h'_V(F)$ has image $(F \wedge F) \Delta_X - \Delta_Y F$ in ${}^{\mathbb{Z}/2}\omega^0\{X; Y \wedge Y\}$.

We may also use the Adams isomorphism [1, Theorem 5.3]

$${}^{\mathbb{Z}/2}\omega^0\{X; S(LV)_+ \wedge (Y \wedge Y)\} \cong \omega^0\{X; S(LV)_+ \wedge_{\mathbb{Z}/2} (Y \wedge Y)\}$$

to regard $h'_V(F)$ as a nonequivariant class

$$h''_V(F) \in \omega^0\{X; S(LV)_+ \wedge_{\mathbb{Z}/2} (Y \wedge Y)\}.$$

Remark 1.3. Instead of applying the Adams isomorphism, we can use fibre-wise techniques. The unstable map $h_V(F)$ lifts to an equivariant fibrewise pointed map over $S(LV)$:

$$S(LV) \times (\Sigma V^+ \wedge X) \rightarrow S(LV) \times (V^+ \wedge LV^+ \wedge (Y \wedge Y)),$$

and we may divide by the free $\mathbb{Z}/2$ -action to get a fibrewise pointed map over the real projective space $P(V) = S(LV)/\mathbb{Z}/2$:

$$P(V) \times (\Sigma V^+ \wedge X) \rightarrow (V \oplus (H \otimes V))_{P(V)}^+ \wedge_{P(V)} (S(LV) \times_{\mathbb{Z}/2} (Y \wedge Y)),$$

where $H = S(LV) \times_{\mathbb{Z}/2} L$ is the Hopf line bundle over $P(V)$ and $\{-\}_{P(V)}^+$ denotes fibrewise one-point compactification over $P(V)$. This fibrewise map represents a stable class in the group

$$\omega_{P(V)}^{-1} \{P(V) \times X; (H \otimes V)_{P(V)}^+ \wedge_{P(V)} (S(LV) \times_{\mathbb{Z}/2} (Y \wedge Y))\},$$

which may be identified by fibrewise Poincaré–Atiyah duality (since $\mathbb{R} \oplus \tau P(V) = H \otimes V$) with

$$\omega^0 \{X; S(LV)_+ \wedge_{\mathbb{Z}/2} (Y \wedge Y)\}.$$

(For the duality theorem, see, for example, [5, Part II, Section 12].)

Remark 1.4. By [3, pp. 60–61] the limit over all finite-dimensional V of the exact sequences in Remark 1.2 is a split exact sequence

$$\omega^0 \{X; (E\mathbb{Z}/2)_+ \wedge_{\mathbb{Z}/2} (Y \wedge Y)\} \xrightarrow{\delta} {}^{\mathbb{Z}/2}\omega^0 \{X; Y \wedge Y\} \xrightarrow{\rho} \omega^0 \{X; Y\}$$

with the fixed point map ρ split by

$$\omega^0 \{X; Y\} \rightarrow {}^{\mathbb{Z}/2}\omega^0 \{X; Y \wedge Y\}; F \mapsto \Delta_Y F.$$

The stable $\mathbb{Z}/2$ -equivariant homotopy class

$$h(F) = (F \wedge F) \Delta_X - \Delta_Y F \in {}^{\mathbb{Z}/2}\omega^0 \{X; Y \wedge Y\}$$

is the image under δ of

$$h'(F) = \varinjlim_V h'_V(F) \in \omega^0 \{X; (E\mathbb{Z}/2)_+ \wedge_{\mathbb{Z}/2} (Y \wedge Y)\}.$$

We shall need some fundamental properties of the geometric Hopf invariant. First, it is an obstruction to desuspension. The Hopf invariant of a suspension is zero, but there is a more precise result for the geometric Hopf invariant.

Given a pointed space $(A, * \in A)$ and $a \neq * \in A$, let $a_1, a_2 \in A \vee A$ denote the two distinct copies of a , and write

$$A \vee_a A = (A \vee A)/(a_1 \sim a_2).$$

Lemma 1.5. (i) *The $\mathbb{Z}/2$ -equivariant maps*

$$i_V : S(LV) \rightarrow \Sigma S(LV)_+; \quad v \mapsto (1/2, v),$$

$$j_V : \Sigma S(LV)_+ \rightarrow LV^+ \vee_0 LV^+; \quad (t, u) \mapsto \begin{cases} [1 - 2t, u]_1 & \text{if } 0 \leq t \leq 1/2, \\ [2t - 1, u]_2 & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

$$k_V : LV^+ \vee_0 LV^+ \rightarrow LV^+; \quad v_1 \mapsto v, v_2 \mapsto v$$

fit into a $\mathbb{Z}/2$ -equivariant cofibration

$$S(LV) \xrightarrow{i_V} \Sigma S(LV)_+ \xrightarrow{j_V} LV^+ \vee_0 LV^+ \xrightarrow{k_V} LV^+$$

with the composite $k_V \circ j_V : \Sigma S(LV)_+ \rightarrow LV^+$ a $\mathbb{Z}/2$ -equivariantly null-homotopic map.

(ii) *The geometric Hopf invariant of $F : V^+ \wedge X \rightarrow V^+ \wedge Y$ is the composite*

$$h_V(F) = (q \vee_0 p)(j_V \wedge 1_{V^+ \wedge X}) :$$

$$\begin{aligned} \Sigma S(LV)_+ \wedge V^+ \wedge X &\rightarrow (LV^+ \vee_0 LV^+) \wedge V^+ \wedge X \\ &\rightarrow LV^+ \wedge V^+ \wedge Y \wedge Y. \end{aligned}$$

(iii) *Suppose that F is the suspension $1_{V^+} \wedge F_0$ of a map $F_0 : X \rightarrow Y$, so that*

$$p = q = 1 \wedge (\Delta_Y \circ F_0) : LV^+ \wedge V^+ \wedge X \rightarrow LV^+ \wedge V^+ \wedge Y \wedge Y.$$

Then

$$h_V(F) = 1 \wedge (k_V \circ j_V) \wedge (\Delta_Y \circ F_0) \simeq *$$

is $\mathbb{Z}/2$ -equivariantly null-homotopic.

The second property, giving a formula for the Hopf invariant of a composition, is suggested by the stable form described in Remark 1.2.

Proposition 1.6 (Composition formula). *Let X, Y and Z be pointed spaces, and suppose that $F : V^+ \wedge X \rightarrow V^+ \wedge Y$ and $G : V^+ \wedge Y \rightarrow V^+ \wedge Z$ are pointed maps. Then*

$$h_V(G \circ F) = h_V(G)[F] + [G \wedge G]h_V(F) \in {}^{\mathbb{Z}/2}\omega^0\{\Sigma S(LV)_+ \wedge X; LV^+ \wedge (Z \wedge Z)\},$$

where

$$[F] \in {}^{\mathbb{Z}/2}\omega^0\{X; Y\} \quad \text{and} \quad [G \wedge G] \in {}^{\mathbb{Z}/2}\omega^0\{Y \wedge Y; Z \wedge Z\}$$

are the stable classes determined by F (with the trivial action of $\mathbb{Z}/2$) and $G \wedge G$.

Lastly, the Hopf invariant satisfies the following sum formula.

Proposition 1.7 (Sum formula). *Let F_+, F_- be maps $V^+ \wedge X \rightarrow V^+ \wedge Y$. Suppose that $v \in S(V)$. Choose a tubular neighbourhood of $\{v, -v\}$ in V and let $\nabla : V^+ \rightarrow V^+ \wedge V^+$ be the associated Pontryagin–Thom map. Then the Hopf invariant of the sum $F = (F_+ \wedge F_-) \circ (\nabla \wedge 1_X)$ is given by*

$$h_V(F) = h_V(F_+) + h_V(F_-) + \iota[(F_+ \wedge F_-) \circ \Delta_X],$$

where the induction homomorphism ι is the composition of the isomorphism

$$\omega^0\{X; Y \wedge Y\} \cong \mathbb{Z}/2 \omega^0\{\Sigma S(Lv)_+ \wedge X; (Lv)^+ \wedge (Y \wedge Y)\}$$

and the map

$$\mathbb{Z}/2 \omega^0\{\Sigma S(Lv)_+ \wedge X; (Lv)^+ \wedge (Y \wedge Y)\} \rightarrow \mathbb{Z}/2 \omega^0\{\Sigma S(LV)_+ \wedge X; LV^+ \wedge (Y \wedge Y)\}$$

induced by the inclusion of the 1-dimensional subspace $\mathbb{R}v \hookrightarrow V$.

The explicit construction of the geometric Hopf invariant is readily extended to the fibrewise theory. Suppose now that $X \rightarrow B$ and $Y \rightarrow B$ are fibrewise pointed spaces over an ENR B . (We shall assume that both are locally fibre homotopy trivial and that the fibres have the homotopy type of CW complexes, finite complexes in the case of X .) Consider a fibrewise pointed map $F : (B \times V^+) \wedge_B X \rightarrow (B \times V^+) \wedge_B Y$. If B is compact, we have a fibrewise geometric Hopf invariant

$$h_V(F) \in \mathbb{Z}/2 \omega_B^0\{(B \times \Sigma S(LV)_+) \wedge_B X; (B \times LV^+) \wedge_B (Y \wedge_B Y)\},$$

and corresponding variants $h'_V(F)$ and $h''_V(F)$. See, for example, [5, Part II, Section 14]. This fibrewise Hopf invariant is an obstruction to fibrewise desuspension. Indeed, suppose that the restriction of F to a closed sub-ENR $A \subseteq B$ is the fibrewise suspension of a map $X_A \rightarrow Y_A$ over A . Then Lemma 1.5(iii) shows how to define a relative fibrewise Hopf invariant in

$$\mathbb{Z}/2 \omega_{(B,A)}^0\{(B \times \Sigma S(LV)_+) \wedge_B X; (B \times LV^+) \wedge_B (Y \wedge_B Y)\},$$

which lifts $h_V(F)$. (One uses the fact that the inclusion of A in B is a cofibration.) When B is not (necessarily) compact and F is a fibrewise suspension outside some compact subspace of B , the same method gives a fibrewise Hopf invariant with compact supports:

$$h_V(F) \in \mathbb{Z}/2 \omega_B^0\{(B \times \Sigma S(LV)_+) \wedge_B X; (B \times LV^+) \wedge_B (Y \wedge_B Y)\}.$$

2. The double point theorem

Let $f : M \looparrowright N$ be a (smooth) immersion of a closed manifold M in a connected manifold N (without boundary) of dimension n , with normal bundle $\nu(f)$, usually abbreviated to ν . We do not require M to be connected, nor that all the components should have the same dimension; the maximum dimension of a component is denoted by m . Let $e : M \rightarrow V$ be a smooth map to a finite-dimensional Euclidean space V such that $e(x) \neq e(y)$ whenever $f(x) = f(y)$ for $x \neq y$. This gives a (smooth) embedding $(e, f) : M \hookrightarrow V \times N$ with normal bundle $V \oplus \nu$. (To be precise, there is a short exact sequence $0 \rightarrow V \rightarrow \nu(e, f) \rightarrow \nu \rightarrow 0$ which is split by a choice of metrics.)

We have an associated Pontryagin–Thom map (defined up to homotopy)

$$F : V^+ \wedge N^+ \rightarrow V^+ \wedge M^\nu.$$

Its geometric Hopf invariant is a stable $\mathbb{Z}/2$ -homotopy class

$$h_V(F) \in {}^{\mathbb{Z}/2}\omega^0\{\Sigma S(LV)_+ \wedge N^+; LV^+ \wedge (M^\nu \wedge M^\nu)\},$$

where $\mathbb{Z}/2$ interchanges the factors of $M^\nu \wedge M^\nu$.

Suppose that the immersion f is self-transverse and that there are no k -tuple points for $k > 2$. (This is the case for a generic immersion if $3m < 2n$.) The *double point set*

$$\mathfrak{D}(f) = \{(x, y) \in M \times M - \Delta(M) \mid f(x) = f(y)\}$$

is then a smooth $\mathbb{Z}/2$ -submanifold of $M \times M$ (of constant dimension $2m - n$ if M is connected), on which $\mathbb{Z}/2$ acts freely, and its normal bundle may be identified with the pullback $j^*\tau N$ of the tangent bundle of N by the map $j : \mathfrak{D}(f) \rightarrow N$ mapping (x, y) to $f(x) = f(y) \in N$. We also have a $\mathbb{Z}/2$ -map $d : \mathfrak{D}(f) \rightarrow LV - \{0\}$ given by $d(x, y) = e(x) - e(y)$, and thus an embedding

$$(d, j) : \mathfrak{D}(f) \hookrightarrow (LV - \{0\}) \times N,$$

with normal bundle $LV \oplus k^*(\nu \times \nu)$, where $k : \mathfrak{D}(f) \rightarrow M \times M$ is the inclusion. The Pontryagin–Thom construction gives a $\mathbb{Z}/2$ -map

$$(LV - \{0\})^+ \wedge N^+ \rightarrow LV^+ \wedge \mathfrak{D}(f)^{k^*(\nu \times \nu)}.$$

Composing with the map induced by k , we get a $\mathbb{Z}/2$ -homotopy class

$$\phi : \Sigma S(LV)_+ \wedge N^+ \rightarrow LV^+ \wedge (M^\nu \wedge M^\nu).$$

Theorem 2.1 (The double point theorem). *The geometric Hopf invariant $h_V(F)$ of the Pontryagin–Thom map F is equal to the $\mathbb{Z}/2$ -equivariant stable homotopy class of the map ϕ determined, as described above, by the double point manifold $\mathfrak{D}(f)$.*

We may also consider the second version of the Hopf invariant

$$h'_V(F) \in {}^{\mathbb{Z}/2}\omega^0\{N^+; S(LV)_+ \wedge (M^\nu \wedge M^\nu)\}.$$

This, too, may be described directly in terms of the double points. The embedding $\mathfrak{D}(f) \hookrightarrow LV \times N$ with normal bundle $LV \oplus k^*(\nu \times \nu)$ provides a Pontryagin–Thom map

$$LV^+ \wedge N^+ \rightarrow LV^+ \wedge \mathfrak{D}(f)^{k^*(\nu \times \nu)}$$

which we compose with the map $\mathfrak{D}(f) \rightarrow S(LV) \times (M \times M)$ given by e and the inclusion k to get

$$\phi' : LV^+ \wedge N^+ \rightarrow LV^+ \wedge S(LV)_+ \wedge (M^\nu \wedge M^\nu).$$

Corollary 2.2. *The second version $h'_V(F)$ of the geometric Hopf invariant is represented by the $\mathbb{Z}/2$ -map ϕ' defined in the text.*

We also have the nonequivariant stable Hopf invariant

$$h''_V(F) \in \omega^0\{N^+; (S(LV) \times_{\mathbb{Z}/2} (M \times M))^{\nu \times \nu}\}.$$

The free $\mathbb{Z}/2$ -manifold $\mathfrak{D}(f)$ is a double cover of the set $\overline{\mathfrak{D}}(f) \subseteq N$ of double points of the immersion f . We have an induced map

$$\overline{\mathfrak{D}}(f) = \mathfrak{D}(f)/\mathbb{Z}/2 \rightarrow S(LV) \times_{\mathbb{Z}/2} (M \times M)$$

and an embedding of $\overline{\mathfrak{D}}(f)$ in N with normal bundle the pullback of $\nu \times \nu$. The Pontryagin–Thom construction gives a map

$$\phi'' : N^+ \rightarrow \overline{\mathfrak{D}}(f)^{\nu \times \nu} \rightarrow (S(LV) \times_{\mathbb{Z}/2} (M \times M))^{\nu \times \nu}.$$

Corollary 2.3. *The stable geometric Hopf invariant $h_V''(F)$ is equal to the stabilization $[\phi'']$ of the class determined by the double point manifold $\overline{\mathfrak{D}}(f) \subseteq N$.*

These results will be obtained as consequences of a more precise fibre-wise theorem, which we describe next.

Let $\mathcal{C} \rightarrow N$ be the space of pairs (x, α) , where $x \in M$ and $\alpha : [0, 1] \rightarrow N$ is a continuous path such that $\alpha(0) = f(x)$, fibred over N by projection to the other endpoint $\alpha(1)$. We have a homotopy equivalence $\pi : \mathcal{C} \rightarrow M$ given by $\pi(x, \alpha) = x$.

The homotopy Pontryagin–Thom map defined by the embedding $(e, f) : M \hookrightarrow V \times N$ as described in [4, Section 6] (and in [5]) is a pointed map

$$\tilde{F} : N \times V^+ \rightarrow (N \times V^+) \wedge_N \mathcal{C}_N^{\pi^* \nu}$$

with compact supports over N . (To be exact, the space \mathcal{C} in [4] is the space of pairs (x, β) , where $x \in M$ and $\beta : [0, 1] \rightarrow V \times N$ is a path starting at $\beta(0) = (e(x), f(x))$. We omit here the redundant component in the contractible space V .) We may then form the fibrewise geometric Hopf invariant

$$h_V(\tilde{F}) \in {}^{\mathbb{Z}/2} \omega_N^0 \{N \times \Sigma S(LV)_+; (N \times LV^+) \wedge_N (\mathcal{C}_N^{\pi^* \nu} \wedge_N \mathcal{C}_N^{\pi^* \nu})\},$$

again as a stable $\mathbb{Z}/2$ -equivariant map with compact supports over N .

Now the fibre product $\mathcal{C} \times_N \mathcal{C}$ of pairs $((x, \alpha), (y, \beta))$ such that $\alpha(1) = \beta(1)$ may be identified, by splicing α to the reversed path β , with the space $\mathbf{h}\text{-}\widehat{\mathfrak{D}}(f)$ of triples (x, y, γ) with $x, y \in M$ and $\gamma : [-1, 1] \rightarrow N$ a continuous path from $\gamma(-1) = f(x)$ to $\gamma(1) = f(y)$ projecting to $\gamma(0) \in N$. (Thus $\gamma(t)$ is $\alpha(1+t)$ if $-1 \leq t \leq 0$, $\beta(1-t)$ if $0 \leq t \leq 1$.) It has an action of $\mathbb{Z}/2$ in which the involution interchanges x and y and reverses the path γ . The double point set $\mathfrak{D}(f)$ is included in $\mathbf{h}\text{-}\widehat{\mathfrak{D}}(f)$ as the space of constant paths.

Let \mathcal{D} be the space of pairs $((x, y), \alpha)$, where $(x, y) \in \mathfrak{D}(f)$ and $\alpha : [0, 1] \rightarrow N$ is a path such that $\alpha(0) = f(x) = f(y)$. This corresponds to the subspace of points $(x, y, \gamma) \in \mathbf{h}\text{-}\widehat{\mathfrak{D}}(f)$ with $\gamma(-t) = \gamma(t)$. The fibrewise Pontryagin–Thom construction on $\mathfrak{D}(f) \hookrightarrow (LV - \{0\}) \times N$ gives a fibrewise map

$$N \times \Sigma S(LV)_+ \rightarrow (N \times LV^+) \wedge_N \mathcal{D}_N^{\nu \times \nu},$$

which we compose with the inclusion

$$\mathcal{D}_N^{\nu \times \nu} \hookrightarrow \mathbf{h}\text{-}\widehat{\mathfrak{D}}(f)_N^{\nu \times \nu}$$

to get an equivariant fibrewise map

$$\tilde{\phi} : N \times \Sigma S(LV)_+ \rightarrow (N \times LV^+) \wedge_N \mathbf{h}\text{-}\widehat{\mathfrak{D}}(f)_N^{\nu \times \nu}.$$

Theorem 2.4 (Homotopy double point theorem). *The fibrewise geometric Hopf invariant*

$$h_V(\tilde{F}) \in {}^{\mathbb{Z}/2}_c \omega_N^0 \left\{ N \times \Sigma S(LV)_+; (N \times LV^+) \wedge_N h\text{-}\widehat{\mathfrak{D}}(f)_N^{\nu \times \nu} \right\}$$

of the homotopy Pontryagin–Thom map \tilde{F} is equal to the fibrewise stable class of the map $\tilde{\phi}$ determined by the double points of f .

The dual version is a class

$$h'_V(\tilde{F}) \in {}^{\mathbb{Z}/2}_c \omega_N^0 \left\{ N \times S^0; (N \times S(LV)_+) \wedge_N h\text{-}\widehat{\mathfrak{D}}(f)_N^{\nu \times \nu} \right\}.$$

There is also a nonequivariant form. The stable Hopf invariant $h''_V(\tilde{F})$ lies in

$${}_c \omega_N^0 \{ N \times S^0; (S(LV) \times_{\mathbb{Z}/2} h\text{-}\widehat{\mathfrak{D}}(f))_N^{\nu \times \nu} \},$$

and this group can be identified with

$$\tilde{\omega}_0 \left((S(LV) \times_{\mathbb{Z}/2} h\text{-}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \right)$$

by fibrewise Poincaré–Atiyah duality. (For a general treatment of fibrewise duality see, for example, [5, Part II, Section 12]. The duality theorem required here is stated in [4] as Proposition 4.1.)

The Pontryagin–Thom construction applied to the double point manifold $\overline{\mathfrak{D}}(f) = \mathfrak{D}(f)/(\mathbb{Z}/2)$ equipped with the map $\overline{\mathfrak{D}}(f) \rightarrow S(LV) \times_{\mathbb{Z}/2} h\text{-}\widehat{\mathfrak{D}}(f)$, given by the inclusion $\mathfrak{D}(f) \rightarrow h\text{-}\widehat{\mathfrak{D}}(f)$, and the map $\mathfrak{D}(f) \rightarrow S(LV)$, given by e (via d), produces a stable homotopy class

$$\tilde{\phi}'' : S^0 \rightarrow (S(LV) \times_{\mathbb{Z}/2} h\text{-}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N}.$$

Corollary 2.5. *We have*

$$h''_V(\tilde{F}) = \tilde{\phi}'' \in \tilde{\omega}_0 \left((S(LV) \times_{\mathbb{Z}/2} h\text{-}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \right).$$

Before turning to the proof of Theorem 2.4, we explain how the fibrewise Hopf invariant $h_V(\tilde{F})$ determines the simpler invariant $h_V(F)$.

Lemma 2.6. *The Hopf invariant $h_V(F)$ is the image of $h_V(\tilde{F})$ under the composition*

$$\begin{aligned} & {}^{\mathbb{Z}/2}_c \omega^0 \left\{ N \times \Sigma S(LV)_+; (N \times LV^+) \wedge_N h\text{-}\widehat{\mathfrak{D}}(f)_N^{\nu \times \nu} \right\} \\ & \rightarrow {}^{\mathbb{Z}/2}_c \omega^0 \left\{ N^+ \wedge \Sigma S(LV)_+; LV^+ \wedge h\text{-}\widehat{\mathfrak{D}}(f)^{\nu \times \nu} \right\} \\ & \rightarrow {}^{\mathbb{Z}/2}_c \omega^0 \left\{ N^+ \wedge \Sigma S(LV)_+; LV^+ \wedge (M^\nu \wedge M^\nu) \right\} \end{aligned}$$

of the homomorphism defined by collapsing the basepoint sections over N to a point and that induced by the projection $\pi \times \pi : h\text{-}\widehat{\mathfrak{D}}(f) = \mathcal{C} \times_N \mathcal{C} \rightarrow M \times M$.

Proof. This is easily seen from the explicit construction of the geometric Hopf invariant. \square

In a similar manner, $h'_V(F)$ and $h''_V(F)$ are the images of the refined invariants $h'_V(\tilde{F})$ and $h''_V(\tilde{F})$ under homomorphisms defined by collapsing basepoint sections over N and projecting from $\mathrm{h}\text{-}\widehat{\mathfrak{D}}(f)$ to $M \times M$.

Remark 2.7. This construction also provides the π -equivariant Hopf invariant considered, in greater detail, in [6].

(i) Suppose that π is a discrete group and that $q : \tilde{N} \rightarrow N$ is a principal π -bundle (for example, a universal covering space with π the fundamental group of N). We let $\widetilde{M} = f^*\tilde{N}$ be the induced bundle over M ; thus

$$\widetilde{M} = \{(x, z) \in M \times \tilde{N} \mid f(x) = q(z)\}.$$

We may define a map $\mathrm{h}\text{-}\widehat{\mathfrak{D}}(f) \rightarrow (\widetilde{M} \times \widetilde{M})/\pi$ as follows. Given $(x, y, \gamma) \in \mathrm{h}\text{-}\widehat{\mathfrak{D}}(f)$ (so $x, y \in M$, $\gamma : [-1, 1] \rightarrow N$, $\gamma(-1) = f(x)$, $\gamma(1) = f(y)$), lift γ to a path $\tilde{\gamma}$ in \tilde{N} , determined up to multiplication by an element of π . The π -orbit of $((x, \tilde{\gamma}(-1)), (y, \tilde{\gamma}(1))) \in \widetilde{M} \times \widetilde{M}$ is independent of the choice of the lift. The Hopf invariant $h''_V(\tilde{F})$ thus gives us an element of

$$\omega^0\{N^+; (S(LV) \times_{\mathbb{Z}/2} ((\widetilde{M} \times \widetilde{M})/\pi))^{\nu \times \nu}\},$$

where $\nu \times \nu$ is lifted from $M \times M$ to $(\widetilde{M} \times \widetilde{M})/\pi$ by the obvious projection.

(ii) Let $(f, b) : N \rightarrow X$ be a normal map from an n -dimensional manifold N to an n -dimensional geometric Poincaré complex X . Let \tilde{X} be the universal cover of X , let $\tilde{N} = f^*\tilde{X}$ be the pullback cover of N and let $\pi = \pi_1(X)$. The Umkehr $\mathbb{Z}[\pi]$ -module chain map

$$f^! : C(\tilde{X}) \simeq C(\tilde{X})^{n-*} \xrightarrow{\tilde{f}^*} C(\tilde{N})^{n-*} \simeq C(\tilde{N})$$

is induced by a π -equivariant geometric Umkehr map $F : \Sigma^\infty \tilde{X}_+ \rightarrow \Sigma^\infty \tilde{N}_+$ π -equivariant S -dual to $T(\tilde{b}) : T(\tilde{\nu}_N) \rightarrow T(\tilde{\nu}_X)$. The Wall surgery obstruction $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi])$ was identified in [19] with the cobordism class of an n -dimensional quadratic Poincaré complex $(C(f^!), \psi)$ over $\mathbb{Z}[\pi]$, with ψ the evaluation of the “quadratic construction” ψ_F on the fundamental class $[X] \in H_n(X)$. In [6] we shall express ψ_F in terms of the π -equivariant geometric Hopf invariant of F .

(iii) See [12] for an application of the π -equivariant Hopf invariant to diagonals in geometric Poincaré complexes.

In view of the relations between the various Hopf invariants, it will suffice to prove Theorem 2.4 in order to establish Theorem 2.1 and their sundry corollaries.

Proof of Theorem 2.4. Writing $\bar{\nu}$ for the normal bundle of $\overline{\mathfrak{D}}(f)$ in N , choose a tubular neighbourhood $D(\bar{\nu}) \hookrightarrow N$ of $\overline{\mathfrak{D}}(f)$. For $(x, y) \in \mathfrak{D}(f)$, the fibre of $\bar{\nu}$ at $f(x) = f(y)$ is $\nu_x \oplus \nu_y$. We identify $f^{-1}(\overline{\mathfrak{D}}(f))$ with $\mathfrak{D}(f)$ by projecting to the first factor. Then the inverse image of the tubular neighbourhood $D(\bar{\nu})$ is a tubular neighbourhood $D(\nu')$ of $\mathfrak{D}(f)$ in M , where $\nu'_x = \nu_y$. The normal bundle ν restricted to $D(\nu')$ is then identified with the restriction of ν to $\mathfrak{D}(f)$.

We may assume that e vanishes outside the tubular neighbourhood $D(\nu')$. The homotopy Pontryagin–Thom map is then a V -fold suspension outside $D(\bar{\nu})$. By Proposition 1.5, the fibrewise Hopf invariant is canonically null-homotopic outside $D(\bar{\nu})$. The Hopf invariant $h_V(\tilde{F})$ is thus represented by a fibrewise map which is zero outside the tubular neighbourhood. In this way we localize $h_V(\tilde{F})$ to an element of

$$\mathbb{Z}/2 \omega_{(D(\bar{\nu}), S(\bar{\nu}))}^0 \left\{ N \times \Sigma S(LV)_+; (N \times LV^+) \wedge_N \mathbf{h}\widehat{\mathfrak{D}}(f)_N^{\nu \times \nu} \right\}$$

constructed from the immersion data in a neighbourhood of the double points.

The local data consists simply of the double cover $\mathfrak{D}(f) \rightarrow \overline{\mathfrak{D}}(f)$ and the vector bundle ν over $\mathfrak{D}(f)$. The bundle ν' is the pullback of ν under the covering involution and $\bar{\nu}$ over $\overline{\mathfrak{D}}(f)$ is the push-forward of ν . The “local M ” is the total space of ν' over $\mathfrak{D}(f)$, and the “local N ” is the total space of $\bar{\nu}$ over $\overline{\mathfrak{D}}(f)$. The immersion f is given by the projection $\mathfrak{D}(f) \rightarrow \overline{\mathfrak{D}}(f)$. We also need the map e and, without loss of generality, we may assume that it is determined by a $\mathbb{Z}/2$ -map $\mathfrak{D}(f) \rightarrow S(LV)$ (extended radially on $D(\nu')$ to taper to zero).

In the fibre over $\{x, y\} \in \overline{\mathfrak{D}}(f)$, the manifold M is the disjoint union

$$(\{0\} \times \nu_y) \sqcup (\nu_x \times \{0\}) \hookrightarrow \nu_x \oplus \nu_y$$

immersed in the fibre of N by projection, with the double point at $(0, 0)$. Write $e(x) = v$, $e(y) = -v$. The Hopf invariant is calculated by the sum formula of Proposition 1.7. In the notation used there, we take $X = \nu_x^+ \times \nu_y^+$, $Y = \nu_x^+ \vee \nu_y^+$, and the maps F_+ and F_- are suspensions of the compositions of the projection to ν_x^+ or ν_y^+ , respectively, and the inclusion of the wedge summand. The Hopf invariants of the suspensions F_+ and F_- vanish and the Hopf invariant of the sum is determined by the product $X \rightarrow Y \times Y$, so by the projection $\nu_x^+ \times \nu_y^+ \rightarrow \nu_x^+ \wedge \nu_y^+$.

The same computation performed fibrewise over $\overline{\mathfrak{D}}(f)$ gives the localized fibrewise Hopf invariant as the image of the element in

$$\omega_{(D(\bar{\nu}), S(\bar{\nu}))}^0 \left\{ D(\bar{\nu}) \times S^0; \bar{\nu}_{D(\bar{\nu})}^+ \right\}$$

given by the inclusion of $D(\bar{\nu})$ in $\bar{\nu}$. Unravelling the definitions, one sees that this produces $[\phi]$. \square

In the remainder of the paper we shall be concerned with the behaviour of the geometric Hopf invariant $h_V(\tilde{F})$ as one deforms the map $f : M \rightarrow N$. Suppose, to begin, that we have smooth homotopies $f_t : M \rightarrow N$ and $e_t : M \rightarrow V$ such that each f_t is an immersion and such that $(e_t, f_t) : M \rightarrow V \times N$ is an embedding for each $t \in [0, 1]$. We write $f = f_0$, $f' = f_1$ and $e = e_0$, $e' = e_1$. Then we have, up to homotopy, an isomorphism $\nu' = \nu(f_1) \rightarrow \nu(f_0) = \nu$ between the normal bundles of the immersions and a fibre homotopy equivalence $\mathbf{h}\widehat{\mathfrak{D}}(f') \rightarrow \mathbf{h}\widehat{\mathfrak{D}}(f)$ over N . In the standard terminology, the homotopy f_t is a *regular homotopy* from f to f' and the

homotopy (e_t, f_t) is an *isotopy* from (e, f) to (e', f') . Let F' be the map obtained from the homotopy Pontryagin–Thom construction on (e', f') .

Proposition 2.8 (Regular homotopy/isotopy invariance). *The fibrewise Hopf invariants $h_V(\tilde{F}')$ and $h_V(\tilde{F})$ correspond under the induced isomorphism:*

$$\begin{aligned} & \mathbb{Z}/2 \omega_N^0 \left\{ N \times \Sigma S(LV)_+; (N \times LV^+) \wedge_N h\text{-}\widehat{\mathfrak{D}}(f')_N^{\nu' \times \nu'} \right\} \\ & \rightarrow \mathbb{Z}/2 \omega_N^0 \left\{ N \times \Sigma S(LV)_+; (N \times LV^+) \wedge_N h\text{-}\widehat{\mathfrak{D}}(f)_N^{\nu \times \nu} \right\}. \end{aligned}$$

Proof. This follows from the homotopy invariance of the geometric Hopf invariant, which in turn follows easily from the explicit construction described in the introduction. \square

3. Immersions and embeddings

The vanishing of the stable Hopf invariant $h_V(\tilde{F})$ is a necessary condition for the existence of a regular homotopy f_t and isotopy (e_t, f_t) such that $f' = f_1$ is an embedding. In the opposite direction, we first record the following.

Proposition 3.1. *Suppose that $m < 2(n - 1)$ and that $h_V(\tilde{F}) = 0$. Then \tilde{F} is homotopic (through maps with compact support over N) to the V -fold suspension of a map determined by a section $N \rightarrow C_N^{\pi^* \nu}$.*

Proof. This is a consequence of the fibrewise EHP-sequence (see [5, Part II, Proposition 14.39]), applicable because $0 \leq 3(n - m - 1) - (n + 0)$. \square

To proceed further, we shall assume that the dimension of V is large, $\dim V > 2m$. This guarantees that M can be embedded in V , and we may take the map $e : M \rightarrow V$ to be an embedding. In a metastable range, the fibrewise Hopf invariant of \tilde{F} is the precise obstruction to the existence of a regular homotopy from the immersion $f : M \looparrowright N$ to an embedding.

Theorem 3.2 (Haefliger [8], Hatcher and Quinn [10]). *Suppose that $3m < 2(n - 1)$ and $\dim V > 2m$. Then*

$$\begin{aligned} h_V''(\tilde{F}) & \in \tilde{\omega}_0 \left((S(LV) \times_{\mathbb{Z}/2} h\text{-}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \right) \\ & = \tilde{\omega}_0 \left((E\mathbb{Z}/2 \times_{\mathbb{Z}/2} h\text{-}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \right) \end{aligned}$$

vanishes if and only if f is homotopic through immersions to an embedding of M into N .

We define $h\text{-}\mathfrak{D}(f)$ to be the subspace of $h\text{-}\widehat{\mathfrak{D}}(f)$ consisting of the triples (x, y, γ) such that $x \neq y$ and we call $h\text{-}\mathfrak{D}(f)$ the space of *homotopy double points* of the immersion f . The Hopf bundle $E\mathbb{Z}/2 \times_{\mathbb{Z}/2} L$ over $B\mathbb{Z}/2$ will be denoted by H .

Lemma 3.3. *Let $\mathfrak{L}(f)$ be the complement of $\mathfrak{h}\text{-}\mathfrak{D}(f)$ in $\mathfrak{h}\text{-}\widehat{\mathfrak{D}}(f)$. Then one has a homotopy cofibration sequence:*

$$\begin{aligned} (E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \mathfrak{h}\text{-}\mathfrak{D}(f))^{\nu \times \nu - \tau N} &\rightarrow (E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \mathfrak{h}\text{-}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \\ &\rightarrow (E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \mathfrak{L}(f))^{H \otimes \tau M + \nu \times \nu - \tau N}, \end{aligned}$$

in which the first map is given by the inclusion of $\mathfrak{h}\text{-}\mathfrak{D}(f)$ in $\mathfrak{h}\text{-}\widehat{\mathfrak{D}}(f)$ and the second by the homotopy Pontryagin–Thom construction on the diagonal submanifold M in $M \times M$.

As a space over M , $\mathfrak{L}(f)$ is the pullback by $f : M \rightarrow N$ of the free loop space of N , $\text{map}(S^1, N) \rightarrow N$, fibred over N by evaluation at $1 \in S^1 (\subseteq \mathbb{C})$.

Proof. The vector bundle $L \otimes \tau M$, corresponding to $H \otimes \tau M$ over $B\mathbb{Z}/2$, is the normal bundle of the diagonal inclusion of M in $M \times M$. Choose a $\mathbb{Z}/2$ -equivariant tubular neighbourhood $D(L \otimes \tau M) \hookrightarrow M \times M$ of the diagonal in $M \times M$. The inclusion of the subspace of $\mathfrak{h}\text{-}\mathfrak{D}(f)$ consisting of the triples (x, y, γ) such that $(x, y) \notin B(L \otimes \tau M)$ into $\mathfrak{h}\text{-}\mathfrak{D}(f)$ is a homotopy equivalence.

The argument used in the proof of Lemma 6.1 in [4] then shows that we have a homotopy cofibre sequence. \square

Corollary 3.4. *The inclusion induces an isomorphism*

$$\tilde{\omega}_0 \left((E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \mathfrak{h}\text{-}\mathfrak{D}(f))^{\nu \times \nu - \tau N} \right) \rightarrow \tilde{\omega}_0 \left((E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \mathfrak{h}\text{-}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \right)$$

provided that $m < n - 1$.

Proof. This follows at once from the long exact sequence of the cofibration. \square

The involution on $\mathfrak{h}\text{-}\mathfrak{D}(f)$ is free, and hence, writing $\mathfrak{h}\text{-}\overline{\mathfrak{D}}(f)$ for the quotient $\mathfrak{h}\text{-}\mathfrak{D}(f)/(\mathbb{Z}/2)$, we have an isomorphism

$$\tilde{\omega}_0 \left((E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \mathfrak{h}\text{-}\mathfrak{D}(f))^{\nu \times \nu - \tau N} \right) \rightarrow \tilde{\omega}_0 \left(\mathfrak{h}\text{-}\overline{\mathfrak{D}}(f)^{\nu \times \nu - \tau N} \right).$$

Until now, we have thought of $\mathfrak{h}\text{-}\widehat{\mathfrak{D}}(f)$, which arose as the fibre product $\mathcal{C} \times_N \mathcal{C}$, as a space over N . It also fibres over $M \times M$ and the fibrewise space $\mathfrak{h}\text{-}\widehat{\mathfrak{D}}(f) \rightarrow M \times M$ is $\mathbb{Z}/2$ -equivariantly locally fibre homotopy trivial. (Compare [4, Definition 2.3].) In the same way, $\mathfrak{h}\text{-}\mathfrak{D}(f)$ is fibred over the complement $M \times M - \Delta(M)$ and $\mathfrak{h}\text{-}\overline{\mathfrak{D}}(f)$ is fibred over $(M \times M - \Delta(M))/\mathbb{Z}/2$.

Proof of Theorem 3.2. We shall use results and terminology from [4, Section 7], which derive from the work of Koschorke [16] and Klein and Williams [13].

Suppose, first, that M is connected. Put $\tilde{B} = M \times M - B(L \otimes \tau M)$; it is a manifold with a free $\mathbb{Z}/2$ -action. Let B be the manifold $\tilde{B}/\mathbb{Z}/2$ with boundary over the projective bundle, $\partial B = P(\tau M)$. Let E be the bundle $(\tilde{B} \times (N \times N))/\mathbb{Z}/2$ over B and $Z \subseteq E$ the diagonal subbundle $B \times N = (\tilde{B} \times N)/\mathbb{Z}/2$. The fibrewise normal bundle of the inclusion $Z \hookrightarrow E$ is $H \otimes \tau N$, where H is the line bundle over B associated with the double cover. The $\mathbb{Z}/2$ -equivariant square $f \times f : \tilde{B} \rightarrow N \times N$ defines a section s of $E \rightarrow B$,

which, if the tubular neighbourhood is chosen to be sufficiently small, has the property that $s(x) \notin Z_x$ for $x \in \partial B$. Together, the fibre bundle $E \rightarrow B$, the subbundle $Z \rightarrow B$ and the section s constitute what is called in [4] an intersection problem. In the language used there, s is nowhere null on the boundary ∂B and the homotopy null-set $\text{h-Null}(s)$, fibred over B , is easily identified with the restriction of $\text{h-}\mathfrak{D}(f) \rightarrow (M \times M - \Delta(M))/\mathbb{Z}/2$. The inclusion $\text{h-Null}(s) \hookrightarrow \text{h-}\mathfrak{D}(f)$ is, as we have already noted in the proof of Lemma 3.3, a homotopy equivalence.

If f is an embedding, then the section s is nowhere null. Now the homotopy Euler index (see [4, Definition 7.3])

$$\text{h-}\gamma(s; \partial B) \in \tilde{\omega}_0(\text{h-Null}(s)^{H \otimes \tau N - \tau B})$$

is an obstruction to deforming s , through a homotopy constant on ∂B , to a section that is nowhere null. By Proposition 7.4 of [4] it is the precise obstruction if $\dim B < 2(\dim N - 1)$. We conclude that if $\text{h-}\gamma(s; \partial B) = 0$ and $m < n - 1$, then

$$f \times f : M \times M - B(L \otimes \tau M) \rightarrow N \times N$$

is $\mathbb{Z}/2$ -equivariantly homotopic, by a homotopy that is constant on the boundary $SL \otimes \tau M$, to a map into $N \times N - \Delta(N)$. According to Haefliger [8, Théorème 2], if further $3m < 2(n - 1)$, then this is a sufficient condition for $f : M \looparrowright N$ to be homotopic through immersions to an embedding of M into N .

Thus far, the argument is taken from [13, Theorem A.4 and Corollary A.5]. We now relate the index $\text{h-}\gamma(s; \partial B)$ to the geometric Hopf invariant.

Lemma 3.5. *The homotopy Euler index*

$$\text{h-}\gamma(s; \partial B) \in \tilde{\omega}_0(\text{h-Null}(s)^{\nu \times \nu - \tau N})$$

corresponds to the fibrewise Hopf invariant

$$h''_V(\tilde{F}) \in \tilde{\omega}_0 \left((E\mathbb{Z}/2 \otimes_{\mathbb{Z}/2} \text{h-}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \right).$$

Proof. The correspondence is made by the homotopy equivalence $\text{h-Null}(s) \hookrightarrow \text{h-}\mathfrak{D}(f)$ and the isomorphism from Corollary 3.4.

We can assume that f satisfies the conditions for the homotopy double point theorem. Then $h''_V(\tilde{F})$ is represented by the double point manifold $\widehat{\mathfrak{D}}(f)$. By Proposition 7.8 of [4], the homotopy Euler index is represented by the null-set $\text{Null}(s)$, which is exactly $\mathfrak{D}(f)$. \square

Hence the vanishing of $h''_V(\tilde{F})$ implies, in the metastable range, that f is regularly homotopic to an embedding.

This has dealt with the case in which M is connected. Now suppose that M is a disjoint union $M^{(1)} \sqcup M^{(2)}$ and write $f^{(i)}$ for the restriction of f to $M^{(i)}$. Then $\text{h-}\widehat{\mathfrak{D}}(f)$ splits equivariantly over

$$M \times M = (M^{(1)} \times M^{(1)}) \sqcup (M^{(2)} \times M^{(2)}) \sqcup (M^{(1)} \times M^{(2)} \sqcup M^{(2)} \times M^{(1)})$$

as a disjoint union

$$\mathrm{h}\widehat{\mathfrak{D}}(f^{(1)}) \sqcup \mathrm{h}\widehat{\mathfrak{D}}(f^{(2)}) \sqcup (\mathbb{Z}/2 \times \mathrm{h}\mathfrak{J}(f^{(1)}, f^{(2)})),$$

where $\mathrm{h}\mathfrak{J}(f^{(1)}, f^{(2)})$ is the space of triples (x, y, γ) with $(x, y) \in M^{(1)} \times M^{(2)}$ and $\gamma(-1) = f^{(1)}(x)$, $\gamma(1) = f^{(2)}(y)$. The fibrewise Hopf invariant decomposes, according to Proposition 1.7, as a sum of three terms. The first two are the fibrewise Hopf invariants of $f^{(1)}$ and $f^{(2)}$; the third, (12)-component, is a more elementary product obstruction.

Arguing by induction, we may suppose that both $f^{(1)}$ and $f^{(2)}$ are embeddings, intersecting transversely, and that $M^{(2)}$ is connected. We consider a new intersection problem with $B = M^{(2)}$, $E = B \times N$ and $Z = B \times f(M^{(1)})$. Let $s : B \rightarrow E$ be the section $s(x) = (x, f^{(2)}(x))$. This time $\mathrm{h}\text{-Null}(s)$ is $\mathrm{h}\mathfrak{J}(f^{(1)}, f^{(2)})$, and we may identify the homotopy Euler index $\mathrm{h}\gamma(s)$ in the same way with the (12)-component of the fibrewise Hopf invariant, both being represented by the manifold $f(M^{(1)}) \cap f(M^{(2)})$. By Proposition 7.4 of [4] again, the vanishing of the homotopy Euler index implies that $f^{(2)}$ is homotopic to a map into $N - f(M^{(1)})$, because $\dim M^{(2)} < 2(n - \dim M^{(1)} - 1)$. Finally, we may apply [10, Theorem 1.1] to deduce that $f^{(2)}$ is isotopic to an embedding of $M^{(2)}$ into the complement of $f(M^{(1)})$. This inductive step is enough to conclude the proof of Theorem 3.2. \square

Remark 3.6. Suppose that M (as well as N) is connected. Choose basepoints $* \in M$ and $* = f(*) \in N$. Then we can include the loop-space ΩN in $\mathrm{h}\widehat{\mathfrak{D}}(f)$ by mapping a loop $\gamma : [-1, 1] \rightarrow N$, with $\gamma(-1) = * = \gamma(1)$, to $(*, *, \gamma) \in \mathrm{h}\widehat{\mathfrak{D}}(f)$, and the set of path components of $\mathrm{h}\widehat{\mathfrak{D}}(f)$ is in this way identified with the set of double cosets

$$f_*\pi_1(M) \backslash \pi_1(N) / f_*\pi_1(M) = \pi_0(\mathrm{h}\widehat{\mathfrak{D}}(f))$$

with the $\mathbb{Z}/2$ -action given by the group-theoretic inverse. Thus we may identify the set of path components of $S(LV) \times_{\mathbb{Z}/2} \mathrm{h}\widehat{\mathfrak{D}}(f)$, if $\dim V > 1$, with the orbit space of the involution.

Example 3.7. Suppose that $n = 2m$. Then $\tilde{\omega}_0((E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \mathrm{h}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N})$ is a direct sum of groups indexed by $\pi_0(\mathrm{h}\widehat{\mathfrak{D}}(f))/\mathbb{Z}/2$, each component being isomorphic to \mathbb{Z} or $\mathbb{Z}/2\mathbb{Z}$. When M is connected we may label the summands as in Remark 3.6 by equivalence classes of group elements $g \in \pi_1(N)$. Let $w_M : \pi_1(M) \rightarrow \{\pm 1\}$ and $w_N : \pi_1(N) \rightarrow \{\pm 1\}$ be the orientation maps (corresponding to w_1M and w_1N). The g -summand is isomorphic to \mathbb{Z} if and only if for all $a, b \in \pi_1(M)$, (i) if $f_*(a)g = gf_*(b)$, then $w_M(ab) = w_N(f_*(a)) (= w_N(f_*(b)))$, and (ii) if $f_*(a)g = g^{-1}f_*(b)$, then $(-1)^m w_M(ab) = w_N(f_*(a)g)$. In particular, if M is orientable, N is oriented and $f_*\pi_1(M)$ is trivial, then the obstruction group is $\mathbb{Z}[\pi_1(N)]/\langle g - (-1)^m g^{-1} \mid g \in \pi_1(N) \rangle$ and the Hopf invariant $h_V''(\tilde{F})$ is Wall's invariant $\mu(f)$ in the form mentioned in the introduction.

4. Homotopic immersions

We next investigate immersions homotopic, as maps, to the given immersion f . Consider smooth homotopies $f_t : M \rightarrow N$ and $e_t : M \rightarrow V$ such that f_0 and f_1 are immersions and each map $(e_t, f_t) : M \rightarrow V \times N$, for $0 \leq t \leq 1$, is an embedding with normal bundle μ_t . We again write $f = f_0$, $f' = f_1$, and $\nu = \nu(f)$, $\nu' = \nu(f')$. The homotopies determine, up to a homotopy, a bundle isomorphism $a : V \oplus \nu = \mu_1 \rightarrow \mu_0 = V \oplus \nu$ and a $\mathbb{Z}/2$ -equivariant fibre homotopy equivalence $\mathrm{h}\widehat{\mathfrak{D}}(f') \rightarrow \mathrm{h}\widehat{\mathfrak{D}}(f)$ over $M \times M$.

There is an associated class

$$\theta(e_t, f_t) \in \tilde{\omega}_0 \left((P(V) \times M)^{H \otimes \nu - \tau M} \right)$$

which vanishes if each f_t is an immersion. It is constructed as follows. Let $v_0 : V \rightarrow V \oplus \nu$ be the inclusion and let $v_1 : V \rightarrow V \oplus \nu' = \mu_1 \rightarrow \mu_0 = V \oplus \nu$ be the composition of the inclusion with the isomorphism a . Now we have a stable cohomotopy difference class

$$\delta(v_0, v_1) \in \tilde{\omega}^{-1} \left((M \times P(V))^{-H \otimes (V \oplus \nu)} \right),$$

which is the metastable obstruction to deforming v_0 to v_1 through vector bundle monomorphisms; see Section 6 for details. We define $\theta(e_t, f_t)$ to be the dual class in stable homotopy. (Recall that $\tau P(V) \oplus \mathbb{R} = H \otimes V$.)

Using the inclusion $M \hookrightarrow \mathrm{h}\widehat{\mathfrak{D}}(f) : x \mapsto (x, x, \gamma)$, where γ is the constant loop at $f(x)$, we get a map

$$i : \tilde{\omega}_0 \left((P(V) \times M)^{H \otimes \nu - \tau M} \right) \rightarrow \tilde{\omega}_0 \left((S(LV) \times_{\mathbb{Z}/2} \mathrm{h}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \right).$$

Remark 4.1. The map $M \rightarrow \mathrm{h}\widehat{\mathfrak{D}}(f)$ picks out (under the assumption that M is connected) a component $\mathrm{h}\widehat{\mathfrak{D}}(f)_0$ of $\mathrm{h}\widehat{\mathfrak{D}}(f)$, which is preserved by $\mathbb{Z}/2$. Hence i maps into the summand

$$\tilde{\omega}_0 \left((S(LV) \times_{\mathbb{Z}/2} \mathrm{h}\widehat{\mathfrak{D}}(f)_0)^{\nu \times \nu - \tau N} \right).$$

The link between the difference class θ and the Hopf invariant is forged by a generalization of the classical description of the Hopf invariant on the image of the J -homomorphism.

Proposition 4.2. *Let $a : V \oplus \nu' \rightarrow V \oplus \nu$ be a vector bundle isomorphism over M . The fibrewise one-point compactification of a is a map of sphere bundles*

$$A : (M \times V^+) \wedge_M (\nu')_M^+ \rightarrow (M \times V)^+ \wedge \nu_M^+$$

over M . Then the fibrewise Hopf invariant

$$h_V(A) \in {}^{\mathbb{Z}/2}\omega_M^0 \{ (M \times \Sigma S(LV)_+) \wedge_M (\nu')_M^+; (M \times LV^+) \wedge_M (\nu_M^+ \wedge_M \nu_M^+) \}$$

of A coincides up to sign, under the identifications described below, with the difference class

$$\delta(v_0, v_1) \in \tilde{\omega}^{-1} \left((M \times P(V))^{-H \otimes (V \oplus \nu)} \right)$$

of the monomorphisms $v_0, v_1 : V \rightarrow V \oplus \nu$ given, respectively, by the inclusion of the first factor and the composition of the inclusion $V \rightarrow V \oplus \nu'$ with a .

Proof. The fibrewise smash product $\nu_M^+ \wedge_M \nu_M^+ = (\nu \oplus \nu)_M^+$ with the action of $\mathbb{Z}/2$ which interchanges the factors is equivariantly homeomorphic (by the construction κ_V in Section 1) to $(\nu \oplus L\nu)_M^+ = \nu_M^+ \wedge (L\nu)_M^+$. The isomorphism $a : V \oplus \nu' \rightarrow V \oplus \nu$ gives a stable fibre homotopy equivalence $(\nu')_M^+ \rightarrow \nu_M^+$. Taken together, these equivalences allow us to think of $h_V(A)$ as an element of the group

$$\mathbb{Z}/2 \omega_M^0 \{M \times \Sigma S(LV)_+; (LV \oplus L\nu)_M^+\},$$

which is then identified with

$$\mathbb{Z}/2 \tilde{\omega}^{-1} \left((M \times S(LV))^{-(LV \oplus L\nu)} \right) = \tilde{\omega}^{-1} \left((M \times P(V))^{-H \otimes (V \oplus \nu)} \right).$$

Both $h_V(A)$ and $\delta(v_0, v_1)$ are defined by difference constructions. The proof that they coincide follows from a direct comparison of the definitions. \square

Theorem 4.3 (Homotopy/isotopy variation). *The Hopf invariant*

$$h_V''(\tilde{F}') \in \tilde{\omega}_0 \left((S(LV) \times_{\mathbb{Z}/2} \mathbf{h}\widehat{\mathfrak{D}}(f'))^{\nu' \times \nu' - \tau N} \right)$$

corresponds to

$$h_V''(\tilde{F}) + i\theta(e_t, f_t) \in \tilde{\omega}_0 \left((S(LV) \times_{\mathbb{Z}/2} \mathbf{h}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \right).$$

Proof. Recollect that \tilde{F} is a fibrewise map with compact supports over N :

$$N \times V^+ \rightarrow (N \times V^+) \wedge_N \mathcal{C}_N^{\pi^* \nu}.$$

Up to homotopy, the map \tilde{F}' associated with (e', f') is the composition of \tilde{F} with the map

$$(N \times V^+) \wedge_N \mathcal{C}_N^{\pi^* \nu} = \mathcal{C}_N^{\pi^*(V \oplus \nu)} \rightarrow \mathcal{C}_N^{\pi^*(V \oplus \nu')} = (N \times V^+) \wedge_N \mathcal{C}_N^{\pi^* \nu'}$$

obtained by lifting the bundle isomorphism $a^{-1} : V \oplus \nu \rightarrow V \oplus \nu'$ over M via $\pi : \mathcal{C} \rightarrow M$. In the notation of Proposition 4.2 we have $\tilde{F} \simeq \pi^* A \circ \tilde{F}'$.

The fibrewise version of the composition formula (Proposition 1.6) expresses the difference $\alpha = h_V(\tilde{F}) - h_V(\tilde{F}')$ in terms of A . Using Proposition 4.2 and the explicit form of the geometric Hopf invariant, one sees that α lies in the image of the diagonal map

$$\begin{aligned} \Delta_* : \mathbb{Z}/2 \omega_N^0 \left\{ N \times \Sigma S(LV)_+; (N \times LV^+) \wedge_N \mathcal{C}_N^{\pi^*(\nu \oplus L\nu)} \right\} \\ \rightarrow \mathbb{Z}/2 \omega_N^0 \left\{ N \times \Sigma S(LV)_+; (N \times LV^+) \wedge_N \left(\mathcal{C}_N^{\pi^* \nu} \wedge_N \mathcal{C}_N^{\pi^* \nu} \right) \right\}. \end{aligned}$$

This may be rewritten in dual form as

$$\tilde{\omega}_0 \left((P(V) \times \mathcal{C})^{\pi^*(\nu \oplus H \otimes \nu - \tau N)} \right) \rightarrow \tilde{\omega}_0 \left((S(LV) \times_{\mathbb{Z}/2} \mathbf{h}\widehat{\mathfrak{D}}(f))^{\nu \times \nu - \tau N} \right).$$

But $\pi : \mathcal{C} \rightarrow M$ is a homotopy equivalence. Hence Δ_* is just another manifestation of the map i in the statement of the theorem.

Now the tubular neighbourhood of M in $V \times N$ gives a proper map $B(\nu) \rightarrow N$. To calculate α , which is concentrated on the diagonal, we can thus lift from N to $B(\nu)$. Here we have a fibrewise problem over M , and the identification of α is achieved by Proposition 4.2. \square

Remark 4.4. In the stable range $\dim V > 2m$, where the maps e_t are redundant, we can use the methods of the previous section to give an alternative proof of Theorem 4.3.

Theorem 4.5 (Hirsch [11]). *Suppose that $3m < 2n - 1$ and $\dim V > 2m$. Then two immersions f and f' are homotopic through immersions if and only if the associated difference class θ in $\tilde{\omega}_0((P(V) \times M)^{H \otimes \nu - \tau M})$ is zero.*

Proof. The derivative of the immersion f gives a vector bundle monomorphism $df : \tau M \rightarrow f^* \tau N$ over M . According to Hirsch [11], for $m < n$ immersions $f' : M \rightarrow N$ together with a homotopy f_t from $f = f_0$ to $f' = f_1$ are classified by homotopy classes of vector bundle monomorphisms $\tau M \rightarrow f^* \tau N$. In the metastable range $m + 1 < 2(n - m)$, that is, $3m < 2n - 1$, immersions with a homotopy to f are thus classified by

$$\begin{aligned} \tilde{\omega}^{-1} \left(P(\tau M)^{-H \otimes f^* \tau N} \right) &= \tilde{\omega}_0 \left(P(\tau M)^{H \otimes (f^* \tau N - \tau M) - \tau M} \right) \\ &= \tilde{\omega}_0 \left(P(\tau M)^{H \otimes \nu - \tau M} \right). \end{aligned}$$

Assuming that $\dim V > 2m$ we may fix an embedding $e : M \hookrightarrow V$. The derivative of e includes τM in the trivial bundle $M \times V$ and gives an isomorphism

$$\tilde{\omega}_0 \left(P(\tau M)^{H \otimes \nu - \tau M} \right) \rightarrow \tilde{\omega}_0 \left((M \times P(V))^{H \otimes \nu - \tau M} \right). \quad \square$$

5. Immersions close to an embedding

Consider the special case of a closed manifold M of (constant) dimension m and a real vector bundle ν of dimension $n - m$ over M . Working in the metastable range $3m < 2n - 1$, we take N to be the total space of ν and $f : M \rightarrow N$ to be the embedding given by the zero section of the vector bundle. As $e : M \rightarrow V$ we may take the constant map 0.

Let $v_0 : V \hookrightarrow V \oplus \nu$ be the inclusion of the first factor. Suppose that $v_1 : V \hookrightarrow V \oplus \nu$ is another inclusion, which we may assume to be isometric. Thus v_0 and v_1 give sections of the bundle $O(V, V \oplus \nu)$ whose fibre at $x \in M$ is the Stiefel manifold of orthogonal linear maps $v : V \rightarrow V \oplus \nu_x$. Let $X_1(V, \nu)$ be the subbundle with fibre consisting of those linear maps v such that $v + i_x$ has kernel of dimension 1, where $i_x : V \rightarrow V \oplus \nu_x$ is the inclusion of the first factor. In the range of dimensions that we are considering, for a generic smooth section v_1 of $O(V, V \oplus \nu) \rightarrow M$, the kernel of $(v_1)_x + i_x$ is nowhere of dimension greater than 1, v_1 is transverse to the subbundle $X_1(V, \nu)$ and $v_1^{-1}(X_1(V, \nu))$ is a submanifold Z of M of dimension $2m - n$, equipped with a map $Z \rightarrow P(V)$ classifying the 1-dimensional kernel. The

normal bundle of Z in M is identified with $H \otimes \nu$ and Z represents the element $\delta \in \tilde{\omega}_0((M \times P(V))^{H \otimes \nu - \tau M})$ dual to $\delta(v_0, v_1)$. More details are provided in the appendix (Section 6).

We want to construct an immersion f' close to the zero section f with double point set Z . More precisely, we shall construct homotopies f_t and e_t , with $f_0 = f$, $f_1 = f'$ and each (e_t, f_t) an embedding, such that $\mathfrak{D}(f') = Z$ and $\theta(e_t, f_t) = \delta$.

Choose an open tubular neighbourhood $\Omega = H \otimes \nu \hookrightarrow M$ of Z . Since $\dim \nu = n - m > 2m - n = \dim Z$, we can split off a trivial line from the restriction of ν to Z as $\nu|_Z = \mathbb{R} \oplus \zeta$.

Whitney gave in [24], for a Euclidean space U , an explicit “punctured figure-of-eight” immersion:

$$w : \mathbb{R} \oplus U \rightarrow (\mathbb{R} \oplus U) \times (\mathbb{R} \oplus U)$$

with double points at $(\pm 1, 0)$. In slightly modified form, it may be written as

$$w(s, y) = ((1 - \lambda(s, y))s, y, -\lambda(s, y), \lambda(s, y)sy),$$

where $\lambda(s, y) = \psi(s^2 + \|y\|^2)$ and $\psi : [0, \infty) \rightarrow \mathbb{R}$ is a smooth, nonnegative, monotonic decreasing function, with $\psi(1) = 1$, $\psi'(1) = -1/2$ and $\psi(r) = 0$ for $r \geq 2$. (In [24], $\psi(r) = 2/(1 + r)$.) The derivative at the double point $(\pm 1, 0)$ is

$$\frac{\partial w}{\partial s} = (1, 0, \pm 1, 0), \quad \frac{\partial w}{\partial y} = (0, 1, 0, \pm 1).$$

Writing $w(s, y)$ in the form $(s, y, 0, 0) + \lambda(s, y)(-s, 0, -1, sy)$, we see that $w(s, y) = (s, y, 0, 0)$ for $\|(s, y)\|$ large. The immersion w has $\mathbb{Z}/2$ -symmetry as an equivariant map

$$w : L \oplus LU \rightarrow (L \oplus LU) \times (\mathbb{R} \oplus U).$$

The two double points are distinguished by the $\mathbb{Z}/2$ -map

$$c : L \oplus LU \rightarrow L$$

given by $c(s, y) = \lambda(s, y)s$.

Now Whitney's construction, applied on the fibres of ζ , gives an immersion

$$\begin{aligned} f' : \Omega = H \otimes \nu &= H \oplus (H \otimes \zeta) \\ &\rightarrow (H \oplus (H \otimes \zeta)) \times (\mathbb{R} \oplus \zeta) = (H \otimes \nu) \times \nu = \nu|_{\Omega} \subseteq N \end{aligned}$$

of the open subset Ω of M into the total space of the restriction of ν to Ω . We extend f' to the whole of M to coincide with the zero section f outside a compact subspace of Ω . Its double point set $\mathfrak{D}(f')$ is the double-cover $S(H|Z)$ of Z in Ω . The map c composed with the inclusion of H into the trivial bundle $\Omega \times V$ and the projection to V gives a map

$$e' : \Omega = H \oplus (H \otimes \zeta) \rightarrow V,$$

which is zero outside a compact subset of Ω and can be extended by 0 to a map $e' : M \rightarrow V$ which distinguishes the double point pairs. The required homotopies e_t and f_t , $t \in [0, 1]$, joining e to e' and f to f' are defined by replacing λ in the definition of e' and f' by $t\lambda$. One checks that (e_t, f_t) is an embedding for all t , but that f_t fails to be an immersion when $t = 1/\psi(0)$.

Theorem 5.1. *Suppose that $3m < 2n - 1$. Then Whitney's construction described above produces, for any given element $\delta \in \tilde{\omega}_0((M \times P(V))^{H \otimes \nu - \tau M})$, a homotopy (e_t, f_t) with $\theta(e_t, f_t) = \delta$.*

Proof. By construction, the double point manifold $\mathfrak{D}(f') = S(H|Z)$ represents δ . The assertion thus follows from the double point theorem (Theorem 2.4) for (e', f') in conjunction with the homotopy variation theorem (Theorem 4.3). \square

Remark 5.2. The same construction may be used to modify a general immersion $f : M \looparrowright N$, and map $e : M \rightarrow V$, in the complement of $\mathfrak{D}(f)$. We can insert Z in the complement, because $2(2m - n) < m$.

Example 5.3. Whitney's construction gives the classical immersions of the sphere $S^m \looparrowright S^{2m}$ close to the equatorial inclusion and $S^m \looparrowright S^m \times S^m$ close to the diagonal.

The early work of Smale [21, 22] has been followed by a vast literature on the homotopy-theoretic properties of immersions, including [7, 9, 10, 11, 14, 18, 20].

6. Appendix: Monomorphisms of vector bundles

Let η and ξ be smooth real vector bundles, of dimension n and r respectively, over a closed m -manifold M . We shall describe the differential-topological classification of homotopy classes of vector bundle monomorphisms $\eta \rightarrow \xi$ in the metastable range $m + 1 < 2(n - r)$.

Suppose that $v_0, v_1 : \eta \hookrightarrow \xi$ are two vector bundle monomorphisms. Doing homotopy theory, we may assume that ξ and η have positive-definite inner products and that the monomorphisms are isometric embeddings. Then v_0 and v_1 are sections of the bundle $O(\eta, \xi)$ whose fibre at $x \in M$ is the Stiefel manifold of orthogonal linear maps $v : \eta_x \rightarrow \xi_x$. Topological obstruction theory gives a difference class

$$\delta(v_0, v_1) \in \tilde{\omega}^{-1}(P(\eta)^{-H \otimes \xi}),$$

where $P(\eta)$ is the projective bundle of η and H is the Hopf line bundle. This arises as follows. A section of $O(\eta, \xi)$ determines a nowhere zero section of $H \otimes \xi$ over $P(\eta)$, as follows. An element $\ell \in P(\eta_x)$ is a line $\ell \subseteq \eta_x$. A monomorphism $v : \eta_x \rightarrow \xi_x$ gives an embedding of ℓ in ξ_x and so a nonzero vector in $\ell^* \otimes \xi_x$, which is the fibre of $H \otimes \xi$. Then $\delta(v_0, v_1)$ is defined as the difference class $\delta(s_0, s_1)$ of the two nowhere zero sections s_0 and s_1 of

the vector bundle ξ over $P(\eta)$ constructed in this way from v_0 and v_1 . We may assume that s_0 and s_1 are sections of the sphere bundle $S(H \otimes \xi)$. Write $s_t = (1-t)s_0 + ts_1$ for $0 \leq t \leq 1$. Then $\delta(s_0, s_1)$ is represented explicitly by the map, over $P(\eta)$,

$$\bar{s} : ([0, 1], \partial[0, 1]) \times P(\eta) \rightarrow (D(H \otimes \xi), S(H \otimes \xi))$$

given by the homotopy s_t . In the metastable range, the vector bundle monomorphisms v_0 and v_1 are homotopic if and only if $\delta(v_0, v_1) = 0$. (See, for example, [3, 5, 4].)

Thus far, the theory is topological. We now use Poincaré duality for the manifold $P(\eta)$ to identify $\tilde{\omega}^{-1}(P(\eta)^{-H \otimes \xi})$ with $\tilde{\omega}_0(P(\eta)^{H \otimes (\xi - \eta) - \tau M})$. (Up to homotopy, the stable tangent bundle is given by an isomorphism $\mathbb{R} \oplus \tau P(\eta) \cong (H \otimes \eta) \oplus \tau M$.) Assuming that the monomorphisms v_0 and v_1 are smooth, we shall represent the dual obstruction class by a submanifold Z of M together with a map $Z \rightarrow P(\eta)$ and appropriate normal bundle information. The monomorphism v_0 will play a special role in the description; to emphasize this, we write $i = v_0$ for the preferred embedding $i : \eta \hookrightarrow \xi$ and write ν for the orthogonal complement of $i(\eta)$ in ξ . Let $X_k(\eta, \nu)$, for $k \geq 1$, be the subbundle of $O(\eta, \xi) = O(\eta, \eta \oplus \nu)$ with fibre consisting of those linear maps v such that $i_x + v$ has kernel of dimension k . By Lemma 6.3 below, we may assume, if $m+1 < 2(n-r)$, that v_1 never meets $X_k(\eta, \nu)$ for $k > 1$ and is transverse to the subbundle $X_1(\eta, \nu)$. The inverse image $v_1^{-1}(X_1(\eta, \nu))$ is, therefore, a submanifold Z of M of dimension $m+r-n$, equipped with a section $Z \rightarrow P(\eta)$ classifying the 1-dimensional kernel. The normal bundle of Z in M is identified, by Lemma 6.3, with $H \otimes \nu$.

Proposition 6.1. *The submanifold Z described above, with the line bundle H classified by the section of $P(\eta)$ over Z and the isomorphism between the normal bundle and $H \otimes \nu$, represents the dual of $\delta(v_0, v_1)$ in $\tilde{\omega}_0(P(\eta)^{H \otimes \nu - \tau M})$.*

Proof. Consider the sections s_0 and s_1 of $S(H \otimes \xi)$ over $P(\eta)$ associated with v_0 and v_1 as in the definition of $\delta(v_0, v_1)$. The section \bar{s} of $D(H \otimes \xi)$ over $[0, 1] \times P(\eta)$ given by the homotopy $s_t = (1-t)s_0 + ts_1$ is transverse to the zero section and its zero-set is precisely $\{\frac{1}{2}\} \times Z$. The normal bundle is $\mathbb{R} \oplus \tau_M P(\eta) \oplus (H \otimes \nu)$, where $\tau_M P(\eta)$ is the tangent bundle along the fibres of $P(\eta) \rightarrow M$, and this is identified with $(H \otimes \eta) \oplus (H \otimes \nu) = H \otimes \xi$. Hence, Z with the normal bundle data represents the stable homotopy class dual to $\delta(s_0, s_1) = \delta(v_0, v_1)$. (This is the classical representation of the dual Euler class of a vector bundle by the zero-set of a generic smooth section.) \square

Remark 6.2. A more symmetric treatment may be given by looking at sections of the fibre product $O(\eta, \xi) \times_M O(\eta, \xi)$ and the subbundles with fibre consisting of the pairs (u, v) such that $\dim \ker(u+v) = k$.

The properties of v_1 required in the proof of Proposition 6.1 follow from the next lemma, in which the Lie algebra of the orthogonal group $O(V)$ of a Euclidean vector space V is written as $\mathfrak{o}(V)$.

Lemma 6.3. *Let V and W be finite-dimensional orthogonal vector spaces. For $0 \leq k \leq \dim V$, let $X_k(V, W)$ be the subspace of the Stiefel manifold $O(V, V \oplus W)$ consisting of the maps v such that $i + v$, where i is the inclusion of the first summand $V \hookrightarrow V \oplus W$, has kernel of dimension k . Then $X_k(V, W)$ is a submanifold diffeomorphic to the total space of the vector bundle $\mathfrak{o}(\zeta^\perp) \oplus \text{Hom}(\zeta^\perp, W)$ over the Grassmann manifold $G_k(V)$ of k -planes in V , where ζ^\perp is the orthogonal complement in V of the canonical k -dimensional vector bundle ζ over $G_k(V)$. Its normal bundle is naturally identified with $\mathfrak{o}(\zeta) \oplus \text{Hom}(\zeta, W)$.*

Proof. This can be established by using the (generalized) Cayley transform, which is written down explicitly in [5, Part II, Lemma 13.13]. The restriction of the normal bundle of the embedded submanifold to the subspace $G_k(V)$ is naturally identified with $\mathfrak{o}(\zeta) \oplus \text{Hom}(\zeta, W)$. The normal bundle itself is naturally identified with the pullback by parallel translation. \square

In particular, the submanifold $X_k(\eta, \nu)$ considered above has codimension $(n - r)k + k(k - 1)/2$ in $O(\eta, \eta \oplus \nu)$. So, if $k \geq 2$, the codimension is at least $2(n - r) + 1$. The condition $m < 2(n - r) + 1$ ensures that a generic section of $O(\eta, \eta \oplus \nu)$ is transverse to $X_1(\eta, \nu)$ and disjoint from $X_k(\eta, \nu)$ for $k > 1$.

Remark 6.4. In [15] Koschorke gave an intermediate representation of the dual of $\delta(v_0, v_1)$ by a submanifold of $(0, 1) \times M$. Consider the section \bar{v} of $\text{Hom}(\eta, \xi)$ over $[0, 1] \times M$ given by the homotopy $v_t = (1 - t)v_0 + tv_1$. Let $Y_k(\eta, \xi)$ be the subbundle of $\text{Hom}(\eta, \xi)$ with fibre consisting of the linear maps with kernel of dimension k ; it has codimension $(n - r)k + k^2$, which is greater than or equal to $2(n - r) + 4 > m + 1$. Suppose that \bar{v} can be deformed, by a homotopy through maps coinciding with v_0 and v_1 at the endpoints, to a smooth section \bar{v}' , that never meets $Y_k(\eta, \xi)$ for $k > 1$ and meets $Y_1(\eta, \xi)$ transversely. This is always possible if $m < 2(n - r) + 3$. The inverse image of $Y_1(\eta, \xi)$ is then a submanifold Z of $(0, 1) \times M$ of dimension $m + r - n$ equipped with a map $Z \rightarrow P(\eta)$ given by the 1-dimensional kernel. This data, too, represents the dual of $\delta(v_0, v_1)$.

Acknowledgment

We thank Mark Grant for pointing out the relevance of [10].

References

- [1] J. F. Adams, *Prerequisites (on equivariant stable homotopy) for Carlsson's lecture*. In: Proc. 1982 Aarhus Topology Conf., Springer Lecture Notes 1051, 1984, 483–532.
- [2] M. Boardman and B. Steer, *Hopf invariants*. Comment. Math. Helv. **42** (1967), 217–224.

- [3] M. C. Crabb, *$\mathbb{Z}/2$ -Homotopy Theory*. LMS Lecture Notes 44, Cambridge University Press, Cambridge, 1980.
- [4] M. C. Crabb, *The homotopy coincidence index*. J. Fixed Point Theory Appl. **7** (2010), 1–32.
- [5] M. C. Crabb and I. M. James, *Fibrewise Homotopy Theory*. Springer, Berlin, 1998.
- [6] M. C. Crabb and A. A. Ranicki, *The geometric Hopf invariant and surgery theory*. In preparation. Preliminary announcement available at <http://www.maths.ed.ac.uk/aar/slides/geohopf.pdf>.
- [7] J. P. Dax, *Étude homotopique des espaces de plongements*. Ann. Sci. École Norm. Sup. (4) **5** (1972), 303–377.
- [8] A. Haefliger, *Plongements différentiables dans le domaine stable*. Comment. Math. Helv. **37** (1962/1963), 155–176.
- [9] A. Haefliger and M. W. Hirsch, *Immersions in the stable range*. Ann. of Math. (2) **75** (1962), 231–241.
- [10] A. Hatcher and F. Quinn, *Bordism invariants of intersections of submanifolds*. Trans. Amer. Math. Soc. **200** (1974), 327–344.
- [11] M. W. Hirsch, *Immersions of manifolds*. Trans. Amer. Math. Soc. **93** (1959), 242–276.
- [12] J. R. Klein, *Poincaré complex diagonals*. Math. Z. **258** (2008), 587–607.
- [13] J. R. Klein and E. B. Williams, *Homotopical intersection theory. I* Geom. Topol. **11** (2007), 939–977.
- [14] J. R. Klein and E. B. Williams, *Homotopical intersection theory. II. Equivariance*. Math. Z. **264** (2010), 849–880.
- [15] U. Koschorke, *Vector Fields and Other Vector Bundle Morphisms—A Singularity Approach*. Lecture Notes in Mathematics 847, Springer, Berlin, 1981.
- [16] U. Koschorke, *Nielsen coincidence theory in arbitrary codimensions*. J. reine angew. Math **598** (2006), 211–236.
- [17] U. Koschorke and B. Sanderson, *Geometric interpretations of the generalized Hopf invariant*. Math. Scand. **41** (1977), 199–217.
- [18] B. Li, R. Liu and P. Zhang, *A simplified approach to embedding problem in normal bordism framework*. Systems Sci. Math. Sci. **5** (1992), 180–192.
- [19] A. A. Ranicki, *The algebraic theory of surgery. II. Applications* Proc. Lond. Math. Soc. (3) **40** (1980), 193–283.
- [20] H. A. Salomonsen, *Bordism and geometric dimension*. Math. Scand. **32** (1973), 87–111.
- [21] S. Smale, *Regular curves on Riemannian manifolds*. Trans. Amer. Math. Soc. **87** (1958), 492–512.
- [22] S. Smale, *The classification of immersions of spheres in Euclidean spaces*. Ann. of Math. (2) **69** (1959), 327–344.
- [23] C. T. C. Wall, *Surgery on Compact Manifolds*. 2nd ed., AMS, Providence, RI, 1999.

- [24] H. Whitney, *The self-intersections of a smooth n -manifold in $2n$ -space*. Ann. of Math. (2) **45** (1944), 220–246.

Michael Crabb
Institute of Mathematics
University of Aberdeen
Aberdeen AB24 3UE
Scotland, UK
e-mail: m.crabb@maths.abdn.ac.uk

Andrew Ranicki
School of Mathematics
University of Edinburgh
Edinburgh, EH9 3JZ
Scotland, UK
e-mail: a.ranicki@ed.ac.uk