

Differential Topology

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Differential topology may be defined as the study of those properties of differentiable manifolds which are invariant under diffeomorphism (differentiable homeomorphism). Typical problems falling under this heading are the following:

- 1) Given two differentiable manifolds, under what conditions are they diffeomorphic?
- 2) Given a differentiable manifold, is it the boundary of some differentiable manifold-with-boundary?
- 3) Given a differentiable manifold, is it parallelisable?

All these problems concern more than the topology of the manifold, yet they do not belong to differential geometry, which usually assumes additional structure (e.g., a connection or a metric).

The most powerful tools in this subject have been derived from the methods of algebraic topology. In particular, the theory of characteristic classes is crucial, where-by one passes from the manifold M to its tangent bundle, and thence to a cohomology class in M which depends on this bundle.

These notes are intended as an introduction to the subject; we will go as far as possible without bringing in algebraic topology. Our two main goals are

- a) Whitney's theorem that a differentiable n -manifold can be embedded as a closed subset of the euclidean space \mathbb{R}^{2n+1} (see §1.32); and
- b) Thom's theorem that the non-orientable cobordism group \mathcal{N}^n is isomorphic to a certain stable homotopy groups (see §3.15).

Chapter I is mainly concerned with approximation theorems. First the basic definitions are given and the inverse function theorem is exploited. (§1.1 – 1.12). Next two local approximation theorems are proved, showing that a given map can be approximated by one of maximal rank. (§1.13 – 1.21). Finally locally finite coverings are used to derive the corresponding global theorems: namely Whitney's embedding theorem and Thom's transversality lemma (§1.35).

Chapter II is an introduction to the theory of vector space bundles, with emphasis on the tangent bundle of a manifold.

Chapter III makes use of the preceding material in order to study the cobordism group \mathcal{N}^n .

Chapter I *Embeddings and Immersions of Manifolds*

Notation. If x is in the euclidean space \mathbb{R}^n , the coordinate of x are denoted by (x^1, \dots, x^n) . Let $\|x\| = \max |x^i|$; let $C^n(r)$ denote the set of x such that $\|x\| < r$; and $C^n(x_0, r)$ the set of x such that $\|x - x_0\| < r$. The closure of a cube C is denoted by \bar{C} .

A real valued function $f(x^1, \dots, x^n)$ is **differentiable** if the partials of f of all orders exist and are continuous (i.e., “differentiable” means C^∞). A map $f = (f^1, \dots, f^p) : U \rightarrow \mathbb{R}^p$ (where U is an open set, in \mathbb{R}^n) is differentiable if each of the coordinate functions f^1, \dots, f^p is differentiable. Df denotes the Jacobian matrix of f ; one verifies that $D(gf) = Dg \cdot Df$. The notation $\partial(f^1, \dots, f^p)/\partial(x^1, \dots, x^n)$ is also used. If $n = p$, $|Df|$ denotes the determinant.

1.1 Definition. A **topological n -manifold** M^n is a Hausdorff space with a countable basis which is locally homeomorphic to \mathbb{R}^n .

A **differentiable structure** \mathcal{D} on a topological manifold M^n is a collection of real-valued functions, each defined on an open subset of M^n such that:

- 1) For every point p of M^n there is a neighbourhood U of p and a homeomorphism h of U onto an open subset of \mathbb{R}^n such that a function f , defined on the open subset W of U , is in \mathcal{D} if and only if fh^{-1} is differentiable.
- 2) If U_i are open sets contained in the domain of f and $U = \cup U_i$, then $f|_U \in \mathcal{D}$ if and only if $f|_{U_i}$ is in \mathcal{D} , for each i .

A **differentiable manifold** M^n is a topological manifold provided with a differentiable structure \mathcal{D} ; the elements of \mathcal{D} are called the **differentiable functions** on M^n . Any open set U and homeomorphism h which satisfy the requirement of 1) above are called a **coordinate system** on M^n .

Notation. A coordinate system is sometimes denoted by the coordinate functions:
 $h(p) = (u^1(p), \dots, u^n(p))$.

1.2 Alternate definition. Let a collection (U_i, h_i) be given, where h_i is a homeomorphism of the open subset U_i of M^n onto an open subset of \mathbb{R}^n , such that

- a) the U_i 's cover M^n ;
- b) $h_j h_i^{-1}$ is a differentiable map on $h_i(U_i \cap U_j)$, for all i, j .

Define a **coordinate system** as an open set U and homeomorphism h of U onto an open subset of \mathbb{R}^n such that $h_i h^{-1}$ and $h h_i^{-1}$ are differentiable on $h(U \cap U_i)$ and $h_i(U \cap U_i)$ respectively, for each i .

Define a **differentiable structure** on M^n as the collection of all such coordinate systems. A function f , defined on the open set V , is **differentiable** if fh^{-1} is differentiable on $h(U \cap V)$, for all coordinate systems (U, h) .

One shows readily that these two definitions are entirely equivalent.

1.3 Definition. Let M_1, M_2 be differentiable manifolds. If U is an open subset of M_1 , $f: U \rightarrow M_2$ is **differentiable** if for every differentiable function g on M_2 , gf is differentiable on M_1 .

If $A \subset M_1$, a function $f: A \rightarrow M_2$ is **differentiable** if it can be extended to a differentiable function defined on a neighbourhood U of A .

$f: M_1 \rightarrow M_2$ is a **diffeomorphism** if f and f^{-1} are defined and differentiable.

(A coordinate system (U, h) on M^n is then an open set U in M^n and a diffeomorphism h of U onto an open set in \mathbb{R}^n .)

If $A \subset M$, we have just defined the notion of differentiable function for subsets of A . Suppose that A is locally diffeomorphic to \mathbb{R}^k : this collection is easily shown to be a differentiable structure on A .

In this case, A is said to be a **differentiable submanifold** of M .

The following lemma is familiar from elementary calculus.

1.4. Lemma. *Let $f: C^n(r) \rightarrow \mathbb{R}^n$ satisfy the condition $|\partial f^i / \partial x^j| \leq b$ for all i, j . Then $\|f(x) - f(y)\| \leq bn\|x - y\|$, for all $x, y \in \overline{C^n(r)}$.*

1.5. Theorem (Inverse Function Theorem). *Let U be an open subset of \mathbb{R}^n , let $f: U \rightarrow \mathbb{R}^n$ be differentiable, and let Df be non-singular at x_0 . Then f is a diffeomorphism of some neighbourhood of x_0 onto some neighbourhood of $f(x_0)$.*

Proof: We may assume $x_0 = f(x_0) = 0$, and that $Df(x_0)$ is the identity matrix.

Let $g(x) = f(x) - x$, so that $Dg(0)$ is the zero matrix. Choose $r > 0$ so that $x \in U$ and $Df(x)$ is non-singular and $|\partial g^i / \partial x^j| \leq 1/2n$, for all x with $\|x\| < r$.

Assertion. *If $y \in C^n(r/2)$, there is exactly one $x \in C^n(r)$ such that $f(x) = y$.*

By the previous lemma,

$$\|g(x) - g(x_0)\| \leq (1/2)\|x - x_0\| \text{ on } C^n(r). \tag{*}$$

Let us define $\{x_n\}$ inductively by $x_0 = 0, x_1 = y, x_{n+1} = y - g(x_n)$. This is well-defined, since $x_n - x_{n-1} = g(x_{n-2}) - g(x_{n-1})$ so that

$$\|x_n - x_{n-1}\| \leq (1/2)\|x_{n-2} - x_{n-1}\| \leq \|y\|/2^{n-1};$$

and thus $\|x_n\| \leq 2\|y\|$ for each n . Hence the sequence $\{x_n\}$ converges to a point x with $\|x\| \leq 2\|y\|$, so that $x \in C^n(r)$. Then $x = y - g(x)$, so that $f(x) = y$. This proves the existence of x . To show uniqueness, note that if $f(x) = f(x_1) = y$, then $g(x_1) - g(x) = x - x_1$, contradicting (*).

Hence $f^{-1}: C^n(r/2) \rightarrow C^n(r)$ exists. Note that

$$\|f(x) - f(x_1)\| \geq \|x - x_1\| - \|g(x) - g(x_1)\| \geq (1/2)\|x - x_1\|$$

so that $\|y - y_1\| \geq (1/2)\|f^{-1}(y) - f^{-1}(y_1)\|$. Hence f^{-1} is continuous; the image $C^n(r/2)$ of under f^{-1} is open because it equals $C^n(r) \cap f^{-1}(C^n(r/2))$, the intersection of two open sets.

To show that f^{-1} is differentiable, note that

$$f(x) = f(x_1) + Df(x_1) \cdot (x - x_1) + h(x, x_1),$$

where $(x - x_1)$ is written as a column matrix and the dot stands for matrix multiplication. Here $h(x, x_1) / \|x - x_1\| \rightarrow 0$ as $x \rightarrow x_1$. Let A be the inverse matrix of $Df(x_1)$. Then

$$\begin{aligned} A \cdot (f(x) - f(x_1)) &= (x - x_1) + A \cdot h(x, x_1), \text{ or} \\ A \cdot (y - y_1) + A \cdot h_1(y, y_1) &= f^{-1}(y) - f^{-1}(y_1), \end{aligned}$$

where $h(y, y_1) = -h(f^{-1}(y), f^{-1}(y_1))$. Now

$$h_1(y, y_1) / \|y - y_1\| = -[h(x, x_1) / \|x - x_1\|] (\|x - x_1\| / \|y - y_1\|).$$

Since $\|x - x_1\| / \|y - y_1\| \leq 2, h_1(y, y_1) / \|y - y_1\| \rightarrow 0$ as $y \rightarrow y_1$. Hence

$$D(f^{-1}) = A = (Df)^{-1}.$$

This means that $(Df)^{-1}$ is obtained as the composition of the following maps:

$$C^n(r/2) \xrightarrow{f^{-1}} C^n(r) \xrightarrow{Df} GL(n) \xrightarrow{\text{matrix inversion}} GL(n);$$

where $GL(n)$ denotes the set of non-singular $n \times n$ matrices, considered as a subspace of n^2 -dimensional euclidean space. Since f^{-1} is continuous and Df and matrix inversion are C^∞ , $(Df)^{-1}$ is continuous, i.e., f^{-1} is C^1 . In general, if f^{-1} is C^k , then by this argument $(Df)^{-1}$ is also, i.e., f^{-1} is of class C^{k+1} . This completes the proof. \square

1.6. Lemma. *Let U be an open subset of \mathbb{R}^n , let $f: U \rightarrow \mathbb{R}^p$ ($n \leq p$), $f(0) = 0$, and let $Df(0)$ have rank n . Then there exists a diffeomorphism g of one neighbourhood of the origin in \mathbb{R}^p onto another so that $g(0) = 0$ and $gf(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$, in some neighbourhood of the origin.*

Proof: Since $\partial(f^1, \dots, f^p)/\partial(x^1, \dots, x^n)$ has rank n , we may assume that

$$\partial(f^1, \dots, f^p)/\partial(x^1, \dots, x^n)$$

is the submatrix which is non-singular. Define $F: U \times \mathbb{R}^{p-n} \rightarrow \mathbb{R}^p$ by the equation

$$F(x^1, \dots, x^p) = f(x^1, \dots, x^n) + (0, \dots, 0, x^{n+1}, \dots, x^p).$$

F is an extension of f , since $F(x^1, \dots, x^n, 0, \dots, 0) = f(x^1, \dots, x^n)$.

DF is non-singular at the origin, since its determinant everywhere equals

$$|\partial(f^1, \dots, f^p)/\partial(x^1, \dots, x^n)|,$$

which is non-zero. Hence F has a local inverse g , so that g maps one neighbourhood of the origin in \mathbb{R}^p onto another, and

$$gF(x^1, \dots, x^p) = (x^1, \dots, x^p)$$

and hence

$$gf(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0). \quad \square$$

1.7. Corollary. *Let A^k be a differentiable sub-manifold of M^n . Given $x \in A^k$, there is a coordinate system (U, h) on M^n about x , such that $h(U \cap A) = h(U) \cap \mathbb{R}^k$ (where \mathbb{R}^k is considered as the subspace $\mathbb{R}^k \times 0$ of $\mathbb{R}^k \times \mathbb{R}^n = \mathbb{R}^n$).*

Proof: Let (U_i, h_i) be a coordinate system on M^n about x ; by hypothesis, there is a differentiable map f of a neighbourhood V of x in M^n into \mathbb{R}^k such that $f|_{V \cap A} = f_1$ is a diffeomorphism whose range is an open set W in \mathbb{R}^k . We may assume $U_1 = V$, and $h_1(x) = f(x) = 0$.

Now $fh_1^{-1}h_1^{-1}$ is the identity on W , so that its Jacobian, which equals $D(fh_1^{-1}), D(h_1^{-1})$ is non-singular. Hence $D(h_1^{-1})$ has rank k , so that by the previous lemma, there is a diffeomorphism g of some neighbourhood $V_1 \subset h_1(U_1)$ of 0 onto another such that $g(0) = 0$ and $gh_1^{-1}(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0)$. Then $U = h_1^{-1}(V_1)$ and $h = gh_1$ will satisfy the requirement of the lemma. \square

1.8. Lemma. *Let U be an open subset of \mathbb{R}^n , let $f: U \rightarrow \mathbb{R}^p$, $f(0) = 0$, ($n \geq p$), and let $Df(0)$ have rank p . Then there is a diffeomorphism h of some neighbourhood of the origin in \mathbb{R}^n onto another such that $h(0) = 0$ and $fh(x^1, \dots, x^n) = (x^1, \dots, x^p)$.*

Proof: We may assume $\partial(f^1, \dots, f^p)/\partial(x^1, \dots, x^p)$ is non-singular at 0, since $Df(0)$ has rank p . Define $F: U \rightarrow \mathbb{R}^n$ by the equation

$$F(x^1, \dots, x^n) = (f^1(x), \dots, f^p(x), x^{p+1}, \dots, x^p).$$

Then $DF(0)$ is non-singular; let h be the local inverse of F . Let g project \mathbb{R}^n onto the subspace \mathbb{R}^p ; $f = gF$. Then

$$fh(x^1, \dots, x^n) = gFh(x^1, \dots, x^n) = g(x^1, \dots, x^n) = (x^1, \dots, x^p). \quad \square$$

1.9. Exercise. Let U be an open subset of \mathbb{R}^n , $f: U \rightarrow \mathbb{R}^p$, $f(0) = 0$; and let $Df(x)$ have rank k for all x in U . Then there are local diffeomorphisms h and g of \mathbb{R}^n and \mathbb{R}^p respectively such that

$$gh(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0).$$

1.10. Definition. If $f: M_1 \rightarrow M_2$, the **rank** of f , written $\text{rank}(f)$, at x is the rank of $D(h_2fh_1^{-1})$ at $h_1(x)$, where (U_1, h_1) and (U_2, h_2) are coordinate systems about x and $f(x)$, respectively. The differentiable map $f: M_1^n \rightarrow M_2^p$ is an **immersion** if $\text{rank}(f) = n$ everywhere ($n \leq p$). It is an **embedding** if it is also a homeomorphism into.

If $f: M_1^n \rightarrow M_2^p$, then $y \in M_2^p$ is a **regular value** of f if $\text{rank}(f) = p$ on the entire set $f^{-1}(y)$. Otherwise, y is a **critical value**. (If $y \notin f(M_1^n)$, y is, by definition, a regular value of f .)

1.11. Exercise. If A is a differentiable submanifold of M , the inclusion $A \rightarrow M$ is an embedding and conversely if $f: M_1 \rightarrow M$ is an embedding then $f(M_1)$ is a differentiable submanifold.

1.12. Exercise. If y is a regular value of $f: M_1^n \rightarrow M_2^p$, then $f^{-1}(y)$ is a differentiable submanifold of M_1^n of dimension $n - p$ (or empty).

1.13. Definition. A subset A of \mathbb{R}^n has **measure zero** if it may be covered by a countable collection of cubes $C^n(x, r)$ having arbitrarily small total volume. In such a case, $\mathbb{R}^n \setminus A$ is everywhere dense (i.e., it intersects every non-empty open set).

1.14. Lemma. Let U be an open subset of \mathbb{R}^n ; let $f: U \rightarrow \mathbb{R}^n$ be differentiable. If $A \subset U$ has measure zero, so does $f(A)$.

Proof: Let C be any cube with $\bar{C} \subset U$. Let b denote the maximum of $|\partial f / \partial x^j|$ on \bar{C} for all i, j . By 1.4, $\|f(x) - f(y)\| \leq bn\|x - y\|$ for $x, y \in \bar{C}$.

Now $A \cap C$ has measure zero; let us cover $A \cap C$ by cubes $C(x_i, r_i)$ with closure contained in C , such that $\sum_{i=1, \infty} r_i^n < \varepsilon$. Then $f(C(x_i, r_i)) \subset C(f(x_i), bnr_i)$, so that $f(A \cap C)$ is covered by cubes of total volume $b^n n^n \sum_{i=1, \infty} r_i^n < b^n n^n \varepsilon$. Hence $f(A \cap C)$ has measure zero.

Since A can be covered by countably many such cubes C , $f(A)$ has measure zero. □

1.15. Corollary. If $f: U \rightarrow \mathbb{R}^n$ be differentiable, where U is an open subset of \mathbb{R}^n and $n < p$, then $f(U)$ has measure zero.

Proof: Project $U \times \mathbb{R}^{p-n}$ onto U and apply f . Since $U \times 0$ has measure zero in \mathbb{R}^p , so does $f(U)$. □

1.16. Definition. If $A \subset M$, M has **measure zero** if $h(A \cap U)$ has measure zero for every coordinate system (U, h) .

1.17. Corollary. If $f: M_1^n \rightarrow M_2^p$ is differentiable and $n < p$, then $f(M_1^n)$ has measure zero.

1.18. Definition. Let $\mathcal{M}(p, n)$ denote the space of $p \times n$ matrices, with the differentiable structure of the euclidean space \mathbb{R}^{pn} . Let $\mathcal{M}(p, n; k)$ denote the subspace consisting of matrices of rank k .

Thus $\mathcal{M}(p, n; n)$ is an open subset of $\mathcal{M}(p, n)$ if $p \geq n$; the determinantal criterion for rank proves this. More generally, we have:

1.19. Lemma. $\mathcal{M}(p, n; k)$ is a differentiable submanifold of $\mathcal{M}(p, n)$ of dimension $k(p + n - k)$, where $k \leq \min(p, n)$.

Proof: Let $E_0 \in \mathcal{M}(p, n; k)$; we may assume that E_0 is of the form, $\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$, where A_0 is a non-singular $k \times k$ matrix. There is an $\varepsilon > 0$ such that if all the entries of $A - A_0$ are less than ε , A must also be non-singular. Let U consist of all matrices in $\mathcal{M}(p, n)$ of the form $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, with all the entries of $A - A_0$ are less than ε .

Then E is in $\mathcal{M}(p, n; k)$ if and only if $D = CA^{-1}B$: for the matrix

$$\begin{bmatrix} I_k & 0 \\ X & I_{p-k} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ XA + C & XB + D \end{bmatrix}$$

has the same rank as E . If $X = -CA^{-1}$, this matrix is

$$\begin{bmatrix} A & B \\ 0 & CA^{-1}B + D \end{bmatrix}.$$

If $D = CA^{-1}B$, this matrix has rank k . The converse also holds, for if any element of $-CA^{-1}B + D$ is different from zero, this matrix has rank $> k$.

Let W be the open set in euclidean space of dimension

$(pn - (p - k)(n - k)) = k(p + n - k)$
 consisting of matrices $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ with all the entries of $A - A_0$ are less than ε . The map

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \rightarrow \begin{bmatrix} A & B \\ 0 & CA^{-1}B + D \end{bmatrix}$$

is then a diffeomorphism of W onto the neighbourhood $U \cap \mathcal{M}(p, n; k)$ of E_0 . □

1.20. Theorem. Let U be an open set in \mathbb{R}^n , and let $f: U \rightarrow \mathbb{R}^p$ be differentiable, where $p \geq 2n$. Given $\varepsilon > 0$, there is a $p \times n$ matrix $A = (a^i_j)$ with each $|a^i_j| < \varepsilon$, such that $g(x) = f(x) + A \cdot x$ is an immersion (x written as a column matrix.)

Proof: $Dg(x) = Df(x) + A$; we would like to choose A in such a way that $Dg(x)$ has rank n for all x . I.e., A should be of the form $Q - Df$, where Q has rank n .

We define $F_k: \mathcal{M}(p, n; k) \times U \rightarrow \mathcal{M}(p, n)$ by the equation

$$F_k(Q, x) = Q - Df(x).$$

Now F_k is a differentiable map, and the domain of F_k has dimension $k(p + n - k) + n$. As long as $k < n$, this expression is monotonic in k (its partial derivative with respect to k is $p + n - 2k$). Hence the domain of F_k has dimension not greater than

$$(n - 1)(p + n - (n - 1)) + n = (2n - p) + pn - 1$$

for $k < n$. Since $p \geq 2n$, this dimension is strictly less than $pn = \dim(\mathcal{M}(p, n))$.

Hence the image of F_k has measure zero in $\mathcal{M}(p, n)$, so that there is an element A of $\mathcal{M}(p, n)$, arbitrarily close to the zero matrix, which is not in the image of F_k for $k = 0, \dots, n - 1$. Then $A + Df(x) = Dg(x)$ has rank n , for each x . □

1.21. Theorem. *Let U be an open subset of \mathbb{R}^n ; and let $f: U \rightarrow \mathbb{R}^p$ be differentiable. Given $\varepsilon > 0$, there are matrices $A(p \times n)$ and $B(p \times 1)$ with entries less than ε in absolute value such that*

$$g(x) = f(x) + A \cdot x + B$$

has the origin as a regular value.

Remark. The following much more delicate result has been proved by [Sard, A.]: *The set of critical values of any differentiable map has measure zero.*

Proof of 1.21. Note that the theorem is trivial if $p > n$, since then $f(U)$ has measure zero, and we may choose $A = 0$ and B small in such a way that 0 is not in the image of g .

Assume $p \leq n$. We wish $Dg(x_0) = Df(x_0) + A$ to have rank p , where x_0 ranges over all points such that

$$g(x_0) = 0 = f(x_0) + A \cdot x_0 + B.$$

Hence A is of the form $Q - Df(x)$, and B is of the form $-f(x) - A \cdot x$, where Q is to have rank p . We define $F_k: \mathcal{M}(p, n; k) \times U \rightarrow \mathcal{M}(p, n) \times \mathbb{R}^p$ by the equation

$$F_k(Q, x) = (Q - Df(x), -f(x) - (Q - Df(x)) \cdot x).$$

Then F_k is differentiable. If $k < p$, the dimension of its domain is not greater than $(p - 1)((p + n - (n - 1)) + n) = p + pn - 1$. Hence the image of F_k , $k = 0, \dots, p - 1$ has measure zero; so that there is a point (A, B) arbitrarily close to the origin which is not in any such image set. This completes the proof. □

1.22. Definition. A covering of a topological space X is **locally-finite** if every point has a neighbourhood which intersects only finitely many elements of the covering. A **refinement** of a covering of X is a second covering each element of which is contained in an element of the first covering. A Hausdorff space is **paracompact** if every open covering has a locally-finite open refinement.

If X is paracompact, and $\{U_\alpha\}$ is an open covering, there is a locally-finite open covering $\{V_\alpha\}$ with $V_\alpha \subset U_\alpha$ for each α . For let $\{W_\beta\}$ be a locally-finite refinement of $\{U_\alpha\}$; choose $\alpha(\beta)$ so that $W_\beta \subset U_{\alpha(\beta)}$ for each β . Set $V_{\alpha 0} = U_{\alpha(\beta) = \alpha 0} W_\beta$. Given a neighbourhood intersecting only finitely many W_β , it intersects only finitely many V_α as well.

1.23. Theorem. *A locally compact Hausdorff space having a countable basis is paracompact.*

Proof: Let X be paracompact and let U_1, U_2, \dots be a basis for X with \bar{U}_i compact with each i . There exists a sequence A_1, A_2, \dots of compact sets whose union is X , such that $A_i \subset \text{Int}A_{i+1}$: set $A_1 = \bar{U}_1$. Given A_i compact, let k be the smallest integer such that A_i is contained in $U_1 \cup \dots \cup U_k$; Let A_{i+1} equal the closure of this set union \bar{U}_{i+1} .

Let \mathcal{O} be an open covering of X . Cover the compact set $A_{i+1} \setminus \text{Int}A_i$ by a finite number of open sets V_1, \dots, V_n where each V_i is contained in some element of \mathcal{O} , and in the open set $\text{Int}A_{i+2} \setminus A_{i-1}$. Let P_i denote the collection $\{V_1, \dots, V_n\}$, and let $P = P_0 \cup P_1 \cup \dots$. P refines \mathcal{O} , and since any compact closed neighbourhood C is contained in some A_i , C can intersect only finitely many elements of P . □

1.24. Exercise. Prove that a paracompact space is normal. (First prove that it is regular.)

1.25. Theorem. Let M^n be a differentiable manifold, $\{U_\alpha\}$ an open covering of M^n . There is a collection (V_j, h_j) of coordinate systems on M^n such that

- 1) $\{V_j\}$ is a locally-finite refinement of $\{U_\alpha\}$.
- 2) $h_j(V_j) = C^n(3)$.
- 3) If $W_j = h_j^{-1}(C^n(1))$, then $\{W_j\}$ covers M^n .

Proof: The proof proceeds along lines similar to the previous one. The only difference is that one chooses the V_j to satisfy 2), and makes sure that the sets $h_j^{-1}(C^n(1))$ also cover $A_{i+1} \setminus \text{Int}A_i$. \square

1.26. We wish to construct a C^∞ function $\varphi(x^1, \dots, x^n)$ such that $\varphi = 1$ on $\overline{C^n(1)}$, $0 < \varphi < 1$ on $C^n(2) \setminus \overline{C^n(1)}$, $\varphi = 0$ on $\mathbb{R}^n \setminus C^n(2)$.

This function may be defined by the equation $\varphi(x^1, \dots, x^n) = \prod_{i=1, n} \psi(x^i)$, where

$$\psi(x) = \lambda(2+x) \cdot \lambda(2-x) / [\lambda(2+x) \cdot \lambda(2-x) + \lambda(x-1) + \lambda(-x-1)]$$

and

$$\lambda(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Note that the denominator in the expression for ψ is always positive, and that

$$\begin{aligned} \psi(x) &= 1 & \text{for } |x| \leq 1 \\ 0 < \psi(x) < 1 & \text{if } 1 < |x| < 2 \\ \psi(x) &= 0 & \text{if } |x| \geq 2. \end{aligned}$$

1.27. Definition. Let $f, g : X \rightarrow Y$, where Y is metrisable, and let $\delta(x)$ be a positive continuous function defined on X . Then g is a δ -**approximation to f** if $d(f(x), g(x)) < \delta(x)$ for all x . [If one takes the δ -approximation to f to be a neighbourhood of f in the function space $F(X, Y)$, this imposes a topology on the function space, independent of the metric on Y provided X, Y are paracompact.]

1.28. Theorem. Given a differentiable map $f : M^n \rightarrow \mathbb{R}^p$ where $p \geq 2n$, and a continuous positive function δ on M^n , there exists an immersion $g : M^n \rightarrow \mathbb{R}^p$ which is a δ -approximation to f . If $\text{rank } f = n$ on the closed set N , we may choose $g|N = f|N$.

Proof: Note that $\text{rank } f = n$ on a neighbourhood U of N . Cover M^n by U and $M^n \setminus N$. Let (V_j, h_j) be a refinement of this covering, constructed as in 1.25. As before, $h_i(\overline{W}_i) = C^n(1)$ and $h_i(V_i) = C^n(3)$. Let $h_j(U_j) = C^n(2)$. Let the V_i be so indexed with positive and negative integers that those V_i with non-positive indices are the ones contained in U . Let $\varepsilon_1 = \min$ of $\delta(x)$ on the compact set \overline{U}_i . Set $f_0 = f$. Given $f_{k-1} : M^n \rightarrow \mathbb{R}^p$, having rank n on $N_{k-1} = \cup_{j < k} W_j$, consider $f_{k-1} h_k^{-1} : C^n(3) \rightarrow \mathbb{R}^p$. Let A be a $p \times n$ matrix; let $F_A : C^n(3) \rightarrow \mathbb{R}^p$ be defined by the equation

$$F_A(x) = f_{k-1} h_k^{-1}(x) + \varphi(x) A \cdot (x),$$

where (x) is written (as usual) as a column matrix $(n \times 1)$; A is yet to be chosen; and $\varphi(x)$ is the function defined in 1.26.

First, we want $F_A(x)$ to have rank n on the set $K = h_k(N_{k-1} \cap \overline{U}_k)$; we are given that $f_{k-1} h_k^{-1}$ has rank n on K . Thus

$$D(F_A(x)) = D(f_{k-1} h_k^{-1}(x)) + A \cdot (x) \cdot D\varphi(x) + \varphi(x) A.$$

($D\varphi$ is a $1 \times n$ matrix.) The map of $K \times \mathcal{M}(p, n)$ into $\mathcal{M}(p, n)$ which carries (x, A) into $D(F_A(x))$ is continuous. It carries $K \times (0)$ into the open subset $\mathcal{M}(p, n; n)$ of $\mathcal{M}(p, n)$. Hence if A is sufficiently small, this map will carry $K \times A$ into $\mathcal{M}(p, n; n)$; our first requirement is that A be this small.

Secondly, we require A to be small enough that $\|A \cdot(x)\| < \varepsilon_k/2^k$ for all $x \in C^n(3)$.

Finally, by 1.20, A may be chosen arbitrarily small so that $f_{k-1}h_k^{-1}(x) + A \cdot(x)$ has rank n on $C^n(2)$.

Let A be chosen to satisfy this requirement.

We then define $f_k : M^n \rightarrow \mathbb{R}^p$ by the equation:

$$f_k(y) = \begin{cases} f_{k-1}(y) + \varphi(h_k(y))A \cdot h_k(y) & \text{for } y \in V_k \\ f_{k-1}(y) & \text{for } y \in M \setminus \bar{U}_k. \end{cases}$$

These definitions agree on the overlapping domains, so that f_k is differentiable. By the first condition on A , it has rank n on N_{k-1} ; by the third condition it has rank n on \bar{W}_k . By the second condition, f_k is a $\delta/2^k$ approximation to f_{k-1} .

We define $g(x) = \lim_{k \rightarrow \infty} f_k(x)$. Since the covering V_k is locally-finite, all the f_k agree on a given compact set for k sufficiently large; it follows that g is differentiable and has rank n everywhere. It is also a δ -approximation to f . □

1.29. Lemma. *If $p > 2n$, any immersion $f : M^n \rightarrow \mathbb{R}^p$ can be δ -approximated by a 1 - 1 immersion g . If f is 1 - 1 in a neighbourhood U of the closed set N , we may choose $g \mid N = f \mid N$.*

Proof: Choose a covering $\{U_\alpha\}$ of M^n such that $f \mid U_\alpha$ is an embedding (possible by 1.6). Let (V_i, h_i) be the locally-finite refinement constructed in 1.25; let $\varphi(x)$ be the function constructed in 1.26.

Let

$$\varphi_1(y) = \begin{cases} \varphi(h_1(y)) & \text{for } y \in V_i \\ 0 & \text{for other } y. \end{cases}$$

Then φ_1 is differentiable. As before, we assume (V_i, h_i) refines the covering $(U, M^n \setminus N)$ and that those V_i with non-positive indices are the ones contained in U .

Let $f_0 = f$. Given the immersion $f_{k-1} : M^n \rightarrow \mathbb{R}^p$, we define f_k by the equation

$$f_k(y) = f_{k-1}(y) + \varphi_k(y)b_k,$$

where b_k is a point of \mathbb{R}^p yet to be chosen. By the argument of the previous theorem, if b_k is chosen sufficiently small, f_k will have rank n everywhere. The first requirement is that b_k be this small; the second requirement is that b_k be small enough that f_k be a $\delta/2^k$ approximation to f_{k-1} .

Finally, let N^{2n} be the open subset of $M^n \times M^n$ consisting of pairs (y, y_0) , with $\varphi_k(y) \neq \varphi_k(y_0)$.

Consider the differentiable map

$$(y, y_0) \mapsto -[f_{k-1}(y) - f_{k-1}(y_0)] / [\varphi_k(y) - \varphi_k(y_0)]$$

from N^{2n} into \mathbb{R}^p . Since $2n < p$, the image of N^{2n} has measure zero, so that b_k may be chosen arbitrarily small and **not** in this image. It follows that $f_k(y) = f_k(y_0)$ if and only if $\varphi_k(y) = \varphi_k(y_0)$ and $f_{k-1}(y) = f_{k-1}(y_0)$ ($k > 0$).

Define $g(y) = \lim_{k \rightarrow \infty} f_k(y)$. If $g(y) = g(y_0)$ and $y \neq y_0$, it would follow that $f_{k-1}(y) = f_{k-1}(y_0)$ and $\varphi_k(y) = \varphi_k(y_0)$ for all $k > 0$. The former condition implies that $f(y) = f(y_0)$, so that y and y_0 cannot belong to any one set U_i . Because of the latter condition, this means that neither is in any set U_i for $i > 0$. Hence, they lie in U , contradicting the fact that f is 1 - 1 on U . □

1.30. Definition. Let $f : M^n \rightarrow \mathbb{R}^p$. The **limit set** $L(f)$ is the set of $y \in \mathbb{R}^p$ such that $y = \lim f(x_n)$ for

some sequence $\{x_1, x_2, \dots\}$ which has no limit point on M^n .

Exercise. Show the following:

- 1) $f(M^n)$ is a closed subset of \mathbb{R}^p if and only if $L(f) \subset f(M^n)$
- 2) f is a topological embedding if and only if f is 1 - 1 and $L(f) \cap f(M^n)$ is vacuous.

1.31. Lemma. *There exists a differentiable map $f: M^n \rightarrow \mathbb{R}$ with $L(f)$ empty.*

Proof: Let (V_i, h_i) and φ be chosen as in 1.25 and 1.26 with i ranging over positive integers; let

$$\varphi_i(y) = \begin{cases} \varphi(h_i(y)) & \text{if } y \in V_i \\ 0 & \text{otherwise.} \end{cases}$$

Define $f(y) = \sum_j (\varphi_j(y))$. This sum is finite, since V_i is a locally-finite covering. If $\{x_i\}$ is a set of points of M^n having no limit point, only finitely many lie in any compact subset of M^n . Given m , there is an integer i such that x_i is not in $\overline{W}_1 \cup \dots \cup \overline{W}_m$. Hence $x_i \in \overline{W}_j$ for some $j > m$, whence $f(x_i) > m$. Thus the sequence $f(x_m)$ cannot converge. □

1.32. Corollary. *Every M^n can be differentiably embedded in \mathbb{R}^{2n+1} as a closed subset.*

Proof: Let $f: M^n \rightarrow \mathbb{R} \subset \mathbb{R}^{2n+1}$ differentiably, with $L(f) = 0$. Set $\delta(x) \equiv 1$, and let g be a 1 - 1 immersion which is a δ -approximation to f . Then $L(g)$ is empty, so that g is a homeomorphism.

1.33. Definition. Let $f: M^n \rightarrow N^p$ be differentiable. Let N_1^{p-q} be a differentiable submanifold of N^p . Let $f(x) \in N_1^{p-q}$. Let (u^1, \dots, u^n) be a coordinate system about x ; and let (v^1, \dots, v^p) be a coordinate system about $f(x)$ such that on N_1^{p-q} , $v^1 = \dots = v^q = 0$ (see 1.6). Consider the condition that $\partial(v^1, \dots, v^q)/\partial(u^1, \dots, u^n)$ has rank q at x . This is the **transverse regularity condition for f and N_1^{p-q} at x** . [Exercise: Show that this condition is independent of coordinate system.]

Note that the set of points on which the transverse regularity condition is satisfied is an open subset of $f^{-1}(N_1^{p-q})$; f is said to be **transverse regular on N_1^{p-q}** if the condition is satisfied for each x in $f^{-1}(N_1^{p-q})$.

1.34. Lemma. *If $f: M^n \rightarrow N^p$ is transverse regular on N_1^{p-q} then $f^{-1}(N_1^{p-q})$ is a differentiable submanifold of dimension $n - q$ (or is empty).*

Proof: Let π project \mathbb{R}^p onto its first q components; $\pi: \mathbb{R}^p \rightarrow \mathbb{R}^q$. If $(V, h) = (v^1, \dots, v^p)$ is the coordinate system hypothesised in 1.33, then

$$N_1^{p-q} \cap V = h^{-1}\pi^{-1}(0)$$

where 0 denotes the origin in \mathbb{R}^q ; and $f^{-1}(N_1^{p-q} \cap V) = (\pi h f)^{-1}(0)$. Since $\pi h f$ has rank q at $x \in f^{-1}(N_1^{p-q} \cap V)$, the origin is a regular value of $\pi h f$. Hence $(\pi h f)^{-1}(0)$ is a differentiable submanifold of M^n of dim $n - q$ (see 1.12). □

1.35. Theorem. *Let $f: M^n \rightarrow N^p$ be differentiable; let N_1^{p-q} be a closed subset of M^n such that the transverse regularity condition for f and N_1^{p-q} holds at each x in $A \cap f^{-1}(N_1^{p-q})$. Let δ be a positive continuous function on M^n . There exists a differentiable map $g: M^n \rightarrow N^p$ such that*

- 1) g is a δ -approximation to f ,
- 2) g is transverse regular on N_1^{p-q} , and

$$3) \quad g|_A = f|_A.$$

Proof: There is a neighbourhood U of A in M^n such that f satisfies the transverse regularity condition on $U \cap f^{-1}(N_1^{p-q})$. Cover N^p by $N^p \setminus N_1^{p-q} = Y_0$ and coordinate system (Y_i, η_i) for $i > 0$; with coordinate functions (v^1, \dots, v^n) such that $v^1 = \dots = v^p = 0$ on N_1^{p-q} . Now the open sets $f^{-1}(Y_i)$ cover M^n , as do the open sets $U, M^n \setminus A$. Let (V_j, h_j) be a refinement of both coverings, constructed as in 1.25. Recall that $h_j(V_j) = C^n(3), h_j(U_j) = C^n(2), h_j(W_j) = C^n(1)$, and the W_j cover M^n . The V_j are to be indexed with positive and negative integers so that those V_j which are contained in U are the ones with non-positive indices.

Let φ be as in 1.26, and define

$$\varphi_i(x) = \begin{cases} \varphi(h_i(x)) & \text{for } x \in V_i \text{ and} \\ 0 & \text{elsewhere.} \end{cases}$$

For each j choose $i(j) \geq 0$ so that $f(V_j)$ is contained in $Y_{i(j)}$.

Set $f_0 = f$. Suppose f_{k-1} is defined and satisfies the transverse regularity condition for N_1^{p-q} at each point of the intersection of $f_{k-1}^{-1}(N_1^{p-q})$ with $\bigcup_{j < k} \overline{W}_j$. Furthermore suppose that $f_{k-1}^{-1}(\overline{U}_j) \subset Y_{i(j)}$ for each j . Setting $i = i(k)$, it follows in particular that $f_{k-1}^{-1}(\overline{U}_k) \subset Y_i$.

Consider

$$\pi \eta_i f_{k-1} h_k^{-1} : C^n(2) \rightarrow \mathbb{R}^q;$$

By 1.21, there is an arbitrarily small affine function $L(x) = A \cdot (x) + B$ such that when added to the previous function, the resulting map has the origin as a regular value. Consider \mathbb{R}^q as the first q coordinates in \mathbb{R}^p , and define

$$f_k(x) = \begin{cases} \eta_i^{-1}(\eta_i f_{k-1}(x) + L(h_k(x) \varphi_k(x))) & \text{for } x \text{ in a neighbourhood of } \overline{U}_k \\ f_{k-1}(x) & \text{for } x \text{ in } M^n \setminus U_k. \end{cases}$$

Here L is yet to be chosen. Of course, we must choose L small enough that

$$\eta_i f_{k-1} + L \varphi_k$$

lies in $C^n(1)$ for $x \in \overline{U}_k$, in order that k_i^{-1} may be applied to it. This is the first requirement on L . Secondly, we choose L small enough that f_k is a $\delta/2^k$ approximation to f_{k-1} . Thirdly choose L small enough so that $f_k(\overline{U}_j)$ is contained in $Y_{i(j)}$ for each j . This is possible since only a finite number of the sets \overline{U}_j can intersect \overline{U}_k .

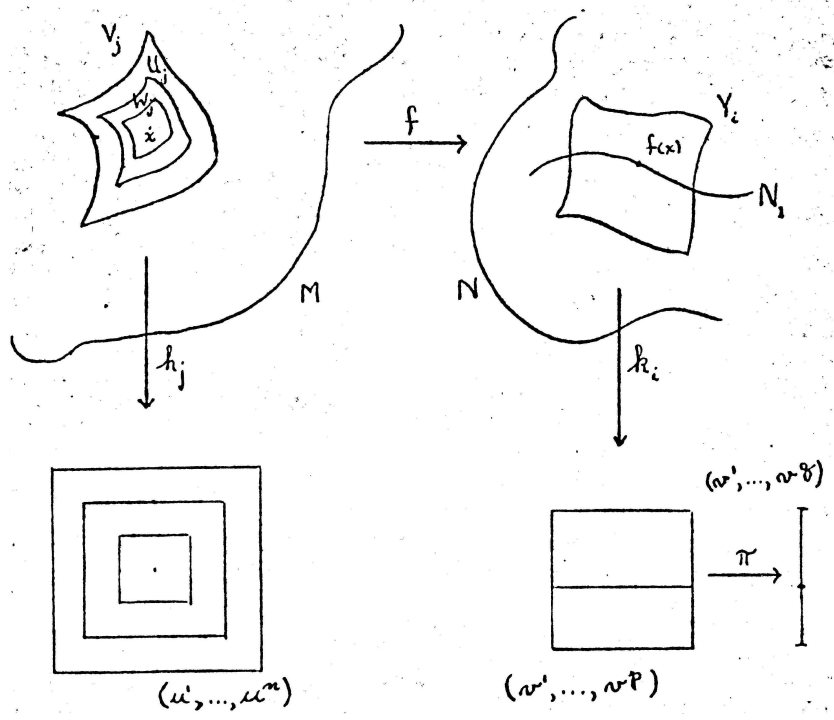
Now f_k by definition satisfies the transverse regularity condition for N_1^{p-q} at each point of $f_k^{-1}(N_1^{p-q}) \cap \overline{W}_k$. We want to choose L small enough that the condition is satisfied at each point of this intersection of $f_k^{-1}(N_1^{p-q})$ with $\bigcup_{j < k} \overline{W}_j$. It is sufficient to consider the intersection of this set with \overline{U}_k ; let this intersection be denoted by K . Consider the function which maps the pair (x, L) ($x \in K$) into

$$(f_k(x), D(\pi \eta_i f_{k-1} h_k^{-1}) \cdot (h_k(x))) \in N_1^{p-q} \times \mathcal{M}(q, n).$$

This function is continuous and carries $K \times (0)$ into the set

$$[(N^p \setminus N_1^{p-q}) \times \mathcal{M}(q, n)] \cup [N_1^{p-q} \times \mathcal{M}(q, n; q)],$$

which is open in $N_1^{p-q} \times \mathcal{M}(q, n)$. Hence for L sufficiently small, (K, L) is carried into this set, so that f_k satisfies the transverse regularity condition for N_1^{p-q} at each point of $f_k^{-1}(N_1^{p-q}) \cap (\bigcup_{j < k} \overline{W}_j)$. We define $g(x) = \lim_{k \rightarrow \infty} f_k(x)$, as usual. □



Chapter II Vector Space Bundles

2.1 Definition. An n -dimensional **real vector space bundle** ζ is a triple (π, a, s) where $\pi : E \rightarrow B$ is an onto continuous map between Hausdorff spaces that satisfy the following:

- 1) $F_b = \pi^{-1}(b)$, called a **fibre**, is an n -dimensional real vector space with $s : R \times E \rightarrow E$ carrying $R \times F_b$ into F_b , and $a : U(F_b \times F_b) \subset E \times E \rightarrow U(F_b)$ carrying $F_b \times F_b$ into F_b , as scalar product and vector addition, respectively.
- 2) (Local triviality) For each $b \in B$, there is a neighbourhood U of b and a homeomorphism $\varphi : U \times \mathbb{R}^n \rightarrow \pi^{-1}(U)$ such that φ is a vector space isomorphism of $b' \times \mathbb{R}^n \cong F_{b'}$, for each $b' \in U$.

If in 2) the neighbourhood U may be taken as all B , the bundle is said to be the **trivial bundle**.

If ζ, η are n -dimensional and p -dimensional vector space bundles, respectively, we define the **product bundle** $\zeta \times \eta$ as follows:

$$\begin{aligned} E(\zeta \times \eta) &= E(\zeta) \times E(\eta) \\ B(\zeta \times \eta) &= B(\zeta) \times B(\eta) \\ (\pi \times \lambda)(x, y) &= ((\pi(x), \lambda(y))) \end{aligned}$$

where π, λ are the projections in ζ, η respectively and $F_b(\zeta \times \eta)$ has the usual product structures for vector spaces.

If U is a subset of $B(\zeta)$, then $\zeta|U$ denotes the bundle $\pi : \pi^{-1}(U) \rightarrow U$. It is called the **restriction** of the bundle to U .

2.2 Definition. Let M^n be a differentiable manifold and let x_0 be in M^n . A **tangent vector** at x_0 is an operation X which assigns to each differentiable function f defined in a neighbourhood U of x_0 , a real number, that is, $X : \mathcal{O}(U) \rightarrow \mathbb{R}$. The following conditions must be satisfied:

- 1) If g is a restriction of f , $X(g) = X(f)$.
- 2) $X(cf + dg) = cX(f) + dX(g)$ for $c, d \in \mathbb{R}$
- 3) $X(f \cdot g) = X(f) \cdot g(x_0) + f(x_0) \cdot X(g)$, where the dot means ordinary real multiplication.

Then $X(1) = X(1 \cdot 1) = X(1) + X(1)$, by 3). Hence $X(1) = 0$ and $X(c)$ also = 0, by 2).

If one thinks of a tangent vector as being the velocity vector of a curve lying in the manifold, then $X(f)$ is merely the derivative of f with respect to the parameter of the curve. This is made more precise below.

2.3 Lemma. Let (u^1, \dots, u^n) be a coordinate system about x . Let X be a tangent vector at x . Then X may be written uniquely as a linear combination of the operators $\partial/\partial u^i$:

$$X = \sum \alpha^i \partial/\partial u^i.$$

Proof: We assume $u(x)$ is the origin. Given any $f(u^1, \dots, u^n)$ define

$$g(u^1, \dots, u^n) = \begin{cases} [f(u^1, \dots, u^n) - f(0, u^2, \dots, u^n)] / u^1 & \text{if } u^1 \neq 0 \\ \partial f(0, u^2, \dots, u^n) / \partial u^1 & \text{if } u^1 = 0. \end{cases}$$

To see that g is differentiable, note that

$$g(0, u^2, \dots, u^n) = \int_{[0,1]} [\partial f(0, u^2, \dots, u^n) / \partial u^1] dt.$$

(Then $f(u^1, \dots, u^n) = u^1 g_1(u^1, \dots, u^n) + f(0, u^2, \dots, u^n)$.) Similarly,

$$f(0, u^2, \dots, u^n) = u^2 g_2(u^2, \dots, u^n) + f(0, 0, u^3, \dots, u^n),$$

where $g_2(0) = \partial f / \partial u^2(0)$. Finally we have $f(u^1, \dots, u^n) = \sum u^i g_i + f(0)$, where $g_i(0) = \partial f / \partial u^i(0)$. Thus

$$X(f) = \sum X(u^i) g_i(0) + 0 \cdot X(f(0)) = \sum \alpha^i \partial f / \partial u^i(0),$$

where $\alpha^i = X(u^i)$.

Remark. If (v^1, \dots, v^n) is another coordinate system about x , and $X = \sum \beta^j \partial / \partial v^j$, then $\alpha^i = X(u^i) = \sum \beta^j \partial u^i / \partial v^j$. The α^i are called the components of the vector X with respect to the coordinate system (u^1, \dots, u^n) .

2.4 Alternate definition. A tangent vector at x is an assignment to every coordinate system (u^1, \dots, u^n) about x of an element $(\alpha^1, \dots, \alpha^n)$ of \mathbb{R}^n , with the requirement that if (β^j) is assigned to the system (v^1, \dots, v^n) , then $\alpha^i = \sum \beta^j \partial u^i / \partial v^j$. The derivation operator X is then defined as $\sum \alpha^i \partial / \partial u^i$. One checks readily that

- a) $X(f)$ is independent of the coordinate system used, and
- b) $X(f)$ satisfies requirements 1), 2), and 3) for a tangent vector.

2.5. Definition. For each x in M^n , the tangents at x form an n -dimensional vector space (by 2.3, the operations $\partial / \partial u^i$ form a basis). Let the totality of these be denoted by $E(\tau)$; define $\pi : E(\tau) \rightarrow M^n$ as mapping all the tangent vectors X at x_0 into x_0 . The local product structure around $x_0 \in U$ is given by $\varphi_U : U \times \mathbb{R}^n \rightarrow E(\tau)$, where $(U, h) = (u^1, \dots, u^n)$ is a coordinate system on M^n , and φ_U is defined as follows:

$$\varphi_U(x_0, a^1, \dots, a^n) = \text{tangent vector } X = \sum \alpha^i \partial / \partial u^i \text{ at } x_0.$$

Since φ_U is to be a homeomorphism, this structure imposes a topology on $E(\tau)$; since $\varphi_V^{-1} \varphi_U$ is a homeomorphism on $(U \cap V) \times \mathbb{R}^n$, this topology is unambiguously determined. One checks immediately that φ_U gives us a vector space bundle isomorphism for each fibre.

Indeed, $\varphi_V^{-1} \varphi_U$ is a C^∞ map on $(U \cap V) \times \mathbb{R}^n$, so that $E(\tau)$ is a differentiable manifold of dimension $2n$ (using definition 1.2 of a differentiable manifold). The map π is differentiable of rank n . This bundle τ is called the **tangent bundle** of M^n .

2.6. Definition. If $f : M_1^n \rightarrow M_2^m$, there is an induced map $df : E(\tau_1) \rightarrow E(\tau_2)$ defined as follows: $df(X) = Y$, where $Y(g) = X(gf)$. If X is a vector at x_0 , Y is a vector at $f(x_0)$. This is clearly linear on each fibre; it is called the **derivative** map.

If (U, h) and (V, k) are coordinate systems about $x_0, f(x_0)$ respectively, and $(\alpha^i), (\beta^j)$ are the respective components of X and Y with respect to these coordinate systems, then $(\beta^j) = D(kfh^{-1})(\alpha^i)$ where the vector components are written as column matrices, as usual.

2.7. Definition. Let ξ, η be two n -dimensional vector bundles. A **bundle map** $f : \xi \rightarrow \eta$ is a continuous map of $E(\xi)$ into $E(\eta)$ which carries each fibre isomorphically onto a fibre. The induced map $f_B : B(\xi) \rightarrow B(\eta)$ is automatically continuous.

If $B(\xi) = B(\eta)$ and the induced map is the identity, f is said to be an **equivalence**. Note that if f is an equivalence, it is a homeomorphism: Locally f is just a map $U \times \mathbb{R}^n \rightarrow V \times \mathbb{R}^n$. The projection of f^{-1} into the factor U is continuous, because f_B^{-1} is the identity. But f may be given by a non-singular

matrix function of $x \in U$; f^{-1} is the inverse of this matrix, so that the projection of f^{-1} into the factor \mathbb{R}^n is continuous. Hence f^{-1} is continuous.

If there is an equivalence of ζ onto η , we write $\zeta \simeq \eta$.

2.8. Lemma. *Given a bundle η with projection map $\lambda : E(\eta) \rightarrow B(\eta)$, and a map $f : B_1 \rightarrow B(\eta)$, there is a bundle $\pi : E_1 \rightarrow B_1$ and a bundle map $g : E_1 \rightarrow E(\eta)$ such that $\lambda g = f\pi$. Furthermore, E_1 is unique up to an equivalence.*

$$\begin{array}{ccc} & g & \\ E_1 & \rightarrow & E(\eta) \\ \pi \downarrow & & \downarrow \lambda \\ B_1 & \xrightarrow{f} & B(\eta) \end{array}$$

Remark. E_1 is called the **induced bundle** by f and is often denoted by $f^*\eta$.

Proof: Let E_1 be that subset of $B_1 \times E(\eta)$ consisting of points (b, e) such that $f(b) = \lambda(e)$. Define $\pi(b, e) = b$; $g(b, e) = e$. To show that E_1 is a vector space bundle, let $\varphi : V \times \mathbb{R}^n \rightarrow E(\eta)$ be a product neighbourhood in $E(\eta)$, and let $f(U) \subset V$. Then define $\varphi_1 : U \times \mathbb{R}^n \rightarrow E_1$ by $\varphi_1(b, x) = (b, \varphi(b, x))$. Then φ_1 is continuous and 1 - 1; its image equals $\pi^{-1}(U)$. Its inverse φ_1^{-1} carries (b, e) into $(b, p\varphi^{-1}(e))$, where p is the natural projection $V \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, hence it is continuous. The map g is an isomorphism on each fibre.

Now suppose $g' : E' \rightarrow E(\eta)$ is a bundle map, where $\pi' : E' \rightarrow B_1$ is a bundle and $\lambda g' = f\pi'$. We map $E' \rightarrow E_1$ by mapping

$$e' \mapsto (\pi'(e'), g'(e')) \in E_1.$$

Because g' is an isomorphism on each fibre, so is this map; and it induces the identity on the base space. Hence it is an equivalence.

$$\begin{array}{ccccc} & g & & g' & \\ E_1 & \rightarrow & E(\eta) & \leftarrow & E' \\ \pi \downarrow & & \downarrow \lambda & & \downarrow \pi' \\ B_1 & \xrightarrow{f} & B(\eta) & \leftarrow & B_1 \end{array}$$

2.9. Definition. Let ζ, η be two bundles over B . The **Whitney sum** $\zeta \oplus \eta$ is a bundle defined as the induced bundle $d^*(\zeta \times \eta)$ for $d : B \rightarrow B \times B$ be the diagonal map and the product bundle $E(\zeta) \times E(\eta) \rightarrow B \times B$.

$$\begin{array}{ccc} \zeta \oplus \eta = d^*(\zeta \times \eta) & \rightarrow & E(\zeta) \times E(\eta) \\ \downarrow & & \downarrow \\ B & \xrightarrow{d} & B \times B \end{array}$$

The proof of the following is left as an exercise.

- a) the fibre over b in $\zeta \oplus \eta$ is $F_b(\zeta) \times F_b(\eta)$, so that $\dim(\zeta \oplus \eta) = \dim \zeta + \dim \eta$,
- b) \oplus is commutative: $\zeta \oplus \eta \simeq \eta \oplus \zeta$,
- c) \oplus is associative: $(\zeta \oplus \eta) \oplus \varsigma \simeq \zeta \oplus (\eta \oplus \varsigma)$.

2.10. Definition. If ζ, η are bundles over B , then $g : E(\zeta) \rightarrow E(\eta)$ is a **homomorphism** if

- 1) it maps each fibre linearly into a fibre,
- 2) the induced map on B is the identity.

Note that an equivalence is both a bundle map and a homomorphism. An **embedding** of bundles is a 1 - 1 homomorphism.

2.11. Theorem. *If $f: E(\xi) \rightarrow E(\eta)$ maps each fibre linearly into a fibre, then f may be factored into a homomorphism followed by a bundle map.*

Proof: Let π_1, π_2 be the projections in ξ, η , respectively.

Let $f_B: B(\xi) \rightarrow B(\eta)$ be the map induced by f . Let $E_1 = f_B^* \eta$ be the bundle induced by f_B ; let g be the bundle map $E_1 \rightarrow E(\eta)$ and π be the projection $E_1 \rightarrow B(\eta)$.

$$\begin{array}{ccccc} & h & & g & \\ E(\xi) & \rightarrow & E_1 & \rightarrow & E(\eta) \\ \downarrow \pi_1 & & \downarrow \pi & & \downarrow \pi_2 \\ B(\xi) & \xrightarrow{i} & B(\xi) & \xrightarrow{f_B} & E(\eta) \end{array}$$

Define $h: E(\xi) \rightarrow B(\xi) \times E(\eta)$ by the equation $h(e) = (\pi_1(e), f(e))$. The image of h actually lies in that subset of $B(\xi) \times E(\eta)$ which is E_1 ; then h is a homomorphism. From the definition $f = gh$. □

2.12. Lemma. *Let ξ, η be bundles over B of dimensions n, p , respectively; let $g: \xi \rightarrow \eta$ be a homomorphism. If g is onto, then the kernel (g) is a bundle. If g is 1 - 1, then the cokernel (g), i.e., the quotient $\eta / \text{image } (g)$, is a bundle.*

Proof: Suppose g is 1 - 1 (i.e., has rank n when restricted to each fibre.) In $E(\eta)$, we define $e \sim e'$ if $e - e'$ exists and is in the image of g . We identify the elements of these equivalence classes; the resulting identification space is defined to be $E(\eta / g(\xi))$. It is a bundle over B with projection naturally defined and each fibre is a vector space of dimension $p - n$. We need only to show the existence of a local product structure.

Let U be an open set in B , with $\xi|U$ equivalent to $U \times \mathbb{R}^n$ and $\eta|U$ equivalent to $U \times \mathbb{R}^p$. Let g_0 denote the homomorphism of $U \times \mathbb{R}^n \rightarrow U \times \mathbb{R}^p$ induced by g . Now $(\eta / g(\xi))|U$ is equivalent to the quotient $U \times \mathbb{R}^p / g_0(U \times \mathbb{R}^n)$, so that it suffices to show that this latter quotient is locally a product.

g_0 is given by a matrix $M(b) \in \mathcal{M}(p, n)$ which depends continuously on the point $b \in U$. Given b_0 , we may assume that in a neighbourhood U_0 of b_0 , the first n rows are independent. We define $h: U_0 \times \mathbb{R}^n \times \mathbb{R}^{p-n} \rightarrow U \times \mathbb{R}^p$ as the linear function on whose matrix (non-singular) is

$$\left[\begin{array}{c|c} M(b) & 0 \\ \hline & I_{p-n} \end{array} \right]$$

The image of $U_0 \times \mathbb{R}^n \times 0$ under h is just $g_0(U_0 \times \mathbb{R}^n)$; since h is an equivalence, it induces an equivalence of

$$U_0 \times \mathbb{R}^{p-n} \simeq U_0 \times \mathbb{R}^n \times \mathbb{R}^{p-n} / U_0 \times \mathbb{R}^n \times 0 \text{ onto } U_0 \times \mathbb{R}^p / g_0(U_0 \times \mathbb{R}^n).$$

Secondly, suppose g is onto (i.e., it has rank p on each fibre.) $E(g^{-1}(0))$ is defined as that subset of $E(\xi)$ consisting of points e with $g(e) = 0$. Again, we need to show the existence of a local product structure. Let U, g_0 , and $M(b)$ be as above. Given b_0 , we may assume that the first p columns of are independent in the neighbourhood U_0 of b_0 . We define $h: U_0 \times \mathbb{R}^n \rightarrow U_0 \times \mathbb{R}^p \times \mathbb{R}^{n-p}$ by the matrix function

$$\left[\begin{array}{c|c} M(b) & \\ \hline 0 & I_{p-n} \end{array} \right]$$

Now h followed by the natural projection of $U_0 \times \mathbb{R}^p \times \mathbb{R}^{n-p}$ onto $U_0 \times \mathbb{R}^p$ equals $g_0|U$. Hence h^{-1} maps $U_0 \times 0 \times \mathbb{R}^{n-p}$ onto $g_0^{-1}(U_0 \times 0)$; since h is an equivalence, so is the restriction of h^{-1} to $U_0 \times 0 \times \mathbb{R}^{n-p}$. □

Remark. If g is onto, $\zeta/g^{-1}(0)$ is a bundle, being the quotient of the inclusion homomorphism $g^{-1}(0) \rightarrow \zeta$. If g is 1 - 1, $g(\zeta)$ is a bundle, being the kernel of the projection homomorphism $\eta \rightarrow g(\zeta)$.

2.13. Definition. If φ is a non-negative function on B , the **support** of φ is the closure of the set of x with $\varphi(x) > 0$. A **partition of unity** is a collection $\{\varphi_\alpha\}$ of non-negative functions on B , such that the sets $\{C_\alpha\} = \{\text{support}(\varphi_\alpha)\}$ form a locally-finite covering of B , and $\sum \varphi_\alpha(x) = 1$ (this is a finite sum for each x .)

2.14. Lemma. Let B be a normal space; $\{U_\alpha\}$ a locally-finite open covering of B . Then there is a partition of unity $\{\varphi_\alpha\}$ with $\text{support}(\varphi_\alpha) \subset U_\alpha$ for each α .

Proof: First, we show that there is an open covering $\{V_\alpha\}$ of B with $\bar{V}_\alpha \subset U_\alpha$ for each α . Assume that U_α are indexed by a set of ordinals (well-ordering theorem.) Let V_α be defined for all $\alpha < \beta$ and assume that the sets V_α along with the sets U_α for $\alpha \geq \beta$ cover B . Consider the set $A(\beta) = B \setminus \bigcup_{\alpha < \beta} V_\alpha \setminus \bigcup_{\alpha > \beta} U_\alpha$. Then $A(\beta) \subset U_\beta$. Let V_β be an open set containing the closed set $A(\beta)$, with $\bar{V}_\beta \subset U_\beta$ (normality.) This completes the construction of the V_α .

Now let g_α be a function which is positive on \bar{V}_α and 0 outside U_α (normality again.) Define $\varphi_{\alpha 0}(x) = g_\alpha(x) / \sum g_\alpha(x)$. Since $\{U_\alpha\}$ is locally-finite, the sum in the denominator is finite and positive, so $\{\varphi_\alpha\}$ is well-defined. □

Remark¹. If B is a differentiable manifold, φ_α may be chosen to be differentiable: Cover B with coordinate systems (V_i, h_i) as in 1.25 refining the covering $U_\alpha, B \setminus \bar{V}_\alpha$. Let $\varphi_i(y) = \varphi_i(h_i(y))$ for $y \in V_i$, and $\varphi_i(y) = 0$ otherwise (φ as in 1.26.) Let $g_\alpha(y) = \sum \varphi_i(y)$, where the sum extends over all i such that $V_i \subset U_\alpha$.

2.15. Lemma. Let B be paracompact and let $0 \rightarrow \zeta \xrightarrow{i} \eta \xrightarrow{\varphi} \zeta \rightarrow 0$ be an exact sequence of homomorphism of bundles. Then there is equivalence $f: \eta \rightarrow \zeta \oplus \zeta$, with f_i the natural inclusion and φf^{-1} the natural projection.

Proof: Let $\dim \zeta = n$; $\dim \eta = p$.

We first construct a Riemannian metric on η (i.e., a continuous inner product in $E(\eta)$.) Let $\{U_\alpha\}$ be a locally-finite covering of B with $\eta|U_\alpha$ trivial; let g_α be the corresponding projection of $\eta|U_\alpha$ onto \mathbb{R}^{n+p} . Let $\{\varphi_\alpha\}$ be a partition of unity with $\text{support}(\varphi_\alpha) \subset U_\alpha$.

If e, e' are in $E(\eta)$ and $\pi(e) = \pi(e')$, define $e \cdot e' = \sum \varphi_\alpha(\pi(e)) g_\alpha(e) \cdot g_\alpha(e')$, where the dot on the right hand side is the ordinary scalar product in \mathbb{R}^{n+p} . This is a finite sum; it satisfies the axioms for a scalar product.

The way we use the Riemannian metric is to break η up into $iE(\zeta)$ and its orthogonal complement. Let ζ' be the image of ζ in η and let $E(\zeta')$ be defined as that subset of consisting of elements which are orthogonal to $iE(\zeta)$. In order to show that ζ' has a local product structure, consider the homomorphism

$$h: \eta \rightarrow \zeta'$$

which sends each vector into its orthogonal projections in ζ' . [Verification that h is continuous. Over any coordinate neighbourhood U we can choose a basis a_1, \dots, a_n for the fibre of ζ' . Then the

¹ See Appendix, Proposition A.

function h carries $v \in E(\eta)$ into $\sum t_j a_j \in E(\zeta') \subset E(\eta)$, where $t_j = \sum B_{jk}(v \cdot a_k)$ and where (B_{jk}) denotes the inverse matrix to $(a_j \cdot a_k)$.] Since h is onto, its kernel ζ' is again a vector space bundle.

Now the bundle $i(\zeta) = \zeta'$ is equivalent to ζ . It remains to show that ζ' is equivalent to ζ and that η is equivalent to $\zeta' \oplus \zeta'$. The former follows immediately from the fact that $\varphi|_{\zeta'}$ is a homomorphism; from rank considerations it must be 1 - 1 and onto as well. The latter follows by noting that $E(\zeta' \oplus \zeta')$ is defined as the subset of $E(\zeta') \times E(\zeta')$ consisting of points (e_1, e_2) such that $\pi(e_1) = \pi(e_2)$. Consider the map f of $E(\zeta' \oplus \zeta')$ into $E(\eta)$ obtained by taking (e_1, e_2) into their sum in $E(\eta)$ (their sum exists because e_1 and e_2 lie in the same fibre.) This is clearly a homomorphism; from rank considerations, it must be 1 - 1 and onto. □

2.16. Definition. Let M_1, M_2 be differentiable manifolds and let $f: M_1 \rightarrow M_2$ be an immersion. The **normal bundle** ν_f is defined as follows:

Let τ_1, τ_2 be the tangent bundles of M_1, M_2 respectively. By 2.11, the map $df: E(\tau_1) \rightarrow E(\tau_2)$ may be factored into a homomorphism h of $E(\tau_1)$ into $E(f^*\tau_2)$ followed by a bundle map g . Now h is a 1 - 1 homomorphism because f is an immersion; hence by 2.12, $f^*\tau_2 / \text{image}(h)$ is a bundle over M_1 . It is called the normal bundle ν_f .

Then $0 \rightarrow \tau_1 \rightarrow f^*\tau_2 \rightarrow \nu_f \rightarrow 0$ is an exact sequence of homomorphisms, so that by 2.15, $f^*\tau_2$ is equivalent to $\tau_1 \oplus \nu_f$. Indeed, given a Riemannian metric on $f^*\tau_2$, ν_f is equivalent to the orthogonal complement of the image of τ_1 .

Let us consider the case $M_2 = \mathbb{R}^{n+p}$, where $\dim M_1 = n$. Then τ_2 is the trivial bundle, so that $f^*\tau_2$ is as well. (*Proof:* If $f: B_1 \rightarrow B(\eta)$ and η is trivial, so is $f^*\eta$. We have the diagram

$$\begin{array}{ccc} B \times \mathbb{R}^n & & \\ & \downarrow \pi & \\ f: B_1 \rightarrow B & & \end{array}$$

$E(f^*\eta)$ is defined as that subset of $B_1 \times (B \times \mathbb{R}^n)$ consisting of points (b_1, b, x) such that $f(b_1) = (b, x)$, i.e., of all points $(b_1, f(b_1), x)$. If we map this into (b_1, x) , we obtain an equivalence of $f^*\eta$ with the bundle $B_1 \times \mathbb{R}^n \rightarrow B_1$.

Thus $\tau_1 \oplus \nu_f$ is equivalent to a trivial bundle. In what follows, we investigate the following question: Given ζ , does there exist an η with $\zeta \oplus \eta$ trivial? Using 1.28, this is always the case for ζ the tangent bundle of an n -manifold, and indeed η may be chosen also to have dimension n . A more general answer appears in 2.19.

2.17. Definition. Let $f: M_1^n \rightarrow M_2^p$; If f has rank p at every point of M_1 , it is said to be **regular**. If f is regular, the homomorphism $h: \tau_1 \rightarrow f^*\tau_2$ given by 2.11 is an onto map. By 2.12, the kernel of h is a bundle α_f . It is called the **bundle along the fibre**.

Note that $f^{-1}(y)$ is a submanifold of M_1 of dimension $n - p$ (by 1.12 or 1.34.) The inclusion i_y of $f^{-1}(y)$ into M_1 induces an inclusion di_y of its tangent bundle into τ_1 . The kernel of h consists precisely of the vectors which are in the image of some di_y , i.e., the vectors tangent to the submanifolds $f^{-1}(y)$ are the ones carried into 0 by h .

One has the exact sequence $0 \rightarrow \alpha_f \rightarrow \tau_1 \xrightarrow{g} f^*\tau_2 \rightarrow 0$, so that by 2.15, τ_1 is equivalent to $\alpha_f \oplus f^*\tau_2$.

2.18. Definition. A bundle ζ is of **finite type** if B is normal and may be covered by a finite number of neighbourhoods U_1, \dots, U_k such that $\zeta|_{U_i}$ is trivial for each i .

2.19. Lemma. ζ is of finite type if B is compact, or paracompact finite dimensional.

Proof: The former statement is clear; let us consider the latter. By definition, the dimension of B is not greater than n if every covering has an open refinement such that

$$\text{no point of } B \text{ is contained in more than } n + 1 \text{ elements of the refinement.} \quad (*)$$

It is a standard theorem of topology that an n -manifold has dimension n in this sense.

Cover B by open sets U , with $\zeta|U$ trivial; let $\{V_\alpha\}$ be an open refinement of this covering satisfying (*). By 1.22, we may assume that $\{V_\alpha\}$ is locally-finite as well. Let $\{\varphi_\alpha\}$ be a partition of unity with $\text{support}(\varphi_\alpha) \subset V_\alpha$ for each α (2.14.)

Let A_i be the set of unordered $(i + 1)$ -tuple of distinct elements of the index set of $\{\varphi_\alpha\}$. Given a in A_i , where $a = \{\alpha_0, \dots, \alpha_n\}$, let W_{ia} be the set of all x such that $\varphi_\alpha(x) < \min \{\varphi_{\alpha_0}(x), \dots, \varphi_{\alpha_n}(x)\}$ for all $\alpha \neq \alpha_0, \dots, \alpha_n$. Each set W_{ia} is open, and $W_{ia} \cap W_{ib} = \emptyset$ if $a \neq b$. Also W_{ia} is contained in the intersection of the supports of $\varphi_{\alpha_0}(x), \dots, \varphi_{\alpha_n}(x)$, and hence in some set V_α . If we set X_i equal to the union of all sets W_{ia} , for fixed i , $\zeta|X_i$ is trivial. Note that $\zeta|W_{ia}$ is trivial and W_{ia} are disjoint.

Finally, the sets X_0, \dots, X_n cover B . Given x in B , x is contained in at most $n + 1$ of the sets V_α , so that at most $n + 1$ of the functions φ_α are positive at x . Since some φ_α is positive at x , x is contained in one of the sets W_{ia} for $0 \leq i \leq n$.

[The intuitive idea of the proof is as follows: Consider an n -dimensional simplicial complex, with φ_α the barycentric coordinate of x with respect to the vertex α . The sets $W_{0\alpha}$ will be disjoint neighbourhoods of the vertices, the sets W_{1a} disjoint neighbourhoods of the open 1-simplices, and so on.] \square

2.20. Theorem. *If ζ is of finite type, there is a bundle η such that $\zeta \oplus \eta$ is trivial.*

Proof: We proceed by showing that ζ may be embedded in a trivial bundle $B \times \mathbb{R}^m$, so that we have

the exact sequence $0 \rightarrow \zeta \rightarrow B \times \mathbb{R}^m \xrightarrow{i} B \times \mathbb{R}^m / i(\zeta) \rightarrow 0$ by 2.12. The theorem then follows from 2.15. (Paracompactness is not needed since the trivial bundle clearly has a Riemannian metric.)

Cover B by finitely many neighbourhoods U_1, \dots, U_k with $\zeta|U_i$ trivial for each i . Let $\varphi_1, \dots, \varphi_k$ be a partition of unity with $\text{support}(\varphi_i) \subset U_i$ for each i (2.14). Let f_i denote the equivalence of $E(\zeta|U_i)$ onto $U_i \times \mathbb{R}^n$; let f_i^1, \dots, f_i^n denote the coordinate functions of its projection into \mathbb{R}^n .

We define $h : E(\zeta) \rightarrow B \times \mathbb{R}^{mk}$ as follows:

$$h(e) = (\pi(e), (\varphi_i \pi(e)) : f_i^j(e)) \quad i = 1, \dots, k; \quad j = 1, \dots, n$$

(no summation indicated.) This is well-defined, since $\varphi_i \pi(e) = 0$ unless $e \in E(\zeta|U_i)$. It is clearly a homomorphism, since each f_i^j is linear on $E(\zeta|U_i)$. To show that it is 1 - 1, let $e \neq 0$. Then for some i , $\varphi_i \pi(e) > 0$. Since f_i is an equivalence, $f_i^j(e) \neq 0$ for some j . Hence $h(e) \neq (\pi(e), 0)$ as desired. \square

2.21. Definition. The bundle ζ is *s-equivalent*² to η if there are trivial bundles σ^p, σ^n such that $\zeta \oplus \sigma^p \cong \eta \oplus \sigma^n$.

Here $\sigma^p = B \times \mathbb{R}^p$. Symmetry and reflexivity are clear. To show transitivity, assume $\zeta \oplus \sigma^p \cong \eta \oplus \sigma^q$ and $\eta \oplus \sigma^r \cong \zeta \oplus \sigma^s$. Then $\zeta \oplus \sigma^p \oplus \sigma^r \cong \zeta \oplus \sigma^s \oplus \sigma^q$.

Remark: *s*-equivalence differs from equivalence. E.g., consider the two-sphere S^2 in \mathbb{R}^3 .

Then $\tau^2 \oplus \nu^1 \cong \sigma^3$. The normal bundle ν^1 is easily seen to be trivial; but it is a classical theorem of topology that τ^2 is not (it does not admit a non-zero cross-section.) Hence τ^2 is *s*-trivial, but not trivial.

² Short for “stably equivalent”.

2.22. Theorem. *The set of s -equivalence classes of vector space bundles of finite type over B forms an abelian group under \oplus^3 .*

Proof: To avoid logical difficulties, we consider only subbundles of $B \times \mathbb{R}^m$, for all m . This suffices, since any bundle of finite type may be embedded in some $B \times \mathbb{R}^m$, by 2.20. The class o^p of trivial bundles is the identity element. The existence of inverses is the substance of 2.20. □

2.23. Corollary. *Given two immersions of the differentiable manifold M in euclidean space, their normal bundles are s -equivalent.* □

2.24. Definition. M^n is a π -manifold if M may be embedded in some \mathbb{R}^{n+p} so that its normal bundle is trivial.

This is equivalent to the requirement that τ^n be s -trivial; Let τ^n be s -trivial. If we take some immersion of M into \mathbb{R}^{n+p} , then $\tau^n \oplus \nu^p$ is trivial by 2.16, so that ν^p is s -trivial, i.e., $\nu^p \oplus o^q = o^{p+q}$ for some q . Consider the composite immersion $M \rightarrow \mathbb{R}^{n+p} \subset \mathbb{R}^{n+p+q}$. The normal bundle of M in \mathbb{R}^{n+p+q} is just $\nu^p \oplus o^q$, which is trivial.

Conversely, if ν^p is trivial for some immersion, then τ^n is s -trivial because $\tau^n \oplus \nu^p$ is trivial.

2.25. Definition. Let $G_{p,n}$ denote the set of all n -dimensional vector subspaces of \mathbb{R}^{n+p} (i.e., all n -dimensional hyperplanes through the origin.) It is called the **Grassman manifold** of n -planes in $n+p$ space.

Its topology is obtained as follows; Consider $\mathcal{M}(n, n+p; n)$; we identify two elements of this set if the hyperplane spanned by their row vectors are the same. $G_{p,n}$ is in 1 - 1 correspondence with this identification space, and is given the identification topology. Let ρ be the projection

$$\rho : \mathcal{M}(n, n+p; n) \rightarrow G_{p,n}.$$

Now $\rho(A) = \rho(B)$ if and only if $A = CB$ for some non-singular $n \times n$ matrix C : The hyperplane $\rho(A)$ consists of all points $(x^1, \dots, x^{n+p}) \in \mathbb{R}^{n+p}$ which equal $(c^1, \dots, c^n) \cdot A$ for some choice of constants c^i . If $\rho(A) = \rho(B)$, then

$$\begin{aligned} (1, 0, \dots, 0) \cdot A &= (c^1, \dots, c^n) \cdot B \\ (0, 1, \dots, 0) \cdot A &= (c^1, \dots, c^n) \cdot B \\ &\dots &= &\dots \\ (0, 0, \dots, 1) \cdot A &= (c^1, \dots, c^n) \cdot B \end{aligned}$$

for some choice of c^i . Then $IA = CB$, where C has rank n because A does. The converse is clear.

(a) $G_{p,n}$ is locally euclidean. Let $A \in \mathcal{M}(n, n+p; n)$; after permuting the columns, we may assume $A = (P, Q)$ where P is $n \times n$ and non-singular. Let U be the set of all such A ; it is an open set in $\mathcal{M}(n, n+p; n)$, being the inverse image of the non-zero reals under the continuous map $(P, Q) \rightarrow \det P$. If $\rho(P, Q) = \rho(R, S)$, where P is non-singular, then $(P, Q) = (CR, CS)$ for some non-singular C . Hence R is necessarily non-singular; it follows that $\rho^{-1}(\rho(U)) = U$, so that $\rho(U)$ is open in $G_{p,n}$ (by definition of the identification topology.)

We show $\rho(U)$ homeomorphic with \mathbb{R}^{pn} . Define $\varphi : U \rightarrow \mathbb{R}^{pn}$ by $\varphi(P, Q) = P^{-1}Q$. If $\rho(P, Q) = \rho(R, S)$

3 The resulting abelian group is called the K -group of B . For more on this, see "Vector Bundles and K -Theory" by Allen Hatcher in his homepage <http://www.math.cornell.edu/~hatcher/#ATI>.

then $(P, Q) = (CR, CS)$, so that

$$P^{-1}Q = (CR)^{-1}(CS) = R^{-1}S.$$

Hence φ induces a continuous map $\varphi_0 : \rho(U) \rightarrow \mathbb{R}^{pn}$. Define $\psi : \mathbb{R}^{pn} \rightarrow \rho(U)$ by $\psi(Q) = \rho(I, Q)$ where Q is an $n \times p$ matrix. One checks immediately that ψ and φ_0 are inverse of each other.

$$\begin{array}{ccc} \mathcal{M}(n, n+p; n) & \supset & U \\ \downarrow \rho & & \downarrow \rho \quad \varphi_0 \searrow \varphi \\ G_{p,n} & \supset & \rho(U) \xleftrightarrow{\psi} \mathbb{R}^{pn} \end{array}$$

(b) To show that $G_{p,n}$ is Hausdorff, we show that φ maps every compact set into a closed set (this will clearly suffice.) Let K be a compact subset of \mathbb{R}^{pn} ; we show $\varphi^{-1}(K)$ is closed in $\mathcal{M}(n, n+p; n)$. $\varphi^{-1}(K)$ consists of all matrices (P, Q) with P non-singular and $P^{-1}Q \in K$. Let $(P, Q) \in \mathcal{M}(n, n+p; n)$ be the limit of the sequence $\{(P_i, Q_i)\}$ of elements of $\varphi^{-1}(K)$. Since K is compact, some subsequence of the sequence $\{\varphi(P_i, Q_i)\} = \{P_i^{-1}Q_i\}$ converges to a point R of K . Then the corresponding subsequence of the sequence $\{Q_i\}$ converges to PR , so that $C = P(I, R)$. Since (P, Q) has rank n it follows that P is non-singular, so that $(P, Q) \in \varphi^{-1}(K)$, as desired.

Hence $G_{p,n}$ is a manifold of dimension pn .

(c) $G_{p,n}$ is a differentiable manifold and ρ is a differentiable map. A function f on the open set V in $G_{p,n}$ belongs to the differentiable structure \mathcal{D} if $f\rho$ is differentiable. To show that this satisfies the condition for a differentiable structure, we show that $(\rho(U), \varphi_0)$, as defined in (a), is a coordinate system. Let f be defined on $V \subset \rho(U)$. Given $Q \in \mathbb{R}^{pn}$, $f\varphi_0^{-1}(Q) = f\rho(I, Q)$ so that $f\varphi_0^{-1}$ is differentiable if $f\rho$ is. Conversely, given $(P, Q) \in V$, $f\rho(P, Q) = f\varphi_0^{-1}\varphi_0\rho(P, Q) = f\varphi_0^{-1}(P^{-1}Q)$, so that $f\rho$ is differentiable if $f\varphi_0^{-1}$ is.

(d) $G_{p,n}$ is compact. Let L be the subset of $\mathcal{M}(n, n+p; n)$ consisting of matrices whose rows are orthonormal vectors. L is a closed and bounded subset of $\mathbb{R}^{n(n+p)}$. Since $\rho(L) = G_{p,n}$ (the Gram-Schmidt orthogonalisation process proves this), $G_{p,n}$ is compact.

(e) $G_{p,n}$ is diffeomorphic to $G_{n,p}$. Geometrically, the homeomorphism h is defined as carrying each hyperplane into its orthogonal complement. It is clearly 1 - 1; to show it is differentiable we use the coordinate system $(\rho(U), \varphi_0)$ defined in (a). Let g map U into $\mathcal{M}(n, n+p; n)$ by carrying (P, Q) into $(-(P^{-1}Q)^t, I_p)$; it is differentiable (t denotes transpose.) The row space of (P, Q) is the same as that of $(I_n, P^{-1}Q)$, while the row vectors of this matrix are orthogonal to those of $(-(P^{-1}Q)^t, I_p)$ (multiply the one by the transpose of the other.) Hence g induces $h | \rho(U)$, so that the latter is differentiable.

2.26. Definition. Let $E(\gamma_p^n)$ be defined as that subsets of $G_{p,n} \times \mathbb{R}^{n+p}$ consisting of pairs (H, x) where x is a vector lying in the hyperplane H . It is called the **universal bundle** (for reasons we shall see.) The projection π maps (H, x) into H ; the fibre is thus an n -dimensional subspace of \mathbb{R}^{n+p} .

γ_p^n is an n -dimensional vector space bundle over $G_{p,n}$. We need to show the existence of a local product structure. Let $(\rho(U), \varphi_0)$ be a coordinate neighbourhood on $G_{p,n}$, as in (a) above. We define $h : \rho(U) \times \mathbb{R}^n \rightarrow \pi^{-1}\rho(U)$ as carrying $(H, (x^1, \dots, x^n))$ into $(x^1, \dots, x^n) \cdot (I_n, Q)$ where $Q = \varphi_0(H)$. This is a vector in the hyperplane H ; h is clearly an isomorphism on each fibre. Its inverse is continuous, since it sends $(H, (y^1, \dots, y^{n+p}))$ in $G_{p,n} \times \mathbb{R}^{n+p}$ into $(H, (y^1, \dots, y^n))$ in $\rho(U) \times \mathbb{R}^n$.

2.27. Definition. ζ is a **differentiable vector space bundle** if $E(\zeta)$ and $B(\zeta)$ are differentiable manifolds, and if the homeomorphisms

$$U \times \mathbb{R}^n \rightarrow \pi^{-1}(U)$$

which specify the local product structure can be chosen as diffeomorphisms.

It follows that $\pi : E \rightarrow B$ is differentiable of maximum rank. Note that B can be differentially embedded in E by mapping b into the 0-vector of F_b . The normal bundle of this embedding is just ζ .

Examples of differentiable bundles include the tangent bundles of a manifold, the normal bundle of an immersed manifold, and the universal bundle γ_p^n above. In the latter case, $E(\gamma_p^n)$ is embedded differentially in $G_{p,n} \times \mathbb{R}^{n+p}$.

2.28. Theorem. Let ζ^n be an n -dimensional vector space bundle. The following conditions are equivalent:

- (a) ζ is of finite type.
- (b) There is a bundle η^p such that $\zeta^n \oplus \eta^p$ is trivial.
- (c) There is a bundle map $\zeta^n \rightarrow \gamma_p^n$ for some p . (Thus the terminology “universal bundle“ for γ_p^n .)

Proof: We have already shown that (a) \implies (b) (2.20); the bundle η^p there constructed has dimension $n(k-1)$, where k is the number of elements in the covering U_1, \dots, U_k of $B(\zeta) = B$ such that $\zeta|U_i$ is trivial.

(b) \implies (c): Condition (b) means that ζ^n may be embedded in the trivial bundle $B(\zeta) \times \mathbb{R}^{n+p}$; let f be this embedding. We wish to define g and g_B in the following diagram:

$$\begin{array}{ccc} E(\zeta) & \xrightarrow{g} & E(\gamma_p^n) \\ \pi \downarrow & & \downarrow \\ B(\zeta) & \xrightarrow{g_B} & G_{p,n} \end{array}$$

Since f is a 1 - 1 homomorphism, $f(F_b)$ is the cartesian product of b and an n -dimensional hyperplane H^n in \mathbb{R}^{n+p} ; let $g_B(b) \equiv H^n$. If $e \in F_b$, then $f(e) = (b, x)$ where x is a vector in the hyperplane H^n ; let $g(e) = (H^n, x)$ in $G_{p,n} \times \mathbb{R}^{n+p}$. Then $g(e)$ actually lies in the subset of $G_{p,n} \times \mathbb{R}^{n+p}$ which constitutes $E(\gamma_p^n)$. From rank considerations, g is automatically an isomorphism on each fibre.

It remains to show that g is continuous. Locally, g just looks like a map $U \times \mathbb{R}^n \rightarrow G_{p,n} \times \mathbb{R}^{n+p}$. We factor it into a continuous map $h : U \times \mathbb{R}^n \rightarrow \mathcal{M}(n, n+p; n) \times \mathbb{R}^{n+p}$ followed by the projection $\rho \times 1$ into $G_{p,n} \times \mathbb{R}^{n+p}$. Locally, f looks like a map $U \times \mathbb{R}^n \rightarrow B \times \mathbb{R}^{n+p}$. Let e_1, \dots, e_n be a basis for \mathbb{R}^n ; we define $h(b, x)$ as $(A, p_2 f(b, x))$. Here p_2 projects $B \times \mathbb{R}^{n+p}$ onto its second factor and A is the matrix having $p_2 f(b, e_1), \dots, p_2 f(b, e_n)$ as its rows. Then h is continuous and $(\rho \times 1)h$ equals g . (Note: The converse assertion, (c) implies (b), can be proved by the same argument.)

(c) \implies (a): Being compact, $G_{p,n}$ is covered by a finitely many neighbourhoods U_i with $\gamma_p^n|U_i$ trivial. (In fact, $(n+p)! / n!p!$ neighbourhoods will suffice.) If f is a bundle map $\zeta^n \rightarrow \gamma_p^n$ then the sets $\{f_B^{-1}(U_i) = V_i\}$ cover B , and $\zeta|V_i$ is equivalent to the bundle induced by $f_B : V_i \rightarrow G_{p,n}$ (the uniqueness part of 2.8.) Then $\zeta|V_i$ is trivial (since it is induced from a trivial bundle.) □

Chapter III The Cobordism Theory of Thom

3.1. Definition. An n -manifold with boundary Q is a Hausdorff space with a countable basis which is locally homeomorphic with \mathbb{H}^n (the subset of \mathbb{R}^n such that $x^1 \geq 0$.) The **boundary** ∂Q is that subset of corresponding to \mathbb{R}^{n-1} under the local homeomorphism (\mathbb{R}^{n-1} being the subset of \mathbb{R}^n with $x^1 = 0$.) ∂Q is well-defined, since the image of an open set in \mathbb{R}^n under a homeomorphism of it into \mathbb{R}^n must be open (Brouwer theorem on invariance of domain.) It is clear that ∂Q is an $(n-1)$ -manifold.

A **differential structure** \mathcal{D} on Q is a collection of real-valued functions f defined on open subsets of Q such that

- 1) Every point of Q has an open neighbourhood U and a homeomorphism h of U into an open subset of \mathbb{H}^n , such that f is in \mathcal{D} if and only if fh^{-1} is differentiable. (f is defined on an open subset of U ; fh^{-1} differentiable means that it may be extended to a neighbourhood of $h(U)$ in \mathbb{R}^n so as to be differentiable.)
- 2) If U_i are open sets contained in the domain of f and $U = \cup U_i$, then $f|_U \in \mathcal{D}$ if and only if $f|_{U_i} \in \mathcal{D}$ for each i .

As before, (U, h) is called a **coordinate system** on Q , and one can define differentiable structure alternatively by means of coordinate systems.

We impose an additional condition on \mathcal{D} in 3.2.

3.2. Definition. Let M_1, M_2 be compact differentiable n -manifolds. They are said to be in the same **cobordism class** ($M_1 \sim M_2$) if there is a compact differentiable $n+1$ manifold-with-boundary Q such that ∂Q is diffeomorphic with the disjoint union of M_1 and M_2 (denoted by $M_1 + M_2$.)

Symmetry and reflexivity of this relation are clear. To show transitivity, we impose the additional condition on \mathcal{D} that there is a neighbourhood U of ∂Q in Q which is diffeomorphic with $\partial Q \times [0, 1)$, the diffeomorphism being the identity on $\partial Q \times 0$. This is redundant, but we assume it to avoid proving it⁴. Transitivity follows:

Let $M_1 + M_2$ be diffeomorphic with ∂Q_1 and $M_2 + M_3$ be diffeomorphic with ∂Q_2 ; let h_1, h_2 be the diffeomorphisms. We form a new space Q_3 from $Q_1 \cup Q_2$ by identifying each point of $h_1(M_2)$ with its image under $h_2 h_1^{-1}$. There is then a homeomorphism of $M_2 \times (-1, 1)$ into this space which equals h_1 when restricted to $M_2 \times 0$, and is a diffeomorphism of $M_2 \times [0, (-1)^i)$ into Q_i for $i = 1, 2$. (It is derived from the postulated "product neighbourhoods" $\partial Q_i \times [0, 1)$.) If this is taken to be a coordinate system on Q_3 , Q_3 becomes a differentiable manifold-with-boundary, and $M_1 + M_3$ is diffeomorphic with ∂Q_3 . Q_1 and Q_2 diffeomorphic with subsets of Q_3 .

3.3. Definition. As usual, there are logical difficulties involved in considering these cobordism classes. One way of avoiding them is to consider only manifolds-with-boundary embedded in some euclidean space \mathbb{R}^n : If Q_1 is a differentiable manifold-with-boundary and $Q_2 = \partial Q_1 \times [0, 1)$, then the space Q_3 constructed in the preceding paragraph is a differentiable manifold, so that it may be embedded in some euclidean space. Hence Q_1 may so be embedded.

With these restrictions, the set of cobordism classes of n -manifolds forms an abelian group (denoted

⁴ This fact is called the **smooth collaring theorem**. See Appendix, Proposition B for a proof.

by \mathcal{N}^n) under the operation $+$ (disjoint union.) If $M_1 \sim M_1'$ and $M_2 \sim M_2'$, this means that $M_i + M_i'$ is diffeomorphic with ∂Q_i . Then $(M_1 + M_2) + (M_1' + M_2')$ is diffeomorphic with $\partial(Q_1 \cup Q_2)$, so that $M_1 + M_2 \sim M_1' + M_2'$ and the operation $+$ is well-defined on cobordism classes. The zero element is the vacuous manifold or the n -sphere (or ∂Q , where Q is any compact differentiable $(n + 1)$ -manifold-with-boundary.) The remaining axioms are clear. Note that $M + M$ is diffeomorphic with $\partial(M \times [0, 1])$, so that every element is of order 2.

The groups \mathcal{N}^n are called the (non-orientable) **cobordism groups**. Let \mathcal{N} denote the direct sum $\mathcal{N}^0 \oplus \mathcal{N}^1 \oplus \mathcal{N}^2 \oplus \dots$. There is a bilinear symmetric pairing of $\mathcal{N}^i, \mathcal{N}^j$ into \mathcal{N}^{i+j} , i.e., a homomorphism of $\mathcal{N}^i \otimes \mathcal{N}^j$ into \mathcal{N}^{i+j} induced by the operation of cartesian product.

First, $(M_1 + M_2) \times M_3 = (M_1 \times M_3) + (M_2 \times M_3)$ by definition of cartesian product. Second, if $M_1 \sim 0$, i.e., $M_1 = \partial Q$, then $M_1 + M_2$ is diffeomorphic with $\partial(Q \times M_2)$, so that $M_1 + M_2 \sim 0$.

Since $M_1 + M_2 \sim M_2 + M_1$, and since $M_1 \times p \sim M_1$ (where p is a point-manifold), this pairing makes \mathcal{N} into a (graded) commutative ring with unit. Indeed, it is a graded algebra over the field $\mathbb{Z}/2\mathbb{Z}$.

3.4. Remark. The general result of Thom is the following

Theorem. \mathcal{N} is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ with one generator in each positive dimension except those of the form $2^m - 1$. If n is even, projective n -space is a generator.

This theorem means that there are compact manifolds M^2, M^4, M^6, \dots such that every compact manifold is in the cobordism class of a disjoint union of products of these manifolds, and that there are no relations among the generators (except commutativity and associativity of products.)

Thom's procedure is to show that \mathcal{N}^n is isomorphic with the $(n + k)^{\text{th}}$ homotopy group of a certain space T_k , and then to compute these homotopy groups. We shall consider only the first of these two problems in the present notes.

3.5. Definition. Let h be an embedding of the differentiable manifold M^n in \mathbb{R}^{n+k} ; consider the normal bundle of this embedding. Using the standard Riemannian metric for the tangent bundle to \mathbb{R}^{n+k} , this normal bundle is equivalent to the orthogonal complement of the image in the tangent bundle of \mathbb{R}^{n+k} of the tangent bundle of M^n (2.16); this complement we denote by ν^k . Define e as the canonical map of $E(\nu^k)$ into \mathbb{R}^{n+k} which maps the vector ν normal to x into its end point. (Described differently, one maps the tangent bundle to \mathbb{R}^{n+k} into itself canonically by mapping the vector ν , based at x , into the point $\nu + x$ of \mathbb{R}^{n+k} . This map is differentiable; its restriction to $E(\nu^k)$ is the map e .)

Consider M^n as the zero vectors of $E(\nu^k)$. Then we have the

3.6. Theorem. There is a neighbourhood of M^n in $E(\nu^k)$ which is mapped diffeomorphically onto a neighbourhood of M^n in \mathbb{R}^{n+k} .

Proof: Note that e is differentiable, and that it has rank $n + k$ at points of $M^n \subset E(\nu^k)$. (This is easily checked by computing the derivative matrix of e with respect to a local coordinate system.) Hence e has rank $n + k$ in some neighbourhood of M^n in $E(\nu^k)$, so that it is a local homeomorphism at points of M^n : It maps a neighbourhood of each $x \in M^n$ homeomorphically onto a neighbourhood of $e(x)$. We then appeal to the topological

Lemma. Let X, Y be Hausdorff spaces with countable bases and X be locally compact. If $f: X \rightarrow Y$ is a local homeomorphism and the restriction of f to the closed subset A is a homeomorphism, then f is a homeomorphism on some neighbourhood V of A .

This lemma is proved as follows:

- 1) If A is compact, the lemma holds. For otherwise, there would be points x, y arbitrarily close to A such that $f(x) = f(y)$. Since A has a compact neighbourhood, we may choose sequences $\{x_n\}, \{y_n\}$ converging to x, y respectively, in A such that $x_n \neq y_n$ and $f(x_n) = f(y_n)$. Hence $f(x) = f(y)$ so that $x = y, f$ being a homeomorphism on A . But then f is not a local homeomorphism at x .
- 2) Let A_0 be a compact subset of A . Then there is a neighbourhood U_0 of A_0 such that \bar{U}_0 is compact and f is a homeomorphism on $\bar{U}_0 \cup A_0$: It will suffice for f to be 1 - 1, since f is a local homeomorphism. By (1), let V_0 be a neighbourhood of A_0 so that $f|_{V_0}$ is 1 - 1. If no neighbourhood of A_0 in V_0 satisfies the requirement for U_0 , there is a sequence $\{x_n\}$ of $X \setminus A$ converging to $x \in A_0$ with $f(x_n) \in f(A)$. Choose $y_n \in A$ with $f(x_n) = f(y_n)$. Since f is continuous, $\{f(y_n)\}$ converges to $f(x)$; since f is a homeomorphism on A , $\{y_n\}$ converges to x . Since $x_n \neq y_n$, this contradicts the fact that f is a local homeomorphism at x .
- 3) Express A as the union of an ascending sequence of compact sets $A_1 \subset A_2 \subset \dots$. Let V_1 be a neighbourhood of A_1 such that \bar{V}_1 is compact and f is a homeomorphism on $\bar{V}_1 \cup A$ (by (2).) Given V_i a neighbourhood of A_i satisfying these conditions, consider the set $\bar{V}_i \cup A_{i+1}$. It is a compact subset of $\bar{V}_i \cup A$, and f is a homeomorphism on $\bar{V}_i \cup A$. Hence by (2) there is a neighbourhood V_{i+1} of $\bar{V}_i \cup A_{i+1}$ with \bar{V}_{i+1} compact, such that f is a homeomorphism on $\bar{V}_{i+1} \cup A_{i+1}$. We proceed by induction: f is 1 - 1 on $V = \cup V_{i+1}$, so that it is a homeomorphism on V (being a local homeomorphism-onto.) □

3.7. Corollary. Any differentiable submanifold of \mathbb{R}^{n+k} is a differentiable neighbourhood retract.

Proof: The projection of $E(\nu^k) \rightarrow M^n$ induces (under e) a differentiable map of a neighbourhood of M^n in \mathbb{R}^{n+k} onto M^n which is the identity on M^n . □

3.8. Definition. Let ζ be a vector space bundle with $B(\zeta)$ compact; Let $T(\zeta)$ denote the 1-point compactification of $E(\zeta)$. It is called the **Thom space** of ζ . Let ∞ denote the added point.

Let ζ have a Riemannian metric. Let $T_\varepsilon(\zeta)$ be obtained from $E(\zeta)$ by identifying all vectors of length greater than or equal to ε to a point. Let $\alpha(x)$ be a C^∞ function with $\alpha'(x) \geq 0$ which equals 1 in a neighbourhood of $x = 0$ and $\rightarrow \infty$ as $x \rightarrow 1$. The map of $E(\zeta)$ into $T(\zeta)$ which carries the vector e into the vector $e\alpha(\|e\|/\varepsilon)$ induces a homeomorphism of $T_\varepsilon(\zeta)$ onto $T(\zeta)$ which is a diffeomorphism on the set $E_\varepsilon(\zeta)$, consisting of vectors of length less than ε . The fact that B is compact is used here.

3.9. Definition. Let the compact manifold M^n be embedded in \mathbb{R}^{n+k} . ν^k is given the Riemannian metric of \mathbb{R}^{n+k} ; by 3.6 there is a neighbourhood of M^n in \mathbb{R}^{n+k} which is diffeomorphic to the subset $E_{2\varepsilon}(\nu^k)$ of $E(\nu^k)$. Such a neighbourhood is called a **tubular neighbourhood** of M^n .

By 3.8, we see that $T(\nu^k)$ is homeomorphic with the space obtained from \mathbb{R}^{n+k} by collapsing the exterior of the ε -neighbourhood of M^n to a point.

We will need three lemmas concerning approximation by differentiable functions.

3.10. Lemma. Let A be a closed subset of the differentiable manifold M^n , let $f: M^n \rightarrow \mathbb{R}^m$ be differentiable on A . Let δ be a positive continuous function on M^n . There exists $g: M^n \rightarrow \mathbb{R}^m$ such that

- 1) g is differentiable,
- 2) g is a δ -approximation to f ,

3) $g|_A = f|_A$.

Proof: It suffices to prove this lemma in the case $m = 1$.

Given $x \in A$, $f|_A$ may be extended to a differentiable function f_x in a neighbourhood N_x of x . Let N_x be chosen small enough that $|f_x(y) - f(y)| < \delta(y)$ for all $y \in N_x$.

Given $x \in M^n \setminus A$, choose a neighbourhood N_x of x small enough that $|f(y) - f(x)| < \delta(y)$ for all $y \in N_x$. Define $f_x(y) \equiv f(x)$ for $y \in N_x$.

Let $\{\varphi_\alpha\}$ be a differentiable partition of unity with $\text{support}(\varphi_\alpha)$ contained in some N_x , say $N_{x(\alpha)}$, for each α . Define $g(y) = \sum_\alpha \varphi_\alpha(y) f_{x(\alpha)}(y)$. One checks the conditions of the lemma easily. \square

More generally:

3.11. Lemma. *Let $f: M_1 \rightarrow M_2$ be a continuous map of differentiable manifolds which is differentiable on the closed subset A of M_1 . Let $\varepsilon(x) > 0$ be given; and give M_2 the metric determined by some embedding $M_2 \subset \mathbb{R}^p$. Then there exists a differentiable map $g: M_1 \rightarrow M_2$ such that*

- 1) g is differentiable,
- 2) g is an ε -approximation to f ,
- 3) $g|_A = f|_A$.

Proof: There is a neighbourhood U of M_2 in \mathbb{R}^p of which is a differentiable retract (3.7.) Let ρ be the differentiable retraction of U onto M_2 . Let $\delta(x)$ be a positive function on M_2 so chosen that the cubical neighbourhood of $f(x)$ of radius $\delta(x)$ lies in U , and so that its image under ρ has radius less than $\varepsilon(x)$. Let $f_1: M_1 \rightarrow \mathbb{R}^p$ be a differentiable map which is a δ -approximation to f , such that $f_1|_A = f|_A$ (by 3.10.) Define $g(x) = \rho(f_1(x))$. \square

3.12. Lemma. *Let $f: M_1 \rightarrow M_2$ be a continuous map of differentiable manifolds; let the metric on M_2 be obtained by embedding it in some euclidean space. Given $\varepsilon(x)$, there is a $\delta(x)$ such that if $g: M_1 \rightarrow M_2$ is a δ -approximation to f , g is homotopic to f under a homotopy $F(x, t)$ with*

- 1) $F(x, t) = f(x)$ for any x such that $g(x) = f(x)$ and
- 2) $F(x, t)$ is a ε -approximation to f for any t .

Proof: Let U, ρ , and $\delta(x)$ be chosen as in 3.11. Let $g: M_1 \rightarrow M_2$ be a δ -approximation to f . Then the line segment from $g(x)$ to $f(x)$ lies in U , so that

$$F(x, t) = \rho(tg(x) + (1 - t)f(x))$$

is well defined. Furthermore $F(x, t)$ is an ε -approximation to $f(x)$ for any t . \square

3.13. Definition. We wish to define a homomorphism $\lambda: \pi_{n+k}(T(\xi^k), \infty) \rightarrow \mathcal{N}^n$ where \mathcal{N}^n is the cobordism class of the base space for $T(\xi^k)$. To this end we need some preparation:

Let ξ^k be a differentiable vector space bundle with $B(\xi)$ compact and m -dimensional; let $E(\xi^k)$ be given a metric by embedding it as a closed differentiable submanifold in some euclidean space (it is an $(m + k)$ -manifold.)

Given an element of $\pi_{n+k}(T(\xi^k), \infty)$, let it be represented by the map

$$f: (\bar{C}_{n+k}, \partial\bar{C}_{n+k}) \rightarrow (T(\xi^k), \infty),$$

where \bar{C}_{n+k} is the closed cube $[0, 1]^{n+k}$ and $\partial\bar{C}_{n+k}$ is the boundary. Let U denote the open subset

$f^{-1}(E(\xi^k))$ of C_{n+k} . Let $g : U \rightarrow E(\xi^k)$ be a differentiable δ -approximation to $f|U$, where δ is so chosen that $\delta < 1$ and g is homotopic to f , the homotopy F also being a 1-approximation to f . (This ensures that F will be continuous if we define $F(x, t) = \infty$ for $x \in \overline{C_{n+k}} \setminus U$.)

Now g may be approximated in turn by a differentiable map $h : U \rightarrow E(\xi^k)$ which is transverse regular on the submanifold $B(\xi^k)$ of $E(\xi^k)$. We choose the approximation close enough to h , the homotopy H being a 1-approximation to g for each t . Extend h to $\overline{C_{n+k}}$ by defining $h(x) = \infty$ for $x \in \overline{C_{n+k}} \setminus U$. Then h is in the homotopy class of f .

$h^{-1}(B(\xi^k))$ is a differentiable submanifold M^n of U which is closed in $\overline{C_{n+k}}$, and thus compact.

3.14. Theorem. Define $\lambda : \pi_{n+k}(T(\xi^k), \infty) \rightarrow \mathcal{N}^n$ by assigning the cobordism class $[M^n] \in \mathcal{N}^n$ to the homotopy class $[h] \in \pi_{n+k}(T(\xi^k), \infty)$. Then λ is a well-defined homomorphism.

Proof: Let $H : (\overline{C_{n+k}} \times I, \partial\overline{C_{n+k}} \times I) \rightarrow (T(\xi^k), \infty)$ be a homotopy between $h_0 = H(x, 0)$ and $h_1 = H(x, 1)$. Let h_0, h_1 satisfy the conditions

- 1) h_i is differentiable on $h_i^{-1}(E(\xi^k))$
- 2) h_i is transverse regular on $B(\xi^k)$. ($i = 0, 1$.)

We wish to show that $h_0^{-1}(B)$ and $h_1^{-1}(B)$ belong to the same cobordism class.

We may assume that $H(x, t) = H(x, 0)$ for $t \leq 1/3$, and $H(x, t) = H(x, 1)$ for $t \geq 2/3$. Let $U = H^{-1}(E(\xi^k)) \cap [\overline{C_{n+k}} \times (0, 1)]$; then U is an open subset of \mathbb{R}^{n+k+1} . Let $G : U \rightarrow E(\xi^k)$ be a differentiable 1-approximation to H which equals H on the closed subset A , where $A = U \cap [\overline{C_{n+k}} \times (0, 1/4] \cup [3/4, 1]$. (See 3.11. H is differentiable on A .)

Now G satisfies the transverse regularity condition for $B(\xi^k)$ at points in A (since h_0 and h_1 are transverse regular on $B(\xi^k)$) so that by 1.35 there is a differentiable map $F : U \rightarrow E(\xi^k)$ which equals G on A , is transverse regular on $B(\xi^k)$, and is a 1-approximation to G . Because F is a 2-approximation to H , it remains continuous if we define $F(x, t) = \infty$ for $(x, t) \in (\overline{C_{n+k}} \times (0, 1)) \setminus U$. Because F equals H on A , it remains continuous if we define $F(x, t) = H(x, t)$ for $t = 0, 1$. Hence $F^{-1}(B)$ is a compact subset of $\overline{C_{n+k}}$, being closed and bounded.

Because $F|U$ is transverse regular on B , $(F|U)^{-1}(B)$ is a differentiable $(n+1)$ -submanifold of $\overline{C_{n+k}} \times (0, 1)$. Then

$$(F|U)^{-1}(B) \cap \overline{C_{n+k}} \times t = \begin{cases} h_0^{-1}(B) \times t & \text{for } t \in [0, 1/4], \\ h_1^{-1}(B) \times t & \text{for } t \in [3/4, 1]. \end{cases}$$

Hence $F^{-1}(B)$ is a differentiable manifold-with-boundary whose boundary is $h_0^{-1}(B) + h_1^{-1}(B)$. Thus λ is well-defined.

It is trivial to show λ is a homomorphism, because the sum in \mathcal{N}^n is derived from disjoint union of representative manifolds. □

3.15. Theorem. If ξ^k is the universal bundle γ_m^k where $k \geq n+1$, $m \geq n$ then $\lambda : \pi_{n+k}(T(\xi^k), \infty) \rightarrow \mathcal{N}^n$ is onto.

Proof: Let M^n be a compact manifold; let $k \geq n+1$. Let M^n be embedded in C_{n+k} (1.32); let ν^k be the normal bundle of this embedding. The Riemannian metric on $E(\nu^k)$ is that derived from the natural scalar product on the tangent bundle to \mathbb{R}^{n+k} , in which ν^k is contained.

By 3.6, for small ε the subset of $E_{2\varepsilon}(\nu^k)$ of $E(\nu^k)$ is diffeomorphic with a tubular neighbourhood of M^n in C_{n+k} ; let U be the image of $E_\varepsilon(\nu^k)$.

Let p_1 project \bar{C}_{n+k} onto the space obtained from \bar{C}_{n+k} by identifying $\bar{C}_{n+k} \setminus U$ to a point (denoted by $\bar{C}_{n+k} / (\bar{C}_{n+k} \setminus U)$).

Let p_2 be the diffeomorphism of U onto $E_\varepsilon(v^k)$, followed by the map of $E(v^k)$ into $T_\varepsilon(v^k)$ which identifies all vectors of length $\geq \varepsilon$ (3.8.) p_2 is then extended by mapping $\bar{C}_{n+k} \setminus U$ into ∞ .

Let p_3 be the homeomorphism of $T_\varepsilon(v^k)$ onto $T(v^k)$ constructed in 3.8. The composite map $p_3 p_2 p_1$ is a diffeomorphism of U onto $E(v^k)$.

Finally, let p_4 be the bundle map of v^k into γ_m^k induced from the embedding of M^n in $\mathbb{R}^{n+k} \subset \mathbb{R}^{m+k}$.

Because both fibres have dimension k , this map satisfies the transverse regularity condition for $G_{k,m}$ at each point of M^n . Extend p_4 in the obvious way to map $T(v^k)$ into $T(\gamma_m^k)$.

Let $g = p_4 p_3 p_2 p_1$. Then $g : \partial \bar{C} \rightarrow \infty$. Let $\mu(M^n)$ denote the homotopy class of g in $\pi_{n+k}(T(\zeta^k), \infty)$.

Now g is transverse regular on $G_{k,m}$ and $M^n = g^{-1}(G_{k,m})$. By definition, the cobordism class of M^n is the image of $\mu(M^n)$ under λ , so that $\lambda \mu(M^n) = [M^n]$. \square

3.16. Theorem. *If ζ^k is the universal bundle γ_m^k where $k \geq n + 2$, $m > n$ then λ is one-to-one.*

Proof: Given an element of $\pi_{n+k}(T(\zeta^k), \infty)$, we may suppose it represented by a map

$$f : (\bar{C}_{n+k}, \partial \bar{C}_{n+k}) \rightarrow (T(\zeta^k), \infty)$$

which is differentiable on $f^{-1}(E)$ and transverse regular on $G_{m,k}$ (by 3.13.) Let $M^n = f^{-1}(G_{m,k})$; we wish to show that if M^n is the boundary of an $(n + 1)$ -manifold-with-boundary Q , then f is homotopic to the constant map.

M^n is a submanifold of C_{n+k} ; let its normal bundle be v^k . Let ε be chosen so that $E_{2\varepsilon}(v^k)$ is diffeomorphic with the 2ε -neighbourhood of M^n ; let U_ε be the image of the vectors of $E_\varepsilon(v^k)$. Impose a Riemannian metric on γ_m^k ; let δ be so chosen that $\|x\| \geq \varepsilon$ implies $\|f(x)\| \geq \delta$ for $x \in E(v^k)$.

Step 1. f is homotopic to a map f_1 such that

- 1) f_1 is differentiable on $f_1^{-1}(E)$ and transverse regular on $G_{m,k}$.
- 2) $f = f_1$ on $M^n = f^{-1}(G_{m,k})$.
- 3) f_1 carries everything outside U_ε into ∞ .

Define $F : E(\gamma_m^k) \rightarrow T(\gamma_n^k)$ by the equation $F(e, t) = e\alpha(t\|e\| / \delta)$, where α is the function defined in 3.8. Let $f_1(x) = F(f(x), 1)$.

Step 2. By the diffeomorphism of $U_{2\varepsilon}$ with $E_{2\varepsilon}$, f_1 induces a map \bar{f}_1 of $\bar{E}_\varepsilon(v^k)$ into $T(\gamma_n^k)$ which carries $\partial(E_\varepsilon)$ into ∞ . Any homotopy of \bar{f}_1 which leaves $\partial(E_\varepsilon)$ at ∞ induces a homotopy of f_1 .

Now \bar{f}_1 is homotopic to a map \bar{f}_2 such that

- 1) \bar{f}_2 is differentiable on $\bar{f}_2^{-1}(E)$ and transverse regular on $G_{m,k}$.
- 2) $\bar{f}_2 = \bar{f}_1$ on $M^n = f^{-1}(G_{m,k})$.
- 3) \bar{f}_2 is locally a bundle map in some neighbourhood of M^n .

The homotopy leaves $\partial(E_\varepsilon)$ at ∞ .

Consider $G : \bar{E}(\gamma_m^k) \times I \rightarrow T(\gamma_n^k)$ defined by the equation $G(e, t) = \bar{f}_1(te) / t$. As $t \rightarrow 0$, $G(e, t)$ approaches a limit which is non-zero if $e \neq 0$ (since \bar{f}_1 is differentiable and transverse regular.) It is easily seen to be a bundle map. It will not suffice for our purpose, since it does not carry $\partial(E_\varepsilon) \times I$ into ∞ . Choose $\delta > 0$ so that $\|x\| \geq \varepsilon$ implies $\|G(x, t)\| \geq \delta$ for $x \in E(v^k)$, $t \in I$, and define

$$H(e, t) = [G(e, t)]\alpha(-\|G(e, t)\| / \delta).$$

If we set $\bar{f}_2 = H(e, 0)$, then \bar{f}_2 is a bundle map for $\|e\|$ small (since $\alpha(x) = 1$ for x small.) The map $H(e, 1) = f_1(e)\alpha(\|f_1(e)\| / \delta)$ does not equal f_1 , but it is homotopic to f_1 , the homotopy leaving $\partial(E_e)$ at ∞ . The homotopy is defined by the equation

$$K(e, t) = \bar{f}_1(e)\alpha(t\|f_1(e)\| / \delta), \text{ as in Step 1.}$$

Step 3. Let Q be the $n + 1$ manifold-with-boundary such that $M^n = \partial Q$. Let h be a diffeomorphism of $M^n \times [0, 1]$ into Q which carries $M^n \times 0$ onto ∂Q .

Define $h_1 : Q \rightarrow C_{n+k} \times I$ as follows:

$$h_1(x) = h(y, t) \text{ if } x = h(y, t) \text{ where } (y, t) \in (M^n, [0, 1/2]).$$

$$h_1(x) = p, \text{ where } p \text{ is some fixed point interior to } C_{n+k} \times I \text{ if } x \notin \text{image } h.$$

$$h_1(x) = (1 - \beta(t))h(y, 1/2) + \beta(t)p, \text{ where } \beta \text{ is a } C^\infty \text{ function with } \beta'(t) \geq 0, \beta(t) = 0 \text{ in a neighbourhood of } t = 1/2 \text{ and } \beta(t) = 1 \text{ in a neighbourhood of } t = 1 \text{ if } x = h(y, t) \text{ where } (y, t) \in (M^n, [1/2, 1]).$$

h_1 is a differentiable map of $\text{Int } Q$ into $\text{Int}(C_{n+k} \times I)$; and h_1 is a 1 - 1 immersion in a neighbourhood of ∂Q . Since $\dim(C_{n+k} \times I) > 2(n + 1)$, h_1 may be approximated by a 1 - 1 immersion h_2 which equals h_1 in a neighbourhood of ∂Q (by 1.29.) It may be extended to an embedding of Q into $C_{n+k} \times I$. (Since Q is compact, a 1 - 1 immersion is automatically an embedding.) Let Q now be considered as this subset of $C_{n+k} \times I$.

Step 4. We have a map f_2 of $\bar{C}_{n+k} \times 0$ into $T(\gamma_n^k)$ which is a bundle map when restricted to a small tubular neighbourhood of $M^n \times 0$ in $C_{n+k} \times 0$. We extend it to $\bar{C}_{n+k} \times [0, b]$ for b small in a trivial way. Suppose there exists a map g of the ε' -neighbourhood N of Q in $C_{n+k} \times I$ into $T(\gamma_n^k)$ which equals f_2 in some neighbourhood of ∂Q in $C_{n+k} \times I$ and maps each point of $N \setminus Q$ into a non-zero vector in $E(\gamma_n^k)$. Our theorem then follows: Let δ be so chosen that, if the distance(x, Q) $\geq \varepsilon'/2$, then $\|g(x)\| \geq \delta$.

Define $g_1 : C_{n+k} \times I \rightarrow T(\gamma_n^k)$ by the equation

$$g_1(x, s) = \begin{cases} g(x, \varepsilon)\alpha(\|g(x, s)\| / \delta) & \text{for } (x, s) \in N, \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

The restriction of g_1 to $C_{n+k} \times 0$ does not equal the map f_2 , but it is homotopic to f_2 , by the same technique as used at the end of Step 2. g_1 is the homotopy required for our theorem.

To show that the extension g exists, we refer to Steenrod, "Fibre Bundles" (Princeton University Press, 1951.) According to §19.4 and §19.7 of this book, the principal bundle associated with γ_n^k is an m -universal bundle. That is: given a vector space bundle ζ^k over a complex of dimension $\leq m$, any bundle map of ζ^k , restricted to a subcomplex, into γ_n^k can be extended throughout ζ^k . We will assume the well known result that Q can be triangulated. The dimension $n + 1$ of Q is $\leq m$. Hence any bundle map of the normal bundle ν^k of Q , restricted to a polyhedral neighbourhood of ∂Q , into γ_n^k can be extended throughout ν^k .

Applying this result to the map f_2 , this completes the proof of 3.16. □

Letting T_k stand for the union of the Thom spaces $T(\gamma_n^k) \subset T(\gamma_{n+1}^k) \subset \dots$, in the weak topology, Theorem 3.15 and 3.16 imply the following.

3.17. Theorem. *The cobordism group N^n is canonically isomorphic to the stable homotopy group $\pi_{n+k}(T_k)$, for $k \geq n + 2$.* □

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Appendix⁵

In this appendix we give a proof for the smooth collaring theorem. Our exposition follows Dirk Schütz. (See “Lecture06_handout.pdf” in the “material” for MAGIC002, in “courses” listed in the page “<http://maths.dept.shef.ac.uk/magic/courses.php>”.)

First we show that partitions of unity allow us to glue together smooth functions which are only defined on subsets of a differentiable manifold M .

Proposition A: *Let $\{U_\alpha\}$ be an open cover of the differentiable manifold M and $\{\varphi_\alpha\}$ a partition of unity with $\text{support}(\varphi_\alpha) \subset U_\alpha$. For every α , assume that $f_\alpha : U_\alpha \rightarrow \mathbb{R}^k$ is a smooth function. Then $f : M \rightarrow \mathbb{R}^k$ defined by*

$$f(x) = \sum_\alpha f_\alpha(x)\varphi_\alpha(x)$$

is a well defined smooth function.

Proof: Observe that $f_\alpha \cdot \varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^k$ has support contained in $\text{support}(\varphi_\alpha)$, so can be extended to a smooth function on M . Also, by the local finiteness, the formula for f is locally just a finite sum, so smoothness follows. \square

The same procedure can be used to extend vector fields defined on each V_i to a vector field on M .

Proposition B (Smooth Collaring Theorem): *Let M be a compact differentiable manifold with boundary. Then there exists an embedding $i : \partial M \times [0, 1) \rightarrow M$ with $i(x, 0) = x$ for all $x \in \partial M$.*

Proof: Let U_1, \dots, U_k be a finite covering of M by coordinate charts, and let $\{\varphi_i : U_i \rightarrow [0, 1]\}$ be a partition of unity subordinate to this cover.

Case I: U_i is diffeomorphic to an open set of \mathbb{R}^n . Define a vector field v_i on U_i to be identically zero.

Case II: U_i contains boundary points. Let $\varphi_i : U_i \rightarrow U_i'$ be a chart, and define a vector field v_i on U_i such that the induced vector field on $U_i' \subset \mathbb{H}^n$ is constant $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$.

We get a vector field on M by using the partition of unity. Let Φ be the corresponding flow. As M is compact, and since the vector field is chosen on the boundary so that it is not possible to flow “out” of the manifold, we get a smooth flow

$$\Phi : M \times [0, \infty) \rightarrow M$$

It is easy to check that $\Phi | \partial M \times [0, 1)$ is the desired embedding. \square

⁵ Added by the transcriber.