Differential Topology

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Differential topology may be defined as the study of those properties of differentiable manifolds which are invariant under diffeomorphism (differentiable homeomorphism). Typical problem falling under this heading are the following:

1) Given two differentiable manifolds, under what conditions are they diffeomorphic?
2) Given a differentiable manifold, is it the boundary of some differentiable manifold-with-boundary?
3) Given a differentiable manifold, is it parallelisable?

All these problems concern more than the topology of the manifold, yet they do not belong to differential geometry, which usually assumes additional structure (e.g., a connection or a metric).

The most powerful tools in this subject have been derived from the methods of algebraic topology. In particular, the theory of characteristic classes is crucial, where-by one passes from the manifold M to its tangent bundle, and thence to a cohomology class in M which depends on this bundle.

These notes are intended as an introduction to the subject; we will go as far as possible without bringing in algebraic topology. Our two main goals are

a) Whitney's theorem that a differentiable n-manifold can be embedded as a closed subset of the euclidean space $\mathbb{R}^{2n+1}$ (see §1.32); and
b) Thom's theorem that the non-orientable cobordism group $\mathcal{N}^n$ is isomorphic to a certain stable homotopy groups (see §3.15).

Chapter I is mainly concerned with approximation theorems. First the basic definitions are given and the inverse function theorem is exploited. (§1.1 – 1.12). Next two local approximation theorems are proved, showing that a given map can be approximated by one of maximal rank. (§1.13 – 1.21). Finally locally finite coverings are used to derive the corresponding global theorems: namely Whitney's embedding theorem and Thom's transversality lemma (§1.35).

Chapter II is an introduction to the theory of vector space bundles, with emphasis on the tangent bundle of a manifold.

Chapter III makes use of the preceding material in order to study the cobordism group $\mathcal{N}^n$.

Chapter I *Embeddings and Immersions of Manifolds*

Notation. If x is in the euclidean space $\mathbb{R}^n$, the coordinate of x are denoted by $(x^1, \ldots, x^n)$. Let $\|x\| = \max |x^i|$; let $C^n(r)$ denote the set of x such that $\|x\| < r$; and $C^n(x_0, r)$ the set of x such that $\|x - x_0\| < r$. The closure of a cube C is denoted by $\overline{C}$.

A real valued function $f(x^1, \ldots, x^n)$ is ***differentiable*** if the partials of f of all orders exist and are continuous (i.e., "differentiable" means C^∞). A map $f = (f^1, \ldots, f^p) : U \to \mathbb{R}^p$ (where U is an open set, in $\mathbb{R}^n$) is differentiable if each of the coordinate functions $f^1, \ldots, f^p$ is differentiable. Df denotes the Jacobian matrix of f; one verifies that $D(gf) = Dg{\cdot}Df$. The notation $\partial(f^1, \ldots, f^p)/\partial(x^1, \ldots, x^n)$ is also used. If $n = p$, $|Df|$ denotes the determinant.

1.1 ***Definition.*** A ***topological n-manifold*** M^n is a Hausdorff space with a countable basis which is locally homeomorphic to $\mathbb{R}^n$.

A ***differentiable structure*** $\mathcal{D}$ on a topological manifold M^n is a collection of real-valued functions, each defined on an open subset of M^n such that:

1) For every point p of M^n there is a neighbourhood U of p and a homeomorphism h of U onto an open subset of $\mathbb{R}^n$ such that a function f, defined on the open subset W of U, is in $\mathcal{D}$ if and only if fh^{-1} is differentiable.
2) If U_i are open sets contained in the domain of f and $U = \cup U_i$, then $f \mid U \in \mathcal{D}$ if and only if $f \mid U_i$ is in $\mathcal{D}$, for each i.

A ***differentiable manifold*** M^n is a topological manifold provided with a differentiable structure $\mathcal{D}$; the elements of $\mathcal{D}$ are called the ***differentiable functions*** on M^n. Any open set U and homeomorphism h which satisfy the requirement of 1) above are called a ***coordinate system*** on M^n.

Notation. A coordinate system is sometimes denoted by the coordinate functions: $h(p) = (u^1(p), \ldots, u^n(p))$.

1.2 ***Alternate definition.*** Let a collection (U_i, h_i) be given, where h_i is a homeomorphism of the open subset U_i of M^n onto an open subset of $\mathbb{R}^n$, such that

a) the U_i's cover M^n;
b) $h_j h_i^{-1}$ is a differentiable map on $h_i(U_i \cap U_j)$, for all i, j.

Define a ***coordinate system*** as an open set U and homeomorphism h of U onto an open subset of $\mathbb{R}^n$ such that $h_i h^{-1}$ and hh_i^{-1} are differentiable on $h(U \cap U_i)$ and $h_i(U \cap U_i)$ respectively, for each i.

Define a ***differentiable structure*** on M^n as the collection of all such coordinate systems. A function f, defined on the open set V, is ***differentiable*** if fh^{-1} is differentiable on $h(U \cap V)$, for all coordinate systems (U, h).

One shows readily that these two definitions are entirely equivalent.

1.3 ***Definition.*** Let M_1, M_2 be differentiable manifolds. If U is an open subset of M_1, $f : U \to M_2$ is ***differentiable*** if for every differentiable function g on M_2, gf is differentiable on M_1.

If $A \subset M_1$, a function $f : A \to M_2$ is ***differentiable*** if it can be extended to a differentiable function defined on a neighbourhood U of A.

$f : M_1 \to M_2$ is a ***diffeomorphism*** if f and f^{-1} are defined and differentiable.
(A coordinate system (U, h) on M^n is then an open set U in M^n and a diffeomorphism h of U onto an open set in $\mathbb{R}^n$.)

If $A \subset M$, we have just defined the notion of differentiable function for subsets of A. Suppose that A is locally diffeomorphic to $\mathbb{R}^k$: this collection is easily shown to be a differentiable structure on A. In this case, A is said to be a ***differentiable submanifold*** of M.

The following lemma is familiar from elementary calculus.

1.4. ***Lemma.*** *Let* $f : C^n(r) \to \mathbb{R}^n$ *satisfy the condition* $|\partial f^i/\partial x^j| \le b$ *for all* i, j. *Then* $||f(x) - f(y)|| \le bn||(x - y)||$, *for all* $x, y \in \overline{C}^n(r)$.

1.5. ***Theorem*** (**Inverse Function Theorem**). *Let U be an open subset of* $\mathbb{R}^n$, *let* $f : U \to \mathbb{R}^n$ *be differentiable*, *and let Df be non-singular at* x_0. *Then f is a diffeomorphism of some neighbourhood of* x_0 *onto some neighbourhood of* $f(x_0)$.

Proof: We may assume $x_0 = f(x_0) = 0$, and that $Df(x_0)$ is the identity matrix.
Let $g(x) = f(x) - x$, so that $Dg(0)$ is the zero matrix. Choose $r > 0$ so that $x \in U$ and $Df(x)$ is non-singular and $|\partial g^i/\partial x^j)| \le 1/2n$, for all x with $||x|| < r$.

Assertion. If $y \in C^n(r/2)$, *there is exactly one* $x \in C^n(r)$ *such that* $f(x) = y$:

By the previous lemma,

$$||g(x) - g(x_0)|| \le (½)||x - x_0|| \text{ on } C^n(r). \qquad (*)$$

Let us define $\{x_n\}$ inductively by $x_0 = 0$, $x_1 = y$, $x_{n+1} = y - g(x_n)$. This is well-defined, since $x_n - x_{n-1} = g(x_{n-2}) - g(x_{n-1})$ so that

$$||x_n - x_{n-1}|| \le (½)||x_{n-2} - x_{n-1}|| \le ||y||/2^{n-1};$$

and thus $||x_n|| \le 2||y||$ for each n. Hence the sequence $\{x_n\}$ converges to a point x with $||x|| \le 2||y||$, so that $x \in C^n(r)$. Then $x = y - g(x)$, so that $f(x) = y$. This proves the existence of x. To show uniqueness, note that if $f(x) = f(x_1) = y$, then $g(x_1) - g(x) = x - x_1$, contradicting (*).
Hence $f^{-1} : C^n(r/2) \to C^n(r)$ exists. Note that

$$||f(x) - f(x_1)|| \ge ||x - x_1|| - ||g(x) - g(x_1)|| \ge (½)||x - x_1||$$

so that $||y - y_1|| \ge (½)||f^{-1}(y) - f^{-1}(y_1)||$. Hence f^{-1} is continuous; the image $C^n(r/2)$ of under f^{-1} is open because it equals $C^n(r) \cap f^{-1}(C^n(r/2))$, the intersection of two open sets.
To show that f^{-1} is differentiable, note that

$$f(x) = f(x_1) + Df(x_1)\cdot(x - x_1) + h(x, x_1),$$

where $(x - x_1)$ is written as a column matrix and the dot stands for matrix multiplication. Here $h(x, x_1) / ||x - x_1|| \to 0$ as $x \to x_1$. Let A be the inverse matrix of $Df(x_1)$. Then

$$A\cdot(f(x) - f(x_1)) = (x - x_1) + A\cdot h(x, x_1), \quad \text{or}$$
$$A\cdot(y - y_1) + A\cdot h_1(y, y_1) = f^{-1}(y) - f^{-1}(y_1),$$

where $h(y, y_1) = -h(f^{-1}(y), f^{-1}(y_1))$. Now

$$h_1(y, y_1) / ||y - y_1|| = -[h(x, x_1) / ||x - x_1||](||x - x_1|| / ||y - y_1||).$$

Since $||x - x_1|| / ||y - y_1|| \le 2$, $h_1(y, y_1) / ||y - y_1|| \to 0$ as $y \to y_1$. Hence

$$D(f^{-1}) = A = (Df)^{-1}.$$

This means that $(Df)^{-1}$ is obtained as the composition of the following maps:

$$C^n(r/2) \xrightarrow{f^{-1}} C^n(r) \xrightarrow{Df} \mathrm{GL}(n) \xrightarrow{\text{matrix inversion}} \mathrm{GL}(n);$$

where GL(n) denotes the set of non-singular $n \times n$ matrices, considered as a subspace of n^2-dimensional euclidean space. Since f^{-1} is continuous and Df and matrix inversion are C^∞, $(Df)^{-1}$ is continuous, i.e., is f^{-1} is C^1. In general, if f^{-1} is C^k, then by this argument $(Df)^{-1}$ is also, i.e., f^{-1} is of class C^{k+1}. This completes the proof. □

1.6. *Lemma.* *Let U be an open subset of $\mathbb{R}^n$, let $f: U \to \mathbb{R}^p$ $(n \le p)$, $f(0) = 0$, and let $Df(0)$ have rank n. Then there exists a diffeomorphism g of one neighbourhood of the origin in $\mathbb{R}^p$ onto another so that $g(0) = 0$ and $gf(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$, in some neighbourhood of the origin.*

***Proof*:** Since $\partial(f^1, \ldots, f^p)/\partial(x^1, \ldots, x^n)$ has rank n, we may assume that

$$\partial(f^1, \ldots, f^n)/\partial(x^1, \ldots, x^n)$$

is the submatrix which is non-singular. Define $F : U \times \mathbb{R}^{p-n} \to \mathbb{R}^p$ by the equation

$$F(x^1, \ldots, x^p) = f(x^1, \ldots, x^n) + (0, \ldots, 0, x^{n+1}, \ldots, x^p).$$

F is an extension of f, since $F(x^1, \ldots, x^n, 0, \ldots, 0) = f(x^1, \ldots, x^n)$.
DF is non-singular at the origin, since its determinant everywhere equals

$$|\partial(f^1, \ldots, f^n)/\partial(x^1, \ldots, x^n)|,$$

which is non-zero. Hence F has a local inverse g, so that g maps one neighbourhood of the origin in $\mathbb{R}^p$ onto another, and

$$gF(x^1, \ldots, x^p) = (x^1, \ldots, x^p)$$

and hence

$$gf(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0).$$ □

1.7. *Corollary.* *Let A^k be a differentiable sub-manifold of M^n. Given $x \in A^k$, there is a coordinate system (U, h) on M^n about x, such that $h(U \cap A) = h(U) \cap \mathbb{R}^k$ (where $\mathbb{R}^k$ is considered as the subspace $\mathbb{R}^k \times 0$ of $\mathbb{R}^k \times \mathbb{R}^k = \mathbb{R}^n$).*

***Proof*:** Let (U_i, h_i) be a coordinate system on M^n about x; by hypothesis, there is a differentiable map f of a neighbourhood V of x in M^n into $\mathbb{R}^k$ such that $f \mid V \cap A = f_1$ is a diffeomorphism whose range is an open set W in $\mathbb{R}^k$. We may assume $U_1 = V$, and $h_1(x) = f(x) = 0$.
Now $fh_1^{-1}h_1f^{-1}$ is the identity on W, so that its Jacobian, which equals $D(fh_1^{-1})_, D(h_1f^{-1})$ is non-singular. Hence $D(h_1f^{-1})$ has rank k, so that by the previous lemma, there is a diffeomorphism g of some neighbourhood $V_1 \subset h_1(U_1)$ of 0 onto another such that $g(0) = 0$ and $gh_1f_1^{-1}(x^1, \ldots, x^k) = (x^1, \ldots, x^k, 0, \ldots, 0)$. Then $U = h_1^{-1}(V_1)$ and $h = gh_1$ will satisfy the requirement of the lemma. □

1**.8. *Lemma.*** *Let U be an open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}^p$, $f(0) = 0$, $(n \ge p)$, and let $Df(0)$ have rank p. Then there is a diffeomorphism h of some neighbourhood of the origin in $\mathbb{R}^n$ onto another such that $h(0) = 0$ and $fh(x^1, \ldots, x^n) = (x^1, \ldots, x^p)$.*

***Proof*:** We nay assume $\partial(f^1, \ldots, f^p)/\partial(x^1, \ldots, x^p)$ is non-singular at 0, since $Df(0)$ has rank p. Define $F : U \to \mathbb{R}^n$ by the equation

$$F(x^1, \ldots, x^n) = (f^1(x), \ldots, f^p(x), x^{p+1}, \ldots, x^p).$$

Then $DF(0)$ is non-singular; let h be the local inverse of F. Let g project $\mathbb{R}^n$ onto the subspace $\mathbb{R}^p$; $f = gF$. Then

$$fh(x^1, \ldots, x^n) = gFh(x^1, \ldots, x^n) = g(x^1, \ldots, x^n) = (x^1, \ldots, x^p).$$ □

1.9. ***Exercise.*** Let U be an open subset of $\mathbb{R}^n$, $f: U \to \mathbb{R}^p$, $f(0) = 0$; and let $Df(x)$ have rank k for all x in U. Then there are local diffeomorphisms h and g of $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively such that

$$gfh(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0).$$

1.10. ***Definition.*** If $f: M_1 \to M_2$, the ***rank*** of f, written rank(f), at x is the rank of $D(h_2 f h_1^{-1})$ at $h_1(x)$, where (U_1, h_1) and (U_2, h_2) are coordinate systems about x and $f(x)$, respectively. The differentiable map $f: M_1^n \to M_2^p$ is an ***immersion*** if rank(f) = n everywhere ($n \le p$). It is an ***embedding*** if it is also a homeomorphism into.

If $f: M_1^n \to M_2^p$, then $y \in M_2^p$ is a ***regular value*** of f if rank(f) = p on the entire set $f^{-1}(y)$. Otherwise, y is a ***critical value***. (If $y \notin f(M_1^n)$, y is, by definition, a regular value of f.)

1.11. ***Exercise.*** If A is a differentiable submanifold of M, the inclusion $A \to M$ is an embedding and conversely if $f: M_1 \to M$ is an embedding then $f(M_1)$ is a differentiable submanifold .

1.12. ***Exercise.*** If y is a regular value of $f: M_1^n \to M_2^p$, then $f^{-1}(y)$ is a differentiable submanifold of M_1^n of dimension $n - p$ (or empty).

1.13. ***Definition.*** A subset A of $\mathbb{R}^n$ has ***measure zero*** if it may be covered by a countable collection of cubes $C^n(x, r)$ having arbitrarily small total volume. In such a case, $\mathbb{R}^n \setminus A$ is everywhere dense (i.e., it intersects every non-empty open set).

1.14. ***Lemma.*** *Let U be an open subset of* $\mathbb{R}^n$; let $f: U \to \mathbb{R}^n$ *be differentiable. If $A \subset U$ has measure zero, so does $f(A)$.*

Proof: Let C be any cube with $\overline{C} \subset U$. Let b denote the maximum of $|\partial f^i/\partial x^j|$ on $\overline{C}$ for all i, j. By 1.4, $\|f(x) - f(y)\| \le bn\|x - y\|$ for $x, y \in \overline{C}$.
Now $A \cap C$ has measure zero; let us cover $A \cap C$ by cubes $C(x_i, r_i)$ with closure contained in C, such that $\sum_{i=1,\infty} r_i^n < \varepsilon$. Then $f(C(x_i, r_i)) \subset C(f(x_i), bnr_i)$, so that $f(A \cap C)$ is covered by cubes of total volume $b^n n^n \sum_{i=1,\infty} r_i^n < b^n n^n \varepsilon$. Hence $f(A \cap C)$ has measure zero.
Since A can be covered by countably many such cubes C, $f(A)$ has measure zero. □

1.15. ***Corollary.*** If $f: U \to \mathbb{R}^n$ *be differentiable, where U is an open subset of* $\mathbb{R}^n$ *and $n < p$, then $f(U)$ has measure zero.*

Proof: Project $U \times \mathbb{R}^{p-n}$ onto U and apply f. Since $U \times 0$ has measure zero in $\mathbb{R}^p$, so does $f(U)$. □

1.16. ***Definition.*** If $A \subset M$, M has ***measure zero*** if $h(A \cap U)$ has measure zero for every coordinate system (U, h).

1.17. ***Corollary.*** If $f: M_1^n \to M_2^p$ *is differentiable and $n < p$, then $f(M_1^n)$ has measure zero.*

1.18. ***Definition.*** Let $\mathcal{M}(p, n)$ denote the space of $p \times n$ matrices, with the differentiable structure of the euclidean space $\mathbb{R}^{pn}$. Let $\mathcal{M}(p, n; k)$ denote the subspace consisting of matrices of rank k. Thus $\mathcal{M}(p, n; n)$ is an open subset of $\mathcal{M}(p, n)$ if $p \ge n$; the determinantal criterion for rank proves this. More generally, we have:

1.19. *Lemma.* *$\mathcal{M}(p, n; k)$ is a differentiable submanifold of $\mathcal{M}(p, n)$ of dimension $k(p + n - k)$, where $k \leq \min(p, n)$.*

Proof: Let $E_0 \in \mathcal{M}(p, n; k)$; we may assume that E_0 is of the form, $\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}$, where A_0 is a non-singular $k \times k$ matrix. There is an $\varepsilon > 0$ such that if all the entries of $A - A_0$ are less than ε, A must also be non-singular. Let U consist of all matrices in $M(p, n)$ of the form $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, with all the entries of $A - A_0$ are less than ε.

Then E is in $\mathcal{M}(p, n; k)$ if and only if $D = CA^{-1}B$: for the matrix

$$\begin{bmatrix} I_k & 0 \\ X & I_{p-k} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ XA + C & XB + D \end{bmatrix}$$

has the same rank as E. If $X = -CA^{-1}$, this matrix is

$$\begin{bmatrix} A & B \\ 0 & CA^{-1}B + D \end{bmatrix}.$$

If $D = CA^{-1}B$, this matrix has rank k. The converse also holds, for if any element of $-CA^{-1}B + D$ is different from zero, this matrix has rank $> k$.

Let W be the open set in euclidean space of dimension

$$(pn - (p - k)(n - k)) = k(p + n - k)$$

consisting of matrices $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ with all the entries of $A - A_0$ are less than ε. The map

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \rightarrow \begin{bmatrix} A & B \\ 0 & CA^{-1}B + D \end{bmatrix}$$

is then a diffeomorphism of W onto the neighbourhood $U \cap \mathcal{M}(p, n; k)$ of E_0. □

1.20. *Theorem.* *Let U be an open set in $\mathbb{R}^n$, and let $f : U \rightarrow \mathbb{R}^p$ be differentiable, where $p \geq 2n$. Given $\varepsilon > 0$, there is a $p \times n$ matrix $A = (a^i_j)$ with each $|a^i_j| < \varepsilon$, such that $g(x) = f(x) + A{\cdot}x$ is an immersion (x written as a column matrix.)*

***Proof*:** $Dg(x) = Df(x) + A$; we would like to choose A in such a way that $Dg(x)$ has rank n for all x. I.e., A should be of the form $Q - Df$, where Q has rank n.

We define $F_k : \mathcal{M}(p, n; k) \times U \rightarrow \mathcal{M}(p, n)$ by the equation

$$F_k(Q, x) = Q - Df(x).$$

Now F_k is a differentiable map, and the domain of F_k has dimension $k(p + n - k) + n$. As long as $k < n$, this expression is monotonic in k (its partial derivative with respect to k is $p + n - 2k$). Hence the domain of F_k has dimension not greater than

$$(n - 1)(p + n - (n - 1)) + n = (2n - p) + pn - 1$$

for $k < n$. Since $p \geq 2n$, this dimension is strictly less than $pn = \dim(\mathcal{M}(p, n))$.

Hence the image of F_k has measure zero in $\mathcal{M}(p, n)$, so that there is an element A of $\mathcal{M}(p, n)$, arbitrarily close to the zero matrix, which is not in the image of F_k for $k = 0, \ldots, n - 1$. Then $A + Df(x) = Dg(x)$ has rank n, for each x. □

1.21. *Theorem.* *Let U be an open subset of $\mathbb{R}^n$; and let $f : U \to \mathbb{R}^p$ be differentiable. Given $\varepsilon > 0$, there are matrices $A(p \times n)$ and $B(p \times 1)$ with entries less than ε in absolute value such that*

$$g(x) = f(x) + A{\cdot}x + B$$

has the origin as a regular value.

Remark. The following much more delicate result has been proved by [Sard, A.]: *The set of critical values of any differentiable map has measure zero.*

***Proof of* 1.21.** Note that the theorem is trivial if $p > n$, since then $f(U)$ has measure zero, and we may choose $A = 0$ and B small in such a way that 0 is not in the image of g.
Assume $p \leq n$. We wish $Dg(x_0) = Df(x_0) + A$ to have rank p, where x_0 ranges over all points such that

$$g(x_0) = 0 = f(x_0) + A{\cdot}x_0 + B.$$

Hence A is of the form $Q - Df(x)$, and B is of the form $-f(x) - A{\cdot}x$, where Q is to have rank p.
We define $F_k : \mathcal{M}(p, n; k) \times U \to \mathcal{M}(p, n) \times \mathbb{R}^p$ by the equation

$$F_k(Q, x) = (Q - Df(x), -f(x) - (Q - Df(x)){\cdot}x).$$

Then F_k is differentiable. If $k < p$, the dimension of its domain is not greater than $(p - 1)((p + n - (n - 1)) + n = p + pn - 1$. Hence the image of F_k, $k = 0, \ldots, p - 1$ has measure zero; so that there is a point (A, B) arbitrarily close to to the origin which is not in any such image set. This completes the proof. □

1.22. *Definition.* A covering of a topological space X is ***locally-finite*** if every point has a neighbourhood which intersects only finitely many elements of the covering. A ***refinement*** of a covering of X is a second covering each element of which is contained in an element of the first covering. A Hausdorff space is ***paracompact*** if every open covering has a locally-finite open refinement.

If X is paracompact, and $\{U_\alpha\}$ is an open covering, there is a locally-finite open covering $\{V_\alpha\}$ with $V_\alpha \subset U_\alpha$ for each α. For let $\{W_\beta\}$be a locally-finite refinement of $\{U_\alpha\}$; choose $\alpha(\beta)$ so that $W_\beta \subset U_{\alpha(\beta)}$ for each β. Set $V_{\alpha 0} = U_{\alpha(\beta) = \alpha 0} W_\beta$. Given a neighbourhood intersecting only finitely many W_β, it intersects only finitely many V_α as well.

1.23. *Theorem.* *A locally compact Hausdorff space having a countable basis is paracompact.*

***Proof*:** Let X be paracompact and let $U_1, U_2, \ldots$ be a basis for X with $\overline{U}_i$ compact with each i. There exists a sequence $A_1, A_2, \ldots$ of compact sets whose union is X, such that $A_i \subset \mathrm{Int}A_{i+1}$: set $A_1 = \overline{U}_1$. Given A_i compact, let k be the smallest integer such that A_i is contained in $U_1 \cup \ldots \cup U_k$; Let A_{i+1} equal the closure of this set union $\overline{U}_{i+1}$.
Let $\mathcal{O}$ be an open covering of X. Cover the compact set $A_{i+1} \setminus \mathrm{Int}A_i$ by a finite number of open sets $V_1, \ldots V_n$ where each V_i is contained in some element of $\mathcal{O}$, and in the open set $\mathrm{Int}A_{i+2} \setminus A_{i-1}$. Let P_i denote the collection $\{V_1, \ldots V_n\}$, and let $P = P_0 \cup P_1 \cup \ldots$. P refines $\mathcal{O}$, and since any compact closed neighbourhood C is contained in some A_i, C can intersect only finitely many elements of P. □

1.24. *Exercise.* Prove that a paracompact space is normal. (First prove that it is regular.)

1.25. *Theorem.* *Let M^n be a differentiable manifold, $\{U_\alpha\}$ an open covering of M^n. There is a collection (V_j, h_j) of coordinate systems on M^n such that*

1) $\{V_j\}$ is a locally-finite refinement of $\{U_\alpha\}$.
2) $h_j(V_j) = C^n(3)$.
3) If $W_j = h_j^{-1}((C^n(1))$, then $\{W_j\}$ covers M^n.

***Proof*:** The proof proceeds along lines similar to the previous one. The only difference is that one chooses the V_j to satisfy 2), and makes sure that the sets $h_j^{-1}((C^n(1))$ also cover $A_{i+1} \setminus \text{Int}A_i$. □

1.26. We wish to construct a C^∞ function $\varphi(x^1, \ldots, x^n)$ such that $\varphi = 1$ on $\overline{C}^n(1)$, $0 < \varphi < 1$ on $C^n(2) \setminus \overline{C}^n(1)$, $\varphi = 0$ on $\mathbb{R}^n \setminus C^n(2)$.
This function may be defined by the equation $\varphi(x^1, \ldots, x^n) = \prod_{i=1,n}\psi(x^i)$, where

$$\psi(x) = \lambda(2 + x)\cdot\lambda(2 - x) / [\lambda(2 + x)\cdot\lambda(2 - x) + \lambda(x - 1) + \lambda(-x - 1)]$$

and

$$\lambda(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Note that the denominator in the expression for ψ is always positive, and that

$$\begin{aligned} \psi(x) &= 1 \quad \text{for} \quad |x| \leq 1 \\ 0 < \psi(x) &< 1 \quad \text{if} \quad 1 < |x| < 2 \\ \psi(x) &= 0 \quad \text{if} \quad |x| \geq 2. \end{aligned}$$

1.27. *Definition.* Let $f, g : X \to Y$, where Y is metrisable, and let $\delta(x)$ be a positive continuous function defined on X. Then g is a ***δ-approximation to*** f if $d(f(x), g(x)) < \delta(x)$ for all x. [If one takes the δ-approximation to f to be a neighbourhood of f in the function space $F(X, Y)$, this imposes a topology on the function space, independent of the metric on Y provided X, Y are paracompact.]

1.28. *Theorem.* *Given a differentiable map $f : M^n \to \mathbb{R}^p$ where $p \geq 2n$, and a continuous positive function δ on M^n, there exists an immersion $g : M^n \to \mathbb{R}^p$ which is a δ-approximation to f. If rank $f = n$ on the closed set N, we may choose $g \mid N = f \mid N$.*

***Proof*:** Note that rank $f = n$ on a neighbourhood U of N. Cover M^n by U and $M^n \setminus N$. Let (V_j, h_j) be a refinement of this covering, constructed as in 1.25. As before, $h_i(\overline{W}_i) = C^n(1)$ and $h_i(V_i) = C^n(3)$. Let $h_j(U_j) = C^n(2)$. Let the V_i be so indexed with positive and negative integers that those V_i with non-positive indices are the ones contained in U. Let ε_1 = min of $\delta(x)$ on the compact set $\overline{U}_i$.
Set $f_0 = f$. Given $f_{k-1} : M^n \to \mathbb{R}^p$, having rank n on $N_{k-1} = \cup_{j<k} W_i$, consider $f_{k-1}h_k^{-1} : C^n(3) \to \mathbb{R}^p$.
Let A be a $p \times n$ matrix; let $F_A : C^n(3) \to \mathbb{R}^p$ be defined by the equation

$$F_A(x) = f_{k-1}h_k^{-1}(x) + \varphi(x)A\cdot(x),$$

where (x) is written (as usual) as a column matrix $(n \times 1)$; A is yet to be chosen; and $\varphi(x)$ is the function defined in 1.26.
First, we want $F_A(x)$ to have rank n on the set $K = h_k(N_{k-1} \cap \overline{U}_k)$; we are given that $f_{k-1}h_k^{-1}$ has rank n on K. Thus

$$D(F_A(x)) = D(f_{k-1}h_k^{-1}(x)) + A\cdot(x)\cdot D\varphi(x) + \varphi(x)A.$$

($D\varphi$ is a $1 \times n$ matrix.) The map of $K \times \mathcal{M}(p, n)$ into $\mathcal{M}(p, n)$ which carries (x, A) into $D(F_A(x))$ is continuous. It carries $K \times (0)$ into the open subset $\mathcal{M}(p, n; n)$ of $\mathcal{M}(p, n)$. Hence if A is sufficiently small, this map will carry $K \times A$ into $\mathcal{M}(p, n; n)$; our first requirement is that A be this small.
Secondly, we require A to be small enough that $\|A\cdot(x)\| < \varepsilon_k/2^k$ for all $x \in C^n(3)$.
Finally, by 1.20, A may be chosen arbitrarily small so that $f_{k-1}h_k^{-1}(x) + A\cdot(x)$ has rank n on $C^n(2)$. Let A be chosen to satisfy this requirement.
We then define $f_k : M^n \to \mathbb{R}^p$ by the equation:

$$f_k(y) = \begin{cases} f_{k-1}(y) + \varphi(h_k(y))A\cdot h_k(y) & \text{for } y \in V_k \\ f_{k-1}(y) & \text{for } y \in M \setminus \overline{U}_k. \end{cases}$$

These definitions agree on the overlapping domains, so that f_k is differentiable. By the first condition on A, it has rank n on N_{k-1}; by the third condition it has rank n on $\overline{W}_k$. By the second condition, f_k is a $\delta/2^k$ approximation to f_{k-1}.
We define $g(x) = \lim_{k\to\infty} f_k(x)$. Since the covering V_k is locally-finite, all the f_k agree on a given compact set for k sufficiently large; it follows that g is differentiable and has rank n everywhere. It is also a δ-approximation to f. □

1.29. *Lemma.* *If $p > 2n$, any immersion $f : M^n \to \mathbb{R}^p$ can be δ-approximated by a* 1 - 1 *immersion g. If f is* 1 - 1 *in a neighbourhood U of the closed set N, we may choose $g \mid N = f \mid N$.*

***Proof*:** Choose a covering $\{U_\alpha\}$ of M^n such that $f \mid U_\alpha$ is an embedding (possible by 1.6). Let (V_i, h_i) be the locally-finite refinement constructed in 1.25; let $\varphi(x)$ be the function constructed in 1.26. Let

$$\varphi_1(y) = \begin{cases} \varphi(h_1(y)) & \text{for } y \in V_i \\ 0 & \text{for other } y. \end{cases}$$

Then φ_1 is differentiable. As before, we assume (V_i, h_i) refines the covering $(U, M^n \setminus N)$ and that those V_i with non-positive indices are the ones contained in U.
Let $f_0 = f$. Given the immersion $f_{k-1} : M^n \to \mathbb{R}^p$, we define f_k by the equation

$$f_k(y) = f_{k-1}(y) + \varphi_k(y)b_k,$$

where b_k is a point of $\mathbb{R}^p$ yet to be chosen. By the argument of the previous theorem, if b_k is chosen sufficiently small, f_k will have rank n everywhere. The first requirement is that b_k be this small; the second requirement is that b_k be small enough that f_k be a $\delta/2^k$ approximation to f_{k-1}.
Finally, let N^{2n} be the open subset of $M^n \times M^n$ consisting of pairs (y, y_0), with $\varphi_k(y) \neq \varphi_k(y_0)$.
Consider the differentiable map

$$(y, y_0) \mapsto -[f_{k-1}(y) - f_{k-1}(y_0)] \,/\, [\varphi_k(y) - \varphi_k(y_0)]$$

from N^{2n} into $\mathbb{R}^p$. Since $2n < p$, the image of N^{2n} has measure zero, so that b_k may be chosen arbitrarily small and ***not*** in this image. It follows that $f_k(y) = f_k(y_0)$ if and only if $\varphi_k(y) = \varphi_k(y_0)$ and $f_{k-1}(y) = f_{k-1}(y_0)$ $(k > 0)$.
Define $g(y) = \lim_{k\to\infty} f_k(y)$. If $g(y) = g(y_0) =$ and $y \neq y_0$, it would follow that $f_{k-1}(y) = f_{k-1}(y_0)$ and $\varphi_k(y) = \varphi_k(y_0)$ for all $k > 0$. The former condition implies that $f(y) = f(y_0)$, so that y and $= y_0$ cannot belong to any one set U_i. Because of the latter condition, this means that neither is in any set U_i for $i > 0$. Hence, they lie in U, contradicting the fact that f is 1 - 1 on U. □

1.30. *Definition*. Let $f : M^n \to \mathbb{R}^p$. The ***limit set*** $L(f)$ is the set of $y \in \mathbb{R}^p$ such that $y = \lim f(x_n)$ for some sequence $\{x_1, x_2, \ldots\}$ which has no limit point on M^n.

Exercise. Show the following:

1) $f(M^n)$ is a closed subset of $\mathbb{R}^p$ if and only if $L(f) \subset f(M^n)$
2) f is a topological embedding if and only if f is 1 - 1 and $L(f) \cap f(M^n)$ is vacuous.

1.31. ***Lemma.*** *There exists a differentiable map* $f : M^n \to \mathbb{R}$ *with* $L(f)$ *empty.*

***Proof*:** Let (V_i, h_i) and φ be chosen as in 1.25 and 1.26 with i ranging over positive integers; let

$$\varphi_i(y) = \begin{cases} \varphi(h_i(y)) & \text{if } y \in V_i \\ 0 & \text{otherwise.} \end{cases}$$

Define $f(y) = \sum_j (j\varphi_j(y))$. This sum is finite, since V_i is a locally-finite covering. If $\{x_i\}$ is a set of points of M^n having no limit point, only finitely many lie in any compact subset of M^n. Given m, there is an integer i such that x_i is not in $\overline{W}_1 \cup \ldots \cup \overline{W}_m$. Hence $x_i \in \overline{W}_j$ for some $j > m$, whence $f(x_i) > m$. Thus the sequence $f(x_m)$ cannot converge. □

1.32. ***Corollary.*** *Every* M^n *can be differentiably embedded in* $\mathbb{R}^{2n+1}$ *as a closed subset.*

***Proof*:** Let $f : M^n \to \mathbb{R} \subset \mathbb{R}^{2n+1}$ differentiably, with $L(f) = 0$. Set $\delta(x) \equiv 1$, and let g be a 1 - 1 immersion which is a δ-approximation to f. Then $L(g)$ is empty, so that g is a homeomorphism.

1.33. ***Definition.*** Let $f : M^n \to N^p$ be differentiable. Let N_1^{p-q} be a differentiable submanifold of N^p. Let $f(x) \in N_1^{p-q}$. Let $(u^1, \ldots, u^n)$ be a coordinate system about x; and let $(v^1, \ldots, v^p)$ be a coordinate system about $f(x)$ such that on N_1^{p-q}, $v^1 = \cdots = v^p = 0$ (see 1.6). Consider the condition that $\partial(v^1, \ldots, v^q)/\partial(u^1, \ldots, u^n)$ has rank q at x. This is the ***transverse regularity condition for*** f ***and*** N_1^{p-q} ***at*** x. [Exercise: Show that this condition is independent of coordinate system.]

Note that the set of points on which the transverse regularity condition is satisfied is an open subset of $f^{-1}(N_1^{p-q})$; f is said to be ***transverse regular on*** N_1^{p-q} if the condition is satisfied foe each x in $f^{-1}(N_1^{p-q})$.

1.34. ***Lemma.*** *If* $f : M^n \to N^p$ *is transverse regular on* N_1^{p-q} *then* $f^{-1}(N_1^{p-q})$ *is a differentiable submanifold of dimension* $n - q$ (*or is empty*).

***Proof*:** Let π project $\mathbb{R}^p$ onto its first q components; $\pi : \mathbb{R}^p \to \mathbb{R}^q$. If $(V, h) = (v^1, \ldots, v^p)$ is the coordinate system hypothesised in 1.33, then

$$N_1^{p-q} \cap V = h^{-1}\pi^{-1}(0)$$

where 0 denotes the origin in $\mathbb{R}^q$; and $f^{-1}(N_1^{p-q} \cap V) = (\pi h f)^{-1}(0)$. Since $\pi h f$ has rank q at $x \in f^{-1}(N_1^{p-q} \cap V)$, the origin is a regular value of $\pi h f$. Hence $(\pi h f)^{-1}(0)$ is a differentiable submanifold of M^n of dim $n - q$ (see 1.12). □

1.35. ***Theorem.*** *Let* $f : M^n \to N^p$ *be differentiable; let* N_1^{p-q} *be a closed subset of* M^n *such that the transverse regularity condition for* f *and* N_1^{p-q} *holds at each* x *in* $A \cap f^{-1}(N_1^{p-q})$. *Let* δ *be a positive continuous function on* M^n. *There exists a differentiable map* $g : M^n \to N^p$ *such that*

1) *g is a δ-approximation to f,*
2) *g is transverse regular on N_1^{p-q}, and*

3) $g \mid A = f \mid A$.

Proof: There is a neighbourhood U of A in M^n such that f satisfies the transverse regularity condition on $U \cap f^{-1}(N_1^{p-q})$. Cover N^p by $N^p \setminus N_1^{p-q} = Y_0$ and coordinate system (Y_i, η_i) for $i > 0$; with coordinate functions $(v^1, \dots, v^n)$ such that $v^1 = \cdots = v^p = 0$ on N_1^{p-q}. Now the open sets $f^{-1}(Y_i)$ cover M^n, as do the open sets U, $M^n \setminus A$. Let (V_j, h_j) be a refinement of both coverings, constructed as in 1.25. Recall that $h_j(V_j) = C^n(3)$, $h_j(U_j) = C^n(2)$, $h_j(W_j) = C^n(1)$, and the W_j cover M^n. The V_j are to be indexed with positive and negative integers so that those V_j which are contained in U are the ones with non-positive indices.
Let φ be as in 1.26, and define

$$\varphi_i(x) = \begin{cases} \varphi(h_i(x)) & \text{for } x \in V_i \text{ and} \\ 0 & \text{elsewhere.} \end{cases}$$

For each j choose $i(j) \geq 0$ so that $f(V_j)$ is contained in $Y_{i(j)}$.
Set $f_0 = f$. Suppose f_{k-1} is defined and satisfies the transverse regularity condition for N_1^{p-q} at each point of the intersection of $f_{k-1}^{-1}(N_1^{p-q})$ with $\cup_{j<k}\overline{W}_j$. Furthermore suppose that $f_{k-1}^{-1}(\overline{U}_j) \subset Y_{i(j)}$ for each j. Setting $i = i(k)$, it follows in particular that $f_{k-1}^{-1}(\overline{U}_k) \subset Y_i$.
Consider

$$\pi\, \eta_i f_{k-1} h_k^{-1} : C^n(2) \to \mathbb{R}^q;$$

By 1.21, there is an arbitrarily small affine function $L(x) = A\cdot(x) + B$ such that when added to the previous function, the resulting map has the origin as a regular value. Consider $\mathbb{R}^q$ as the first q coordinates in $\mathbb{R}^p$, and define

$$f_k(x) = \begin{cases} \eta_i^{-1}(\eta_i f_{k-1}(x) + L(h_k(x)\, \varphi_k(x)) & \text{for } x \text{ in a neighbourhood of } \overline{U}_k \\ f_{k-1}(x) & \text{for } x \text{ in } M^n \setminus U_k. \end{cases}$$

Here L is yet to be chosen. Of course, we must choose L small enough that

$$\eta_i f_{k-1} + L\varphi_k$$

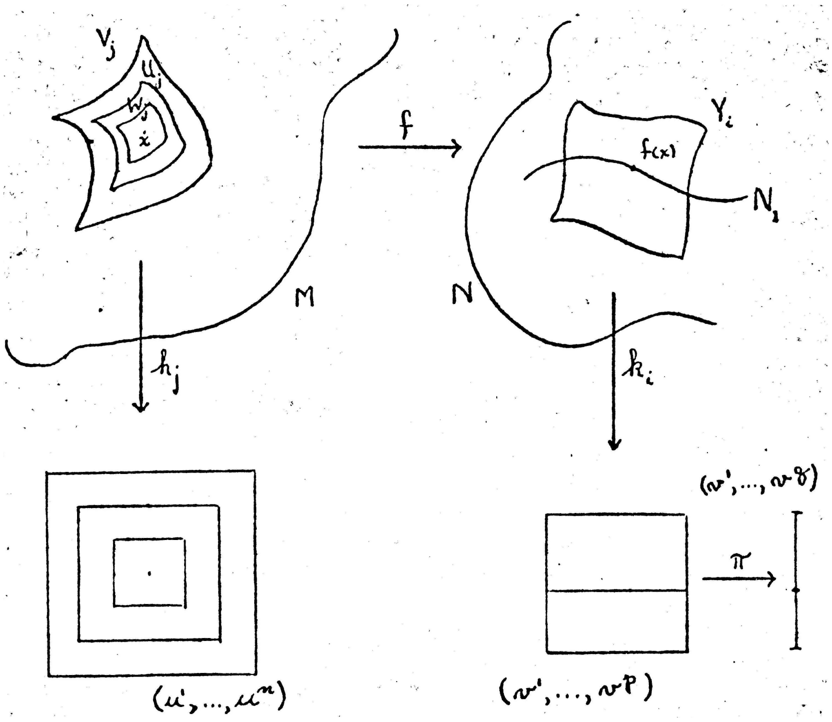
lies in $C^n(1)$ for $x \in \overline{U}_k$, in order that k_i^{-1} may be applied to it. This is the first requirement on L. Secondly, we choose L small enough that f_k is a $\delta/2^k$ approximation to f_{k-1}. Thirdly choose L small enough so that $f_k(\overline{U}_j)$ is contained in $Y_{i(j)}$ for each j. This is possible since only a finite number of the sets $\overline{U}_j$ can intersect $\overline{U}_k$.
Now f_k by definition satisfies the transverse regularity condition for N_1^{p-q} at each point of $f_k^{-1}(N_1^{p-q}) \cap \overline{W}_k$. We want to choose L small enough that the condition is satisfied at each point of this intersection of $f_k^{-1}(N_1^{p-q})$ with $\cup_{j<k}\overline{W}_j$. It is sufficient to consider the intersection of this set with $\overline{U}_k$; let this intersection be denoted by K. Consider the function which maps the pair (x, L) $(x \in K)$ into

$$(f_k(x), D(\pi\, \eta_i f_{k-1} h_k^{-1})\cdot(h_k(x)) \in N_1^{p-q} \times \mathcal{M}(q, n).$$

This function is continuous and carries $K \times (0)$ into the set

$$[(N^p \setminus N_1^{p-q}) \times \mathcal{M}(q, n)] \cup [N_1^{p-q} \times \mathcal{M}(q, n; q)],$$

which is open in $N_1^{p-q} \times \mathcal{M}(q, n)$. Hence for L sufficiently small, (K, L) is carried into this set, so that f_k satisfies the transverse regularity condition for N_1^{p-q} at each point of $f_k^{-1}(N_1^{p-q}) \cap (\cup_{j<k}\overline{W}_j)$.
We define $g(x) = \lim_{k\to\infty} f_k(x)$, as usual. □


V_j
U_j
W_j
x
M
f
N
Y_i
f(x)
N_1
h_j
k_i
(u^1, ..., u^n)
(v^1, ..., v^p)
(v^1, ..., v^q)
π

Chapter II *Vector Space Bundles*

2.1 ***Definition.*** An n-dimensional ***real vector space bundle*** ξ is a triple (π, a, s) where $\pi : E \to B$ is an onto continuous map between Hausdorff spaces that satisfy the following:

1) $F_b = \pi^{-1}(b)$, called a ***fibre,*** is an n-dimensional real vector space with $s : R \times E \to E$ carrying $R \times F_b$ into F_b, and $a : U(F_b \times F_b) \subset E \times E \to U(F_b)$ carrying $F_b \times F_b$ into F_b, as scalar product and vector addition, respectively.
2) (Local triviality) For each $b \in B$, there is a neighbourhood U of b and a homeomorphism $\varphi : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that φ is a vector space isomorphism of $b' \times \mathbb{R}^n \cong F_{b'}$, for each $b' \in U$.

If in 2) the neighbourhood U may be taken as all B, the bundle is said to be the ***trivial bundle***.

If ξ, η are n-dimensional and p-dimensional vector space bundles, respectively, we define the ***product bundle*** $\xi \times \eta$ as follows:

$$E(\xi \times \eta) = E(\xi)\times E(\eta)$$
$$B(\xi \times \eta) = B(\xi)\times B(\eta)$$
$$(\pi \times \lambda)(x, y) = ((\pi(x), \lambda(y))$$

where π, λ are the projections in ξ, η respectively and $F_b(\xi \times \eta)$ has the usual product structures for vector spaces.

If U is a subset of $B(\xi)$, then $\xi \mid U$ denotes the bundle $\pi : \pi^{-1}(U) \to U$. It is called the ***restriction*** of the bundle to U.

2.2 ***Definition.*** Let M^n be a differentiable manifold and let x_0 be in M^n. A ***tangent vector*** at x_0 is an operation X which assigns to each differentiable function f defined in a neighbourhood U of x_0, a real number, that is, $X : \mathcal{O}(U) \to \mathbb{R}$. The following conditions must be satisfied:

1) If g is a restriction of f, $X(g) = X(f)$.
2) $X(cf + dg) = cX(f) + dX(g)$ for $c, d \in \mathbb{R}$
3) $X(f{\cdot}g) = X(f){\cdot}g(x_0) + f(x_0){\cdot}X(g)$, where the dot means ordinary real multiplication.

Then $X(1) = X(1{\cdot}1) = X(1) + X(1)$, by 3). Hence $X(1) = 0$ and $X(c)$ also $= 0$, by 2).

If one thinks of a tangent vector as being the velocity vector of a curve lying in the manifold, then $X(f)$ is merely the derivative of f with respect to the parameter of the curve. This is made more precise below.

2.3 ***Lemma.*** *Let* $(u^1, \ldots, u^n)$ *be a coordinate system about* x. *Let* X *be a tangent vector at* x. *Then* X *may be written uniquely as a linear combination of the operators* $\partial/\partial u^i$:

$$X = \sum \alpha^i \partial/\partial u^i.$$

Proof: We assume $u(x)$ is the origin. Given any $f(u^1, \ldots, u^n)$ define

$$g(u^1, \ldots, u^n) = \begin{cases} [f(u^1, \ldots, u^n) - f(0, u^2, \ldots, u^n)] / u^1 & \text{if } u^1 \neq 0 \\ \partial f(0, u^2, \ldots, u^n)/\partial u^1 & \text{if } u^1 = 0. \end{cases}$$

To see that g is differentiable, note that

$$g(0, u^2, \ldots, u^n) = \int_{[0,1]}[\partial f(0, u^2, \ldots, u^n)/\partial u^1]dt.$$

(Then $f(u^1, \ldots, u^n) = u^1 g_1(u^1, \ldots, u^n) + f(0, u^2, \ldots, u^n)$.) Similarly,

$$f(0, u^2, \ldots, u^n) = u^2 g_2(u^2, \ldots, u^n) + f(0, 0, u^3, \ldots, u^n),$$

where $g_2(0) = \partial f/\partial u^2(0)$. Finally we have $f(u^1, \ldots, u^n) = \sum u^i g_i + f(0)$, where $g_i(0) = \partial f/\partial u^i(0)$. Thus

$$X(f) = \sum X(u^i)g_i(0) + 0 \cdot X(g_i) = \sum \alpha^i \partial f/\partial u^i(0),$$

where $\alpha^i = X(u^i)$.

Remark. If $(v^1, \ldots, v^n)$ is another coordinate system about x, and $X = \sum \beta^j \partial/\partial v^j$., then $\alpha^i = X(u^i) = \sum \beta^j \partial u^i/\partial v^j$. The α^i are called the components of the vector X with respect to the coordinate system $(u^1, \ldots, u^n)$.

2.4 ***Alternate definition.*** A tangent vector at x is an assignment to every coordinate system $(u^1, \ldots, u^n)$ about x of an element $(\alpha^1, \ldots, \alpha^n)$ of $\mathbb{R}^n$, with the requirement that if (β^j) is assigned to the system $(v^1, \ldots, v^n)$, then $\alpha^i = \sum \beta^j \partial u^i/\partial v^j$. The derivation operator X is then defined as $\sum \alpha^i \partial/\partial u^i$. One checks readily that

a) $X(f)$ is independent of the coordinate system used, and

b) $X(f)$ satisfies requirements 1), 2), and 3) for a tangent vector.

2.5. ***Definition.*** For each x in M^n, the tangents at x form an n-dimensional vector space (by 2.3, the operations $\partial/\partial u^i$ form a basis). Let the totality of these be denoted by $E(\tau)$; define $\pi : E(\tau) \to M^n$ as mapping all the tangent vectors X at x_0 into x_0. The local product structure around $x_0 \in U$ is given by $\varphi_U : U \times \mathbb{R}^n \to E(\tau)$, where $(U, h) = (u^1, \ldots, u^n)$ is a coordinate system on M^n, and φ_U is defined as follows:

$$\varphi_U(x_0, a^1, \ldots, a^n) = \text{tangent vector } X = \sum \alpha^i \partial/\partial u^i \text{ at } x_0.$$

Since φ_U is to be a homeomorphism, this structure imposes a topology on $E(\tau)$; since $\varphi_V^{-1}\varphi_U$ is a homeomorphism on $(U \cap V) \times \mathbb{R}^n$, this topology is unambiguously determined. One checks immediately that φ_U gives us a vector space bundle isomorphism for each fibre.

Indeed, $\varphi_V^{-1}\varphi_U$ is a C^∞ map on $(U \cap V) \times \mathbb{R}^n$, so that $E(\tau)$ is a differentiable manifold of dimension $2n$ (using definition 1.2 of a differentiable manifold). The map π is differentiable of rank n. This bundle τ is called the ***tangent bundle*** of M^n.

2.6. ***Definition.*** If $f : M_1^n \to M_2^m$, there is an induced map $df : E(\tau_1) \to E(\tau_2)$ defined as follows: $df(X) = Y$, where $Y(g) = X(gf)$. If X is a vector at x_0, Y is a vector at $f(x_0)$. This is clearly linear on each fibre; it is called the ***derivative*** map.

If (U, h) and (V, k) are coordinate systems about $x_0, f(x_0)$ respectively, and (α^i), (β^j) are the respective components of X and Y with respect to these coordinate systems, then $(\beta^j) = D(kfh^{-1})(\alpha^i)$ where the vector components are written as column matrices, as usual.

2.7. ***Definition.*** Let ξ, η be two n-dimensional vector bundles. A ***bundle map*** $f : \xi \to \eta$ is a continuous map of $E(\xi)$ into $E(\eta)$ which carries each fibre isomorphically onto a fibre. The induced map $f_B : B(\xi) \to B(\eta)$ is automatically continuous.

If $B(\xi) = B(\eta)$ and the induced map is the identity, f is said to be an ***equivalence***. Note that if f is an equivalence, it is a homeomorphism: Locally f is just a map $U \times \mathbb{R}^n \to V \times \mathbb{R}^n$. The projection of f^{-1} into the factor U is continuous, because f_B^{-1} is the identity. But f may be given by a non-singular matrix function of $x \in U$; f^{-1} is the inverse of this matrix, so that the projection of f^{-1} into the factor $\mathbb{R}^n$ is continuous. Hence f^{-1} is continuous.

If there is an equivalence of ξ onto η, we write $\xi \simeq \eta$.

2.8. *Lemma.* *Given a bundle η with projection map $\lambda : E(\eta) \to B(\eta)$, and a map $f : B_1 \to B(\eta)$, there is a bundle $\pi : E_1 \to B_1$ and a bundle map $g : E_1 \to E(\eta)$ such that $\lambda g = f\pi$. Furthermore, E_1 is unique up to an equivalence.*

$$\begin{array}{ccc} E_1 & \xrightarrow{g} & E(\eta) \\ \pi\downarrow & & \downarrow\lambda \\ B_1 & \xrightarrow[f]{} & B(\eta) \end{array}$$

Remark. E_1 is called the ***induced bundle*** by f and is often denoted by $f^*\eta$.

***Proof*:** Let E_1 be that subset of $B_1 \times E(\eta)$ consisting of points (b, e) such that $f(b) = \lambda(e)$. Define $\pi(b, e) = b$; $g(b, e) = e$. To show that E_1 is a vector space bundle, let $\varphi : V \times \mathbb{R}^n \to E(\eta)$ be a product neighbourhood in $E(\eta)$, and let $f(U) \subset V$. Then define $\varphi_1 : U \times \mathbb{R}^n \to E_1$ by $\varphi_1(b, x) = (b, \varphi((b), x))$. Then φ_1 is continuous and 1 - 1; its image equals $\pi^{-1}(U)$. Its inverse φ_1^{-1} carries (b, e) into $(b, p\varphi^{-1}(e))$, where p is the natural projection $V \times \mathbb{R}^n \to \mathbb{R}^n$, hence it is continuous. The map g is an isomorphism on each fibre.

Now suppose $g' : E' \to E(\eta)$ is a bundle map, where $\pi' : E' \to B_1$ is a bundle and $\lambda g' = f\pi'$. We map $E' \to E_1$ by mapping

$$e' \mapsto (\pi'(e'), g'(e')) \in E_1.$$

Because g' is an isomorphism on each fibre, so is this map; and it induces the identity on the base space. Hence it is an equivalence.

$$\begin{array}{ccccc} E_1 & \xrightarrow{g} & E(\eta) & \xleftarrow{g'} & E' \\ \pi\downarrow & & \downarrow\lambda & & \downarrow\pi' \\ B_1 & \xrightarrow[f]{} & B(\eta) & \xleftarrow[f']{} & B_1 \end{array}$$

2.9. *Definition.* Let ξ, η be two bundles over B. The ***Whitney sum*** $\xi \oplus \eta$ is a bundle defined as the induced bundle $d^*(\xi \times \eta)$ for $d : B \to B \times B$ be the diagonal map and the product bundle $E(\xi) \times E(\eta) \to B \times B$.

$$\begin{array}{ccc} \xi \oplus \eta = d^*(\xi \times \eta) & \to & E(\xi) \times E(\eta) \\ \downarrow & & \downarrow \\ B & \xrightarrow[d]{} & B \times B \end{array}$$

The proof of the following is left as an exercise.

a) the fibre over b in $\xi \oplus \eta$ is $F_b(\xi) \times F_b(\eta)$, so that $\dim(\xi \oplus \eta) = \dim \xi + \dim \eta$,

b) $\oplus$ is commutative: $\xi \oplus \eta \simeq \eta \oplus \xi$,

c) $\oplus$ is associative: $(\xi \oplus \eta) \oplus \varsigma \simeq \xi \oplus (\eta \oplus \varsigma)$.

2.10. *Definition.* If ξ, η are bundles over B, then $g : E(\xi) \to E(\eta)$ is a ***homomorphism*** if

1) it maps each fibre linearly into a fibre,

2) the induced map on B is the identity.

Note that an equivalence is both a bundle map and a homomorphism. An ***embedding*** of bundles is a 1 - 1 homomorphism.

2.11. ***Theorem.*** *If $f : E(\xi) \to E(\eta)$ maps each fibre linearly into a fibre, then f may be factored into a homomorphism followed by a bundle map.*

***Proof*:** Let π_1, π_2 be the projections in ξ, η, respectively.
Let $f_B : B(\xi) \to B(\eta)$ be the map induced by f. Let $E_1 = f_B{}^*\eta$ be the bundle induced by f_B; let g be the bundle map $E_1 \to E(\eta)$ and π be the projection $E_1 \to B(\eta)$.

$$\begin{array}{ccccc} E(\xi) & \xrightarrow{h} & E_1 & \xrightarrow{g} & E(\eta) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_2} \\ B(\xi) & \xrightarrow[i]{} & B(\xi) & \xrightarrow[f_B]{} & E(\eta) \end{array}$$

Define $h : E(\xi) \to B(\xi) \times E(\eta)$ by the equation $h(e) = (\pi_1(e), f(e))$. The image of h actually lies in that subset of $B(\xi) \times E(\eta)$ which is E_1; then h is a homomorphism. From the definition $f = gh$. □

2.12. ***Lemma.*** *Let ξ, η be bundles over B of dimensions n, p, respectively; let $g : \xi \to \eta$ be a homomorphism. If g is onto, then the kernel (g) is a bundle. If g is* 1 - 1, *then the cokernel (g), i.e., the quotient η / image (g), is a bundle.*

***Proof*:** Suppose g is 1 - 1 (i.e., has rank n when restricted to each fibre.) In $E(\eta)$, we define $e \sim e'$ if $e - e'$ exists and is in the image of g. We identify the elements of these equivalence classes; the resulting identification space is defined to be $E(\eta / g(\xi))$. It is a bundle over B with projection naturally defined and each fibre is a vector space of dimension $p - n$. We need only to show the existence of a local product structure.
Let U be an open set in B, with $\xi \mid U$ equivalent to $U \times \mathbb{R}^n$ and $\eta \mid U$ equivalent to $U \times \mathbb{R}^p$. Let g_0 denote the homomorphism of $U \times \mathbb{R}^n \to U \times \mathbb{R}^p$ induced by g. Now $(\eta / g(\xi)) \mid U$ is equivalent to the quotient $U \times \mathbb{R}^p / g_0(U \times \mathbb{R}^n)$, so that it suffices to show that this latter quotient is locally a product.
g_0 is given by a matrix $M(b) \in \mathcal{M}(p, n)$ which depends continuously on the point $b \in U$. Given b_0, we may assume that in a neighbourhood U_0 of b_0, the first n rows are independent. We define $h : U_0 \times \mathbb{R}^n \times \mathbb{R}^{p-n} \to U \times \mathbb{R}^p$ as the linear function on whose matrix (non-singular) is

$$\left[\begin{array}{c|c} M(b) & \begin{array}{c} 0 \\ \hline I_{p-n} \end{array} \end{array}\right]$$

The image of $U_0 \times \mathbb{R}^n \times 0$ under h is just $g_0(U_0 \times \mathbb{R}^n)$; since h is an equivalence, it induces an equivalence of

$$U_0 \times \mathbb{R}^{p-n} \simeq U_0 \times \mathbb{R}^n \times \mathbb{R}^{p-n} / U_0 \times \mathbb{R}^n \times 0 \text{ onto } U_0 \times \mathbb{R}^p / g_0(U_0 \times \mathbb{R}^n).$$

Secondly, suppose g is onto (i.e., it has rank p on each fibre.) $E(g^{-1}(0))$ is defined as that subset of $E(\xi)$ consisting of points e with $g(e) = 0$. Again, we need to show the existence of a local product structure. Let U, g_0, and $M(b)$ be as above. Given b_0, we may assume that the first p columns of are independent in the neighbourhood U_0 of b_0. We define $h : U_0 \times \mathbb{R}^n \to U_0 \times \mathbb{R}^p \times \mathbb{R}^{n-p}$ by the matrix function

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{M(b)} \\ \hline 0 & I_{p-n} \end{array}\right]$$

Now h followed by the natural projection of $U_0 \times \mathbb{R}^p \times \mathbb{R}^{n-p}$ onto $U_0 \times \mathbb{R}^p$ equals $g_0 \mid U$. Hence h^{-1} maps $U_0 \times 0 \times \mathbb{R}^{n-p}$ onto $g_0^{-1}(U_0 \times 0)$; since h is an equivalence, so is the restriction of h^{-1} to $U_0 \times 0 \times \mathbb{R}^{n-p}$. □

Remark. If g is onto, $\xi / g^{-1}(0)$ is a bundle, being the quotient of the inclusion homomorphism $g^{-1}(0) \to \xi$. If g is 1 - 1, $g(\xi)$ is a bundle, being the kernel of the projection homomorphism $\eta \to g(\xi)$.

2.13. *Definition.* If φ is a non-negative function on B, the ***support*** of φ is the closure of the set of x with $\varphi(x) > 0$. A ***partition of unity*** is a collection $\{\varphi_\alpha\}$ of non-negative functions on B, such that the sets $\{C_\alpha\} = \{\text{support}(\varphi_\alpha)\}$ form a locally-finite covering of B, and $\sum\varphi_\alpha(x) = 1$ (this is a finite sum for each x.)

2.14. *Lemma.* *Let B be a normal space; $\{U_\alpha\}$ a locally-finite open covering of B. Then there is a partition of unity $\{\varphi_\alpha\}$ with* support$(\varphi_\alpha) \subset U_\alpha$ *for each α.*

***Proof*:** First, we show that there is an open covering $\{V_\alpha\}$ of B with $\overline{V}_\alpha \subset U_\alpha$ for each α. Assume that U_α are indexed by a set of ordinals (well-ordering theorem.) Let V_α be defined for all $\alpha < \beta$ and assume that the sets V_α along with the sets U_α for $\alpha \geq \beta$ cover B. Consider the set $A(\beta) = B \setminus \cup_{\alpha<\beta}V_\alpha \setminus \cup_{\alpha>\beta}U_\alpha$. Then $A(\beta) \subset U_\beta$. Let V_β be an open set containing the closed set $A(\beta)$, with $\overline{V}_\beta \subset U_\beta$ (normality.) This completes the construction of the V_α.

Now let g_α be a function which is positive on $\overline{V}_\alpha$ and 0 outside U_α (normality again.) Define $\varphi_{\alpha 0}(x) = g_{\alpha 0}(x) / \sum g_\alpha(x)$. Since $\{U_\alpha\}$ is locally-finite, the sum in the denominator is finite and positive, so $\{\varphi_\alpha\}$ is well-defined. □

***Remark*[1].** If B is a differentiable manifold, φ_α may be chosen to be differentiable: Cover B with coordinate systems (V_i, h_i) as in 1.25 refining the covering U_α, $B \setminus \overline{V}_\alpha$. Let $\varphi_i(y) = \varphi_i(h_i(y))$ for $y \in V_i$, and $\varphi_i(y) = 0$ otherwise (φ as in 1.26.) Let $g_\alpha(y) = \sum\varphi_i(y)$, where the sum extends over all i such that $V_i \subset U_\alpha$.

2.15. *Lemma.* *Let B be paracompact and let $0 \to \xi \xrightarrow{i} \eta \xrightarrow{\varphi} \zeta \to 0$ be an exact sequence of homomorphism of bundles. Then there is equivalence $f : \eta \to \xi \oplus \zeta$, with fi the natural inclusion and φf^{-1} the natural projection.*

***Proof*:** Let $\dim \xi = n$; $\dim \zeta = p$.
We first construct a Riemannian metric on η (i.e., a continuous inner product in $E(\eta)$.) Let $\{U_\alpha\}$ be a locally-finite covering of B with $\eta \mid U_\alpha$ trivial; let g_α be the corresponding projection of $\eta \mid U_\alpha$ onto $\mathbb{R}^{n+p}$. Let $\{\varphi_\alpha\}$ be a partition of unity with support$(\varphi_\alpha) \subset U_\alpha$.

If e, e' are in $E(\eta)$ and $\pi(e) = \pi(e')$, define $e \cdot e' = \sum_\alpha \varphi_\alpha(\pi(e))g_\alpha(e)\cdot g_\alpha(e')$, where the dot on the right hand side is the ordinary scalar product in $\mathbb{R}^{n+p}$. This is a finite sum; it satisfies the axioms for a scalar product.

The way we use the Riemannian metric is to break η up into $iE(\xi)$ and its orthogonal complement. Let ξ' be the image of ξ in η and let $E(\zeta')$ be defined as that subset of consisting of elements which are orthogonal to $iE(\xi)$. In order to show that ζ' has a local product structure, consider the homomorphism

$$h : \eta \to \zeta'$$

which sends each vector into its orthogonal projections in ξ'. [Verification that h is continuous. Over any coordinate neighbourhood U we can choose a basis $a_1, \ldots a_n$ for the fibre of ξ'. Then the

1 See Appendix, Proposition A.

function h carries $v \in E(\eta)$ into $\sum t_j a_j \in E(\xi') \subset E(\eta)$, where $t_j = \sum B_{jk}(v \cdot a_k)$ and where (B_{jk}) denotes the inverse matrix to $(a_j \cdot a_k)$.] Since h is onto, its kernel ζ' is again a vector space bundle.

Now the bundle $i(\xi) = \xi'$ is equivalent to ξ. It remains to show that ξ' is equivalent to ζ and that η is equivalent to $\xi' \oplus \zeta'$. The former follows immediately from the fact that $\varphi \mid \zeta'$ is a homomorphism; form rank considerations it must be 1 - 1 and onto as well. The latter follows by noting that $E(\xi' \oplus \zeta')$ is defined as the subset of $E(\xi') \times E(\zeta')$ consisting of points (e_1, e_2) such that $\pi(e_1) = \pi(e_2)$. Consider the map f of $E(\xi' \oplus \zeta')$ into $E(\eta)$ obtained by taking (e_1, e_2) into their sum in $E(\eta)$ (their sum exists because e_1 and e_2 lie in the same fibre.) This is clearly a homomorphism; from rank considerations, it must be 1 - 1 and onto. □

2.16. *Definition*. Let M_1, M_2 be differentiable manifolds and let $f: M_1 \to M_2$ be an immersion. The ***normal bundle*** ν_f is defined as follows:

Let τ_1, τ_2 be the tangent bundles of M_1, M_2 respectively. By 2.11, the map $df: E(\tau_1) \to E(\tau_2)$ may be factored into a homomorphism h of $E(\tau_1)$ into $E(f^*\tau_2)$ followed by a bundle map g. Now h is a 1 - 1 homomorphism because f is an immersion; hence by 2.12, $f^*\tau_2$ / image (h) is a bundle over M_1. It is called the normal bundle ν_f.

Then $0 \to \tau_1 \to f^*\tau_2 \to \nu_f \to 0$ is an exact sequence if homomorphisms, so that by 2. 15, $f^*\tau_2$ is equivalent to $\tau_1 \oplus \nu_f$. Indeed, given a Riemannian metric on $f^*\tau_2$, ν_f is equivalent to the orthogonal complement of the image of τ_1.
Let us consider the case $M_2 = \mathbb{R}^{n+p}$, where dim $M_1 = n$. Then τ_2 is the trivial bundle, so that $f^*\tau_2$ is as well. (*Proof*: If $f: B_1 \to B(\eta)$ and η is trivial, so is $f^*\eta$. We have the diagram

$$\begin{array}{c} B \times \mathbb{R}^n \\ \downarrow \pi \\ f: B_1 \to B \end{array}$$

$E(f^*\eta)$ is defined as that subset of $B_1 \times (B \times \mathbb{R}^n)$ consisting of points (b_1, b, x) such that $f(b_1) = (b, x)$, i.e., of all points $(b_1, f(b_1), x)$. If we map this into (b_1, x), we obtain an equivalence of $f^*\eta$ with the bundle $B_1 \times \mathbb{R}^n \to B_1$.
Thus $\tau_1 \oplus \nu_f$ is equivalent to a trivial bundle. In what follows, we investigate the following question: Given ξ, does there exist an η with $\xi \oplus \eta$ trivial? Using 1.28, this is always the case for ξ the tangent bundle of an n-manifold, and indeed η may be chosen also to have dimension n. A more general answer appears in 2.19.

2.17. *Definition*. Let $f: M_1{}^n \to M_2{}^p$; If f has rank p at every point of M_1, it is said to be ***regular***. If f is regular, the homomorphism $h: \tau_1 \to f^*\tau_2$ given by 2.11 is an onto map. By 2.12, the kernel of h is a bundle α_f. It is called the ***bundle along the fibre***.

Note that $f^{-1}(y)$ is a submanifold of M_1 of dimension $n - p$ (by 1.12 or 1.34.) The inclusion i_y of $f^{-1}(y)$ into M_1 induces an inclusion di_y of its tangent bundle into τ_1. The kernel of h consists precisely of the vectors which are in the image of some di_y, i.e., the vectors tangent to the submanifolds $f^{-1}(y)$ are the ones carried into 0 by h.

One has the exact sequence $0 \to \alpha_f \to \tau_1 \to f^*\tau_2 \xrightarrow{g} 0$, so that by 2.15, τ_1 is equivalent to $\alpha_f \oplus f^*\tau_2$.

2.18. *Definition*. A bundle ξ is of ***finite type*** if B is normal and may be covered by a finite number of neighbourhoods $U_1, \ldots U_k$ such that $\xi \mid U_i$ is trivial for each i.

2.19. *Lemma*. *ξ is of finite type if B is compact, or paracompact finite dimensional.*

Proof: The former statement is clear; let us consider the latter. By definition, the dimension of B is not greater than n if every covering has an open refinement such that

no point of B is contained in more than $n + 1$ elements of the refinement. (*)

It is a standard theorem of topology that an n-manifold has dimension n in this sense.
Cover B by open sets U, with $\xi \mid U$ trivial; let $\{V_\alpha\}$ be an open refinement of this covering satisfying (*). By 1.22, we may assume that $\{V_\alpha\}$ is locally-finite as well. Let $\{\varphi_\alpha\}$ be a partition of unity with support$(\varphi_\alpha) \subset V_\alpha$ for each α (2.14.)

Let A_i be the set of unordered $(i + 1)$-tuple of distinct elements of the index set of $\{\varphi_\alpha\}$. Given a in A_i, where $a = \{\alpha_0, \ldots, \alpha_n\}$, let W_{ia} be the set of all x such that $\varphi_\alpha(x) < \min \{\varphi_{\alpha 0}(x), \ldots, \varphi_{\alpha n}(x)\}$ for all $\alpha \neq \alpha_1, \ldots, \alpha_i$. Each set W_{ia} is open, and $W_{ia} \cap W_{ib} = \emptyset$ if $a \neq b$. Also W_{ia} is contained in the intersection of the supports of $\varphi_{\alpha 0}(x), \ldots, \varphi_{\alpha i}(x)$, and hence in some set V_α. If we set X_i equal to the union of all sets W_{ia}, for fixed i, $\xi \mid X_i$ is trivial. Note that $\xi \mid W_{ia}$ is trivial and W_{ia} are disjoint.

Finally, the sets $X_0, \ldots, X_n$ cover B. Given x in B, x is contained in at most $n + 1$ of the sets V_α, so that at most $n + 1$ of the functions φ_α are positive at x. Since some φ_α is positive at x, x is contained in one of the sets W_{ia} for $0 \leq i \leq n$.
[The intuitive idea of the proof is as follows: Consider an n-dimensional simplicial complex, with φ_α the barycentric coordinate of x with respect to the vertex α. The sets W_{0a} will be disjoint neighbourhoods of the vertices, the sets W_{1a} disjoint neighbourhoods of the open 1-simplices, and so on.] □

2.20. *Theorem.* *If ξ is of finite type, there is a bundle η such that $\xi \oplus \eta$ is trivial.*

Proof: We proceed by showing that ξ may be embedded in a trivial bundle $B \times \mathbb{R}^m$, so that we have the exact sequence $0 \to \xi \xrightarrow{i} B \times \mathbb{R}^m \to B \times \mathbb{R}^m / i(\xi) \to 0$ by 2.12. The theorem then follows from 2.15. (Paracompactness is not needed since the trivial bundle clearly has a Riemannian metric.)
Cover B by finitely many neighbourhoods $U_1, \ldots, U_k$ with $\xi \mid U_i$ trivial for each i. Let $\varphi_1, \ldots, \varphi_k$ be a partition of unity with support$(\varphi_i) \subset U_i$ for each i (2.14). Let f_i denote the equivalence of $E(\xi \mid U_i)$ onto $U_i \times \mathbb{R}^n$; let $f_i^1, \ldots, f_i^n$ denote the coordinate functions of its projection into $\mathbb{R}^m$.
We define $h : E(\xi) \to B \times \mathbb{R}^{mk}$ as follows:

$$h(e) = (\pi(\mathrm{e}), (\varphi_i \, \pi(e)) \cdot f_i^j(e)) \quad i = 1, \ldots, k; \quad j = 1, \ldots, n$$

(no summation indicated.) This is well-defined, since $\varphi_i \, \pi(e) = 0$ unless $e \in E(\xi \mid U_i)$. It is clearly a homomorphism, since each f_i^j is linear on $E(\xi \mid U_i)$. To show that it is 1 - 1, let $e \neq 0$. Then for some i, $\varphi_i \pi(e) > 0$. Since f_i is an equivalence, $f_i^j(e) \neq 0$ for some j. Hence $h(e) \neq (\pi(\mathrm{e}), 0)$ as desired. □

2.21. *Definition.* The bundle ξ is ***s-equivalent***[2] to η if there are trivial bundles o^p, o^n such that $\xi \oplus o^p \cong \eta \oplus o^n$.

Here $o^p = B \times \mathbb{R}^p$. Symmetry and reflexivity are clear. To show transitivity, assume $\xi \oplus o^p \cong \eta \oplus o^q$ and $\eta \oplus o^r \cong \zeta \oplus o^s$. Then $\xi \oplus o^p \oplus o^r \cong \zeta \oplus o^s \oplus o^q$.

Remark: s-equivalence differs from from equivalence. E.g., consider the two-sphere S^2 in $\mathbb{R}^3$. Then $\tau^2 \oplus \nu^1 \cong o^3$. The normal bundle ν^1 is easily seen to be trivial; but it is a classical theorem of topology that τ^2 is not (it does not admit a non-zero cross-section.) Hence τ^2 is s-trivial, but not trivial.

2 Short for "stably equivalent".

2.22. *Theorem.* *The set of s-equivalence classes of vector space bundles of finite type over B forms an abelian group under* $\oplus$[3].

***Proof*:** To avoid logical difficulties, we consider only subbundles of $B \times \mathbb{R}^m$, for all m. This suffices, since any bundle of finite type may be embedded in some $B \times \mathbb{R}^m$, by 2.20. The class o^p of trivial bundles is the identity element. The existence of inverses is the substance of 2.20. □

2.23. *Corollary.* *Given two immersions of the differentiable manifold M in euclidean space, their normal bundles are s-equivalent.* □

2.24. *Definition.* M^n is a ***π-manifold*** if M may be embedded in some $\mathbb{R}^{n+p}$ so that its normal bundle is trivial.

This is equivalent to the requirement that τ^n be s-trivial; Let τ^n be s-trivial. If we take some immersion of M into $\mathbb{R}^{n+p}$, then $\tau^n \oplus \nu^p$ is trivial by 2.16, so that ν^p is s-trivial, i.e., $\nu^p \oplus o^q = o^{p+q}$ for some q. Consider the composite immersion $M \to \mathbb{R}^{n+p} \subset \mathbb{R}^{n+p+q}$. The normal bundle of M in $\mathbb{R}^{n+p+q}$ is just $\nu^p \oplus o^q$, which is trivial.
Conversely, if ν^p is trivial for some immersion, then τ^n is s-trivial because $\tau^n \oplus \nu^p$ is trivial.

2.25. *Definition.* Let $G_{p,n}$ denote the set of all n-dimensional vector subspaces of $\mathbb{R}^{n+p}$ (i.e., all n-dimensional hyperplanes through the origin.) It is called the ***Grassman manifold*** of n-planes in $n + p$ space.
Its topology is obtained as follows; Consider $\mathcal{M}(n, n+p; n)$; we identify two elementss of this set if the hyperplane spanned by their row vectors are the same. $G_{p,n}$ is in 1 - 1 correspondence with this identification space, and is given the identification topology. Let ρ be the projection

$$\rho : \mathcal{M}(n, n+p; n) \to G_{p,n}.$$

Now $\rho(A) = \rho(B)$ if and only if $A = CB$ for some non-singular $n \times n$ matrix C: The hyperplane $\rho(A)$ consists of all points $(x^1, \ldots, x^{n+p})\ \mathbb{R}^{n+p}$ which equal $(c^1, \ldots, c^n)\cdot A$ for some choice of constants c^i. If $\rho(A) = \rho(B)$, then

$$\begin{aligned}(1, 0, \ldots, 0)\cdot A &= (c^1{}_1, \ldots, c^n{}_1)\cdot B\\ (0, 1, \ldots, 0)\cdot A &= (c^1{}_2, \ldots, c^n{}_2)\cdot B\\ \cdots &= \cdots\\ (0, 0, \ldots, 1)\cdot A &= (c^1{}_n, \ldots, c^n{}_n)\cdot B\end{aligned}$$

for some choice of $c^j{}_i$. Then $IA = CB$, where C has rank n because A does. The converse is clear.

(a) $G_{p,n}$ is locally euclidean. Let $A \in \mathcal{M}(n, n+p; n)$; after permuting the columns, we may assume $A = (P, Q)$ where P is $n \times n$ and non-singular. Let U be the set of all such A; it is an open set in $\mathcal{M}(n, n+p; n)$, being the inverse image of the non-zero reals under the continuous map $(P, Q) \to \det P$. If $\rho(P, Q) = \rho(R, S)$, where P is non-singular, then $(P, Q) = (CR, CS)$ for some non-singular C. Hence R is necessarily non-singular; it follows that $\rho^{-1}(\rho(U)) = U$, so that $\rho(U)$ is open in $G_{p,n}$ (by definition of the identification topology.)

We show $\rho(U)$ homeomorphic with $\mathbb{R}^{pn}$. Define $\varphi : U \to \mathbb{R}^{pn}$ by $\varphi(P, Q) = P^{-1}Q$. If $\rho(P, Q) = \rho(R, S)$

3 The resulting abelian group is called the K-group of B. For more on this, see "*Vector Bundles and K-Theory*" by Allen Hatcher in his homepage http://www.math.cornell.edu/~hatcher/#ATI.

then $(P, Q) = (CR, CS)$, so that

$$P^{-1}Q = (CR)^{-1}(CS) = R^{-1}S.$$

Hence φ induces a continuous map $\varphi_0 : \rho(U) \to \mathbb{R}^{pn}$. Define $\psi : \mathbb{R}^{pn} \to \rho(U)$ by $\psi(Q) = \rho(I, Q)$ where Q is an $n \times p$ matrix. One checks immediately that ψ and φ_0 are inverse of each other.

$$\begin{array}{ccccc} \mathcal{M}(n, n+p; n) & \supset & U & & \\ \downarrow\rho & & \downarrow\rho & \varphi_0 & \searrow\varphi \\ G_{p,n} & \supset & \rho(U) & \longleftrightarrow & \mathbb{R}^{pn} \\ & & & \psi & \end{array}$$

(b) To show that $G_{p,n}$ is Hausdorff, we show that maps every compact set into a closed set (this will clearly suffice.) Let K be a compact subset of $\mathbb{R}^{pn}$; we show $\varphi^{-1}(K)$ is closed in $\mathcal{M}(n, n+p; n)$. $\varphi^{-1}(K)$ consists of all matrices (P, Q) with P non-singular and $P^{-1}Q \in K$. Let $(P, Q) \in \mathcal{M}(n, n+p; n)$ be the limit of the sequence $\{(P_i, Q_i)\}$ of elements of $\varphi^{-1}(K)$. Since K is compact, some subsequence of the sequence $\{\varphi(P_i, Q_i)\} = \{P_i^{-1}Q_i\}$ converges to a point R of K. Then the corresponding subsequence of the sequence $\{Q_i\}$ converges to PR, so that $C = P(I, R)$. Since (P, Q) has rank n it follows that P is non-singular, so that $(P, Q) \in \varphi^{-1}(K)$, as desired.
Hence $G_{p,n}$ is a manifold of dimension pn.

(c) $G_{p,n}$ is a differentiable manifold and ρ is a differentiable map. A function f on the open set V in $G_{p,n}$ belongs to the differentiable structure $\mathcal{D}$ if $f\rho$ is differentiable. To show that this satisfies the condition for a differentiable structure, we show that $(\rho(U), \varphi_0)$, as defined in (a), is a coordinate system. Let f be defined on $V \subset \rho(U)$. Given $Q \in \mathbb{R}^{pn}$, $f\varphi_0^{-1}(Q) = f\rho(I, Q)$ so that $f\varphi_0^{-1}$ is differentiable if $f\rho$ is. Conversely, given $(P, Q) \in V$, $f\rho(P, Q) = f\varphi_0^{-1}\varphi_0\rho(P, Q) = f\varphi_0^{-1}(P^{-1}Q)$, so that $f\rho$ is differentiable if $f\varphi_0^{-1}$ is.

(d) $G_{p,n}$ is compact. Let L be the subset of $\mathcal{M}(n, n+p; n)$ consisting of matrices whose rows are orthonormal vectors. L is a closed and bounded subset of $\mathbb{R}^{n(n+p)}$. Since $\rho(L) = G_{p,n}$ (the Gram-Schmidt orthogonalisation process proves this), $G_{p,n}$ is compact.

(e) $G_{p,n}$ is diffeomorphic to $G_{n,p}$. Geometrically, the homeomorphism h is defined as carrying each hyperplane into its orthogonal complement. It is clearly 1 - 1; to show it is differentiable we use the coordinate system $(\rho(U), \varphi_0)$ defined in (a). Let g map U into $\mathcal{M}(n, n+p; n)$ by carrying (P, Q) into $(-(P^{-1}Q)^\tau, I_p)$; it is differentiable (τ denotes transpose.) The row space of (P, Q) is the same as that of $(I_n, P^{-1}Q)$, while the row vectors of this matrix are orthogonal to those of $(-(P^{-1}Q)^\tau, I_p)$ (multiply the one by the transpose of the other.) Hence g induces $h \mid \rho(U)$, so that the latter is differentiable.

2.26. *Definition*. Let $E(\gamma_p{}^n)$ be defined as that subsets of $G_{p,n} \times \mathbb{R}^{n+p}$ consisting of pairs (H, x) where x is a vector lying in the hyperplane H. It is called the ***universal bundl***e (for reasons we shall see.) The projection π maps (H, x) into H; the fibre is thus an n-dimensional subspace of $\mathbb{R}^{n+p}$.

$\gamma_p{}^n$ is an n-dimensional vector space bundle over $G_{p,n}$. We need to show the existence of a local product structure. Let $(\rho(U), \varphi_0)$ be a coordinate neighbourhood on $G_{p,n}$, as in (a) above. We define $h : \rho(U) \times \mathbb{R}^n \to \pi^{-1}\rho(U)$ as carrying $(H, (x^1, \ldots, x^n))$ into $(x^1, \ldots, x^n)\cdot(I_n, Q)$ where $Q = \varphi_0(H)$. This is a vector in the hyperplane H; h is clearly an isomorphism on each fibre. Its inverse is continuous, since it sends $(H, (y^1, \ldots, y^{n+p}))$ in $G_{p,n} \times \mathbb{R}^{n+p}$ into $(H, (y^1, \ldots, y^n))$ in $\rho(U) \times \mathbb{R}^n$.

2.27. *Definition.* ξ is a ***differentiable vector space bundle*** if $E(\xi)$ and $B(\xi)$ are differentiable manifolds, and if the homeomorphisms

$$U \times \mathbb{R}^n \to \pi^{-1}(U)$$

which specify the local product structure can be chosen as diffeomorphisms.

It follows that $\pi : E \to B$ is differentiable of maximum rank. Note that B can be differentiably embedded in E by mapping b into the 0-vector of F_b. The normal bundle of this embedding is just ξ.
Examples of differentiable bundles include the tangent bundles of a manifold, the normal bundle of an immersed manifold, and the universal bundle γ_p^n above. In the latter case, $E(\gamma_p^n)$ is embedded differentiably in $G_{p,n} \times \mathbb{R}^{n+p}$.

2.28. *Theorem.* *Let ξ^n be an n-dimensional vector space bundle. The following conditions are equivalent*:

(a) *ξ is of finite type.*

(b) *There is a bundle η^p such that $\xi^n \oplus \eta^p$ is trivial.*

(c) *There is a bundle map $\xi^n \to \gamma_p^n$ for some p.* (*Thus the terminology* "*universal bundle*" *for γ_p^n.*)

Proof: We have already shown that (a) $\Longrightarrow$ (b) (2.20); the bundle η^p there constructed has dimension $n(k-1)$, where k is the number of elements in the covering $U_1, \ldots, U_k$ of $B(\xi) = B$ such that $\xi \mid U_i$ is trivial.

(b) $\Longrightarrow$ (c): Condition (b) means that ξ^n may be embedded in the trivial bundle $B(\xi) \times \mathbb{R}^{n+p}$; let f be this embedding. We wish to define g and g_B in the following diagram:

$$\begin{array}{ccc} E(\xi) & \xrightarrow{g} & E(\gamma_p^n) \\ \pi\downarrow & & \downarrow \\ B(\xi) & \xrightarrow[g_B]{} & G_{p,n} \end{array}$$

Since f is a 1 - 1 homomorphism, $f(F_b)$ is the cartesian product of b and an n-dimensional hyperplane H^n in $\mathbb{R}^{n+p}$; let $g_B(b) \equiv H^n$. If $e \in F_b$, then $f(e) = (b, x)$ where x is a vector in the hyperplane H^n; let $g(e) = (H^n, x)$ in $G_{p,n} \times \mathbb{R}^{n+p}$. Then $g(e)$ actually lies in the subset of $G_{p,n} \times \mathbb{R}^{n+p}$ which constitutes $E(\gamma_p^n)$. From rank considerations, g is automatically an isomorphism on each fibre.

It remains to show that g is continuous. Locally, g just looks like a map $U \times \mathbb{R}^n \to G_{p,n} \times \mathbb{R}^{n+p}$. We factor it into a continuous map $h : U \times \mathbb{R}^n \to \mathcal{M}(n, n+p; n) \times \mathbb{R}^{n+p}$ followed by the projection $\rho \times 1$ into $G_{p,n} \times \mathbb{R}^{n+p}$. Locally, f looks like a map $U \times \mathbb{R}^n \to B \times \mathbb{R}^{n+p}$. Let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$; we define $h(b, x)$ as $(A, p_2 f(b, x))$. Here p_2 projects $B \times \mathbb{R}^{n+p}$ onto its second factor and A is the matrix having $p_2 f(b, e_1), \ldots, p_2 f(b, e_n)$ as its rows. Then h is continuous and $(\rho \times 1)h$ equals g. (Note: The converse assertion , (c) implies (b), can be proved by the same argument.)

(c) $\Longrightarrow$ (a): Being compact, $G_{p,n}$ is covered by a finitely many neighbourhoods U_i with $\gamma_p^n \mid U_i$ trivial. (In fact, $(n+p)! \,/\, n!p!$ neighbourhoods will suffice.) If f is a bundle map $\xi^n \to \gamma_p^n$ then the sets $\{f_B^{-1}(U_i) = V_i\}$ cover B, and $\xi \mid V_i$ is equivalent to the bundle induced by $f_B : V_i \to G_{p,n}$ (the uniqueness part of 2.8.) Then $\xi \mid V_i$ is trivial (since it is induced from a trivial bundle.) □

Chapter III *The Cobordism Theory of Thom*

3.1. *Definition.* An ***n-manifold with boundary*** Q is a Hausdorff space with a countable basis which is locally homeomorphic with $\mathbb{H}^n$ (the subset of $\mathbb{R}^n$ such that $x^1 \geq 0$.) The ***boundary*** ∂Q is that subset of corresponding to $\mathbb{R}^{n-1}$ under the local homeomorphism ($\mathbb{R}^{n-1}$ being the subset of $\mathbb{R}^n$ with $x^1 = 0$.) ∂Q is well-defined, since the image of an open set in $\mathbb{R}^n$ under a homeomorphism of it into $\mathbb{R}^n$ must be open (Brouwer theorem on invariance of domain.) It is clear that ∂Q is an $(n-1)$-manifold.
A ***differential structure*** $\mathcal{D}$ on Q is a collection of real-valued functions f defined on open subsets of Q such that

1) Every point of Q has an open neighbourhood U and a homeomorphism h of U into an open subset of $\mathbb{H}^n$, such that f is in $\mathcal{D}$ if and only if fh^{-1} is differentiable. (f is defined on an open subset of U; fh^{-1} differentiable means that it may be extended to a neighbourhood of $h(U)$ ***in*** $\mathbb{R}^n$ so as to be differentiable.)

2) If U_i are open sets contained in the domain of f and $U = \cup U_i$, then $f \mid U \in \mathcal{D}$ if and only if $f \mid U_i \in \mathcal{D}$ for each i.

As before, $(U, h)$is called a ***coordinate system*** on Q, and one can define differentiable structure alternatively by means of coordinate systems.

We impose an additional condition on $\mathcal{D}$ in 3.2.

3.2. *Definition.* Let M_1, M_2 be compact differentiable n-manifolds. They are said to be in the same ***cobordism class*** ($M_1 \sim M_2$) if there is a compact differentiable $n + 1$ manifold-with-boundary Q such that ∂Q is diffeomorphic with the disjoint union of M_1 and M_2 (denoted by $M_1 + M_2$.)

Symmetry and reflexivity of this relation are clear. To show transitivity, we impose the additional condition on $\mathcal{D}$ that there is a neighbourhood U of ∂Q in Q which is diffeomorphic with $\partial Q \times [0, 1)$, the diffeomorphism being the identity on $\partial Q \times 0$. This is redundant, but we assume it to avoid proving it[4]. Transitivity follows:
Let $M_1 + M_2$ be diffeomorphic with ∂Q_1 and $M_2 + M_3$ be diffeomorphic with ∂Q_2; let h_1, h_2 be the diffeomorphisms. We form a new space Q_3 from $Q_1 \cup Q_2$ by identifying each point of $h_1(M_2)$ with its image under $h_2 h_1^{-1}$. There is then a homeomorphism of $M_2 \times (-1, 1)$ into this space which equals h_1 when restricted to $M_2 \times 0$, and is a diffeomorphism of $M_2 \times [0, (-1)^i)$ into Q_i for $i = 1, 2$. (It is derived from the postulated "product neighbourhoods" $\partial Q_i \times [0, 1)$.) If this is taken to be a coordinate system on Q_3, Q_3 becomes a differentiable manifold-with-boundary, and $M_1 + M_3$ is diffeomorphic with ∂Q_3. Q_1 and Q_2 diffeomorphic with subsets of Q_3.

3.3. *Definition.* As usual, there are logical difficulties involved in considering these cobordism classes. One way of avoiding them is to consider only manifolds-with-boundary embedded in some euclidean space $\mathbb{R}^n$: If Q_1 is a differentiable manifold-with-boundary and $Q_2 = \partial Q_1 \times [0, 1)$, then the space Q_3 constructed in the preceding paragraph is a differentiable manifold, so that it may be embedded in some euclidean space. Hence Q_1 may so be embedded.

With these restrictions, the set of cobordism classes of n-manifolds forms an abelian group (denoted

4 This fact is called the ***smooth collaring theorem***. See Appendix, Proposition B for a proof.

by $\mathcal{N}^n$) under the operation + (disjoint union.) If $M_1 \sim M_1'$ and $M_2 \sim M_2'$, this means that $M_i + M_i'$ is diffeomorphic with ∂Q_i. Then $(M_1 + M_2) + (M_1' + M_2')$ is diffeomorphic with $\partial(Q_1 \cup Q_2)$, so that $M_1 + M_2 \sim M_1' + M_2'$ and the operation + is well-defined on cobordism classes. The zero element is the vacuous manifold or the n-sphere (or ∂Q, where Q is any compact differentiable $(n + 1)$-manifold-with-boundary.) The remaining axioms are clear. Note that $M + M$ is diffeomorphic with $\partial(M \times [0, 1])$, so that every element is of order 2.

The groups $\mathcal{N}^n$ are called the (non-orientable) ***cobordism groups***. Let $\mathcal{N}$ denote the direct sum $\mathcal{N}^0 \oplus \mathcal{N}^1 \oplus \mathcal{N}^2 \oplus \cdots$. There is a bilinear symmetric pairing of $\mathcal{N}^i$, $\mathcal{N}^j$ into $\mathcal{N}^{i+j}$, i.e., a homomorphism of $\mathcal{N}^i \otimes \mathcal{N}^j$ into $\mathcal{N}^{i+j}$ induced by the operation of cartesian product.
First, $(M_1 + M_2) \times M_3 = (M_1 \times M_3) + (M_2 \times M_3)$ by definition of cartesian product. Second, if $M_1 \sim 0$, i.e., $M_1 = \partial Q$, then $M_1 + M_2$ is diffeomorphic with $\partial(Q \times M_2)$, so that $M_1 + M_2 \sim 0$.
Since $M_1 + M_2 \sim M_2 + M_1$, and since $M_1 \times p \sim M_1$ (where p is a point-manifold), this pairing makes $\mathcal{N}$ into a (graded) commutative ring with unit. Indeed, it is a graded algebra over the field $\mathbb{Z}/2\mathbb{Z}$.

3.4. ***Remark.*** The general result of Thom is the following

Theorem. *$\mathcal{N}$ is a polynomial algebra over $\mathbb{Z}/2\mathbb{Z}$ with one generator in each positive dimension except those of the form $2^m - 1$. If n is even, projective n-space is a generator.*

This theorem means that there are compact manifolds M^2, M^4, M^5, … such that every compact manifold is in the cobordism class of a disjoint union of products of these manifolds, and that there are no relations among the generators (except commutativity and associativity of products.)
Thom's procedure is to show that $\mathcal{N}^n$ is isomorphic with the $(n + k)^{th}$ homotopy group of a certain space T_k, and then to compute these homotopy groups. We shall consider only the first of these two problems in the present notes.

3.5. ***Definition.*** Let h be an embedding of the differentiable manifold M^n in $\mathbb{R}^{n+k}$; consider the normal bundle of this embedding. Using the standard Riemannian metric for the tangent bundle to $\mathbb{R}^{n+k}$, this normal bundle is equivalent to the orthogonal complement of the image in the tangent bundle of $\mathbb{R}^{n+k}$ of the tangent bundle of M^n (2.16); this complement we denote by ν^k. Define e as the canonical map of $E(\nu^k)$ into $\mathbb{R}^{n+k}$ which maps the vector v normal to at x into its end point. (Described differently, one maps the tangent bundle to $\mathbb{R}^{n+k}$ into itself canonically by mapping the vector v, based at x, into the point $v + x$ of $\mathbb{R}^{n+k}$. This map is differentiable; its restriction to $E(\nu^k)$ is the map e.)
Consider M^n as the zero vectors of $E(\nu^k)$. Then we have the

3.6. ***Theorem.*** *There is a neighbourhood of M^n in $E(\nu^k)$ which is mapped diffeomorphically onto a neighbourhood of M^n in $\mathbb{R}^{n+k}$.*

***Proof*:** Note that e is differentiable, and that it has rank $n + k$ at points of $M^n \subset E(\nu^k)$. (This is easily checked by computing the derivative matrix of e with respect to a local coordinate system.) Hence e has rank $n + k$ in some neighbourhood of M^n in $E(\nu^k)$, so that it is a local homeomorphism at points of M^n: It maps a neighbourhood of each $x \in M^n$ homeomorphically onto a neighbourhood of $e(x)$. We then appeal to the topological

Lemma. *Let X, Y be Hausdorff spaces with countable bases and X be locally compact. If $f : X \to Y$ is a local homeomorphism and the restriction of f to the closed subset A is a homeomorphism, then f is a homeomorphism on some neighbourhood V of A.*

This lemma is proved as follows:

1) If A is compact, the lemma holds. For otherwise, there would be points x, y arbitrarily close to A such that $f(x) = f(y)$. Since A has a compact neighbourhood, we may choose sequences $\{x_n\}$, $\{y_n\}$ converging to x, y respectively, in A such that $x_n \neq y_n$ and $f(x_n) = f(y_n)$. Hence $f(x) = f(y)$ so that $x = y$, f being a homeomorphism on A. But then f is not a local homeomorphism at x.
2) Let A_0 be a compact subset of A. Then there is a neighbourhood U_0 of A_0 such that $\overline{U}_0$ is compact and f is a homeomorphism on $\overline{U}_0 \cup A_0$: It will suffice for f to be 1 - 1, since f is a local homeomorphism. By (1), let V_0 be a neighbourhood of A_0 so that is $f\,|\,\overline{V}_0$ 1 - 1. If no neighbourhood of A_0 in V_0 satisfies the requirement for U_0, there is a sequence $\{x_n\}$of $X \setminus A$ converging to $x \in A_0$ with $f(x_n) \in f(A)$. Choose $y_n \in A$ with $f(x_n) = f(y_n)$. Since f is continuous, $\{f(y_n)\}$ converges to $f(x)$; since f is a homeomorphism on A, $\{y_n\}$converges to x. Since $x_n \neq y_n$, this contradicts the fact that f is a local homeomorphism at x.
3) Express A as the union of an ascending sequence of compact sets $A_1 \subset A_2 \subset \cdots$. Let V_1 be a neighbourhood of A_1 such that $\overline{V}_1$ is compact and f is a homeomorphism on $\overline{V}_1 \cup A$ (by (2).) Given V_i a neighbourhood of A_i satisfying these conditions, consider the set $\overline{V}_i \cup A_{i+1}$. It is a compact subset of $\overline{V}_i \cup A$, and f is a homeomorphism on $\overline{V}_i \cup A$. Hence by (2) there is a neighbourhood V_{i+1} of $\overline{V}_i \cup A_{i+1}$ with $\overline{V}_{i+1}$ compact, such that f is a homeomorphism on $\overline{V}_i \cup A_{i+1}$. We proceed by induction: f is 1 - 1 on $V = \cup V_{i+1}$, so that it is a homeomorphism on V (being a local homeomorhism-onto.) □

3.7. *Corollary.* *Any differentiable submanifold of* $\mathbb{R}^{n+k}$ *is a differentiable neighbourhood retract.*

Proof: The projection of $E(\nu^k) \to M^n$ induces (under e) a differentiable map of a neighbourhood of M^n in $\mathbb{R}^{n+k}$ onto M^n which is the identity on M^n. □

3.8. *Definition.* Let ξ be a vector space bundle with $B(\xi)$ compact; Let $T(\xi)$ denote the 1-point compactification of $E(\xi)$. It is called the ***Thom space*** of ξ. Let ∞ denote the added point.

Let ξ have a Riemannian metric. Let $T_\varepsilon(\xi)$ be obtained from $E(\xi)$ by identifying all vectors of length greater than or equal to ε to a point. Let $\alpha(x)$ be a C^∞ function with $\alpha'(x) \geq 0$ which equals 1 in a neighbourhood of $x = 0$ and $\to \infty$ as $x \to 1$. The map of $E(\xi)$ into $T(\xi)$ which carries the vector e into the vector $e\alpha(\|e\| / \varepsilon)$ induces a homeomorphism of $T_\varepsilon(\xi)$ onto $T(\xi)$ which is a diffeomorphism on the set $E_\varepsilon(\xi)$, consisting of vectors of length less than ε. The fact that B is compact is used here.

3.9. *Definition.* Let the compact manifold M^n be embedded in $\mathbb{R}^{n+k}$. ν^k is given the Riemannian metric of $\mathbb{R}^{n+k}$; by 3.6 there is a neighbourhood of M^n in $\mathbb{R}^{n+k}$ which is diffeomorphic to the subset $E_{2\varepsilon}(\nu^k)$ of $E(\nu^k)$. Such a neighbourhood is called a ***tubular neighbourhood*** of M^n.

By 3.8, we see that $T(\nu^k)$ is homeomorphic with the space obtained from $\mathbb{R}^{n+k}$ by collapsing the exterior of the ε-neighbourhood of M^n to a point.

We will need three lemmas concerning approximation by differentiable functions.

3.10. *Lemma.* *Let A be a closed subset of the differentiable manifold M^n, let $f : M^n \to \mathbb{R}^m$ be differentiable on A. Let δ be a positive continuous function on M^n. There exists $g : M^n \to \mathbb{R}^m$ such that*

1) *g is differentiable,*
2) *g is a δ-approximation to f,*

3) $g \mid A = f \mid A$.

Proof: It suffices to prove this lemma in the case $m = 1$.
Given $x \in A$, $f \mid A$ may be extended to a differentiable function f_x in a neighbourhood N_x of x. Let N_x be chosen small enough that $|f_x(y) - f(y)| < \delta(y)$ for all $y \in N_x$.
Given $x \in M^n \setminus A$, choose a neighbourhood N_x of x small enough that $|f(y) - f(x)| < \delta(y)$ for all $y \in N_x$. Define $f_x(y) \equiv f(x)$ for $y \in N_x$.
Let $\{\varphi_\alpha\}$ be a differentiable partition of unity with support(φ_α) contained in some N_x, say $N_{x(\alpha)}$, for each α. Define $g(y) = \sum_\alpha \varphi_\alpha(y) f_{x(\alpha)}(y)$. One checks the conditions of the lemma easily. □

More generally:

3.11. *Lemma.* *Let $f : M_1 \to M_2$ be a continuous map of differentiable manifolds which is differentiable on the closed subset A of M_1. Let $\varepsilon(x) > 0$ be given; and give M_2 the metric determined by some embedding $M_2 \subset \mathbb{R}^p$. Then there exists a differentiable map $g : M_1 \to M_2$ such that*

1) *g is differentiable,*
2) *g is an ε-approximation to f,*
3) $g \mid A = f \mid A$.

Proof: There is a neighbourhood U of M_2 in $\mathbb{R}^p$ of which is a differentiable retract (3.7.) Let ρ be the differentiable retraction of U onto M_2. Let $\delta(x)$ be a positive function on M_2 so chosen that the cubical neighbourhood of $f(x)$ of radius $\delta(x)$ lies in U, and so that its image under ρ has radius less than $\varepsilon(x)$. Let $f_1 : M_1 \to \mathbb{R}^p$ be a differentiable map which is a δ-approximation to f, such that $f_1 \mid A = f \mid A$ (by 3.10.) Define $g(x) = \rho(f_1(x))$. □

3.12. *Lemma.* *Let $f : M_1 \to M_2$ be a continuous map of differentiable manifolds; let the metric on M_2 be obtained by embedding it in some euclidean space. Given $\varepsilon(x)$, there is a $\delta(x)$ such that if $g : M_1 \to M_2$ is a δ-approximation to f, g is homotopic to f under a homotopy $F(x, t)$ with*

1) *$F(x, t) = f(x)$ for any x such that $g(x) = f(x)$ and*
2) *$F(x, t)$ is a an ε-approximation to f for any t.*

Proof: Let U, ρ, and $\delta(x)$ be chosen as in 3.11. Let $g : M_1 \to M_2$ be a δ-approximation to f. Then the line segment from $g(x)$ to $f(x)$ lies in U, so that

$$F(x, t) = \rho(tg(x) + (1 - t)f(x))$$

is well defined. Furthermore $F(x, t)$ is an ε-approximation to $f(x)$ for any t. □

3.13. *Definition.* We wish to define a homomorphism $\lambda : \pi_{n+k}(T(\xi^k), \infty) \to \mathcal{N}^n$ where $\mathcal{N}^n$ is the cobordism class of the base space for $T(\xi^k)$. To this end we need some preparation:

Let ξ^k be a differentiable vector space bundle with $B(\xi)$ compact and m-dimensional; let $E(\xi^k)$ be given a metric by embedding it as a closed differentiable submanifold in some euclidean space (it is an $(m + k)$-manifold.)
Given an element of $\pi_{n+k}(T(\xi^k), \infty)$, let it be represented by the map

$$f : (\overline{C}_{n+k}, \partial\overline{C}_{n+k}) \to (T(\xi^k), \infty),$$

where $\overline{C}_{n+k}$ is the closed cube $[0, 1]^{n+k}$ and $\partial\overline{C}_{n+k}$ is the boundary. Let U denote the open subset

$f^{-1}(E(\xi^k))$ of C_{n+k}. Let $g : U \to E(\xi^k)$ be a differentiable δ-approximation to $f \mid U$, where δ is so chosen that $\delta < 1$ and g is homotopic to f, the homotopy F also being a 1-approximation to f. (This ensures that F will be continuous if we define $F(x, t) = \infty$ for $x \in \overline{C}_{n+k} \setminus U$.)

Now g may be approximated in turn by a differentiable map $h : U \to E(\xi^k)$ which is transverse regular on the submanifold $B(\xi^k)$ of $E(\xi^k)$. We choose the approximation close enough to h, the homotopy H being a 1-approximation to g for each t. Extend h to $\overline{C}_{n+k}$ by defining $h(x) = \infty$ for $x \in \overline{C}_{n+k} \setminus U$. Then h is in the homotopy class of f.
$h^{-1}(B(\xi^k))$ is a differentiable submanifold M^n of U which is closed in $\overline{C}_{n+k}$, and thus compact.

3.14. *Theorem.* *Define* $\lambda : \pi_{n+k}(T(\xi^k), \infty) \to \mathcal{N}^n$ *by assigning* the cobordism class $[M^n] \in \mathcal{N}^n$ *to the homotopy class* $[h] \in \pi_{n+k}(T(\xi^k), \infty)$. *Then* λ *is a well-defined homomorphism.*

***Proof*:** Let $H : (\overline{C}_{n+k} \times I, \partial\overline{C}_{n+k} \times I) \to (T(\xi^k), \infty)$ be a homotopy between $h_0 = H(x, 0)$ and $h_1 = H(x, 1)$. Let h_0, h_1 satisfy the conditions

1) h_i is differentiable on $h_i^{-1}(E(\xi^k))$
2) h_i is transverse regular on $B(\xi^k)$. ($i = 0, 1$.)

We wish to show that $h_0^{-1}(B)$ and $h_1^{-1}(B)$ belong to the same cobordism class.
We may assume that $H(x, t) = H(x, 0)$ for $t \leq 1/3$, and $H(x, t) = H(x, 1)$ for $t \geq 2/3$. Let $U = H^{-1}(E(\xi^k)) \cap [\overline{C}_{n+k} \times (0, 1)]$; then U is an open subset of $\mathbb{R}^{n+k+1}$. Let $G : U \to E(\xi^k)$ be a differentiable 1-approximation to H which equals H on the closed subset A, where $A = U \cap [\overline{C}_{n+k} \times (0, 1/4] \cup [3/4, 1)]$. (See 3.11. H is differentiable on A.)

Now G satisfies the transverse regularity condition for $B(\xi^k)$ at points in A (since h_0 and h_1 are transverse regular on $B(\xi^k)$) so that by 1.35 there is a differentiable map $F : U \to E(\xi^k)$ which equals G on A, is transverse regular on $B(\xi^k)$, and is a 1-approximation to G. Because F is a 2-approximation to H, it remains continuous if we define $F(x, t) = \infty$ for $(x, t) \in (\overline{C}_{n+k} \times (0, 1)) \setminus U$. Because F equals H on A, it remains continuous if we define $F(x, t) = H(x, t)$ for $t = 0, 1$. Hence $F^{-1}(B)$ is a compact subset of $\overline{C}_{n+k}$, being closed and bounded.

Because $F \mid U$ is transverse regular on B, $(F \mid U)^{-1}(B)$ is a differentiable $(n + 1)$-submanifold of $\overline{C}_{n+k} \times (0, 1)$. Then

$$(F \mid U)^{-1}(B) \cap \overline{C}_{n+k} \times t = \begin{cases} h_0^{-1}(B) \times t & \text{for } t \in [0, 1/4], \\ h_1^{-1}(B) \times t & \text{for } t \in [3/4, 1]. \end{cases}$$

Hence $F^{-1}(B)$ is a differentiable manifold-with-boundary whose boundary is $h_0^{-1}(B) + h_1^{-1}(B)$. Thus λ is well-defined.
It is trivial to show λ is a homomorphism, because the sum in $\mathcal{N}^n$ is derived from disjoint union of representative manifolds. □

3.15. *Theorem.* *If* ξ^k *is the universal bundle* γ_m^k *where* $k \geq n + 1$, $m \geq n$ *then* $\lambda : \pi_{n+k}(T(\xi^k), \infty) \to \mathcal{N}^n$ *is onto.*

***Proof*:** Let M^n be a compact manifold; let $k \geq n + 1$. Let M^n be embedded in C_{n+k} (1.32); let ν^k be the normal bundle of this embedding. The Riemannian metric on $E(\nu^k)$ is that derived from the natural scalar product on the tangent bundle to $\mathbb{R}^{n+k}$, in which ν^k is contained.
By 3.6, for small ε the subset of $E_{2\varepsilon}(\nu^k)$ of $E(\nu^k)$ is diffeomorphic with a tubular neighbourhood of M^n in C_{n+k}; let U be the image of $E_\varepsilon(\nu^k)$.

Let p_1 project $\overline{C}_{n+k}$ onto the space obtained from $\overline{C}_{n+k}$ by identifying $\overline{C}_{n+k} \setminus U$ to a point (denoted by $\overline{C}_{n+k} / (\overline{C}_{n+k} \setminus U)$).
Let p_2 be the diffeomorphism of U onto $E_\varepsilon(\nu^k)$, followed by the map of $E(\nu^k)$ into $T_\varepsilon(\nu^k)$ which identifies all vectors of length $\geq \varepsilon$ (3.8.) p_2 is then extended by mapping $\overline{C}_{n+k} \setminus U$ into ∞.
Let p_3 be the homeomorphism of $T_\varepsilon(\nu^k)$ onto $T(\nu^k)$ constructed in 3.8. The composite map $p_3p_2p_1$ is a diffeomorphism of U onto $E(\nu^k)$.
Finally, let p_4 be the bundle map of ν^k into γ_m^k induced from the embedding of M^n in $\mathbb{R}^{n+k} \subset \mathbb{R}^{m+k}$.
Because both fibres have dimension k, this map satisfies the transverse regularity condition for $G_{k,m}$ at each point of M^n. Extend p_4 in the obvious way to map $T(\nu^k)$ into $T(\gamma_m^k)$.

Let $g = p_4p_3p_2p_1$. Then $g : \partial\overline{C} \to \infty$. Let $\mu(M^n)$ denote the homotopy class of g in $\pi_{n+k}(T(\xi^k), \infty)$.
Now g is transverse regular on $G_{k,m}$ and $M^n = g^{-1}(G_{k,m})$. By definition, the cobordism class of M^n is the image of $\mu(M^n)$ under λ, so that $\lambda\mu(M^n) = [M^n]$. □

3.16. *Theorem*. *If ξ^k is the universal bundle γ_m^k where $k \geq n + 2$, $m > n$ then λ is one-to-one.*

Proof: Given an element of $\pi_{n+k}(T(\xi^k), \infty)$, we may suppose it represented by a map

$$f : (\overline{C}_{n+k}, \partial\overline{C}_{n+k}) \to (T(\xi^k), \infty)$$

which is differentiable on $f^{-1}(E)$ and transverse regular on $G_{m,k}$ (by 3.13.) Let $M^n = f^{-1}(G_{m,k})$; we wish to show that if M^n is the boundary of an $(n + 1)$-manifold-with-boundary Q, then f is homotopic to the constant map.

M^n is a submanifold of C_{n+k}; let its normal bundle be ν^k. Let ε be chosen so that $E_{2\varepsilon}(\nu^k)$ is diffeomorphic with the 2ε-neighbourhood of M^n; let U_ε be the image of the vectors of $E_\varepsilon(\nu^k)$.
Impose a Riemannian metric on γ_m^k; let δ be so chosen that $||x|| \geq \varepsilon$ implies $||f(x)|| \geq \delta$ for $x \in E(\nu^k)$.

***Step* 1**. f is homotopic to a map f_1 such that

1) f_1 is differentiable on $f_1^{-1}(E)$ and transverse regular on $G_{m,k}$.
2) $f = f_1$ on $M^n = f^{-1}(G_{m,k})$.
3) f_1 carries everything outside U_ε into ∞.

Define $F : E(\gamma_m^k) \to T(\gamma_n^k)$ by the equation $F(e, t) = e\alpha(t||e|| / \delta)$, where α is the function defined in 3.8. Let $f_1(x) = F(f(x), 1)$.

***Step* 2**. By the diffeomorphism of $U_{2\varepsilon}$ with $E_{2\varepsilon}$, f_1 induces a map $\bar{f}_1$ of $\overline{E}_\varepsilon(\nu^k)$ into $T(\gamma_n^k)$ which carries $\partial(E_\varepsilon)$ into ∞. Any homotopy of $\bar{f}_1$ which leaves $\partial(E_\varepsilon)$ at ∞ induces a homotopy of f_1.
Now $\bar{f}_1$ is homotopic to a map $\bar{f}_2$ such that

1) $\bar{f}_2$ is differentiable on $\bar{f}_2^{-1}(E)$ and transverse regular on $G_{m,k}$.
2) $\bar{f}_2 = \bar{f}_1$ on $M^n = f^{-1}(G_{m,k})$.
3) $\bar{f}_2$ *is* locally a bundle map in some neighbourhood of M^n.

The homotopy leaves $\partial(E_\varepsilon)$ at ∞.

Consider $G : \overline{E}(\gamma_m^k) \times I \to T(\gamma_n^k)$ defined by the equation $G(e, t) = \bar{f}_1(te) / t$. As $t \to 0$, $G(e, t)$ approaches a limit which is non-zero if $e \neq 0$ (since $\bar{f}_1$ is differentiable and transverse regular.) It is easily seen to be a bundle map. It will not suffice for our purpose, since it does not carry $\partial(E_\varepsilon) \times I$ into ∞. Choose $\delta > 0$ so that $||x|| \geq \varepsilon$ implies $||G(x, t)|| \geq \delta$ for $x \in E(\nu^k)$, $t \in I$, and define

$$H(e, t) = [G(e, t)]\alpha(-||G(e, t)|| / \delta).$$

If we set $\bar{f}_2 = H(e, 0)$, then $\bar{f}_2$ is a bundle map for $\|e\|$ small (since $\alpha(x) = 1$ for x small.) The map $H(e, 1) = \bar{f}_1(e)\alpha(\|\bar{f}_1(e)\| / \delta)$ does not equal $\bar{f}_1$, but it is homotopic to $\bar{f}_1$, the homotopy leaving $\partial(E_\varepsilon)$ at ∞. The homotopy is defined by the equation

$$K(e, t) = \bar{f}_1(e)\alpha(t\|\bar{f}_1(e)\| / \delta), \text{ as in Step 1.}$$

***Step* 3**. Let Q be the $n + 1$ manifold-with-boundary such that $M^n = \partial Q$. Let h be a diffeomorphism of $M^n \times [0, 1]$ into Q which carries $M^n \times 0$ onto ∂Q.
Define $h_1 : Q \to C_{n+k} \times I$ as follows:

$h_1(x) = h(y, t)$ if $x = h(y, t)$ where $(y, t) \in (M^n, [0, ½])$.
$h_1(x) = p$, where p is some fixed point interior to $C_{n+k} \times I$ if $x \notin$ image h.
$h_1(x) = (1- \beta(t))h(y, ½) + \beta(t)p$, where β is a C^∞ function with $\beta'(t) \geq 0$, $\beta(t) = 0$ in a neighbourhood of $t = ½$ and $\beta(t) = 1$ in a neighbourhood of $t = 1$ if $x = h(y, t)$ where $(y, t) \in (M^n, [½, 1])$.

h_1 is a differentiable map of Int Q into Int$(C_{n+k} \times I)$; and h_1 is a 1 - 1 immersion in a neighbourhood of ∂Q. Since $\dim(C_{n+k} \times I) > 2(n + 1)$, h_1 may be approximated by a 1 - 1 immersion h_2 which equals h_1 in a neighbourhood of ∂Q (by 1.29.) It may be extended to an embedding of Q into $C_{n+k} \times I$. (Since Q is compact, a 1 - 1 immersion is automatically an embedding.) Let Q now be considered as this subset of $C_{n+k} \times I$.

***Step* 4**. We have a map f_2 of $\bar{C}_{n+k} \times 0$ into $T(\gamma_n^k)$ which is a bundle map when restricted to a small tubular neighbourhood of $M^n \times 0$ in $C_{n+k} \times 0$. We extend it to $\bar{C}_{n+k} \times [0, b)$ for b small in a trivial way. Suppose there exists a map g of the ε'-neighbourhood N of Q in $C_{n+k} \times I$ into $T(\gamma_n^k)$ which equals f_2 in some neighbourhood of ∂Q in $C_{n+k} \times I$ and maps each point of $N \setminus Q$ into a non-zero vector in $E(\gamma_n^k)$. Our theorem then follows: Let δ be so chosen that, if the distance$(x, Q) \geq \varepsilon'/2$, then $\|g(x)\| \geq \delta$.

Define $g_1 : C_{n+k} \times I \to T(\gamma_n^k)$ by the equation

$$g_1(x, s) = \begin{cases} g(x, \varepsilon)\alpha(\|g(x, s)\| / \delta) & \text{for } (x, s) \in N, \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

The restriction of g_1 to $C_{n+k} \times 0$ does not equal the map f_2, but it is homotopic to f_2, by the same technique as used at the end of Step 2. g_1 is the homotopy required for our theorem.

To show that the extension g exists, we refer to Steenrod, "Fibre Bundles" (Princeton University Press, 1951.) According to §19.4 and §19.7 of this book, the principal bundle associated with γ_n^k is an m-universal bundle. That is: given a vector space bundle ξ^k over a complex of dimension $\leq m$, any bundle map of ξ^k, restricted to a subcomplex, into γ_n^k can be extended throughout ξ^k. We will assume the well known result that Q can be triangulated. The dimension $n + 1$ of Q is $\leq m$. Hence any bundle map of the normal bundle ν^k of Q, restricted to a polyhedral neighbourhood of ∂Q, into γ_n^k can be extended throughout ν^k.

Applying this result to the map f_2, this completes the proof of 3.16. □

Letting T_k stand for the union of the Thom spaces $T(\gamma_n^k) \subset T(\gamma_{n+1}^k) \subset \cdots$, in the weak topology, Theorem 3.15 and 3.16 imply the following.

3.17. *Theorem.* *The cobordism group $\mathcal{N}^n$ is canonically isomorphic to the stable homotopy group $\pi_{n+k}(T_k)$, for $k \geq n + 2$.* □

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Appendix[5]

In this appendix we give a proof for the smooth collaring theorem. Our exposition follows Dirk Schütz. (See "Lecture06_handout.pdf" in the "material" for MAGIC002, in "courses" listed in the page "http://maths.dept.shef.ac.uk/magic/courses.php".)

First we show that partitions of unity allow us to glue together smooth functions which are only defined on subsets of a differentiable manifold M.

Proposition A: *Let* $\{U_\alpha\}$ *be an open cover of the differentiable manifold* M *and* $\{\varphi_\alpha\}$ *a partition of unity with* support$(\varphi_\alpha) \subset U_\alpha$. *For every* α, *assume that* $f_\alpha : U_\alpha \to \mathbb{R}^k$ *is a smooth function. Then* $f : M \to \mathbb{R}^k$ defined by

$$f(x) = \textstyle\sum_\alpha f_\alpha(x)\varphi_\alpha(x)$$

is a well defined smooth function.

Proof: Observe that $f_\alpha \cdot \varphi_\alpha : U_\alpha \to \mathbb{R}^k$ has support contained in support(φ_α), so can be extended to a smooth function on M. Also, by the local finiteness, the formula for f is locally just a finite sum, so smoothness follows. □

The same procedure can be used to extend vector fields defined on each V_i to a vector field on M.

Proposition B (***Smooth Collaring Theorem***): *Let* M *be a compact differentiable manifold with boundary. Then there exists an embedding* $i : \partial M \times [0, 1) \to M$ *with* $i(x, 0) = x$ *for all* $x \in \partial M$.

Proof: Let $U_1, \ldots, U_k$ be a finite covering of M by coordinate charts, and let $\{\varphi_i : U_i \to [0, 1]\}$ be a partition of unity subordinate to this cover.
Case I: U_i is diffeomorphic to an open set of $\mathbb{R}^n$. Define a vector field v_i on U_i to be identically zero.
Case II: U_i contains boundary points. Let $\varphi_i : U_i \to U_i'$ be a chart, and define a vector field v_i on U_i such that the induced vector field on $U_i' \subset \mathbb{H}^n$ is constant $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$.

We get a vector field on M by using the partition of unity. Let Φ be the corresponding flow. As M is compact, and since the vector field is chosen on the boundary so that it is not possible to flow "out" of the manifold, we get a smooth flow

$$\Phi : M \times [0, \infty) \to M$$

It is easy to check that $\Phi \mid \partial M \times [0, 1)$ is the desired embedding. □

5 Added by the transcriber.