APPENDIX.

1. Demonstration of Theorems A, B, C, &c., page 484 of *Transactions*, Royal Society of Edinburgh, vol. xxxii. By Rev. T. P. Kirkman, M.A., F.R.S.

1. In the circle of an unifilar knot every crossing a is read twice, once in an odd and once in an even place; and the thread is supposed to pass under and over itself alternately at successive crossings

Every contiguous duad of the circle is a different edge of the knot. Let every mid-edge round the circle be dotted on the right.

Let $$128 \ldots a213 \ldots 5, \quad \ldots \quad (A)$$

of $2n$ terms 1, 2, &c., where 51 is a contiguous duad, be the circle of an unifilar $_{n}$N of n crossings. We see that 1 and 2 are the crossings of a 2-gon; for no mesh of $_{n}$N but a 2-gon can have two summits joined by two different edges of the mesh. (A) is the circle of any unifilar which has the duads 12 and 21.

Let us write the above thus, omitting only the crossing 1, and simply reversing one of the sequences between 1 and 1—

$$28 \ldots a25 \ldots 3, \quad \ldots \quad (B)$$

where 32 is a contiguous duad. What does this mean, when in the projection of $_{n}$N the 2-gon is shrunk up to a point 2, the edges 12 and 21 disappearing?

It is the circle of an unifilar $_{n-1}$M of $n-1$ crossings, which has every duad of (A) except 12 and 21; for it has 51 and 13 because it has 52 and 23, 2 and 1 being now the same point.

2. In (A) as we walk from 1 to 2, and later from 2 to 1, we make the circuit of the 2-gon 12 in the same direction round it; therefore our two dots will be both inside or both outside of it. This 12 is an *even* 2-gon; and every even 2-gon mn of an unifilar is known by a glance at the circle, by its exhibition of mn and nm.

3. We have demonstrated the following

Theorem A.—Every unifilar knot $_{n}$N of n crossings, which has an even 2-gon, can be reduced to an unifilar ${}_{n-1}$M of $n-1$ crossings by shrinking up that 2-gon to a point.

4. Let the sequence of $2n$ terms

$$128 \ldots a123 \ldots 5 \quad . \quad . \quad . \qquad \text{(C)}$$

be the circle of an unifilar ${}_n$P of n crossings.

This ${}_n$P has an *odd* 2-gon 12; for since we pass over both its threads from 1 to 2, it must have one dot within and one without it. Any unifilar that has an odd 2-gon is represented by (C). We now write this, omitting the duads 12 and no other, and simply reversing 3 . . . 5—one sequence between 2 and 1;

$$28 \ldots a15 \ldots 3 \quad . \quad . \quad . \qquad \text{(D).}$$

What means this, when from the projection of ${}_n$P we have deleted the two edges, and consequently the two crossings of the 2-gon 12?

5. The edges of the crossings 1 and 2 in (C) are 12, 1a, 12, 15 and 21, 28, 21, 23. These make angles 212 and 51a vertically opposite, and 121 and 823 vertically opposite, where 51a and 823 are angles of the $(4+r)$-gon F′ laid bare by deletion of the 2-gon 12. The points 1 and 2 of these angles are merely bends or creases at the mid-points of the edges 5a and 83 of the $(2+r)$-gon F. Effacing the creases 1 and 2, (D) becomes

$$8 \ldots a5 \ldots 3, \quad . \quad . \quad . \qquad \text{(E);}$$

which contains every crossing of (C) but 1 and 2, and every duad of (C) except 12, 1a, 15, 23, 28, and has besides the new edges 83 and 5a of the $(2+r)$-gon F. This (E) is the circle of an unifilar ${}_{n-2}$Q of $n-2$ crossings. We have thus proved

Theorem B.—If any unifilar knot ${}_n$P of n crossings has an odd 2-gon, the knot is reduced to an unifilar ${}_{n-2}$Q of $n-2$ crossings by the deletion of the two edges, and consequently of the two crossings of that 2-gon.

6. Let

$$\begin{array}{l} . \; . \; d765r \ldots p567s \\ . \; . \; d765r \ldots p765s \quad . \quad . \quad . \qquad \text{(F)} \end{array}$$

be two circles of two unifilars of n crossings. In both of them 56 and 67 are contiguous 2-gons, having a common crossing 6—that is, both knots have a double 2-gon (a plural flap) 567. Let the first 7 in each be in an even place; then the second 7 is in an odd place, art. 1. Let the thread cross over itself in the even places. In both circles (F) then the thread, passing under itself at d, 6, and r, goes onwards till it passes over itself at p, 6, and s, proceeding till it comes under itself again at d. Let now the double flap 765 in the projection of each circle be shrunk to a point 7, the four edges of its two 2-gonal loops 65 and 67 disappearing. These circles thus become the two unifilars of $n-2$ crossings,

$$\begin{array}{l} \ldots d7r \ldots p7s \ldots \\ \ldots d7r \ldots p7s \ldots \end{array} \quad . \quad . \quad \text{(G)},$$

which of course differ exactly as do the circles (F) in the portions omitted. The thread in each goes under at d, over at 7, under at r, and so on its course till it passes over at p, under at 7, over at s . . . and finally comes again under at d, as before the shrinking. And this is true whatever be the crossings d, r, p, s, in 2-gons or not, and whether these four be or not crossings in like meshes on the knots. Thus is proved

Theorem C.—If any unifilar knot $_nR$ of n crossings has a double flap, the two contiguous terminal 2-gons of a $(2+i)$-ple flap, $(i \geqq 0)$, it is reduced, by the shrinking up of the two 2-gons to a point, to an unifilar S of $n-2$ crossings.

7. Returning to the even 2-gon 12 of art. 1, it is plain that at its crossings it is covertical with two meshes whose edges, meeting at 1 and at 2, will be dotted both of each pair inside or outside its mesh, according as the 2-gon 12 has its dots both outside or both inside of it. And it is equally plain that every crossing r of a knot has two covertical angles about it so evenly dotted, and another covertical pair which have both one, and only one, dot inside those angles about r. We may call the latter pair the *odd* angles, and the former the *even* angles about r.

In the unifilar $_{n-1}M$ whose circle is (B), art. 1, we can reverse the process by which $_{n-1}M$ is obtained from $_nN$, *i.e.*, we can, by restoring in the projection of $_{n-1}M$ the deleted 2-gon 12, construct upon it $_nN$. In the circle (A) $a28$ and 315 are even angles about the crossings 2 and 1. In (B), the circle of $_{n-1}M$, we read along 32 and next along 28, dotting both on the right, and later along $a2$ and 25, dotting both on the right; *i.e.*, $a28$ and 325 are even coverticals about 2.

By making these two covertical with a 2-gon we construct $_n$N, whose circle is (A), and thus demonstrate

Theorem AA.—If at any crossing r of an unifilar knot of $n-1$ crossings we make the two even angles of r covertical with a 2-gon, thus adding an edge to each collateral of the 2-gon, we construct an unifilar of n crossings by one of its even 2-gons.

This is the constructing converse of the Theorem A, art. 3.

8. Let

$$v \ldots fcdr \ldots bam \ldots s$$

be the circle of an unifilar of k crossings, in which cd and ba are edges of the mesh H, both dotted inside H at their mid-points α in cd and β in ba. The circle is read

$$v \ldots fc(\alpha)dr \ldots b(\beta)am \ldots s,$$

say from c above to d below with the dot (α) on the right in H, till we come to the crossing b, and proceed from b below to a above past the dot (β) on the right in the same H. In H draw the 2-gon pq from p between (α) and d to q between b and (β). We now read from c above, not to d but to p below, having the dot (α) on our right in H; then crossing the upper edge pq of the 2-gon, we proceed along the lower pq to q, planting a dot (ϵ) inside the 2-gon on that pq; at q, again crossing the upper pq, we proceed to a past (β) on our right in H, next to m, &c., completing the smaller circle

$$v \ldots fcpqam \ldots s,$$

which contains neither the upper edge of the 2-gon pq, nor any of the crossings $dr \ldots b$. We have constructed a bifilar knot of $k+2$ crossings, and demonstrated

Theorem D.—If in any mesh H of an unifilar knot $_n$T we connect by a 2-gon two mid-edges that are dotted either both inside or both outside of H, we complete a bifilar $_{n+2}$U.

Observe that the dotting of art. 1 may be done either on the right or on the left of every edge.

9. If in the unifilar $_{n-2}$Q of art. 5 we replace in the $(2+r)$-gonal face F the 2-gon 12 effaced from $_n$P, art. 4, we reconstruct the unifilar $_n$P. It follows that of the mid-points 1 and 2 of that face F on $_{n-2}$Q one, and only one, is dotted inside F. For if otherwise

we should construct (Theo. D) by the 2-gon 12, not $_n$P unifilar, but a bifilar $_n$U. Thus we have proof of

Theorem BB.—If in the projection of any $(2+r)$-gonal mesh F of an unifilar of $n-2$ crossings we connect by a 2-gon the mid-points of two edges of which one, and only one, is dotted (art. 2) inside F, we construct an unifilar of n crossings by one of its odd 2-gons.

This is the constructing converse of Theorem B, art. 5, and is true when $r \gtreqless 0$.

10. The constructing converse of Theorem C is

Theorem CC.—If in any unifilar knot of $n-2$ crossings we make, at any projected crossing r, either pair of opposed angles covertical with a double flap, adding two edges to each of the other pair of opposed meshes about r, we construct an unifilar of n crossings by one of its plural flaps.

No base which has a plural flap, not fixed, can be operated upon by Theorem AA or BB, unless the operation abolishes the plural flap. And every flap, single or plural, is fixed, if its deletion lays bare a section through two edges only. Such a fixed flap cannot compete for the lead, nor hinder an operation by AA or BB, which does not abolish the fixture. Every construction is by a leading flap (*vide* my paper, xvii.), and the leader has the most 2-gons.

2. On the Twists of Listing and Tait. By the Rev. Thomas P. Kirkman, M.A., F.R.S.

In the figure 1 following, the knot $_9Aj$ has a triangular section P*rr* cutting away on each side of it a $(3+r)$-gonal mesh, and $_9Ar^2$ has such a section R*pp*, through one crossing only. Make in these knots creases at *rr* that approach to meet at R in 2, and creases *pp* that meet at P in 2. In 3, 2 is prepared for a rotation through two right angles about the fixed axis PR, through the crossing P and the crease-kiss R, or, as a more learned man would say, through the decussation P and the plicatorial osculation R—taking 2 for $_9Aj$: exchange here P and R if 2 is $_9Ar^2$. In 4 the rotation is effected, undoing the crossing P of 3, which has become a kiss P′ in 4, while the kiss R of 3 has become the crossing R′ in 4, if 3 is $_9Aj$: exchange here P and R, if 3 is $_9Ar^2$. In 5 the crease kiss is