Δ-SETS I: HOMOTOPY THEORY

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Introduction

IN this paper we develop the homotopy theory of semi-simplicial complexes which do not have degeneracy operators; we call such a complex a Δ-set. In the original study of semi-simplicial theory it was natural to introduce degeneracies, since the canonical example—the singular complex—has degeneracies, also the definition of the product of two semi-simplicial complexes appears simpler with degeneracies than without (see § 3). However, a semi-simplicial complex without degeneracies is geometrically a simpler object than one with degeneracies, and we will show that the Kan condition [see (**1**)] can be used to replace the use of degeneracies in the usual approach [see e.g. (**2**)]. This paper arose out of our previous work (**4**) in which we defined Δ-groups which had no degeneracy homomorphisms (and no natural degeneracy functions).

Our main result, Theorem 5.3, is a strong relative 'simplicial approximation' theorem for Kan Δ-sets:

Suppose $Z \subset Y$ is a pair of Δ-sets and X is a Kan Δ-set. Suppose given a map $f\colon |Y| \to |X|$ such that $f\,|\,|Z|$ is the realization of a Δ-map. Then f is homotopic rel$|Z|$ to the realization of a Δ-map $f'\colon Y \to X$.

The theorem implies the equivalence of the homotopy categories of Kan Δ-sets and CW-complexes. We also have an approximation theorem in which X is not assumed to be Kan and Y is allowed to be derived away from J (5.1). Both theorems are deduced painlessly from Zeeman's relative simplicial approximation theorem for simplicial complexes (**8**), and some elementary collapsing lemmas.

The material is organized as follows. In §§ 1 and 2 we compare Δ-sets, CSS-sets, and CW-complexes, and the realization functors. In § 3 products of Δ-sets are introduced and elementary properties proved. § 4 contains the collapsing lemmas needed for the main theorems in § 5. As a consequence of 5.3 we show that a Kan Δ-set always admits a system of degeneracies! § 6 is devoted to homotopy theory, in particular homotopies of polyhedra in Δ-sets are defined—a concept which originated from (**4**). In § 7 we prove a polyhedral lifting property for a Kan fibration of Kan sets and in §§ 8 and 9 we show how minimal complexes and function spaces may be treated in the absence of degeneracies.

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In the second half of this paper we will apply our results to Δ-groups and Δ-monoids in which it may not be possible to install a system of degeneracy homomorphisms.

1. Semi-simplicial complexes

Let Δ^n be the standard n-simplex in R^{n+1} with vertices $v_0,\ldots, v_n$, where $v_i = (0,\ldots, 0, 1, 0,\ldots)$ with 1 occurring in the $(i+1)$th place.

Then css is the category with objects Δ^n ($n = 0, 1, 2,\ldots$) and whose morphisms are the simplicial maps determined by order-preserving vertex maps. Define Δ to be the subcategory of css determined by the injective maps.

A css-set (pointed css-set, css-group, etc.) is a contravariant functor from css to the category of sets (pointed sets, groups, etc.). Replacing css by Δ gives definitions of Δ-sets, etc. A css-set or Δ-set is often referred to as a *complex*. If X is a css- or Δ-set then $X^{(k)} = X(\Delta^k)$ is the set of *k-simplexes* of X. A map $\lambda\colon \Delta^s \to \Delta^k$ in css induces $\lambda^\sharp = X(\lambda)$: $X^{(k)} \to X^{(s)}$. If λ is injective then $\lambda^\sharp$ is called a *face map* and $\lambda^\sharp(\sigma^k)$ is called a *face* of σ^k, otherwise $\lambda^\sharp$ is a *degeneracy operator* and $\lambda^\sharp(\sigma^k)$ a *degeneracy* of σ^k. A simplex τ is *degenerate* if it is a degeneracy of some σ, otherwise τ is *non-degenerate*. A Δ-set is *locally finite* if $\sigma \in X \Rightarrow \sigma$ is a face of only finitely many simplexes in X. It is also convenient to denote the set $\bigcup \{X^{(k)} \mid k$ a non-negative integer$\}$ by X. A simplicial complex K is *ordered* if a partial ordering of the vertices of K is given so that the vertices of any simplex of K are totally ordered. Then K determines a Δ-set also denoted by K and a css-set denoted by $\mathbf{K}$ defined as follows.

$$K^{(n)} = \{f \mid f\colon \Delta^n \to K \text{ is injective, order-preserving, and simplicial}\},$$
$$\mathbf{K}^{(n)} = \{f \mid f\colon \Delta^n \to K \text{ is simplicial and order-preserving}\}.$$

$\lambda^\sharp f$ is defined to be $f \circ \lambda$.

In particular with these definitions Δ^s now denotes both a subspace of R^{s+1} and a Δ-set. In the latter case $(\Delta^s)^{(q)} = \{\mu \mid \mu\colon \Delta^q \to \Delta^s, \mu \in \Delta\}$.

Now suppose that X, Y are css- or Δ-sets. A Δ-map or css-map $f\colon X \to Y$ is a natural transformation of functors. This means that we have commutative diagrams

$$\begin{tikzcd} X^{(k)} \arrow[r, "\lambda^\sharp"] \arrow[d, "f^{(k)}"] & X^{(s)} \arrow[d, "f^{(s)}"] \\ Y^{(k)} \arrow[r, "\lambda^\sharp"] & Y^{(s)}, \end{tikzcd}$$

where $f^{(n)} = f(\Delta^n)$. We get a category of Δ-sets denoted by $\mathbf{\Delta}$ and the category of css-sets denoted by **css**. There is a forgetful functor

$F\colon \mathbf{css} \to \boldsymbol{\Delta}$ defined in the obvious way and if K is an ordered complex there is an inclusion $K \subset F(\mathbf{K})$. Now suppose K, L are ordered complexes and $f\colon K \to L$ is an order-preserving simplicial map, then a css-map $f\colon \mathbf{K} \to \mathbf{L}$ is defined by $f(\sigma) = f \circ \sigma$. If in addition the map $f\colon K \to L$ is injective on simplexes then it may be regarded as a Δ-map.

In particular a morphism $\lambda\colon \Delta^s \to \Delta^k$ in Δ may be regarded as a morphism in $\boldsymbol{\Delta}$. Then $\lambda(\mu) = \lambda \circ \mu$.

If X is a pointed Δ-set or css-set we denote the base simplex in dimension k by $*_k$. The base simplexes form a subcomplex $* \subset X$.

A group complex G is pointed by the identities $e_k \in G^{(k)}$ and a css-set X can be pointed at any vertex $*_0 \in X^{(0)}$ by setting $*_k = \mu^\sharp *_0$ where $\mu\colon \Delta^k \to \Delta^0$ is the unique map.

If X is a complex the subcomplex of X generated by $\sigma \in X^{(k)}$ is denoted by $\boldsymbol{\sigma}$ and the subcomplex generated by all (proper) faces of σ is denoted by $\dot{\boldsymbol{\sigma}}$. The simplex σ determines a *characteristic map* $\tilde{\sigma}\colon \Delta^k \to X$ defined by $\tilde{\sigma}(\mu) = \mu^\sharp(\sigma)$. In the css case $\tilde{\sigma}\colon \boldsymbol{\Delta}^k \to X$.

$\delta_i\colon \Delta^{k-1} \to \Delta^k$ is the morphism of Δ such that $v_i \notin \text{image}(\delta_i)$ and $\delta_i^\sharp$ is usually denoted by ∂_i. It is an easy exercise to show that any face map factors into a product of ∂_is.

The *i-th horn* $\Lambda_{n,i}$ of Δ^n is the subcomplex of Δ^n defined by

$$\Lambda_{n,i} = \Delta^n - \{1_n\} - \{\partial_i 1_n\},$$

where $1_n = 1_{\Delta^n}\colon \Delta^n \to \Delta^n$.

LEMMA 1.1. *Every $\phi\colon \Delta^n \to \Delta^k$ in* css *factors uniquely as $\phi_1 \circ \phi_2$ with ϕ_2 surjective and ϕ_1 injective.*

Proof. There is a unique order-preserving isomorphism $\Psi\colon \Delta^m \to \text{im}(\phi)$ and we must have $\phi_2 = \Psi^{-1} \circ \phi$ and $\phi_1 = \text{incl.} \circ \Psi$.

A left adjoint G for $F\colon \mathbf{css} \to \boldsymbol{\Delta}$ is defined by

$$G(X)^{(k)} = \{(\mu, \sigma) \mid \sigma \in X^{(r)},\ \mu \in \text{css, and } \mu\colon \Delta^k \to \Delta^r \text{ is surjective}\},$$

and by setting $\lambda^\sharp(\mu, \sigma) = (\phi_2, \phi_1^\sharp \sigma)$ where $\phi = \mu\lambda$ and $\phi = \phi_1 \circ \phi_2$ is the factoring of Lemma 1.1.

If $f\colon X \to Y$ is a Δ-map then $G(f)\colon G(X) \to G(Y)$ is defined by $G(f)(\mu, \sigma) = (\mu, f(\sigma))$. We leave the reader to check that G is a functor and we prove adjointness after some further definitions and lemmas.

Definition. Let Y be css, then the *core of* Y is the Δ-subset

$$\text{Core}(Y) \subset F(Y)$$

consisting of the non-degenerate simplexes of Y and their faces. We say Y is ndc if there are no degenerate simplexes in its core.

The following lemma is well known and easily proved.

LEMMA 1.2 (Eilenberg–Zilber). *Let Y be* CSS *and let $\sigma \in Y^{(n)}$. Then there exists a non-degenerate simplex τ_0 and a surjective μ_0 such that $\sigma = \mu_0^\sharp \tau_0$, and if also $\sigma = \mu_1^\sharp \tau_1$ with τ_1 non-degenerate and μ_1 surjective then $\mu_0 = \mu_1$ and $\tau_0 = \tau_1$.*

Remark 1.3. $G(X)$ is ndc and its core consists of simplexes $(1_n, \sigma^n)$ and may be identified with X. Each simplex (μ, τ^n) is uniquely written as $\mu^\sharp(1_n, \tau^n)$.

PROPOSITION 1.4. *Suppose $g, f\colon Y_0 \to Y_1$ are* CSS *and $g(\tau) = f(\tau)$ if τ is non-degenerate. Then $g = f$.*

Proof. Let $\sigma \in Y_0$. Then by 1.2, $\sigma = \mu^\sharp\tau$ with τ non-degenerate and $g(\sigma) = g(\mu^\sharp\tau) = \mu^\sharp g(\tau) = \mu^\sharp f(\tau) = f(\mu^\sharp\tau) = f(\sigma)$.

Now define $\theta\colon G(\mathrm{Core}(Y)) \to Y$ by $\theta(\mu, \sigma) = \mu^\sharp\sigma$. Then θ is a CSS map, since
$$\theta\lambda^\sharp(\mu, \sigma) = \theta(\phi_2, \phi_1^\sharp\sigma) = \phi_2^\sharp\phi_1^\sharp\sigma = \lambda^\sharp\mu^\sharp\sigma = \lambda^\sharp\theta(\mu, \sigma),$$
where $\mu\lambda = \phi_1\phi_2$ is the decomposition of Lemma 1.1.

PROPOSITION 1.5. *$\theta\colon G(\mathrm{Core}(Y)) \to Y$ is onto for any Y and an isomorphism if Y is ndc.*

Proof. Let $\sigma \in Y$. Then $\sigma = \mu^\sharp\tau$ with τ non-degenerate by 1.2 and $\theta(\mu, \tau) = \sigma$, which proves that θ is onto. Suppose now Y is ndc and $\theta(\mu_0, \tau_0) = \theta(\mu_1, \tau_1)$. Then $\mu_0^\sharp\tau_0 = \mu_1^\sharp\tau_1$ and $\mu_0 = \mu_1$, $\tau_0 = \tau_1$ by 1.2.

Suppose $f\colon X \to FY$ is a Δ-map. We define the *adjoint* CSS-map $\hat{f}\colon GX \to Y$ by $\hat{f}(\mu, \sigma) = \mu^\sharp f(\sigma)$.

Remark 1.6. Observe that by combining 1.3 and 1.4 we may regard $\hat{f}$ as the unique extension of f.

THEOREM 1.7. *The map $\phi\colon \{X, FY\} \to \{GX, Y\}$, defined by $\phi(f) = \hat{f}$, is an adjunction morphism.*

Proof. Since f is obtained from $\hat{f}$ by restriction it is clear that ϕ is injective. Let $g\colon GX \to Y$. Then $g = \hat{f}$ for some f by Remark 1.6 and so ϕ is surjective. We leave naturality to the reader.

2. The realization functors

Let **cw** denote the category of CW-complexes. In this section we introduce two more pairs of adjoint functors so that there are commutative diagrams

$$\begin{tikzcd} \boldsymbol{\Delta} \arrow[rr, "G"] \arrow[dr, "|\,|"'] & & \mathbf{css} \arrow[dl, "|\,|_M"] \\ & \mathbf{cw} & \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\Delta} & & \mathbf{css} \arrow[ll, "F"'] \\ & \mathbf{cw} \arrow[ul, "F_0 S"] \arrow[ur, "S"'] & \end{tikzcd}$$

Here S is the well-known singular complex functor, $||$ and $||_M$ are *realization* functors.

Let Y be css. Then $|Y|_M$ is formed from the disjoint union

$$\bigcup \{\{\sigma^n\} \times \Delta^n \mid \sigma \in Y\}$$

by identifying pairs $(\sigma, \lambda(x))$ with $(\lambda^\sharp\sigma, x)$. If X is a Δ-set then $|X|$ is similarly defined. Then $|X|$ is a CW-complex having one cell for each simplex in X and $|Y|_M$ is a CW-complex having one cell for each non-degenerate simplex of Y. (The functor $||_M$ was introduced by Milnor in (**3**) and the reader is referred to this paper and to (**2**) for unproved facts about $||_M$.) Now let $f\colon W \to Z$ be a Δ-map (resp. css-map), then

$$|f|\colon |W| \to |Z| \quad (\text{resp. } |f|_M\colon |W|_M \to |Z|_M)$$

is defined by

$$|f|[\sigma^n, x] = [f(\sigma^n), x] \quad (\text{resp. } |f|_M[\sigma^n, x] = [f(\sigma^n), x]).$$

In particular $|f|$ is a homeomorphism when restricted to the interior of a cell of $|W|$. Further, $||_M$ and S (and similarly $||$ and $F \circ S$) are adjoint.

PROPOSITION 2.1. *Let X be a* css*-set and Y a Δ-set. Then $|FX|$ and $|X|_M$ have the same homotopy type, and $|GY|_M$ and $|Y|$ are homeomorphic.*

Proof. Let $\sigma \in X^{(n)}$. Then $\sigma = \mu^\sharp\tau$, where μ is surjective and τ is non-degenerate, by 1.2. Now define $g(\sigma, x) = (\tau, \mu(x))$, where $x \in \Delta^n$. Then g respects identifications and determines a map $g\colon |FX| \to |X|_M$. It is now a standard exercise to show that g induces isomorphisms of homology and fundamental groups. The result then follows from Whitehead's theorem (**6**). It follows from the definitions that $|GY|_M$ and $|Y|$ have the same cell structure and hence in particular are homeomorphic.

3. The products $X \otimes Y$ and $X \times Y$

Let X and Y be Δ-sets (resp. css-sets). Then define $X \times Y$ by

$$(X \times Y)^{(n)} = X^{(n)} \times Y^{(n)} \quad \text{and} \quad \lambda^\sharp(x, y) = (\lambda^\sharp x, \lambda^\sharp y).$$

Then $X \times Y$ is a Δ-set (resp. css-set) called the *product of X with Y*.

Now let X and Y be Δ-sets. The *geometric product of X with Y, $X \otimes Y$*, is defined by

$$X \otimes Y = \operatorname{Core}(G(X) \times G(Y)).$$

Remarks 3.1. If $X = \Delta^1$ and Y is a Δ-set then $X \times Y$ has simplexes only in dimensions 0 and 1 and is consequently not a very interesting object. This example provides motivation for the geometric product. We shall show later (5.10) that if X and Y are both Kan then $|X \times Y|$ has the same homotopy type as $|X \otimes Y|$.

Suppose that K is an ordered simplicial complex with vertices $\{\alpha_i\}$. Consider now $(\alpha_{i_0},\ldots,\alpha_{i_r})$ with $\alpha_{i_0} \leqslant \ldots \leqslant \alpha_{i_r}$. Then $(\alpha_{i_0},\ldots,\alpha_{i_r})$ corresponds to the simplex $(\mu,\sigma) \in GK = \mathbf{K}$, where $\sigma\mu(v_k) = \alpha_{i_k}$. Further, $(\alpha_{i_0},\ldots,\alpha_{i_r})$ is in $K \subset \mathbf{K}$ if and only if $\alpha_{i_s} \neq \alpha_{i_{s+1}}$ $(0 \leqslant s < r)$.

We now define an ordered simplicial complex $P_{n,m}$ such that as a space $P_{n,m} = \Delta^n \times \Delta^m \subset R^{n+m+2}$ and a typical r-simplex τ of $GP_{n,m}$ is denoted by $((v_{i_0}, v_{j_0}),\ldots,(v_{i_r}, v_{j_r}))$, where $i_s \leqslant i_{s+1}$ and $j_s \leqslant j_{s+1}$ $(0 \leqslant s < r)$. Further, τ is in $P_{n,m}$ if and only if $i_s \neq i_{s+1}$ or $j_s \neq j_{s+1}$ for each s $(0 \leqslant s < r)$.

A css isomorphism $\phi\colon G\Delta^n \times G\Delta^m \to GP_{n,m}$ is defined by

$$\phi((\mu,\lambda),(\mu',\lambda')) = ((\lambda\mu(v_0), \lambda'\mu'(v_0)),\ldots,(\lambda\mu(v_r), \lambda'\mu'(v_r))),$$

for each r-simplex $((\mu,\lambda),(\mu',\lambda'))$ of $G\Delta^n \times G\Delta^m$. It follows that ϕ restricts to a Δ-isomorphism $\Delta^n \otimes \Delta^m \to P_{n,m}$.

Let X, Y be Δ-sets and let $\sigma^n \in X$ and $\tau^m \in Y$. Then there is the canonical map $G(\Delta^n \otimes \Delta^m) \to G(X) \times G(Y)$ which restricts to a map $\Delta^n \otimes \Delta^m \to X \otimes Y$. From this and the above discussion we see that $X \otimes Y$ may be defined by taking a copy of the prism $P_{n,m}$ for each $\sigma \in X^n$ and $\tau \in Y^m$ and then making identifications. There is a canonical homeomorphism of $|\Delta^n \otimes \Delta^m|$ with $|\Delta^n| \times |\Delta^m|$ and we have

THEOREM 3.2. *Let X, Y be Δ-sets. Then the map*

$$\theta\colon G(X \otimes Y) \to G(X) \times G(Y)$$

is a css *isomorphism and $|X \otimes Y|$ is canonically homeomorphic with the* cw*-complex $|X| \times |Y|$. Further, if either X or Y is locally finite then the product topology on $|X| \times |Y|$ coincides with the* cw *topology.*

4. Subdivisions and collapsing

It will be convenient to confuse a Δ-set X with the complex $|X|$ equipped with its characteristic maps $|\tilde{\sigma}|\colon |\Delta^n| \to |X|$. Then X_1 is a subdivision of X if $|X_1| = |X|$ and if for each $\sigma \in X_1^{(n)}$ there exists a $\tau \in X^{(m)}$ (some $m \geqslant 0$) and a linear embedding $e\colon |\Delta^n| \to |\Delta^m|$ so that $|\tilde{\tau}| \circ e = |\tilde{\sigma}|$.

Note in particular that if K is a simplicial complex and X, X_1 are obtained from K by ordering vertices, then X is a subdivision of X_1 and conversely!

Recall that a derived subdivision of a simplicial complex may be defined inductively (in increasing dimensions) by replacing a simplex by the cone on its derived boundary or by itself. This definition readily extends to Δ-sets—order the cone point later than all vertices in the boundary. By iterating r times we get an rth-derived. If every simplex is replaced

by the cone on its boundary then we have *the* 1*st derived* dX, and again by iterating we have the *r-th derived* d^rX.

Suppose $X \subset Y$ and let Y' be the derived of Y obtained by replacing a simplex by itself when possible, subject to the condition that $dX \subset Y'$. We refer to Y' as *Y derived at X*. There is a simplicial map of a derived X' to X defined by mapping a vertex of X' to the last vertex of the smallest simplex of X in which it lies.

Note that if we begin with a simplicial complex K then the vertices of dK are partially ordered so that dK may be regarded as a Δ-set. Conversely if we begin with a Δ-set X then d^2X may be regarded as a simplicial complex, since after deriving twice $i \neq j$ implies $\partial_i \sigma \neq \partial_j \sigma$. In particular if X is locally finite then $|X| = |d^2X|$ is a polyhedron (in the sense of (8)) in a natural way.

Recall now that if $f\colon \Delta^n \to \Delta^r$ is a simplicial map then we can define a simplicial complex M_f, *the mapping cylinder* of f [see (5) 259]. Further, there are natural disjoint inclusions Δ^n, $d\Delta^n$, $\Delta^r \subset M_f$ and each vertex of M_f is in the image of one of these inclusions and no simplex of M_f has vertices both in Δ^n and in Δ^r, so that M_f becomes a Δ-set by ordering all the vertices of $d\Delta^n$ later than those of Δ^n and Δ^r.

We now generalize this construction. Let X and Y be Δ-sets. A map $f\colon |X| \to |Y|$ is *simplicial* if for each $\sigma \in X$ there is a simplicial map f_σ and a commutative diagram

$$\begin{array}{ccc} |X| & \xrightarrow{f} & |Y| \\ \uparrow{\scriptstyle |\tilde\sigma|} & & \uparrow{\scriptstyle |\tilde\tau|} \\ \Delta^n & \xrightarrow{f_\sigma} & \Delta^r, \end{array}$$

where $\tau = f(\sigma)$.

The *mapping cylinder* M_f is the Δ-set obtained from the disjoint union $\bigcup \{M_{f_\sigma} \mid \sigma \in X\}$ by identifying the mapping cylinder of $f_\sigma \mid \lambda(\Delta^m)$ with the mapping cylinder of $f_{\lambda^\#\sigma}$ for each σ and $\lambda\colon \Delta^m \to \Delta^n$ in Δ.

Then there are inclusions X, dX, $Y \subset M_f$. In particular M_1, where $1 = 1_X\colon X \to X$, is obtained from $|X| \times I$ by inductively deriving the 'prisms' $|\sigma| \times I$ at their barycentres. Also there is a folding map $\nu\colon M_1 \to M_1$ which identifies $X \otimes \{0\}$ with $X \otimes \{1\}$ and restricts to the identity on dX.

We now extend the notion of collapsing for simplicial complexes [see e.g. (5) 247] to Δ-sets. Suppose a Δ-set X contains a simplex σ which is not the face of any other simplex and τ is a free face of σ, i.e. τ is the face of no other simplex except σ. Then $Y = X - \{\sigma, \tau\}$ is obtained from X by *collapsing* σ *from* τ. Then Y is a subcomplex of X and we

write $X \searrow^e Y$. We say X *collapses* to Z, and write $X \searrow Z$, if there exist subcomplexes X_i ($i = 1, \ldots, n$ for some n) of X such that

$$X \searrow^e X_1 \searrow^e X_2 \ldots \searrow^e X_n = Z.$$

LEMMA 4.1. *If $X \searrow Y$ and X' is a derived subdivision of X inducing Y', then $X' \searrow Y'$.*

Proof. By induction we may suppose $X \searrow^e Y$ by collapsing σ from τ. Corresponding to $|\tilde{\sigma}|\colon |\Delta^n| \to |X|$, there is a derived map

$$|\tilde{\sigma}'|\colon |\Delta^{n'}| \to |X'|.$$

Then by the result for simplicial complexes [see (9)], $\Delta^{n'}$ collapses on to $Q = |\tilde{\sigma}'|^{-1}|Y'|$. But $|\tilde{\sigma}'|$ is clearly injective on the complement of Q and this collapse induces the required collapse $X' \searrow Y'$.

LEMMA 4.2. *If $X \searrow Y$ then there exists a subdivision X' of X, so that $Y \subset X'$, and a simplicial retraction $r\colon |X'| \to |Y|$.*

Proof. The result for simplicial complexes adapts as in the proof of 4.1 above.

Recall that $\boldsymbol{\sigma}$ denotes the Δ-set generated by a simplex $\sigma \in X$, and $\dot{\boldsymbol{\sigma}}$ denotes the union of the proper faces of σ.

LEMMA 4.3. *The complex $\boldsymbol{\sigma} \otimes I$ derived at one end collapses to either end together with $\dot{\boldsymbol{\sigma}} \otimes I$.*

LEMMA 4.4. *If $f\colon \boldsymbol{\sigma} \to \boldsymbol{\tau}$ is simplicial and onto then $M_f \searrow M_g$, where $g = f \,|\, \dot{\boldsymbol{\sigma}}$.*

Proofs. In both cases we may assume that $\sigma = \Delta^n$ ($\tau = \Delta^r$ in Lemma 4.4) as in the proof of 4.1. Then 4.3 follows from 4.1 by using the cylindrical collapse of $\Delta^n \otimes I$ [see (9)]. For Lemma 4.4 we use the Whitehead collapse [see (5)].

Remark 4.5. The collapse of 4.4 can be taken to respect the half-way level, i.e. the collapses which meet vertices of $\boldsymbol{\sigma} \subset M_f$ can be performed first.

5. Simplicial approximation and the generalized extension condition

First we prove a simplicial approximation theorem in the category $\boldsymbol{\Delta}$.

THEOREM 5.1. *Suppose $J \subset Y$ and X are Δ-sets, $f\colon |Y| \to |X|$ is a continuous map, and $f \,|\, |J| = |g|$, where $g\colon J \to X$ is a simplicial map. Then there exists a simplicial map $f'\colon |Y'| \to |X|$ and a homotopy $H\colon f \simeq f' \operatorname{rel} |J|$, where Y' is a subdivision of Y and $J \subset Y'$.*

Proof

Case 1. Assume Y is finite. Let Y_n denote Y derived n times but J only twice so that $d^2J \subset Y_n$. By the relative simplicial approximation theorem in (8) there is an n and a simplicial map f_1: $|Y_n| \to |d^2X|$ which is homotopic rel$|J|$ to f. Now let $\hat{X}$ (resp. $\hat{J}$) denote $M_{1|X}$ (resp. $M_{1|J}$) derived twice at the 0-end and let $\bar{Y}$ denote $M_{1|Y}$ derived at the 0-end so that Y_n is there. Then $\hat{J} \subset \bar{Y}$ (see diagram 1).

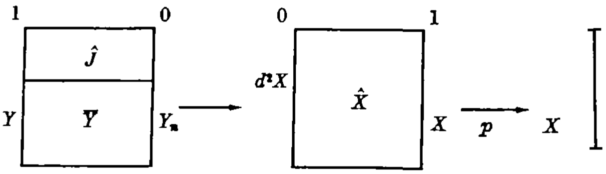

DIAGRAM 1

Let p: $|\hat{X}| \to |X \otimes I| \to |X|$ be the obvious simplicial composition. Now $\bar{Y} \searrow (\hat{J} \cup Y_n)$ by Lemma 4.4 and so by Lemma 4.2 there is a subdivision $\bar{Y}'$ of $\bar{Y}$, so that $\hat{J} \cup Y_n \subset \bar{Y}'$, and a simplicial retraction r: $|\bar{Y}'| \to |\hat{J} \cup Y_n|$. Define f': $|Y'| \to |X|$ by

$$f'(x) = \begin{cases} pf_1 r(x) & \text{if } r(x) \in \text{domain } f_1, \\ pgr(x) & \text{if } r(x) \notin \text{domain } f_1. \end{cases}$$

It is easy to check that f' has the desired properties.

General case. This follows from Case 1 by induction over the skeleta of Y.

Let X be a Δ-set. Then X satisfies the *extension condition* for the pair of Δ-sets (Z, W) if every Δ-map f: $W \to X$ extends over Z.

X is *Kan* if X satisfies the extension condition for the pairs $(\Delta^n, \Lambda_{n,i})$. A css-set Y is Kan if FY is Kan. We then have by an easy induction

PROPOSITION 5.2. *A Kan Δ-set has the extension condition for each Δ-pair (W, Z) such that $W \searrow Z$.*

THEOREM 5.3. *Suppose (Y, J) is a pair of Δ-sets and X is a Kan Δ-set. Suppose given f: $|Y| \to |X|$ such that $f \mid |J| = |g|$, where g: $J \to X$ is a Δ-map. Then there exists a Δ-map f': $Y \to X$ and a homotopy*

$$H: f \simeq |f'| \operatorname{rel} |J|.$$

Proof

Case 1. *Y finite and g: $J \subset X$ an inclusion.* As in the proof of 5.1 we have $\hat{J} \subset \bar{Y}$, $\hat{X}$ and Y, $Y_n \subset \bar{Y}$ and d^2X, $X \subset \hat{X}$. Let (M, M_1) be the mapping cylinders of f_1 and and 1_J. By identifying the ends of M with

the 0-ends of $\bar{Y}$ and $\hat{X}$ we have a Δ-set Z pictured in diagram 2 (the two copies of $\hat{J}$ in the diagram should really be identified).

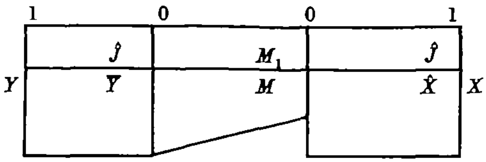

DIAGRAM 2

We describe a Δ-retraction of Z on X. The restriction of this retraction to Y will give the desired Δ-map $f'\colon Y \to X$. First observe that the bottom half of M_1 (right-hand half in diagram 2) together with $\hat{J} \subset \hat{X}$ collapses to J by Lemma 4.4 (see also Remark 4.5). By using the folding map $v\colon M_1 \to M_1$, the collapse, and 5.2 it is clear how to define the retraction on $\hat{J} \cup M_1$. By Lemmas 4.1 and 4.4 there is a collapse $Z \searrow M_1 \cup \hat{J} \cup X$ and a final application of 5.2 can be made to complete the definition of the retraction. It is now easy to check that f' has all the desired properties.

Case 2. *Y finite.* Define $\tilde{Y} = Y/\sim$ where $\sigma_1 \sim \sigma_2$ if $\sigma_1, \sigma_2 \in J$ and $g(\sigma_1) = g(\sigma_2)$. Then f factors via $|\tilde{Y}|$ and Case 1 may be used.

General case. This follows from Case 2 by induction over the skeleta of Y.

COROLLARY 5.4. *A Kan Δ-set satisfies the* generalized extension condition (GEC), *i.e. X satisfies the extension condition for pairs (W, Z) such that $|Z|$ is a retract of $|W|$.*

We can also use Theorem 5.3 to show that a Kan Δ-set admits degeneracy operators. First we prove two lemmas.

LEMMA 5.5. *Let X be a Δ-set and $\sigma \in X^{(n)}$. Then there is a deformation retraction $r\colon |FG(\boldsymbol{\sigma})| \to |FG(\dot{\boldsymbol{\sigma}}^n) \cup \boldsymbol{\sigma}^n|$, where $\boldsymbol{\sigma}$ is included in $FG(\sigma)$ by the map $\phi^{-1}(1_{G\sigma})$.*

Proof. It is enough to show that the inclusion
$$i\colon |FG(\dot{\boldsymbol{\sigma}}^n) \cup \boldsymbol{\sigma}^n| \subset |FG(\boldsymbol{\sigma})|$$
is a homotopy equivalence. One simply checks that i induces isomorphisms on π_1 and $H_*(\ , Z)$ using well-known techniques; the result then follows from Whitehead's theorem (6).

LEMMA 5.6. *A Δ-set X admits degeneracy operators if and only if there exists a Δ-retraction $r\colon FG(X) \to X$ satisfying $r(\lambda, r(\mu, \sigma)) = r(\mu\lambda, \sigma)$ for all λ, μ, σ.*

Proof. Suppose X admits degeneracy operators. Then $r(\lambda, \sigma) = \lambda^\sharp\sigma$ gives the required retraction. Conversely, given a retraction r, define $\lambda^\sharp\sigma$ to be $r(\lambda_2, \lambda_1^\sharp\sigma)$ where $\lambda = \lambda_1\lambda_2$ is the factoring provided by Lemma 1.1. Now let $\lambda\colon \Delta^n \to \Delta^m$, $\mu\colon \Delta^m \to \Delta^s$. Then by 1.1 there is a commutative diagram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "\lambda"] \arrow[dr, "\lambda_2"'] & & \Delta^m \arrow[rr, "\mu"] \arrow[dr, "\mu_2"'] & & \Delta^s \\
& \Delta^p \arrow[ur, "\lambda_1"'] \arrow[dr, "\phi_2"'] & & \Delta^q \arrow[ur, "\mu_1"'] & \\
& & \Delta^t \arrow[ur, "\phi_1"'] & &
\end{tikzcd}$$

where λ_2, ϕ_2, μ_2 are surjective and λ_1, ϕ_1, μ_1 are injective. We must show $\lambda^\sharp\mu^\sharp = (\mu\lambda)^\sharp$. From definitions we have

$$\begin{aligned}
\lambda^\sharp(\mu^\sharp\sigma^s) &= \lambda^\sharp r(\mu_2, \mu_1^\sharp\sigma) \\
&= r(\lambda_2, \lambda_1^\sharp r(\mu_2, \mu_1^\sharp\sigma)) \\
&= r(\lambda_2, r\lambda_1^\sharp(\mu_2, \mu_1^\sharp\sigma)) \\
&= r(\lambda_2, r(\phi_2, (\mu, \phi)^\sharp\sigma)),
\end{aligned}$$

but $(\mu\lambda)^\sharp\sigma = r(\phi_2\lambda_2, (\mu_1\phi_1)^\sharp\sigma)$ and the result follows from the condition on r.

THEOREM 5.7. *A Kan Δ-set X admits a system of degeneracy operators.*

Proof. We define inductively $d_i\colon FG(\mathrm{Sk}^i X) \to X$ and

$$r_i\colon FG(\mathrm{image}(d_i)) \to \mathrm{image}(d_i) \subset X,$$

where $\mathrm{Sk}^i X$ denotes the ith skeleton of X, so that $r_i(\lambda, r_i(\mu, \sigma)) = r_i(\mu\lambda, \sigma)$ whenever this makes sense and d_i (resp. r_i) extends d_{i-1} (resp. r_{i-1}). The induction begins by taking $d_{-1} = r_{-1} =$ the empty map. Suppose now that r_{n-1} has been defined and $\sigma \in (X^{(n)} - \mathrm{im}\, r_{n-1})$. Let

$$\theta_\sigma\colon |FG(\boldsymbol{\sigma})| \to |FG(\dot{\boldsymbol{\sigma}}) \cup \boldsymbol{\sigma}|$$

be the retraction of Lemma 5.5. Now apply Theorem 5.3 with Y, J, X, f, g replaced by $FG(\boldsymbol{\sigma})$, $FG(\dot{\boldsymbol{\sigma}}) \cup \boldsymbol{\sigma}$, X, $r_{n-1}\theta_\sigma \cup 1_\sigma$, $r_{n-1} \cup 1_\sigma$ respectively to get a map $g_\sigma\colon FG(\boldsymbol{\sigma}) \to X$. Then define $d_n(\mu, \sigma) = g_\sigma(\mu, \sigma)$ and $r_n(\mu, d_n(\lambda, \sigma)) = d_n(\lambda\mu, \sigma)$. Finally define $r = \bigcup \{r_i\}$. Then r satisfies the condition of Lemma 5.6 and the theorem is proved.

Remark 5.8. The proof of 5.7 shows that $|X| \subset |FG(X)|$ is a deformation retract, since the maps θ_σ used in the proof were deformation retractions. A specific deformation retraction can also be defined using formula (20.8) on p. 104 of (7), even in the case when X is not Kan.

COROLLARY 5.9. *A Kan Δ-set can be based at any vertex.*

Proof. Let $*_0 \in X^{(0)}$ be the vertex. Then introduce degeneracies and define $*_k = \mu^{\sharp} *_0$, where $\mu: \Delta^k \to \Delta^0$ is the non-empty map.

COROLLARY 5.10. *If X and Y are Kan Δ-sets then $|X \times Y|$ has the same homotopy type as $|X \otimes Y|$.*

Proof. Introduce degeneracies. Then $|X \times Y| \simeq |X \times Y|_M$ and $|X| \times |Y| \simeq |X|_M \times |Y|_M$ by Proposition 2.1. Finally

$$|X \times Y|_M \cong |X|_M \times |Y|_M \quad \text{by (3),}$$

and

$$|X \otimes Y| \cong |X| \times |Y| \quad \text{by 3.2.}$$

6. Homotopy of Δ-sets

Definition. Δ-maps $f_0, f_1: X \to Y$ are *homotopic*, $f_0 \simeq f_1$, if they are restrictions of a map $F: X \otimes I \to Y$.

THEOREM 6.1. *Suppose Y is a Kan Δ-set. Then maps f_0, f_1 are homotopic if and only if their realizations $|f_0|$, $|f_1|$ are homotopic.*

Proof. Since $|X \otimes I| \cong |X| \times |I|$, it is sufficient to show that $|f_0| \simeq |f_1|$ implies $f_0 \simeq f_1$. This follows from Theorem 5.3.

Remark 6.2. If a homotopy $F: |f_0| \simeq |f_1|$ is already a realization on a subcomplex of X, i.e. $F \mid |Z| \times |I| = |G|$ for some $G: Z \otimes I \to Y$, $z \subset X$, then the resulting homotopy $f_0 \simeq f_1$ may be assumed to extend G.

There is also a version of 6.1 for maps of pairs of Δ-sets, etc.

COROLLARY 6.3. *Homotopy of Δ-maps is an equivalence relation when the range involved is a Kan Δ-set.*

Using 6.1 and 6.3 we can define a category $h\boldsymbol{\Delta}$ with objects Kan Δ-sets and $\mathrm{Morph}(X, Y) = [X, Y]$, the set of homotopy classes of Δ-maps.

Now recall that a *polyhedron* P is a topological space, denoted by $|P|$, together with a maximal family $\mathscr{P}$ of homeomorphisms $t: |K| \to |P|$, where K is a locally finite simplicial complex (and there is no loss in assuming K ordered) satisfying: $t_1, t_2 \in \mathscr{P}$ implies $t_2^{-1} t_1$ is PL. The elements of $\mathscr{P}$ are called *triangulations* of P.

Definition. Let X be a Δ-set and let P be a polyhedron. Then a *map* $(f, t): P \to X$ is a triangulation $t: |K| \to |P|$ and a Δ-map $f: K \to X$. Maps (f_0, t_0) and (f_1, t_1) are *homotopic* if there exists a map

$$(F, T): P \times I \to X$$

so that the appropriate restrictions yield (f_0, t_0) and (f_1, t_1).

Homotopy is easily proved to be an equivalence relation where the range is Kan and we denote the set of equivalence classes of maps $P \to X$

by $[P, X]$. The next theorem shows that in representing a homotopy class or a homotopy there is freedom of choice of the triangulation involved. The theorem also has relative versions.

THEOREM 6.4. *Let* $t_i\colon |K_i| \to |P|$ $(i = 0, 1)$ *be triangulations of the polyhedron* P *and let* $\alpha \in [P, X]$, *where* X *is a Kan* Δ-*set. Then*

(i) α *is represented by some* (f_0, t_0),

(ii) *if* $(f_0, t_0) \simeq (f_1, t_1)$ *and* $t\colon |K| \to |P \times I|$ *extends* t_0 *and* t_1 *then there exists* $f\colon K \to X$ *extending* f_0 *and* f_1.

Proof. This is an application of Theorem 5.3. For (i) one also needs Lemma 2.5 of (**4**).

Remark 6.5. Let Y be a locally finite Δ-set so that $|Y|$ is a polyhedron in a natural way and denote it by P_Y to avoid confusion. It now follows from 6.1 and 6.4 that the obvious maps $[|Y|, |X|] \leftarrow [Y, X] \rightarrow [P_Y, X]$ are bijections.

Now let X, Y be Kan Δ-sets pointed by $* \subset Y \subset X$. We define homotopy groups by

$$\pi_n(X, Y, *) = \pi_n(|X|, |Y|, |*_0|),$$

$$\pi_n(Y, *) = \pi_n(|Y|, |*_0|).$$

As is usual we simply write $\pi_n(X, Y)$ and $\pi_n(Y)$ if X and Y are connected. It follows from a relative version of 6.5 that we could also have defined $\pi_n(X, Y, *)$ to be $[\Sigma^n, \Sigma^n_+, \Sigma^n_-; X, Y, *]$, where Σ^n is the polyhedron defined by $\Sigma^n = \{x \in R^{n+1}\colon |x_i| = 1 \text{ for some } i\ (1 \leqslant i \leqslant n+1) \text{ and } 0 \leqslant |x_j| \leqslant 1$ for all $j\ (1 \leqslant j \leqslant n+1)\}$ and

$$\Sigma_n^+ = \{x \in \Sigma^n\colon x_{n+1} \geqslant 0\}, \qquad \Sigma_n^- = \{x \in \Sigma^n\colon x_{n+1} \leqslant 0\}.$$

Similarly we could define $\pi_n(X, *)$ to be $[I^n, \Sigma^{n-1}; X, *]$, where

$$I^n = \{x \in R^n\colon |x_i| \leqslant 1 \text{ for each } i\}.$$

There is another definition of homotopy groups [see (**2**) 7] which we refer to as the Kan definition. Although it refers to css-sets, degeneracies are not needed for the definition and again using 6.5 one readily shows that the result agrees with our definitions.

THEOREM 6.6. *Let* X *and* Y *be connected pointed Kan* Δ-*sets and let* $f\colon X \to Y$ *be a pointed* Δ-*map which induces isomorphisms*

$$f_*\colon \pi_n(X) \to \pi_n(Y) \quad \textit{for all } n \geqslant 0.$$

Then f *is a homotopy equivalence.*

Proof. From Whitehead's theorem we have that $|f|\colon |X| \to |Y|$ is a homotopy equivalence. Let $g\colon |Y| \to |X|$ be a homotopy inverse.

Use 5.3 to homotope g to $|g'|$ where $g'\colon Y \to X$ is a Δ-map, and again to replace $H\colon |f|\,|g'| \simeq 1$ and $G\colon |g'|\,|f| \simeq 1$ by homotopies in $\mathbf{\Delta}$.

Remark 6.7. Theorem 6.6 gives simple conditions for $X \subset Y$ to be a deformation retract. For example, the condition used in (4) § 5: any Δ-map $\Lambda_{n,i} \to Y$ which carries $\dot\Lambda_{n,i}$ into X extends to a Δ-map $\Delta^n \to Y$ which carries $\partial_i \Delta^n$ into X. This can now be interpreted as saying that a typical element of $\pi_n(Y, X)$ is zero.

THEOREM 6.8. *Let* hcw *denote the category of* CW-*complexes and homotopy classes of maps. Then* $||\colon h\mathbf{\Delta} \to h\mathrm{cw}$ *is a natural equivalence.*

Proof. There is no loss in assuming that all Δ-sets and CW-complexes are connected.

There are adjunction morphisms $i_X\colon X \to S|X|$ and $j_Y\colon |S(Y)| \to Y$ [see (3)], and since $j_{|X|} \circ |i_X| = \mathrm{id}_{|X|}$, it suffices to prove that i_X is a homotopy equivalence or, by 6.6, that $i_*\colon \pi_n(X) \to \pi_n(S|X|)$ is an isomorphism. Now i_* is a monomorphism and to see that i_* is onto it is convenient to use the combinatorial definition of $\pi_n(\ )$ and let

$$[f_0] \in \pi_n(S|X|),$$

so that $f_0\colon (I^n, \Sigma^{n-1}) \to (S|X|, i_X(*))$. Then

$$j_{|X|} \circ |f_0|\colon (|I^n|, |\Sigma^{n-1}|) \to (|X|, |*|),$$

and this is homotopic rel$|\Sigma^{n-1}|$ to the realization of a polyhedral map $|f_1|\colon (|I^n|, |\Sigma^{n-1}|) \to (|X|, |*|)$. We claim that $i_*[f_1] = [f_0]$. To see this, triangulate the domain of the above homotopy extending the given triangulations on the ends. The adjoint of the result is the required homotopy $f_0 \simeq i_X f_1$.

We now prove a similar result for $||_M\colon h\mathrm{css} \to h\mathrm{cw}$ (this result is well known, see for example (2)). First we show that $F\colon h\mathrm{css} \to h\mathbf{\Delta}$ is a natural equivalence. Unfortunately its adjoint $G\colon h\mathbf{\Delta} \to h\mathrm{css}$ is not well-defined since $G(X)$ may not be Kan even if X is Kan. An example is provided by $X = D_n$, the complex of § 8. The situation is easily remedied:

Definition. Let X be a Δ-set and define $H^1(X) \supset X$ to be the Δ-set obtained from X by adjoining an n-simplex to X for each Δ-map $f\colon \Lambda_{n,i} \to X$, and inductively define $H^n(X) = H^1(H^{n-1}(X))$. Let $H(X)$ be the union $\bigcup H^n(X)$. If $f\colon X \to Y$ then it is clear how to define $H(f)\colon H(X) \to H(Y)$ and H becomes a functor—the *horn* functor. Further, $H(X)$ is clearly Kan.

THEOREM 6.9. $F\colon h\mathrm{css} \to h\mathbf{\Delta}$ *is an equivalence of categories and* HG *is an inverse for* F.

Proof. Suppose Y is Kan. Then any css-map $GX \to Y$ admits an extension $HGX \to Y$ which is easily seen to be unique up to homotopy, and by 1.7 we have a bijection $\phi: [X, FY] \to [HGX, Y]$ and maps $f: X \to FGHX$, $g: HGFY \to Y$. It is enough to show that f and g are homotopy equivalences. The first follows easily from Remark 5.8, the obvious deformation retraction $|FHGX| \to |FGX|$, and Theorem 6.8. For the second we use the Whitehead theorem in **css** [see (**2**)] so that all we need prove is that $g_x: \pi_n(HGFY) \to \pi_n(Y)$ is an isomorphism for each n. (There is no loss in assuming Y connected.) But using the Kan definition of $\pi_n(\ )$ we can forget degeneracies, and by 6.8 again it is enough to show that $|FGHFY| \to |FY|$ is a homotopy equivalence, and this is a special case, $X = FY$, of the result already proved.

COROLLARY 6.10. $||_M$: h**css** $\to h$**cw** *is an equivalence and* $S(\ )$ *is an inverse.*

Proof. This follows from 6.8, 6.9, the deformation retraction $|HY| \to |Y|$, and commutativity in the diagrams of § 2.

7. The homotopy lifting property

A Δ-map $\Pi: E \to B$ has the *extension lifting property*, ELP, for a pair (W, Z) if given $f: W \to B$ and $\tilde{f}_1: Z \to E$ such that $\Pi \circ \tilde{f}_1 = f \mid Z$, then there exists an $\tilde{f}: W \to E$ such that $\Pi\tilde{f} = f$ and $\tilde{f} \mid Z = \tilde{f}_1$.

$\Pi: E \to B$ is a Kan fibration if Π has the ELP for $(\Delta^n, \Lambda_{n,i})$ ($n \geqslant 0$, $0 \leqslant i \leqslant n$).

PROPOSITION 7.1. *A Kan fibration has the ELP for* (W, Z) *if* $W \searrow Z$.

Proof. This follows by induction on the collapse.

Remark 7.2. In particular we may take $(W, Z) = (Z \otimes I, Z \otimes \{0\})$ so that homotopies in **Δ** may be lifted.

PROPOSITION 7.3. *A Kan fibration of Kan Δ-sets has the ELP for pairs* (W, Z) *with the property that* $(|W|, |Z|)$ *is isomorphic, as a polyhedron, with* $(I^n \times I, I^n \times \{0\})$.

Proof. Let $C(X)$ denote the cone on X. Extend $\tilde{f}_1$ to $\tilde{f}_2: C(Z) \to E$, by the GEC. Let $f_2 = \Pi\tilde{f}_2$. Now extend $f \cup f_2$ to $f_3: C(W) \to B$ by the GEC. Now $C(W) \searrow C(Z)$ and so, by 7.1, f_3 lifts to $\tilde{f}_3: C(W) \to E$. Then $\tilde{f} = \tilde{f}_3 \mid W$ is the required lift of f.

Remark 7.4. From 7.3 one easily proves that if $\Pi: E \to B$ is a pointed Δ-map of Kan sets and $F = \Pi^{-1}(*)$ then $\Pi_*: \pi_n(E, F) \to \pi_n(B)$ is an isomorphism. Thus we have the usual exact sequence of a fibration.

THEOREM 7.5. *A Kan fibration of Kan Δ-sets has the ELP for pairs (W, Z) such that $(|W|, |Z|) = (P\times I, P)$ for some polyhedron P.*

COROLLARY 7.6. *A Kan fibration of Kan sets has the homotopy lifting property for polyhedral maps.*

Proof. Let J be another copy of I (to avoid confusion) and triangulate $P\times I\times J$ as follows. Use W on $P\times I\times\{0\}$ and on $P\times I\times\{1\}$ use a stellar subdivision W' of W such that $|\sigma|\times I$ is a subcomplex of W' for each $\sigma\in Z$ [see (9) Lemma 4]. Now extend over $W\otimes J$ by deriving at the halfway level. Let the whole triangulation be $\hat{W}$ and the restriction to $P\times\{0\}\times J$ be $\hat{Z}$.

Now extend $f\colon W\to B$ to $\hat{f}\colon \hat{W}\to B$ using the Kan condition and the collapse $\hat{W}\searrow W$ (cf. 4.3). We lift $\hat{f}$ and this lifts f as required. First lift $\hat{f}\,|\,\hat{Z}$ using 7.1 and the collapse $\hat{Z}\searrow Z$. Then lift $\hat{f}\,|\,W'$ by inductive use of 7.2. Finally lift $\hat{f}$ using 7.1 and the collapse $\hat{W}\searrow W'\cup\hat{Z}$.

8. The minimal complex

Throughout this section X denotes a Kan Δ-set.

Definition. We define a *minimal complex* $X_0\subset X$ as follows. Choose one 0-simplex from each component of X. The result is $X_0^{(0)}$. Suppose now that $X_0^{(n-1)}$ has been defined. We say that simplexes $\sigma,\tau\in X^{(n)}$ are equivalent and write $\sigma\sim\tau$ if $\partial_i\sigma=\partial_i\tau$ $(0\leqslant i\leqslant n)$ and there is a homotopy $h_t\colon|\Delta^n|\to|X|\operatorname{rel}|\dot{\Delta}^n|$ with $h_0=|\tilde{\sigma}|$, $h_1=|\tilde{\tau}|$ the characteristic maps of σ and τ. Define $X_0^{(n)}$ by choosing a simplex from each equivalence class which contains simplexes σ with $\partial_i\sigma\in X_0^{(n-1)}$ for each i.

It follows from 5.3 that the equivalence relation can also be defined as follows: let ${}^0\Delta^{n+1}$, ${}^1\Delta^{n+1}$ be copies of Δ^{n+1} and let D_{n+1} be the result of identifying ${}^0\Lambda_{n+1,0}$ with ${}^1\Lambda_{n+1,0}$. Then there are inclusions

$$i_k\colon \Delta^n \xrightarrow{\delta_0} \Delta^{n+1} \cong {}^k\Delta^{n+1}\subset D_{n+1}\quad \text{for } k=0,1.$$

We now say that σ is equivalent to τ if there exists $f\colon D_{n+1}\to X$ extending $\tilde{\tau}i_0^{-1}$ and $\tilde{\sigma}i_1^{-1}$.

LEMMA 8.1. *If $X_0\subset X$ is minimal then it is Kan.*

Proof. Let $f\colon\Lambda_{n,i}\to X_0$ be a Δ-map. Since X is Kan there is an extension $f_1\colon\Delta^n\to X$. Now consider $|f_1\circ\delta_i|\colon|\Delta^{n-1}|\to|X|$. This is homotopic $\operatorname{rel}(\delta,(\dot{\Delta}^{n-1}))$ to a map $|f_2|\colon|\Delta^{n-1}|\to|X_0|$, by the definition of X_0. Extend the homotopy to a homotopy $\operatorname{rel}|\dot{\Delta}^n|$ of $|\Delta^n|$. Let the resulting map be $f_3\colon|\Delta^n|\to X$. Finally use the definition of X_0 once more to homotope f_3 to the realization of the required map

$$|f_4|\colon|\Delta^n|\to|X_0|.$$

LEMMA 8.2. *Let $X_0 \subset X$ be a minimal complex. Then X_0 is a deformation retract of X.*

Proof. Let $f\colon (\Lambda_{n,i}, \dot{\Lambda}_{n,i}) \to (X, X_0)$ be a Δ-map. Then there is an extension $f_1\colon \Delta^n \to X$. By the definition of X_0 and by 5.3 we may assume $f_1 \delta_0(\Delta^{n-1}) \subset X_0$. The result then follows from 6.7.

THEOREM 8.3. *Suppose X and Y are Kan Δ-sets of the same homotopy type and $X_0 \subset X$, $Y_0 \subset Y$ are minimal. Then any homotopy equivalence $f\colon X_0 \to Y_0$ is an isomorphism.*

Proof. Let $f\colon X_0 \to Y_0$ be a homotopy equivalence and let $g\colon Y_0 \to X_0$ be a homotopy inverse. We show that $fg = 1$ and $gf = 1$, which proves the result. Let $\sigma^n \in Y_0$ and suppose inductively that $fg \mid \dot{\sigma} = 1$. Then $fg(\sigma) \sim \sigma$ by the HEP and the fact that $fg \simeq 1$. It follows that $fg(\sigma) = \sigma$ as required.

COROLLARY 8.4. *Any two minimal complexes of X are isomorphic.*

Proof. This follows from 8.2 and the theorem.

COROLLARY 8.5. *There is a* 1:1 *correspondence between minimal complexes and homotopy types.*

COROLLARY 8.6. *Suppose that $X_0 \subset X$ is minimal and $X_0' \subset X_0$ is a homotopy equivalence so that in particular X_0' is Kan. Then $X_0' = X_0$.*

Proof. Let $X_0'' \subset X_0'$ be minimal. Then by 8.2 and the theorem $X_0'' \subset X_0' \subset X_0$ is an isomorphism.

9. Function spaces

Definition. Let X, Y be Δ-sets. A Δ-set X^Y is defined as follows. A typical k-simplex is a Δ-map $\sigma\colon Y \otimes \Delta^k \to X$. Face maps are defined by restriction.

THEOREM 9.1. *If X is Kan then so is X^Y.*

Proof. A Δ-map $f\colon \Lambda_{n,i} \to X^Y$ corresponds to a map $f'\colon Y \otimes \Lambda_{n,i} \to X$ and this extends over $Y \otimes \Delta^n$ by the GEC. This is sufficient.

The following theorem may be regarded as generalizing 1.7.

THEOREM 9.2. *If X is* CSS *and Y a Δ-set then the Δ-sets $F(X^{GY})$ and $(FX)^Y$ are isomorphic.*

Proof. A k-simplex of $(FX)^Y$ is a Δ-map $Y \otimes \Delta^k \to FX$. By 1.6 and 3.2 this corresponds to a CSS-map $GY \times G\Delta^k \to X$, i.e. a k-simplex of X^{GY}. Since this correspondence commutes with face maps we have the result on forgetting degeneracies in X^{GY}.

REFERENCES

1. D. M. Kan, 'Abstract homotopy III', *Proc. Nat. Acad. Sci.* 42 (1956) 419–21.
2. P. May, *Semisimplicial objects in algebraic topology* (Van Nostrand, 1967).
3. J. W. Milnor, 'The realisation of a semisimplicial complex', *Ann. of Math.* 65 (1957) 357–62.
4. C. P. Rourke and B. J. Sanderson, 'Block bundles: I', ibid. 87 (1968) 1–28.
5. J. H. C. Whitehead, 'Simplicial spaces, nuclei and *m*-groups', *Proc. London Math. Soc.* 45 (1939) 243–337.
6. —— 'Combinatorial homotopy: I', *Bull. Am. Math. Soc.* 55 (1949) 213–45.
7. —— 'A certain exact sequence', *Ann. of Math.* 52 (1950) 51–110.
8. E. C. Zeeman, 'Relative simplicial approximation', *Proc. Camb. Phil. Soc.* 60 (1964) 39–43.
9. —— *Seminar on combinatorial topology* (I.H.E.S. Paris, 1963–6).

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Δ-SETS II: BLOCK BUNDLES AND BLOCK FIBRATIONS

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Introduction

IN 'Δ-sets I' we showed how to handle semi-simplicial complexes without degeneracies. In this paper we apply some of the results to semi-simplicial groups and monoids. Our results have application in the theory of block bundles. This paper is organized as follows:

In § 1 we define a principal G-bundle where G is an arbitrary Kan Δ-group and we construct a Kan classifying space for such bundles. The construction is based on Heller's method in (**3**). We then define a G-bundle over a polyhedron and deduce a classification of concordance classes of such bundles. Examples of G-bundles over polyhedra are principal bundles of block bundles.

In § 2 we give a considerably more general definition of a block bundle than has been given elsewhere [cf. (**1**), (**4**), (**5**), (**6**)]. The base is an arbitrary Δ-set, the fibre an arbitrary topological space F, and the group an arbitrary Kan subgroup of $\tilde{\mathrm{Top}}(F)$. We construct a universal block bundle of this type. We then show that amalgamation and subdivision of such bundles is a formal consequence of the fact that the group is Kan and satisfies a natural 'amalgamation' condition. This recovers results proved geometrically by ourselves [in (**6**)] and others [in particular by Casson in (**1**)].

In § 3 we introduce a new kind of homotopy bundle, namely a 'block fibration'. This is the correct homotopy analogue of a block bundle, a block bundle being itself an example. We construct a universal block fibration and classify block fibrations within block homotopy equivalence. Since a Serre fibration gives a block fibration in a natural way (and conversely) this classification recovers the one in (**8**). The idea of a block fibration was arrived at while trying to understand the Serre fibration associated with a block bundle [see (**6**) § 5] and the notion eliminates previous difficulties. See also (**1**) for a construction of the associated Serre fibration.

We end this introduction with a short discussion of the foundations of block bundle theory. The block bundles we define here (in § 2) all

have a local block triviality condition, in other words we assume the existence of 'charts', and our results show that there is a good 'theory' of such bundles as a formal consequence of the definition. On the other hand, the PL block bundles we defined in (6) § 1 did not have charts and we proved the existence of charts geometrically using relative regular neighbourhoods. This approach is good when the fibre is a polyhedral cone and the base is a polyhedron and then charts exist by a similar argument to (6) § 7 using Cohen's regular neighbourhood theory (2).

0. Definitions

We refer to (7) as I. Notation and definitions are as in I, and we recall the principal ones.

Δ is the category with objects Δ^n ($n = 0, 1,\ldots$) and maps order-preserving injective simplicial maps. A Δ-set, -group, -monoid is a contravariant functor from Δ to the category of sets, groups, monoids. If G is a Δ-group, we denote by e_n the identity in $G^{(n)} = G(\Delta^n)$. An ordered simplicial complex K is regarded as a Δ-set by letting $K^{(n)}$ be the set of order-preserving injective simplicial maps $\Delta^n \to K$ and defining $K(\lambda)$ for $\lambda \in \mathrm{Map}(\Delta)$ by composition. If X is a Δ-set and $\sigma \in X^{(n)}$ is an n-simplex then the *characteristic map* $\tilde{\sigma}\colon \Delta^n \to X$ is defined by $\tilde{\sigma}(\lambda) = \lambda^\#(\sigma)$ and we denote by $\boldsymbol{\sigma}$ the subcomplex of X consisting of σ together with all its faces (here we write $\lambda^\#$ for $X(\lambda)$, as usual, and use 'complex' synonymously with 'Δ-set'). $\dot{\boldsymbol{\sigma}} \subset \boldsymbol{\sigma}$ consists of all faces of σ.

$\delta_i\colon \Delta^{n-1} \to \Delta^n$ is the map in Δ which fails to cover the ith vertex and we write ∂_i for $\delta_i^\#$. $\Lambda_{n,i} = \dot{\Delta}^n - \delta_i(\Delta^{n-1})$ is the ith *horn*.

If $K \supset L$ are Δ-sets we write $K \searrow^{s} L$ if $K - L$ consists of two simplexes σ, τ with $\tau = \partial_i(\sigma)$, some i, and σ, τ not the faces of other simplexes of K. We write $K \searrow L$ if $K \searrow^{s} K_1 \searrow^{s} \ldots \searrow^{s} K_n = L$ and we write $K \searrow 0$ if $K \searrow$ a vertex.

1. Principal bundles

Let G be a Kan Δ-group. A Δ-map $E \times G \to E$ is a *free action* of G on E if

(i) $(\sigma g_1)g_2 = \sigma(g_1 g_2)$,

(ii) $\sigma e_n = \sigma$,

(iii) $\sigma g_1 = \sigma g_2 \Leftrightarrow g_1 = g_2$,

for all $\sigma \in E^{(n)}$, $g_1, g_2 \in G^{(n)}$ ($n \geqslant 0$).

A *principal G-bundle* ξ/B with *base* B, where B is a Δ-set, consists of

(i) a surjective Δ-map $p\colon E(\xi) \to B$, the *projection* of ξ,

(ii) a free G-action $E \times G \to E$ over p (i.e. $p(\sigma g) = p(\sigma)$ for all σ, g), such that $p^{-1}(\sigma) = \tau G^{(n)}$ whenever $p(\tau) = \sigma \in B^{(n)}$. In other words B is canonically isomorphic to the orbit space of E under the action of G.

Principal G-bundles $\xi_0, \xi_1/B$ are *isomorphic* if there is a Δ-isomorphism h: $E_0 \to E_1$ which commutes with the projection and with the action of G (i.e. $p_0 = p_1 h$ and $h(\sigma g) = h(\sigma)g$ for all σ, g).

ξ/B is *trivial* if it is isomorphic with the trivial G-bundle ε/B, given by $E(\varepsilon) = B \times G$, $(\sigma, g_1)g_2 = (\sigma, g_1 g_2)$, and $p(\sigma, g) = \sigma$.

Given ξ/B and $B_0 \subset B$ define the restriction $\xi|B_0$ by $E(\xi \mid B_0) = p^{-1}(B_0)$ with induced action and projection.

A *bundle map* f: $\xi_1 \to \xi_2$ is a pair of Δ-maps such that the following diagram commutes:

$$\begin{tikzcd} E(\xi_1) \arrow[r, "f_E"] \arrow[d] & E(\xi_2) \arrow[d] \\ B_1 \arrow[r, "f_B"] & B_2 \end{tikzcd}$$

and f_E commutes with the action of G.

If ξ/B is a principal G-bundle and f: $X \to B$ is a Δ-map then we define the *induced bundle* $f^*(\xi)/X$ by $E(f^*(\xi)) \subset X \times E(\xi)$ consists of pairs (σ, τ) such that $f(\sigma) = p(\tau)$, $(\sigma, \tau)g = (\sigma, \tau g)$, and $p_{f^*(\xi)}(\sigma, \tau) = \sigma$. Then $(\hat{f}, f)$ is a bundle map, where $\hat{f}(\sigma, \tau) = \tau$.

The following easy proposition is left to the reader:

PROPOSITION 1.1. *Let f: $\xi_0 \to \xi_1$ be a bundle map. Then there is a unique isomorphism h: $E(f_B^*(\xi_1)) \to E(\xi_0)$ so that*

$$\begin{tikzcd} E(f_B^*(\xi_1)) \arrow[dr, "\hat{f}_B"] \arrow[dd, "h"'] & \\ & E(\xi_1) \\ E(\xi_0) \arrow[ur, "f_B"'] & \end{tikzcd}$$

commutes.

Now for each Δ-set B let $PG(B)$ denote the set of isomorphism classes of principal G-bundles with base B, and for each Δ-map f: $B_0 \to B_1$ let $PG(f)$: $PG(B_1) \to PG(B_0)$ be induced by f^*.

Then $PG(\ )$ becomes a contravariant functor on the category $\mathbf{\Delta}$. Our aim is to represent $PG(\ )$.

PROPOSITION 1.2. *If p: $E \to B$ is the projection of a principal G-bundle then p is a Kan fibration.*

(Note that neither E nor B is assumed to be Kan.)

Proof. We must find an h' which makes the diagram below commute:

$$\begin{tikzcd} \Lambda_{n,k} \arrow[r, "h"] \arrow[d, phantom, "\cap"] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "f"] \arrow[ur, dashed, "h'"] & B \end{tikzcd}$$

Let $\hat{f}\colon \Delta^n \to E$ be any lifting of f ($\hat{f}$ is the characteristic map of any simplex in $p^{-1}(f(id_{\Delta^n}))$. Let $e\colon \Lambda_{n,k} \to G$ be defined by $\hat{f}(x)e(x) = h(x)$. e is easily seen to be a Δ-map. Since G is Kan, e extends to $e'\colon \Delta^n \to G$. Then h' defined by $h'(x) = \hat{f}(x)e'(x)$ is the required map.

Corollary 1.3. *Suppose ξ_i/Δ^n are principal G-bundles ($i = 0, 1$) and $h\colon \xi_0 | \Lambda_{n,i} \to \xi_1 | \Lambda_{n,i}$ is an isomorphism. Then h extends to an isomorphism h_1 of ξ_0 with ξ_1.*

Proof. Let $s\colon \Delta^n \to E(\xi_0)$ be a section (a lifting of $id|_{\Delta^n}$). Then $h \circ s|\colon \Lambda_{n,i} \to E(\xi_1)$ extends to a section $s_1\colon \Delta^n \to E(\xi_1)$ by 1.2. Define $h_1(s(\sigma)g) = s_1(\sigma)g$ for each $\sigma \in \Delta^n$.

Corollary 1.4. *If $K \searrow L$ and if ξ_i/K ($i = 0, 1$) are G-bundles then any bundle isomorphism $h\colon \xi_0 | L \to \xi_1 | L$ extends to an isomorphism $h'\colon \xi_0 \to \xi_1$.*

Proof. Suppose the collapse is elementary across σ^n from $\tau^{n-1} = \partial_i \sigma^n$. Let $\bar{\sigma}\colon \Delta^n \to K$ be the characteristic map for σ. Then h defines an isomorphism

$$\bar{h}\colon \bar{\sigma}^*(\xi_0) | \Lambda_{n,i} \to \bar{\sigma}^*(\xi_1) | \Lambda_{n,i}.$$

$\bar{h}$ extends over Δ^n by 1.3 and this defines an extension of h since σ and τ are not identified with faces of other simplexes of K. The result now follows by induction on the collapse.

Corollary 1.5. *If $K \searrow 0$ then any ξ/K is trivial.*

Corollary 1.6. *If $K \searrow L$ and if ξ/L is a G-bundle then there is a G-bundle ξ_1/K with $\xi_1 | L = \xi/L$.*

Proof. Suppose the collapse is elementary across σ^n from $\tau^{n-1} = \partial_i \sigma^n$, and $g\colon \Lambda_{n,i} \to L$ is the restriction of $\bar{\sigma}$. $g^*(\xi)$ is trivial by 1.5 and we can define ξ_1 by attaching ε/Δ^n to ξ by $\hat{g}\colon E(g^*(\xi)) \to E(\xi)$. The general result follows by induction.

Corollary 1.7. *Suppose that ξ, $\eta/B \otimes I$ are two principal G-bundles and that $h\colon E(\xi \mid B \otimes \{0\}) \to E(\eta \mid B \otimes \{0\})$ is an isomorphism. Then h extends to an isomorphism of ξ with η.*

Proof. This follows from 1.4 by induction over the skeleta of B using the fact that $\Delta^n \otimes I \searrow \Delta^n \otimes \{0\} \cup \dot{\Delta}^n \times I$, which implies (cf. I, § 4) that for $\sigma \in B$, $\boldsymbol{\sigma} \otimes I \searrow \boldsymbol{\sigma} \otimes \{0\} \cup \dot{\boldsymbol{\sigma}} \otimes I$.

Remark. 1.7 will imply (see 1.11 below) that $\xi \mid B \otimes \{0\} \cong \xi \mid B \otimes \{1\}$, since it will follow from the existence of a Kan classifying space that there is a bundle over $B \otimes I$ with ends isomorphic to $\xi \mid B \otimes \{0\}$.

Construction of the universal bundle. We will construct a principal bundle γ with projection π_γ: $EG \to BG$ so that

(i) both EG and BG are Kan complexes,

(ii) EG is contractible.

We define $EG^{(n)}$. For any Δ-set K, denote by K_0 the graded set of simplexes of K (i.e. forget the face operators). Define $EG^{(n)}$ to be the set of graded functions $\Delta^n_0 \to G_0$.

Then $EG^{(n)}$ is a group, since we can multiply two graded functions by multiplying images in G_0, and we can identify $G^{(n)}$ with the subgroup of $EG^{(n)}$ corresponding to Δ-maps $\Delta^n \to G$ (a Δ-map determines a graded function on forgetting face maps).

We now define face operators in EG, making it a Δ-group. Let λ: $\Delta^r \to \Delta^n$ be a face map. Then we have the corresponding map of graded sets λ_0: $\Delta^r_0 \to \Delta^n_0$ and we define for $\sigma \in G^{(n)}$,

$$\lambda^\# \sigma = \sigma\lambda_0 \colon \Delta^r_0 \to G_0.$$

The reader will have no trouble checking that $\lambda^\#$ is a homomorphism and that this makes $G \subset EG$ a Δ-subgroup.

Observe that if K is a Δ-set then Δ-maps $K \to EG$ can be identified with graded functions $K_0 \to G_0$ and hence:

OBSERVATION 1.8. *Any Δ-map $\dot{\Delta}^n \to EG$ possesses an extension to Δ^n.*

For a graded function $\dot{\Delta}^n_0 \to G_0$ clearly possesses an extension to Δ^n_0.

COROLLARY 1.9. *EG is Kan and contractible.*

Proof. Extend a Δ-map $\Lambda_{n,i} \to EG$ in two stages using 1.8. Extend first over $\partial_i \Delta^n$, then over Δ^n. The second part now follows from 1.8 and I, 6.6.

Define $BG = EG/G$ (i.e. the Δ-set of right cosets of G in EG) and let π_γ: $EG \to BG$ be the natural projection. Then we have defined a principal G-bundle γ/BG with $E(\gamma) = EG$ and it follows from 1.2 and 1.9 that BG is Kan.

PROPOSITION 1.10. *Let $L \subset K$ be Δ-sets and ξ/K a G-bundle. Any bundle map $f\colon \xi \mid L \to \gamma$ extends over ξ.*

Proof. By induction over the skeleta of $K - L$ we may assume $L, K = \dot{\sigma}^n, \sigma^n$. Let $\eta/\Delta^n = \tilde{\sigma}^*(\xi)$. We extend $h = f \circ \hat{\sigma}\colon \eta \mid \dot{\Delta}^n \to \gamma$ to $h'\colon \eta \to \gamma$, and this determines the required extension of f. Let

$$s\colon \Delta^n \to E(\eta)$$

be a section; $h \circ s\colon \dot{\Delta}^n \to EG$ extends to $s_1\colon \Delta^n \to EG$ by 1.8. Now define $h'(s(\sigma)g) = s_1(\sigma)g$ for $\sigma \in \Delta^n$.

COROLLARY 1.11. *Suppose $\xi/B \otimes I$ is a principal G-bundle. Then $\xi \mid B \otimes \{0\} \cong \xi \mid B \otimes \{1\}$.*

Proof. By 1.7 it is only necessary to find a bundle $\eta/B \otimes I$ with $\eta \mid B \otimes \{i\} \cong \xi \mid B \otimes \{0\}$ ($i = 0, 1$). Let $f\colon \xi \mid B \otimes \{0\} \to \gamma$ be a bundle map (from 1.10) and let $h\colon B \otimes I \to BG$ be a homotopy of f_B to itself (see I, 6.1). Then $\eta = h^*(\gamma)$ is the required bundle.

Now for each Δ-map $f\colon K \to BG$ define $T(f) \in PG(K)$ to be the class of $f^*(\gamma)$. By 1.11 $T(f)$ depends only on the homotopy class of f. T is then a natural transformation from $[\ , BG]$ to $PG(\ )$ and is an isomorphism of sets by 1.10. ($[\ , BG]$ is regarded as a functor via I, 6.1.) We have proved:

THEOREM 1.12. *The natural transformation*

$$T\colon [\ , BG] \to PG(\ )$$

defined by $T[f] = [f^*(\gamma)]$ is a natural equivalence of functors on the category of Δ-sets.

Remarks. (1) The construction of γ is clearly functorial on the category of Δ-groups.

(2) If $H \subset G$ is a Δ-subgroup then one has a fibration (up to homotopy type)

$$G/H \to BH \to BG.$$

For factor the universal bundle of G by H and use the fact that

$$EG/H \simeq BH$$

from the classification theorem (cf. 3.18).

(3) Given a Kan fibration

$$G_1 \subset G_2 \xrightarrow{\pi} G_3$$

of Kan Δ-groups with π a homomorphism, then there is a corresponding fibration

$$BG_1 \subset BG_2 \xrightarrow{B\pi} BG_3$$

of classifying spaces. That $B\pi$ is a Kan fibration follows from the commutative diagram

$$\begin{array}{ccc} EG_2 & \xrightarrow{E\pi} & EG_3 \\ \downarrow & & \downarrow \\ BG_2 & \xrightarrow{B\pi} & BG_3 \end{array}$$

in which the other three maps are all Kan fibrations ($E\pi$ trivially from definitions, the vertical maps by 1.2). Then the reader may readily identify the fibre of $B\pi$ with BG_1.

(4) There is a natural identification of $B(G_1 \times G_2)$ with $B(G_1) \times B(G_2)$, since there is a natural identification of $E(G_1 \times G_2)$ with $E(G_1) \times E(G_2)$.

Principal bundles over polyhedra. Let P be a polyhedron and G a Kan Δ-group. A *G-bundle* ξ/P is an ordered triangulation K of P and a principal G-bundle ξ/K. $\xi_0, \xi_1/P$ are *equivalent* if there is a G-bundle $\eta/P \times I$ such that $\eta \mid P \times \{i\} \cong \xi_i$ ($i = 0, 1$). Let $G(P)$ denote the set of equivalence classes.

PROPOSITION 1.13. *The function* $T\colon [P; BG] \to G(P)$ *defined by* $T(f) = f^*(\gamma)$ *is a bijection.*

Proof. This follows at once from 1.12 and the definition of polyhedral maps $P \to BG$ (see I, § 6).

Now $[\ ; BG]$ is a functor on the category of polyhedra and continuous maps (see I, 6.1) and we make $G(\ )$ into a functor by insisting that T be a natural equivalence. This defines the induced class $f^*(\xi)/P$ for a map $f\colon P \to Q$ and G-bundle ξ/Q. There is a direct construction of $f^*(\xi)$, as follows.

Let ξ be defined over K and find a triangulation L of P and a simplicial map $f_1\colon L \to K$ homotopic to f. Let M_{f_1} be the simplicial mapping cylinder. Then $M_{f_1} \searrow K$ and hence ξ extends over M_{f_1}, uniquely up to isomorphism, by 1.4 and 1.6. Let this extension be ξ_1. Then $\xi_1 \mid L$ is in the class $f^*(\xi)$ since the classifying map for $\xi_1 \mid L$ is homotopic to f composed with the classifying map for ξ.

COROLLARY 1.14. *If K is a Δ-set such that $|K|$ is homotopy equivalent to P then $PG(K)$ is isomorphic to $G(P)$. In case $|K| = P$ the isomorphism is the natural one.*

Proof. This follows from 1.13 and the results of I, § 6.

Remark. 1.14 recovers (6) 3.3 since for a Δ-set K, $|F(\mathbf{K})| \simeq |K|$. However, the results of the present paper show that (6) 3.3 is irrelevant to block bundle theory.

2. Application to block bundles

Let K be a Δ-set. The associated category of K, denoted by $\mathfrak{K}$, is defined by

$$\mathrm{Ob}(\mathfrak{K}) = \bigcup_n K^{(n)},$$

$$\mathrm{Map}\,\mathfrak{K}(\tau,\sigma) = \{(\lambda,\tau,\sigma) \mid \lambda^{\#}\sigma = \tau\} \quad \text{for } \sigma, \tau \in K.$$

Composition of maps in $\mathfrak{K}$ is just composition of the corresponding face relations.

Now suppose $f\colon K \to L$ is a Δ-map. Then we associate to f the functor $\mathfrak{f}\colon \mathfrak{K} \to \tilde{\mathbf{L}}$ defined by $\mathfrak{f} = f$ on objects and $\mathfrak{f}(\lambda,\tau,\sigma) = (\lambda, f\tau, f\sigma)$, which is a map in $\tilde{\mathbf{L}}$ since f is a Δ-map.

A *K-space* Q is a functor from $\mathfrak{K}$ to the category of topological spaces and embeddings. If $f\colon L \to K$ is a Δ-map then we define the L-space $f^*(Q)$ to be $Q \circ \mathfrak{f}$.

Thus for each $\sigma \in K^{(n)}$ we have the Δ^n-space $Q_\sigma = \tilde{\boldsymbol{\sigma}}^*(Q)$ where we write $\tilde{\boldsymbol{\sigma}}$ for the functor associated to the characteristic map of σ.

Associate to each K-space Q a topological space $|Q|$ defined by

$$|Q| = \bigcup Q(\sigma)/Q(\mathrm{Map}\,\mathfrak{K}),$$

i.e. identify the $Q(\sigma)$ via the family of embeddings $Q(\mathrm{Map}\,\mathfrak{K})$.

A *map* of K-spaces (or K-map) $f\colon Q_1 \to Q_2$ is a natural transformation of functors where the range category is enlarged to include all maps (rather than just embeddings), i.e. f consists of maps $f_\sigma\colon Q_1(\sigma) \to Q_2(\sigma)$ for each $\sigma \in K$ such that the obvious diagrams commute. A *K-homeomorphism* is a K-map in which each f_σ is a homeomorphism. A K-map f determines a map $|f|\colon |Q_1| \to |Q_2|$ in the obvious way.

For any Δ-set K and topological space F, define the *trivial* K-space $\varepsilon(K,F)$ by $\varepsilon(\sigma) = \Delta^n \times F$ for each $\sigma \in K^{(n)}$ and

$$\varepsilon(\lambda,\sigma,\tau) = \lambda \times id\colon \Delta^n \times F \to \Delta^r \times F$$

for each $\sigma \in K^{(n)}$, $\tau \in K^{(r)}$ with $\sigma = \lambda^{\#}\tau$. Then $|\varepsilon|$ can be naturally identified with $|K| \times F$, if we endow the latter with the identification topology. We often write $K \times F$ for $\varepsilon(K,F)$.

Remarks. (1) If Q is a Δ^n-space then the natural map $Q(1_n) \to |Q|$ is a homeomorphism, since each $Q(\lambda)$ is the domain of a unique embedding in $Q(1_n)$. Thus the natural map $Q(\sigma) \to |Q_\sigma|$ is a homeomorphism for each $\sigma \in K$, where Q is any K-space, and we can identify the two.

(2) If K is a simplicial complex and Q is a K-space, then the natural maps $\pi_\sigma\colon Q(\sigma) \to |Q|$ are all embeddings for $\sigma \in K$. This is for similar

reasons to example (1). Hence Q is determined up to K-homeomorphism by the space $|Q|$ and the family of subspaces

$$\pi_\sigma(Q(\sigma)) \subset |Q| \quad (\sigma \in K).$$

(3) Generalizing (2) to Δ-sets, define for each $\sigma \in K$ the *characteristic map* for Q at σ to be the natural map

$$\pi_\sigma\colon Q(\sigma) \to |Q|.$$

Using the identification of (2) we can regard π_σ as a map

$$\pi_\sigma\colon |Q_\sigma| \to |Q|.$$

Then the K-space Q is determined up to K-homeomorphism by the space $|Q|$, the Δ^n-spaces Q_σ for each $\sigma \in K$, and the characteristic maps $|Q_\sigma| \to |Q|$. Compare this with the idea of a Δ-set as a CW-complex $|K|$ together with a set of characteristic maps for the cells of $|K|$ (cf. I, § 4).

A *block bundle* with base K and fibre F is a K-space ξ such that for each $\sigma \in K^{(n)}$ there is a Δ^n-homeomorphism $\Delta^n \times F \to \xi_\sigma$. We usually write $E(\xi)$ for $|\xi|$ and $\beta_\sigma(\xi)$ for $\xi(\sigma)$ (the 'block' over σ).

An *isomorphism* $h\colon \xi_1 \to \xi_2$ of block bundles is simply a K-homeomorphism.

Notice that when K is simplicial, ξ is determined up to isomorphism by $E(\xi)$ and the natural embeddings $\pi_\sigma\colon \beta_\sigma \to E(\xi)$ [see Remark (2) above, and compare (6)]. When K is a Δ-set, ξ/K may be regarded as being made up of block bundles over simplexes, with a recipe for gluing.

Notice that if ξ/K is a block bundle and $f\colon L \to K$ is a Δ-map then $f^*(\xi)$ is a block bundle, the *induced* bundle.

A block bundle ξ is *trivial* if it is isomorphic with the trivial bundle $\varepsilon(K, F)$ defined above.

A *chart* for ξ at $\sigma \in K^{(n)}$ is an isomorphism

$$h_\sigma\colon \varepsilon(\Delta^n, F) \to \xi_\sigma.$$

An *atlas* for ξ is a family $\mathscr{H} = \{h_\sigma \mid \sigma \in K\}$ of charts.

Now let $\tilde{\mathrm{Top}}(F)$ be the Δ-group in which a typical n-simplex is a self-isomorphism of $\varepsilon(\Delta^n, F)$ and the group operation is composition. Face operators are defined by the diagram

$$\begin{tikzcd} \Delta^r \times F \arrow[r, "\lambda \times id"] \arrow[d, "\lambda^\# \sigma"'] & \Delta^n \times F \arrow[d, "\sigma"] \\ \Delta^r \times F \arrow[r, "\lambda \times id"] & \Delta^n \times F \end{tikzcd}$$

i.e. 'by restriction'.

Suppose that $\tau^r = \lambda^\# \sigma^n$ is in K and that h_σ, h_τ are charts for ξ at σ, τ.

Then h_σ, h_τ are *related* by the element $\mu \in \tilde{\mathrm{T}}\mathrm{op}(F)^{(r)}$ defined by the commutative diagram

$$\begin{tikzcd} \Delta^r \times F \arrow[rr, "h_\tau"] \arrow[d, "\mu"] & & \xi_\tau \arrow[d, "\xi(\lambda,\tau,\sigma)"] \\ \Delta^r \times F \arrow[r, "\lambda \times id"] & \Delta^n \times F \arrow[r, "h_\sigma"] & \xi_\sigma \end{tikzcd}$$

If h_σ, h_τ belong to an atlas $\mathscr{H}$, then we write $\mu = \mathscr{H}(\lambda, \tau, \sigma)$.

PROPOSITION 2.1. *If ξ/K is a block bundle with atlas $\mathscr{H}$ then ξ is isomorphic with the bundle ξ_1 defined by*

$$\xi_1(\sigma) = \Delta^n \times F \quad \textit{for each } \sigma \in K^{(n)},$$

$$\xi_1(\lambda, \tau, \sigma) = (\lambda \times id) \circ \mathscr{H}(\lambda, \tau, \sigma) \quad \textit{for each } (\lambda, \tau, \sigma) \in \mathrm{Map}(\mathbf{\bar{K}}).$$

Proof. The homeomorphisms $|h_\sigma|: \xi_1(\sigma) \to \xi(\sigma)$ clearly determine an isomorphism.

2.1 shows that all the information about ξ is contained in the set $\mathscr{H}(\mathrm{Map}(\mathbf{\bar{K}})) \subset \tilde{\mathrm{T}}\mathrm{op}(F)$. This motivates the next definition.

Let $A(F)$ be a Kan subgroup of $\tilde{\mathrm{T}}\mathrm{op}(F)$. An *$A(F)$-block bundle* with base K is a pair $(\xi, \mathscr{H}(\xi))$ where ξ is a block bundle over K with fibre F, and $\mathscr{H}$ is an atlas for ξ such that $\mathscr{H}(\mathrm{Map}(\mathbf{\bar{K}})) \subset A(F)$.

An $A(F)$-isomorphism of $A(F)$-bundles $(\xi_1, \mathscr{H}_1)$, $(\xi_2, \mathscr{H}_2)$ is an isomorphism $g: \xi_1 \to \xi_2$ such that for each $\sigma \in K^{(n)}$ the element $\mu_\sigma \in \tilde{\mathrm{T}}\mathrm{op}(F)^{(n)}$, determined by the following diagram, lies in $A(F)^{(n)}$:

$$\begin{tikzcd} \Delta^n \times F \arrow[r, "(h_\sigma)_1"] \arrow[d, "\mu_\sigma"] & \xi_1(\sigma) \arrow[d, "g(\sigma)"] \\ \Delta^n \times F \arrow[r, "(h_\sigma)_2"] & \xi_2(\sigma). \end{tikzcd}$$

Note that the elements μ_σ determine the isomorphism g.

From now on we will confuse ξ with the pair $(\xi, \mathscr{H})$ and write 'ξ is an $A(F)$-block bundle'.

An *$A(F)$-chart* for ξ at σ is an $A(F)$-isomorphism

$$g_\sigma: \xi(\Delta^n, F) \to \xi_\sigma,$$

where $\varepsilon(K, F)$ has the natural $A(F)$-structure (charts being the identity maps). In other words g_σ is simply a composition

$$\varepsilon(\Delta^n, F) \xrightarrow{\mu} \varepsilon(\Delta^n, F) \xrightarrow{h_\sigma} \xi_\sigma,$$

where $\mu \in A(F)^{(n)}$ and $h_\sigma \in \mathscr{H}(\xi)$.

We associate to ξ the principal $A(F)$-bundle $P(\xi)/K$ defined by

$$E(P(\xi))^{(n)} = \{g_\sigma \mid g_\sigma \text{ is an } A(F)\text{-chart for } \xi \text{ at } \sigma \in K^{(n)}\}.$$

$\lambda^{\#}(g_\sigma)$ is the element determined by

$$\begin{tikzcd} \Delta^r \times F \arrow[r, "\lambda^{\#}(g_\sigma)"] \arrow[d, "\lambda \times id"] & \xi(\tau) \arrow[d, "\xi(\lambda,\tau,\sigma)"] \\ \Delta^n \times F \arrow[r, "g_\sigma"] & \xi(\sigma), \end{tikzcd}$$

where $\tau = \lambda^{\#}\sigma$.

In other words, if $\xi(\lambda^{\#}\sigma)$ is regarded as a subspace of $\xi(\sigma)$ then $\lambda^{\#}(g_\sigma)$ is defined 'by restriction'. It is easy to check that if $g_\sigma = h_\sigma \circ \mu$ then $\lambda^{\#}(g_\sigma) = h_\tau \mathscr{H}^{-1}(\lambda, \sigma, \tau) \circ \lambda^{\#}(\mu)$, and hence lies in $E(P(\xi))^{(r)}$.

Define $E(P/\xi) \times A(F) \to E(P/(\xi))$ by $(g_\sigma, \mu) \to g_\sigma \circ \mu$. Finally define $\pi\colon E(P(\xi)) \to K$ by $\pi(g_\sigma) = \sigma$.

It is trivial to check that isomorphic $A(F)$-bundles yield isomorphic principal $A(F)$-bundles.

We now define the functor $A(F)(\ )$ by letting $A(F)(K)$ be the set of isomorphism classes of $A(F)$-block bundles with base K and setting $A(F)(f)[\xi] = [f^*(\xi)]$, where $f\colon L \to K$ and $f^*(\xi)$ is given the induced $A(F)$-structure: for each $\sigma \in L$ define $h_\sigma = h_{f\sigma}$ where $h_{f\sigma} \in \mathscr{H}(\xi)$ and $h_\sigma \in \mathscr{H}(f^*(\xi))$. This makes sense since $f^*\xi(\sigma) = \xi(f\sigma)$ by definition.

This makes $A(F)(\ )$ into a functor and it is easy to check that

$$P(\ )\colon A(F)(\ ) \to PA(F)(\ ),$$

where $P(K)[\xi] = [P(\xi)]$, is a natural transformation.

THEOREM 2.2. *$P(\ )$ is a natural equivalence of functors on* $\mathbf{\Delta}$.

Proof. We need to show that $P(K)$ is an isomorphism for each K. To prove it onto, let ξ/K be a principal $A(F)$-bundle and choose a pseudo-section $s\colon K \to E(\xi)$, i.e. a function $s\colon K_0 \to E(\xi)_0$ such that $p \circ s = id_K$. Then for each $(\lambda, \tau, \sigma) \in \mathrm{Map}(\bar{\mathbf{K}})$ define the element $\mathscr{H}(\lambda, \tau, \sigma) \in A(F)^{(r)}$ by $s(\tau)\mathscr{H}(\lambda, \tau, \sigma) = \lambda^{\#}(s(\sigma))$. The elements $\mathscr{H}(\lambda, \tau, \sigma)$ determine an $A(F)$-block bundle ξ_1, with identity charts, as in the proof of 2.1 and it is trivial to check that $P(\xi_1) \simeq \xi$. 2.1 implies that ξ_1 depends only on ξ and hence $P(K)$ is an isomorphism of sets, as required.

COROLLARY 2.3. *There is a universal $A(F)$-block bundle $\gamma_{A(F)}$ over $BA(F)$.*

Proof. Let $\gamma_{A(F)}$ be the bundle defined by 2.2 such that $P(\gamma)$ is the universal principal bundle. Then the universality of γ is clear.

Subdivision and amalgamation. We proceed to give formal proofs of the usual results on subdivision and amalgamation.

We say $A(F)$ satisfies the *amalgamation condition* (a.c.) if, given a

linear ordered triangulation J of Δ^n and an $A(F)$-bundle isomorphism $f\colon \varepsilon(J,F) \to \varepsilon(J,F)$, the element $\sigma \in \tilde{\mathrm{T}}\mathrm{op}(F)^{(n)}$ defined by the diagram

$$\begin{tikzcd} |\varepsilon(J,F)| \arrow[r, "|f|"] & |\varepsilon(J,F)| \arrow[d, "id"] \\ \Delta^n \times F \arrow[u, "id"] \arrow[r, "|\sigma|"'] & \Delta^n \times F \end{tikzcd}$$

lies in $A(F)^{(n)}$.

Now let K' be a subdivision of the Δ-set K (see I, § 4). Then for each $\sigma \in K^{(n)}$ we have a linear triangulation J_σ of Δ^n and a Δ-map $f_\sigma\colon J_\sigma \to K'$ such that $|f_\sigma| = |\tilde{\sigma}|$, and for each face relation (λ, τ, σ) a simplicial inclusion $J_\lambda\colon J_\tau \to J_\sigma$.

Now let ξ/K' be an $A(F)$-block bundle. Define a block bundle ξ_1/K, the *amalgamation* of ξ, by letting $\xi_1(\sigma) = |f_\sigma^*(\xi)|$ and defining $\xi_1(\lambda,\tau,\sigma)$ by the diagram

$$\begin{tikzcd} |f_\sigma^*(\xi)| \arrow[r, "\hat{f}_\sigma"] & |\xi| \\ |f_\tau^*(\xi)| \arrow[u, "\xi_1(\lambda,\tau,\sigma)"] \arrow[ur, "\hat{f}_\tau"'] & \end{tikzcd}$$

where $\hat{f}_\sigma$, $\hat{f}_\tau$ are the natural maps. In other words, factor f_τ as $f_\sigma \circ J_\lambda$ and then $\xi_1(\lambda,\tau,\sigma)$ is the bundle map from $f_\tau^*(\xi) = J_\lambda^*(f_\sigma^*(\xi))$ to $f_\sigma^*(\xi)$.

If $A(F)$ satisfies the a.c. then we can give ξ_1 an $A(F)$-structure by choosing an atlas $\mathscr{H}$ with $h_\sigma \in \mathscr{H}$ defined by

$$\begin{tikzcd} |\varepsilon(t_\sigma, F)| \arrow[r, "|\tau\sigma|"] & |f_\sigma^*(\xi)| \\ \Delta^n \times F \arrow[u, "id"] \arrow[ur, "|h_\sigma|"'] & \end{tikzcd}$$

when t_σ is any $A(F)$-trivialization.

Now there is a bijection

$$q\colon A(F)(K') \to A(F)(K)$$

given by 2.3 and I, 5.3, since $|K'| = |K|$.

THEOREM 2.4. *As a block bundle without group (or equivalently as a* $\tilde{\mathrm{T}}\mathrm{op}(F)$*-bundle), $q(\xi)$ is the class given by amalgamating ξ. If, further, $A(F)$ satisfies the a.c. then $q(\xi)$ is given as an $A(F)$-bundle by amalgamating ξ.*

Proof. We show that if $A(F)$ has the a.c. then $q(\xi)$ is given by amalgamating ξ. The first part of the theorem then follows by taking $A(F) = \tilde{\mathrm{T}}\mathrm{op}(F)$.

Let ξ_1 be the amalgamation of ξ, and let J be a subdivision of $K \otimes I$

which has K' on one end and K on the other. Extend ξ/K' to an $A(F)$-bundle η/J and let $\eta_1/K \otimes I$ be the amalgamation. Then η_1 has ξ_1 on one end and ζ, say, on the other, while η has ζ and ξ on its ends. It follows that ξ and ξ_1 bound a bundle α, say, on $L = J \cup_K K \otimes I$. But the classifying map for α gives a homotopy between those for ξ and ξ_1 and it follows that $q(\xi) \cong \xi$, as required.

Let the converse process to amalgamation be *subdivision*.

COROLLARY 2.5. *Subdivisions exist uniquely if $A(F)$ satisfies the a.c.*

Block bundles over polyhedra. Define an $A(F)$-block bundle ξ over a polyhedron P to be an ordered triangulation K of P and an $A(F)$-block bundle ξ/K.

$\xi_0, \xi_1/P$ are *concordant* if there is an ordered triangulation J of $P \times I$ and an $A(F)$-block bundle η/J with $\eta \mid P \times \{i\} \cong \xi_i$ $(i = 0, 1)$.

ξ_0, ξ_1 are *equivalent* if they have isomorphic subdivisions (this is an equivalence relation only if $A(F)$ satisfies the a.c.; see 2.6 below for the proof).

Denote by $A(F)(P)$ the set of concordance classes of $A(F)$-bundles over P (this is coherent with § 1 using 2.2).

COROLLARY 2.6. *Let K triangulate P. The natural map*

$$A(F)(K) \to A(F)(P)$$

is an isomorphism of sets. Further, if $A(F)$ has the a.c. then ξ_0, ξ_1 are equivalent iff they are concordant.

Proof. The first part follows at once from 1.14 and 2.2. Suppose ξ_0, ξ_1 have isomorphic subdivisions. Then they are concordant by 2.4. Conversely if $\xi_0/K_0, \xi_1/K_1$ are concordant, then let J be a common subdivision of K_0, K_1 and let ξ_0', ξ_1' be subdivisions given by 2.5. Then $\xi_0' \cong \xi_1'$ by 2.4.

Remarks. Induced block bundles are defined for a topological map $f: P \to Q$ using 2.2 and 1.12 ($PG(\ )$ is a homotopy functor) but for a direct construction we could use the construction given below 1.13.

3. Block fibrations

Let K be a Δ-set. A *K-complex* Q is a functor from $\tilde{K}$ to the category of cw-complexes and embeddings of subcomplexes which satisfies the following intersection condition for each pair $(\lambda_1, \tau_1, \sigma), (\lambda_2, \tau_2, \sigma) \in \mathrm{Map}(\tilde{K})$:

$$Q(\lambda_1, \tau_1, \sigma)(Q(\tau_1)) \cap Q(\lambda_2, \tau_2, \sigma)(Q(\tau_2)) = \begin{cases} Q(\lambda_3, \tau_3, \sigma)(Q(\tau_3)) & \text{if } \operatorname{Im} \lambda_1 \cap \operatorname{Im} \lambda_2 = \operatorname{Im} \lambda_3, \\ \varnothing & \text{if } \operatorname{Im} \lambda_1 \cap \operatorname{Im} \lambda_2 = \varnothing. \end{cases}$$

A K-map $f\colon Q_1 \to Q_2$ of K-complexes is a map as K-spaces. There is an obvious notion of homotopy of such maps.

PROPOSITION 3.1. *A K-map $f\colon Q_1 \to Q_2$ of K-complexes is a K-homotopy equivalence iff each $f(\sigma)\colon Q_1(\sigma) \to Q_2(\sigma)$ is a homotopy equivalence of* CW*-complexes.*

Proof. 'Only if' is obvious. To prove 'if' define the K-complex M_f by $M_f(\sigma) = M_{f(\sigma)}$, the CW mapping cylinder, with the obvious embeddings. We show that $Q_1 \times \{0\} \subset M_f$ is a strong K-deformation retract and the result follows. Denote $Q_1(\mathbf{K}^n)$, etc., by Q_1^n, etc. Suppose inductively that $Q_1^{n-1} \times \{0\} \subset M_f^{n-1}$ is a strong K^{n-1}-deformation retract and prove the same for n. Then, using the intersection condition, we can work separately for each $\sigma \in K^{(n)}$.

Define the subcomplex $\dot{Q}(\sigma) \subset Q(\sigma)$ to be the union of the images of $Q(\lambda, \tau, \sigma)$ for $\lambda \in \mathrm{Map}(\tilde{\Delta}^n)$ $(\lambda \neq id)$. Then denoting $f(\sigma) \mid \dot{Q}(\sigma)$ by g we have a deformation retract defined by the $r_t(\tau)$ $(\tau < \sigma)$:

$$r_t\colon M_g \to M_g,$$

such that
$$r_t \mid \dot{Q}(\sigma) \times \{0\} = id$$

and
$$r_t(M_g) \subset \dot{Q}(\sigma) \times \{0\}.$$

Now consider the inclusions

$$Q(\sigma) \times \{0\} \subset Q(\sigma) \times \{0\} \cup M_g \subset M_f(\sigma).$$

$Q(\sigma) \times \{0\} \subset M_f(\sigma)$ is a homotopy equivalence by hypothesis, and the first inclusion is a homotopy equivalence since $M_g \cap Q(\sigma) \times \{0\} = \dot{Q}(\sigma) \times \{0\}$. It follows that the second inclusion is a homotopy equivalence and hence that there is a strong deformation retraction

$$\tilde{r}_t\colon M_f(\sigma) \to M_f(\sigma)$$

of $M_f(\sigma)$ to $Q(\sigma) \times \{0\} \cup M_g$.

Extend r_t to M_f by defining $r_t \mid Q(\sigma) \times \{0\} = id$ and then using the HEP for CW-complexes.

Now define $r_t(\sigma)\colon M_f(\sigma) \to M_f(\sigma)$ to be $r_t \circ \tilde{r}_t$. Then $r_t(\sigma)$ is a strong deformation retract compatible with the inclusions of $Q(\tau)$ for each $\tau = \lambda^{\sharp}\sigma$ and hence continues the induction.

A *block fibration* with base K and fibre the CW-complex F is a K-complex ξ such that for each $\sigma \in K^{(n)}$ there is a Δ^n-homotopy equivalence $\Delta^n \times F \to \xi_\sigma$.

A *block homotopy equivalence* of block fibrations is simply a K-homotopy equivalence. ξ is *block homotopy trivial* if it is block homotopy equivalent to $\varepsilon(K, F)$.

Note that for each Δ-map $f: L \to K$ we have the induced fibration $f^*(\xi)$ defined, as before, to be $\xi \circ \bar{f}$. If $L \subset K$ then we write $\xi \mid L$ for $i^*(\xi)$.

PROPOSITION 3.2. *Given ξ/Δ^n a block fibration and a block homotopy trivialization*

$$t: \Lambda_{n,i} \times F \to \xi \mid \Lambda_{n,i}$$

then there is an extension of t to

$$t_1: \Delta^n \times F \to \xi.$$

Proof. Since ξ is a block fibration, there is a block homotopy equivalence $h: \Delta^n \times F \to \xi$. Let $h_1: \xi \to \Delta^n \times F$ be an inverse to h (cf. 3.1).

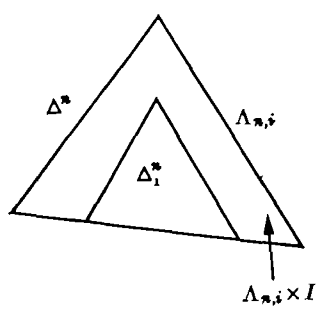
Now $h_1 \circ t: \Lambda_{n,i} \times F \to \Delta^n \times F$ extends to a block homotopy equivalence g, say, of $\Delta^n \times F$ with itself since we can write Δ^n as $\Lambda_{n,i} \times I$.

Then consider $h \circ g: \Delta^n \times F \to E(\xi)$. This is a block homotopy equivalence and $h \circ g \mid \Lambda_{n,i} \times F = h \circ h_1 \circ t$ is block homotopic to t via the homotopy M, say,

$$M: \Lambda_{n,i} \times F \times I \to \xi \mid \Lambda_{n,i},$$

since h and h_1 are block homotopy inverses.

Now write $\Delta^n = \Delta_1^n \cup \Lambda_{n,i} \times I$ when the latter is a collar neighbourhood of $\Lambda_{n,i}$. Δ_1^n is a 'smaller' n-simplex:



Then define t_1 to be M on $\Lambda_{n,i} \times F \times I$ and to be $h \circ g \circ q$ on Δ_1^n, where $q: \Delta_1^n \times F \to \Delta^n \times F$ is the product of the linear identification with the identity map on F.

COROLLARY 3.3. *Given block fibrations ξ_i/Δ^n ($i = 1, 2$) and a block homotopy equivalence $h: \xi_1 \mid \Lambda_{n,i} \to \xi_2 \mid \Lambda_{n,i}$, then h extends to a block homotopy equivalence of ξ_1 with ξ_2.*

Proof. Choose a block homotopy trivialization $g: \Delta^n \times F \to \xi_1$ and an inverse g_1 for g. Then $h \circ g$ extends to h_1, say, by 3.2 and we can homotope h_1 to extend h by the HEP for cw-complexes.

The next three corollaries follow from 3.3 in the same way that 1.4–1.6 followed from 1.3:

COROLLARY 3.4. *If $K \searrow L$ and if ξ_i/K $(i = 1, 2)$ are block fibrations then any block homotopy equivalence $\xi_1 \mid L \to \xi_2 \mid L$ extends to one of ξ_1 with ξ_2.*

COROLLARY 3.5. *If $K \searrow 0$ then any ξ/K is block homotopy trivial.*

COROLLARY 3.6. *If ξ, $\eta/K \otimes I$ are block fibrations then any block homotopy equivalence $\xi \mid K \otimes \{0\} \to \eta \mid K \otimes \{0\}$ extends to one of ξ with η.*

Let ξ/K be a block fibration where K is a Δ-set. A *chart* for ξ at σ is a block homotopy equivalence h_σ: $\Delta^n \times F \to \xi_\sigma$. We now define the associated principal bundle $P(\xi)$ of ξ:

$$P(\xi)^{(n)} = \{h_\sigma \mid h_\sigma \text{ is a chart for } \xi \text{ at } \sigma \in K^{(n)}\},$$

face operators in $P(\xi)$ are defined as in § 2 ('by restriction'), and π: $P(\xi) \to K$ is defined by $\pi(h_\sigma) = \sigma$.

PROPOSITION 3.7. *π: $P(\xi) \to K$ is a Kan fibration.*

Proof. Let $\sigma \in K$ and suppose given a lift f: $\Lambda_{n,i} \to P(\xi)$ for $\tilde{\sigma} \mid \Lambda_{n,i}$. We have to extend f to a lift for $\tilde{\sigma}$. Now f can be identified with a block homotopy equivalence

$$f_1: \Lambda_{n,i} \times F \to \xi_\sigma \mid \Lambda_{n,i},$$

which we can extend to a block homotopy equivalence

$$f_2: \Delta^n \times F \to \xi_\sigma.$$

Define $f(1_n) = f_2$. This extends f over Δ^n, as required.

The prolongation construction. Let ξ/K be a block fibration and f: $\dot{\Delta}^n \to P(\xi)$ a Δ-map. We will construct a new block fibration ξ_1/K_1 which extends ξ. Write $g = \pi f$: $\dot{\Delta}^n \to K$ and define $K_1 = K \cup_g \Delta^n$, i.e. define $K_1^{(n)} = K^{(n)} \cup \{\sigma\}$ and $\lambda^\#(\sigma) = g(\lambda)$ for λ: $\Delta^r \to \Delta^n$. The other simplexes and face maps in K_1 are those in K.

Next write f_1: $\dot{\Delta}^n \times F \to g^*(\xi)$ for the block homotopy equivalence determined by f and define $\xi_1(\sigma) = |g^*(\xi)| \cup_{f_1} \Delta^n \times F$. Then there are natural inclusions of $\xi(\tau)$ in $\xi_1(\sigma)$ for $\tau = \lambda^\# \sigma$ and this determines $\xi_1(\lambda, \tau, \sigma)$. On the rest of K_1 let ξ_1 equal ξ.

Define $\operatorname{Prol}^1(\xi)$ to be the block fibration obtained from ξ by this construction applied to every Δ-map $\dot{\Delta}^n \to P(\xi)$.

Define $\operatorname{Prol}^n(\xi) = \operatorname{Prol}^1(\operatorname{Prol}^{n-1}(\xi))$ and $\underset{\sigma}{\operatorname{Prol}}(\xi) = \bigcup_n \operatorname{Prol}^n(\xi)$.

PROPOSITION 3.8. *The base $B\operatorname{Prol}(\xi)$ of $\operatorname{Prol}(\xi)$ is Kan, and $P(\operatorname{Prol}(\xi))$ is Kan and contractible.*

Proof. Each Δ-map $\dot{\Delta}^n \to P(\operatorname{Prol}(\xi))$ has an extension to Δ^n by the construction of $\operatorname{Prol}^1(\ )$. The proposition now follows from 3.7 exactly as in § 1 (see 1.8, etc.).

COROLLARY 3.9. *Given a block fibration $\xi/K \otimes I$, there is a block homotopy equivalence $\xi \mid K \otimes \{0\} \to \xi \mid K \otimes \{1\}$.*

Proof. By Corollary 3.6 we only have to show the existence of $\xi/K \otimes I$ with $\xi_1/K \otimes \{i\}$ ($i = 1, 2$) both block homotopy equivalent to $\xi/K \times \{0\}$. But consider the inclusion

$$i\colon K \times \{0\} \to B\,\mathrm{Prol}(\xi/K \times \{0\}).$$

Then by 3.8 there is a homotopy j, say, of i to itself.

$$\xi_1 = j^*(\mathrm{Prol}(\xi/K \times \{0\}))$$

now satisfies the requirements.

The classifying block fibration. Let $\tilde{G}(F)$ be the Δ-monoid of which a typical n-simplex is a self block homotopy equivalence of $\Delta^n \times F$. Then for any block fibration ξ/K we have an action $P(\xi) \times \tilde{G}(F) \to P(\xi)$ over K defined by composition.

Let ε^0 denote the trivial block fibration over Δ^0 and define

$$B\tilde{G}(F) = B\,\mathrm{Prol}(\varepsilon^0), \qquad E\tilde{G}(F) = P(\mathrm{Prol}(\varepsilon^0)).$$

Define inductively a base complex $* \subset B\tilde{G}(F)$ so that

$$\mathrm{Prol}(\varepsilon^0) \mid * = \varepsilon(*, F),$$

by letting $*_0 = \Delta^0$ and $*_n$ be the simplex in $B\,\mathrm{Prol}(\varepsilon^0)$ defined by the Δ-map $\dot{\Delta}^n \to P(\mathrm{Prol}^{n-1}(\varepsilon^0))$ determined by the identity maps

$$\Delta^r \times F \to \mathrm{Prol}^{n-1}(\varepsilon^0)(*_r)$$

for each $r < n$.

PROPOSITION 3.10. *$\pi\colon E\tilde{G}(F) \to B\tilde{G}(F)$ is a principal $\tilde{G}(F)$-fibration with contractible total space.*

(For a Kan Δ-monoid A, a principal A-fibration is a Kan fibration $\pi\colon E \to B$ of based Kan Δ-sets with B connected and $\pi^{-1}(*) = A$, and an action $E \times A \to E$ over B which extends the multiplication in $\pi^{-1}(*)$.)

Proof. This is clear from 3.8 and the choice of base-point in $B\tilde{G}(F)$.

Now denote $\mathrm{Prol}(\varepsilon^0)$ by $\gamma/B\tilde{G}(F)$, and let $Bf_F(K)$ be the set of block homotopy equivalence classes of block fibrations with base K and fibre F. Then $Bf_F(\ )$ is a functor on **Δ** via the induced block fibration, and we have a natural transformation

$$T\colon [\ ; B\tilde{G}(F)] \to Bf_F(\ ),$$

defined by $T(f) = f^*(\gamma)$, which is well defined in $Bf_F(K)$ by 3.9.

THEOREM 3.11. *T is a natural equivalence of functors on **Δ**.*

Proof. We have to show that $T(K)\colon [K; B\tilde{G}(F)] \to Bf_F(K)$ is a bijection.

$T(K)$ is onto. Let ξ/K be a block fibration and define ξ^+/K^+ to be the disjoint union $\xi \cup \varepsilon^0$. Consider the diagram

$$\begin{tikzcd}
K \arrow[r, phantom, "\overset{i_1}{\subset}"] \arrow[d, phantom, "\cap"] & B\,\mathrm{Prol}(\xi) \arrow[d, phantom, "\cap i_2"] & \\
K^+ \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup"] & B\,\mathrm{Prol}(\xi^+) \arrow[d, "h"', shift right=2] \arrow[d, phantom, "\cup i_\varepsilon", shift left=3] & P(\mathrm{Prol}(\xi^+)) \arrow[l, "\pi^+"'] \arrow[d, phantom, "\cup"] \\
\Delta^0 \arrow[r, phantom, "\subset"] & B\tilde{G}(F) & E\tilde{G}(F). \arrow[l, "\pi"']
\end{tikzcd}$$

Then i_ε is a homotopy equivalence of Kan Δ-sets, since π, π^+ are both projections of principal $\tilde{G}(F)$-bundles with contractible total spaces. Choose a homotopy inverse h to i_ε. Now let $\alpha = h \circ i_2 \circ i_1$ and $\xi' = \alpha^*(\gamma)$. Then $i_\varepsilon \circ \alpha \simeq i_2 \circ i_1$ since $i_3 \circ h \simeq id$. Hence $\alpha^*(\gamma) = (i_\varepsilon \circ \alpha)^* \,\mathrm{Prol}(\xi_+)$ is block homotopy equivalent to $(i_2 \circ i_1)^* \,\mathrm{Prol}(\xi^+) = \xi$, as required.

$T(K)$ is 1–1. Use a similar argument to the above after noting that ξ is block homotopy equivalent to η iff they bound over $K \otimes I$. 'If' is 3.9 and 'only if' follows by gluing to η the bundle ξ_1 (constructed in the proof of 3.9), via the block homotopy equivalence.

Subdivision and amalgamation. This part is so similar to the corresponding part of § 2, that we just sketch it.

Let K' be a subdivision of K and ξ/K' a block fibration. Define the amalgamation ξ_1/K of ξ by letting $\xi_1(\sigma) = |f_\sigma^*(\xi)|$ [see the notation in § 2], and defining $\xi_1(\lambda, \tau, \sigma)$ exactly as in § 2.

ξ_1 is a block fibration since $f_\sigma^*(\xi)$ is block homotopy trivial for each $\sigma \in K$ (from Theorem 3.11) and the trivialization

$$J_\sigma \times F \to |f_\sigma^*(\xi)|$$

then respects the Δ^n-structure on each of these spaces, hence is a block homotopy trivialization of $(\xi_1)_\sigma$ by 3.1.

THEOREM 3.12. *The bijection*

$$q\colon Bf_F(K') \to Bf_F(K)$$

is determined by amalgamation.

Proof. 2.5 readily adapts to prove 3.12.

COROLLARY 3.13. *Given ξ/K, there exists η/K' such that η_1/K is block homotopy equivalent to ξ.*

Remark. One would not expect to do better than 3.13 and obtain $\eta_1 = \xi$, since ξ might not have enough cells to subdivide.

The long exact sequence. Let ξ/K be a block fibration. We define a

projection $p\colon |\xi| \to |K|$ which is a map of K-spaces and is unique up to K-homotopy. Suppose p defined on $|\xi^{n-1}|$. Then the extension to any $\xi(\sigma)$ $(\sigma \in K^{(n)})$ follows from the contractibility of Δ^n and the HEP, as does the uniqueness.

THEOREM 3.14. *For any vertex $v \in K$, $p^*(|\xi|, |\xi(v)|) \to (|K|, v)$ induces isomorphisms of homotopy groups.*

COROLLARY 3.15. *There is a long exact homotopy sequence*

$$\ldots \to \pi_n(F) \to \pi_n|\xi| \to \pi_n|K| \to \ldots$$

Proof. This follows at once from 3.14, since $|\xi(v)| \simeq F$.

COROLLARY 3.16. *There is a bijection between $Bf_F(K)$ and the set of f.h.e. classes of Serre fibrations with base $|K|$ and fibre having the homotopy type of F. The bijection is given by the process of 'making p Serre'* [see (8) for example].

Proof. Construct the inverse to 'making p Serre' as follows. Let $\xi/|K|$ be a Serre fibration and inductively replace $E(\tilde{\sigma}^*(\xi))$ by cw-complexes of the same homotopy type [see e.g. (8)]. This defines the required block fibration up to b.h.e. Then the two functions are inverse by the usual argument, using 3.14.

Remark. 3.16 and 3.12 recover Stasheff's main theorem in (8), since **Δ** and **cw** are homotopy equivalent (see I, § 6).

Proof of 3.14. We first replace (K, v) by Kan complexes. Let $H(K) \supset K$ be the complex obtained from K by the 'horn ∞' functor determined by repeatedly attaching simplexes along all horns in K [see I, § 6], and let $* \subset H(v)$ be a base complex containing v. Then $(|K|, v) \subset (|H(K)|, |*|)$ is a homotopy equivalence of pairs. Now extend ξ to $\eta/H(K)$ by the usual method (extend the inclusion $K \subset B\,\mathrm{Prol}(\xi)$ over $H(K)$ and then take induced block fibrations).

To complete the proof of 3.14 we now require a lemma.

LEMMA 3.17. *$(|\xi|, |\xi(v)|) \subset (|\eta|, |\eta|*|)$ is a homotopy equivalence of pairs.*

Proof. If $K \searrow L$ and if ξ/K is a block fibration then it is trivial to construct a deformation retract of $|\xi|$ on $|\xi|L|$. It follows by induction that $|\xi| \subset |\eta|$ is a homotopy equivalence. But $|\xi(v)| \subset |\eta|*|$ is a homotopy equivalence since $|*|$ is contractible.

We may now complete the proof of 3.14 as follows. By 3.17 we have to show that

$$p\colon (|\eta|, |\eta|*|) \to (|H(K)|, |*|)$$

induces isomorphisms of homotopy groups.

p_* *is onto*. Let $f\colon (I^n, \dot{I}^n) \to (|H(K)|, |*|)$ represent an element of π_n. By I, 5.3 we can approximate f by a Δ-map $f_1\colon K \to H(K)$ where K is a simplicial complex and $K \searrow 0$. It is then easy to lift f_1 to $|\eta|$ up to homotopy within cells.

p_* is 1–1. The proof of this part is similar.

Block bundles and block fibrations. We now examine the connection between block bundles and block fibrations. We first observe in analogy to remark (2) after Theorem 1.12, that there is a fibration (up to homotopy type):

$$\tilde{G}(F)/A(F) \xrightarrow{i} BA(F) \xrightarrow{\pi} B\tilde{G}(F), \tag{3.18}$$

which is obtained by factoring the first two terms of

$$\tilde{G}(F) \subset E\tilde{G}(F) \to B\tilde{G}(F)$$

by $A(F)$, which acts freely. Then $E\tilde{G}(F)/A(F) \simeq BA(F)$ from the classification theorem.

We proceed to identify $\tilde{G}(F)/A(F)$ as the classifying space for the theory of '$A(F)$-block bundles with block homotopy trivialization' as is suggested by the fibration 3.18.

We consider pairs (ξ, t) where ξ/K is an $A(F)$-block bundle and $t\colon \xi \to \epsilon(K, F)$ is a block homotopy trivialization. (ξ_1, t_1) is *isomorphic* to (ξ_2, t_2) if there is an $A(F)$-isomorphism $h\colon \xi_1 \to \xi_2$ such that $t_2 \circ h = t_1$.

PROPOSITION 3.19. *There is a bijection between the set of Δ-maps $K \to \tilde{G}(F)/A(F)$ and the set of isomorphism classes of $A(F)$-block bundles base K with block homotopy trivialization.*

Proof. Given $t\colon \xi \to \epsilon(K, F)$, define $t'\colon K \to \tilde{G}(F)/A(F)$ by

$$t'(\sigma) = \{gh_\sigma \mid h_\sigma \text{ is a chart for } \xi \text{ at } \sigma\}.$$

This is readily proved to induce the bijection.

Now let $(\gamma, t_\gamma)/\tilde{G}(F)/A(F)$ be the $A(F)$-block bundle with trivialization given by 3.19 and the identity map. Define induced bundles in the natural manner—$f^*(\xi, t) = (f^*\xi, t \circ f)$—and then the bijection of 3.19 is clearly induced by sending f to $f^*(\gamma, t_\gamma)$.

Next define $(\xi_1, t_1), (\xi_2, t_2)$ to be *equivalent* if there is an isomorphism $h\colon \xi_1 \to \xi_2$ so that the diagram

$$\begin{tikzcd}
\xi_1 \arrow[d, "h"'] \arrow[dr, "t_1"] & \\
\xi_2 \arrow[r, "t_2"'] & \epsilon
\end{tikzcd}$$

commutes up to block homotopy.

PROPOSITION 3.20. *(ξ_0, t_0), (ξ_1, t_1) are equivalent iff there is a pair $(\eta, t)/K \otimes I$ with $(\eta, t) \mid K \otimes \{i\} \cong (\xi_i, t_i)$ $(i = 0, 1)$.*

Proof. 'If' is obvious. To prove 'only if', find $\eta/K \otimes I$ with ends isomorphic to ξ_i by the usual method and use the given homotopy (and the fact that if $\zeta/\Delta^n \times I$ then $|\zeta| \cong |\zeta|\Delta^n \times \{0\}| \times I$) to construct compatible maps
$$t_\sigma\colon |f_\sigma^*(\eta)| \to \Delta^n \times I \times F$$
for each $\sigma \in K$, where $f_\sigma\colon \Delta^n \otimes I \to K \otimes I$ is the map which covers the characteristic maps for σ.

We have to deform the t_σ to make them preserve the block structures of $f_\sigma^*(\eta)$ for each σ, but there is no obstruction to doing this inductively.

Now let $A(F)^t(K)$ denote the set of equivalence classes of $A(F)$-block bundles over K with block homotopy trivializations and make $A(F)^t(\ )$ into a functor via the induced bundle construction.

THEOREM 3.21. *The natural transformation*
$$T\colon [\ ;\tilde{G}(F)/A(F)] \to A(F)^t(\ )$$
given by $T[f] = [f^*(\gamma, t)]$ is a natural equivalence.

Proof. This follows from 3.19 and 3.20.

Remark. We can easily identify i and π in (3.18) as the maps which classify γ as an $A(F)$-bundle and the classifying block bundle as a block fibration respectively.

REFERENCES

1. A. Casson, 'Block bundles with arbitrary fibres' (to appear).
2. M. M. Cohen, 'A general theory of relative regular neighbourhoods', *Trans. Am. Math. Soc.* 136 (1969) 189–229.
3. A. Heller, 'Homotopy resolutions of semi-simplicial complexes', ibid. 80 (1955) 299–344.
4. M. Kato, 'Combinatorial prebundles: I', *Osaka J. Math.* 4 (1967) 289–303.
5. C. Morlet, 'Les Voisinages tubulaires des variétés semi-linéaires', *C.R. Acad. Sci. Paris*, 262 (1966) 740–3.
6. C. P. Rourke and B. J. Sanderson, 'Block bundles: I', *Ann. of Math.* 87 (1968) 1–28.
7. —— —— 'Δ-sets: I. Homotopy theory', *Quart. J. of Math.* (Oxford) 22 (1971), 321–38.
8. J. Stasheff, 'A classification theorem for fibre spaces', *Topology*, 2 (1963) 239–46.

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