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Journal of Pure and Applied Algebra 198 (2005) 105–121

JOURNAL OF PURE AND APPLIED ALGEBRA

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Quadratic functions on torsion groups

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Received 5 April 2004; received in revised form 5 August 2004
Available online 19 November 2004
Communicated by C. Kassel

Abstract

We investigate classification results for general quadratic functions on torsion abelian groups. Unlike the previously studied situations, general quadratic functions are allowed to be inhomogeneous or degenerate. We study the discriminant construction which assigns, to an integral lattice with a distinguished characteristic form, a quadratic function on a torsion group. When the associated symmetric bilinear pairing is fixed, we construct an affine embedding of a quotient of the set of characteristic forms into the set of all quadratic functions and determine explicitly its cokernel. We determine a suitable class of torsion groups so that quadratic functions defined on them are classified by the stable class of their lift. This refines results due to A.H. Durfee, V. Nikulin, and E. Looijenga and J. Wahl. Finally, we show that on this class of torsion groups, two quadratic functions q, q' are isomorphic if and only if they have equal associated Gauss sums and there is an isomorphism between the associated symmetric bilinear pairings b_q and $b_{q'}$ which sends d_q to $d_{q'}$, where d_q is the homomorphism defined by $d_q(x) = q(x) - q(-x)$. This generalizes a classical result due to V. Nikulin. Our results are elementary in nature and motivated by low-dimensional topology.

MSC: 15A63; 11E81; 11E39

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0022-4049/$ - see front matter © 2004 Published by Elsevier B.V.
doi:10.1016/j.jpaa.2004.10.011

0. Introduction

A quadratic function q on an abelian group G is a map, with values in an abelian group such that the map $b\colon (x, y) \mapsto q(x + y) - q(x) - q(y)$ is $\mathbb{Z}$-bilinear. Such a map q satisfies $q(0) = 0$. If, in addition, q satisfies the relation $q(nx) = n^2 q(x)$ for all $n \in \mathbb{Z}$ and $x \in G$, then q is homogeneous. In general, a quadratic function cannot be recovered from the associated bilinear pairing b. Homogeneous quadratic functions on torsion groups first appeared as quadratic enhancements of the linking pairing on the torsion subgroup of the $(2n - 1)$–th homology group of an oriented $(4n - 1)$–manifold. Typically, these quadratic enhancements appear in topology when the manifold is equipped with a framing [1,8,10]. They were used as a fundamental ingredient in the classification up to regular homotopy of immersed surfaces in $\mathbb{R}^3$ [12]. They were extensively studied from the algebraic viewpoint of Witt and Grothendieck groups, see for instance [5–7]. However, there are topological motivations to consider inhomogeneous enhancements of the linking pairing [3,9]. It is also convenient to consider possibly degenerate quadratic functions [2]. The motivation for considering *general* quadratic functions stems from our work on closed Spin^c-manifolds of dimension 3 and their finite type invariants [4].

This paper studies and gives classification results for quadratic functions on torsion abelian groups with values in $\mathbb{Q}/\mathbb{Z}$. These results are relatively well known in the case of *nondegenerate* symmetric bilinear pairings and *homogeneous* quadratic functions. However, the authors have not succeeded in finding in the literature the general results for quadratic functions.

To describe the first result, we review (Section 2.2) a construction, known as the discriminant construction. This construction assigns to a symmetric bilinear lattice M equipped with a certain element (called a characteristic form) $c \in \mathrm{Hom}(M, \mathbb{Z})$, a quadratic function on a torsion abelian group, with values in $\mathbb{Q}/\mathbb{Z}$. The first result is an embedding theorem (Theorem 2.10) which describes the quadratic functions on a torsion abelian group with values in $\mathbb{Q}/\mathbb{Z}$, come from a characteristic form (for a fixed symmetric bilinear pairing). Unlike the previously studied situations, the quadratic functions obtained by the discriminant construction may be degenerate and the torsion abelian group may be infinite.

The second result addresses the problem of classifying up to isomorphism the quadratic functions arising from the discriminant construction. This problem is closely related to the stable classification of symmetric bilinear lattices equipped with characteristic forms. More precisely, we prove (Section 3) that the stable equivalence on lattices is equivalent to a particular class of isomorphism between the associated quadratic functions, which we determine (Theorem 3.2). The main application of this result has a particularly simple form: the stable equivalence can be realized by adding 1-dimensional unimodular lattices (Corollary 3.5). This generalizes a classical result for lattices without distinguished characteristic forms.

While the second result theoretically solves the classification problem of a vast class of quadratic functions (which includes all nondegenerate quadratic functions when the group is finite), or at least reduces it to a stable classification problem for lattices with characteristic forms, it does not seem to allow to decide concretely whether two quadratic functions are isomorphic. This leads to the problem of classification of quadratic functions by means of invariants. We propose a solution to the problem on finite abelian groups by giving a

short proof (Section 4, Theorem 4.1) that the isomorphism class of a quadratic function q on a finite abelian group G is determined by the isomorphism class of the associated bilinear pairing b equipped with a distinguished element $d_q \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$, called the homogeneity defect of q, and the Gauss sum $\sum_{x\in G} \exp(2\pi i q(x))$ associated to q. This result is then extended to a more general class of torsion groups. The particular case of homogeneous quadratic functions ($d_q = 0$) on a finite group G was proved by Nikulin by an induction on the number of generators of G [11, Theorem 1.11.3].

These three results were applied to the classification of degree 0 finite type invariants of Spin^c 3-manifolds [4].

1. Preliminaries

Let G be an abelian group. A set X on which G acts freely and transitively is an *affine space over* G. For such actions, the left multiplicative notation is used. Any bilinear pairing $b\colon G\times G' \to H$, where G, G' and H are abelian groups, has a left (resp. right) adjoint map $\widehat{.b}\colon G \to \mathrm{Hom}(G', H)$ (resp. $\widehat{b.}\colon G' \to \mathrm{Hom}(G, H)$), defined by $\widehat{.b}(x)(y) = b(x, y)$ (resp. $\widehat{b.}(y)(x) = b(x, y)$), $x \in G$, $y \in G'$. We say that the bilinear pairing b is *left nondegenerate* (resp. *left nonsingular* or *left unimodular*) if its left adjoint map is injective (resp. bijective). We define similarly right nondegenerate and right nonsingular. The bilinear pairing b is *nondegenerate* (resp. *nonsingular*) if b is both left and right nondegenerate. In the case when $G = G'$ and b is symmetric, $\widehat{.b} = \widehat{b.}$ is denoted by $\widehat{b}$. Throughout the paper, the abelian group G will be a finitely generated free module over $R = \mathbb{Z}$ or $\mathbb{Q}$, or a torsion group. We set $C_G = R$ if G is free and$C_G = \mathbb{Q}/\mathbb{Z}$ if G is torsion. It is convenient to define the dual of G by $G^* = \mathrm{Hom}(G, C_G)$.

A map $q : G \to \mathbb{Q}/\mathbb{Z}$ on a torsion abelian group G is a *quadratic function* if the associated map $b_q : G \times G \to \mathbb{Q}/\mathbb{Z}$, defined by $b_q(x, y) = q(x + y) - q(x) - q(y)$, is a bilinear pairing. Note that this definition implies that $q(0) = 0$ for any quadratic function q. A quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$ satisfying $q(nx) = n^2 q(x)$ for all $x \in G$ and all $n \in \mathbb{Z}$ is said to be *homogeneous*. We say that a map $q : G \to \mathbb{Q}/\mathbb{Z}$ is *quadratic over* a symmetric bilinear pairing $b\colon G \times G \to \mathbb{Q}/\mathbb{Z}$ if $b_q = b$. A quadratic function q is *nondegenerate* (resp. *nonsingular* or *unimodular*) if b_q is. The difference between two quadratic functions q, q' over the same bilinear pairing lies in G^*. In particular, the set $\mathrm{Quad}(b)$ of all quadratic functions $q\colon G \to \mathbb{Q}/\mathbb{Z}$ over a symmetric bilinear pairing $b\colon G \times G \to \mathbb{Q}/\mathbb{Z}$ is affine over the group G^*. The group G acts also on $\mathrm{Quad}(b)$ via the adjoint map $\widehat{b}\colon G \to G^*$:

$$G \times \mathrm{Quad}(b) \to \mathrm{Quad}(b), \quad (\alpha, q) \mapsto \alpha \cdot q = q + \widehat{b}(\alpha).$$

If q is nonsingular, then $\mathrm{Quad}(b)$ is affine over the group G. To any $q \in \mathrm{Quad}(b)$, we associate the quadratic functions $-q \in \mathrm{Quad}(-b)$ and $\bar{q} \in \mathrm{Quad}(b)$, respectively, by $(-q)(x) = -q(x)$ and $\bar{q}(x) = q(-x)$, $x \in G$. Clearly the maps $q \mapsto -q$ and $q \mapsto \bar{q}$ are involutive bijections. The *homogeneity defect* d_q of a quadratic function $q \in \mathrm{Quad}(b)$ is defined by $d_q = q - \bar{q} \in G^*$. Since

$$n^2 q(x) - q(nx) = \frac{n(n-1)}{2}\, d_q(x),$$

a quadratic function q is homogeneous if and only if $d_q = 0$.

Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function with $b_q = b$. Let $\psi : G' \to G$ be a group isomorphism. We define $\psi^* b : G' \times G' \to \mathbb{Q}/\mathbb{Z}$ to be the symmetric bilinear pairing defined by

$$\psi^* b(x, y) = b(\psi(x), \psi(y)) \quad \text{for all } x, y \in G'.$$

Similarly we define the quadratic function $\psi^* q : G' \to \mathbb{Q}/\mathbb{Z}$ by

$$\psi^* q(x) = q(\psi(x)) \quad \text{for all } x \in G'.$$

We say that two symmetric bilinear pairings $b,\ b'$ (resp. two quadratic functions q, q') defined on G and G' with values in $\mathbb{Q}/\mathbb{Z}$ are *isomorphic*, and we write $b \sim b'$ (resp. $q \sim q'$), if there exists a group isomorphism $\psi : G' \to G$ such that $\psi^* b = b'$ (resp. $\psi^* q = q'$). An isomorphism $\psi : G' \to G$ induces a bijective correspondence between $\mathrm{Quad}(b)$ and $\mathrm{Quad}(\psi^* b)$. In particular, the subgroup $\mathrm{Iso}(b)$ of automorphisms of G preserving b acts on $\mathrm{Quad}(b)$: $\psi \cdot q = \psi^* q$.

2. Presentations of quadratic functions

2.1. Lattices and their characteristic forms

A *lattice* M is a finitely generated free abelian group. A (*symmetric*) *bilinear lattice* (M, f) is a symmetric bilinear form $f : M \times M \to \mathbb{Z}$ on a lattice M. Consider the vector space $V = M \otimes \mathbb{Q}$ over $\mathbb{Q}$. The dimension of V is finite and equal to the rank of M. Any bilinear lattice (M, f) gives rise by extension of scalars to a bilinear pairing $f_{\mathbb{Q}} : V \times V \to \mathbb{Q}$. Let $M^\sharp = \{x \in V : f_{\mathbb{Q}}(x, M) \subset \mathbb{Z}\}$ be the *dual lattice* for (M, f). Clearly $M \subset M^\sharp$. More generally, for any subgroup N of V, we can define $N^\sharp = \{x \in V : f_{\mathbb{Q}}(x, N) \subset \mathbb{Z}\}$. A *fractional* (resp. *integral*) *Wu class* for (M, f) is an element $w \in V$ (resp. an element $w \in M$) such that

$$\forall x \in M, \quad f(x, x) - f_{\mathbb{Q}}(w, x) \in 2\mathbb{Z}.$$

The set of fractional (resp. integral) Wu classes is denoted by $\mathrm{Wu}^{\mathbb{Q}}(f)$ (resp. $\mathrm{Wu}(f)$) and is contained in $M^\sharp$. Furthermore, $\mathrm{Wu}^{\mathbb{Q}}(f)$ is an affine space over $2M^\sharp$.

A *characteristic form* for f is an element $c \in M^*$ satisfying

$$\forall x \in M, \quad f(x, x) - c(x) \in 2\mathbb{Z}.$$

The set $\mathrm{Char}(f)$ of characteristic forms for f is an affine space over $\mathrm{Hom}(M, 2\mathbb{Z})$. Since $x \mapsto f(x, x) \bmod 2\mathbb{Z}$ is a homomorphism, $\mathrm{Char}(f)$ is not empty. Each fractional Wu class gives a characteristic form by the equivariant map $w \mapsto f_{\mathbb{Q}}(w, -)|_M$, $\mathrm{Wu}^{\mathbb{Q}}(f) \to \mathrm{Char}(f)$.

Lemma 2.1. *If f is nondegenerate, then the map $w \mapsto f_{\mathbb{Q}}(w, -)|_M$ is a bijective correspondence between* $\mathrm{Wu}^{\mathbb{Q}}(f)$ *and* $\mathrm{Char}(f)$.

Proof. Since f is nondegenerate, $\widehat{f} : M \to M^*$ is injective. Then $\widehat{f_{\mathbb{Q}}} : V \to V^* = \mathrm{Hom}(V, \mathbb{Q})$ is bijective. Hence, the map $x \mapsto f_{\mathbb{Q}}(x, -)|_M$, $\mathrm{Wu}^{\mathbb{Q}}(f) \to \mathrm{Char}(f)$ is also bijective. □

The quotient $\bar{M} = M/\mathrm{Ker}\,\widehat{f}$ is finitely generated free. Hence, the short exact sequence

$$0 \to \mathrm{Ker}\,\widehat{f} \xrightarrow{i} M \xrightarrow{p} \bar{M} \to 0 \tag{2.1}$$

is split. Any section s of p induces an isomorphism

$$(M, f) \simeq (\bar{M}, \bar{f}) \oplus (\mathrm{Ker}\,\widehat{f}, 0),$$

where $\bar{f} : \bar{M} \times \bar{M} \to \mathbb{Z}$ is the nondegenerate pairing induced by f.

Lemma 2.2. *There is an injection* $p^*|_{\mathrm{Char}(\bar{f})} : \mathrm{Char}(\bar{f}) \to \mathrm{Char}(f)$ *induced by p, and any section s of p induces an affine retraction* $s^*|_{\mathrm{Char}(f)} : \mathrm{Char}(f) \to \mathrm{Char}(\bar{f})$ *for* $p^*|_{\mathrm{Char}(\bar{f})}$.

Proof. Since $p \circ s = \mathrm{Id}_{\bar{M}}$, we have $s^* \circ p^* = \mathrm{Id}_{\bar{M}^*}$. Moreover, one easily verifies that $p^*(\mathrm{Char}(\bar{f})) \subset \mathrm{Char}(f)$ and $s^*(\mathrm{Char}(f)) \subset \mathrm{Char}(\bar{f})$. $\square$

2.2. The discriminant construction

Suppose we are given a bilinear lattice (M, f) as above. Consider the torsion group $G_f = M^\sharp/M$ and the symmetric bilinear pairing

$$L_f : G_f \times G_f \to \mathbb{Q}/\mathbb{Z}$$

defined by

$$L_f([x], [y]) = f_{\mathbb{Q}}(x, y) \bmod \mathbb{Z}, \quad x, y \in M^\sharp. \tag{2.2}$$

Since $M^{\sharp\sharp} = M + \mathrm{Ker}\,\widehat{f_{\mathbb{Q}}}$, the radical of L_f is

$$\mathrm{Ker}\,\widehat{L_f} = (M + \mathrm{Ker}\,\widehat{f_{\mathbb{Q}}})/M = \mathrm{Ker}\,\widehat{f_{\mathbb{Q}}}/\mathrm{Ker}\,\widehat{f} = (\mathrm{Ker}\,\widehat{f}) \otimes \mathbb{Q}/\mathbb{Z}. \tag{2.3}$$

In particular, L_f is nondegenerate if and only if f is nondegenerate.

Consider the torsion subgroup $\mathrm{T\,Coker}\,\widehat{f}$ of $\mathrm{Coker}\,\widehat{f}$. The adjoint map $\widehat{f_{\mathbb{Q}}} : V \to V^*$ restricted to $M^\sharp$ induces a canonical epimorphism $B_f : G_f \to \mathrm{T\,Coker}\,\widehat{f}$. Hence there is a short exact sequence

$$0 \to \mathrm{Ker}\,\widehat{L_f} \to G_f \to \mathrm{T\,Coker}\,\widehat{f} \to 0. \tag{2.4}$$

It follows that the symmetric bilinear form L_f factors to a nondegenerate pairing

$$\lambda_f : \mathrm{T\,Coker}\,\widehat{f} \times \mathrm{T\,Coker}\,\widehat{f} \to \mathbb{Q}/\mathbb{Z}.$$

It also follows from (2.4) that $G_f = 0$ if and only if $M^\sharp = M$ if and only if f is unimodular. The map $B_f : G_f \to \mathrm{T\,Coker}\,\widehat{f}$ induces a canonical isomorphism $(\mathrm{T\,Coker}\,\widehat{f}, \lambda_f) \simeq (G_{\bar{f}}, L_{\bar{f}})$. In general, observe that a section s of p in (2.1) induces a section of (2.4). Hence there is a (noncanonical) orthogonal decomposition:

$$(G_f, L_f) \simeq (\mathrm{T\,Coker}\,\widehat{f}, \lambda_f) \oplus (\mathrm{Ker}\,\widehat{L_f}, 0). \tag{2.5}$$

For future use, we notice that the converse of our previous observation holds.

Lemma 2.3. *Any section of* $B_f : G_f \to \mathrm{T\,Coker}\,\widehat{f}$ *is induced by a section of* $p : M \to \bar{M}$.

Proof. The underlying map of the affine map from sections of p to section of B_f is the natural homomorphism $\mathrm{Hom}(\bar{M}, \mathrm{Ker}\,\widehat{f}) \to \mathrm{Hom}(G_{\bar{f}}, \mathrm{Ker}\,\widehat{L}_f)$. It suffices to lift $\alpha \in \mathrm{Hom}(G_{\bar{f}}, \mathrm{Ker}\,\widehat{L}_f)$ to a map $\tilde{\alpha} \in \mathrm{Hom}(\bar{M}^\sharp, \mathrm{Ker}\,\widehat{f_{\mathbb{Q}}})$; the restriction $\tilde{\alpha}|_{\bar{M}}$ will be the desired section of p. Since $\bar{M}^\sharp$ is free, the lift exists. □

Suppose next that (M, f, c) is a bilinear lattice equipped with a characteristic form $c \in M^*$. Denote by $c_{\mathbb{Q}} : V \to \mathbb{Q}$ the linear extension of c. We associate to (M, f, c) a quadratic function $\phi_{f,c} : G_f \to \mathbb{Q}/\mathbb{Z}$ over L_f by

$$\phi_{f,c}([x]) = \tfrac{1}{2}(f_{\mathbb{Q}}(x,x) - c_{\mathbb{Q}}(x)) \bmod \mathbb{Z}, \quad x \in M^\sharp.$$

Observe that the homogeneity defect $d_{\phi_{f,c}} \in G_f^* = \mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$ is given by

$$d_{\phi_{f,c}}([x]) = -c_{\mathbb{Q}}(x) \bmod \mathbb{Z}, \quad x \in M^\sharp. \tag{2.6}$$

Definition 2.4. The triple (M, f, c) is said to be a *presentation* of the quadratic function $\phi_{f,c}$ on the torsion group G_f. The assignation $(M, f, c) \mapsto (G_f, \phi_{f,c})$ is called the *discriminant* construction.

Lemma 2.5. *The discriminant construction preserves orthogonal sums*.

Remark 2.6. The following conditions are seen to be equivalent from (2.3) and (2.4):

- f is nondegenerate;
- L_f is nondegenerate;
- G_f is a finite product of cyclic groups;
- Coker $\widehat{f}$ is a finite product of cyclic groups;
- G_f and Coker $\widehat{f}$ are isomorphic.

If one of these conditions is satisfied, $B_f : G_f \to \mathrm{Coker}\,\widehat{f}$ is an isomorphism which induces a bijective correspondence $\mathrm{Quad}(\lambda_f) \simeq \mathrm{Quad}(L_f)$. Hence $\phi_{f,c}$ factors via B_f to a nondegenerate quadratic function over λ_f. More generally, $\phi_{f,c}$ factors via B_f to a nondegenerate quadratic function over λ_f if and only if c can be taken to be the image of a fractional Wu class (cf. Lemma 2.1). This particular case coincides with the usual discriminant construction, as presented for example in [2, Section 1.2.1].

The map $(M, f, c) \mapsto (G_f, \phi_{f,c})$, from the monoid of nondegenerate bilinear lattices equipped with characteristic forms to the monoid of isomorphism classes of nondegenerate quadratic functions on finite groups, is known to be surjective [14].

Remark 2.7. There exists a discriminant construction which applies to arbitrary Dedekind rings (but produces only *homogeneous* quadratic functions) instead of $\mathbb{Z}$, see [5]. This leads to the following

Question. Does there exist a generalization of the discriminant construction producing from lattices over Dedekind domains quadratic functions on torsion modules over Dedekind

domains? (The notion of quadratic function should be generalized first.) More generally, can one define a localization of quadratic functions? (For homogeneous quadratic functions, see [6].)

2.3. Properties of the discriminant construction

Consider the bilinear map $M^* \times M^\sharp \to \mathbb{Q}$ defined by $(\alpha, x) \mapsto \alpha_F(x)$. This map induces a bilinear pairing

$$\langle -, - \rangle : \operatorname{Coker} \widehat{f} \times G_f \to \mathbb{Q}/\mathbb{Z}.$$

Lemma 2.8. *The bilinear pairing* $\langle -, - \rangle : \operatorname{Coker} \widehat{f} \times G_f \to \mathbb{Q}/\mathbb{Z}$ *is left nondegenerate* (*respectively left nonsingular if and only if f is nondegenerate*) *and right nonsingular*.

Proof. Consider the left adjoint map $\operatorname{Coker} \widehat{f} \to \operatorname{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$, $[\alpha] \mapsto \langle [\alpha], - \rangle \pmod{\mathbb{Z}}$. Let $[\alpha]$ lie in the kernel, that is, $\alpha_{\mathbb{Q}}(M^\sharp) \subseteq \mathbb{Z}$. To prove that $[\alpha] = 0$, it is sufficient to show that there exists $v \in M$ such that $\alpha = \widehat{f}(v)$. Assume first that f is nondegenerate. Then $\widehat{f_{\mathbb{Q}}}$ is bijective, so there is an element $v \in V$ such that $\alpha_{\mathbb{Q}} = \widehat{f_{\mathbb{Q}}}(v)$. Since $\alpha_{\mathbb{Q}}(M^\sharp) = f_{\mathbb{Q}}(v, M^\sharp) \subset \mathbb{Z}$, this means that $v \in M^{\sharp\sharp} = M$. To deal with the general case (f possibly degenerate), we consider the nondegenerate symmetric bilinear pairing $\bar{f}$ induced by f on the lattice $\bar{M} = M/\operatorname{Ker} \widehat{f}$. We shall use the following observation.

Sublemma 1. *Let N be a free abelian group and let* $W = N \otimes \mathbb{Q}$. *Let* $\gamma \in \operatorname{Hom}_{\mathbb{Q}}(W, \mathbb{Q})$. *We have* $\gamma = 0$ *if and only if* $\gamma(W) \subseteq \mathbb{Z}$.

Recall that $\operatorname{Ker} \widehat{f}$ is a free subgroup of M and $\operatorname{Ker} \widehat{f_{\mathbb{Q}}} = (\operatorname{Ker} \widehat{f}) \otimes \mathbb{Q}$. We have $\alpha_{\mathbb{Q}}(\operatorname{Ker} \widehat{f_{\mathbb{Q}}}) \subseteq \alpha_{\mathbb{Q}}(M^\sharp) \subseteq \mathbb{Z}$. Apply the sublemma above with $N = \operatorname{Ker} \widehat{f}$ to $\alpha_{\mathbb{Q}}|_{\operatorname{Ker} \widehat{f_{\mathbb{Q}}}}$ yields $\alpha_{\mathbb{Q}}(\operatorname{Ker} \widehat{f_{\mathbb{Q}}}) = 0$. In particular, $\alpha(\operatorname{Ker} \widehat{f}) = 0$. Hence, α induces a form $\bar{\alpha} : \bar{M} \to \mathbb{Z}$ such that $\bar{\alpha}(\bar{M}^\sharp) \subseteq \mathbb{Z}$. Applying the previous (nondegenerate) case to $\bar{\alpha}$ yields an element $\bar{v} \in \bar{M}$ such that $\bar{\alpha} = \widehat{\bar{f}}(\bar{v})$. Any lift $v \in M$ of $\bar{v}$ satisfies $\alpha = \widehat{f}(v)$. So $[\alpha] = 0$ as desired.

Suppose that f is nondegenerate. By Remark 2.6, G_f (resp. $\operatorname{Coker} \widehat{f}$) is a product of cyclic groups $(1/n)\mathbb{Z}/\mathbb{Z}$ (resp. $\mathbb{Z}/n\mathbb{Z}$). The injective left adjoint map $\operatorname{Coker} \widehat{f} \to \operatorname{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$ is between two groups of the same finite order, so must be surjective as well. Suppose that f is not nondegenerate. Then $\operatorname{Coker} \widehat{f}$ (resp. G_f) has a summand $\mathbb{Z}$ (resp. a summand which is $\mathbb{Q}/\mathbb{Z}$). It suffices to show that the map $\mathbb{Z} \to \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, $r \mapsto r\, \mathrm{Id}_{\mathbb{Q}/\mathbb{Z}}$ is not surjective. Recall that $\mathbb{Q}/\mathbb{Z} = \bigoplus_p A_p$, where the direct sum is over all primes and A_p is the subgroup consisting of elements $x \in \mathbb{Q}/\mathbb{Z}$ such that $p^k x = 0$ for some k. Consider the map $\pi_p : A_p \to A_p$ which is multiplication by p. Then the product $\prod_p \pi_p \in \prod_p \operatorname{Hom}(A_p, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is not in the image.

Consider the right adjoint map $G_f \to \operatorname{Hom}(\operatorname{Coker} \widehat{f}, \mathbb{Q}/\mathbb{Z})$. An element $[x] \in G_f$ lies in the kernel if and only if $\alpha_{\mathbb{Q}}(x) = 0 \bmod \mathbb{Z}$ for all $\alpha \in M^*$. In particular, for a unimodular bilinear pairing g on M, $g_{\mathbb{Q}}(v, x) = 0 \bmod \mathbb{Z}$ for all $v \in V$. Unimodularity of g implies that $x \in M$. Thus $[x] = 0$.

We now prove that the right adjoint map is surjective. Consider the canonical isomorphism $V \to V^{**}$, $v \mapsto \langle -, v \rangle$. This map sends $M^\sharp$ onto $N = \{\langle -, w \rangle \in V^{**} : \langle \widehat{f}(M), w \rangle \subseteq \mathbb{Z}\}$

which is a subgroup of V^{**}. The subgroup $N^0 = \{x \in \mathrm{Hom}(M^*, \mathbb{Q}) : \langle \widehat{f}(M), x\rangle \subseteq \mathbb{Z}\}$ embeds in N by linear extension over $\mathbb{Q}$: $M^* \to M^* \otimes \mathbb{Q} = V^*$. Any $x \in N^0$ induces a homomorphism $\mathrm{Coker}\,\widehat{f} \to \mathbb{Q}/\mathbb{Z}$, hence there is an induced homomorphism $N^0 \to \mathrm{Hom}(\mathrm{Coker}\,\widehat{f}, \mathbb{Q}/\mathbb{Z})$, which is obviously surjective (since M^* is free). Hence any $\varphi \in \mathrm{Hom}(\mathrm{Coker}\,\widehat{f}, \mathbb{Q}/\mathbb{Z})$ lifts to an element $x \in N^0 \subseteq N$ where $x = \langle -, v\rangle$ for some $v \in M^\sharp$.Thus $\varphi = \langle -, [v]\rangle$. □

Lemma 2.9. *The cokernel of the inclusion* $\mathrm{T\,Coker}\,\widehat{f} \hookrightarrow \mathrm{Coker}\,\widehat{f}$ *is* $(\mathrm{Ker}\,\widehat{f})^*$.

Proof. Since $M/\mathrm{Ker}\,\widehat{f}$ is free, the inclusion $\mathrm{Ker}\,\widehat{f} \hookrightarrow M$ induces a surjective homomorphism $M^* \to (\mathrm{Ker}\,\widehat{f})^*$ which sends $\widehat{f}(M)$ to 0. Hence we obtain a well-defined surjective homomorphism $\mathrm{Coker}\,\widehat{f} \to (\mathrm{Ker}\,\widehat{f})^*$, the kernel of which, since $(\mathrm{Ker}\,\widehat{f})^*$ is free of same rank as $\mathrm{Coker}\,\widehat{f}$, consists of all torsion elements. □

By (2.3), any homomorphism $\mathrm{Ker}\,\widehat{f} \to \mathbb{Z}$ induces by tensoring with $\mathbb{Q}/\mathbb{Z}$ a homomorphism $\mathrm{Ker}\,\widehat{L}_f \to \mathbb{Q}/\mathbb{Z}$. Denote by $j_f : (\mathrm{Ker}\,\widehat{f})^* \to (\mathrm{Ker}\,\widehat{L}_f)^*$ the corresponding homomorphism.

Let $c \in \mathrm{Char}(f)$. Observe that $\phi_{f,c+2\hat{f}(u)} = \phi_{f,c}$ for any $u \in M$. Hence $\phi_{f,c}$ depends on c only mod $2\widehat{f}(M)$. Furthermore, the abelian group $M^*/\widehat{f}(M) = \mathrm{Coker}\,\widehat{f}$ acts freely and transitively on $\mathrm{Char}(f)/2\widehat{f}(M)$ by

$$[\alpha]\cdot[c] = [c+2\alpha] \in \mathrm{Char}(f)/2\widehat{f}(M), \quad \alpha \in M^*, \quad c \in \mathrm{Char}(f).$$

The following result is the main result of this section.

Theorem 2.10. *The map* $c \mapsto \phi_{f,c}$ *induces an affine embedding*

$$\phi_f : \mathrm{Char}(f)/2\widehat{f}(M) \hookrightarrow \mathrm{Quad}(L_f)$$

over the opposite of the group monomorphism $.\langle -,-\rangle : \mathrm{Coker}\,\widehat{f} \hookrightarrow \mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$. *Furthermore,*

$$\mathrm{Coker}\,\phi_f \simeq \mathrm{Coker}\,j_f = \frac{(\mathrm{Ker}\,\widehat{L}_f)^*}{j_f((\mathrm{Ker}\,\widehat{f})^*)}$$

and given $q \in \mathrm{Quad}(L_f)$, *the following two assertions are equivalent*:

- $q \in \mathrm{Im}\,\phi_f$;
- $q|_{\mathrm{Ker}\,\widehat{L}_f} \in \mathrm{Im}\,j_f$.

As a consequence of Lemma 2.8, we have:

Corollary 2.11. *The map* ϕ_f *is bijective if and only if f is nondegenerate*.

Combining Corollary 2.11 and Lemma 2.2, we obtain the following result.

Corollary 2.12. *For any section s of $p: M \to \overline{M}$, the map*

$$\mathrm{Char}(f) \to \mathrm{Quad}(\lambda_f), \quad c \mapsto \phi_{\bar{f},s^*|_{\mathrm{Char}(f)}(c)}$$

is onto.

Remark 2.13. This construction applies in particular when (M, f) is the homology group $H_2(X)$ of a compact connected oriented simply-connected 4-manifold X, equipped with its symmetric bilinear intersection pairing. Then λ_f can be identified with the torsion linking pairing on $\mathrm{T}H_1(\partial X)$ (up to sign depending on the orientation of $(X, \partial X)$). For details, see for example [2,5]. Furthermore G_f identifies with $H_2(\partial X; \mathbb{Q}/\mathbb{Z})$ and the affine space $\mathrm{Char}(f)/2\widehat{f}(M)$ identifies with the affine space of Spin^c-structures on ∂X. See [4] for further details about the last point.

Proof of Theorem 2.10. Let us prove that the map ϕ_f is indeed affine over the homomorphism stated above. Let $\alpha \in M^*$ and $x \in M^\sharp$: $\phi_{f,[\alpha]\cdot[c]}([x]) - \phi_{f,[c]}([x]) = \phi_{f,c+2\alpha}([x]) - \phi_{f,c}([x]) = -\alpha_{\mathbb{Q}}(x) \bmod \mathbb{Z}$. Hence $\phi_{f,[\alpha]\cdot[c]}([x]) = \phi_{f,[c]}([x]) - \langle[\alpha], [x]\rangle$. The fact that ϕ_f is injective follows from the fact that the map $.\langle -, -\rangle : \mathrm{Coker}\, \widehat{f} \to \mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$ is injective and this was proved in Lemma 2.8.

Next, we proceed to determine Coker ϕ_f. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathrm{T}\,\mathrm{Coker}\,\widehat{f}) & \longrightarrow & \mathrm{Coker}\,\widehat{f} & \longrightarrow & (\mathrm{Ker}\,\widehat{f})^* & \longrightarrow & 0 \\
 & & \simeq \Big\downarrow \widehat{\lambda}_f & & \Big\downarrow .\langle -,- \rangle & & \Big\downarrow j_f & & \\
0 & \longrightarrow & (\mathrm{T}\,\mathrm{Coker}\,\widehat{f})^* & \longrightarrow & \mathrm{Hom}\,(G_f, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & (\mathrm{Ker}\,\widehat{L}_f)^* & \longrightarrow & 0
\end{array}$$

The first row is given by Lemma 2.9. The second row is obtained from the short exact sequence (2.4) by applying the exact functor $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ (since $\mathbb{Q}/\mathbb{Z}$ is divisible). It follows from the definitions that the diagram is commutative.

A standard argument (for instance, the “snake lemma”) applied to the diagram leads to Coker $.\langle -, -\rangle \simeq$ Coker j_f.

Finally we prove the last statement of the theorem. By definition, $c(\mathrm{Ker}\,\widehat{f}) \subseteq 2\mathbb{Z}$. In particular, $\phi_{f,c}|_{\mathrm{Ker}\,\widehat{L}_f} : \mathrm{Ker}\,\widehat{L}_f \to \mathbb{Q}/\mathbb{Z}$ is a homomorphism given by

$$\phi_{f,c}([x]) = -\tfrac{1}{2}\, c_{\mathbb{Q}}(x) \bmod \mathbb{Z}, \quad x \in M + \mathrm{Ker}\,\widehat{f}_{\mathbb{Q}} \subseteq M^\sharp. \tag{2.7}$$

Hence $\phi_{f,c}|_{\mathrm{Ker}\,\widehat{L}_f} = j_f(-\frac{1}{2}c) \in \mathrm{Im}\, j_f$. Conversely, let $q \in \mathrm{Quad}(L_f)$ such that $q|_{\mathrm{Ker}\,\widehat{L}_f} \in \mathrm{Im}\, j_f$. Pick $c \in \mathrm{Char}(f)$. Then the restriction homomorphism $(G_f)^* \to (\mathrm{Ker}\,\widehat{L}_f)^*$ induced by $\mathrm{Ker}\,\widehat{L}_f \hookrightarrow G_f$ sends $q - \phi_{f,c}$ into $\mathrm{Im}\, j_f$. Since

$$\mathrm{Coker}\, .\langle -, -\rangle \simeq \mathrm{Coker}\, j_f,$$

there is an element $[\alpha] \in \mathrm{Coker}\,\widehat{f}$ such that $q - \phi_{f,c} = \langle[\alpha], -\rangle$. Since the map ϕ_f is affine over $\mathrm{Coker}\,\widehat{f} \hookrightarrow \mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$, it follows that $q = \phi_{f,[-\alpha]\cdot[c]}$. $\square$

Corollary 2.14. *A homomorphism* $h : G_f \to \mathbb{Q}/\mathbb{Z}$ *is in the image of* $.\langle -, - \rangle : \mathrm{Coker}\, \widehat{f} \to \mathrm{Hom}(G_f, \mathbb{Q}/\mathbb{Z})$ *if and only if* $h|_{\mathrm{Ker}\, \widehat{L}_f} : \mathrm{Ker}\, \widehat{L}_f \to \mathbb{Q}/\mathbb{Z}$ *lifts to a homomorphism* $\mathrm{Ker}\, \widehat{f} \to \mathbb{Z}$.

Proof. Set $h' = h|_{\mathrm{Ker}\, \widehat{L}_f}$. By definition, h' lifts to $\mathrm{Ker}\, \widehat{f}$ if and only if $h' \in \mathrm{Im}\, j_f$. Now apply Theorem 2.10. $\square$

We complete this section with a simple observation on $\mathrm{Coker}\, \phi_f$. Let p be a prime. Let $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^k\mathbb{Z}$ be the ring of p-adic numbers. Set $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ where the product is taken over all primes. The map $n \mapsto (n \bmod p^k)_{k \geqslant 0}$ defines a natural embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$. Consequently, there is a diagonal embedding $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}$. Since $\mathbb{Q}/\mathbb{Z}$ is a direct limit of finitely generated subgroups and since any finitely generated torsion group decomposes as a direct sum of cyclic subgroups, we see the following:

Lemma 2.15. *There is an isomorphism* $\psi : \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \simeq \widehat{\mathbb{Z}}$.

Denote by j the embedding $\mathbb{Z} \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}),\ 1 \mapsto \mathrm{Id}_{\mathbb{Q}/\mathbb{Z}}$. The diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[d] \arrow[dr, "j"] & \\ \widehat{\mathbb{Z}} \arrow[r, "\psi"'] & \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \end{tikzcd}$$

is commutative. Recall, from (2.3), that j_f is the canonical injection $\mathrm{Hom}(\mathrm{Ker}\, \widehat{f}, \mathbb{Z}) \to \mathrm{Hom}(\mathrm{Ker}\, \widehat{f} \otimes \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. Since $\mathrm{Ker}\, \widehat{f}$ is free abelian, we may regard (by means of Lemma 2.15) j_f simply as the diagonal embedding $\delta : \mathrm{Ker}\, \widehat{f} \to \mathrm{Ker}\, \widehat{f} \otimes \widehat{\mathbb{Z}}$. Therefore,

$$\mathrm{Coker}\, \phi_f \simeq \mathrm{Coker}\, j_f \simeq \frac{(\mathrm{Ker}\, \widehat{f}) \otimes \widehat{\mathbb{Z}}}{\delta(\mathrm{Ker}\, \widehat{f})}. \tag{2.8}$$

3. The stable classification theorem

The goal is to generalize a result due to Wall and Durfee [15, Corollary 1]; [5, Theorem 4.1]. We define a natural notion of stable equivalence on lattices equipped with a characteristic form. The resulting stable classification problem is shown to be essentially equivalent to the classification of the quadratic functions induced by the discriminant construction (Section 2).

There is a natural notion of isomorphism among triples (M, f, c) defined by bilinear lattices with characteristic forms: we say that two triples (M, f, c) and (M', f', c') are isomorphic (denoted $(M, f, c) \simeq (M', f', c')$) if there is an isomorphism $\psi : M \to M'$ such that $\psi^* f' = f$ and $\psi^* c' = c \bmod 2\widehat{f}(M)$. All such triples form a monoid for the orthogonal sum $\oplus$. Two triples (M, f, c) and (M', f', c') are said to be *stably equivalent* if they become isomorphic after stabilizations with some unimodular lattices, that is,

there is an isomorphism (called a *stable equivalence*) between $(M, f, c) \oplus (U, g, u)$ and $(M', f', c') \oplus (U', g', u')$ for some unimodular lattices (U, g, u) and (U', g', u') equipped with characteristic forms u and u', respectively. Since, as we have seen in Section 2, unimodular lattices induce trivial discriminant bilinear forms and quadratic functions, a stable equivalence between two triples induces an isomorphism of the corresponding quadratic functions. We are interested in whether the converse holds and to what extent. In fact, a positive answer is provided in the case of nondegenerate lattices.

Proposition 3.1. *Two nondegenerate symmetric bilinear lattices (M, f, c) and (M', f', c') equipped with characteristic forms are stably equivalent if and only if their associated quadratic functions $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ are isomorphic. In fact, any isomorphism ψ between their associated quadratic functions $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ lifts to a stable equivalence between (M, f, c) and (M', f', c').*

Proof. It is sufficient to prove the second statement. There is an obvious notion of stable equivalence for symmetric bilinear lattices (simply forget the characteristic form). Consider the symmetric bilinear lattices (M, f) and (M', f'). The hypothesis implies that $L_{f'} = \psi^* L_f$. It follows from [5, Proof of Theorem 4.1] that there is stable equivalence $\mathfrak{s}$ between (M, f) and (M', f') inducing ψ. Explicitly, $\mathfrak{s}$ is an isomorphism $(M, f) \oplus (U, g) \simeq (M', f') \oplus (U', g')$, where (U, g) and (U', g') are unimodular lattices. In particular, $\mathfrak{s}$ induces an affine isomorphism $\mathrm{Char}(f \oplus g) \simeq \mathrm{Char}(f' \oplus g')$. Let $u \in \mathrm{Char}(g)$. We have $\mathfrak{s}(c \oplus u) = c'' \oplus u' \in \mathrm{Char}(f' \oplus g') = \mathrm{Char}(f') \times \mathrm{Char}(g')$. We have to show that $c'' = c' \bmod 2\widehat{f'}(M')$. By hypothesis, $\phi_{f',c'} \circ \psi = \phi_{f,c}$. Since ψ is induced by $\mathfrak{s}$, we have $\phi_{f',c'} = \phi_{f',\mathfrak{s}(c)}$. The desired result follows from Theorem 2.10. □

In the general case, we have to deal with the potential degeneracy of lattices. The first observation is that it is *not true* that any isomorphism between $(G_f, \phi_{f,c})$ and $(G_{f'}, \phi_{f',c'})$ will lift to a stable equivalence between (M, f, c) and (M', f', c'). In fact, the simplest counterexample is given by the degenerate bilinear lattice $(\mathbb{Z}, 0, 0)$ (with 0 as characteristic form). We have $G_0 = \mathbb{Q}/\mathbb{Z}$, $\phi_{0,0}([x]) = 0$. Trivially any automorphism of $\mathbb{Q}/\mathbb{Z}$ is an automorphism of the degenerate quadratic function $(\mathbb{Q}/\mathbb{Z}, 0)$. However, not every automorphism of $\mathbb{Q}/\mathbb{Z}$ lifts to an automorphism of $\mathbb{Z}$.

For each bilinear lattice (M, f), Lemma 2.8 gives an isomorphism $\langle -, - \rangle_\cdot : G_f \to \mathrm{Hom}(\mathrm{Coker}\, \widehat{f}, \mathbb{Q}/\mathbb{Z})$. Therefore any isomorphism $\psi : \mathrm{Coker}\, \widehat{f} \to \mathrm{Coker}\, \widehat{f'}$ induces an isomorphism $\psi^\sharp : G_{f'} \to G_f$. The map

$$\psi \mapsto \psi^\sharp, \quad \mathrm{Iso}(\mathrm{Coker}\, \widehat{f}, \mathrm{Coker}\, \widehat{f'}) \to \mathrm{Iso}(G_{f'}, G_f)$$

is injective (but not surjective in general: see Lemma 3.4 below). We now state the main theorem of this section.

Theorem 3.2. *Two bilinear lattices (M, f, c) and (M', f', c') with characteristic forms are stably equivalent if and only if there exists an element*

$$\psi^\sharp \in \mathrm{Im}(\mathrm{Iso}(\mathrm{Coker}\, \widehat{f}, \mathrm{Coker}\, \widehat{f'}) \to \mathrm{Iso}(G_{f'}, G_f))$$

such that the associated quadratic functions $(G_f, \phi_{f,c})$ *and* $(G_{f'}, \phi_{f',c'})$ *are isomorphic via* $\psi^\sharp$. *Furthermore, any such isomorphism between* $(G_{f'}, \phi_{f',c'})$ *and* $(G_f, \phi_{f,c})$ *lifts to a stable equivalence between* (M, f, c) and (M', f', c').

In the sequel, (M, f, c) and (M', f', c') denote bilinear lattices with characteristic forms. Set $N_f = \operatorname{Ker} \widehat{L}_f$ and $N_{f'} = \operatorname{Ker} \widehat{L}_{f'}$.

Lemma 3.3. *Any isomorphism* $\psi : \operatorname{Coker} \widehat{f} \to \operatorname{Coker} \widehat{f'}$ *induces an isomorphism* $\operatorname{Ker} \widehat{f'} \to \operatorname{Ker} \widehat{f}$. *Furthermore, if* $\psi([c]) = [c']$ *then* ψ *induces an isomorphism from the triple* $(\operatorname{Ker} \widehat{f'}, 0, c'|_{\operatorname{Ker} \widehat{f'}})$ *onto the triple* $(\operatorname{Ker} \widehat{f}, 0, c|_{\operatorname{Ker} \widehat{f}})$.

Proof. Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{T}\operatorname{Coker} \widehat{f} \arrow[r] \arrow[d, "\psi|_{\mathrm{T}\operatorname{Coker} \widehat{f}}"', "\simeq"] & \operatorname{Coker} \widehat{f} \arrow[r] \arrow[d, "\simeq"', "\psi"] & (\operatorname{Ker} \widehat{f})^* \arrow[r] \arrow[d, "{[\psi]}"] & 0 \\ 0 \arrow[r] & \mathrm{T}\operatorname{Coker} \widehat{f}' \arrow[r] & \operatorname{Coker} \widehat{f}' \arrow[r] & (\operatorname{Ker} \widehat{f}')^* \arrow[r] & 0 \end{tikzcd}$$

where the two exact rows are given by Lemma 2.9 and where $[\psi]$ is induced by ψ and is an isomorphism. Applying $\operatorname{Hom}(-, \mathbb{Z})$ to $[\psi]$ yields an isomorphism $(\operatorname{Ker} \widehat{f'})^{**} = \operatorname{Ker} \widehat{f'} \to (\operatorname{Ker} \widehat{f})^{**} = \operatorname{Ker} \widehat{f}$. For the second statement, the image of $[c] \in \operatorname{Coker} \widehat{f}$ in $(\operatorname{Ker} \widehat{f})^*$ is just $c|_{\operatorname{Ker} \widehat{f}}$. Thus $\psi([c]) = [c']$ implies that $[\psi](c|_{\operatorname{Ker} \widehat{f}}) = c'|_{\operatorname{Ker} \widehat{f'}}$, as desired. □

Lemma 3.4. *Let* $\Psi \in \operatorname{Iso}(G_{f'}, G_f)$. *Then, the following assertions are equivalent*:

(1) *there exists* $\psi \in \operatorname{Iso}(\operatorname{Coker} \widehat{f}, \operatorname{Coker} \widehat{f'})$ such that $\Psi = \psi^\sharp$;
(2) $\Psi(N_{f'}) = N_f$ *and the map* $\Psi|_{N_{f'}} : N_{f'} \to N_f$ *lifts to an isomorphism* $\operatorname{Ker} \widehat{f'} \to \operatorname{Ker} \widehat{f}$.

Proof. (1) $\Longrightarrow$ (2): Lemma 3.3 gives an isomorphism $\operatorname{Hom}([\psi], \mathbb{Z}) : \operatorname{Ker} \widehat{f'} \to \operatorname{Ker} \widehat{f}$. Since $\Psi = \psi^\sharp$ corresponds to $\operatorname{Hom}(\psi, \mathbb{Q}/\mathbb{Z})$ via the right adjoint map $\langle -, - \rangle_{\cdot}$, $\operatorname{Hom}([\psi], \mathbb{Z})$ is a lift of $\Psi|_{N_{f'}} : N_{f'} \to N_f$.

(2) $\Longrightarrow$ (1): for $[\alpha] \in \operatorname{Coker} \widehat{f}$, consider the homomorphism

$$h : [x] \mapsto \langle [\alpha], \Psi([x]) \rangle, \quad G_{f'} \to \mathbb{Q}/\mathbb{Z}.$$

Since by hypothesis, $\Psi|_{N_{f'}}$ lifts to $\operatorname{Ker} \widehat{f'}$, the map $h|_{N_{f'}}$ also lifts to a homomorphism $\operatorname{Ker} \widehat{f'} \to \mathbb{Z}$. Hence by Corollary 2.14 applied to $h|_{N_{f'}}$, we obtain that $h : G_{f'} \to \mathbb{Q}/\mathbb{Z}$ lies in the image of $_{\cdot}\langle -, - \rangle : \operatorname{Coker} \widehat{f'} \to \operatorname{Hom}(G_{f'}, \mathbb{Q}/\mathbb{Z})$. Thus there exists $[\beta] \in \operatorname{Coker} \widehat{f'}$ such that $h = \langle [\beta], - \rangle$. In other words,

$$\forall [x] \in G_{f'}, \quad \langle [\alpha], \Psi([x]) \rangle = \langle [\beta], [x] \rangle \in \mathbb{Q}/\mathbb{Z}.$$

The assignment $\psi : [\alpha] \mapsto [\beta]$ is additive and is bijective since Ψ is bijective. By construction, $\Psi = \psi^\sharp$. $\square$

Proof of Theorem 3.2. The nondegenerate case is treated by Proposition 3.1. Consider now the general case. Let $\mathfrak{s}$ be a stable equivalence between lattices, that is, an isomorphism of symmetric bilinear lattices $\mathfrak{s}: (M', f', c') \oplus (U', g', u') \to (M, f, c) \oplus (U, g, u)$ where (U', g', u') and (U, g, u) are unimodular lattices equipped with characteristic forms. The map $\mathfrak{s}$ induces via the discriminant construction an isomorphism $\Psi : G_{f'} \to G_f$ between $\phi_{f',c'}$ and $\phi_{f,c}$ since unimodular lattices induce trivial quadratic functions. The map s also induces in the obvious way an isomorphism $\psi :$ Coker $\widehat{f} \to$ Coker $\widehat{f'}$ and it is easily verified that $\psi^\sharp = \Psi$.

Conversely, suppose that the given isomorphism Ψ between $(G_{f'}, \phi_{f',c'})$ and $(G_f, \phi_{f,c})$ is induced by an isomorphism $\psi :$ Coker $\widehat{f} \to$ Coker $\widehat{f'}$. First, the homogeneity defects are preserved by Ψ: $d_{\phi_{f',c'}} = d_{\phi_{f,c}} \circ \Psi$. Since $\Psi = \psi^\sharp$, it follows from (2.6) that $\psi([c]) = [c']$. By Lemma 3.3, the induced map Ker $\widehat{f'} \to$ Ker $\widehat{f}$ is an isomorphism between (Ker $\widehat{f'}$, $0, c'|_{\mathrm{Ker}\ \widehat{f'}}$) and (Ker $\widehat{f}$, $0, c|_{\mathrm{Ker}\ \widehat{f}}$).

Let $(\bar{M}, \bar{f})$ be the nondegenerate lattice induced by f (see (2.1)). Choose a section s of the canonical projection $p : M \to \bar{M}$. Lemma 2.2 yields a characteristic form $\bar{c} = s^*|_{\mathrm{Char}(f)}(c) \in \mathrm{Char}(\bar{f})$. The section s of p induces a map $s : G_{\bar{f}} \to G_f$ such that $\phi_{\bar{f},\bar{c}} = \phi_{f,c} \circ s$. Then $s' = \Psi^{-1} \circ s \circ (\psi|_{\mathrm{T(Coker}\ \widehat{f})})^{-1}$ is a section of $B_{f'} : G_{f'} \to \mathrm{T(Coker}\ \widehat{f'})$. Note that by Lemma 2.3, $s'$is induced by a section of $p' : M' \to \bar{M}'$, again denoted s'. If we set $\bar{c'} = (s')^*|_{\mathrm{Char}(f')}(c') \in \mathrm{Char}(\bar{f}')$, we find that $\phi_{\bar{f}',\bar{c}'} = \phi_{f',c'} \circ s'$. It follows from the that $\phi_{\bar{f}',\bar{c}'} \circ \psi|_{\mathrm{T(Coker}\ \widehat{f})} = \phi_{\bar{f},\bar{c}}$. Thus $\phi_{\bar{f}',\bar{c}'} \simeq \phi_{\bar{f},\bar{c}}$.Since these quadratic functions are nondegenerate, Proposition 3.1 applies: the isomorphism $\psi|_{\mathrm{T(Coker}\ \widehat{f})}$ lifts to a stable equivalence between the lattices $(\bar{M}, \bar{f}, \bar{c})$ and $(\bar{M}', \bar{f}', \bar{c}')$.

Finally, there is a stable equivalence between $(M, f, c) \simeq (\bar{M}, \bar{f}, \bar{c}) \oplus (\mathrm{Ker}\ \widehat{f}, 0, c|_{\mathrm{Ker}\ \widehat{f}})$ and $(M', f', c') \simeq (\bar{M}', \bar{f}', \bar{c}') \oplus (\mathrm{Ker}\ \widehat{f'}, 0, c'|_{\mathrm{Ker}\ \widehat{f'}})$ where the isomorphisms are induced by the sections s and s', respectively. By construction, the isomorphism $G_{f'} \to G_f$ induced by this stable equivalence is Ψ. $\square$

Consider two bilinear pairings, denoted ± 1, defined on $\mathbb{Z}$ by $(1, 1) \mapsto \pm 1$, and both equipped with the Wu class $1 \in \mathbb{Z}$. For $n \in \mathbb{N}$, we denote by $n(\mathbb{Z}, \pm 1, 1)$ an n-fold orthogonal sum of some copies of $(\mathbb{Z}, \pm 1, 1)$. The next result says that the stable equivalence in Theorem 3.2 can be realized by adding orthogonal summands of $(\mathbb{Z}, 1, 1)$ and $(\mathbb{Z}, -1, 1)$.

Corollary 3.5. *Let (M, f, c) and (M', f', c') be two symmetric bilinear lattices equipped with characteristic forms. We have*: $(G_f, \phi_{f,c}) \simeq (G_{f'}, \phi_{f',c'})$ *if and only if there exists* $n, n' \in \mathbb{N}$ *such that* $(M, f, c) \oplus n(\mathbb{Z}, \pm 1, 1) \simeq (M', f', c') \oplus n'(\mathbb{Z}, \pm 1, 1)$.

Proof. One direction is obvious. For the converse, we apply Theorem 3.2. We obtain unimodular lattices (U, g) and (U', g'), equipped with characteristic forms s and s', respectively, such that there is an isomorphism sending $(M, f, c) \oplus (U, g, u)$ onto $(M', f', c') \oplus (U', g', u')$. By stabilizing if necessary with $(\mathbb{Z}, 1)$ and $(\mathbb{Z}, -1)$, we may assume that as a

symmetric bilinear pairing (U, g) is indefinite[1] and not even[2]. Then a theorem asserts that (U, g) is isomorphic to an orthogonal sum of copies of $(\mathbb{Z}, 1)$ and $(\mathbb{Z}, -1)$ [13]. It follows that (U, g, u) is isomorphic to a sum of copies of $(\mathbb{Z}, 1, 1)$ and $(\mathbb{Z}, -1, 1)$. (Here we use the fact that any odd integer $a \in \mathbb{Z}$ induces a characteristic form for $(\mathbb{Z}, \pm 1)$ and any such triple $(\mathbb{Z}, \pm 1, a)$ is isomorphic to $(\mathbb{Z}, \pm 1, 1)$. More generally, it follows from Section 2.1 that any unimodular bilinear lattice (U, g) has only one characteristic form modulo $2\widehat{g}(U)$.) A similar observation holds for (U', g', u'). We conclude that there is a stable equivalence between (M, f, c) and (M', f', c') involving only a stabilization with copies of $(\mathbb{Z}, 1, 1)$ and $(\mathbb{Z}, -1, 1)$, which is the desired result. $\square$

Remark 3.6 (*and erratum*). This is a generalization of [2, Lemma 2.1(b)] (whose proof is unfortunately incorrect).

4. A complete system of invariants

We present a complete system of invariants for a certain class of quadratic functions on torsion abelian groups, including the nondegenerate quadratic functions defined on finite abelian groups.

First, we assume that the abelian group G is finite. Let S^1 be the multiplicative group of complex numbers of absolute value 1. For any quadratic function $q : G \to \mathbb{Q}/\mathbb{Z}$, we define the Gauss sum

$$\gamma(q) = |G|^{-1/2} \sum_{x \in G} \exp(2\pi \mathrm{i} q(x)) \in \mathbb{C}.$$

It is not difficult to see that $\gamma(q) \in S^1$ if q is nondegenerate. An important observation is the relation

$$\gamma(\alpha \cdot q) = \mathrm{e}^{-2\pi \mathrm{i} q(\alpha)} \gamma(q), \quad \alpha \in G. \tag{4.1}$$

Recall that b_q (resp. $d_q = q - \overline{q} \in G^*$) is the bilinear pairing (resp. the homogeneity defect) associated to q. The following result is the main result of this section.

Theorem 4.1. *Two nondegenerate quadratic functions $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ on finite abelian groups are isomorphic if and only if there is an isomorphism $\psi : G \to G'$ such that $\psi^* b_{q'} = b_q$, $\psi^* d_{q'} = d_q$ and $\gamma(q') = \gamma(q)$.*

That two isomorphic quadratic functions have same Gauss sums and isomorphic associated bilinear pairings and homogeneity defects is straightforward. The nonobvious part lies in the converse.

Since homogeneous quadratic functions have trivial homogeneity defect, we deduce the following result (see [11, Theorem 1.11.3]).

[1] i.e. g takes both positive and negative values.

[2] even means that $g(x, x) \in 2\mathbb{Z}$ for all $x \in U$.

Corollary 4.2 (*Nikulin* [11]). *Two nondegenerate homogeneous quadratic functions $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ on finite abelian groups are isomorphic if and only if there is an isomorphism $\psi : G \to G'$ such that $\psi^* b_{q'} = b_q$ and $\gamma(q') = \gamma(q)$.*

The key step in the proof of Theorem 4.1 is the following fundamental lemma.

Lemma 4.3 (*Fundamental Lemma*). *Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function on a torsion abelian group G. If $\alpha \in G$ is an element of order 2 such that $q(\alpha) = 0$, then $\alpha \cdot q \sim q$.*

Proof of Theorem 4.1 from Lemma 4.3. With no loss of generality, we may assume that $G = G'$, $b_q = b_{q'}$, $d_q = d_{q'}$ and $\gamma(q) = \gamma(q')$ and then show that $q \sim q'$. The equality $b_q = b_{q'}$ implies (cf. Section 1) that $q' = \alpha \cdot q$ for some $\alpha \in G$. The equality $d_q = d_{q'}$ implies $2\alpha = 0$. The equality $\gamma(q) = \gamma(q')$, together with (4.1) imply that $q(\alpha) = 0$. Therefore Lemma 4.3 applies. □

Hence it remains to prove Lemma 4.3.

Proof of the Fundamental Lemma. If we were just looking for a permutation σ of G such that $q \circ \sigma = \alpha \cdot q$, we could simply choose the involution $(x \mapsto x + \alpha)$ since $q(x + \alpha) = q(x) + b_q(x, \alpha) + q(\alpha) = q(x) + b_q(\alpha, x) = (\alpha \cdot q)(x)$, $x \in G$. So it is sufficient to find $\phi \in \mathrm{End}(G)$ such that

$$(\mathrm{Id}_G + \phi)^2 = \mathrm{Id}_G \quad \text{and} \quad q(x + \phi(x)) = q(x + \alpha) \quad \text{for any } x \in G. \tag{4.2}$$

Since α has order 2, the map $\widehat{b_q}(\alpha) \in G^*$ is of order 2. For any $x \in G$, define $n(x) \in \mathbb{Z}/2\mathbb{Z}$ by

$$\frac{n(x)}{2} = b_q(\alpha, x) \in \mathbb{Q}/\mathbb{Z}.$$

Consider the map $\phi : G \to G$ defined by

$$\phi(x) = n(x)\alpha, \quad x \in G.$$

It is a well-defined (independent of the lift in $\mathbb{Z}$ of $n(x) \in \mathbb{Z}/2\mathbb{Z}$) endomorphism of G. Using the fact that $q(\alpha) = 0$, we readily verify that ϕ satisfies (4.2). □

We now deduce a more general version of Theorem 4.1 which allows for some degeneracy. Consider the following commutative diagram of extensions of abelian groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{i} & G & \xrightarrow{p} & B & \longrightarrow & 0 \\
 & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle \psi|_A} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle \psi} & & {\scriptstyle\simeq}\big\downarrow{\scriptstyle [\psi]} & & \\
0 & \longrightarrow & A' & \xrightarrow{i'} & G' & \xrightarrow{p'} & B' & \longrightarrow & 0.
\end{array}$$

Here $[\psi] : B \to B'$ is the isomorphism induced by ψ.

Definition 4.4. Given a diagram as above, two sections s and s' of p and p', respectively, are said ψ*-compatible*, or simply *compatible*, if $s' = \psi \circ s \circ [\psi]^{-1}$.

$$\begin{tikzcd} G \arrow[r, "p"'] \arrow[d, "\psi"'] & B \arrow[l, bend right, "s"'] \arrow[d, "{[\psi]}"] \\ G' \arrow[r, "p'"] & B' \arrow[l, bend left, "s'"] \end{tikzcd}$$

Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic function on a torsion abelian group. Clearly G can be regarded as the extension of $A=$ Ker $\widehat{b_q}$ by $B = G/\mathrm{Ker}\,\widehat{b_q}$. We shall say that (G, q) meets the *finiteness condition* if the following two conditions are satisfied:

- $G/\mathrm{Ker}\,\widehat{b_q}$ is finite;
- the extension G of Ker $\widehat{b_q}$ by $G/\mathrm{Ker}\,\widehat{b_q}$ is split.

Example 4.5. Let (M, f, c) be a symmetric bilinear lattice equipped with a characteristic form c. Then, the quadratic function $(G_f, \phi_{f,c})$ meets the finiteness condition since by (2.5), the short exact sequence (2.4) is split.

For any given section s of the projection $p : G \to G/\mathrm{Ker}\,\widehat{b_q}$, $q \circ s$ is a nondegenerate quadratic function on $G/\mathrm{Ker}\,\widehat{b_q}$. Note that the finiteness condition implies that $\gamma(q \circ s)$ is well-defined for any section s. In the sequel, we denote by $r_q : \mathrm{Ker}\,\widehat{b_q} \to \mathbb{Q}/\mathbb{Z}$ the homomorphism $q|_{\mathrm{Ker}\,\widehat{b_q}}$.

Corollary 4.6. *Two quadratic functions $q : G \to \mathbb{Q}/\mathbb{Z}$ and $q' : G' \to \mathbb{Q}/\mathbb{Z}$ satisfying the finiteness condition are isomorphic if and only if there is an isomorphism $\psi : G \to G'$ such that $\psi^* b_{q'} = b_q$, $\psi^* d_{q'} = d_q$, $\psi^* r_{q'} = r_q$ and $\gamma(q' \circ s') = \gamma(q \circ s)$ for ψ-compatible sections s and s'.*

Proof. Consider the nondegenerate quadratic functions $q_1 = q \circ s$ and $q_1' = q' \circ s'$. The isomorphism ψ induces an isomorphism $[\psi] : G/\mathrm{Ker}\,\widehat{b_q} \to G'/\mathrm{Ker}\,\widehat{b_{q'}}$. It is readily checked that $[\psi]^* b_{q_1'} = b_{q_1}$, $[\psi]^* d_{q_1'} = d_{q_1}$ and $\gamma(q_1) = \gamma(q_1')$. By Theorem 4.1, q_1 and q_1' are isomorphic. We deduce that $(G, q) \simeq (G/\mathrm{Ker}\,\widehat{b_q}, q_1) \oplus (\mathrm{Ker}\,\widehat{b_q}, r_q)$ is isomorphic to $(G', q') \simeq (G'/\mathrm{Ker}\,\widehat{b_{q'}}, q_1') \oplus (\mathrm{Ker}\,\widehat{b_{q'}}, r_{q'})$. $\square$

Remark 4.7. Theorem 4.1 does not say that if $\psi^* b_{q'} = b_q$, $\psi^* d_{q'} = d_q$ and $\gamma(q') = \gamma(q)$, then $\psi^* q' = q$. In general, the isomorphism between q and q' will be different from ψ. However, how the isomorphism between q and q' will differ from ψ can be read off from the proof of the Fundamental Lemma.

Acknowledgements

F.D. was supported by a E.U. Marie Curie Research Fellowship (HMPH-2001-01174).

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