Notes on Spinors

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Introduction

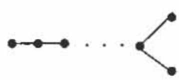
In these notes, we collect the properties of spinors in various dimensions and, over $\mathbb{R}$, for spaces of various signatures. Such information is needed to discuss the possible supersymmetries in various dimensions (super Poincaré groups), and the possible Lorentz invariant kinetic and mass terms for fermions in lagrangians.

The exposition owes a lot to Bourbaki's treatment in Alg. Ch. 9, and through Bourbaki to C. Chevalley's "The algebraic theory of spinors". The super Brauer group of §3 was first considered by C. T. C. Wall (1963), under the name "graded Brauer group" and with a different, but equivalent, definition. I have learned of the analogy between spinorial and oscillator representations (2.3, 2.5) from a lecture R. Howe gave in 1978. The uniform treatment in 6.1 of the Minkowski signature cases was inspired by a conversation with Witten.

CHAPTER 1
Overview

1.1. A *quadratic vector space* is a vector space given with a nondegenerate quadratic form Q. Let V be a complex quadratic vector space. If $\dim(V) \geq 3$, the complex spin group $\mathrm{Spin}(V)$ is the universal covering of the special orthogonal group $\mathrm{SO}(V)$. For V of any dimension ≥ 1, it is a double covering of $\mathrm{SO}(V)$. For $\dim(V) = 1$ or 2, it is described in the table below.

For $n \geq 3$, the Dynkin diagram of the group $\mathrm{Spin}(n)$ is:



$n = 2r$

$n = 2r + 1$

where r is the rank. The spin or semi-spin representations are the fundamental representations corresponding to the vertex, or vertices, at the right of the diagram. They are either self-dual, or permuted by duality. If one restricts the spin representation of $\mathrm{Spin}(2r+1)$ to $\mathrm{Spin}(2r)$, one obtains the sum of the two semi-spin representations of $\mathrm{Spin}(2r)$. Each of the two semi-spin representations of $\mathrm{Spin}(2r)$ restricts as the spin representation of $\mathrm{Spin}(2r-1)$. For a proof, see 5.1.

The Dynkin diagram makes clear the low rank exceptional isomorphisms. We list in the next table $n = 1, 2, 3, 4, 5$ or 6, the Dynkin diagram, a group G to which $\mathrm{Spin}(n)$ is isomorphic, the description of spin or semi-spin representations as representations of G, and the description of the defining representation of $\mathrm{SO}(n) = \mathrm{Spin}(n)/(\mathbb{Z}/(2))$ as a representation of G.

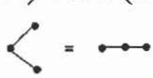
n	diagram	G	(semi-)spin	defining orthogonal
1	none	$\mathbb{Z}/2$	nontrivial character	trivial character
2	none	$\mathbb{G}_m$	characters z, z^{-1}	$z^2 \oplus z^{-2}$
3	$\bullet$	$\mathrm{SL}(2)$	defining V	adjoint $= \mathrm{Sym}^2(V)$
4	$:$	$\mathrm{SL}(2) \times \mathrm{SL}(2)$	defining V_1, V_2	$V_1 \otimes V_2$
5		$\mathrm{Sp}(4)$	defining V	$\wedge^2 V/\text{fixed line}$
6		$\mathrm{SL}(4)$	defining V, and V^*	$\wedge^2 V$

Table 1.1.1

The symmetric bilinear form on the defining orthogonal representation is given by

$n = 3$: Killing form $\mathrm{Tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y)$, or
$\langle v^2, w^2 \rangle = \psi(v,w)^2$, for ψ the symplectic (= volume) form of V.

$n = 4$: $\langle v_1 \otimes v_2,\ w_1 \otimes w_2 \rangle = \psi(v_1, w_1)\psi_2(v_2, w_2)$.

$n = 5$: $\langle v_1 \wedge v_2,\ w_1 \wedge w_2 \rangle = \psi(v_1, w_1)\psi(v_2, w_2) - \psi(v_1, w_2)\psi(v_2, w_1)$.

$n = 6$: $\langle v_1 \wedge v_2,\ w_1 \wedge w_2 \rangle = v_1 \wedge v_2 \wedge w_1 \wedge w_2 \in \wedge^4 V = \mathbb{C}$.

For $n = 8$, the Dynkin diagram makes clear that the vector and the two semi-spinor representations play symmetric roles (triality). The octonionic model for them, given in 6.5, makes this symmetry explicit.

1.2. Let V be a real quadratic vector space, with complexification $V_{\mathbb{C}}$. The real form V of $V_{\mathbb{C}}$ defines real forms $\mathrm{SO}(V)$ and $\mathrm{Spin}(V)$ of the complex algebraic groups $\mathrm{SO}(V_{\mathbb{C}})$ and $\mathrm{Spin}(V_{\mathbb{C}})$. For $\dim V \geq 3$, the group of real points of $\mathrm{Spin}(V)$ is connected, and is the unique nontrivial double covering of the connected component of the group of real points of $\mathrm{SO}(V)$.

The *spinorial representations* of $\mathrm{Spin}(V)$ are the real representations which, after extensions of scalars to $\mathbb{C}$, become sums of spin or semi-spin representations. The dual of a spinorial representation is again spinorial.

We will be interested in the following kind of morphisms of representations.

(1.2.1) symmetric morphisms $S \otimes S \to V$, for S spinorial. Such morphisms enter into the construction of super Minkowski spaces and super Poincaré groups.

(1.2.2) $V \otimes S_1 \to S_2$, for S_1 and S_2 spinorial. Such morphisms enter into the construction of Dirac operators $\not{D}$ between spin bundles.

(1.2.3) $S \otimes S \otimes V \to$ trivial. Such morphisms enter into kinetic terms $\psi \not{D} \psi$ in Lagrangians.

(1.2.4) $S \otimes S \to$ trivial. Such morphisms enter into mass terms $\psi M \psi$ in Lagrangians.

1.3. The nature of the spinorial representations and of the morphisms (1.2.1), (1.2.2), (1.2.3), (1.2.4) is controlled by the signature and the dimension modulo 8. We explain those two modulo 8 periodicities in 1.4 and 1.5 and, with more details, in §3 and §4.

1.4. The signature modulo 8 determines how many nonisomorphic irreducible spinorial representations there are, and whether they are real, complex or quaternionic. As (V, Q) and $(V, -Q)$ define the same spin group, only the signature taken up to sign matters.

Let p and q be the number of $+$ and $-$ signs of Q, in an orthogonal basis. The signature $p-q$ and the dimension $p+q$ have the same parity. If they are odd, there is over the complex numbers a unique irreducible spinorial representation. If they are even, there are two. It follows that over the reals there is in the odd case a unique irreducible spinorial representation. It is either real or quaternionic. In the

even case, either there are two, real or quaternionic, or there is only one, and it is complex. Here is a table showing for which signatures each case occurs:

signature $p-q$ mod 8	real, complex or quaternionic
0	$\mathbb{R}$, $\mathbb{R}$
1 or 7	$\mathbb{R}$
2 or 6	$\mathbb{C}$
3 or 5	$\mathbb{H}$
4	$\mathbb{H}$, $\mathbb{H}$

Table 1.4.1

1.5. Over the complex numbers, the dimension modulo 8 controls the existence and symmetry properties of the morphisms (1.2.1), (1.2.2), (1.2.3) and (1.2.4).

For n odd, the spin representation S of $\mathrm{Spin}(n)$ is orthogonal or symplectic. There is a morphism of representations from $S \otimes S$ to the defining representation V; it is unique, hence it is symmetric or antisymmetric. Here and below, "unique" is understood projectively: uniqueness up to a scalar factor of a nonzero morphism.

For n even, the semi-spin representations S^+ and S^- are orthogonal, symplectic, or duals of each other. There are unique morphisms of representations $V \otimes S^+ \to S^-$ and $V \otimes S^- \to S^+$, and no morphism $V \otimes S^+ \to S^+$ or $V \otimes S^- \to S^-$. If S^+ and S^- are duals of each other, the morphism $V \otimes S^+ \to S^-$ (resp. $V \otimes S^- \to S^+$) corresponds to a morphism $S^+ \otimes S^+ \to V$ (resp. $S^- \otimes S^- \to V$), which is either symmetric or antisymmetric. If S^+ and S^- are self-dual, they correspond to a morphism $S^+ \otimes S^- \to V$.

Here is a table showing in which dimension each case occurs:

n mod 8	forms on spinors	symmetry of spinors, spinors $\to V$
0	S^+ and S^- orthogonal	$S^+ \otimes S^- \to V$
1	orthogonal	symmetric
2	S^+ dual to S^-	symmetric (on S^+, and on S^-)
3	symplectic	symmetric
4	S^+ and S^- sympletic	$S^+ \otimes S^- \to V$
5	symplectic	antisymmetric
6	S^+ dual to S^-	antisymmetric (on S^+, and on S^-)
7	orthogonal	antisymmetric

Table 1.5.1

1.6. To go from 1.5 to information over $\mathbb{R}$, one proceeds as follows.

(A) When the irreducible complex spinorial representations are real, i.e. admit a real form, the real story is the same as the complex story.

(B) Suppose now that they are quaternionic. After extension of scalars to $\mathbb{C}$, the quaternions become a 2×2 matrix algebra: $\mathbb{H}_{\mathbb{C}} = \mathrm{End}(W)$, the real form $\mathbb{H}$ of $\mathrm{End}(W)$ being induced by an antilinear automorphism σ of W, with $\sigma^2 = -1$, and the conjugation $*$ of $\mathbb{H}$ being the transposition relative to any symplectic form ψ_W on W. The form ψ_W can and will be chosen to be real, in the sense that

$\psi_W(\sigma x, \sigma y) = \overline{\psi_W(x, y)}$. The $\psi_W(x, \sigma x)$ are then real, and ψ_W can and will be normalized so that $\psi_W(x, \sigma x) > 0$ for $x \neq 0$.

If S is an irreducible real representation, with field of endomorphisms $\mathbb{H}$, the complexification $S_\mathbb{C}$ of S is acted upon by $\mathrm{End}(W)$ and hence can be written as

$$S_\mathbb{C} = S_0 \otimes W ,$$

for $S_0 = \mathrm{Hom}_{\mathrm{End}(W)}(W, S_\mathbb{C})$. The complex conjugation σ_S of $S_\mathbb{C}$ can be written $\sigma_S = \sigma_0 \otimes \sigma$, with σ: $S_0 \to S_0$ $\mathbb{C}$-antilinear and $\sigma_0^2 = -1$.

The complex representation S_0 is irreducible, and 1.5 applies to it. If S_0 is orthogonal, $\psi(X, Y) \mapsto \phi^\sigma(X, Y) := \overline{\phi(\sigma x, \sigma y)}$ is a $\mathbb{C}$-antilinear involution on the one-dimensional space of symmetric bilinear forms on S_0. If ϕ is real, i.e. if $\phi = \phi^\sigma$, $\phi \otimes \psi_W$ on $S_\mathbb{C}$ is the complexification of a symplectic form on S, relative to which the conjugation $*$ of $\mathbb{H}$ is transposition.

Similarly, if S_0 is orthogonal, S is symplectic. If two irreducible spinorial representations S^+ and S^- are quaternionic, if S_0^+ and S_0^- are in duality, so are S^+ and S^-. The same applies to pairings with values in V.

(C) If the irreducible spinorial representation S is complex, i.e. if its commutant is $\mathbb{C}$, by complexification it becomes the sum of the two semi-spinorial representations. If they are duals of each other (resp. orthogonal, resp. symplectic), S admits an invariant Hermitian (resp. complex orthogonal, complex symplectic) form.

1.7. In *Minkowski signature,* $(+,-,-,\dots)$, and with a positive light cone chosen on V, if S is an irreducible spinorial representation, there is up to a real factor a unique symmetric bilinear maps B: $S \otimes S \to V$. It can be normalized so that $Q(s) := \frac{1}{2} B(s, s)$ takes values in the closed positive cone. Such a Q is now unique up to a positive real factor.

If S is complex, B is the real part of a Hermitian bilinear form with values in $V_\mathbb{C}$: if J is the complex structure, Q is invariant under the circle group of the $\exp(\theta J)$ $(\theta \in \mathbb{R})$. After extension of scalars to $\mathbb{C}$, S becomes the sum of the two semi-spinorial representations, and B corresponds to a pairing $S^+ \otimes S^- \to V$.

If S is quaternionic, the quaternions with absolute value one preserve Q. After extension of scalars to $\mathbb{C}$, S becomes the sum $S_0 \otimes W$ of two copies of an irreducible spinorial representation S_0, and B becomes the tensor product of an antisymmetric pairing $S_0 \otimes S_0 \to V_\mathbb{C}$ with a scalar alternating form $W \otimes W \to \mathbb{C}$. The quaternions become $\mathrm{End}(W)$.

The next table gives the nature of irreducible spinorial representations, as a function of n mod 8, for $\mathrm{Spin}(1, n-1)$. If $\mathrm{Spin}(1, n-1)$ has two irreducible spinorial representations, they have isomorphic restrictions to $\mathrm{Spin}(1, n-2)$. The last column of the table describes the restriction of an irreducible representation of $\mathrm{Spin}(1, n-1)$ to $\mathrm{Spin}(1, n-2)$. It is either an irreducible spinorial representation, or twice it, or

the sum of two distinct spinorial representations. Notation: S, $2S$, $S^+ + S^-$.

n mod 8	nature	restriction to $\mathrm{SO}(1, n-2)$
1	$\mathbb{R}$	S
2	$\mathbb{R}$, $\mathbb{R}$ (S^+ and S^-, in duality)	S
3	$\mathbb{R}$	$S^+ + S^-$
4	$\mathbb{C}$	$2S$
5	$\mathbb{H}$	$2S$
6	$\mathbb{H}$, $\mathbb{H}$ (S^+ and S^-, in duality)	S
7	$\mathbb{H}$	$S^+ + S^-$
8	$\mathbb{C}$	S

The cases $n = 3$, 4, 6 and 10 are particularly interesting. The spin groups acting on an irreducible spinorial representation S are respectively $\mathrm{SL}(2, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{C})$, $\mathrm{SL}(2, \mathbb{H})$ and deserving to be called $\mathrm{SL}(2, \mathbb{O})$. In those dimensions, the spin group acts transitively on non-zero spinors and by the symmetric pairing $S \otimes S \to V$, $s \otimes s$ is mapped to an isotropic vector.

CHAPTER 2
Clifford Modules

2.1. Let k, V and Q be a commutative ring, a k-module and a quadratic form on V. The *Clifford algebra* $C(V,Q)$ is the associative k-algebra with unit generated by the k-module V with the relations

$$x^2 = Q(x) \cdot 1 \tag{2.1.1}$$

for $x \in V$. This is Bourbaki's definition (Alg. Ch. 9 §9, n° 1). Some authors prefer to use as defining relations $x^2 = -Q(x) \cdot 1$. We will often write $C(V)$, or $C(Q)$, instead of $C(V,Q)$.

It results immediately from this definition that

(A) $C(Q)$ is mod 2 graded, the image of V being odd. In other words: $C(Q)$ is a super algebra. Indeed, the defining relations are in the even part of the tensor algebra on V. We will write $p(x)$ for the parity of a homogeneous element x of $C(Q)$.

(B) The algebra $C(Q)$ admits a unique anti-involution β which is the identity on the image of V. Indeed, the opposite algebra $C(Q)^0$ is a solution of the same universal problem as $C(Q)$ is. "Opposite" is taken in the ungraded sense, not in the super sense. Bourbaki's terminology: β is the principal antiautomorphism of $C(Q)$. By definition, $\beta(xy) = \beta(y)\beta(x)$.

If we apply (2.1.1) to $x+y$, x and y and take the difference, we obtain the polarized form of (2.1.1):

$$xy + yx = \Phi(x,y) \cdot 1 \tag{2.1.2}$$

for Φ the bilinear form $Q(x+y) - Q(x) - Q(y)$ associated to Q.

When 2 is invertible in k, (2.1.1) is equivalent to (2.1.2): take $x = y$ in (2.1.2). In general, if $X \subset V$ generates V, (2.1.1) is implied by (2.1.1) for $x \in X$, and (2.1.2) for $x \neq y$ in X. This makes it clear that

(C) The formation of $C(Q)$ is compatible with extensions of scalars.

(D) If (V,Q) is the orthogonal direct sum of (V',Q') and (V'',Q''), the Clifford algebra $C(Q)$ is the tensor product, in the sense of super algebras (Supersymmetry 1.1 (1.1.5)), of $C(Q')$ and $C(Q'')$. If β' and β'' are the principal antiautomorphisms of $C(Q')$ and $C(Q'')$, the principal antiautomorphism β of $C(Q)$ is given by

$$\beta(x \otimes y) = (-1)^{p(x)p(y)} \beta(x) \otimes \beta(y) .$$

Indeed, $C(Q)$ is generated by the k-modules V' and V'', with the relations (2.1.1) for $x \in V'$ or $x \in V''$ and the relations (2.1.2) for $x \in V'$ and $y \in V''$. For $x \in V'$ and $y \in V''$, (2.1.2) means that x and y supercommute, and the first assertion of (D) follows. For the second, one observes that for x in $C(V')$ and y in $C(V'')$, one has

$$\begin{aligned}\beta(x \otimes y) &= \beta((x \otimes 1)(1 \otimes y)) = \beta(1 \otimes y)\beta(x \otimes 1) \\ &= (1 \otimes \beta(y))(\beta(x) \otimes 1) = (-1)^{p(x)p(y)}\beta(x) \otimes \beta(y) \ .\end{aligned}$$

(E) The identity of V extends to an isomorphism from the opposite of the super algebra $C(Q)$ to the super algebra $C(-Q)$.

Indeed, each defining relation $x \cdot x = Q(x) \cdot 1$ is replaced by its opposite $x \cdot x = -Q(x) \cdot 1$.

We now take $k = \mathbb{C}$. From now on, we assume V non reduced to 0.

Proposition 2.2 (Bourbaki Alg. Ch. 9 §9, no. 4). *Let V be a complex quadratic vector space.*

(i) *If V is of dimension $2n > 0$ the super algebra $C(V)$ is isomorphic to $\underline{\mathrm{End}}\,(S)$, with S of dimension $2^{n-1}|2^{n-1}$.*

(ii) *Let D be the super algebra $\mathbb{C}[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$. If V is of dimension $2n+1$, $C(V)$ is isomorphic to $\underline{\mathrm{End}}_D(D^N) \sim D \otimes M_N(\mathbb{C})$ for $N = 2^n$.*

Proof of (i). The quadratic space V is isomorphic to $L \oplus L^\vee$ with $\dim L = n$ and $Q(\ell + \alpha) = \langle \ell, \alpha \rangle$. Take $S = \wedge^* L^\vee$. One defines

$$C(L \oplus L^\vee) \to \underline{\mathrm{End}}\,(S) \tag{2.2.1}$$

by mapping α in $L^\vee$ to the odd endomorphism $\alpha\wedge$, and ℓ in L to ι_ℓ, the odd derivation of the exterior algebra $\wedge^* L$ (viewed as a commutative super algebra) for which $\iota_\ell(\alpha) = \langle \ell, \alpha \rangle$. A decomposition of L as a direct sum of lines L_i induces a dual decomposition of $L^\vee$ as the sum of the dual lines $L^\vee{}_i$, and a decomposition of V as the orthogonal direct sum of the hyperbolic planes $V_i := L_i \oplus L^\vee{}_i$. The super algebra $C(V)$ is the tensor product of the super algebra $C(V_i)$, the graded vector space $S = \wedge^* L^\vee$ is the tensor product of the graded vector spaces $S_i := \wedge^* L^\vee{}_i$, and (2.2.1) is the tensor product of the similarly defined morphisms $C(V_i) \to \underline{\mathrm{End}}\,(S_i)$. To prove that (2.2.1) is an isomorphism, it hence suffices to check it when $\dim(L) = 1$, a case left to the reader.

Proof of (ii). For $\dim(V) = 1$, $C(V)$ is isomorphic to D. For $\dim(V) \geq 3$, we write V as the orthogonal direct sum of a line V_1 and of V_{2n}, of dimension $2n$. For S of dimension $2^{n-1}|2^{n-1}$, $C(V)$ is isomorphic to $\underline{\mathrm{End}}\,(S) \otimes D \sim \underline{\mathrm{End}}_D(S \otimes D) \sim \underline{\mathrm{End}}_D(D^N)$, the D-module ΠD being isomorphic to the D-module D.

2.3. The construction used in the proof of 2.2. (i) is an odd analogue of that of the Schrödinger representation of Heisenberg groups, or Lie algebras. The analogy goes as follows. Let V be symplectic, rather than orthogonal, over a field k. Provide $k \oplus V$ with the Lie algebra structure for which k is central and for which the bracket of elements of V is given by $[v_1, v_2] = \psi(v_1, v_2)$. It is the Heisenberg Lie algebra

$\mathcal{H}$ associated to V. Write V as a direct sum $L \oplus L^\vee$, with $\psi(\ell' + \alpha', \ell'' + \alpha'') = \alpha''(\ell') - \alpha'(\ell'')$. The Lie algebra $\mathcal{H}$ acts on the vector space $\mathrm{Sym}^*(L^\vee)$, the algebra of polynomial functions on L, with $1 \in k$ acting as the identity, α in $L^\vee$ acting by multiplication by α, and ℓ in L acting by ∂_ℓ. This turns $\mathrm{Sym}^*(L^\vee)$ into a module over the universal envelopping algebra $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$. Let 1 be the unit element of $\mathcal{U}(\mathcal{H})$, and $1_\mathcal{H}$ be the central element $1 \in k$ of $\mathcal{H}$. Both act as the identity on $\mathrm{Sym}^*(L^\vee)$, which is hence a module over the quotient $\mathcal{U}_1(\mathcal{H})$ of $\mathcal{U}(\mathcal{H})$ by the relation $1 = 1_\mathcal{H}$. If we repeat this in the super case, for a purely odd super vector space V, the symplectic structure is, in concrete terms, a symmetric bilinear form on the underlying vector space. The sum $k \oplus V$ is a super Lie algebra $\mathcal{H}$, and $\mathcal{U}_1(\mathcal{H})$ is the Clifford algebra $C(V)$.

In the even case, $\mathcal{U}_1(\mathcal{H})$ maps isomorphically to the algebra of all polynomial differential operators acting on $\mathrm{Sym}^*(L^\vee)$. In the odd case, the analogue is all endomorphisms of $\wedge^* L^\vee$. Physicists use the same terminology in both cases: 1 in $\mathrm{Sym}^*(L^\vee)$ (resp. $\wedge^* L^\vee$) is the vacuum, $L^\vee$ acts by creation operators, and L by annihilation operators.

The Poincaré-Birkhoff-Witt theorem implies, in the even case, that $\mathrm{Sym}^*(V) \xrightarrow{\sim} \mathrm{Gr}\,\mathcal{U}_1(\mathcal{H})$, where $\mathcal{U}_1(\mathcal{H})$ is filtered by the images of the $\bigoplus_{i \le n} V^{\otimes i}$. In the odd case, it implies that $\wedge^* V \xrightarrow{\sim} \mathrm{Gr}\, C(V)$.

2.4. We return to the case of a complex quadratic vector space V. By 2.2 (i), if the dimension of V is even, the Clifford algebra $C(V)$ is isomorphic to a matrix algebra, hence has up to isomorphism a unique simple module S. The group of automorphisms of S is the multiplicative group $\mathbb{C}^*$. The module S admits a mod 2 grading $S = S^+ \oplus S^-$ compatible with the grading of $C(V)$, unique up to parity change. From the super point of view, $C(V)$ is a super matrix algebra, hence has exactly two isomorphism classes of simple super modules, exchanged by parity change. The even part $C^+(V)$ of $C(V)$ is $\mathrm{End}(S^+) \times \mathrm{End}(S^-)$. As $S^+, S^- \neq \{0\}$, the center Z of $C^+(V)$ is $\mathbb{C} \times \mathbb{C}$, and the two isomorphism classes of simple super modules are distinguished by the character through which Z acts on S^+.

It follows that any automorphism g of $C(V)$ can be extended to an automorphism of ($C(V)$, module S). The extension is unique up to an automorphism $\lambda \in \mathbb{C}^*$ of S. If g respects the grading of $C(V)$, the extension respects or permutes the homogeneous components $S^\pm$ of S, depending on the action of g on the center of $C^+(V)$.

In particular, $\mathrm{O}(V)$ acts projectively on S. The parity respecting subgroup is $\mathrm{SO}(V)$.

The projective action of the group $\mathrm{SO}(V)$ on S induces a projective action of its Lie algebra $\mathfrak{so}(V)$. In algebraic language, this can be expressed as follows. The construction of the projective action can be repeated after extension of scalars from $\mathbb{C}$ to the ring of dual numbers $B := \mathbb{C}[\varepsilon]/(\varepsilon^2)$, and interpreting $\mathfrak{so}(V)$ as $\mathrm{Ker}(\mathrm{SO}(V)(B) \to \mathrm{SO}(V)(\mathbb{C}))$, one obtains a projective action $\rho\colon \mathfrak{so}(V) \to \mathrm{End}(S)/\mathbb{C}$ of $\mathfrak{so}(V)$ on S. Equivalent description: as $C(V)$ is a matrix algebra, any derivation of $C(V)$ is inner. In particular, the derivation defined by x in $\mathfrak{so}(V)$ can be written as $a \mapsto [f(x), a]$. As this derivation is even, $f(x)$ is in $C^+(V)$. It is well defined up an additive constant, and $\rho(x)$ is multiplication by $f(x)$. As $C(Q)$ is, as an ordinary Lie algebra, the Lie algebra product of $\mathbb{C}$ and of $[C(V), C(V)]$, f can be normalized to be a Lie algebra morphism from $\mathfrak{so}(V)$ to $[C(V), C(V)]$ (usual bracket). Here is a direct construction of f, so normalized.

Identifying $\mathfrak{so}(V)$ with $\wedge^2(V)$ by

$$x \wedge y \longmapsto \text{endomorphism } (v \mapsto \Phi(y,v)x - \Phi(x,v)y) \text{ of } V \ , \tag{2.4.1}$$

one defines f to be

$$f\colon\ x \wedge y \longmapsto \frac{1}{2}(xy - yx)\ . \tag{2.4.2}$$

We have to check that for v in V, $[f(x \wedge y), v]$ is given by (2.4.1). As $xy + yx$ is a scalar, one has indeed, for $\{\ ,\ \}$ the super bracket in $C(V)$

$$\begin{aligned}[f(x \wedge y), v] &= [xy, v] = \{xy, v\} = x\{y,v\} - \{x,v\}y \\ &= \Phi(y,v)x - \Phi(x,v)y\ .\end{aligned}$$

If the dimension of V is odd, $C^+(V)$ is a matrix algebra and the same argument shows that $\mathfrak{so}(V)$ acts projectively on a simple $C^+(V)$-module S^+ (unique up to isomorphism). This action lifts uniquely to a Lie algebra morphism from $\mathfrak{so}(V)$ to $[C^+(V), C^+(V)] = C^+(V) \cap [C(V), C(V)]$. This lifting is again given by (2.4.1), (2.4.2).

Variant: if the dimension of V is odd, the super algebra $C(V)$ has up to isomorphism a unique simple super module S. The group of (even) automorphisms is $\mathbb{C}^*$. The group $\mathrm{O}(V)$ acts projectively on S (respecting the mod 2 grading), and so does the Lie algebra $\mathfrak{so}(V)$. The action is by multiplication by $f(D)$, with f as in (2.4.2).

2.5. In the spirit of 2.3, the construction used in 2.4 is analogous to the one by which, using the uniqueness of an irreducible representation of the Heisenberg commutation relations, one gets on the representation space a projective action of a real symplectic group — or an actual representation of its double covering, the metaplectic group. This is, however, an analytic story, involving infinite dimensional Hilbert spaces, as evidenced by the fact that the metaplectic double covering of the real symplectic group is not algebraic. The Lie algebra story, however, has a purely algebraic analogue. With the notations of 2.3, the symmetrized product gives a vector space isomorphism

$$\mathrm{Sym}^*(V) \xrightarrow{\sim} \mathfrak{U}_1(k \oplus V) \tag{2.5.1}$$

The image of $\mathrm{Sym}^2(V)$ by this map is a Lie algebra normalizing V, and it is identified by its action on V with the Lie algebra of the symplectic group.

Assume V is the dual of E, so that $\mathrm{Sym}^*(V)$ is the polynomial functions on E. If we replace ψ by $t\psi$, and transport the product on the corresponding $\mathfrak{U}_1(k \oplus V)$ to $\mathrm{Sym}^*(V)$ by (2.5.1), we obtain on $\mathrm{Sym}^*(V)$ a product $*_t$ depending on t. It comes from a $k[t]$-algebra structure on $\mathrm{Sym}^*(V) \otimes k[t]$.

One has

$$f *_t g = fg + \tfrac{1}{2}\{f,g\}t + O(t^2),$$

for $\{\ ,\ \}$ the Poisson bracket on the symplectic manifold E. A weakened version of the fact that $\mathrm{Sym}^*(V) \subset \mathfrak{U}_1(k \oplus V)$ acts on V as Lie $\mathrm{Sp}(V)$ is the fact that the Hamiltonian vector fields on E given by quadratic functions are the infinitesimal symplectic transformations.

2.6. We return to the case of a complex quadratic vector space V. The Lie algebra morphism (2.4.1), (2.4.2) from $\mathfrak{so}(V)$ to $C^+(V)$ induces an algebra morphism

$$\mathcal{U}\mathfrak{so}(V) \to C^+(V)\,. \tag{2.6.1}$$

For x, y in V, $\frac{1}{2}(xy - yx)$ is in the image of $\mathfrak{so}(V)$; it differs from xy by the constant $\frac{1}{2}(xy + yx) = \frac{1}{2}\phi(x,y)$, and it follows that (2.6.1) is onto. The *spinorial* representations S of $\mathfrak{so}(V)$ are those for which $\mathcal{U}\mathfrak{so}(V) \to \mathrm{End}(S)$ factors through $C^+(V)$. By (2.6.1), they are identified with $C^+(V)$-modules. By 2.2, if S is a simple super $C(V)$-module, the simple spinorial representations are isomorphic to

for n even: S^+ or S^-;

for n odd: S^+ (isomorphic to S^-).

For real quadratic vector spaces, *spinorial* representations are those which become spinorial after extension of scalars to $\mathbb{C}$. They are identified by (2.6.1) to $C^+(V)$-modules.

The *spinorial representations* of the spin group $\mathrm{Spin}(V)$ are those representations obtained by integrating the spinorial representations of the Lie algebra $\mathfrak{so}(V)$.

Remark. If S is a super module over $C(V)$, the induced $C^+(V)$-module structure turns S^+ and S^- into spinorial representations of $\mathfrak{so}(V)$. The maps $V \otimes S^\pm \to S^\mp$ induced by the module structure are morphisms of representations. Indeed, for $a \in \mathfrak{so}(V) \subset C^+(V)$, $a(vs) = [a,v]s + v(as)$.

If V is odd-dimensional, let Z be the commutant of $C^\cdot(V)$ in $C(V)$ and let Z^- be its odd part. It is one-dimensional. The multiplication by a generator z of Z^- is an isomorphism of representations $S^\pm \to S^\mp$.

2.7. In the same way that, in 2.4, the Lie algebra $\mathfrak{so}(V)$ is identified with a sub-Lie algebra of $C^+(V)$, the spin group $\mathrm{Spin}(V)$ can be realized as a subgroup of the multiplicative group of $C^+(V)$. Let G be the group of invertible elements in $C^+(V)$ or $C^-(V)$ which normalize V. We let it act on V by

$$\rho(g):\ v \longmapsto (-1)^{p(g)} g v g^{-1}.$$

This action respects the quadratic form $Q(v) = v^2$ on V. If g fixes V, g is in the center (in the super sense) of $C(V)$, hence is a scalar (3.4 and 3.3.1): we have an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G \longrightarrow \mathrm{O}(V). \tag{2.7.1}$$

Elements w of V with $Q(w) \neq 0$ are in $G^- := G \cap C^-(Q)$, and $\rho(w)$ is the reflection relative to the hyperplane orthogonal to w. As each element of $\mathrm{O}(V)$ is a product of reflections, the sequence (2.7.1) is exact on the right. As a product of k reflections has determinant $(-1)^k$, it follows from (2.7.1) that $G^+ := G \cap C^+(Q)$ maps onto $\mathrm{SO}(V)$, while G^- maps onto $\mathrm{O}(V) - \mathrm{SO}(V)$. We have

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G^+ \longrightarrow \mathrm{SO}(V) \longrightarrow 1. \tag{2.7.2}$$

If $g \in G$, applying the principal antiautomorphism β to the defining relation

$$gv = (-1)^{p(g)} vg,$$

we see that $\beta(g) \in G$ with $\rho(\beta(g)) = \rho(g)^{-1}$. It follows that $g\beta(g)$ is in $\mathbb{C}^*$, and $g \mapsto g\beta(g)$ is a homomorphism from G to $\mathbb{C}^*$: $gh\beta(gh) = g(h\beta(h))\beta(g) = g\beta(g)h\beta(h)$. The *Spin group* is the kernel of

$$g\beta(g) : G^+ \longrightarrow \mathbb{C}^*. \tag{2.7.3}$$

On $\mathbb{C}^* \subset G^+$, $g\beta(g)$ is the squaring map. It follows that the group $\mathrm{Spin}(V)$ is a double covering of $\mathrm{SO}(V)$: we have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathrm{Spin} \arrow[r] \arrow[d] & \mathrm{SO} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[dr, "\lambda^2"'] & G^+ \arrow[r] \arrow[d, "g\beta(g)"] & \mathrm{SO} \arrow[r] & 1 \\
& & \mathbb{C}^* & &
\end{tikzcd} \tag{2.7.4}$$

For real quadratic vector spaces, or over any field k, the groups of k-points of the algebraic group G^+ and $\mathrm{Spin}(V)$ maintain the same description, but in (2.7.4) $\mathrm{Spin}(k) \to \mathrm{SO}(k)$ is in general not surjective, as $\lambda \mapsto \lambda^2$: $k^* \to k^*$ isn't.

CHAPTER 3
Reality of Spinorial Representations and Signature Modulo 8

3.1. Let V be a real quadratic vector space. By 2.6, the reality properties of the irreducible spinorial representations of $\mathfrak{so}(V)$ depend only on the structure of the $\mathbb{R}$-algebra $C^+(V)$. As what we care about is the category of $C^+(V)$-modules, and more precisely the number of isomorphism classes of simple modules, and their fields of endomorphisms, $C^+(V)$ matters only up to Morita equivalence. It will be convenient to consider the full super algebra $C(V)$, and to encode its properties in terms of the super Brauer group (C.T.C. Wall (1963)) whose properties we review in 3.2 to 3.5.

3.2. Fix a ground field k of characteristic $\neq 2$. A super k-algebra A is a (super) *division algebra* if $A \neq 0$ and if every nonzero homogeneous element a of A is invertible. Every super module M over a super division algebra is free: if A is purely even, it is the sum of A-vector spaces in even and odd degree; if A is not purely even, a basis of M^+ over the even part A^+ of A (which is an ordinary division algebra) is a basis of M over A.

From now on, we will consider only finite dimensional super algebras over k.

Example. If k is algebraically closed, the only finite dimensional super division algebras A over k are, up to isomorphism, k itself and $D := k[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$. The super algebra D is not super commutative. Its super center is reduced to k. It is isomorphic to the opposite super algebra D^0, which is $k[\varepsilon^0]$ with ε^0 odd and $(\varepsilon^0)^2 = -1$. Left and right multiplications turn the super vector space D into a $D \otimes D^0$-module, where $\otimes$ is the super tensor product of super algebras (Supersymmetry (1.1.5)), and

$$D \otimes D^0 \xrightarrow{\sim} \underline{\mathrm{End}}_k(D)\ , \tag{3.2.1}$$

a $1|1$ matrix algebra. Indeed, in the basis $1, \varepsilon$ of D, the matrix of multiplication by $\varepsilon \otimes 1$ (resp. $1 \otimes \varepsilon^0$) is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$). Those matrices generate $M_{1|1}$ and both sides of (3.2.1) have the same dimension. Over an algebraically closed field, D hence appears as a square root of the matrix algebra $M_{1|1}$.

3.3. A (finite dimensional) super algebra A over k is *central simple* if $A \neq 0$ and, after extension of scalars to an algebraic closure $\bar{k}$ of k, it becomes isomorphic to

a matrix algebra $M_{r|s}$, or to $M_{r|s} \otimes D$. The latter is isomorphic to $M_{r+s} \otimes D$, the super D-module $k^{r|s} \otimes D$ being isomorphic to D^{r+s}. The following statements about an algebra are true as soon as they are true after an extension of scalars. To check them for central simple algebras, it hence suffices to check them after extension of scalars to an algebraic closure $\bar{k}$ of k.

(3.3.1) The super center is reduced to k.

(3.3.2) As in (3.2.1), $A \otimes A^0 \xrightarrow{\sim} \underline{\mathrm{End}}_k(A)$.

(3.3.3) For any A-module $L \neq 0$, with commutant $B := \underline{\mathrm{End}}_A(L)$, B is central simple and one has

$$A \xrightarrow{\sim} \underline{\mathrm{End}}_B(L) .$$

The super center of a tensor product is the tensor product of the super centers, and this reduces (3.3.1) to the cases of $M_{r|s}$ and D. The map (3.3.2) for a tensor product is the tensor product of the maps (3.3.2) for the factors, and this reduces (3.3.2) to (3.2.1) and to (3.3.2) for $M_{r|s}$, i.e. for $\underline{\mathrm{End}}_k(V)$ for $V = k^{r|s}$. This case is checked by identifying the $\underline{\mathrm{End}}(V) \otimes \mathrm{End}(V)^0$-module $\underline{\mathrm{End}}(V)$ to the $\underline{\mathrm{End}}(V) \otimes \mathrm{End}(V^\vee)^0$-module $V \otimes V^\vee$. If $A = \underline{\mathrm{End}}(V)$, any A-module is of the form $V \otimes W$, and (3.3.3) results from the commutant being $\underline{\mathrm{End}}(W)$. If $A = D \otimes \mathrm{End}_k(V)$, for V a purely even k-vector space, any A-module is of the form $D \otimes V \otimes W$ for W a purely even k-vector space, and (3.3.3) results from the commutant being $D^0 \otimes \mathrm{End}_k(W)$.

It also results from (3.2.1) that the tensor product (in the super sense) of central simple algebras is central simple.

3.4. Let A be a central simple algebra. If S is a simple super module, and f a homogeneous endomorphism, the kernel and images of f are submodules. If $f \neq 0$, f is hence invertible: the commutant $B = \underline{\mathrm{End}}_A(S)$ is a super division algebra (super Schur's lemma). By (3.3.3), A is isomorphic to $M_{r|s} \otimes B^0$ for some r and s. If S' is another simple module, $\underline{\mathrm{Hom}}_A(S, S')$ is compatible with extension of scalars, hence, as one sees over $\bar{k}$, nonzero. Picking a homogeneous element in $\mathrm{Hom}_A(S, S')$, one sees that S and S' are isomorphic, possibly up to parity change. It follows that their commutants are isomorphic.

Two central simple algebras are said to be *similar* if the commutants of their simple modules are isomorphic, i.e. if they are both of the form $M_{r|s} \otimes B$ for the same central simple division algebra B.

The *super Brauer group* $\mathrm{s\,Br}(k)$ is the set of similarity classes of central simple algebras over k, with product the tensor product. That it is a (commutative) group follows from (3.3.2).

Let B be a central simple division algebra, and $A = M_{r|s} \otimes B$. If A is not purely even, then

(a) if B is purely even, $A^+ \simeq M_r(B) \times M_s(B)$,

(b) if B is not purely even, $A \simeq M_n \otimes B$ with $n = r + s$ and

$$A^+ \simeq M_n \otimes B^+ .$$

In both cases, the Morita equivalence class of A^+ depends only on the class of A in $\mathrm{s\,Br}(k)$.

If V is a quadratic vector space over k, $C(V)$ is central simple. Indeed, after extension of scalars to $\bar{k}$, $C(V)$ becomes isomorphic to the tensor product of $\dim(V)$ copies of D. If V is hyperbolic, $C(V)$ is a matrix algebra. This can be checked as in the proof of 2.2 (i), or writing $(V,Q) = (V_1, Q_1) \oplus (V_1, -Q_1)$, it can be deduced from (3.3.2) and 2.1 (D)(E). It then follows from 2.1 (D) that $V \mapsto C(V)$ induces a homomorphism

$$W(k) \to \mathrm{s\,Br}(k) \tag{3.4.1}$$

from the Witt group of k to the super Brauer group.

Remark 3.5. Using arguments parallel to those of Bourbaki Alg. Ch. 8, one can show that for A a nonzero finite dimensional super algebra over k, the following conditions are equivalent.

(i) A is central simple;

(ii) $A \otimes A^0 \xrightarrow{\sim} \mathrm{End}_k(A)$;

(iii) The super center of A is reduced to k, and the super A-module A is semi-simple (equivalently: as an ungraded algebra, A is semi-simple).

(iv) A is isomorphic to $M_{r|s} \otimes L$ for some r and s and some super division algebra L over k with super center reduced to k.

Proposition 3.6 (C. T. C. Wall (1963) p. 195). *The super Brauer group of $\mathbb{R}$ is cyclic of order 8, generated by the class of $\mathbb{R}[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$. For V a real quadratic vector space, $C(V)$ is similar to $\mathbb{R}[\varepsilon]^{\otimes \mathrm{sgn}(V)}$. The similarity class of $C(V)$ and the Morita equivalence class of $C^+(V)$ (for $V \neq 0$) hence depend only on the signature of V modulo 8.*

That $\mathrm{s\,Br}(\mathbb{R})$ is of order 8 results from the following lemma, valid for any field k of characteristic $\neq 2$.

Lemma 3.7 (C. T. C. Wall (1963) Th. 3). *The group $\mathrm{s\,Br}(k)$ is an iterated extension of $\mathbb{Z}/2$ by k^*/k^{*2} by the ordinary Brauer group of k.*

Proof. The morphism $\mathrm{s\,Br}(k) \to \mathbb{Z}/2$ is given by extension of scalars to $\bar{k}$, for $\bar{k}$ an algebraic closure of k. By 3.2 (example), one has indeed $\mathrm{s\,Br}(\bar{k}) = \mathbb{Z}/2$.

The kernel $\mathrm{s\,Br}(k)'$ corresponds to central simple super algebras A which by extension of scalars to $\bar{k}$ become isomorphic to a matrix algebra $M_{r|s}$. The algebra $A_{\bar{k}}$, isomorphic to $M_{r|s}$, has two isomorphism classes of simple modules, exchanged by parity change. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on the set $I(A_{\bar{k}})$ of those isomorphism classes, defining

$$\alpha_A \colon \mathrm{Gal}(\bar{k}/k) \to \mathbb{Z}/2\ .$$

Using that the simple $(A' \otimes A'')_{\bar{k}}$-modules are the tensor product of simple $A'_{\bar{k}}$ and $A''_{\bar{k}}$ modules, one checks that $\alpha_{A \otimes A'} = \alpha_A + \alpha_{A'}$ and $A \mapsto \alpha_A$ induces a homomorphism

$$\mathrm{s\,Br}(k)' \to \mathrm{Hom}(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2) = k^*/k^{*2}\ , \tag{3.7.1}$$

where to u in k^* corresponds the homomorphism giving the action of Galois on $\pm\sqrt{u}$.

If A is purely even, $A_{\bar{k}}$ is isomorphic to $M_n(\bar{k})$, one element of $I(A)$ is purely even, the other purely odd, and α_A is trivial. If A is not purely even, $A_{\bar{k}}$ is isomorphic to $M_{r|s}(\bar{k})$ with $r, s > 0$, the center Z^+ of A^+ is of dimension 2, becoming isomorphic to $\bar{k}\times\bar{k}$ over $\bar{k}$, and $I(A)$ is in bijection with the two characters $Z^+ \to \bar{k}$ of Z^+, a simple module S corresponding to the character by which Z^+ acts on S^+. For A the Clifford algebra $C(k^2, x^2 - ay^2)$, one has $Z^+ = A^+ = \langle 1, e_1e_2\rangle$ and $(e_1e_2)^2 = -e_1^2e_2^2 = a$. This shows that (3.7.1) is onto.

The kernel $\mathrm{s\,Br}(k)''$ of (3.7.1) corresponds to the central simple algebras, for which $A_{\bar{k}}$ is isomorphic to $M_{r|s}$ and for which either A is purely even or A^+ has a center isomorphic to $k \times k$. If A is a super division algebra, A must be an ordinary, purely even, division algebra, and $\mathrm{s\,Br}(k)'' = \mathrm{Br}(k)$.

3.8 Proof of 3.6. For A a central simple algebra over $\mathbb{R}$, let t be the trace for the underlying ungraded algebra, let sgn^+ (resp. sgn^-) be the signatures of the quadratic forms $t(xy)$ on A^+ (resp. A^-), and define

$$s(A) := \mathrm{sgn}^+ + \mathrm{sgn}^- i \in \mathbb{Z}[i] \subset \mathbb{C}\ .$$

For a tensor product $A_1 \otimes A_2$, t vanishes on the $A_1^\pm \otimes A_2^\pm$ other than $A_1^+ \otimes A_2^+$, where it is the tensor product of t for A_1 and A_2. On $A_1^\pm \otimes A_2^\pm$, $t(xy)$ is related to the tensor product of the bilinear forms $t_1(x_1y_1)$ and $t_2(x_2y_2)$ by

$$\begin{aligned} &A_1^+ \otimes A_2^+, A_1^+ \otimes A_2^- \text{ or } A_1^- \otimes A_2^+ : && t(xy) = t_1(x_1y_1) \otimes t_2(x_2y_2) \\ &A_1^- \otimes A_2^- : && t(xy) = -t_1(x_1y_1) \otimes t_2(x_2y_2), \end{aligned}$$

the latter because $(x_1 \otimes y_1)\cdot(x_2 \otimes y_2) = -(x_1x_2)\otimes(y_1y_2)$.

The signature being multiplicative, and changing sign when the bilinear form is replaced by its opposite, we get

$$s(A_1 \otimes A_2) = s(A_1)s(A_2).$$

For a matrix algebra $M_{r,s}$, off-diagonals contribute an isotropic form, and

$$s(M_{r,s}) = r + s$$

For $\mathbb{R}[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$, one has $s = 1 + i$, of order 8 in $\mathbb{C}^*/(\mathbb{R}^+)^*$. As $\mathrm{s\,Br}(\mathbb{R})$ is of order 8, it follows that $\mathrm{s\,Br}(\mathbb{R})$ is cyclic of order 8 with $\mathbb{R}[\varepsilon]$ as generator. The remaining statements in 3.6 result from 3.4.

3.9. Let us identify $\mathrm{s\,Br}(\mathbb{R})$ with the set of isomorphism classes of super division algebras over $\mathbb{R}$. Write ε (resp. ε^0) for an odd quantity with $\varepsilon^2 = 1$ (resp. $(\varepsilon^0)^2 = -1$), and write $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ for $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ in purely even degree. The bijection 3.6 between $\mathrm{s\,Br}(\mathbb{R})$ and $\mathbb{Z}/8$ is as follows.

$$\begin{aligned} &0 \longmapsto \mathbb{R} && 4 \longmapsto \mathbb{H} \\ &1 \longmapsto \mathbb{R}[\varepsilon] && 5 \longmapsto \mathbb{H}\otimes\mathbb{R}[\varepsilon] \\ &2 \longmapsto \mathbb{C}[\varepsilon], \varepsilon z = \bar{z}\varepsilon, && 6 \longmapsto \mathbb{C}[\varepsilon^0], \varepsilon z = \bar{z}\varepsilon^0, \\ &3 \longmapsto \mathbb{H}\otimes\mathbb{R}[\varepsilon^0] && 7 \longmapsto \mathbb{R}[\varepsilon^0] \end{aligned}$$

The image of 2 is the class of $\mathbb{R}[\varepsilon_1, \varepsilon_2]$ with $\varepsilon_i^2 = 1$, $\varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1$. One has $(\varepsilon_1\varepsilon_2)^2 = -1$ and the claim for 2 follows. Passing to the opposite algebra, we get the image of 6. As the class of $\mathbb{H}$ is of order 2, it must be the image of 4. One concludes by using that $5 = 4 + 1$ and that n and $-n$ correspond to opposite algebras.

By 3.4, to justify 1.4, it suffices to read the even part of the super algebras in this table.

CHAPTER 4
Pairings and Dimension Modulo 8, over $\mathbb{C}$

Let V be a complex quadratic vector space, and S a spinorial representation of $\mathrm{Spin}(V)$ or, what amounts to the same, a $C^+(V)$-module (2.6). Let β be the principal antiautomorphism of $C(V)$ (2.1 (B)).

Proposition 4.1. *A bilinear form* $(\ ,\ )$ *on* S *is invariant by* $\mathrm{Spin}(V)$ *if and only if for* a *in* $C^+(V)$,

$$(as,t) = (s,\beta(a)t)\ . \tag{4.1.1}$$

If (4.1.1) holds, for $g \in G^+ \subset C^+(V)$ (2.7), one has

$$(gs,gt) = (s,\beta(g)gt) = \beta(g)g(s,t)\ . \tag{4.1.2}$$

In particular if g is in the spin group, i.e. if $\beta(g)g = 1$, g leaves $(\ ,\ )$ invariant.

Conversely, if G leaves $(\ ,\ )$ invariant, its Lie algebra $\mathfrak{so}(V) \subset C^+(V)$ leaves it invariant too: for x in $\mathfrak{so}(V)$,

$$(xs,t) + (s,xt) = 0\ .$$

As $\beta(x) = -x$ (a consequence of (2.4.1) (2.4.2)), this gives (4.1.1) for $a = x$ and one concludes by using that $\mathfrak{so}(V)$ generates $C^+(V)$ (2.6).

4.2 Variants. (i) If S is a $C(V)$-module, and if a bilinear form $(\ ,\ )$ on S is such that $(as,t) = (s,\beta(a)t)$, for a in $C(V)$, the same proof shows that for g in the subgroup G of $C(V)^*$ (2.6), (4.1.2) holds. In particular, $(\ ,\ )$ is invariant by the kernel of $\beta(g)g$: $G \to \mathbb{C}^*$. The converse holds. The condition $(as,t) = (s,\beta(a)t)$ for a in $C(V)$ is equivalent to

$$(vs,t) = (s,vt) \qquad \text{for } v \text{ in } V\ . \tag{4.2.1}$$

(ii) Let V be a complex quadratic vector space. We have $Q(v) = \langle v,v\rangle$ and we use the symmetric bilinear form $\langle\ ,\ \rangle$ to identify V with its dual. Let S and $S^\vee$ be vector spaces in duality, provided with Γ: $S \otimes S \to V$ and Γ': $S^\vee \otimes S^\vee \to V$, both symmetric. The self-duality of V allows us to identify Γ and Γ' with γ: $V \otimes S \to S^\vee$ and γ': $V \otimes S^\vee \to S$. Assume that γ and γ' turn $S \oplus S^\vee$ into a $C(V)$-module. The vector spaces S and $S^\vee$ become $C^+(V)$-modules, hence representations of $\mathrm{Spin}(V)$.

Claim: the duality pairing $S \otimes S^\vee \to \mathbb{C}$, Γ and Γ' are morphisms of representations.

By the remark in 2.6, γ and γ' are morphisms of representations of $\mathrm{Spin}(V)$, so that it suffices to check that the duality pairing is a morphism too. Define, on $S \oplus S^\vee$, $\varepsilon(s+s', t+t') = s'(t) + t'(s)$. The symmetry of Γ and Γ' translates as

$$\varepsilon(vs, t) = \langle \Gamma(s,t), v\rangle = \varepsilon(s, vt)$$
$$\varepsilon(vs', t') = \langle \Gamma'(s',t')v\rangle = \varepsilon(s', vt)\ ,$$

and (i) gives the G-invariance of ε, a fortiori the $\mathrm{Spin}(V)$-invariance of the duality pairing of S with S'.

4.3. It follows that the pairings between irreducible spinorial representations of $\mathrm{Spin}(V)$ are encoded in $(C^+(V), \beta)$. As in §3, it is convenient to consider the full Clifford algebra.

This leads to considering central simple super algebras A, endowed with an even involution β: $A \to A$ which is an antiautomorphism of the underlying ungraded algebra:

(4.3.1) $$\beta(xy) = \beta(y)\beta(x).$$

The tensor product is defined as follows: take the tensor product of super algebras, and define β as extending the involutions of the factors:

(4.3.2) $$\beta(a \otimes b) = (-1)^{p(a)p(b)}\beta(a) \otimes \beta(b).$$

Tensor product is associative and commutative. For Clifford algebras, endowed with their principal antiautomorphisms, it corresponds to the orthogonal direct sum of quadratic vector spaces.

4.4 Example. Let S be a super vector space, and $A := \underline{\mathrm{End}}(S)$. Antiautomorphisms β of the underlying ungraded algebra correspond to nondegenerate bilinear forms on the ordinary vector space underlying S, taken up to a factor, by

$$(ax, y) = (x, \beta(a)y).$$

Involutive β correspond to symmetric or antisymmetric forms. Even β correspond to even or odd forms.

Suppose given two super vector spaces S_i $(i = 1, 2)$ provided with even or odd, symmetric or antisymmetric bilinear forms. Let (A, β) be the tensor product of the corresponding (A_i, β). One has

$$A = A_1 \otimes A_2 = \underline{\mathrm{End}}(S_1) \otimes \underline{\mathrm{End}}(S_2) \xrightarrow{\sim} \underline{\mathrm{End}}(S_1 \otimes S_2).$$

We leave it to the reader to check that the involution β of A corresponds to the following bilinear form on $S_1 \otimes S_2$:

(4.4.1) $$(s_1 \otimes s_2, t_1 \otimes t_2) + (-1)^{(p(s_1)+p(t_1))p(s_2)}(s_1, s_2)(t_1, t_2).$$

If the form $(\ ,\ )_i$ on S_i is of parity p_i and of sign s_i (+ for symmetric, − for antisymmetric), the parity and sign of the form (4.4.1) on $S_1 \otimes S_2$ are given by

(4.4.2) $$p = p_1 + p_2, \quad s = s_1 s_2 (-1)^{p_1 p_2}\ .$$

4.5 Remark. (4.3.1) flouts the sign rule. This could be repaired by considering rather γ defined by

$$\begin{aligned}\gamma(a) &:= \beta(a) \text{ for } a \text{ even},\\ &:= i\beta(a) \text{for } a \text{ odd} .\end{aligned}$$

The condition (4.3.1) translates into $\gamma(ab) = (-1)^{p(a)p(b)}\gamma(b)\gamma(a)$, involutivity of β translates as the square of γ being the parity automorphism "multiplication by $(-1)^{p(a)}$", and (4.3.2) becomes $\gamma = \gamma' \otimes \gamma''$.

4.6. Let us call (A, β) neutral if A is of the form $\underline{\mathrm{End}}\,(S)$ and if β is given as in 4.4 by an even symmetric form (,) on S: the sum of symmetric forms on S^+ and S^-.

Define (A, β) and (B, β) to be *similar* if for suitable neutrals (M, β) and (N, β), one has

$$(A, \beta) \otimes (M, \beta) \simeq (B, \beta) \otimes (N, \beta).$$

Similarity is an equivalence relation, and is stable under tensor product. Let $B(\mathbb{C})$ be the set of similarity classes. Tensor product induces on $B(\mathbb{C})$ a composition law associative, commutative and with unit. The following proposition shows that it is a group.

Proposition 4.7. *Let (A, β) be as in 4.3, and let Tr be the trace of the underlying ungraded algebra. Let α be the parity automorphism $a \mapsto (-1)^{p(a)}a$. Then $(A, \beta) \otimes (A^0, \alpha\beta)$ is neutral. More precisely, by the isomorphism*

$$A \otimes A^0 \xrightarrow{\sim} \underline{\mathrm{End}}(A)$$

and its involution corresponds to the even symmetric form $\mathrm{Tr}(x\beta(y))$ on A.

Proof. We have

$$\mathrm{Tr}((a \otimes b \cdot x)\beta(y)) = (-1)^{p(b)p(x)}\mathrm{Tr}(axb\beta(y)) = (-1)^{p(b)p(x)}\mathrm{Tr}(xb\beta(y)a) .$$

As $b\beta(y)a = \beta(\beta(a)y\beta(b)) = (-1)^{p(b)p(y)}\beta(\beta(a) \otimes \beta(b) \cdot y)$, this equals

$$= (-1)^{p(b)(p(x)+p(y))}\mathrm{Tr}(x \cdot \beta(\beta(a) \otimes \beta(b) \cdot y) .$$

Both sides are zero if $p(a) + p(b) + p(x) + p(y)$ is odd. We hence may replace $p(b)(p(x) + p(y))$ by $p(b)(p(a) + p(b))$, giving

$$= \mathrm{Tr}(x.\beta((-1)^{p(a)p(b)}\beta(a) \otimes \alpha\beta(b) \cdot y),$$

and $a \otimes b \mapsto (-1)^{p(a)p(b)}\beta(a) \otimes \alpha\beta(b)$ is the tensor product of β and $\alpha\beta$.

Proposition 4.8. *The group $B(\mathbb{C})$ is cyclic of order 8, generated by the class of the Clifford algebra $C(V)$ for $\dim(V) = 1$.*

Proof. We first consider the kernel $B'(\mathbb{C})$ of the morphism "forgetting β" from $B(\mathbb{C})$ to $\mathrm{s\,Br}(\mathbb{C}) = \mathbb{Z}/2$, i.e. we consider the (A, β) for which A is a matrix algebra $M_{r|s}$. By 4.4, there are four similarity classes of them, characterized by a parity p and a sign s: the parity, and the symmetry, of a bilinear form on a simple A-module defining β. The product is given by (4.4.2), showing that $B'(\mathbb{C})$ is cyclic of order 4, with generators the elements for which the parity p is odd.

The Clifford algebra $C(V)$ with $\dim(V) = 1$ maps to a generator of $\mathrm{s\,Br}(\mathbb{C})$. It remains to check that its square, the Clifford algebra $C(W)$ with $\dim(W) = 2$, is a generator of $B'(\mathbb{C})$. The algebra $C(W)$ is a matrix algebra $M_{1|1} = \mathrm{End}(\mathbb{C}^{1,1})$ with W as its odd part. The transposition relative to the odd symmetric form xy on $\mathbb{C}^{1,1}$ fixes W, hence is the principal antiautomorphism, giving for $C(W)$ the sign $+$ and the parity 1: the parity is odd as required. Applying (4.4.2), we also see that for $\dim(V)$ even, the parity p and the sign s of $C(V)$ are given in terms of $\dim(V)$ mod 8 by the table

	dimension mod 8	s	p
	0	$+$	0
	2	$+$	1
	4	$-$	0
(4.8.1)	6	$-$	1

4.9. Suppose V odd-dimensional, and let S be the even part of a simple super $C(V)$-module. Let Z be the center of the ungraded algebra $C(V)$, and z be a generator of Z^-. Then, $v, s \mapsto zvs$ is a morphism of representations of the Spin group: $V \otimes S \to S$.

If V is even-dimensional and S a simple super $C(V)$-module, the Clifford multiplication induces morphisms of representations of the Spin group: $V \otimes S^+ \to S^-$ and $V \otimes S^- \to S^+$.

We now check the uniqueness claim in 1.5 that, up to a scalar factor, these morphisms are the only morphisms $V \otimes S_1 \to S_2$, for S_1 and S_2 irreducible spinorial representations. If V is of dimension 1 or 2, this is clear by inspection, using the description 1.1. If $\dim V \neq 2$, the representation V is irreducible. As $\mathrm{Hom}(V \otimes S_1, S_2) = \mathrm{Hom}(V, \underline{\mathrm{Hom}}\,(S_1, S_2))$, it suffices to check the

Lemma 4.9.1. (i) *For $d := \dim(V)$ odd, the irreducible representation V occurs exactly once in $\underline{\mathrm{End}}\,(S)$.*

(ii) *For d even, V occurs exactly twice in $\underline{\mathrm{End}}\,(S)$.*

Proof. By the odd analogue of the Poincaré-Birkhoff-Witt Theorem, the Clifford algebra $C(V)$ is, as a representation of $\mathrm{O}(V)$, isomorphic to $\wedge^* V$, with $C^+(V)$ isomorphic to $\wedge^{2*} V$.

If d is odd, $\underline{\mathrm{End}}\,(S) \simeq C^+(V) \simeq \wedge^{2*} V$. As $\wedge^i V \simeq \wedge^{d-i} V$ as a representation of $\mathrm{SO}(V)$, this can be rewritten

$$S \otimes S = \bigoplus \wedge^i V \quad (\text{sum for } 0 \leq i \leq (d-1)/2)\ .$$

The $\wedge^i V$ for $i \leq (d-1)/2$ are irreducible, nonisomorphic representations: they are the trivial representation, and all fundamental representations except the spinorial one. The representation V occurs only once, for $i = 1$.

If d is even, $\underline{\mathrm{End}}(S) \simeq C(V) \simeq \wedge^* V$, which decomposes as follows into irreducible representations: twice $\bigoplus_{i<d/2} \wedge^i V$ and the two constituents of $\wedge^{d/2} V$ (the eigenspace of $*$). The representation V occurs twice.

We now check the table in 1.5 for $d := \dim(V)$ even. Let $(\ ,\ )$ be the nondegenerate bilinear form on S for which

$$(vs, t) = (s, vt) \tag{4.9.2}$$

for v in V. It is unique up to a scalar factor. Its parity and sign (4.4) are given by (4.8.1). If of odd parity, it makes of S^+ the dual of S^-. If even, it gives a nondegenerate form on S^+ and S^-, of sign given by (4.8.1). This confirms the column "forms on spinors" in 1.5.

The morphism of representations (2.6, Remark)

$$V \otimes S \to S \tag{4.9.3}$$

induced by the Clifford module structure of S can, as V is self dual, be reinterpreted as a morphism from $S \otimes S^\vee$ to V, or, using $(\ ,\ )$ to identify S and $S^\vee$, as a morphism of representations

$$\Gamma\colon S \otimes S \to V\ . \tag{4.9.4}$$

Its defining characteristic property is that

$$\langle \Gamma(s,t), v\rangle = (vs, t)\ . \tag{4.9.5}$$

As (4.9.3) is odd, the parity of (4.9.4) is opposite to that of $(\ ,\ )$. By (4.9.2), its sign (symmetric, or antisymmetric) is the same as that of $(\ ,\ )$ and (4.8.1) confirms the column "symmetry of spinors, spinors $\to V$" of 1.5.

Corollary 4.9.6. *Let L^+ and L^- be spinorial representations of* $\mathrm{Spin}(V)$, *and* $\bullet\colon V \otimes L^\pm \to L^\mp$ *be morphisms of representations which turn $L := L^+ \oplus L^-$ into a $C(V)$-module:*

$$v \bullet v \bullet \ell = Q(v)\ell\ .$$

Then the resulting $C^+(V)$-module structure of $L^\pm$ induces its structure of representation of $\mathrm{Spin}(V)$.

Proof. We first assume that L^+ and L^- are isotypic. Let S be a simple super $C(V)$-module, with its resulting structure of representation of $\mathrm{Spin}(V)$. Replacing S by ΠS if needed, we may assume that $L^\pm$ is a multiple of $S^\pm$: $L^\pm = S^\pm \otimes W^\pm$. As the morphism $V \otimes S^\pm \to S^\mp$ induced by the module structure is the unique morphism of representations from $V \otimes S^\pm \to S^\mp$, the morphisms $V \otimes L^\pm \to L^\mp$ are induced by $\alpha^\pm\colon W^\pm \to W^\mp$. That they turn L into a $C(V)$-module implies that α^+ and α^- are inverses of each other: L, as a representation of $\mathrm{Spin}(V)$ and as a $C(V)$-module, is a multiple of S.

If L^+ or L^- is not isotypic, the dimension of V is even, they are two irreducible spinorial representations S_1 and S_2 and if $L_i^\pm$ is the S_i-isotypic part of $L^\pm$, as there is no morphism $V \otimes S_i \to S_i$ the data $V \otimes L^\pm \to L^\mp$ breaks into the direct sum of $(V \otimes L_1^+ \to L_2^-, V \otimes L_2^- \to L_1^+)$ and of $(V \otimes L_2^+ \to L_1^-, V \otimes L_1^- \to L_2^+)$. This reduces us to the already considered isotypic case.

4.10 Lemma. *Fix (A,β) as in 4.3, with A nontrivial in* $\mathrm{s\,Br}(\mathbb{C})$, *i.e. of the form $A^+\otimes Z$, where A^+ is a purely even matrix algebra and where the center (not super center) Z of A is of dimension $1|1$. Whether β acts by $+1$ or by -1 on Z^- depends only on the similarity class of (A,β).*

Proof. Take (B,β) with $B=\underline{\mathrm{End}}\,(S)$ and β defined, as in 4.4, by a nondegenerate bilinear form on S of parity p and sign s. The involution β of B^+ acts trivially on the center $Z(B^+)$ if $p=0$, nontrivially if $p=1$. The center $Z(B^+)$ is $\mathbb{C}$ if B is purely even, $\mathbb{C}\times\mathbb{C}$ otherwise. Let ν be 1 in the first case, and a nonzero element $(a,-a)$ in the second. One has $\beta(\nu)=(-1)^p\nu$. For z in Z^-, $\nu\otimes z$ commutes with $(B\otimes A)^+$. Indeed, both ν and z commute with $B^+\otimes A^+$, and anticommute with $B^-\otimes A^-$. The negative part $Z^-(B\otimes A)$ of the center of $B\otimes A$ is hence νZ^-, and

$$\beta\Big|Z^-(B\otimes A)=(-1)^p\beta\Big|Z^-(A)\,. \tag{4.10.1}$$

Indeed, $\beta(\nu z)=\beta(z)\beta(\nu)=\beta(\nu)\beta(z)=(-1)^p\nu\beta(z)$. If (B,β) is neutral, $p=0$ and 4.10 follows.

4.11 Corollary. *The similarity class of (A^+,β) depends only on that of (A,β).*

Proof. It is the difference of (A,β) and (Z,β) in $B(\mathbb{C})$.

From 4.10 and 4.11, we get two similarity invariants of (A,β) as in 4.10, i.e. of (A,β) whose class in $B(\mathbb{C})=\mathbb{Z}/8$ is odd: the sign $s(\beta)$ by which β acts on Z^-, and the sign $s(A^+)$ of the form corresponding to (A^+,β) by 4.4. The class 1 in $B(\mathbb{C})=\mathbb{Z}/8$ is represented by $(\mathbb{C}[\varepsilon],\beta)$ with ε odd, $\varepsilon^2=1$ and β the identity. By 4.7, the class -1 is represented by $\mathbb{C}[\varepsilon]$ with $\beta\varepsilon=-\varepsilon$. The classes 0 and 4 can be represented by purely even algebras, and writing any odd element of $\mathbb{Z}/8$ as (0 or 4) $\pm$ 1, one deduces from (4.8.1) the following table:

(4.11.1)

n	$s(A^+)$	$s(\beta)$
1	$+$	$+$
3	$-$	$-$
5	$-$	$+$
7	$+$	$-$

4.12. We now check the table in 1.5 for V odd-dimensional. The Clifford algebra is as in 4.10: $C(V)=C^+(V)\otimes Z$. Let S be a simple super $C(V)$-module. The $C^+(V)$-module S^+ is then the unique irreducible spinorial representation of $\mathrm{Spin}(V)$. On S^+, there is a nondegenerate bilinear form $(\ ,\ )$ for which

$$(as,t)=(s,\beta(a)t) \tag{4.12.1}$$

for a in $C^+(V)$. It is invariant under $\mathrm{Spin}(V)$. Its sign is the sign $s(C^+(V))$ attached to $(C(V),\beta)$ in 4.4, and (4.11.1) confirms the column "forms on spinors" in 1.5.

Multiplication by zv gives a morphism of representations

$$V\otimes S^+\to S^+ \tag{4.12.2}$$

which, as in 4.9, defines

$$S^+ \otimes S^+ \to V \tag{4.12.3}$$

characterized by

$$\langle f(s \otimes t), v \rangle = (zvs, t)\ . \tag{4.12.4}$$

One has $\beta(zv) = \beta(v)\beta(z) = v\beta(z) = \beta(z)v$ and, by (4.12.1), the sign of (4.12.3) is that of $(\ ,\ )$ if $s(\beta) = +$, and is the opposite if $s(\beta) = -$. The table (4.11.1) confirms the column "symmetry of spinors, spinors $\to V$" of 1.5.

CHAPTER 5
Passage to Quadratic Subspaces

In this Chapter we work over $\mathbb{C}$. As promised in §1.1, we first check the

Proposition 5.1. *For $r > 0$, if one restricts the spin representation of* $\mathrm{Spin}(2r+1)$ *to* $\mathrm{Spin}(2r)$*, one obtains the sum of the two semi-spin representations of* $\mathrm{Spin}(2r)$*. Each of the two semi-spin representations of* $\mathrm{Spin}(2r)$ *restricts as the spin representation of* $\mathrm{Spin}(2r-1)$.

Proof. Let V be the orthogonal direct sum of W and of a line L, and put $D := C(L)$. If $\dim(W)$ is odd, one has a tensor decomposition $C(W) = C^+(W) \otimes Z$. The commutant Z of $C^+(W)$ is of dimension $1|1$, and $C(V) = C^+(W) \otimes (Z \otimes D) \simeq C^+(W) \otimes M_{1|1}$. If S is a simple $C^+(W)$-module, $S \otimes \mathbb{C}^{1,1}$ is a simple $C(V)$-module. As a $C^+(W)$-module, this is $S \oplus \Pi S$, and the second statement follows.

Suppose now that $\dim(W)$ is even. For some r, $C(W) \approx M_{r|r}$. If S is a simple $C(W)$-super module, $S \otimes D$ is a simple $C(W) \otimes D$-module. Its even part (the spin representation of $\mathrm{Spin}(V)$) is, as a $C^+(W)$-module, isomorphic to $S^+ \oplus S^-$ and the first statement follows.

5.2. Let $V \neq 0$ be an even-dimensional complex quadratic space, and let S be a simple super $C(V)$-module. Let $(\ ,\ )$ be a nondegenerate bilinear form on S for which

$$(vs, t) = (s, vt)\ . \tag{5.2.1}$$

As we saw in 4.4, this form is unique up to a factor.

Let $v \neq 0$ be an isotropic vector in V. Define $V_1 = v^\perp/\mathbb{C}v$. The square zero endomorphism v of S super commutes with $v^\perp$. Its kernel $\ker(v)$ is hence stable under $v^\perp$. The action of $v^\perp$ on $\mathrm{Ker}(v)$ factors through V_1, and turns $\mathrm{Ker}(v)$ into a $C(V_1)$-module.

Proposition 5.3. *With the notations of 5.2*

(i) $\mathrm{Ker}(v) = \mathrm{Im}(v)$;

(ii) $\mathrm{Ker}(v)$ *is an irreducible representation of* $C(V_1)$;

(iii) *The form* $(\ ,\ )$ *vanishes on* $\mathrm{Ker}(v)$;

(iv) *Define the form* $(\ ,\ )_1$ *on* $\mathrm{Ker}(v) = \mathrm{Im}(v)$ *by* $(s, vt)_1 = (-1)^{p(s)}(s, t)$*. It is nondegenerate. The* $C(V_1)$*-module* $\mathrm{Ker}(v)$*, and* $(\ ,\ )_1$*, obey (5.2.1).*

Proof (i), (ii). Lift V_1 in $v^\perp$ and let H be the orthogonal complement of V_1 in V. The quadratic space V becomes the orthogonal direct sum of V_1 and H, with $\dim(H) = 2$ and $v \in H$. Let S_1 be an irreducible super representation of $C(V_1)$, and let S_H be one of $C(H)$. The super vector space S_H is of dimension $1|1$ and, possibly after parity change and in a suitable basis, the matrix of multiplication by v is

$$v\colon \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \tag{5.3.1}$$

One has $C(V) = C(V_1) \otimes C(H)$ and, possibly after a parity change for S_1, the $C(V)$-module S is $S_1 \otimes S_H$. This reduces (i) to the case of H, clear on (5.3.1), which also shows that the $C(V_1)$-module $\mathrm{Ker}(v)$ is isomorphic to S_1, proving (ii).

(iii) As $(vs, vt) = (s, vvt)$ and $v^2 = Q(v) = 0$, (iii) results from (i).

(iv) As $(\ ,\ )$ is nondegenerate, for $s \neq 0$ in $\mathrm{Ker}(v)$ there exists t such that $(s,t) \neq 0$. By (iii), $t \notin \mathrm{Ker}(v)$ and $(s, vt)_1 \neq 0$: the form $(\ ,\ )_1$ is nondegenerate and it remains to check that it obeys (5.2.1). Indeed, if $t_1 = vt$ and if w is in $v^\perp$, one has

$$\begin{aligned}(ws, t_1) &= (-1)^{p(ws)}(ws, t) = (-1)^{p(ws)}(s, wt) \\ &= (-1)^{p(ws)+p(s)}(s, vwt)_1 = -(s, -wvt) = (s, wt)_1 .\end{aligned}$$

Proposition 5.4. *Under the same assumptions, let $\Gamma\colon S \otimes S \to V$ be defined by (4.9.5). For s, t in* $\mathrm{Ker}(v)$ *one has in V*

$$\Gamma(s,t) = (-1)^{p(s)}(s,t)_1 v . \tag{5.4.1}$$

Proof. We check that both sides have the same inner product with any $w \in V$. If $t = vx$, the left side indeed gives

$$\begin{aligned}(w, \Gamma(s,t)) &= (ws, vx) = (vws, x) = ((vw + wv)s, x) \\ &= (v,w)(s,x) = (-1)^{p(s)}(v,w)(s,t)_1 .\end{aligned}$$

CHAPTER 6
The Minkowski Case

6.1 Theorem. *Let (V, Q) over $\mathbb{R}$ be of signature $(+,-,\ldots,-)$. Let $S_{\mathbb{R}}$ be an irreducible real spinorial representation of* $\mathrm{Spin}(V,Q)$. *The commutant Z of $S_{\mathbb{R}}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $^-$ be the standard anti-involution of Z.*

(i) *Up to a real factor, there exists a unique symmetric morphism $\Gamma\colon S_{\mathbb{R}}\otimes S_{\mathbb{R}} \to V$. It is invariant under the group Z^1 of elements of norm 1 of Z.*

(ii) *For $v \in V$, if $Q(v) > 0$, the form $\langle v, \Gamma(s,t)\rangle$ on $S_{\mathbb{R}}$ is positive or negative definite.*

The set of v for which $Q(v) > 0$ has two connected components. It follows from (ii) that for v in one of them, call it C, the form $\langle v, \Gamma(s,t)\rangle$ is positive definite and that for v in the other, i.e. for $-v$ in C, it is negative definite.

Let d be the dimension of V. The signature is $2-d$ and, by the table 1.4.1, proved in 3.1, 3.9, Z and the complexification of $S_{\mathbb{R}}$ are given as follows as a function of d modulo 8. For d even, S^+ and S^- denote the semi-spinorial representations. For d odd, S denotes the spin representation.

$$
\begin{array}{ll}
2 & : \ \mathbb{R}\,, S^+ \text{ or } S^- \\
3 \text{ or } 1 & : \ \mathbb{R}\,, S \\
4 \text{ or } 0 & : \ \mathbb{C}\,, S^+ + S^- \\
5 \text{ or } 7 & : \ \mathbb{H}\,, 2S \\
6 & : \ \mathbb{H}\,, 2S^+ \text{ or } 2S^-
\end{array}
\tag{6.1.1}
$$

Proof. We first show that if d is congruent to 2 modulo 8, a morphism Γ with the listed properties exists. In this case, the signature is 0 modulo 8, so that $C(V)$ is a matrix algebra. Let $S = S^+ \oplus S^-$ be a simple $C(V)$-module. The possible $S_{\mathbb{R}}$ are S^+ and S^-. The pairing (,) on S, relative to which the principal antiautomorphism β is transposition, is odd and the corresponding $\Gamma\colon S\otimes S \to V$ is even and symmetric ((1.5.1) proved in 4.8, 4.9).

Fix v with $Q(v) > 0$ and let V_1 be its orthogonal complement. By 5.1, the graded module S remains simple as a graded $C(V_1)$-module, and S^+ remains absolutely irreducible as a representation of the group $\mathrm{Spin}(V_1)$. The symmetric bilinear form $\langle v, \Gamma(s,t)\rangle$ on S^+ is not identically zero: if it were zero for one v, it would

be zero for all v by $\mathrm{Spin}(V)$-invariance, and Γ would be zero. This form is invariant under the compact group $\mathrm{Spin}(V_1)$. As S^+ is an irreducible representation of $\mathrm{Spin}(V_1)$, it must be definite. The same applies to S^-.

We now show, in all dimensions, the existence of a morphism $S_{\mathbb{R}} \otimes S_{\mathbb{R}} \to V$ with the properties listed in 6.1 (i) (ii). Embed V in a Minkowski space V_0 of dimension congruent to 2 modulo 8: V_0 is the orthogonal direct sum of V and of a negative definite quadratic space of a suitable dimension. Let pr be the orthogonal projection of V_0 onto V.

Let $S_0 = S_0^+ \oplus S_0^-$ be a simple $C(V_0)$-module. As V_0 is of dimension congruent to 2 modulo 8, there exists a morphism of representations of $\mathrm{Spin}(V_0)$: Γ_0: $S_0^\pm \otimes S_0^\pm \to V_0$, with the properties listed in 6.1 (i)(ii). As $\mathrm{Spin}(V) \subset \mathrm{Spin}(V_0)$, we can by restriction consider S_0 as a $\mathrm{Spin}(V)$-representation. The action of $\mathrm{Spin}(V)$ comes from the structure of $C(V)$-module of S_0, deduced from its structure of $C(V_0)$-module by the inclusion of $C(V)$ in $C(V_0)$. It follows that S_0 is a spinorial representation of $\mathrm{Spin}(V)$. Further, S_0 being a faithful $C(V_0)$-module, and hence a faithful $C(V)$-module, any irreducible real spinorial representation of $\mathrm{Spin}(V)$ occurs as a direct factor of S_0: there exists a morphism of $\mathrm{Spin}(V)$-representations $S_{\mathbb{R}} \hookrightarrow S_0^+$ or $S_{\mathbb{R}} \hookrightarrow S_0^-$. On $S_{\mathbb{R}}$, we now define Γ_1 as the orthogonal projection

$$\Gamma_1(s,t) = \mathrm{pr}\,\Gamma_0(s,t)$$

of Γ_0 on V. From the same properties of Γ_0, it follows that the form Γ_1 is symmetric and that if $Q(v) > 0$ with v in V, the form $\langle v, \Gamma_1(s,t)\rangle = \langle v, \Gamma_0(s,t)\rangle$ on $S_{\mathbb{R}}$ is definite. It remains to average Γ_0 over the compact group K of elements of norm 1 in Z to obtain the required

$$\Gamma(s,t) := \int_K \Gamma_1(ks, kt)dk\ .$$

We now prove the uniqueness claim in (i). Let $S_{\mathbb{C}} := S_{\mathbb{R}} \otimes \mathbb{C}$ be the complexification of $S_{\mathbb{R}}$. It suffices to show that symmetric morphisms of representations $S_{\mathbb{C}} \otimes S_{\mathbb{C}} \to V_{\mathbb{C}}$ are unique up to a (complex) factor.

Case 1. $Z = \mathbb{R}$, that is $S_{\mathbb{C}}$ is an irreducible representation. It is spinorial for d odd, semi-spinorial for d even. If we exclude the case $d = 2$, V is an absolutely irreducible representation, and over $\mathbb{C}$, $V_{\mathbb{C}}$ occurs at most once in $S_{\mathbb{C}} \otimes S_{\mathbb{C}}$ (4.9.1). Uniqueness follows.

The case $d = 2$ is easy to treat directly: V decomposes as $D_1 \oplus D_2$, for D_1 and D_2 the two isotropic lines, and the $S^\pm$ are of dimension one with tensor square isomorphic, respectively, to D_1 and D_2.

Case 2. $Z = \mathbb{H}$, i.e. $S_{\mathbb{C}}$ is twice an irreducible representation S_0: $S_{\mathbb{C}} = S_0 \otimes_{\mathbb{C}} W$ with $\dim(W) = 2$. We know the existence of a symmetric Z^1-invariant Γ. After complexification, the Z^1-invariance amounts to $\mathrm{SL}(W)$-invariance, and means that Γ is the tensor product of Γ_0: $S_0 \otimes S_0 \to V_{\mathbb{C}}$ and of the unique (up to a factor) antisymmetric ψ: $W \otimes W \to \mathbb{C}$. Symmetry of Γ implies antisymmetry of Γ_0: there is a morphism Γ_0: $S_0 \otimes S_0 \to V_{\mathbb{C}}$ (unique up to a factor by 4.9), and it is antisymmetric. Any symmetric Γ: $S_{\mathbb{R}} \otimes S_{\mathbb{R}} \to V$ is, after complexification, a multiple of $\Gamma_0 \otimes \psi$; uniqueness follows.

Case 3. $Z = \mathbb{C}$, i.e. $S_{\mathbb{C}}$ is the sum of two inequivalent (complex conjugate) representations. This can happen only for d even, with $S_{\mathbb{C}} = S^+ \oplus S^-$ the sum of the two semi-spinorial representations. We know the existence of a symmetric Z^1-invariant Γ. After complexification, the Z^1-invariance amounts to saying that Γ is a linear combination of a morphism $S^+ \otimes S^- \to V_{\mathbb{C}}$ and of its symmetric. It follows that there is a morphism of representations Γ_0: $S^+ \otimes S^- \to V_{\mathbb{C}}$. By 4.9, any symmetric morphism Γ: $S \otimes S \to V$ is a multiple of the sum of Γ_0 and of its symmetric. Uniqueness follows.

Corollary 6.2. *Let S be a real spinorial representation, not necessarily irreducible, and Γ: $S \otimes S \to V$ be symmetric and such that for v in the open positive cone C of V, $\langle v, \Gamma(s,t)\rangle$ be positive definite.*

(i) S is the direct sum of irreducible subrepresentations $S^{(\alpha)}$, with $\Gamma = \sum \Gamma^{(\alpha)}$.

(ii) There is a unique $\check{\Gamma}$: $S^\vee \otimes S^\vee \to V$ such that, if Γ and $\check{\Gamma}$ are reinterpreted as morphisms γ: $V \to \underline{\mathrm{Hom}}\,(S, S^\vee)$ and $\check{\gamma}$: $V \to \underline{\mathrm{Hom}}\,(S^\vee, S)$, the Clifford relations

$$\check{\gamma}(v)\gamma(v), \gamma(v)\check{\gamma}(v) = Q(v) \tag{6.2.1}$$

hold. For v in C one has again $\langle v, \check{\Gamma}(s,t)\rangle$ symmetric and positive definite.

In (ii), there is a factor of 2 ambiguity. Our choice here is to write $Q(v) = (v, v)$ and to define γ by

$$\langle \gamma(v)(s), t\rangle = (\Gamma(s,t), v)$$

It results from 4.9.6 that the structure of $\mathrm{Spin}(V)$-representation of S is induced by the $C(V)$-module structure of $S \oplus S^\vee$ defined by (6.2.1).

Proof. If there are two nonisomorphic irreducible real spinorial representations S_1 and S_2, after extension of scalars to $\mathbb{C}$, they must be multiples of irreducible complex spinorial representations S_1' and S_2'. This results from there being at most two such. The existence of nontrivial morphisms $S_i \otimes S_i \to V$ $(i = 1, 2)$ implies over $\mathbb{C}$ that of nontrivial morphisms $S_i' \otimes S_i' \to V_{\mathbb{C}}$. By 4.9.1 (ii), it in turn implies the nonexistence of morphisms $S_1' \otimes S_2' \to V_{\mathbb{C}}$, hence the nonexistence of morphisms $S_1 \otimes S_2 \to V$. The representation S is the direct sum of a multiple $S^{(1)}$ of S_1 and of a multiple $S^{(2)}$ of S_2, and Γ, being a morphism of representations, is the sum of morphisms $\Gamma^{(i)}$: $S^{(i)} \otimes S^{(i)} \to V$. This reduces the proof of (i), and the existence statement in (ii), to the case where S is isotypic: a multiple of an irreducible real representation S_0. Let Z be the field of endomorphisms of S_0. Define $W := \mathrm{Hom}(S_0, S)$. It is a right Z-vector space, and

$$W \otimes_Z S_0 \xrightarrow{\sim} S\ .$$

Fix a symmetric morphism Γ_0: $S_0 \otimes S_0 \to V$ with $\langle \Gamma(s,t), v\rangle$ positive definite for v in C. If Γ: $S \otimes S \to V$ is symmetric, for each w in W, $\Gamma(w \otimes s_0, w \otimes t_0)$ is symmetric too, hence a multiple of Γ_0 (6.1 (i)):

$$\Gamma(w \otimes s_0, w \otimes t_0) = F(w)\Gamma_0(s_0, t_0)\ .$$

Fixing s_0 and t_0, one sees that the function F is quadratic, and the Z^1-invariance of Γ_0 (6.1 (i)) implies that it is invariant by $Z^1 \subset Z^*$. It is hence of the form $\Phi(w, w)$, for some Z-Hermitian form Φ on W. The quadratic function F, or equivalently the Hermitian form Φ, uniquely determines Γ, and any Hermitian Φ is obtained, as one

sees by writing (W,Φ) as an orthogonal sum of subspaces of dimension 1 over Z. The Γ for which $\langle\Gamma(s,t),v\rangle$ is positive definite for v in C correspond to the positive definite forms Φ. Decomposing (W,Φ) as an orthogonal direct sum of subspaces of dimension 1 over Z, we obtain (i).

For v in V, the relation (6.2.1) between Γ and $\check{\Gamma}$ means that, if $Q(v)\neq 0$, $\langle\check{\Gamma}(s',t'),v\rangle$ is $Q(v)^{-1}$ times the inverse of the bilinear form $\langle\Gamma(s,t),v\rangle$. This makes the uniqueness of $\check{\Gamma}$ clear, as well as its symmetry and positivity property. In coordinates, for a basis e_μ of V and e_α of S, (6.2.1) reads

$$\frac{1}{2}\left(\Gamma^{\mu}_{\alpha\beta}\check{\Gamma}^{\nu\beta\gamma}+\text{ same with } \mu,\ \nu \text{ permuted}\right)=g^{\mu\nu}\delta^{\gamma}_{\alpha}\ . \tag{6.2.2}$$

To prove the existence of $\check{\Gamma}$, we begin, as in 6.1, by treating the case where V is of dimension congruent to 2 modulo 8. In this case, if S is irreducible, there are up to scalar factors unique morphisms $\Gamma\colon S\otimes S\to V$ and $\widetilde{\Gamma}\colon S^\vee\otimes S^\vee\to V$; they are deduced from the Clifford module structure of $S\oplus S^\vee$, and (6.2.1) follows, for $\widetilde{\Gamma}$ replaced by a suitable scalar multiple.

Going from V to a subspace as in 6.1, and projecting Γ and $\widetilde{\Gamma}$ orthogonally onto that subspace, we obtain in any dimension systems $(S,\Gamma,\Gamma^\vee)$ as in (ii), where S can be assumed to contain any preassigned irreducible spinorial representation S_1. A decomposition of S as in (i) induces a similar decomposition of $(S^\vee,\Gamma^\vee)$ and we obtain a system $(S_1,\Gamma,\Gamma^\vee)$ as in (ii). By (i), taking direct sums of such systems, we prove existence in (ii).

6.3 Remark. (i) The proof of 6.2 (i) gives us a classification of the (S,Γ) for S a multiple of S_0, with commutant Z. Once Γ_0 for S_0 has been chosen, Γ corresponds to a positive definite Z-Hermitian form on $W:=\mathrm{Hom}(S_0,S)$.

(ii) Let us compute the group of automorphisms of a structure of the following kind: V is a vector space with Minkowski metric $(\ ,\ )$ and positive cone C, S and $S^\vee$ are vector spaces in duality, $\Gamma\colon S\otimes S\to V$ is symmetric and positive: $\langle\Gamma(s,t),v\rangle$ is positive definite for v in C^0, and for some (unique) $\check{\Gamma}\colon S^\vee\otimes S^\vee\to V$, (6.2.1) holds. In fact, we will compute the group of automorphisms acting on V with determinant 1.

By (6.2.1), $S\oplus S^\vee$ is a $C(V)$-module. This turns S and $S^\vee$ into representations of $\mathrm{Spin}(V)$. By 4.2 (ii), the representations S and $S^\vee$ are contragredient, and Γ, $\check{\Gamma}$ are morphisms of representations: $\mathrm{Spin}(V)$ acts by automorphisms. If g acts on V with determinant 1, and respects the positive cone C, its image in $\mathrm{O}(V)$ is in the connected component of $\mathrm{O}(V)$, the image of $\mathrm{Spin}(V)$: we have $g=g_1g_2$, with g_1 in $\mathrm{Spin}(V)$ and g_2 acting trivially on V, hence on $\mathrm{Spin}(V)$. By (i), if S is isotypic: $S=S_0\otimes_Z W$, g_2 is in the unitary group of the Z-Hermitian space W. In the nonisotypic case: S the sum of the $S_i\otimes_{Z_i}W_i$ $(i=1,2)$, it is in the product of two such unitary groups. The group $\mathrm{U}(W)$ (resp. $\mathrm{U}(W_1)\times\mathrm{U}(W_2)$) is called the R-group.

6.4. The cases where V is of dimension $d=3,4,6$ or 10 (i.e. $d-2=2^t$, $t=0,1,2,3$) are particularly interesting. In those dimensions, an irreducible real spinorial representation S is of dimension 2^{t+1}. For $d=3,4,6$, the commutant is respectively $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, S is of dimension 2 over Z and

$$\mathrm{Spin}(1,d-1)\xrightarrow{\sim}\mathrm{SL}(2,Z)\ .$$

6.5. We now explain how the case $d = 10$ is similarly related to the octonion algebra $\mathbb{O}$. To avoid unnecessary signs, we will work in signature $(9,1)$ rather than $(1,9)$. This does not affect the spin group. We write the Minkowski space V as the orthogonal direct sum of W positive definite of dimension 8 and of H hyperbolic of dimension 2. We choose an isotropic basis $\{e, f\}$ of H with $Q(e+f) = 1$.

Here is a model for W, its quadratic form, the irreducible $C(W)$-module $S = S_W^+ \oplus S_W^-$, the Clifford multiplication ρ, quadratic forms on S_W^+ and S_W^- for which $(\rho(w)s, t) = (s, \rho(w)t)$ and the resulting pairing (4.9.4) Γ: $S_W^+ \otimes S_W^- \to W$:

W, S_W^+ and S_W^-: the space $\mathbb{O}$ of octonions;

quadratic forms on W, S_W^+ and S_W^-: the octonion reduced norm $N(x) = x\bar{x}$;

morphisms $W \otimes S_W^+ \xrightarrow{\rho} S_W^-$, $S_W^+ \otimes S_W^- \xrightarrow{\Gamma} V$ and $S_W^- \otimes W = W \otimes S_W^- \xrightarrow{\rho} S_W^+$: $x \otimes y \mapsto \bar{x}\bar{y}$.

One has to check that (a) ρ turns $S_W^+ \oplus S_W^-$ into an irreducible $C(W)$-module, (b) $(\rho(w)s, t) = (s, \rho(w)t)$ and (c) $(w, \Gamma(s,t)) = (\rho(w)s, t)$. The bilinear form $(\ ,\ )$ associated to the octonion norm is $\mathrm{Tr}(x\bar{y})$, where Tr is the octonion reduced trace $x + \bar{x}$. The proof will use that in $\mathbb{O}$, the subalgebra generated by two elements x and y (to which one can add $\bar{x} = \mathrm{Tr}(x)\cdot 1 - x$ and $\bar{y} = \mathrm{Tr}(\bar{y})\cdot 1 - y$) is associative, and that $\mathrm{Tr}(x(yz)) = \mathrm{Tr}((xy)z)$, $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ and $\mathrm{Tr}(\bar{x}) = \mathrm{Tr}(x)$.

Proof of (a). That $\rho(w)^2 = N(w)$ is seen as follows: for w in W and s in S_W^+, $\rho(w)^2 s = (\bar{w}\bar{s})^- \bar{w} = (sw)\bar{w} = s(w\bar{w}) = N(w)s$. For s in S_W^-, one repeats this argument in the opposite algebra $\mathbb{O}^0$. Irreducibility is clear.

Proof of (b). For s in S_W^+ and t in S_W^-, $(\rho(w)s, t) = \mathrm{Tr}(\bar{w}\bar{s}\bar{t})$ while $(s, \rho(w)t) = \mathrm{Tr}(\bar{s}\bar{t}\bar{w})$. For s in S_W^- and t in S_W^+, one repeats this argument in $\mathbb{O}^0$.

Proof of (c). For s in S_W^+ and t in S_W^-, $(w, \Gamma(s,t)) = \mathrm{Tr}(\bar{w}(\bar{s}\bar{t}))$ while $(\rho(w)s, t) = \mathrm{Tr}((\bar{w}\bar{s})\bar{t})$.

In this octonionic description of W, S_W^+ and S_W^-, triality is in evidence.

An irreducible $C(H)$-super module is of dimension $1|1$. Up to parity change, it has a basis α (even), β (odd) in which the matrices of e and f are

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{6.5.1}$$

In those models, the even part of the irreducible super $C(V)$-module $S = (S_W^+ \oplus S_W^-) \otimes \langle \alpha, \beta \rangle$ is

$$S^+ = S_W^+ \alpha \oplus S_W^- \beta = \mathbb{O}\alpha \oplus \mathbb{O}\beta . \tag{6.5.2}$$

An element $v = a + \lambda e + \mu f$ of V is isotropic if $a\bar{a} + \lambda\mu = 0$. An element $\bar{x}\alpha + y\beta$ of S^+ is in the kernel of the multiplication by v if

$$\begin{aligned} \bar{a}x - \lambda y &= 0 \\ ay + \mu x &= 0 . \end{aligned} \tag{6.5.3}$$

Suppose v nonzero and isotropic. If $a \neq 0$, λ and μ are then nonzero and by the isotropy equation $a\bar{a} = -\lambda\mu$, the first equation (6.3.3) implies the second. If $a = 0$, one of λ and μ is 0 and the equations (6.5.3) reduce either to $x = 0$ or to $y = 0$.

In the *octonion plane* $\mathbb{O}^2$, the *octonion lines* are the subspaces $y = ax$ (for a in $\mathbb{O}$), as well as the subspace $x = 0$. We have obtained a model of S^+ as the octonion plane. In this model, the map $\mathbb{R}v \mapsto$ Ker (multiplication by v) is a bijection from the set of isotropic lines in V to the set of octonion lines. The octonion line $y = ax$ corresponds to the line in V spanned by $(\bar{a}, 1, -N(a))$ and the line $x = 0$ to the line spanned by $(0, 0, 1)$.

It is not unreasonable to define $\mathrm{GL}(2, \mathbb{O})$ as the group of $\mathbb{R}$-linear transformations of the octonion plane, which transform octonion lines into octonion lines. The Spin group, as it respects the Clifford multiplication, is contained in $\mathrm{GL}(2, \mathbb{O})$.

Proposition 6.6. *After extension of scalars to $\mathbb{C}$, the algebraic group* $\mathrm{GL}(2, \mathbb{O})$ *becomes the group generated by* $\mathrm{Spin}(V)$ *and the multiplicative group of homotheties.*

Sketch of proof. The map $v \mapsto \mathrm{Ker}(v)$ maps the quadric of isotropic lines to the Grassmannian of 8-dimensional subspaces of $\mathbb{O}^2$. It follows that the Zariski closure in the complex Grassmannian of the space of real octonion lines is a homogeneous space of $\mathrm{Spin}(V)$, and a quadric. The automorphism group of a quadric (viewed as an algebraic variety) is the projective orthogonal group. This gives $\mathrm{GL}(2, \mathbb{O}) \to \mathrm{PO}$. Using the nonassociativity of $\mathbb{O}$, one checks that the only elements of $\mathrm{GL}(2, \mathbb{O})$ respecting each octonion line are the homotheties. It remains to check that $\mathrm{GL}(2, \mathbb{O})$ maps in fact to PSO. There are, over the complex numbers, two kinds of maximal isotropic subspaces of $V_{\mathbb{C}}$, and SO is the subgroup of O which does not permute the two kinds. One of the two kinds is singled out by the property that for L of that kind, the intersection of the octonion lines relative to v in L is not reduced to zero.

The intersection of the group of homotheties and of $\mathrm{Spin}(V) \subset \mathrm{GL}(2, \mathbb{O})$ is the center of $\mathrm{Spin}(V)$. It is the group of 4^{th} roots of 1. There is hence a "determinant"

$$\det\colon \mathrm{GL}(2, \mathbb{O}) \to \text{multiplicative group} ,$$

which on a scalar λ takes the value $\det(\lambda) = \lambda^4$ and, if $\mathrm{SL}(2, \mathbb{O})$ is defined to be the kernel of det, 6.6 gives

$$\mathrm{Spin}(V) = \mathrm{SL}(2, \mathbb{O}) . \tag{6.6.1}$$

6.7 Remark. Over $\mathbb{R}$, the group $\mathrm{Spin}(V)$ acts transitively on $S^+ - \{0\}$. Indeed, it permutes transitively the octonion lines $\mathrm{Ker}(v)$ (v isotropic) and the stabilizer of an octonion line L acts on L as a group of orthogonal similitudes.

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