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GENUS OF ALTERNATING LINK TYPES

BY RICHARD H. CROWELL

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1. Introduction

A link projection[1] is said to be *alternating* iff it is connected and, as one follows along any component of the link, undercrossings and overcrossings alternate. A projection is *trivial* iff it is connected and has no crossings; otherwise it is *non-trivial*. We include trivial projections as alternating. An *alternating link type* is one which has an alternating projection.

The principal result of this paper is Theorem 3.5 which contains the assertion that *the degree of the reduced Alexander polynomial of an alternating link type plus one equals twice its genus plus its multiplicity.* A second result, obtained as an immediate corollary of the same method which ultimately yields (3.5), is Theorem (2.13): *The reduced Alexander polynomial of an alternating link type is an alternating polynomial.* This theorem provides the simplest proof of the existence of non-alternating types.

The image P of a connected, non-trivial projection has a natural decomposition as a graph. The vertices are the crossings, i.e., the images of the undercrossings, and the edges are the open arcs into which the crossings subdivide P. Since we shall have no reason to distinguish a point at infinity, we regard P as a spherical rather than a planar graph. The results of this paper are obtained by studying the image graph of a non-

1. A *link* L of *multiplicity* μ is the union of μ ordered, oriented, and pairwise disjoint topological circles (1-spheres) L_i imbedded in the 3-sphere S^3. Two links L and L' are *equivalent* iff $\mu = \mu'$ and there exists an orientation preserving homeomorphism f of S^3 on itself such that $fL_i = L_i'$ and $f \mid L_i$ is also orientation preserving, $i = 1, \cdots, \mu$. An equivalence class of links is a *link type*. A *knot* is a link of multiplicity $\mu = 1$. For any link L, we may select a " point at infinity " $\infty \in S^3 - L$ and consider a Cartesian coordinate system $R \times R \times R = S^3 - \infty$. The projection p: $S^3 \to S^2$ defined by $p(\infty) = \infty$ and $p(x, y, z) = (x, y)$ is said to be *regular* iff

(i) $p \mid L$ is a homeomorphism except for at most a finite number of double points called *crossings* and

(ii) for each crossing $p(a) = p(b)$, $a, b \in L$, $a \neq b$, L is linear in every sufficiently small neighborhood of a and of b (the one of a and b with the larger z-coordinate is the *overcrossing* and the other is the *undercrossing*). Condition (ii) is just one of several ways of insuring that each double point describes a genuine crossing. By the *link type* of p is meant, of course, the link type of L. A given link type is *tame* iff it possesses a regular projection p. The projection is *connected* iff the image $P = p(L)$ is connected. Finally, in this paper all link projections are assumed to be regular and all link types, tame.

trivial, alternating projection. The connection between knot theory and graph theory is provided by a combinatorial theorem which relates certain minor determinants of a matrix of values assigned to the edges of an oriented graph to the maximal rooted trees of the graph. These determinants, in our application, turn out to be link type invariants. We have referred to this result (cf. paragraph preceding (2.11)) as the matrix-tree theorem and a reference is [4]. An edge of a graph will typically be denoted by the letter "e". Thus if G is a graph, the formula $e \in G$ is understood to read "e is an edge of G".

I should like to take this opportunity to express my sincerest gratitude to Professor R. H. Fox for his encouragement and supervision of this research which formed the principal part of my doctoral thesis at Princeton.

2. The reduced Alexander polynomial

Consider a regular projection of a link L. The undercrossings subdivide L into a set $\underset{\sim}{x}$ of n arbitrarily ordered *overpasses* $x_1, \cdots, x_n$ consisting of oriented open arcs plus those components of L which contain no undercrossing. The orientation of each overpass is that which it inherits from L. The number of overpasses is certainly no less than the number d of crossings. The latter we order arbitrarily and denote by $(1), \cdots, (d)$. Notice that each crossing (i), $i = 1, \cdots, d$, lies in the image of just one overpass, which we denote by $x_{\nu(i)}$. Furthermore, for each $i = 1, \cdots, d$, we define $x_{\lambda(i)}$, and $x_{\rho(i)}$ to be the overpasses to whose images (i) is incident and which are to the left and right, respectively, of (i) when one is looking along $x_{\nu(i)}$ in the direction of its orientation.

Let $F(\underset{\sim}{x})$ be the free group freely generated by $\underset{\sim}{x}$. We define the subset $\underset{\sim}{r} = (r_1, \cdots, r_d)$ of $F(\underset{\sim}{x})$ by

$$(2.1) \qquad r_i = x_{\lambda(i)} x_{\nu(i)} x_{\rho(i)}^{-1} x_{\nu(i)}^{-1} \qquad i = 1, \cdots, d\,.$$

The group presentation $(\underset{\sim}{x} : \underset{\sim}{r})$ can be shown to be a presentation of the fundamental group $\pi(S^3 - L)$; it is called a *Wirtinger presentation* (cf. [8]) determined by the link projection, and x_i and r_i are the Wirtinger *generators* and *relators*, respectively.

If $Z(t)$ is the infinite cyclic, multiplicative group generated by t, the homomorphism $\theta : F(\underset{\sim}{x}) \to Z(t)$ defined by $\theta x_i = t, i = 1, \cdots, n$, has a unique linear extension to a homomorphism $\theta : JF(\underset{\sim}{x}) \to JZ(t)$ of the integral group rings [6]. Where $\| \partial r_i / \partial x_j \|$, $i=1, \cdots, d$ and $j=1, \cdots, n$, is the matrix of free derivatives [7], we denote by

$$(2.2) \qquad A = \| a_{ij} \| = \| \theta(\partial r_i / \partial x_j) \|$$

the image matrix over $JZ(t)$. We call A a *reduced Alexander matrix* of the Wirtinger presentation $(\underline{x} : \underline{r})$. It can be shown that

$$\sum_{j=1}^{n} a_{ij} = 0 \, , \qquad i = 1, \cdots, d \, . \tag{2.3}$$

(2.4) *Any row of A is a linear (over $JZ(t)$) combination of other rows.*

The ring $JZ(t)$ is a Gaussian domain [3] whose units are the elements $\pm t^i$. The g. c. d. of the determinants of all $(n-1)\times(n-1)$ minors of A is the *reduced Alexander polynomial* of the Wirtinger presentation $(\underline{x} : \underline{r})$ and is denoted by $\Delta(t)$(if $n > 1$ and $n - 1 > d$, then $\Delta(t) = 0$; if $n = 1$, then $\Delta(t) = 1$). Notice that $\Delta(t)$ is defined only to within an arbitrary factor $\pm t^i$. From (2.3) and (2.4) it follows (cf. [7, pp. 204, 209]) that

(2.5) *If $1<n = d$, then $\Delta(t)$ is the determinant of any $(n-1) \times (n-1)$ minor of A.*

It is false that the polynomial $\Delta(t)$ is an invariant of the abstract group of $(\underline{x} : \underline{r})$. Nevertheless, it can be shown that

(2.6) *The reduced Alexander polynomial is an invariant of link type.*

If the multiplicity of L is μ and $\Delta(t_1, \cdots, t_\mu)$ is the ordinary Alexander polynomial [11] of L, it is a straightforward matter (using the results of [7] Section 6 and [6] Section 1) to prove that.

(2.7) *If $\mu = 1$, the reduced and ordinary Alexander polynomials are the same. If $\mu > 1$, then*

$$\Delta(t) = (1 - t)\Delta(t, \cdots, t) \, .$$

Since a link projection can always be chosen so that $n = d$, it is a consequence of (2.3) and (2.4) that the reduced Alexander matrix of a link projection whose image is not connected can be assumed to be of the form

$$A = \begin{vmatrix} B & 0 \\ 0 & C \end{vmatrix}$$

where B and C are square and $\det B = \det C = 0$. Hence,

(2.8) *If a link type has a disconnected projection, its polynomial $\Delta(t)$ is zero.*

We define the *degree* of $\Delta(t)$ to be the difference between the greatest and least power of t. This number is obviously unchanged by multiplication of $\Delta(t)$ by a unit factor $\pm t^i$.

For the remainder of this section we shall assume that the link projection under consideration is non-trivial and alternating. As remarked in the introduction, the image P is a graph whose vertices are the crossings $(1), \cdots, (d)$. We define an orientation o on the graph P and an assignment α of either $+1$ or $-t$ to each edge of P as follows: For each $i=1,\cdots, d$, there are two distinct edges e_1 and e_2 of P incident to (i) and contained in the images of $x_{\lambda(i)}$ and $x_{\rho(i)}$ respectively. We set

(2.9) $$\alpha e_1 = 1 \quad \text{and} \quad \alpha e_2 = -t .$$

Notice that it can happen that $x_{\lambda(i)} = x_{\rho(i)}$. If so, an arbitrary one of e_1 and e_2 is assigned the value 1 and the other the value $-t$. In any event, (2.9) holds. The orientation o on e_1 and e_2 is chosen so that *the vertex* (i) *is the terminal endpoint of both* e_1 *and* e_2. It is easy to see that α and o are consistently defined by the above for all edges of P because (and only because) the link projection is alternating and non-trivial. We call o the *alternating orientation* of P; it is, of course, not the orientation which P inherits from the original orientation on the link L. We denote the set of all edges of P whose initial and terminal endpoints with respect to o are (i) and (j) respectively by E_{ij}. The number of edges in E_{ij} is either 0, 1, or 2. Then,

(2.10) *The matrix* $B = \| b_{ij} \|$, $i, j = 1, \cdots, d$, *defined by*

$$b_{ij} = \sum\nolimits_{e \in E_{ij}} \alpha e \qquad i \neq j$$

$$b_{ii} = -\sum\nolimits_{\substack{j=1 \\ j\neq i}}^{d} b_{ji}$$

is the transposed matrix of a reduced Alexander matrix of the link projection.

PROOF. Where δ_{ij} is the usual Kronecker delta, we have

$$b_{ij} = \delta_{\upsilon(i)\lambda(j)} - t\delta_{\upsilon(i)\rho(j)} \qquad i \neq j .$$

Since the projection is alternating and non-trivial, each overpass passes over one and only one undercrossing. Hence, the function $(i) \to x_{\upsilon(i)}$ is a one-one correspondence between the crossings and the overpasses, and we may therefore order them so that $\upsilon(i) = i$. As a result,

$$b_{ij} = \delta_{i\lambda(j)} - t\delta_{i\rho(j)} \qquad i \neq j .$$

Furthermore, the reduced Alexander matrix is square, i.e., $n = d$, and the Wirtinger relators become

$$r_i = x_{\lambda(i)} x_i x_{\rho(i)}^{-1} x_i^{-1} .$$

Consequently (cf. [6]), if $i \neq j$.

$$a_{ij} = \theta\left(\frac{\partial r_i}{\partial x_j}\right) = \delta_{j\lambda(i)} - t\delta_{j\rho(i)} = b_{ji} .$$

By (2.3)

$$a_{ii} = -\sum\nolimits_{\substack{j=1 \\ j\neq i}}^{d} a_{ij} = -\sum\nolimits_{\substack{j=1 \\ j\neq i}}^{d} b_{ji} = b_{ii}$$

and the proof is complete.

Let $Tr(i)$ be the set of all rooted trees of P with respect to the orientation o and with origin (i) which contain all the vertices of P.[2] We denote by Δ_i the determinant of the $(d-1)\times(d-1)$ principal minor obtained by deleting the i^{th} row and column of the matrix B. A direct application of the matrix-tree theorem (cf. Introduction and [4]) yields

$$(2.11)\qquad (-1)^{d-1}\Delta_i = \sum_{T\in Tr(i)} \prod_{e\in T} \alpha e\ .$$

As a corollary of (2.5), (2.10), and (2.11), we obtain

(2.12) THEOREM. *The reduced Alexander polynomial* $\Delta(t)$ *of a non-trivial, alternating link projection is given by*

$$\Delta(t) = \sum_{T\in Tr(i)} \prod_{e\in T} \alpha e\ .$$

The proof of our principal result, Theorem (3.5), is obtained using (2.12) as the basic lemma. An immediate corollary is the rather odd result:

(2.13) THEOREM. *The reduced Alexander polynomial of an alternating link type is an alternating polynomial.*[3]

PROOF. We note that any alternating link type has a non-trivial alternating projection. Since $\alpha e = 1$ or $-t$, any product $\prod_{e\in T} \alpha e$ is of the form $(-1)^n t^n$. A sum of such monomials is an alternating polynomial.

The Alexander polynomial of the knot type 8_{19} (cf. [8]) is $\Delta(t) = t^6 - t^5 + t^3 - t + 1$, and we therefore conclude that it is non-alternating, i.e., there exists no alternating projection of this type. Indeed, any non-alternating polynomial which satisfies Seifert's n.a.s.c. for being the Alexander polynomial of a knot (cf. [9]) may be used to construct a non-alternating type.

3. The genus of a link type

The *genus* $h = h(L)$ of a (tame) link L is the minimum of the genera of all connected, orientable surfaces[4] S tamely imbedded in the 3-sphere

[2] A *rooted tree* of an oriented graph is a subgraph T which is a tree and which contains no two edges with the same terminal endpoint. It is easy to check that if T is not empty, it contains a unique vertex which is the initial endpoint of all edges of T to which it is incident; this vertex is called the *origin* of T.

[3] We define a polynomial $\sum a_n t^n$ to be *alternating* iff $(-1)^{i+j}a_i a_j \geqq 0$, e.g., $4t-3+t^{-1}$, $3t^3+t-7$, t, 1, and 0 are alternating whereas $4t+3+t^{-1}$ and $t^6-t^5+t^3-t+1$ are not. Note that a polynomial is alternating iff any unit $(\pm t^i)$ multiple of it is alternating.

[4] A surface, in our terminology, is *a priori* compact. The *genus* of a connected surface with boundary is by definition the minimum of the genera of all connected boundaryless surfaces in which it can be imbedded. The Euler characteristic χ and genus h of a connected, orientable surface whose boundary has μ components satisfy the equation (cf. [10])

$$\chi = 2 - 2h - \mu\ .$$

An imbedding of a surface S in the 3-sphere S^3 is *tame* iff there exists a triangulation of S^3 such that S is a subcomplex. The Alexander "horned sphere" is an example of a *wild*, i.e. not tame, imbedding of the 2-sphere in S^3 [1].

with boundary $S = L$. Since equivalent links obviously have the same genus, h is an invariant of the link type $\mathcal{L}$ of L, and we may write $h = h(\mathcal{L})$. Seifert has shown [9] how to construct, for any knot K prescribed by a regular projection, a tame imbedding of a connected orientable surface with boundary K. The same procedure is applicable to links: Consider a connected, non-trivial projection p of a link L. The image graph $p = p(L)$ inherits an orientation O from L, which we call the *link orientation* (in contrast to the alternating orientation introduced in the preceding section). A subgraph C of P will be called a *Seifert circuit* iff (i) C is an n-circuit, (ii) C is a cycle with respect to O, and (iii) the inverse image $p^{-1}(C) \cap L$ has n-components.[5] An insect crawling along the edges of P in the O direction which always makes a right or left turn at a crossing will traverse a Seifert circuit (observe first that his path must be closed, next that it must be simple, and, finally, that the inverse images of the edges of the path must be disjoint). Conversely, an insect crawling along a Seifert circuit in the O direction has his route completely prescribed. It follows that

(3.1) *Every edge of P lies in exactly one Seifert circuit.*

Roughly speaking, a surface S with boundary L is now constructed by filling in each of the Seifert circuits $C_1, \cdots, C_f$ with a spanning 2-cell. A more detailed description (due to R. H. Fox) is as follows: In a sufficiently small neighborhood of each of the d undercrossings and d overcrossings the link L is linear. Hence, we may select an open, connected, linear neighborhood of each of these $2d$ points. At each crossing we then connect the neighborhood of the undercrossing to that of the overcrossing lying above it by joining the four endpoints with two straight-line segments. There are two ways of making this hook-up and the choice is dictated by the condition that the endpoints of the projection under p of each segment shall belong to a single Seifert circuit. Let L' be the graph obtained from L by discarding the $2d$ neighborhoods of the undercrossings and overcrossings and adjoining the segments just described. Obviously, L' is an unknotted link of f components $L'_1, \cdots, L'_f$ which are in a natural one-one correspondence with the Seifert circuits. We now choose f distinct planes $z = z_i$, $i = 1, \cdots, f$, lying below L and such that $z_i < z_j$ if pL'_j is contained in the interior of pL'_i. The image $p_iL'_i$ under the projection $p_i(x, y, z) = (x, y, z_i)$ is a simple closed curve in the plane $z = z_i$ and

[5] An *n-circuit* is a graph with $n \geqq 1$ edges whose underlying space is a simple closed curve. Where explicit reference to the number of edges is unnecessary we speak simply of a *circuit*. A graph G is a *cycle* with respect to an orientation iff in the free abelian group generated by the vertices of G, $\sum_{e \in G}$ (terminal endpoint e – initial endpoint e) = 0.

its interior is an open 2-cell D_i. The union E_i of D_i and the points lying between L'_i and $z = z_i$ (i.e., $E_i = D_i \cup \{(x, y, z) \mid \text{for some } \bar{z}, (x, y, \bar{z}) \in L'_i \text{ and } z_i \leqq z < \bar{z}\}$) is also an open 2-cell. Finally, at each crossing we introduce a 2-cell F_i whose boundary consists of the two linear neighborhoods of the undercrossing and overcrossing and the two segments joining their endpoints. It should be clear that the union

$$S = L \cup L' \cup \bigcup_{i=1}^{f} E_i \cup \bigcup_{i=1}^{d} F_i$$

is a surface with boundary L, (Fig. 1). In addition, we have described a cellular decomposition of S into $(f + d)$ 2-cells, $(2d + 4d)$ edges, and $4d$ vertices. Hence, by the Euler-Poincaré formula,

$$\chi = (f + d) - (2d + 4d) + 4d = f - d .$$

Since P is connected, we may conclude that S is also connected ; and, since the 2-cells may be coherently oriented so that the sum of the boundaries is just L with its prescribed orientation, we know that S is orientable. Thus (cf. footnote 4), we have

$$2h(S) + \mu - 1 = d - f + 1 . \tag{3.2}$$

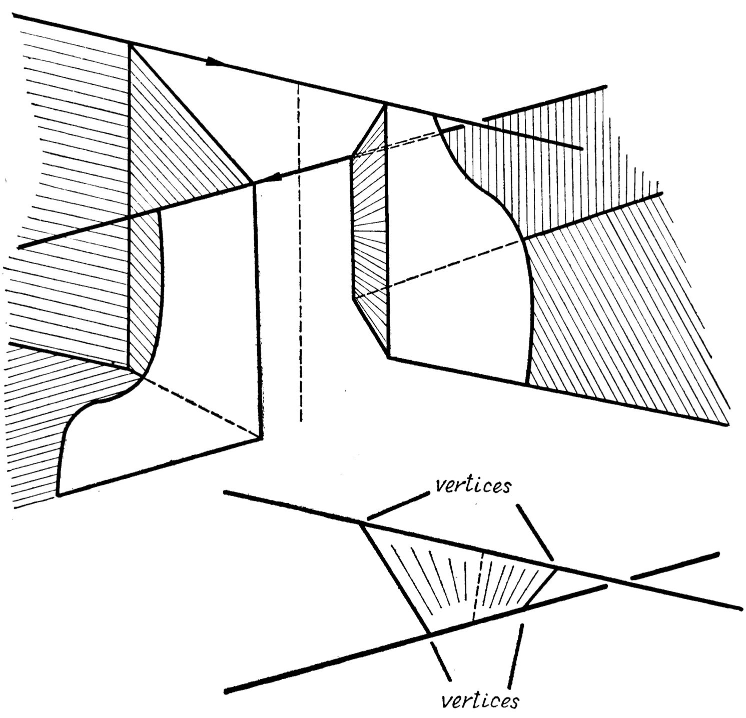

Fig. 1.

Torres [11] has proved that the genus $h(\mathcal{L})$ of a link type $\mathcal{L}$ of multiplicity μ and its reduced Alexander polynomial $\Delta(t)$ satisfy the relation

(3.3) $$\text{degree } \Delta(t) \leqq 2h(\mathcal{L}) + \mu - 1$$

(cf. (2.7) in conjunction with Torres' statement of the above inequality). Since, by definition, $h(\mathcal{L}) \leqq h(S)$, we obtain from (3.3) and (3.2)

(3.4) $$\text{degree } \Delta(t) \leqq 2h(\mathcal{L}) + \mu - 1 \leqq 2h(S) + \mu - 1 = d - f + 1 .$$

The principal result of this paper is that, for a non-trivial, alternating link projection, degree $\Delta(t) \geqq d - f + 1$ and, therefore, the inequalities (3.4) are equalities. Thus, we shall have proved

(3.5) THEOREM. *If $\mathcal{L}$ is an alternating link type of multiplicity μ and genus $h(\mathcal{L})$, then*

$$\text{degree } \Delta(t) = 2h(\mathcal{L}) + \mu - 1$$

where $\Delta(t)$ is the reduced Alexander polynomial of $\mathcal{L}$. Furthermore, the genus $h(S)$ of any Seifert surface constructed with respect to any non-trivial, alternating projection of type $\mathcal{L}$ is minimal, i.e., $h(S) = h(\mathcal{L})$.

An interesting corollary is

(3.6) *An alternating link type cannot be pulled apart*[6] (does not possess a disconnected projection).

PROOF. Suppose $\mathcal{L}$ is alternating and does have a disconnected projection. Then $\mu \geqq 2$, and

$$\text{degree } \Delta(t) = 2h(\mathcal{L}) + \mu - 1 \geqq 1 .$$

By (2.8), $\mathcal{L}$ has polynomial $\Delta(t) = 0$. Since our definition gives the zero polynomial degree zero, we have a contradiction, and the proof is complete.

4. Proof of Theorem (3.5)

Throughout this section we consider an arbitrary non-trivial, alternating link projection with image P. As in the preceding sections, the graph P is assumed to possess link orientation O, edge assignment α (cf., (2.9)), and alternating orientation o. We continue to denote the number of vertices (crossings) of P by d and the number of Seifert circuits of P by f. Since any alternating link type has a non-trivial alternating projection, Theorem (3.5) follows from the inequalities (3.4) if it can be proved that

(4.1) $$\text{degree } \Delta(t) \geqq d - f + 1 .$$

Hence, this inequality will be the object of the discussion which follows.

[6] There are at least two other proofs of this result; cf. [2, 5].

A decomposition of P into the union of edgewise disjoint subgraphs H and K is obtained by defining

$$H = P - \alpha^{-1}(-t) \text{ and } K = P - \alpha^{-1}(1) .$$

Thus, H consists of all vertices of P and all edges which are assigned the value 1 by α, and K consists of the vertices of P and edges assigned the value $-t$. Probably the most important single property of these two subgraphs is the fact (Fig. 2) that

(4.2) *Neither H nor K contains a pair of distinct edges with a common terminal endpoint with respect to the orientation o.*

One important consequence is that every circuit of H or of K is a cycle with respect to o. A simple counting (Fig. 2) yields

(4.3) *Number of edges of H = number of vertices of H = number of K = number of vertices of $K = d$.*
Hence, we obtain

(4.4) *Every component of each of the graphs H and K contains exactly one circuit.*

PROOF. The Euler-Poincaré formula applied to (4.3) shows that the number of components of H does not exceed the number of circuits. If the latter were not distributed one to a component, we could obviously find a pair of edges of H with a common terminal endpoint with respect to o. The same argument holds for K.

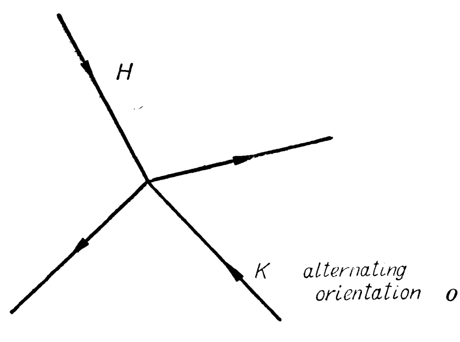

Fig. 2

We define the *characteristic* of any rooted tree T of the oriented graph (P, o), denoted by char T, to be the number of edges of T which lie in K. As a result of Theorem (2.12), the problem of proving the basic inequality (4.1) becomes one of showing that (P, o) has maximal rooted trees of sufficiently high and sufficiently low characteristic.

An arbitrary graph G with an orientation ∂ will be called ∂-connected iff any two vertices u and v of G can be joined by a path coherently oriented with respect to ∂ which runs from u to v.[7] It is a straightforward matter to check that

(4.5) *Any rooted tree of a ∂-connected graph (G, ∂) can be extended to a rooted tree with the same origin which contains all the vertices of G.*

In particular, in any ∂-connected graph (G, ∂) rooted trees with any given vertex as origin which contain all the vertices of G do exist.

(4.6) *The graph (P, o) is o-connected.*

PROOF. It is a consequence of the fact that every vertex of P is of even order and of the configuration of (P, o) at any vertex (Fig. 2) that the boundary of every region of P is a cycle with respect to o. Hence, any path joining two vertices u and v can be replaced by one coherently oriented with respect to o and running from u to v. Since P is connected, the proof is complete.

A Seifert circuit C of P will be called *special* iff one of the two regions into which C divides the 2-sphere contains no edges or vertices of P. We denote the set of all Seifert circuits of P by $\mathscr{F}$ and the set of special Seifert circuits by $\mathscr{F}_S$. For any oriented graph (G, ∂), we denote by $\mathcal{C}(G, \partial)$ the set of all circuits of G which are cycles with respect to ∂. Then,

(4.7) $\mathscr{F} = \mathcal{C}(P, o) \cap \mathcal{C}(P, O)$ and $\mathscr{F}_S = \mathcal{C}(H, o) \cup \mathcal{C}(K, o)$.

PROOF. That the Seifert circuits of P are precisely those circuits which are cycles with respect to both orientations O and o is easily seen from

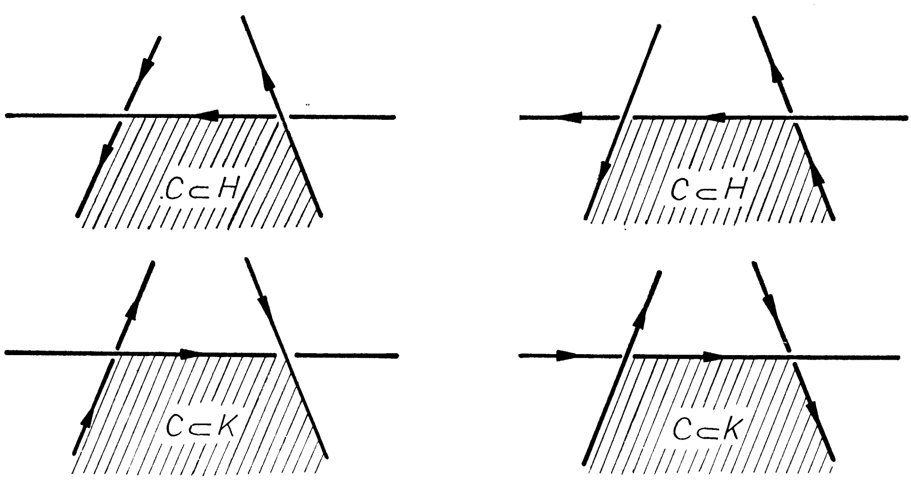

Fig. 3

[7] Such a path is a subgraph L of G such that, with respect to ∂ and in the free abelian group generated by the vertices of G, $\sum_{e \in L}$(terminal endpoint e − initial endpoint e) $= v - u$. Notice that the existence of another path coherently oriented and running from v to u is implied.

Fig. 2 and the characterization in Section 2 of a Seifert circuit as the path of a traveling insect. To check the second equation, we consider, first of all, two consecutive crossings along an arbitrary $C \in \mathcal{F}_S$. Incidentally, since all 1-circuits of P (subgraphs consisting of one edge and one vertex) are contained in $\mathcal{F}_S$ and $\mathcal{C}(H, o) \cup \mathcal{C}(K, o)$ anyway, we may safely assume that C contains at least two edges. The four possible configurations are displayed in Fig. 3, whence it is obvious that consecutive edges of C must lie either both in H or both in K. We conclude that $\mathcal{F}_S \subset \mathcal{C}(H, o) \cup \mathcal{C}(K, o)$.

The converse inclusion is somewhat more subtle. Consider any $C \in \mathcal{C}(H, o)$. We denote the two 2-cells into which C divides the 2-sphere by A and B, of which, say, A is on the left with respect to o. Consider any edge of C, whose initial and terminal endpoints with respect to o we denote by u and v, respectively. As before, we may assume C has at least two edges; so $u \neq v$. The four possible configurations are shown below in Fig. 4, where the edges of C are distinguished by solid line segments. For each of these, on the basis of the definition of H, we have determined the corresponding link orientation at v. The crux of the argument is the fact that, with respect to the link orientation O, the number of edges in the interior of A with an initial endpoint on C must equal the number with a terminal endpoint on C, and the same goes for B. This remark follows from the fact that the image of any component of the original link is a closed curve and must eventually return to its starting point. From Fig. 4, however, one sees that any occurrence of either (ii) or (iii) will spoil the count. Thus, in traversing C one will either make a left turn at every crossing or a right turn at every crossing. This conclusion implies, first, that one of A or B contains none of P and, second, that C is a cycle with respect to O. Hence, $C \in \mathcal{F}_S$. The analogous argument holds for K, and the proof is complete.

The proof of (4.1) is based, as we have remarked, on Theorem 2.12 and proceeds from there by induction on the number N of non-special Seifert circuits, i.e., $N =$ cardinality of $\mathcal{F} - \mathcal{F}_S$. It is interesting that the invariance of the polynomial $\Delta(t)$ enters the induction in an essential way (cf. (4.11)). As a corollary of (4.4) and (4.7), we have

(4.8) $$f = N + p_1(H) + p_1(K) .$$

The first step of the induction is

(4.9) *If $N = 0$, then, for any vertex v of P there exist maximal rooted trees $T(H)$ and $T(K)$ of (P, o) with origin v such that*

$$T(H) \text{ contains } d - p_1(H) \text{ edges of } H$$
$$T(K) \text{ contains } d - p_1(K) \text{ edges of } K .$$

PROOF. Observe, first of all, that every edge of H belongs to a circuit of $\mathcal{C}(H, o)$ and similarly for K. For, if $N = 0$, we have,[8]

$$H \cup K = P = |\mathcal{F}| = |\mathcal{F}_S| = |\mathcal{C}(H, o)| \cup |\mathcal{C}(K, o)| .$$

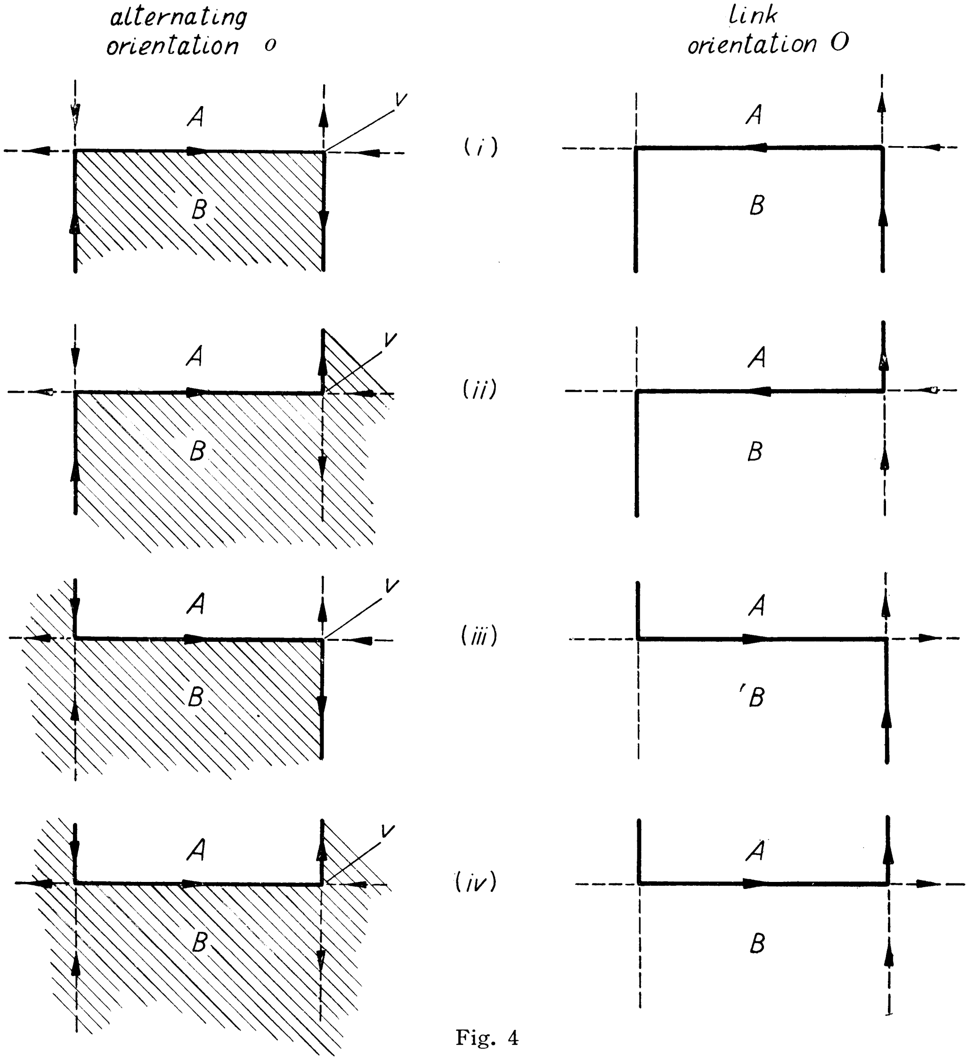

Fig. 4

Each $C \in \mathcal{F}_S$ is the boundary of exactly one 2-cell of the cellular decomposition of the 2-sphere determinable by the spherical graph P. We construct a new spherical graph P' by collapsing to a point the closure of each 2-cell whose boundary is a circuit of $\mathcal{C}(H, o)$. This collapsing is

[8] If Q is any set, by $|Q|$ we mean the union of the members of Q.

possible because (i) the closure of any 2-cell bounded by a circuit is simply-connected and (ii) by virtue of (4.4), the closed 2-cells which are to be collapsed are disjoint from one another. The graph P' inherits an orientation from o which we also denote by o. Since P' is connected, since every vertex is of even order, and since, in addition, the boundary of any 2-cell of P' is obviously a cycle with respect to o, we may conclude that P' is o-connected. Consequently (cf. (4.5)), there exists a maximal rooted tree of (P', o) with origin v', where v' is the vertex into which v has collapsed. If one now pictures P' as being simply P with the circuits of H filled in and constituting the vertices of P', it is apparent that by replacing each vertex of T' by the corresponding circuit of H with one edge removed, we may obtain from T' a maximal rooted tree $T(H)$ of (P, o) with origin v. In view of our initial observation that every edge of H lies in a circuit of H, we see that the number of edges in $T(H) \cap H$ is exactly $d - p_1(H)$, i.e., the total number d of edges of H (cf. (4.3)) minus one for each circuit of H (cf. (4.4)). The analogous argument is valid for the graph K, and the proof of (4.9) is complete.

Next consider

(4.10) *There exist a vertex v of P, maximal rooted trees $T(H)$ and $T(K)$ of (P, o) with origin v, and non-negative integers h and k such that*

$$T(H) \text{ contains at least } d - p_1(H) - h \text{ edges of } H$$
$$T(K) \text{ contains at least } d - p_1(K) - k \text{ edges of } K$$
$$N = h + k\ .$$

Notice that an apparently stronger result is the assertion that the trees and non-negative integers described in (4.10) exist for any vertex v of P. In fact, let us refer to the proposition " For any vertex v of P, there exist maximal rooted trees etc." as the " strong form of Lemma (4.10)". Before proving (4.10), we observe that

(4.11) *The following are equivalent* :
- (i) *Lemma* 4.10 ;
- (ii) degree $\Delta(t) \geqq d - f + 1$;
- (iii) *Strong form of Lemma* 4.10 .

PROOF. (i) implies (ii): Since any maximal tree of P contains $d - 1$ edges,

$$\text{char } T(H) \leqq (d-1) - (d - p_1(H) - h) = p_1(H) + h - 1$$
$$\text{char } T(K) \geqq d - p_1(K) - k\ .$$

Hence,

$$\begin{aligned}\text{char } T(K) - \text{char } T(H) &\geqq d - (p_1(H) + p_1(K) + h + k) + 1 \\ &= d - (p_1(H) + p_1(K) + N) + 1 \,.\end{aligned}$$

By (4.8) and (3.2),

$$\text{char } T(K) - \text{char } T(H) \geqq d - f + 1 \geqq 0 \,.$$

Using Theorem 2.12, we conclude that

$$\text{degree } \Delta(t) \geqq |\, \text{char } T(K) - \text{char } T(H) \,| \geqq d - f + 1 \,.$$

(ii) *implies* (iii): Let v be an arbitrary vertex of P and let $T(H)$ and $T(K)$ be maximal rooted trees of (P, o) with origin v of smallest and largest characteristic, respectively. Then,

$$\text{degree } \Delta(t) = \text{char } T(K) - \text{char } T(H) \geqq d - f + 1 \,.$$

Since a tree cannot contain a circuit, (cf. (4.3) and (4.4))

$$\begin{aligned}&\text{no. edges in } T(H) \cap H \leqq d - p_1(H) \\ &\text{no. edges in } T(K) \cap K \leqq d - p_1(K) \,.\end{aligned}$$

That is, there exist non-negative integers m and n such that

$$\begin{aligned}&\text{no. edges in } T(H) \cap H = d - p_1(H) - m \\ &\text{no. edges in } T(K) \cap K = d - p_1(K) - n \,.\end{aligned}$$

Hence,

$$\begin{aligned}\text{degree } \Delta(t) &= \text{char } T(K) - \text{char } T(H) \\ &= d - p_1(K) - n - (p_1(H) + m - 1) \\ &= d - (p_1(H) + p_1(K) + m + n) + 1 \,.\end{aligned}$$

Hence, by (ii) and (4.8), we obtain

$$N \geqq m + n$$

and we may therefore choose integers $h \geqq m$ and $k \geqq n$ which satisfy the requirements of (iii). That (iii) *implies* (i) is obvious, and the proof of (4.11) is complete.

PROOF OF LEMMA 4.10. By (4.11) and induction on N. Since (4.9) gives the strong form of (4.10) for $N = 0$, we pass immediately to the inductive step $N > 0$. Consequently, there exists at least one non-special Seifert circuit C, (Fig. 5). We construct two new link projections as follows: the image P_1 of one consists of C and that part of P lying in the exterior of C, and the image P_2 of the other consists of C and that part of P lying in the interior of C, (Fig. 6). The orientations O and o on each of P_1 and P_2 are the same as on P. It is obvious that these link projections exist and that both are non-trivial and alternating. The essential feature of this decomposition is, of course, that C is a special Seifert circuit of both P_1 and P_2. Notice that the one simple closed curve $|C|$ is the underlying space of three distinct graphs: (i) the non-special Seifert circuit C of P, (ii) and (iii) the special Seifert circuit C_i of P_i, $i = 1, 2$. That is, the set of vertices of C is the disjoint union of the sets of vertices of C_1 and C_2. We denote the number of vertices, Seifert circuits, non-special Seifert circuits, the graphs H and K, etc., of P_i, $i = 1, 2$, by d_i, f_i, N_i, H_i, K_i, etc. Clearly,

$$\begin{aligned} d_1 + d_2 &= d \\ f_1 + f_2 &= f + 1 \\ N_1 + N_2 &= N - 1 \, . \end{aligned} \tag{1}$$

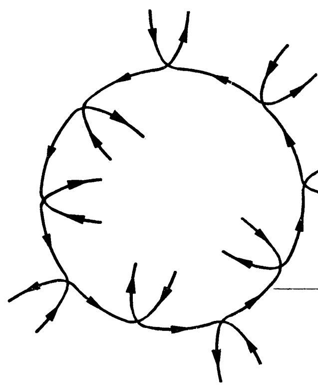

Fig. 5

Since C_i is a special Seifert circuit, we have (cf. (4.7)) that either $C_i \subset H_i$ or $C_i \subset K_i$, $i = 1, 2$. Thus, there are the four possibilities.

(i)	$C_1 \subset H_1$	and	$C_2 \subset K_2$
(ii)	$C_1 \subset K_1$	and	$C_2 \subset H_2$
(iii)	$C_1 \subset H_1$	and	$C_2 \subset H_2$
(iv)	$C_1 \subset K_1$	and	$C_2 \subset K_2$.

The last two are actually impossibilities. To see this, recall the important fact that no two edges belonging to H or to K can have a common terminal point with respect to o (this remark is also true of H_i and K_i). Since

$$H_i - C_i \subset H - C \quad \text{and} \quad K_i - C_i \subset K - C \qquad i = 1, 2,$$

(iii) implies that all edges of $P - C$ which have a terminal point in C must lie in K, and this fact implies $C \subset H$. Similarly, (iv) implies $C \subset K$. Since C is by assumption non-special, neither inclusion is possible (cf. (4.7)).

Either of the remaining possibilities can occur and it makes no difference which. To be specific, we shall assume that it is (i) which is valid. Obviously,

$$p_1(H_1) + p_1(H_2) = p_1(H) + 1 \tag{2}$$
$$p_1(K_1) + p_1(K_2) = p_1(K) + 1 .$$

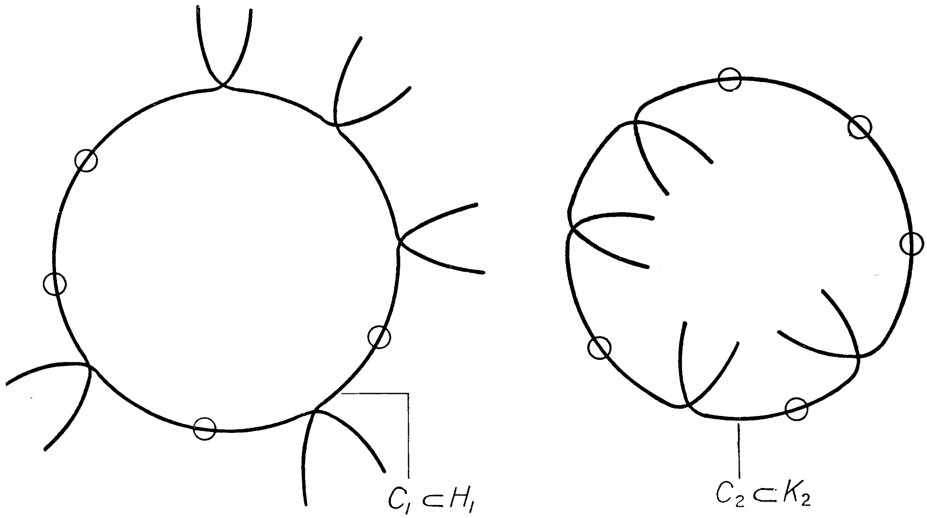

Fig. 6

Select an edge b of C whose terminal endpoint v_1 is a vertex of C_1 and whose initial endpoint v_2 is a vertex of C_2 (with respect to o) ; we denote the edge of C preceding b, i.e., whose terminal endpoint with respect to o is v_2, by a. Since $N_i < N$, $i=1, 2$, the principle of induction and (4.11) imply the strong form of (4.10) for P_1 and P_2. Thus, there exist maximal rooted trees $T(H_i)$ and $T(K_i)$ of (P_i, o) with origin v_i and non-negative integers h_i and k_i such that

$$\begin{aligned} &\text{no. edges in } T(H_i) \cap H_i \geqq d_i - p_1(H_i) - h_i \\ &\text{no. edges in } T(K_i) \cap K_i \geqq d_i - p_1(K_i) - k_i \qquad i = 1, 2 \\ &h_i + k_i = N_i \,. \end{aligned} \tag{3}$$

Let us denote by e_i, $i=1, 2$, the edge of C_i whose terminal endpoint with respect to o is v_i. We contend that $T(H_1)$ and $T(K_2)$ can always be so chosen that

$$C_1 - e_1 \subset T(H_1) \quad \text{and} \quad C_2 - e_2 \subset T(K_2) \,. \tag{4}$$

The proof for $T(H_1)$ is as follows : Suppose e is an edge of C_1 other than e_1 which is not contained in $T(H_1)$. Denote the terminal endpoint of e by v and the other edge whose terminal endpoint is v by g. Since $T(H_1)$ is a maximal tree and there is no other way of getting to v, we have $g \in T(H_1)$. Obviously, $T(H_1)-g$ is the disjoint union of two rooted trees, one with origin v_1 and the other v. The graph $T^1(H_1) = (T(H_1) - g) \cup e$ is thus a maximal rooted tree of (P_1, o) with origin v_1. Finally, since $T^1(H_1)$ contains one more edge of H_1 than does $T(H_1)$, it too has at least $d_1 - p_1(H_1) - h_1$ edges of H_1. This construction may be repeated as often as necessary and the contention is proved for $T(H_1)$. The analogous argument holds for $T(K_2)$. We shall assume, therefore, that conditions (4) are fulfilled.

The construction of $T(H)$ now proceeds easily. By virtue of (4), we see that the graph $T^1(H)$ formed from $T(H_1)$ by deleting all the edges of C_1 is the disjoint union of rooted trees of (P, o) whose origins are just the vertices of C_1. The tree $T(H_2)$ may be regarded as a subgraph of P if C_2 is subdivided to C. Then, the graph $T^2(H) = T(H_2) \cup (H \cap C)$, which is formed from $T(H_2)$ by adding all the edges of $H \cap C$ not already contained in $T(H_2)$, is a rooted tree of (P, o) with origin v_2. To check this last remark, recall that, since $C_2 \subset K_2$, all edges of $P_2 - C_2$ with a terminal point on C_2 lie in H and, therefore, all edges of C with terminal endpoints in C_2 lie in K. Consequently, the addition of edges of $C \cap H$ results in the formation of no cycles nor any pair of distinct edges with a common terminal point. A similar argument shows that $T^2(H)$ contains all the vertices of C. Finally, then, the union

$$T(H) = T^1(H) \cup T^2(H)$$

is a maximal rooted tree of (P, o) with origin v_2. Moreover, where p is the number of edges of C_1, $T(H)$ contains the following number of edges of H

(i)	at least	$d_1 - p_1(H_1) - h_1 - (p - 1)$	from $T^1(H)$
(ii)	at least	$d_2 - p_1(H_2) - h_2$	from $T(H_2)$
(iii)	exactly p		from $C \cap H$.

Since $C_2 \subset K_2$, none of the edges of $T(H_2) \cap H_2$ lies in C_2; hence, no edge is counted twice in (i), (ii), (iii). Thus $T(H)$ contains at least

$$(d_1 + d_2) - (p_1(H_1) + p_1(H_2) - 1) - (h_1 + h_2)$$

edges of H. From (1) and (2), we have

$$\text{no. edges in } T(H) \cap H \geqq d - p_1(H) - (h_1 + h_2)\,. \tag{5}$$

In order to get $T(K)$ we carry out the exact analogue of the construction of $T(H)$. The result is a maximal rooted tree of (P, o) with origin v_1 which we denote by $T'(K)$. The analogue of (5) is

$$\text{no. edges in } T'(K) \cap K \geqq d - p_1(K) - (k_1 + k_2)\,. \tag{6}$$

To shift the origin to v_2 so it will be the same as $T(H)$, we make use of the relative positions of v_1 and v_2 and the edges a and b. It follows from the construction of $T'(K)$ that $a \in T'(K)$ and $b \notin T'(K)$. Hence,

$$T(K) = (T'(K) - a) \cup b$$

is a maximal rooted tree of (P, o) with origin v_2. Since $a \in K$ and $b \in H$, we get, from (6),

$$\text{(7)} \qquad \text{no. edges in } T(K) \cap K \geqq d - p_1(K) - (k_1 + k_2 + 1) \, .$$

Since, by (3) and (1)

$$(h_1 + h_2) + (k_1 + k_2 + 1) = N_1 + N_2 + 1 = N \, ,$$

equations (5) and (7) are exactly the inequalities called for in the statement of Lemma 4.10, and the proof is complete.

The combination of (4.10) and (4.11) yields the inequality degree $\Delta(t) \geqq d - f + 1$, which, as we have observed, implies Theorem 3.5.

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