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THE MASLOV INDEX REVISITED

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Abstract. Let $\mathcal{D}$ be a Hermitian symmetric space of tube type, S its Shilov boundary and G the neutral component of the group of bi-holomorphic diffeomorphisms of $\mathcal{D}$. In the model situation $\mathcal{D}$ is the Siegel disc, S is the manifold of Lagrangian subspaces and G is the symplectic group. We introduce a notion of transversality for pairs of elements in S, and then study the action of G on the set of triples of mutually transversal points in S. We show that there is a finite number of G-orbits, and to each orbit we associate an integer, thus generalizing the *Maslov index*. Using the scalar automorphy kernel of $\mathcal{D}$, we construct a $\mathbb{C}^*$-valued, G-invariant kernel on $\mathcal{D} \times \mathcal{D} \times \mathcal{D}$. Taking a specific determination of its argument and studying its limit when approaching the Shilov boundary, we are able to define a $\mathbb{Z}$-valued, G-invariant kernel for triples of mutually transversal points in S. It is shown to coincide with the Maslov index. Symmetry properties and cocycle properties of the Maslov index are then easily obtained.

Introduction

In the theory of partial differential equations a key role is played by the so-called Maslov index, invented by Maslov and Leray and developed further (see [G-S], [L-V], [M], [Go], [C-L-M]). It has several interesting properties, especially since it is intimately connected with the Segal–Shale–Weil representation of the metaplectic group. But its definition is subtle, and has not been connected to geometric properties of the metaplectic group. In this paper we give a novel way of looking at the Maslov index in order to encode some natural geometry. In particular we use the holomorphic geometry of the Hermitian symmetric space introduced by Siegel as a generalized upper half space, and obtain a natural definition of the Maslov index, which also works for other tube type domains. In this paper, we limit ourselves to the transversal situation, but hope to study the non-transversal case in the future. The infinite-dimensional case could be developed along the same lines as well.

We briefly indicate the setting. Let $\Lambda = \Lambda_r$ be the space of Lagrangian subspaces in a $2r$-dimensional symplectic space, and denote by $\Lambda^3_\top$ the space of triples of transversal Lagrangians. The geometric result, on which the theory of the Maslov index is based,

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is the fact that the symplectic group has a finite number of open orbits in $\Lambda^3_\top$. For a choice of a symplectic basis, it is possible to describe a set of representatives of the orbits.

The Maslov index $\iota : \Lambda^3_\top \to \mathbb{Z}$ is invariant under the symplectic group, so, in some sense, it is enough to know its value on a set of representatives. However, it has symmetry properties and cocycle type properties that are not transparent from this definition.

We thus propose a new approach to the Maslov index, which on the one hand makes the symmetry and cocycle properties almost obvious, and on the other hand works for more general geometric situations.

The space Λ_r is classically known as the Shilov boundary of the *Siegel disc* $\mathcal{D} = \mathcal{D}_r$. In this interpretation, the symplectic group appears as the (neutral component of the) group of bi-holomorphic diffeomorphisms of the Siegel disc. This introduces some holomorphic theory in the picture. In the reference [M], the space Λ_r is also viewed as the Shilov boundary of the domain $\mathcal{D}$ (viewed as the space of complex positive Lagrangians), and extensions to $\mathcal{D}$ of the Maslov index (in connection with the metaplectic representation) are obtained. Our point of view is close, but more geometric. We show that the Maslov index can be defined by using the (scalar) *automorphy kernel* of the domain $\mathcal{D}$, which has a well known covariance property with respect to the action of the symplectic group. It is also strongly related to the Bergman kernel of the domain.

The Siegel disc is holomorphically equivalent (under the Cayley transform) to the *Siegel upper half-space*, which is the tube domain over the cone of positive-definite symmetric matrices. This correspondence extends in some sense to the Shilov boundaries of both domains, and the orbit picture in $\Lambda^3_\top$ under the symplectic group is strongly related to the orbit picture of the linear group $\mathrm{GL}(r,\mathbb{R})$ acting on the space of symmetric matrices by $(g, X) \to gXg^t$ for $g \in \mathrm{GL}(r,\mathbb{R}), X \in \mathrm{Sym}(r,\mathbb{R})$.

Our construction extends to the Shilov boundary S of any Hermitian symmetric space $\mathcal{D}$ of *tube type*. These domains in turn are in one-to-one correspondence with the Euclidean Jordan algebras (see the appendix for a list of such spaces). It is important to remark that the basic geometric fact (the existence of open orbits in $S \times S \times S$) is *not* true for non-tube-type domains.[1]

In Section 1, we present as a pedagogical introduction the case of Λ_1, which is the circle viewed as the boundary of the unit disc $\mathcal{D}_1$ in $\mathbb{C}$. In this case, the Maslov index can be interpreted as an area (with respect to the Poincaré metric), which makes symmetry and cocycle properties even more transparent. Since we wrote this paper, several authors told us that such an interpretation of the Maslov index was known, but no written reference seems to exist. Sections 2, 3 and 4 introduce the tube-type domains, their relation to Euclidean Jordan algebras and derive properties of their Shilov boundary. The main result of our work is Theorem 5.2 which exactly expresses the Maslov index in terms of the intrinsic holomorphic geometry of the tube domain. Section 6 is a kind of functorial property for the Maslov index.

For more information on the Maslov index and applications, see [G-S], [L-V] and bibliographical references there. For more recent work, see [C-L-M], [Go].

[1]One of the referees of this paper pointed out the reference [C], where an invariant for triples on the unit sphere in $\mathbb{C}^2$ (the Shilov boundary of a non-tube-type Hermitian symmetric space) is introduced in a way very similar to ours.

1. The Maslov index on the circle *via* the automorphy kernel for the unit disc

Let $\mathcal{D}$ be the open unit disc in the complex plane,

$$\mathcal{D} = \{z = x + iy \in \mathbb{C}, |z| < 1\},$$

equipped with its Poincaré metric, which is given infinitesimally by

$$ds^2 = (dx^2 + dy^2)/(1 - |z|^2)^2.$$

There is a corresponding Riemannian measure

$$dm = dx\, dy/(1 - |z|^2)^2.$$

Recall that the metric is conformal with respect to the Euclidean metric, and it has constant curvature equal to -1. The geodesics are known to be (segments of) the circles orthogonal to the unit circle $S = \partial\mathcal{D} = \{z \in \mathbb{C}, |z| = 1\}$.

The Poincaré disc has a large group of holomorphic automorphisms. Let

$$G = \mathrm{SU}(1,1) = \left\{ \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix},\ |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

For $g = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \in G$ and $z \in \mathcal{D}$, define $g(z) = (\alpha z + \beta)/(\overline{\beta} z + \overline{\alpha})$. This formula defines an action of G on $\mathcal{D}$.

Let (z, w) be two distinct points in $\mathcal{D}$. Let $T = d(z, w)$ be the Poincaré distance between z and w. Then there is a unique geodesic curve $\gamma(t), t \in [0, T]$ with endpoints $z = \gamma(0)$ and $w = \gamma(T)$. Let us orientate this geodesic *from* z *to* w, and define the *angular variation* $\angle(z, w)$ from z to w as

$$\angle(z, w) = \measuredangle(\dot\gamma(0), \dot\gamma(T)). \tag{1}$$

Of course, this is *not* an invariant notion for the isometries of the Poincaré disc. On the contrary, it measures the Euclidean deviation of the tangent vector to the geodesic, whereas the tangent vector is invariant under parallel transport for the Poincaré metric along the geodesic.

But these quantities are very useful to compute areas of polygonal domains. First recall a general formula for the area. Let D be a bounded simply connected domain with piecewise smooth boundary contained in $\mathcal{D}$. Fix a tangent vector v at some point of the boundary ∂D so that the direction of v corresponds to travelling counterclockwise on ∂D. Let $P_{\partial D} v$ denote the parallel transport of the vector v along the boundary of D. The area of D is related to the angular variation of v under parallel transport by the formula

$$-\int_D dm = \angle(v, P_{\partial D} v). \tag{2}$$

The *minus* sign comes from the value -1 for the curvature. If the boundary is travelled in the opposite direction (clockwise), then the right-hand side of formula still makes sense, and is the opposite of the corresponding quantity for the counterclockwise orientation. Hence it may be used to define the *oriented* area of the domain D.

Assume now $D = T$ is a geodesic triangle (its sides are assumed to be geodesic segments). We have the following formula for the oriented area of T.

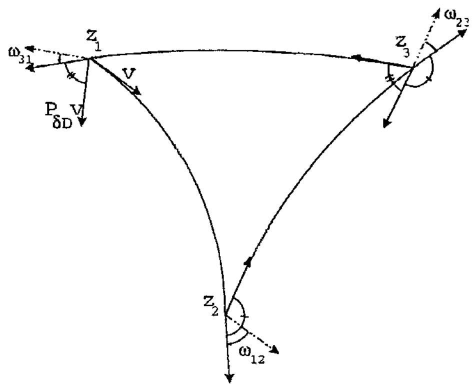

FIGURE 1.

Theorem 1.1. *Let T be a geodesic triangle with summits $z_1, z_2, z_3 \in D$. The oriented area $\mathcal{A}(T)$ is given by*

$$\mathcal{A}(T) = -\big(\angle(z_1,z_2) + \angle(z_2,z_3) + \angle(z_3,z_1)\big). \tag{3}$$

Proof. It is enough to prove the formula when the orientation on the boundary (starting from z_1, going to z_2, z_3 and then back to z_1) is positive. The parallel displacement along a geodesic curve preserves the unit tangent vector, and is a direct isometry between the two tangent planes at two arbitrary points of the curve. So the formula is easily obtained (see Figure 1). Needless to say, this formula is equivalent to the more classical formula giving the area in terms of the angles of the triangle. ☐

Now there is a very convenient way for computing $\angle(z,w)$ for $z, w \in \mathcal{D}$. Consider the points $z' = 1/\overline{z}$ and $w' = 1/\overline{w}$. Then the four points z, w, z', w' are cocyclic, and in fact they belong to the circle containing the geodesic segment from z to w. Let ω be its (Euclidean) center. Then an elementary argument (see Figure 2) shows that

$$\angle(z,w) = \measuredangle(\overrightarrow{\omega z}, \overrightarrow{\omega w}) = \measuredangle(\overrightarrow{z'z}, \overrightarrow{z'w}) + \measuredangle(\overrightarrow{w'z}, \overrightarrow{w'w}) = \arg\frac{w-z'}{z-z'} + \arg\frac{w-w'}{z-w'} = \arg\frac{1-w\overline{z}}{1-\overline{w}z}.$$

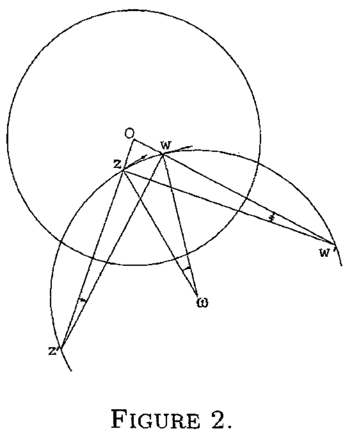

FIGURE 2.

Theorem 1.2. *Let T be a geodesic triangle with summits z_1, z_2, z_3. Then the following formula holds for its oriented area $\mathcal{A}(T)$*

$$\mathcal{A}(T) = \arg \frac{1 - z_1\overline{z_2}}{1 - \overline{z_1}z_2} + \arg \frac{1 - z_2\overline{z_3}}{1 - \overline{z_2}z_3} + \arg \frac{1 - z_3\overline{z_1}}{1 - \overline{z_3}z_1}. \tag{4}$$

There is still another way to write and understand this formula. Let us introduce the *automorphy kernel* defined for $z, w \in \mathcal{D}$ by the formula

$$k(z, w) = 1 - z\overline{w}.$$

Let us also introduce the *automorphy factor*, defined for $z \in \mathcal{D}$ and $g = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \in G$ by

$$j(g, z) = \overline{\beta}z + \overline{\alpha}.$$

The automorphy kernel is holomorphic in the variable z and anti-holomorphic in the variable w. It has the Hermitian symmetry property $\overline{k(w,z)} = k(z, w)$ and transforms under the action of G by the following formula

$$k(g(z), g(w)) = j(g, z)^{-1}\, k(z, w)\, \overline{j(g, w)}^{-1}. \tag{5}$$

Given three distinct points z_1, z_2, z_3 we may form the expression

$$c(z_1, z_2, z_3) = k(z_1, z_2)k(z_2, z_1)^{-1}k(z_2, z_3)k(z_3, z_2)^{-1}k(z_3, z_1)k(z_1, z_3)^{-1}. \tag{6}$$

From the transformation law for k it is easily seen that $c(z_1, z_2, z_3)$ is invariant under the action of G, that is $c(g(z_1), g(z_2), g(z_3)) = c(z_1, z_2, z_3)$, $\forall\, g \in G$. From the Hermitian symmetry property, it is also obvious that $c(z_1, z_2, z_3)$ is a complex number of modulus 1. Now observe that the right-hand side in formula (4) is a specific determination of the argument of $c(z_1, z_2, z_3)$. Notice that for $z, w \in \mathcal{D}$, the complex number $k(z, w) = 1 - z\overline{w}$ always belong to the open right half-plane, and the same is true for its inverse. Then we define $\arg c(z_1, z_2, z_3)$ by adding the *principal determination* of the argument for each of the six factors. With this convention, we can restate the previous theorem.

Theorem 1.3. *Let T be a geodesic triangle with summits z_1, z_2, z_3. Then its oriented area $\mathcal{A}(T)$ is given by the formula*

$$\mathcal{A}(T) = \arg\, c(z_1, z_2, z_3). \tag{7}$$

The formulae we have obtained can be extended to *ideal triangles*. An ideal triangle is a geodesic triangle with summits at the boundary. In more precise terms, let $\zeta_1, \zeta_2, \zeta_3$ be three distinct points on the unit circle $S = \partial\mathcal{D}$, to be thought of as the points at infinity of $\mathcal{D}$, and draw the three (infinite) geodesics joining ζ_1 to ζ_2, ζ_2 to ζ_3 and ζ_3 to ζ_1. Notice that the value of the "angles" is 0, and the area of the triangle is π. More precisely, the oriented area is π if on travelling counterclockwise from ζ_1 to ζ_3 one hits ζ_2, and $-\pi$ on the contrary. Define the *Maslov index* $\iota(\zeta_1, \zeta_2, \zeta_3)$ to be $+1$ in the first case, and -1 in the second case. Then $\iota(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{\pi}\mathcal{A}(\zeta_1, \zeta_2, \zeta_3)$ where $\mathcal{A}(\zeta_1, \zeta_2, \zeta_3)$ denotes the oriented area of the ideal triangle with summits $\zeta_1, \zeta_2, \zeta_3$ in this order.

The formulae we have obtained for the area of geodesic triangles can be extended to give analytic expressions for the Maslov index.

Theorem 1.4. *Let $(\zeta_1, \zeta_2, \zeta_3)$ be three distinct points of the unit circle S. Then the Maslov index $\iota(\zeta_1, \zeta_2, \zeta_3)$ is given by*

$$\iota(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{\pi} \lim \arg c(z_1, z_2, z_3) \tag{8}$$

when $z_1 \to \zeta_1$, $z_2 \to \zeta_2$, $z_3 \to \zeta_3$ "from inside".

The formula shows immediately that the Maslov index is invariant under G. This invariant is somewhat subtle. If one views the unit circle S as the real one-dimensional projective space, then the real projective group operates transitively on triplets of distinct points (a basic result in projective geometry). The real projective group has two connected components, and one way to distinguish them is to introduce an orientation on S. Then the neutral component corresponds to those transformations preserving the orientation. Choosing an orientation of S is tantamount to introducing the complex plane into the picture. The unit circle S is then automatically given an orientation. It is also equivalent to approaching points in S from the interior of $\mathcal{D}$. The Maslov index is the extra information necessary to characterize orbits of G in triplets of distinct points in S.

From the formula (8) it is easy to get the cocycle relation for the Maslov index (the statement and its proof are deferred to Section 5 since it is exactly the same as in the general case).

There are several variations of the Maslov index. Following Hörmander, we may define a *four-points cross-index*. Consider four points (z_1, z_2, z_3, z_4) of $\mathcal{D}$. Define

$$d(z_1, z_2, z_3, z_4) = k(z_1, z_3)k(z_2, z_3)^{-1}k(z_2, z_4)k(z_1, z_4)^{-1}. \tag{9}$$

From the transformation formula for the automorphy kernel, it is easily seen that d is invariant under G. The formula can be extended to points in S, provided they are distinct. For $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ four distinct points of S, using the relation $\overline{\zeta_j} = 1/\zeta_j$, we get $k(\zeta_1, \zeta_3) = 1 - \frac{\zeta_1}{\zeta_3}$, and so on, so $d(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = (1 - \frac{\zeta_1}{\zeta_3})(1 - \frac{\zeta_2}{\zeta_3})^{-1}(1 - \frac{\zeta_2}{\zeta_4})(1 - \frac{\zeta_1}{\zeta_4})^{-1} = \left(\frac{\zeta_1 - \zeta_3}{\zeta_2 - \zeta_3}\right)\Big/\left(\frac{\zeta_1 - \zeta_4}{\zeta_2 - \zeta_4}\right)$, and so

$$d(\zeta_1, \zeta_2, \zeta_3, \zeta_4) = [\zeta_1, \zeta_2, \zeta_3, \zeta_4], \tag{10}$$

where the symbol $[\cdot, \cdot, \cdot, \cdot]$ is used for the (complex) cross-ratio. As this quantity is already invariant by the full complex projective group, no information is gained. But we may consider as before the argument (or rather a specific determination of the argument) of the complex number $d(z_1, z_2, z_3, z_4)$ and then let z_j tend to ζ_j, $1 \le j \le 4$, within $\mathcal{D}$.

To choose a determination of the argument, we use the same convention as before.

Lemma 1.5. *Let z_1, z_2, z_3, z_4 be four points in $\mathcal{D}$. Then the following identities holds:*

$$(d/\overline{d})(z_1, z_2, z_3, z_4) = c(z_1, z_3, z_2)c(z_1, z_2, z_4), \tag{11}$$

$$2 \arg d(z_1, z_2, z_3, z_4) = \arg c(z_1, z_3, z_2) + \arg c(z_1, z_2, z_4). \tag{12}$$

Proof. We have

$$d/\bar{d} = \frac{k(z_1,z_3)}{k(z_3,z_1)}\frac{k(z_3,z_2)}{k(z_2,z_3)}\frac{k(z_2,z_4)}{k(z_4,z_2)}\frac{k(z_4,z_1)}{k(z_1,z_4)}$$

$$= \frac{k(z_1,z_3)}{k(z_3,z_1)}\frac{k(z_3,z_2)}{k(z_2,z_3)}\frac{k(z_2,z_1)}{k(z_1,z_2)}\frac{k(z_1,z_2)}{k(z_2,z_1)}\frac{k(z_2,z_4)}{k(z_4,z_2)}\frac{k(z_4,z_1)}{k(z_1,z_4)}$$

$$= c(z_1,z_3,z_2)c(z_1,z_2,z_4).$$

This shows (11). The proof of (12) is about the same, just by following carefully the convention for the choice of the determination of the argument. Details are left to the reader. □

Now, for four distinct points $\zeta_1,\zeta_2,\zeta_3,\zeta_4$ in S, set

$$\iota(\zeta_1,\zeta_2,\zeta_3,\zeta_4) = \frac{1}{\pi}\lim \arg d(z_1,z_2,z_3,z_4) \tag{13}$$

when $z_j \to \zeta_j$ within $\mathcal{D}$, $1 \le j \le 4$. This Maslov index can be computed using the ideal geodesic quadrangle ($\zeta_1 \to \zeta_3 \to \zeta_2 \to \zeta_4 \to \zeta_1$), and following the variation of the tangent vector at some point under parallel transport, or by evaluating the algebraic area of the quadrangle.

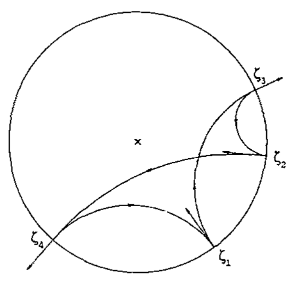

FIGURE 3.

Proposition 1.6. *For any four distinct points $\zeta_1,\zeta_2,\zeta_3,\zeta_4$ in S, we have*

$$\iota(\zeta_1,\zeta_2,\zeta_3,\zeta_4) = \frac{1}{2}\big(\iota(\zeta_1,\zeta_2,\zeta_4) - \iota(\zeta_1,\zeta_2,\zeta_3)\big). \tag{14}$$

This is an easy consequence of Lemma 1.5.

2. Shilov boundary of a Hermitian symmetric space of tube type

The classical theory of the Maslov index deals with the space of Lagrangians. Let E,ω be a real symplectic space of dimension $2r$, and consider the space of Lagrangian subspaces $S = \Lambda(E)$. It is well known that it may be considered as the Shilov boundary of the so-called *Siegel disc*, a Hermitian symmetric space of tube type, thus generalizing the realization of the unit circle as the boundary of the Poincaré disc. Our goal is to generalize the theory of the Maslov index to the Shilov boundary of *any* Hermitian symmetric space of tube-type. In turn the best way to look at these domains is to use the Jordan algebra approach as presented in [F-K], to which we refer for details.

Let V be a Euclidean Jordan algebra with identity element e, which for simplicity we assume to be simple. By complexification, we get a complex Jordan algebra $\mathbb{V}$, with the inner product on V extended to the Hermitian inner product defined by

$$(z, w) = \operatorname{tr}(z\overline{w}).$$

For $z \in \mathbb{V}$, denote by $L(z)$ the linear operator defined by $w \mapsto L(z)w = zw$, and introduce the quadratic representation $P(\cdot)$ and the "square" operator $\square$, defined by

$$P(z) = 2L(z)^2 - L(z^2), \quad z\square w = L(zw) + [L(z)\,, L(w)]$$

where the brackets denote the commutator. Departing slightly from notation in [F-K], denote by $\mathbb{L} = \mathbb{S}\mathrm{tr}(\mathbb{V})$ the structure group of $\mathbb{V}$.

Let
$$S = \{z \in \mathbb{V} \mid \overline{z} = z^{-1}\}.$$

Proposition 2.1. *For $z \in \mathbb{V}$ the following properties are equivalent:*

(i) $z \in S$,
(ii) $[L(z)\,, L(\overline{z})] = 0$ *and* $z\overline{z} = e$,
(iii) $z\square\overline{z} = \mathrm{Id}$,
(iv) $z = \exp(iu)$ *with* $u \in V$,
(v) *There exists a Peirce frame* $(c_j)_{1\le j\le r}$ *of* V *and complex numbers* $(\zeta_j)_{1\le j\le r}$ *of modulus* 1, *such that* $z = \sum_{j=1}^r \zeta_j c_j$.

Proof. (see [F-K] Proposition X.2.3). □

Define the group
$$L(S) = \{g \in \mathrm{GL}(\mathbb{V}) \mid g(S) = S\}.$$

Then $L(S) = \mathbb{L} \cap \mathrm{U}(\mathbb{V})$, where $\mathrm{U}(\mathbb{V})$ denotes the unitary group for the Hermitian inner product on $\mathbb{V}$. Moreover, the stabilizer of e in $L(S)$ coincides with $\mathrm{Aut}(V)$, the group of automorphisms of the real Jordan algebra V (extended as complex linear automorphisms of $\mathbb{V}$). Let U be the identity component of $L(S)$, and let U_e be the stabilizer of e in U. The group U_e contains the identity component K of $\mathrm{Aut}(V)$.

Proposition 2.2. *Let $c_1, c_2, \ldots, c_r$ be a Jordan frame in V. Then every $z \in \mathbb{V}$ can be written in the form $z = u(\sum_{j=1}^r \lambda_j c_j)$ where $u \in U$ and $0 \le \lambda_1 \le \ldots \le \lambda_r$. The scalars $(\lambda_j)_{1\le j\le r}$ are unique and called the spectral values of z.*

For an element $z \in \mathbb{V}$, define its *spectral norm* by

$$|z| = \sup_{1\le j\le r} \lambda_j.$$

It turns out to be a norm on $\mathbb{V}$, invariant under the group U. Introduce the domain $\mathcal{D}$ in $\mathbb{V}$ as the open unit ball for the spectral norm

$$\mathcal{D} = \{z \in \mathbb{V} \mid |z| < 1\}.$$

Proposition 2.3. *The domain $\mathcal{D}$ can be described as*

(i) $\mathcal{D} = \{z \in \mathbb{V} \mid \mathrm{Id} - z\Box\overline{z} \gg 0\}$,

(ii) $\mathcal{D} = \{z \in \mathbb{V} \mid \mathrm{Id} - P(z)P(\overline{z}) \gg 0\}$,

(iii) *$\mathcal{D}$ is the connected component of* 0 *in the set* $\{z \in \mathbb{V} \mid \mathrm{Id} - 2z\Box\overline{z} + P(z)P(\overline{z}) \gg 0\}$.

The main result for what concerns us is

Theorem 2.4. *The Shilov boundary of $\mathcal{D}$ is the set S.*

There is a realization of the domain $\mathcal{D}$ as a tube domain through the *Cayley transform*. First let Ω be the (interior of the) *cone of squares*, i.e., the connected component of the unit e in the set of invertible elements

$$V^\times = \{x \in V \mid \det(x) \neq 0\}.$$

The set Ω is an open, convex, proper, generating, symmetric, homogeneous cone. In particular, let $L(\Omega)$ be the subgroup of linear transformations of V which preserve Ω. Then $L(\Omega)$ is a reductive group, which acts transitively on Ω. The same properties are true for its neutral component, which we denote by L. The stabilizer $K = L_e$ of the point e is a maximal compact subgroup of L and it is the neutral component of $\mathrm{Aut}(V)$, the automorphism group of the Jordan algebra V, and also $K = L \cap \mathrm{O}(V)$, where $\mathrm{O}(V)$ is the orthogonal group for the inner product on V. The space $\Omega \simeq L/K$ is a Riemannian symmetric space. It can be thought of as the noncompact dual of S.

Now form the tube over Ω, namely

$$T_\Omega = \{z = x + iy \in \mathbb{V} \mid y \in \Omega\}.$$

Next define

$$D(p) = \{z \in \mathbb{V} \mid \det(z + ie) \neq 0\}, \quad D(c) = \{w \in \mathbb{V} \mid \det(e - w) \neq 0\},$$

and for z in $D(p)$, w in $D(c)$

$$p(z) = (z - ie)(z + ie)^{-1}, \quad c(w) = i(e + w)(e - w)^{-1}. \tag{15}$$

Proposition 2.5. *The map p is a bijection of $D(p)$ onto $D(c)$, and c, called the Cayley transform, is its inverse. $D(p)$ contains $\overline{T_\Omega}$, the map p induces a biholomorphic isomorphism from T_Ω onto $\mathcal{D}$, and $p(V) = \{z \in S \mid \det(e - z) \neq 0\}$.*

See [F-K] Proposition X.2.3. Both domains T_Ω and $\mathcal{D}$ are holomorphically equivalent, and V can be thought of as the *Shilov boundary* of T_Ω. Its image under the Cayley transform is almost all of S. The complementary set $S \setminus p(V)$ is a "small" set, and corresponds to points at infinity in a compactification of V. This idea will be studied more systematically in the next section.

Let us denote by $G = G(\mathcal{D})$ the neutral component of the group of biholomorphic diffeomorphisms of $\mathcal{D}$. Equipped with the topology of uniform convergence on compact sets, G has a structure of a Lie group. It is a semisimple Lie group. The stabilizer of the element 0 in G is contained in the linear group $\mathrm{GL}(\mathbb{V})$, and can be shown to coincide with the group U. It is also a maximal compact subgroup of G.

To describe more accurately the group G, we use the Cayley transform. Let $G(T_\Omega)$ be the neutral component of the group of biholomorphic diffeomorphisms of T_Ω. The mapping $g \to c^{-1} \circ g \circ c$ is an isomorphism of G onto $G(T_\Omega)$.

Proposition 2.6. *The group $G(T_\Omega)$ is generated by the following transforms:*
(i) *the group N^+ of translations $t_v : z \mapsto z + v$ with $v \in V$,*
(ii) *the complexified action of the group L,*
(iii) *the inversion $s\colon z \mapsto -z^{-1}$.*

The subgroup L normalizes the group of translations. Moreover, the semidirect product $L \ltimes N^+$ is the neutral component of the subgroup of affine biholomorphic diffeomorphisms of T_Ω.

3. The 3-transitivity property on $\mathbb{X}$

Denote by $\mathbb{V}^\times$ the set of invertible elements,

$$\mathbb{V}^\times = \{z \in \mathbb{V} \mid \det(z) \neq 0\}.$$

The determinant (which is the holomorphic extension to $\mathbb{V}$ of the polynomial det on V) is semi-invariant under the action of $\mathbb{L}$:

$$\det(gz) = \chi(g)\det(z), \tag{16}$$

where χ is a certain character of $\mathbb{L}$. Moreover, this property characterizes the elements of the structure group. For $z \in \mathbb{V}^\times$, $P(z)$ is an element of $\mathbb{L}$ and

$$\chi(P(z)) = (\det z)^2. \tag{17}$$

Denote by $\mathbb{N}^+$ the group of translations $t_w : z \mapsto z + w$ for $w \in \mathbb{V}$ and by s the mapping, defined on $\mathbb{V}^\times$ by $z \mapsto -z^{-1}$. It is a rational map, and $\det(z)\, z^{-1}$ is in fact a polynomial mapping from $\mathbb{V}$ into $\mathbb{V}$. At a point $z \in \mathbb{V}^\times$ its differential is equal to $Ds(z) = P(z)^{-1}$. The subgroup $\mathbb{G} = \mathbb{C}\mathrm{o}(\mathbb{V})$ of rational transforms of $\mathbb{V}$ generated by $\mathbb{N}^+$, $\mathbb{L}$, and the map s is a closed (hence Lie) subgroup, called the conformal group of $\mathbb{V}$. If $g \in \mathbb{G}$, and if z is a point where g is defined, then the differential $Dg(z)$ belongs to $\mathbb{L}$ (this is essentially a characteristic property, see Liouville theorem in [B]). Define the scalar *automorphy factor* to be

$$j(g,z) = \chi(Dg(z)). \tag{18}$$

It satisfies the *cocycle property*

$$j(g_1g_2,z) = j(g_1,g_2(z))j(g_2,z). \tag{19}$$

Notice the following formulae:
(i) for $w \in \mathbb{V}, j(t_w,z) = 1$,
(ii) for $g \in \mathbb{S}\mathrm{tr}(\mathbb{V}), j(g,z) = \chi(g)$,
(iii) $j(s,z) = (\det z)^{-2}$.

The first two are obvious, whereas the third is a consequence of (17).

The subgroup $\mathbb{N}^+$ is invariant under the inner automorphisms associated to the elements of $\mathbb{S}\mathrm{tr}(\mathbb{V})$, and hence we can form the semidirect product

$$\mathbb{P}^+ = \mathbb{S}\mathrm{tr}(\mathbb{V}) \ltimes \mathbb{N}^+.$$

This is the subgroup of affine transformations in $\mathbb{G}$, and it is a maximal parabolic subgroup of $\mathbb{G}$. Hence the manifold $\mathbb{X} = \mathbb{G}/\mathbb{P}^+$ is compact, and the mapping $u \mapsto t_u \circ s$ is an imbedding of $\mathbb{V}$ into $\mathbb{X}$ with dense image. Any element $g \in \mathbb{G}$ which was defined as a birational map of $\mathbb{V}$ extends to a diffeomorphism of $\mathbb{X}$. In particular, this is true for the element s, and we define the "point at infinity" by $\infty = s(0)$. Let

$$\mathbb{P}^- = s \circ \mathbb{P}^+ \circ s.$$

Then $\mathbb{P}^-$ can be shown to be exactly the stabilizer of the point 0 in $\mathbb{G}$, whereas the stabilizer of ∞ is $\mathbb{P}^+$.

The transversality is known to be an important ingredient in the theory of the Maslov index. We define now this notion for points in $\mathbb{X}$. Let us first investigate this notion when both points are in $\mathbb{V}$.

Definition 1. Let $z, w \in \mathbb{V}$. Then the points z and w are transversal if one of the following equivalent conditions are satisfied:

(i) $\det(z - w) \neq 0$,
(ii) $\det P(z - w) \neq 0$.

Now recall the important Hua's formula.

Proposition 3.1. (Hua's formula) *Let $z, w \in \mathbb{V}^\times$. Then*

$$\det(s(z) - s(w)) = \det(z)^{-1} \det(z - w) \det(w)^{-1}. \tag{20}$$

Proof. See [F-K], Lemma X.4.4. □

Hua's formula shows the invariance of the notion of transversality under the conformal group.

Proposition 3.2. *Let $z, w \in \mathbb{V}$, and $g \in \mathbb{G}$, such that z and w are transversal, and g is defined at z and w. Then $g(z)$ and $g(w)$ are transversal.*

Proof. If g is a translation, the result is obvious. If $g \in \mathbb{S}\mathrm{tr}(\mathbb{V})$, then $\det(g(z) - g(w)) = \det(g(z - w)) = \chi(g) \det(z - w)$, so the the result again is true. Now if $g = s$, then Hua's formula shows that $\det(s(z) - s(w)) \neq 0$. As these elements generate the group $\mathbb{G}$, the theorem follows. □

This invariance property now allows us to extend the definition of transversality to the compactification $\mathbb{X}$. Namely, given two points z, w in $\mathbb{X}$, it is always possible to find an element $g_0 \in \mathbb{G}$ such that $g_0(z)$ and $g_0(w)$ both belong to $\mathbb{V}$. Then z, w are said to be transversal if $g_0(z)$ and $g_0(w)$ are transversal. Needless to say, one verifies that this condition is independent of the element of $\mathbb{G}$ used to send z and w in $\mathbb{V}$, as a consequence of the invariance result. For a more intrinsic point of view, see [Kh].

As for notation, we write $z \top w$ for a pair of transversal points in $\mathbb{X}$. For $z \in \mathbb{X}$, let

$$\mathbb{X}_z = \{w \in \mathbb{X} \mid w \top z\}.$$

Notice in particular that $\mathbb{X}_\infty = \mathbb{V}$ and that $\mathbb{X}_\infty \cap \mathbb{X}_0 = \mathbb{V}^\times$.

Let $\mathbb{X}^2_\top$ be the open set of transversal points in $\mathbb{X} \times \mathbb{X}$. Clearly $\mathbb{G}$ preserves $\mathbb{X}^2_\top$.

Proposition 3.3. *The group* $\mathbb{G}$ *is transitive on* $\mathbb{X}^2_\top$. *The stabilizer in* $\mathbb{G}$ *of the element* $(0,\infty)$ *is the subgroup* $\mathbb{S}\mathrm{tr}(\mathbb{V})$.

Let

$$\mathbb{X}^3_\top = \{(z_1, z_2, z_3) \in \mathbb{X}\times\mathbb{X}\times\mathbb{X} \mid z_1 \top z_2, z_2 \top z_3, z_3 \top z_1\}.$$

Again $\mathbb{G}$ preserves $\mathbb{X}^3_\top$.

Proposition 3.4. *The group* $\mathbb{G}$ *acts transitively on* $\mathbb{X}^3_\top$. *The stabilizer of the element* $(0,1,\infty)$ *is the automorphism group of* $\mathbb{V}$.

This just reflects the fact that the structure group is transitive on the set of invertible elements (see [F-K], Proposition VIII.3.5). The stabilizer of the unit e in $\mathbb{S}\mathrm{tr}(\mathbb{V})$ is the automorphism group of $\mathbb{V}$.

4. The open orbits in $S^3_\top$

We now want to study a similar problem for the action of G on the Shilov boundary S. The transversality condition for $\sigma, \zeta \in S$ amounts to $\sigma \top \zeta \iff \det(\sigma - \zeta) \neq 0$. From the previous construction and result, we know that this notion is invariant under the action of G. Let

$$S^2_\top = \{(\sigma_1, \sigma_2) \in S \times S \mid \sigma_1 \top \sigma_2\}.$$

Theorem 4.1. *The group* G *operates transitively on* $S^2_\top$. *The stabilizer of the element* $(e, -e)$ *in* G *is the group* $c^{-1} \circ L \circ c$.

Proof. Let $(\sigma, \zeta) \in S$, with $\sigma \top \zeta$. As the group U is transitive on S, we may assume that $\sigma = e$. Next we observe that $\zeta \top e$ implies $\zeta \in D(p)$, and in fact $p(V)$ is exactly the set of transversal elements to e. Now $c(\zeta)$ is an element of V and by an appropriate translation it can be mapped to $0 = c(-e)$. By using the inverse Cayley transform, we get the first statement. For the second statement, we can also transform the problem by using the Cayley transform. Then we look for the stabilizer of 0 in the subgroup of affine transforms in $G(T_\Omega)$. This gives the subgroup of linear transformations in $G(T_\Omega)$ which is equal to L. $\square$

Now let

$$S^3_\top = \{(\sigma_1, \sigma_2, \sigma_3) \in S \times S \times S \mid \sigma_i \top \sigma_j, 1 \le i \neq j \le 3\}.$$

Choose a Peirce decomposition $e = \sum_{j=1}^r c_j$, and for $j = 0, 1, \dots, r$, let

$$\varepsilon_0 = -e, \quad \varepsilon_j = \sum_{i=1}^j c_i - \sum_{i=j+1}^r c_i, \quad \varepsilon_r = e.$$

The space

$$\mathfrak{a} = \bigoplus_{j=1}^r \mathbb{R}L(c_j)$$

is a Cartan subspace of $\mathfrak{l} = \mathrm{Lie}(L) = \mathrm{Lie}(L(\Omega))$. Let

$$A = \exp \mathfrak{a} = \{P(a) \mid a = \sum_{j=1}^r \lambda_j c_j, \forall j, 1 \le j \le r, \lambda_j > 0\}.$$

Proposition 4.2. *There are exactly $r+1$ orbits in $V^\times$ under the action of L. The elements $\varepsilon_j, 0 \le j \le r$, are a set of representatives of all the orbits.*

The result is presumably well known, but for lack of a firm reference let us give a proof. Let x be any element of V. Let $(\lambda_j)_{1\le j\le r}$ be its spectral values, repeated as many times as their multiplicity. Then x is conjugate under $K = \mathrm{Aut}(V)_0$ to $a = \sum_{j=1}^r \lambda_j c_j$. If moreover x is invertible, all λ_j are different from 0. Hence we may separate negative and positive eigenvalues, say $a = \sum_{j=1}^k \lambda_j c_j - \sum_{j=k+1}^r |\lambda_j| c_j$. Now using the action of A, it is possible to map the element a on ε_k. So any element of $V^\times$ is indeed conjugate to one of the ε_j. Conversely, define the signature of an invertible element to be the number of positive spectral values minus the number of negative eigenvalues. The signature is clearly invariant under any automorphism of V, and it is also invariant under the action of $P(x), x \in \Omega$, because Ω is arcwise connected. But now by using the Cartan decomposition of L, any element of $g \in L$ can be written as $g = kP(x)$, where $k \in K$ and $x \in \Omega$, and hence g preserves the signature.

Theorem 4.3. *There are exactly $r+1$ orbits in $S^3_\top$ under the action of G. The family $(e, -e, -i\varepsilon_j)$, $0 \le j \le r$, is an exhaustive family of representatives of the orbits.*

Proof. Given any triplet of transversal points $(\sigma_1, \sigma_2, \sigma_3)$, we have already seen that it is possible to map σ_1 to e and σ_2 to $-e$. Then we apply the Cayley transform to the situation, and use Proposition 4.2 to map $c(\sigma_3)$ to some ε_j. Now an elementary computation shows that $p(\varepsilon_j) = -i\varepsilon_j$. So the result is obtained by using back the Cayley transform.

Let $(\sigma_1, \sigma_2, \sigma_3) \in S^3_\top$. Then define the *Maslov index* of the triplet $(\sigma_1, \sigma_2, \sigma_3)$ to be

$$\iota(\sigma_1, \sigma_2, \sigma_3) = k - (r-k) = 2k - r \tag{21}$$

where k is the unique integer, $0 \le k \le r$, such that $(\sigma_1, \sigma_2, \sigma_3)$ is conjugate under G to the triplet $(e, -e, -i\varepsilon_k)$. $\square$

5. The Maslov index and the automorphy kernel

For $z, w \in \mathbb{V}$, set

$$K(z,w) = (I - 2z\Box\overline{w} + P(z)P(\overline{w})).$$

The operator-valued function $K(z,w)$, when restricted to $\mathcal{D}$ coincides with the *canonical automorphy kernel* of the Hermitian domain $\mathcal{D}$ (in its action on $\mathfrak{p}^+ \simeq \mathbb{V}$, see [Sa]), and for $z, w \in \mathcal{D}$ it is invertible and belongs to $\mathbb{S}\mathrm{tr}(\mathbb{V})$. It is clearly holomorphic in z and antiholomorphic in w, satisfies the symmetry property $K(z,w)^* = K(w,z)$, and the transformation property

$$K(g(z), g(w)) = J(g,z)K(z,w)J(g,w)^* \tag{22}$$

for $g \in G$ and $z, w \in \mathcal{D}$ (see [Sa], Chapter 2, Lemma 5.2).

Define, for $z, w \in \mathcal{D}$ [2]

$$k(z,w) = \chi(K(z,w)).$$

[2]The definition we used in Section 1 for the automorphy kernel was slightly different. The present kernel is the square of the former. This is responsible for the factor $\frac{1}{\pi}$ in (8) instead of $\frac{1}{2\pi}$ in formula (26). A similar remark applies to the automorphy factor.

This satisfies the transformation property

$$k(g(z), g(w)) = j(g,z)k(z,w)\overline{j(g,w)}, \tag{23}$$

for $g \in G$ and $z, w \in \mathcal{D}$.

The quantity $k(z,z)$ is a positive real number for $z \in \mathcal{D}$, and more precisely, if $z = u\sum_{j=1}^{r} \lambda_j c_j$, with $u \in U$ and $0 \leq \lambda_j < 1$, then

$$k(z,z) = \prod_{j=1}^{r}(1-\lambda_j^2)^2. \tag{24}$$

(cf. [F-K], Proposition X.4.5).

As $\mathcal{D} \times \mathcal{D}$ is simply connected, there is a unique continuous determination of the argument of $k(z,w)$ which is 0 on the diagonal $\{z = w\}$, which we denote by arg $k(z,w)$.

For $z_1, z_2, z_3 \in \mathcal{D}$, let

$$c(z_1,z_2,z_3) = k(z_1,z_2)k(z_2,z_1)^{-1}k(z_2,z_3)k(z_3,z_2)^{-1}k(z_3,z_1)k(z_1,z_3)^{-1}. \tag{25}$$

Again, as $\mathcal{D} \times \mathcal{D} \times \mathcal{D}$ is simply connected, there exists a unique continuous determination of the argument of $c(z_1,z_2,z_3)$ which is 0 on $\{z_1 = z_2 = z_3\}$. From (23) we deduce that arg c is invariant under the action of G, namely arg $c(g(z_1), g(z_2), g(z_3)) =$ arg $c(z_1,z_2,z_3)$. In fact, as the function c is invariant, both sides are continuous determinations of the argument of the same function on $\mathcal{D} \times \mathcal{D} \times \mathcal{D}$, and they coincide on the diagonal $\{z_1 = z_2 = z_3\}$, hence everywhere.

Lemma 5.1. *Let $\sigma_1, \sigma_2 \in S$, and assume σ_1 and σ_2 are transversal. Then $K(\sigma_1,\sigma_2)$ is invertible.*

Proof. Taking advantage of the transformation law for K, we may assume $\sigma_1 = e$, and $\sigma_2 = \sum_{j=1}^{r} \zeta_j c_j$, with $|\zeta_j| = 1$ and $\zeta_j \neq 1$ for all $j, 1 \leq j \leq r$ (see Proposition 2.1(v)). Then $K(e,\sigma_2) = \operatorname{Id} - 2L(\overline{\sigma_2}) + 2L(\overline{\sigma_2})^2 - L(\overline{\sigma_2}^2)$ is a diagonal operator with respect to the (complexified) Peirce decomposition of the Jordan algebra $\mathbb{V} = \bigoplus_{1\leq i\leq j\leq r} \mathbb{V}_{i,j}$ and the eigenvalue on $\mathbb{V}_{i,j}$ is $1-(\overline{\zeta}_i+\overline{\zeta}_j)+2((\overline{\zeta}_i+\overline{\zeta}_j)/2)^2-(\overline{\zeta}_i^2+\overline{\zeta}_j^2)/2 = (1-\overline{\zeta}_i)(1-\overline{\zeta}_j) \neq 0$, hence the result. □

From the lemma it follows that the function k can be continuously extended, with values in $\mathbb{C}^*$, to points (σ_1, σ_2), provided σ_1 and σ_2 are transversal. In turn, this implies that the function $c(z_1,z_2,z_3)$ can be extended continuously (with values in $\mathbb{C}^*$) to $S^3_\top$, and the same is true for arg $c(z_1,z_2,z_3)$. So now define

$$\gamma(\sigma_1,\sigma_2,\sigma_3) = \frac{1}{2\pi}\lim \arg c(z_1,z_2,z_3) \tag{26}$$

when $\mathcal{D} \ni z_1 \to \sigma_1$, $\mathcal{D} \ni z_2 \to \sigma_2$ and $\mathcal{D} \ni z_3 \to \sigma_3$. By construction, γ is clearly invariant under G.

Theorem 5.2. *Let $(\sigma_1,\sigma_2,\sigma_3) \in S^3_\top$. Then $\iota(\sigma_1,\sigma_2,\sigma_3) = \gamma(\sigma_1,\sigma_2,\sigma_3)$.*

Proof. Taking advantage of the invariance of γ and ι under G, we may assume that $\sigma_1 = e, \sigma_2 = -e$ and $\sigma_3 = -i\varepsilon_k$ for some $k, 0 \leq k \leq r$, and we need to compute $\gamma(e, -e, -i\varepsilon_k)$. □

Lemma 5.3. *Let* $0 \leq k \leq r$. *Then*

$$\gamma(e, -e, -i\varepsilon_k) = 2k - r. \tag{27}$$

The proof requires the following lemma.

Lemma 5.4. *Let* $(\zeta_j), (\xi_j)$ *for* $1 \leq j \leq r$ *be* r*-uples of complex numbers of a modulus strictly less than* 1. *Then*

$$k(\textstyle\sum_{j=1}^r \zeta_j c_j, \sum_{j=1}^r \xi_j c_j) = \prod_{j=1}^r (1 - \zeta_j \overline{\xi_j})^2. \tag{28}$$

Proof. As both expressions are holomorphic in the ζ_j and antiholomorphic in the ξ_j, it is enough to prove the formula when $\zeta_j = \xi_j$ for all $j, 1 \leq j \leq r$. But then the result is a consequence of (24). □

For the proof of Lemma 5.3 we need to compute $\lim_{t_1 \uparrow 1, t_2 \uparrow 1, t_3 \uparrow 1} \arg c\,(t_1 e, t_2 e, -it_3 \varepsilon_k)$. Now c can be easily computed, using Lemma 5.4. For instance

$$k(-t_2 e, -it_3\varepsilon_k) k(-it_3\varepsilon_k, -t_2 e)^{-1} = \left(\prod_{j=1}^{k} \frac{1 + t_2(it_3)}{1 - t_2(it_3)} \prod_{j=k+1}^{r} \frac{1 + t_2(-it_3)}{1 + t_2(it_3)} \right)^2.$$

As $t \to 1$, $\arg \frac{1+it}{1-it} \to \frac{\pi}{2}$. Hence, as $t_1, t_2, t_3 \to 1$,

$$\arg k(-t_2 e, -it_3\varepsilon_k) k(-it_3\varepsilon_k, -t_2 e)^{-1} \to 2\big(k - (r-k)\big)\frac{\pi}{2} = (2k - r)\pi.$$

The other computations are similar. □

Theorem 5.5. *The Maslov index* ι *is skew-invariant:*

$$\iota(\zeta_{\tau(1)}, \zeta_{\tau(2)}, \zeta_{\tau(3)}) = \operatorname{sgn}(\tau)\, \iota(\zeta_1, \zeta_2, \zeta_3)$$

for any permutation τ *of* $\{1, 2, 3\}$.

Let us prove the result for the transposition which exchanges 1 and 2. Let z_1, z_2, z_3 be three points in $\mathcal{D}$. Then the following identity is easily verified: $c(z_2, z_1, z_3) = c(z_1, z_2, z_3)^{-1}$ and hence

$$\arg c(z_2, z_1, z_3) = -\arg c(z_1, z_2, z_3). \tag{29}$$

Let $(\zeta_1, \zeta_2, \zeta_3) \in S^3_{\top}$. Let (z_1, z_2, z_3) tend to $(\zeta_1, \zeta_2, \zeta_3) \in S^3_{\top}$. Then (29) becomes $\iota(\zeta_2, \zeta_1, \zeta_3) = -\iota(\zeta_1, \zeta_2, \zeta_3)$. The proof is similar for the two other transpositions.

Theorem 5.6. *Let* $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ *be four points of* S, *such that* $\zeta_i \top \zeta_j$ *for any* $i, j, 1 \leq i, j \leq 4, i \neq j$. *Then*

$$\iota(\zeta_1, \zeta_2, \zeta_3) = \iota(\zeta_1, \zeta_2, \zeta_4) + \iota(\zeta_2, \zeta_3, \zeta_4) + \iota(\zeta_3, \zeta_1, \zeta_4). \tag{30}$$

Proof. Let z_1, z_2, z_3, z_4 be four points in $\mathcal{D}$. Then, from the definition of the function c, it is easy to verify the similar cocycle property

$$c(z_1, z_2, z_3) = c(z_1, z_2, z_4) c(z_2, z_3, z_4) c(z_3, z_1, z_4).$$

With the choice of the determination of the argument we have described, it is fairly obvious that we also get $\arg c(z_1, z_2, z_3) = \arg c(z_1, z_2, z_4) + \arg c(z_2, z_3, z_4) + \arg c(z_3, z_1, z_4)$. Then (30) is obtained by letting $z_j \to \zeta_j$, $1 \leq j \leq 4$. □

6. A functorial property of the Maslov index

Let V_1, V_2 be two Euclidean Jordan algebras, which as before are assumed to be simple. We use same notation as before with a subscript 1 or 2. For instance, let r_1 (resp. r_2) be the rank of V_1 (resp. V_2). Recall that a linear map $\Phi : V_1 \to V_2$ is said to be a *Jordan algebra homomorphism* if

$$\Phi(xy) = \Phi(x)\Phi(y), \quad \Phi(e_1) = e_2 \quad \text{for all } x, y \in V_1.$$

Lemma 6.1. *Let $\Phi : V_1 \to V_2$ be a Jordan algebra homomorphism. Then r_2/r_1 is an integer and*

$$\det_2 \Phi(x) = (\det_1 x)^{r_2/r_1} \tag{31}$$

for all $x \in V_1$.

Proof. Let $x \in V_1$ be invertible. Then there exists by definition a polynomial $p \in \mathbb{R}[X]$ such that $xp(x) = e_1$. But as Φ is a Jordan algebra homomorphism, $\Phi(q(x)) = q(\Phi(x))$ for any polynomial $q \in \mathbb{R}[X]$. Hence $\Phi(x)p(\Phi(x)) = e_2$, which shows that $\Phi(x)$ is invertible. Conversely, assume that $\Phi(x)$ is invertible. Then there exists $p \in \mathbb{R}[X]$ such that $\Phi(x)p(\Phi(x)) = e_2$, which implies $\Phi(xp(x)) = e_2 = \Phi(e_1)$. The kernel of Φ is a proper ideal of V_1, so it is $\{0\}$ as V_1 is assumed to be simple. Hence Φ is injective, and $xp(x) = e_1$, proving that x is invertible. The sets $\{x \in V_1 \mid \det_2 \Phi(x) \neq 0\}$ coincides with the set $\{x \in V_1 \mid \det_1 x \neq 0\}$. But as V is assumed to be simple, the polynomial $\det_2$ is absolutely irreducible (cf. [Sp]), and hence there exists an integer ν and a real number A such that $\det_2 \Phi(x) = A\,(\det_1 x)^\nu$, $\forall x \in V_1$. Checking on $x = e_1$ gives $A = 1$. Now $\det_1$ and $\det_2$ are homogeneous polynomials of degree, respectively r_1 and r_2. Hence $\nu = r_2/r_1$. □

Proposition 6.2. *Under the previous assumptions, the $\mathbb{C}$-linear extension of Φ maps $\mathcal{D}_1$ into $\mathcal{D}_2$, and the corresponding automorphy kernels satisfies the relation*

$$\forall z, w \in \mathcal{D}_1, \quad k_2(\phi(z), \phi(w)) = k_1(z, w)^{r_2/r_1}. \tag{32}$$

Proof. Because it is easier, we first derive a similar formula for the tube, and then use the Cayley transform to get the result. As $\Phi(x^2) = \Phi(x)^2$ for any $x \in V_1$, Φ maps the cone Ω_1 into the cone Ω_2, and the $\mathbb{C}$-linear extension of Φ (still denoted by Φ) maps the tube T_{Ω_1} into the tube T_{Ω_2}. For $z, w \in T_{\Omega_1}$, notice that $(z - \overline{w})/2i$ is invertible, and the kernel $l_1(z, w) = \chi_1(P((z - \overline{w})/2i))$ is well defined. It is the analog for the tube T_{Ω_1} of the automorphy kernel for $\mathcal{D}_1$. The same observations are valid for T_{Ω_2}. Using (17) and (31), we get $\chi_2(P((\Phi(z) - \Phi(\overline{w}))/2i)) = \chi_2(P(\Phi((z - \overline{w})/2i))) = \det_2(\Phi((z - \overline{w})/2i))^2 = \det_1((z - \overline{w})/2i)^{2r_2/r_1} = \chi_1(P((z - \overline{w})/2i))^{r_2/r_1}$, hence

$$l_2(\Phi(z), \Phi(w)) = l_1(z, w)^{r_2/r_1}. \tag{33}$$

Next, we transfer these results by Cayley transform to the bounded domains $\mathcal{D}_1$ and $\mathcal{D}_2$. Let c_1 and c_2 be the Cayley transforms as in (15). Then, as Φ is a Jordan algebra homomorphism, $c_2 \circ \Phi = \Phi \circ c_1$ where defined. As Φ maps T_{Ω_1} into T_{Ω_2} Φ maps $\mathcal{D}_1$ into $\mathcal{D}_2$.

Going back to general notation, the automorphy kernel for $\mathcal{D}$ satisfies the relation

$$K(z,w) = P(e-z)^{-1}P\Big(\frac{1}{2i}\big(c(z)-\overline{c(w)}\big)\Big)\,\overline{P(e-w)^{-1}} \tag{34}$$

for $z, w \in \mathcal{D}$ (cf. [FK], Lemma X.4.4). Observe that for $z \in \mathcal{D}$, $e - z$ is invertible. Taking the image under the character χ and using again (17) gives

$$k(z,w) = \det(e-z)^{-2}\; l(c(z), c(w))\; \overline{\det(e-w)^{-2}}. \tag{35}$$

These remarks are valid for both V_1 and V_2. Using again (31), we get from (33) the desired relation (32). □

Theorem 6.3. *Under the same assumptions as above, Φ maps S_1 into S_2, preserves transversality, and the following relation is satisfied:*

$$\iota_2(\Phi(\sigma_1), \Phi(\sigma_2), \Phi(\sigma_3)) = \frac{r_2}{r_1}\,\iota_1(\sigma_1, \sigma_2, \sigma_3) \tag{36}$$

for all $\sigma_1, \sigma_2, \sigma_3$ mutually transverse in S_1.

Using the characterization of the Shilov boundary *(iv)* in Proposition 2.1, it is easy to see that Φ maps S_1 into S_2. The fact that Φ is a Jordan algebra homomorphism implies that transversality is preserved. Finally (36) is obtained from (32) by taking the limit as the points ξ_j tend to σ_j, $1 \le j \le 3$.

Appendix: List of tube type domains and their Shilov boundaries (cf. [F-K], Section X)

V	$\mathbb{V}$	$\mathcal{D} \simeq G/U$	S
$\mathrm{Sym}(r,\mathbb{R})$	$\mathrm{Sym}(r,\mathbb{C})$	$\mathrm{Sp}(2r,\mathbb{R})/U(r)$	$\mathrm{U}(r)/\mathrm{O}(r)$
$\mathrm{Herm}(r,\mathbb{C})$	$\mathrm{Mat}(r,\mathbb{C})$	$\mathrm{SU}(r,r)/\mathrm{S}(\mathrm{U}(r)\times\mathrm{U}(r))$	$\mathrm{U}(r)$
$\mathrm{Herm}(r,\mathbb{H})$	$\mathrm{Skew}(2r,\mathbb{C})$	$\mathrm{SO}^*(4r)/\mathrm{U}(2r)$	$\mathrm{U}(2r)/\mathrm{SU}(r,\mathbb{H})$
$\mathbb{R}\times\mathbb{R}^{q-1}$	$\mathbb{C}\times\mathbb{C}^{q-1}$	$\mathrm{SO}_0(2,q)/\mathrm{SO}(2)\times\mathrm{SO}(q)$	$(\mathrm{U}(1)\times S^{q-1})/\mathbb{Z}_2$
$\mathrm{Herm}(3,\mathbb{O})$	$\mathrm{Mat}(3,\mathbb{O})$	$\mathrm{E}_{7(-25)}/\mathrm{U}(1).\mathrm{E}_6$	$\mathrm{U}(1).\mathrm{E}_6/\mathrm{F}_4$

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