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## ON THE INDEX OF A FIBERED MANIFOLD<sup>1</sup>

S. S. CHERN, F. HIRZEBRUCH, AND J-P. SERRE

**Introduction.** Let  $V$  be a real vector space of dimension  $r$ . Let  $F(x, y) = \langle x, y \rangle$ ,  $x, y \in V$ , be a real-valued symmetric bilinear function. We can find a base  $e_i$ ,  $1 \leq i \leq r$ , in  $V$ , such that

$$(1) \quad F(x, y) = \sum_{i=1}^p x^i y^i - \sum_{i=p+1}^{p+q} x^i y^i$$

where  $x = \sum_{i=1}^r x^i e_i$  and  $y = \sum_{i=1}^r y^i e_i$ .

The number  $p - q$  is called the index of  $F$ , to be denoted by  $\tau(F)$ . It depends only on  $F$ . If  $F$  is nonsingular (i.e.  $p + q = r$ ), then  $\min(p, q)$  equals the maximal dimension of the linear subspaces of  $V$  contained in the "cone"  $F(x, x) = 0$ .

Now let  $M$  be a compact oriented manifold. The index of  $M$  is defined to be zero, if the dimension of  $M$  is not a multiple of 4. If  $M$  has the dimension  $4k$ , consider the cohomology group  $H^{2k}(M)$  with real coefficients. This is a real vector space, and the equation

$$(2) \quad \langle x, y \rangle \xi = x \cup y, \quad x, y \in H^{2k}(M),$$

where  $\xi$  is the generator of  $H^{4k}(M)$  defined by the given orientation of  $M$ , defines a real-valued symmetric bilinear form  $\langle x, y \rangle$  over  $H^{2k}(M)$ . Its index is called the index of  $M$ , to be denoted by  $\tau(M)$ . Reversal of the orientation of  $M$  changes the sign of the index. The form  $\langle x, y \rangle$  defined by (2) is nonsingular, since, by Poincaré's duality theorem, the equation  $x \cup y = 0$  for all  $x \in H^{2k}(M)$  implies  $y = 0$ .

The main purpose of this paper is to prove the theorem:

**THEOREM.** *Let  $E \rightarrow B$  be a fiber bundle, with the typical fiber  $F$ , such that the following conditions are satisfied:*

- (1)  *$E, B, F$  are compact connected oriented manifolds;*
- (2) *The fundamental group  $\pi_1(B)$  acts trivially on the cohomology ring  $H^*(F)$  of  $F$ .*

*Then, if  $E, B, F$  are oriented coherently, so that the orientation of  $E$  is induced by those of  $F$  and  $B$ , the index of  $E$  is the product of the indices of  $F$  and  $B$ , that is,*

$$\tau(E) = \tau(F)\tau(B).$$

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REMARK. We do not know whether condition (2) and the connect-  
edness hypothesis of condition (1) are necessary. For instance, let  $E$   
be an  $n$ -sheeted covering of  $B$  (the spaces  $B$  and  $E$  still being compact  
oriented manifolds); is it true that  $\tau(E) = n\tau(B)$ ? We know the  
answer to be positive only when  $B$  possesses a differentiable structure:  
in that case, according to a theorem of one of us,  $\tau(B)$  (resp.  $\tau(E)$ )  
is equal to the Pontrjagin number  $L(B)$  (resp.  $L(E)$ ) and it is clear  
that  $L(E) = n \cdot L(B)$ .

1. **Algebraic properties of the index of a matrix.** Let  $e_i, 1 \leq i \leq r$ ,  
be a base in  $V$ . A real-valued symmetric bilinear function  $\langle x, y \rangle$  de-  
fines a real-valued symmetric matrix  $C = (c_{ij}), c_{ij} = \langle e_i, e_j \rangle, 1 \leq i, j \leq r$ ,  
and is determined by it. The index of the bilinear function is equal  
to the index  $\tau(C)$  of  $C$ , if we define the latter to be the excess of  
the number of positive eigenvalues over the number of negative eigen-  
values of  $C$ , each counted with its proper multiplicity. We have the  
following properties of the index of a real symmetric matrix:

For a nonsingular  $(r \times r)$ -matrix  $T$  we have

$$(3) \quad \tau(C) = \tau({}^tTCT).$$

Here, as always, we denote by  ${}^tT$  the transpose of  $T$ . For nonsingular  
square matrices  $A, L$  (with  $A$  symmetric) we have

$$(4) \quad \tau \begin{pmatrix} 0 & 0 & L \\ 0 & A & 0 \\ {}^tL & 0 & 0 \end{pmatrix} = \tau \begin{pmatrix} 0 & L \\ {}^tL & 0 \end{pmatrix} + \tau(A) = \tau(A).$$

Here and always we make use of the convention that the index of  
the empty matrix is zero.

To prove (4) it is enough to show that

$$(5) \quad \tau \begin{pmatrix} 0 & L \\ {}^tL & 0 \end{pmatrix} = 0.$$

In this case,  $r$  is even. Put  $r = 2\mu$ . Obviously, the cone  $F(x, x) = 0$  of  
the symmetric bilinear function  $F(x, y)$  belonging to the matrix

$$\begin{pmatrix} 0 & L \\ {}^tL & 0 \end{pmatrix}$$

contains a linear space of dimension  $\mu$ . Thus  $\min(p, q) \geq \mu$ . On the  
other hand,  $p + q = 2\mu$ . Therefore,  $p = q$  and  $\tau = 0$ .

LEMMA 1. *Let  $C$  be a real, symmetric, nonsingular matrix of the form*

$$C = \begin{pmatrix} 0 & & L_0 \\ & \ddots & \\ L_m & & * \end{pmatrix}$$

where  $L_0, \dots, L_m$  are square matrices (empty matrices are admitted) and where  $L_i$  is the transpose of  $L_{m-i}$ . Then

$$\tau(C) = \tau \begin{pmatrix} 0 & & L_0 \\ & \ddots & \\ L_m & & 0 \end{pmatrix} = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \tau(L_n), & \text{if } m = 2n. \end{cases}$$

PROOF. We put

$$(6) \quad C_\lambda = \begin{pmatrix} 0 & & L_0 \\ & \ddots & \\ L_m & & \lambda * \end{pmatrix}, \quad 0 \leq \lambda \leq 1.$$

Since  $\det(C_\lambda) = \pm \prod_{i=0}^m \det(L_i) \neq 0$ , the index  $\tau(C_\lambda)$  is obviously independent of  $\lambda$ , so that  $\tau(C) = \tau(C_1) = \tau(C_0)$ . By (4) we have  $\tau(C_0) = 0$  resp.  $\tau(C_0) = \tau(L_n)$ , q.e.d.

LEMMA 2. Let  $A$  and  $B$  be two square matrices, which are either both symmetric or both skew-symmetric. Then their tensor product  $A \otimes B$  is symmetric, and

$$(7) \quad \tau(A \otimes B) = \tau(A)\tau(B) \text{ or } 0,$$

according as both  $A$  and  $B$  are symmetric or skew-symmetric.

Suppose first that  $A$  and  $B$  are both symmetric. Let  $\alpha_i > 0, \alpha_j < 0, 1 \leq i \leq p, p+1 \leq j \leq p+q$ , be the nonzero eigenvalues of  $A$  and  $\beta_k > 0, \beta_l < 0, 1 \leq k \leq p', p'+1 \leq l \leq p'+q'$  be the nonzero eigenvalues of  $B$ . Then the nonzero eigenvalues of  $A \otimes B$  are  $\alpha_u \beta_t, 1 \leq u \leq p+q, 1 \leq t \leq p'+q'$ . It follows that

$$\tau(A \otimes B) = p p' + q q' - p q' - p' q = \tau(A)\tau(B).$$

Now let  $A$  and  $B$  be both skew-symmetric. By applying (3) to the matrix  $C = A \otimes B$  we can suppose that  $A$  and  $B$  are both of the form

$$\begin{pmatrix} A_1 & & & 0 \\ & \ddots & & \\ & & A_n & \\ 0 & & & 0 \end{pmatrix}$$

where each  $A_i$  is a  $2 \times 2$  block:

$$A_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J.$$

Since

$$\tau \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \tau \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} = 0,$$

we have  $\tau(A \otimes B) = 0$ .

**2. Poincaré rings.** We consider a graded ring  $A$  with the following properties:

(1) In the direct sum decomposition

$$A = \sum_{0 \leq r < \infty} A^r$$

of  $A$  into the subgroups of its homogeneous elements, each  $A^r$  is a real vector space of finite dimension. There exists an  $n$  with  $A^r = 0$  for  $r > n$  and with  $\dim A^n = 1$ .

(2) If  $x \in A^i$ ,  $y \in A^j$  then  $xy \in A^{i+j}$  and

$$xy = (-1)^{ij}yx.$$

Let  $\xi \neq 0$  be a base element of  $A^n$ . Relative to  $\xi$  we define a bilinear pairing  $\langle x, y \rangle$  of  $A^r$  and  $A^{n-r}$  into the real field by the equation

$$\langle x, y \rangle \xi = xy, \quad x \in A^r, y \in A^{n-r}.$$

Let  $i_{n-r}$  be the linear mapping of  $A^{n-r}$  into  $(A^r)^*$ , the dual vector space of  $A^r$ , which assigns to  $y \in A^{n-r}$  the linear function  $\langle x, y \rangle$  on  $A^r$  ( $x \in A^r$ ).

A graded ring  $A$  is called a Poincaré ring if it satisfies (1), (2) and has moreover the following property:

(3) The mapping  $i_{n-r}$  is a bijection of  $A^{n-r}$  onto  $(A^r)^*$ .

A consequence of (3) is

$$\dim A^r = \dim A^{n-r}, \quad 0 \leq r \leq n.$$

The cohomology ring of a compact orientable manifold is a Poincaré ring.

A differentiation in a Poincaré ring  $A$  is a linear endomorphism  $d: A \rightarrow A$ , satisfying the following conditions:

( $\alpha$ )  $dA^r \subset A^{r+1}$ ;

( $\beta$ )  $dd = 0$ ;

( $\gamma$ )  $d(xy) = (dx)y + (-1)^r x(dy)$ , if  $x \in A^r$ ;

( $\delta$ )  $dA^{n-1} = 0$ .

As is well known, such a differentiation defines a derived ring  $A' = d^{-1}(0)/dA$ . If we put  $A'^r = d^{-1}(0) \cap A^r/dA^{r-1}$ , we have the direct sum decomposition

$$A' = \sum_{0 \leq r \leq n} A'^r,$$

and  $A'$  is a graded ring. It is easy to verify that, if  $x' \in A'^i, y' \in A'^j$ , then  $x'y' \in A'^{i+j}$ , and

$$x'y' = (-1)^{ij}y'x'.$$

From the property ( $\delta$ ) of  $d$  we have  $\dim A'^n = 1$ . Thus  $A'$  satisfies (1) and (2) with the same maximal degree  $n$  as  $A$ . We denote the residue class of  $\xi$  in  $A'^n$  by  $\xi'$ . Relative to  $\xi'$  we have the linear mapping

$$i'_{n-r}: A'^{n-r} \rightarrow (A'^r)^*.$$

LEMMA 3. *The derived ring of a Poincaré ring with differentiation is a Poincaré ring, i.e.  $i'_{n-r}$  is bijective.*

It remains to prove that  $A'$  has the property (3) in the definition of a Poincaré ring. Let  $x \in A^r, y \in A^{n-r-1}$ . By property ( $\delta$ ) of  $d$ , we have

$$0 = d(xy) = (dx)y + (-1)^r x(dy).$$

This gives

$$(8) \quad \langle dx, y \rangle = (-1)^{r-1} \langle x, dy \rangle,$$

a relation which is independent of the choice of  $\xi$ . This relation is equivalent to saying that the following diagram is commutative:

$$\begin{array}{ccccc} A^{n-r-1} & \xrightarrow{d} & A^{n-r} & \xrightarrow{d} & A^{n-r+1} \\ \downarrow i_{n-r-1} & & \downarrow i_{n-r} & & \downarrow i_{n-r+1} \\ (A^{r+1})^* & \xrightarrow{(-1)^{r-1}({}^t d)} & (A^r)^* & \xrightarrow{(-1)^r({}^t d)} & (A^{r-1})^* \end{array}$$

where  $(A^r)^*$  is the dual space of  $A^r$ , and  ${}^t d$  is the dual homomorphism of  $d$ . We have the canonical isomorphism

$$(A'^r)^* \cong {}^t d^{-1}(0) \cap (A^r)^*/{}^t d(A^{r+1})^*.$$

The above diagram shows that  $i_{n-r}$  induces an isomorphism, namely  $i'_{n-r}$ , of  $A'^{n-r}$  onto  $(A'^r)^*$ . It follows that  $A'^r$  and  $A'^{n-r}$  are dually paired into the real field relative to the element  $\xi' \in A'^n$ , which is the residue class of  $\xi$ .

In analogy with the index of an oriented manifold we can define the index  $\tau_\xi(A)$  of our Poincaré ring  $A$  relative to  $\xi$ . It is to be zero, if  $n \equiv 0, \text{ mod } 4$ . If  $n = 4k$ ,  $\tau_\xi(A)$  is to be the index of the bilinear function  $\langle x, y \rangle$ ,  $x, y \in A^{2k}$ . Obviously,  $\tau_\xi(A) = \tau_{\xi_1}(A)$ , if  $\xi_1$  is a positive multiple of  $\xi$ .

LEMMA 4. *In a Poincaré ring  $A$  let  $\xi \neq 0$  be a base of  $A^n$ , and let  $\xi' \in A^n$  be the residue class which contains  $\xi$ . Then  $\tau_{\xi'}(A') = \tau_\xi(A)$ .*

It is only necessary to prove the lemma for the case  $n = 4k$ . Let  $Z^{2k} = d^{-1}(0) \cap A^{2k}$ ,  $B^{2k} = dA^{2k-1}$ , and let  $a, b, c$  be the respective dimensions of  $A^{2k}, B^{2k}, Z^{2k}$ . It follows immediately from (8) that each of the two spaces  $B^{2k}$  and  $Z^{2k}$  is the orthogonal of the other with respect to the symmetric form  $\langle x, y \rangle$  of  $A^{2k}$ , whence  $a = b + c$ . We have  $B^{2k} \subset Z^{2k} \subset A^{2k}$ . If  $e_i$  is a base of  $A^{2k}$  such that  $e_i \in B^{2k}$  for  $1 \leq i \leq b$  and  $e_i \in Z^{2k}$  for  $b+1 \leq i \leq c$ , the matrix  $(\langle e_i, e_j \rangle)$  has then the form

$$\begin{pmatrix} 0 & 0 & L \\ 0 & Q & * \\ L & * & * \end{pmatrix},$$

where  $L$  and  $Q$  are square nonsingular matrices, of orders  $b$  and  $c - b$  respectively. Its index is  $\tau_\xi(A)$ , while  $\tau(Q)$  is  $\tau_{\xi'}(A')$ . By Lemma 1, we get therefore  $\tau_{\xi'}(A') = \tau_\xi(A)$ , as contended.

3. **Proof of the theorem.** It suffices to prove the theorem (see Introduction) for the case  $\dim E = 4k$ , which we suppose from now on. We consider the cohomology spectral sequence  $E_r^{p,q}$ ,  $2 \leq r \leq \infty$ , of the bundle  $E \rightarrow B$ , with the real field as the coefficient field. Let

$$E_r^s = \sum_{p+q=s} E_r^{p,q}, \quad E_r = \sum_{0 \leq s} E_r^s, \quad 2 \leq r \leq \infty.$$

Each  $E_r$  is a graded ring, satisfying  $E_r^s E_r^{s'} \subset E_r^{s+s'}$  and also  $E_r^{p,q} E_r^{p',q'} \subset E_r^{p+p',q+q'}$ . It has a differentiation  $d_r$ , such that  $E_{r+1}$  is the derived ring of  $E_r$ . In our case  $d_r$  is trivial for sufficiently large  $r$  and  $E_\infty$ , or  $E_r$  for  $r$  sufficiently large, is the graded ring belonging to a certain filtration of the cohomology ring of the manifold  $E$ . The term  $E_2$  of the spectral sequence is by hypothesis (2) of our theorem isomorphic to  $H^*(B, H^*(F)) = H^*(B) \otimes H^*(F)$ , such that

$$E_2^{p,q} \cong H^p(B, H^q(F)) \cong H^p(B) \otimes H^q(F).$$

If we identify  $E_2^{p,q}$  with  $H^p(B) \otimes H^q(F)$  under this isomorphism, the multiplication in  $E_2$  is given by

$$(b \otimes f)(b' \otimes f') = (-1)^{p'a}(b \cup b') \otimes (f \cup f'),$$

$$b \in H^p(B), \quad b' \in H^{p'}(B), \quad f \in H^q(F), \quad f' \in H^{q'}(F).$$

Let  $m = \dim F$ , so that  $\dim B = 4k - m$ . Since  $B$  and  $F$  are manifolds,  $E_2$  is a Poincaré ring with respect to the grading

$$E_2 = \sum_{0 \leq s < \infty} E_2^s \quad (E_2^s = 0 \text{ for } s > 4k, E_2^{4k} = E_2^{4k-m, m}).$$

The ring  $E_2$  is isomorphic to the cohomology ring of  $B \times F$ .

The orientations of  $B, F$  define a generator  $\xi_2 = \xi_B \otimes \xi_F$  of  $E_2^{4k}$ . Here  $\xi_B$  (resp.  $\xi_F$ ) denotes the generator of  $H^{4k-m}(B)$  (resp.  $H^m(F)$ ) belonging to the orientation of  $B$  (resp.  $F$ ). We wish to prove that

$$\tau_{\xi_2}(E_2) = \tau(B) \cdot \tau(F).$$

We have

$$(9) \quad E_2^{2k} = E_2^{2k, 0} + E_2^{2k-1, 1} + \dots + E_2^{2k-m, m}.$$

Here some of the  $E_2^{p, q}$  might vanish, in particular  $E_2^{p, q} = 0$  if  $p < 0$ . Clearly, for  $x \in E_2^{2k-a, a}$  and  $y \in E_2^{2k-a', a'}$  we have  $xy = 0$  unless

$$q + q' = m.$$

By Poincaré duality in  $B$  and  $F$ , we have

$$\dim E_2^{2k-a, a} = \dim E_2^{2k-m+a, m-a}.$$

Therefore, the symmetric matrix, which defines the bilinear symmetric function over  $E_2^{2k}$ , is, when written in blocks relative to the direct sum decomposition (9), of the form

$$\begin{pmatrix} 0 & & & L_0 \\ & \ddots & & \\ & & \ddots & \\ L_m & & & 0 \end{pmatrix}$$

where the  $L_i$  are nonsingular square matrices, such that  $L_i$  is the transpose of  $L_{m-i}$ . By Lemma 1 we obtain

$$\tau_{\xi_2}(E_2) = 0 \text{ if } m \text{ is odd,} \quad \tau_{\xi_2}(E_2) = \tau(L_{m/2}) \text{ if } m \text{ is even.}$$

In the first case the equation  $\tau_{\xi_2}(E_2) = \tau(B)\tau(F)$  is proved, since  $\tau_{\xi_2}(E_2) = \tau(F) = 0$ . In the latter case we have

$$E_2^{2k-m/2, m/2} = H^{2k-m/2}(B) \otimes H^{m/2}(F),$$



and it is clear that up to the sign  $(-1)^{m/2}$  the matrix  $L_{m/2}$  is the tensor product of the two matrices defining the bilinear forms of  $B$  and  $F$ . If  $m/2$  is odd, both matrices in this tensor product are skew-symmetric, and we have, by Lemma 2,  $\tau(L_{m/2})=0$ ; on the other hand we have  $\tau(B)\tau(F)=0$ , since  $\dim F \not\equiv 0 \pmod{4}$  and thus by definition  $\tau(F)=0$ . If  $m/2$  is even, that is, if  $m \equiv 0 \pmod{4}$ , both matrices are symmetric, and Lemma 2 gives:  $\tau(L_{m/2})=\tau(B)\tau(F)$ . Combining all cases, we get the formula

$$(10) \quad \tau_{\xi_2}(E_2) = \tau(B)\tau(F)$$

in full generality.

The differentiation  $d_2$  of  $E_2$  satisfies the conditions of a differentiation in a Poincaré ring given in §2. In fact,  $\dim E_\infty^{4k} = 1$ , since  $E$  is a manifold of dimension  $4k$ . Therefore,  $\dim E_r^{4k} = 1$  for  $2 \leq r$ . Thus  $d_2$  annihilates  $E_2^{4k-1}$ ; more generally  $d_r$  annihilates  $E_r^{4k-1}$ . It follows by Lemma 3 that  $E_3$  is a Poincaré ring. It has  $d_3$  as differentiation and therefore  $E_4$  is a Poincaré ring etc. Finally,  $E_\infty$  is a Poincaré ring. By Lemma 4 and (10) we get

$$\tau(B)\tau(F) = \tau_{\xi_2}(E_2) = \tau_{\xi_3}(E_3) = \dots = \tau_{\xi_\infty}(E_\infty),$$

where  $\xi_r$  (resp.  $\xi_\infty$ ) is the image of  $\xi_2$  in  $E_r$  (resp.  $E_\infty$ ).

It remains to prove that  $\tau_{\xi_\infty}(E_\infty) = \tau(E)$ . The cohomology ring  $H^*(E)$  is filtered:

$$(11) \quad \begin{aligned} H^*(E) &= D^0 \supset D^1 \supset \dots \supset D^p \supset D^{p+1} \supset \dots, & \bigcap D^p &= 0, \\ D^{p,q} &= D^p \cap H^{p+q}(E), \\ D^{p,q} \cdot D^{p',q'} &\subset D^{p+p',q+q'}. \end{aligned}$$

We have the filtration

$$H^r(E) = D^{0,r} \supset D^{1,r-1} \supset \dots \supset D^{r,0} \supset D^{r+1,-1} = 0$$

and the canonical isomorphism

$$(12) \quad D^{p,q} / D^{p+1,q-1} \cong E_\infty^{p,q}.$$

The ring structure of  $E_\infty$  is induced by that of  $H^*(E)$  by the canonical homomorphisms  $D^{p,q} \rightarrow E_\infty^{p,q}$  (see (12) and (11)). Since  $E_\infty^{4k} = E_\infty^{4k-m,m}$ , (where  $m = \dim F$ ), we have

$$(13) \quad H^{4k}(E) = D^{4k-m,m} \cong E_\infty^{4k-m,m}$$

and

$$(14) \quad D^{4k-i,i} = 0 \quad \text{for } i < m.$$

Earlier we have chosen a generator  $\xi_\infty \in E_\infty^{2k}$ . Under the canonical isomorphism (13)  $\xi_\infty$  goes over in the generator  $\xi_E$  of  $H^{2k}(E)$  belonging to the orientation of  $E$  generated by the given orientations of  $B$  and  $F$  in this order.<sup>2</sup> We now consider the bilinear symmetric function  $\langle x, y \rangle$  over  $H^{2k}(E)$  relative to  $\xi_E$ . Choose a direct sum decomposition of  $H^{2k}(E)$  in linear subspaces,

$$(15) \quad H^{2k}(E) = V_0 + V_1 + V_2 + \dots + V_m$$

such that

$$\sum_{j=0}^q V_j = D^{2k-q, q} \quad (0 \leq q \leq m).$$

Here we use that  $D^{2k-s, s} = D^{2k-m, m}$  for  $s > m$ . By (11) and (14) we have

$$(16) \quad \langle x, y \rangle = 0 \quad \text{for } x \in V_i, y \in V_j \text{ and } i + j < m,$$

and moreover by (13)

$$(17) \quad \langle x, y \rangle = \langle \bar{x}, \bar{y} \rangle, \quad \text{for } x \in V_i, y \in V_j \text{ and } i + j = m,$$

where  $\bar{x}$  (resp.  $\bar{y}$ ) denotes the image (see (12)) of  $x$  (resp.  $y$ ) in  $E_\infty^{2k-i}$  (resp.  $E_\infty^{2k-j}$ ) and where on the right side of this equation stands the symmetric bilinear form over  $E_\infty^{2k}$  relative to  $\xi_\infty$ . Since  $\langle \bar{x}, \bar{y} \rangle = 0$  for  $\bar{x} \in E_\infty^{2k-i, i}, \bar{y} \in E_\infty^{2k-j, j}$ , unless  $i + j = m$ , and since  $E_\infty$  is a Poincaré algebra, we can conclude

$$(18) \quad \dim E_\infty^{2k-i, i} = \dim E_\infty^{2k-m+i, m-i}.$$

The preceding remarks, in particular (16), (17), (18), imply: The matrix of the symmetric bilinear function over  $H^{2k}(E)$  relative to  $\xi_E$  can be written in blocks with respect to the direct sum decomposition (15) in the form

$$\begin{pmatrix} 0 & & & & L_0 \\ & & & & \\ & & & L_1 & \\ & & & \cdot & \\ & & & \cdot & \\ L_m & & & & * \end{pmatrix}$$

<sup>2</sup> This is easy to see when  $E$  is a trivial bundle, in which case it is almost the definition of the orientation of a product of manifolds. The general case can be reduced to this one by comparing the spectral sequence of  $E$  to that of the bundle induced by  $E$  on an open cell of the base, the cohomology being taken with compact carriers.

where the  $L_i$  are nonsingular square matrices and where  $L_i$  is the transpose of  $L_{m-i}$ . Moreover,

$$\begin{pmatrix} 0 & & & L_0 \\ & \ddots & & \\ & & \ddots & \\ L_m & & & 0 \end{pmatrix}$$

is the matrix of the symmetric bilinear function over  $E_\infty^{2k}$  relative to  $\xi_\infty$ . By Lemma 1 we have  $\tau(E) = \tau_{\xi_\infty}(E_\infty)$ . This concludes the proof of our theorem.

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## THE PERIPHERAL CHARACTER OF CENTRAL ELEMENTS OF A LATTICE<sup>1</sup>

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A lattice being a Hausdorff space together with a pair of continuous lattice operations ( $\wedge$  and  $\vee$ ) the content of this note is best exhibited by quoting a corollary to our theorem: *If a compact connected lattice is (topologically) situated in Euclidean  $n$ -space then its center is contained in its boundary.* Thus, far from being "centrally located," the central elements are "peripheral."

The above is a consequence (see [3, p. 273]) of the

**THEOREM.** *If  $L$  is a compact connected lattice, if  $R$  is an  $(n, G)$ -rim [3] for  $L$  and if (i)  $a$  is central [1, p. 27] or if (ii)  $L$  is modular and  $a$  is complemented then  $a \in R$ .*

**PROOF.** The procedure is to introduce an appropriate multiplication into  $L$  so that  $L$  is a semigroup, to show that  $L$  is not simple (in the semigroup sense [3]) and that  $a$  is a left unit. Since  $L$  is compact it has a zero and unit, 0 and 1, as is well-known. Indeed, the set  $\bigcap \{x \vee L \mid x \in L\}$  is easily seen to consist of exactly one element, namely 1. If  $a = 1$  then the hypotheses of Theorem 1 of [3] are fulfilled using the multiplication  $(x, y) \rightarrow x \wedge y$  so that 1 being a unit for the multiplication,  $1 \in R$ . If  $a \neq 1$  let  $x \cdot y = (a' \wedge x) \vee y$ ,  $a'$  being a

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