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A SIMPLE INTRINSIC PROOF OF THE GAUSS-BONNET FORMULA FOR CLOSED RIEMANNIAN MANIFOLDS

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Introduction

C. B. Allendoerfer[1] and W. Fenchel[2] have independently given a generalization of the classical formula of Gauss-Bonnet to a closed orientable Riemannian manifold which can be imbedded in a euclidean space. Recently, Allendoerfer and André Weil[3] extended the formula to a closed Riemannian polyhedron and proved in particular its validity in the case of a general closed Riemannian manifold. In their proof use is still made of the imbedding of a Riemannian cell in a euclidean space. The object of this paper is to offer a direct intrinsic proof of the formula by making use of the theory of vector fields in differentiable manifolds.

The underlying idea of the present proof is very simple, so that a brief summary might be helpful. Let R^n be a closed orientable Riemannian manifold of an even dimension n. According to details to be given below, we define in R^n an intrinsic exterior differential form Ω of degree n, which is of course equal to a scalar invariant of R^n multiplied by the volume element. The formula of Gauss-Bonnet in question asserts that the integral of this differential form over R^n is equal to the Euler-Poincaré characteristic χ of R^n. To prove this we pass from the manifold R^n to the manifold M^{2n-1} of $2n - 1$ dimensions formed by the unit vectors of R^n.[4] In M^{2n-1} we show that Ω is equal to the exterior derivative of a differential form Π of degree $n - 1$. By defining a continuous field of unit vectors over R^n with isolated singular points, we get, as its image in M^{2n-1}, a submanifold V^n of dimension n, and the integral of Ω over R^n is equal to the same integral over V^n. The application of the theorem of Stokes shows that the latter is equal to the integral of Π over the boundary of V^n. Now, the boundary of V^n corresponds exactly to the singular points of the vector field defined in R^n, the sum of whose indices is, by a well-known theorem, equal to χ. With such an interpretation the integral of Π over the boundary of V^n can be evaluated and is easily proved to be equal to χ.

The method can of course be applied to derive other formulas of the same type and, with suitable modifications, to deduce the Gauss-Bonnet formula for a Riemannian polyhedron. We publish this proof, because it is in the present case that the main ideas of our method are most clear. Further results will be given in a forthcoming paper.

§1. Résumé of some fundamental formulas in Riemannian Geometry

Let R^n be a closed orientable differentiable manifold[5] of an even dimension $n = 2p$ and class $r \geqq 4$. In R^n suppose a Riemannian metric be defined, with the fundamental tensor g_{ij}, whose components we suppose to be of class 3. Since we are to deal with multiple integrals, it seems convenient to follow Cartan's treatment of Riemannian Geometry,[6] with the theory of exterior differential forms, instead of the ordinary tensor analysis, playing the dominant rôle. The differential forms which occur below are exterior differential forms.

According to Cartan we attach to each point P of R^n a set of n mutually perpendicular unit vectors $\mathfrak{e}_1, \cdots, \mathfrak{e}_n$, with a certain orientation. Such a figure $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$ is called a frame. A vector $\mathfrak{v}$ of the tangent space of R^n at P can be referred to the frame at P, thus

$$\mathfrak{v} = u_i \mathfrak{e}_i, \tag{1}$$

where the index i runs from 1 to n and repeated indices imply summation. The law of infinitesimal displacement of tangent spaces, as defined by the parallelism of Levi-Civita, is given by equations of the form

$$\begin{cases} dP = \omega_i \mathfrak{e}_i, \\ d\mathfrak{e}_i = \omega_{ij} \mathfrak{e}_j, \qquad \omega_{ij} + \omega_{ji} = 0 \end{cases} \tag{2}$$

where ω_i, ω_{ij} are Pfaffian forms. These Pfaffian forms satisfy the following "equations of structure":

$$\begin{cases} d\omega_i = \omega_j \omega_{ji}, \\ d\omega_{ij} = -\omega_{ik}\omega_{jk} + \Omega_{ij}, \qquad \Omega_{ij} + \Omega_{ji} = 0. \end{cases} \tag{3}$$

In (3) Ω_{ij} are exterior quadratic differential forms and give the curvature properties of the space.

The forms Ω_{ij} satisfy a system of equations obtained by applying to (3) the theorem that the exterior derivatives of the left-hand members are zero. The equations are

$$\begin{cases} \omega_j \Omega_{ji} = 0, \\ d\Omega_{ij} - \omega_{jk}\Omega_{ik} + \omega_{ik}\Omega_{jk} = 0, \end{cases} \tag{4}$$

and are called the Bianchi identities.

For the following it is useful to know how the Ω_{ij} behave when the frame $\mathfrak{e}_1 \cdots \mathfrak{e}_n$ undergoes a proper orthogonal transformation. In a neighborhood of P in which the same system of coordinates is valid let $\mathfrak{e}_1 \cdots \mathfrak{e}_n$ be changed to $\mathfrak{e}_1^* \cdots \mathfrak{e}_n^*$ according to the proper orthogonal transformation:

$$\mathfrak{e}_i^* = a_{ij}\mathfrak{e}_j \tag{5}$$

or

$$\mathfrak{e}_i = a_{ji}\mathfrak{e}_j^*, \tag{5'}$$

where (a_{ij}) is a proper orthogonal matrix, whose elements a_{ij} are functions of the coordinates. Suppose Ω_{ij}^* be formed from the frames $P\mathfrak{e}_1^* \cdots \mathfrak{e}_n^*$ in the same way as Ω_{ij} are formed from $P\mathfrak{e}_1 \cdots \mathfrak{e}_n$. Then we easily find

$$\Omega_{ij}^* = a_{ik}a_{jl}\Omega_{kl}. \tag{6}$$

From (6) we deduce an immediate consequence. Let $\epsilon_{i_1 \cdots i_n}$ be a symbol which is equal to $+1$ or -1 according as $i_1, \cdots, i_n$ form an even or odd permutation of $1, \cdots, n$, and is otherwise zero. Since our space R^n is of even dimension $n = 2p$, we can construct the sum

$$\Omega = (-1)^{p-1} \frac{1}{2^{2p} \pi^p p!} \epsilon_{i_1 \cdots i_{2p}} \Omega_{i_1 i_2} \Omega_{i_3 i_4} \cdots \Omega_{i_{2p-1} i_{2p}}, \tag{7}$$

where each index runs from 1 to n. Using (6), we see that Ω remains invariant under a change of frame (5) and is therefore intrinsic. This intrinsic differential form Ω is of degree n and is thus a multiple of $\omega_1 \cdots \omega_n$. As the latter product (being the volume element of the space) is also intrinsic, we can write

$$\Omega = I \omega_1 \cdots \omega_n, \tag{8}$$

where the coefficient I is a scalar invariant of the Riemannian manifold.

With all these preparations we shall write the formula of Gauss-Bonnet in the following form

$$\int_{R^n} \Omega = \chi, \tag{9}$$

χ being the Euler-Poincaré characteristic of R^n.

§2. The space of unit vectors and a formula for Ω

From the Riemannian manifold R^n we pass now to the manifold M^{2n-1} of dimension $2n - 1$ formed by its unit vectors. M^{2n-1} is a closed differentiable manifold of class $r - 1$. As its local coordinates we may of course take the local coordinates of R^n and the components u_i of the vector $\mathfrak{v}$ in (1), subjected to the condition

$$u_i u_i = 1. \tag{1'}$$

If θ_i are the components of $d\mathfrak{v}$ with respect to the frame $\mathfrak{e}_1 \cdots \mathfrak{e}_n$, we have

$$d\mathfrak{v} = \theta_i \mathfrak{e}_i, \tag{10}$$

where

$$\theta_i = du_i + u_j \omega_{ji} \tag{11}$$

and

$$u_i \theta_i = 0. \tag{12}$$

From (11) we get, by differentiation,

$$d\theta_i = \theta_j \omega_{ji} + u_j \Omega_{ji}. \tag{13}$$

As to the effect of a change of frame (5) on the components u_i, θ_i, it is evidently given by the equations

$$u_i^* = a_{ij} u_j, \qquad \theta_i^* = a_{ij} \theta_j. \tag{14}$$

We now construct the following two sets of differential forms:

$$\Phi_k = \epsilon_{i_1 \cdots i_{2p}} u_{i_1} \theta_{i_2} \cdots \theta_{i_{2p-2k}} \Omega_{i_{2p-2k+1} i_{2p-2k+2}} \cdots \Omega_{i_{2p-1} i_{2p}}, \qquad k = 0, 1, \cdots, p-1, \tag{15}$$

$$\Psi_k = \epsilon_{i_1 \cdots i_{2p}} \Omega_{i_1 i_2} \theta_{i_3} \cdots \theta_{i_{2p-2k}} \Omega_{i_{2p-2k+1} i_{2p-2k+2}} \cdots \Omega_{i_{2p-1} i_{2p}}, \qquad k = 0, 1, \cdots, p-1. \tag{16}$$

The forms Φ_k are of degree $2p - 1$ and Ψ_k of degree $2p$, and we remark that Ψ_{p-1} differs from Ω only by a numerical factor. Using (6) and (14), we see that Φ_k and Ψ_k are intrinsic and are therefore defined over the entire Riemannian manifold R^n.

We shall prove the following recurrent relation:

$$d\Phi_k = \Psi_{k-1} + \frac{2p - 2k - 1}{2(k+1)} \Psi_k, \qquad k = 0, 1, \cdots, p-1, \tag{17}$$

where we define $\Psi_{-1} = 0$. Using the property of skew-symmetry of the symbol $\epsilon_{i_1 \cdots i_{2p}}$ in its indices, we can write

$$\begin{aligned} d\Phi_k = {} & \epsilon_{(i)} du_{i_1} \theta_{i_2} \cdots \theta_{i_{2p-2k}} \Omega_{i_{2p-2k+1} i_{2p-2k+2}} \cdots \Omega_{i_{2p-1} i_{2p}} \\ & + (2p - 2k - 1) \epsilon_{(i)} u_{i_1} d\theta_{i_2} \theta_{i_3} \cdots \theta_{i_{2p-2k}} \Omega_{i_{2p-2k+1} i_{2p-2k+2}} \cdots \Omega_{i_{2p-1} i_{2p}} \\ & - k \epsilon_{(i)} u_{i_1} \theta_{i_2} \cdots \theta_{i_{2p-2k}} d\Omega_{i_{2p-2k+1} i_{2p-2k+2}} \Omega_{i_{2p-2k+3} i_{2p-2k+4}} \cdots \Omega_{i_{2p-1} i_{2p}}, \end{aligned}$$

where $\epsilon_{(i)}$ is an abbreviation of $\epsilon_{i_1 \cdots i_{2p}}$. For the derivatives du_i, $d\theta_i$, $d\Omega_{ij}$ we can substitute their expressions from (11), (13), and (4). The resulting expression for $d\Phi_k$ will then consist of terms of two kinds, those involving ω_{ij} and those not. We collect the terms not involving ω_{ij}, which are

$$\Psi_{k-1} + (2p - 2k - 1) \epsilon_{(i)} u_{i_1} u_j \Omega_{j i_2} \theta_{i_3} \cdots \theta_{i_{2p-2k}} \Omega_{i_{2p-2k+1} i_{2p-2k+2}} \cdots \Omega_{i_{2p-1} i_{2p}}. \tag{18}$$

This expression is obviously intrinsic. Its difference with $d\Phi_k$ is an expression which contains a factor ω_{ij} in each of its terms.

We shall show that this difference is zero. In fact, let P be an arbitrary but fixed point of R^n. In a neighborhood of P we can choose a family of frames $e_1 \cdots e_n$ such that at P,

$$\omega_{ij} = 0.$$

(This process is "equivalent" to the use of geodesic coordinates in tensor notation.) Hence, for this particular family of frames, the expressions (18) and $d\Phi_k$ are equal at P. It follows that they are identical, since both expressions are intrinsic and the point P is arbitrary.

To transform the expression (18) we shall introduce the abbreviations

$$\begin{cases} P_k = \epsilon_{(i)} u_{i_1}^2 \Omega_{i_1 i_2} \theta_{i_3} \cdots \theta_{i_{2p-2k}} \Omega_{i_{2p-2k+1} i_{2p-2k+2}} \cdots \Omega_{i_{2p-1} i_{2p}}, \\ \Sigma_k = \epsilon_{(i)} u_{i_1} u_{i_3} \Omega_{i_3 i_2} \theta_{i_3} \cdots \theta_{i_{2p-2k}} \Omega_{i_{2p-2k+1} i_{2p-2k+2}} \cdots \Omega_{i_{2p-1} i_{2p}}, \\ T_k = \epsilon_{(i)} u_{i_3}^2 \Omega_{i_1 i_2} \theta_{i_3} \cdots \theta_{i_{2p-2k}} \Omega_{i_{2p-2k+1} i_{2p-2k+2}} \cdots \Omega_{i_{2p-1} i_{2p}}, \end{cases} \tag{19}$$

which are forms of degree $2p$. Owing to the relations (1) and (12) there are some simple relations between these forms and Ψ_k. In fact, we can write

$$P_k = \epsilon_{(i)}(1 - u_{i_2}^2 - u_{i_3}^2 - \cdots - u_{i_{2p}}^2)\Omega_{i_1 i_2}\theta_{i_3} \cdots \theta_{i_{2p-2k}}\Omega_{i_{2p-2k+1}i_{2p-2k+2}} \cdots \Omega_{i_{2p-1}i_{2p}}$$
$$= \Psi_k - P_k - 2(p - k - 1)T_k - 2kP_k,$$

which gives

$$\Psi_k = 2(k + 1)P_k + 2(p - k - 1)T \tag{20}$$

Again, we have

$$\Sigma_k = \epsilon_{(i)} u_{i_1}\Omega_{i_3 i_2}(-u_{i_1}\theta_{i_1} - u_{i_2}\theta_{i_2} - u_{i_4}\theta_{i_4} - \cdots - u_{i_{2p}}\theta_{i_{2p}})\theta_{i_4} \cdots$$
$$\theta_{i_{2p-2k}}\Omega_{i_{2p-2k+1}i_{2p-2k+2}} \cdots \Omega_{i_{2p-1}i_{2p}}$$
$$= T_k - (2k + 1)\Sigma_k,$$

and hence

$$T_k = 2(k + 1)\Sigma_k. \tag{21}$$

The expression (18) for $d\Phi_k$ therefore becomes

$$d\Phi_k = \Psi_{k-1} + (2p - 2k - 1)\{P_k + 2(p - k - 1)\Sigma_k\}, \qquad k = 0, 1, \cdots, p - 1.$$

Using (20) and (21), we get the desired formula (17).

From (17) we can solve Ψ_k in terms of $d\Phi_0, d\Phi_1, \cdots, d\Phi_k$. The result is easily found to be

$$\psi_k = \sum_{m=0}^{k} (-1)^m \frac{2^{m+1}(k + 1)k \cdots (k - m + 1)}{(2p - 2k - 1)(2p - 2k + 1) \cdots (2p - 2k + 2m - 1)} d\Phi_{k-m}, \tag{22}$$
$$k = 0, 1, \cdots, p - 1.$$

In particular, it follows that Ω is the exterior derivative of a form Π:

$$\Omega = (-1)^{p-1} \frac{1}{2^{2p}\pi^p p!} \Psi_{p-1} = d\Pi, \tag{23}$$

where

$$\Pi = \frac{1}{\pi^p} \sum_{m=0}^{p-1} (-1)^m \frac{1}{1\cdot 3 \cdots (2p - 2m - 1)m!\,2^{p+m}} \Phi_m. \tag{24}$$

§3. Proof of the Gauss-Bonnet formula

Basing on the formula (24) we shall give a proof of the formula (9), under the assumption that R^n is a closed orientable Riemannian manifold.

We define in R^n a continuous field of unit vectors with a point 0 of R^n as the only singular point.[7] By a well-known theorem the index of the field at 0 is equal to χ, the Euler-Poincaré characteristic of R^n. This vector field defines in M^{2n-1} a submanifold V^n, which has as boundary χZ, where Z is the $(n - 1)$-

dimensional cycle formed by all the unit vectors through 0. The integral of Ω over R^n is evidently equal to the same over V^n. Applying Stokes's theorem, we get therefore

$$(25)\qquad \int_{R^n} \Omega = \int_{V^n} \Omega = \chi \int_Z \Pi = \chi \frac{1}{1\cdot 3 \cdots (2p-1)2^p \pi^p} \int_Z \Phi_0 .$$

From the definition of Φ_0 we have

$$(26)\qquad \Phi_0 = (2p-1)! \sum_{i=1}^{n} (-1)^i \theta_1 \cdots \theta_{i-1} u_i \theta_{i+1} \cdots \theta_{2p} .$$

The last sum is evidently the volume element of the $(2p-1)$-dimensional unit sphere. Therefore

$$\int_Z \Phi_0 = (2p-1)! \frac{2\pi^p}{(p-1)!} .$$

Substituting this into (25), we get the formula (9).

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