$\text{sign}\,\tilde{M} = n\,\text{sign}\,M$ now follows from the regular case applied to the regular coverings

$\tilde{M}_0 \to M$, $\tilde{M}_0 \to \tilde{M}$. $\square$

References

[1] M.F. Atiyah and R. Bott, _A Lefschetz fixed point formula for elliptic complexes: II. Applications_, Ann. of Math. (2) 88(1968), 451-491.

[2] M.F. Atiyah and I.M. Singer, _The Index of elliptic operators: III_, Ann. of Math. (2) 87(1968), 546-604.

[3] A.J. Casson and C. McA. Gordon, _Cobordism of classical knots_, mimeographed notes, Orsay, 1975.

[4] _____, _On slice knots in dimension three_, Proc. Symp. Pure Math. XXXII, Part 2, AMS, 1978, 39-53.

[5] S.S. Chern, F. Hirzebruch and J.P. Serre, _On the index of a fibered manifold_, Proc. Amer. Math. Soc., 8(1957), 587-596.

[6] P.E. Conner and E.E. Floyd, _Differentiable periodic maps_, Ergebnisse der Mathematik und ihrer Grenzgebiete 33, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1964.

[7] P. Gilmer, _Topological proof of the G-signature theorem for G finite_, to appear.

[8] W.C. Hsiang and R.H. Szczarba, _On embedding surfaces in four-manifolds_, Proc. Symp. Pure Math. XXII, AMS, 1971, 97-103.

[9] R.A. Litherland, _Topics in knot theory_, Ph.D. Thesis, University of Cambridge, 1978.

[10] W.S. Massey, _Proof of a conjecture of Whitney_, Pacific J. Math. 31(1969), 143-156.

[11] V.A. Rohlin, _Two-dimensional submanifolds of four-dimensional manifolds_, Functional Anal. Appl. 5(1971), 39-48.

Department of Mathematics
The University of Texas at Austin
Austin, TX 78712

COBORDISM OF CLASSICAL KNOTS

by

A.J. CASSON and C. McA. GORDON

A _knot_ is a smooth submanifold of S^3 which is homeomorphic to S^1. We regard S^3 as an _oriented_ 3-sphere ; the orientation of S^1 is usually immaterial. The knot K is _slice_ if there is a smooth 2-disc $D \subset B^4$ such that $K = \partial D$. Knots K_0, K_1 are _cobordant_ if there is a smoothly embedded annulus in $S^3 \times I$ meeting $S^3 \times \{t\}$ in K_t $(t = 0, 1)$. Addition of _cobordism classes_ of oriented knots is defined by connected sum, giving _cobordism group_ θ_1^3.

We can also construct cobordism groups of knots of S^n in S^{n+2}, say θ_n^{n+2}. It is known that $\theta_n^{n+2} = 0$ if n is even [5], that $\theta_7^9 \cong \theta_{11}^{13} \cong \ldots$ and $\theta_5^7 \cong \theta_9^{11} \cong \theta_{13}^{15} \cong \ldots$. θ_3^5 is isomorphic to a subgroup of θ_7^9 of index 2. There is a natural surjection $\theta_1^3 \to \theta_5^7$; we show that this is _not_ injective. First we describe the known cobordism invariants.

$K \subset S^3$ bounds an oriented surface $F \subset S^3$; we thicken F to an embedding $F \times I \subset S^3$. Given $x, y \in H_1(F)$, put $\alpha(x,y) = $ linking number of $x \times 0$ and $y \times 1$. This defines a bilinear form $\alpha : H_1(F) \times H_1(F) \to \mathbb{Z}$, such that $\alpha(x,y) - \alpha(y,x) = $ intersection number of x and y. We refer to α as a _Seifert form_ for K. Let $g = $ genus of $F = \frac{1}{2}$ (dimension of $H_1(F)$). We say that a Seifert form is null-cobordant if it vanishes on a subgroup of $H_1(F)$ of dimension g.

THEOREM [6]. _If_ K _is slice, then any Seifert form for_ K _is null-cobordant. In higher (odd) dimensions, the analogous condition is necessary and sufficient for_ K _to be slice._

In fact, proving $\theta_1^3 \to \theta_5^7$ is not injective is equivalent to finding a non-slice knot with a null-cobordant Seifert form.

Example.

$$K = $$

t full twists   (t full twists) (-1 full twist)
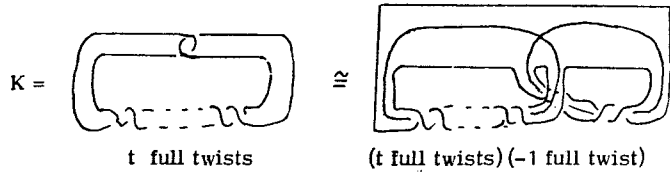
With respect to the basis shown, the Seifert form has matrix $A = \begin{pmatrix} t & 1 \\ 0 & -1 \end{pmatrix}$.

Put $x = \begin{pmatrix} 1 \\ u \end{pmatrix}$; then $x'Ax = t + u - u^2$, which vanishes if $t = u(u-1)$. So K has a null-cobordant Seifert form if $t = 0$, 2 , 6 , 12 , ... We shall show that K is slice if and only if $t = 0$ or 2 .

K is a <u>ribbon knot</u> if it bounds an immersed disc (<u>a ribbon</u>) in S^3 , each of whose singularities is of the type shown (two sheets intersecting in an arc which lies in the interior of one of the sheets).

Examples.

(square knot)   (stevedore's knot : example above, $t = 2$)
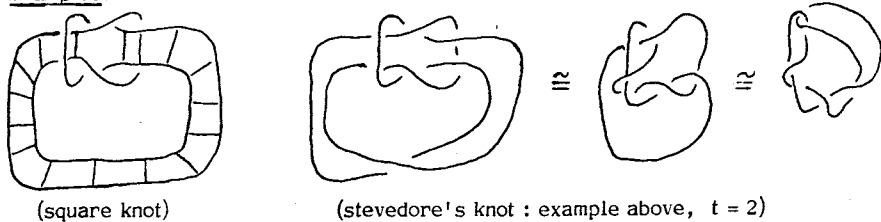
Ribbon knots are slice (push the interior of the ribbon into B^4 , then deform slightly a neighbourhood of each singular arc).

<u>Problem</u> (Fox). Is every slice knot a ribbon knot ?

Our method will prove that certain knots do not bound ribbons without proving them non-slice.

First we give a fairly simple method of showing that knots do not bound ribbons. The only special property of ribbons needed is the following (see [3] and [10]).

<u>LEMMA 1</u>. If 2-<u>disc</u> $D \subset B^4$ <u>is obtained by deforming a ribbon</u>, <u>and</u> $K = \partial D \subset S^3$, <u>then the map</u> $\pi_1(S^3 - K) \to \pi_1(B^4 - D)$ <u>is surjective</u>.

Let K be a knot whose double branched covering is a lens space L . Our method is not confined to these, but calculations have not yet been made in other cases.

<u>Examples</u>. 2-bridge knots (Viergeflechte) [2], [9]

c_1 crossings c_3 c_{n-1}
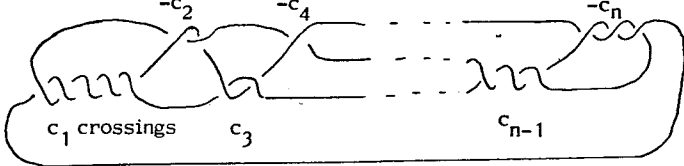
has double branched covering $L_{p,q}$, where $\frac{q}{p}$ has continued fraction $\frac{1}{c_1} + \frac{1}{c_2} + \cdots \frac{1}{c_n}$. (We obtain a knot if p is odd, and a 2-string link if p is even.) This sets up a 1 - 1 correspondance between 2-bridge knots and lens spaces, and we refer to the knot $\frac{q}{p}$. The example given earlier was $2/(4t+1)$, with continued fraction $\frac{1}{2t} + \frac{1}{2}$.

Now we define an invariant of our knot K with double branched covering L . It is in fact a disguised form of a standard Atiyah-Singer invariant of 3-manifolds. Let $\chi : H_1(L) \to \mathbb{C}^*$ be a character with image C_m , the group of mth roots of 1 . We say that χ has order m ; χ is induced by a map $L \to K(C_m, 1)$. Since the bordism group $\Omega_3 K(C_m, 1)$ is finite, rL bounds a compact 4-manifold W over $K(C_m, 1)$, for some $r > 0$. Let $\widetilde{W}$ be the induced m-fold covering of W , so $\partial \widetilde{W} = r\widetilde{L}$ for some m-fold covering $\widetilde{L}$ of L .

Let k the cyclotomic field $\mathbb{Q}(C_m) \subseteq \mathbb{C}$; this is a module over the group ring $\mathbb{Z}[C_m]$. By Maschke's theorem, k is projective over the rational group ring $Q[C_m]$, so k is flat over $\mathbb{Z}[C_m]$. Let $C_*(\widetilde{W})$ be a chain complex for $\widetilde{W}$ (integer coefficients) on which the covering translations induce a $\mathbb{Z}[C_m]$-module structure. Write $H_*(W;k)$ for the homology of $C_*(\widetilde{W}) \otimes_{\mathbb{Z}[C_m]} k$; since k is flat, $H_*(W;k) \cong H_*(\widetilde{W};\mathbb{Z}) \otimes_{\mathbb{Z}[C_m]} k$. Homology with k coefficients will always be twisted in this way.

The intersection pairing $H_2(W;k) \times H_2(W;k) \to k$ is hermitian, so it has a signature $s(W)$. Let $s_0(W)$ be the signature of the intersection pairing on $H_2(W;\mathbb{Q})$ (in other words, the ordinary signature of W) . Define $\sigma(K,\chi) = \frac{1}{r}(s(W) - s_0(W))$; this is clearly independent of r .

Suppose W_1 , W_2 are two manifolds over $K(C_m, 1)$ with $\partial W_1 = \partial W_2 = rL$. Let $X = W_1 \cup (-W_2)$; this is closed manifold over $K(C_m, 1)$, and $(s(W_1) - s_0(W_1)) - (s(W_2) - s_0(W)) = s(X) - s_0(X)$. It is easily proved that s , s_0

define homomorphisms from $\Omega_4 K(C_m, 1)$ to $\mathbb{Z}$. Modulo torsion, $\Omega_4 K(C_m, 1)$ is generated by $\mathbb{C}P^2$ (with the constant map to $K(C_m, 1)$) . Since $s(\mathbb{C}P^2) = s_0(\mathbb{C}P^2) = 1$, $s(X) = s_0(X)$ for any closed 4-manifold X over $K(C_m, 1)$. It follows that $\sigma(K, \chi)$ is independent of the particular null-cobordism W .

THEOREM 1. If K is a ribbon knot whose double branched covering is a lens space L , then : (1) $|H_1(L)|$ is a square, say m^2 ;

(2) If χ is a non-constant character of order dividing m , then $\sigma(K, \chi) = \pm 1$.

Statement (1) is true (and well-known) for slice knots ; it follows from lemmas 2 and 3 below.

LEMMA 2. If K is slice, then L bounds a compact 4-manifold W with $\tilde{H}_*(W; \mathbb{Q}) = 0$.

Proof. Let $K = \partial D$, $D \subset B^4$, and let W be the double covering of B^4 branched over D . Let W' be the infinite cyclic covering of $B^4 - D$. Let $T : \tilde{H}_*(W'); \mathbb{Z}_2 \leftarrow$ be induced by a generator of the group of covering translations.

The exact sequence of the infinite cyclic covering $W' \to W'/T^2$ yields the following exact sequence :

$$\to \tilde{H}_n(W'; \mathbb{Z}_2) \xrightarrow{1-T^2} \tilde{H}_n(W'; \mathbb{Z}_2) \to \tilde{H}_n(W; \mathbb{Z}_2) \to \tilde{H}_{n-1}(W'; \mathbb{Z}_2) \to \dots$$

Similarly, there is an exact sequence :

$$\to \tilde{H}_n(W'; \mathbb{Z}_2) \xrightarrow{1-T} \tilde{H}_n(W'; \mathbb{Z}_2) \to \tilde{H}_n(B^4; \mathbb{Z}_2) \to \tilde{H}_{n-1}(W'; \mathbb{Z}_2) \to \dots .$$

From the second sequence, $1-T$ is an isomorphis. With $\mathbb{Z}_2$ coefficients, $1-T^2 = (1-T)^2$; from the first sequence, $\tilde{H}_*(W; \mathbb{Z}_2) = 0$. Since W is compact, $\tilde{H}_*(W; \mathbb{Q}) = 0$.

LEMMA 3. If the image of $H_1(L)$ in $H_1(W)$ has order m , then $|H_1(L)| = m^2$.

Proof. $H_2(L) = 0$, so we have exact sequence :

$$0 \to H_2(W) \to H_2(W, L) \to H_1(L) \to H_1(W) \to H_1(W, L) \to 0 .$$

By duality and universal coefficient theorems,

$$|H_2(W)| = |H_1(W, L)| \quad \text{and} \quad |H_2(W, L)| = |H_1(W)| ;$$

the result follows. (Note that here it is crucial that we are in dimension 4.)

To prove Theorem 1, recall our hypotheses that K is ribbon and L is a lens space. By lemmas 1 and 3 , $\pi_1(L) \cong \mathbb{Z}_{m^2}$ and $\pi_1(W) \cong \mathbb{Z}_m$. If χ is a character on $H_1(L)$ of order dividing m , then χ factors through $H_1(W)$, so we can use W to compute $\sigma(K, \chi)$.

Let $C_*(\tilde{W})$ be the chain complex corresponding to a cell (or handle) decomposition of W . $C_n(\tilde{W})$ is free over $\mathbb{Z}[C_m]$ with one basis element for each n-cell of W . The complex of k-vector spaces $C_*(\tilde{W}) \otimes_{\mathbb{Z}[C_m]} k$ has Euler characteristic equal to that of W , namely 1 .

Recall that $H_n(W; k) \cong H_n(\tilde{W}; \mathbb{Z}) \otimes_{\mathbb{Z}[C_m]} k$. Since χ is not constant, $H_0(W, k) \cong \mathbb{Z} \otimes_{\mathbb{Z}[C_m]} k = 0$. Since $\tilde{W}$ is simply connected, $H_1(\tilde{W}, \mathbb{Z}) = 0 = H_3(\tilde{W}, \mathbb{Z})$, giving $H_1(W; k) = 0 = H_3(W; k)$. Therefore $H_2(W; k)$ has dimension 1 over k . Similarly $H_1(L; k) = 0 = H_2(L; k)$, so the intersection pairing on $H_2(W; k)$ is non-singular. It follows that $s(W) = \pm 1$, $s_0(W) = 0$, giving $\sigma(K, \chi) = \pm 1$.

Calculation of $\sigma(K, \chi)$.

The rational number $\sigma(K, \chi)$ depends only on the 3-manifold L and the character χ on $H_1(L)$. First, we shows how $\sigma(K, \chi)$ is related to the Atiyah-Singer G-signature.

Let A be a vector space over k , with hermitian form $\varphi : A \times A \to k$. Suppose that the group C_m of mth roots of 1 acts on A , preserving φ . To avoid confusion, we write g_ω for the automorphism of A corresponding to the element ω of C_m . Let $A_s = \{a \in A : g_\omega a = \omega^s a\}$; then $A = A_0 \oplus A_1 \oplus \dots \oplus A_{m-1}$ and g acts on A_s by multiplication by ω^s .

Observe that A_s and A_t are orthogonal with respect to φ if $s \neq t$. For if $x \in A_s$, $y \in A_t$ then $\varphi(x, y) = \varphi(g_\omega x, g_\omega y) = \varphi(\omega^s x, \omega^t y) = \omega^{t-s} \varphi(x, y)$. By diagonalising $\varphi | A_r$, we obtain (non uniquely) $A_s = A_s^+ \oplus A_s^- \oplus A_s^o$, where φ is $\pm$ definite on $A_s^\pm$ and zero on A_s^o . Let $A^\pm = \Sigma A_s^\pm$, $A^o = \Sigma A_s^o$; these are C_m-invariant subspaces with $\varphi^\pm$ definite on $A^\pm$ and zero on A^o .

By definition, the g_ω-signature of φ is :

$$\sigma(\varphi, g_\omega) = \text{trace}(g_\omega | A^+) - \text{trace}(g_\omega | A^-) ,$$

so $\qquad \sigma(\varphi, g_\omega) = \sum_{s=0}^{m-1} \omega^s (\text{signature of } \varphi | A_s) .$

Now suppose $A = H_2(W;k)$ (untwisted for once), and φ is the hermitian extension of the intersection pairing on $H_2(\widetilde{W};\mathbb{Z})$. Recall our earlier construction ; the character χ gave rise to a twisted homology group $H_2(W;k)$, isomorphic to $H_2(\widetilde{W};\mathbb{Z}) \otimes_{\mathbb{Z}[C_m]} k$. This in turn is isomorphic to $A \otimes_{k[C_m]} k$, so

$$H_2(W;k) \cong \sum A_s \otimes_{k[C_m]} k \cong A_1 .$$

The intersection pairing $H_2(\widetilde{W};\mathbb{Z}) \times H_2(\widetilde{W};\mathbb{Z}) \to \mathbb{Z}[C_m]$ is defined by $(x,y) = \sum_{\omega \in C_m} \varphi(x, g_\omega y) \omega^{-1}$. Tensoring with k gives the intersection pairing on $H_2(W;k)$ by the same formula. However, $g_\omega y = \omega y$ for $y \in H_2(W;k) \cong A_1$, so

$$(x,y) = \sum_{\omega \in C_m} \varphi(x, \omega y) \omega^{-1} = m \varphi(x,y) .$$

Suppose $\partial W = rL$; then we defined $\sigma(K,\chi) = \frac{1}{r}(s(W) - s_0(W))$, where :

$$s(W) = \text{signature of intersection form on } H_2(W;k) ;$$
$$= \text{signature of } \varphi | A_1 .$$

Therefore $\qquad r\sigma(K,\chi) + s_0(W) = \text{signature of } \varphi | A_1$, and a similar argument shows that : $\qquad r\sigma(K,\chi^s) + s_0(W) = \text{signature of } \varphi | A_s$.

We can regard $\widetilde{L}$ as a free C_m-manifold, the element ω of C_m corresponding to the covering translation $\chi^{-1}(\omega)$ of $\widetilde{L}$. Atiyah and Singer define the invariant $\alpha(\widetilde{L}, \omega)$ to be $\frac{1}{r}\sigma(\varphi, g_\omega)$. We obtain :

$$\alpha(\widetilde{L}, \omega) = \sum_{s=0}^{m-1} \omega^s (\sigma(K,\chi^s) + \frac{1}{r} s_0(W)) = \sum_{s=0}^{m-1} \omega^s \sigma(K,\chi^s) \quad \text{if } \omega \neq 1 .$$

Since A_s and A_{-s} are related by complex conjugation, $\alpha(\widetilde{L}, \omega)$ is real.

Inverting these equations :

$$\sum_{\omega \neq 1} \alpha(\widetilde{L}, \omega) \omega^{-r} = \sum_{s=0}^{m-1} \sum_{\omega \neq 1} \omega^{s-r} \sigma(K,\chi^s)$$
$$= \sum_{s=0}^{m-1} \sum_{\omega} \omega^{s-r} \sigma(K,\chi^s) - \sum_{s=1}^{m-1} \sigma(K,\chi^s)$$
$$= m \sigma(K,\chi^r) + \sum_{\omega \neq 1} \alpha(\widetilde{L}, \omega) ,$$

so $\qquad \sigma(K,\chi^r) = \frac{1}{m} \sum_{\omega \neq 1} \alpha(\widetilde{L}, \omega)(\omega^{-r} - 1)$

$$= \frac{1}{m} \sum_{s=1}^{m-1} \alpha(\widetilde{L}, e^{2\pi i s/m}) (\cos\frac{2\pi rs}{m} - 1)$$
$$= -\frac{2}{m} \sum_{s=1}^{m-1} \alpha(\widetilde{L}, e^{2\pi i s/m}) \sin^2(\frac{\pi rs}{m})$$

If L is the lens space $L_{mn,q}$, then $L = S^3/C_{mn}$, $\widetilde{L} = S^3/C_n$, where C_{mn} acts on S^3 by the formula $\zeta(z,w) = (\zeta z, \zeta^q w)$. This implies the choice of a particular character χ on $H_1(L)$. Let $p : C_{mn} \to C_m$ be the projection.

A lemma of Hirzebruch [4] states that $\alpha(\widetilde{L}, \omega) = \frac{1}{n} \sum_{\zeta \in p^{-1}(\omega)} \alpha(S^3, \zeta)$. Putting $\zeta = e^{2\pi i s/mn}$, so $\omega = e^{2\pi i s/m}$, we obtain :

$$\sigma(K,\chi^r) = -\frac{2}{mn} \sum_{s=1}^{mn-1} \alpha(S^3, e^{2\pi i s/mn}) \sin^2(\frac{\pi rs}{m})$$

The action of C_{mn} on S^3 extends to a linear action on B^4 with one fixed point at the origin. By the Atiyah-Singer G-signature theorem [1] :

$$\alpha(S^3, e^{2\pi i s/mn}) = \cot\frac{\pi s}{mn} \cot\frac{\pi qs}{mn} ,$$

so $\qquad \sigma(K,\chi^r) = -\frac{2}{mn} \sum_{s=1}^{mn-1} \cot\frac{\pi s}{mn} \cot\frac{\pi qs}{mn} \sin^2(\frac{\pi rs}{m})$

Since q is prime to m , it will suffice to calculate $\sigma(K,\chi^{qr})$. Putting $z = e^{\pi i/mn}$ gives $\sigma(K,\chi^{qr}) = -\frac{1}{2mn} \sum_{s=1}^{mn-1} P(z^s)$, where

$$P(z) = \left(\frac{z+z^{-1}}{z-z^{-1}}\right) \left(\frac{z^q+z^{-q}}{z^q-z^{-q}}\right) (z^{nqr} - z^{-nqr})^2 =$$
$$= (z+z^{-1})(z^q+z^{-q})(z^{nqr-1}+z^{nqr-3}+\ldots+z^{1-nqr})(z^{(nr-1)q}+z^{(nr-3)q}+\ldots+z^{(1-nr)q}) .$$

Since $P(z) = P(z^{-1})$,

$$\sigma(K,\chi^{qr}) = \frac{P(1) + P(-1)}{4mn} - \frac{1}{4mn} \sum_{s=0}^{2mn-1} P(z^s)$$
$$= \frac{2nqr^2}{m} - \frac{1}{2} \sum_t (\text{coefficient of } z^{2mnt} \text{ in } P(z)) .$$

By considering the contribution made by the term $z^{(nr-1-2s)q}$ in the last bracket of $P(z)$, we find :

$$\sum_t \text{(coefficient of } z^{2mnt} \text{ in } P(z)) =$$

$$= \sum_{s=0}^{nr-1} \left(\lceil \frac{qs-1}{mn} \rceil + 1 + \lceil \frac{q(nr-s)}{mn} \rceil + 1 + \lceil \frac{q(nr-s)-1}{mn} \rceil + \lceil \frac{q(s+1)-1}{mn} \rceil + 1 + \right.$$

$$\left. + \lceil \frac{q(nr-s-1)}{mn} \rceil + \lceil \frac{q(s+1)}{mn} \rceil + 1 + \lceil \frac{q(nr-s-1)-1}{mn} \rceil \right) .$$

Let $\Delta(x,y)$ be the triangle whoses vertices has coordinates $(0,0)$, $(x,0)$, (x,y) . Let $\text{int } \Delta(x,y)$ be the number of integer points in $\Delta(x,y)$, where boundary points count $\frac{1}{2}$, $(0,0)$ is not counted and other vertices count $\frac{1}{4}$. The above sum is then $8 \text{ int } \Delta(nr,\frac{qr}{m})$, so we have the formula $\sigma(K,\chi^{qr}) = 4(\text{area } \Delta(nr,\frac{qr}{m}) - \text{int} \Delta(nr,\frac{qr}{m}))$.

COROLLARY. If $\frac{q}{p}$ is a ribbon knot, then $p = m^2$, and , for $r = 1,2,\ldots,m-1$,
$4(\text{area } \Delta(mr,\frac{qr}{m}) - \text{int } \Delta(mr,\frac{qr}{m})) = \pm 1$.

We showed earlier that the knot $K = \frac{2}{4t+1}$ has null-cobordant Seifert form if $t = u(u-1)$, in which case $4t+1 = (2u-1)^2 = m^2$. We easily compute that $\sigma(K,\chi^{2r}) = 4r^2 - 2mr+1$ if $2r < m$. In particular, $\sigma(K,\chi^2) < -1$ if $m > 3$, so $\frac{2}{m^2}$ is a ribbon knot if and only if $m = 3$ (stevedore's knot) or 1 (unknot).

The only known values of q/m^2 satisfying the condition of the corollary are :

(1) $q = km \pm 1$, with k , m coprime ;

(2) $q = (m \pm 1)d$, with $d | 2m \mp 1$;

(3) $q = (m \mp 1)d$ or $(2m \mp 1)(m \pm 1)/d$, with $d/m \pm 1)$ and d odd .

These are all ribbon knots, and for $m \leq 105$, they are the only knots satisfying the condition (this was verified by a Hewlett-Packard calculator).

Now we consider to what extent Theorem 1 holds for slice (instead of ribbon) knots. To push through the proof, we should have to answer the following question.

Problem. Which 3-dimensional lens spaces bound compact 4-manifolds W with $\widetilde{H}_*(W;\mathbb{Q}) = 0$?

The method described above deals with the analogous question for manifolds W with cyclic fundamental group. By using the group-theoretic lemma in [7] , one can

deal with manifolds with $H_1(W)$ cyclic. The general question seems difficult.

We avoid this problem by considering infinite coverings instead of finite branched coverings. Fortunately the result can be expressed in terms of the invariants $\sigma(K,\chi)$ calculable from the double branched covering.

Let $K \subset S^3$ be a knot, and let M be the closed 3-manifold obtained from S^3 by surgery on K . We use the null-homologous framing for K , so $H_*(M;\mathbb{Z}) \cong H_*(S^1 \times S^2; \mathbb{Z})$. We write M_n for the n-fold cyclic covering of M (including $n = \infty$) , and (temporarily) write $(S^3 - K)_n$ for the n-fold cyclic covering of $S^3 - K$. The map $H_1((S^3-K)_n) \to H_1(M_n)$ is clearly surjective, and has kernel generated by a parallel of K . But a parallel of K bounds a surface in $(S^3-K)_n$ (obtained by lifting a Seifert surface for K) , so $H_1(M_n)$ is isomorphic to $H_1((S^3-K)_n)$.

Let L_n be the n-fold branched cyclic covering of K ; we no longer assume that L_n is a lens space. Let $\chi : H_1(L_n) \to \mathbb{C}^*$ be a character of order m . By composition with the maps, $H_1(M_n) \cong H_1((S^3-K)_n) \to H_1(L_n)$, χ induces m-fold coverings $\widetilde{M}_n$ of M_n and $\widetilde{M}_\infty$ of M_∞. Observe that $\widetilde{M}_\infty$ is an Abelian covering of M_n with group $C_m \times C_\infty$.

Since the bordism group $\Omega_3(K(C_m \times C_\infty,1))$ is finite, rM_n bounds a compact 4-manifold V_n over $K(C_m \times C_\infty,1)$, for some r . Let V_∞ , $\widetilde{V}_n$, $\widetilde{V}_\infty$ be the induced coverings. We have diagrams :

$$r \left(\begin{array}{ccc} \widetilde{M}_\infty & \longrightarrow & \widetilde{M}_n \\ \downarrow & & \downarrow \\ M_\infty & \longrightarrow & M_n \end{array} \right) = \partial \left(\begin{array}{ccc} \widetilde{V}_\infty & \longrightarrow & \widetilde{V}_n \\ \downarrow & & \downarrow \\ V_\infty & \longrightarrow & V_n \end{array} \right) .$$

We identify the group ring $\mathbb{Z}[C_m \times C_\infty]$ with the Laurent polynomial ring $\mathbb{Z}[C_m][t,t^{-1}]$. Recall that k is the cyclotomic field $\mathbb{Q}(C_m)$; we write $k(t)$ for the field of rational functions over k . This is a module over $\mathbb{Z}[C_m \times C_\infty]$, easily seen to be flat. Let $C_*(\widetilde{V}_\infty)$ be a chain complex for $\widetilde{V}_\infty$, on which the covering translations induce a $\mathbb{Z}[C_m \times C_\infty]$-module structure. Write $H_*(V_n;k(t))$ for the homology of $C_*(\widetilde{V}_\infty) \otimes_{\mathbb{Z}[C_m \times C_\infty]} k(t)$; since $k(t)$ is flat, this is isomorphic to

$H_*(\widetilde{V}_\infty;\mathbf{Z}) \otimes_{\mathbf{Z}[C_m \times C_\infty]} k(t)$.

According to Wall [11] , there is an intersection pairing

$H_2(V_n;k(t)) \times H_2(V_n; k(t)) \to k(t)$. This is hermitian with respect to the involution

sending $p(t).q(t)$ to $\overline{p}(t^{-1})/\overline{q}(t^{-1})$. We shall soon see that it is non-singular if m is

a prime-power. Again following Wall, we write $L_0(k(t))$ for the Witt group of non-

singular hermitian forms on finite-dimensional $k(t)$-modules. Assuming that the inter-

section pairing is non-singular, it represents an element $t(V_n)$ of $L_0(k(t))$ (since $k(t)$

is a field, this assumption is not really necessary). Let $t_0(V_n)$ be the image of the

intersection pairing on $H_2(V_n;\mathbf{Q})$ in $L_0(k(t))$, and define :

$$\tau(K,\chi) = \frac{1}{r}(t(V_n) - t_0(V_n)) \in L_0(k(t)) \otimes_{\mathbf{Z}} \mathbf{Q} .$$

This is clearly independent of r , and the proof that $\sigma(K,\chi)$ is independent of W

extends to show that $\tau(K,\chi)$ is independent of V_n . One step is to show that

$t(V_n \cup (-V_n')) = t(V_n) - t(V_n')$; this is simplified if the intersection pairing is non-singu-

lar, for then $H_*(M_n;k(t)) = 0$. The next lemma shows this if m is a prime-power ;

it has more important applications to slice knots.

LEMMA 4. Let X be a connected infinite cyclic covering of a finite complex Y . Let

$\widetilde{X}$ be a regular p^r-fold covering of X , with p prime. If $H_*(Y;\mathbf{Z}_p) \cong H_*(S^1;\mathbf{Z}_p)$,

then $H_*(\widetilde{X};\mathbf{Q})$ is finite dimensional.

COROLLARY. If m is a prime-power, then $H_*(M_n;k(t)) = 0$ and the intersection

pairing on $H_2(V_n;k(t))$ is non-singular.

Proof. Let T be a tubular neighbourhood of K and let $X = \overline{S^3 - T}$. By

lemma 4, $H_*(\widetilde{X}_\infty;\mathbf{Q})$ is finite-dimensional. Since $\widetilde{M}_\infty = \widetilde{X}_\infty \cup (m$ copies of $\mathbf{R}^1 \times D^2)$,

$H_*(\widetilde{M}_\infty;\mathbf{Q})$ is finite-dimensional. Therefore :

$$H_*(M_n;k(t)) \cong H_*(\widetilde{M}_\infty;\mathbf{Q}) \otimes_{\mathbf{Z}[C_m \times C_\infty]} k(t) = 0 .$$

The non-singularity follows, using the exact sequence of (V_n,M_n) . To prove lemma 4,

we need two more lemmas.

LEMMA 5. Let $\widetilde{X}$ be a p-fold cyclic covering of X , with p prime. If $H_*(X;\mathbf{Z}_p)$

is finite, then $H_*(\widetilde{X};\mathbf{Z}_p)$ is finite.

Proof. Let C_p be the group of covering translations. The spectral sequence

of the covering has $E^2_{i,j} = H_i(C_p;H_j(\widetilde{X};\mathbf{Z}_p))$ and $\bigoplus_{i+j=k} E^\infty_{i,j} \cong H_k(X;\mathbf{Z}_p)$. It follows

that $E^\infty_{0,k}$ is finite for all k .

Suppose that $H_j(\widetilde{X};\mathbf{Z}_p)$ is finite for all $j < k$. Then $E^2_{i,j}$ is finite for $j < k$.

In particular, the differential $d^r : E^r_{r,k+1-r} \to E^r_{0,k}$ has finite rank for all $r \geq 2$.

Since $E^\infty_{0,k}$ is finite, $E^2_{0,k} = H_0(C_p;H_k(\widetilde{X};\mathbf{Z}_p))$ is finite. Let $T : H_k(\widetilde{X};\mathbf{Z}_p) \to H_k(\widetilde{X};\mathbf{Z}_p)$

be induced by a generator of C_p . Then $E^2_{0,k} = \text{coker}(T-1)$ and $(T-1)^p = T^p - 1 = 0$.

It follows that $\dim H_k(\widetilde{X};\mathbf{Z}_p) \leq p \dim E^2_{0,k}$ is finite. By induction, $H_*(\widetilde{X};\mathbf{Z}_p)$ is

finite.

LEMMA 6. Let X be an infinite cyclic covering of a finite complex. If $H_*(X;\mathbf{Z}_p)$ is

finite for some prime p , then $H_*(X;\mathbf{Q})$ is finite-dimensional.

Proof. The homology of X may be calculated from a chain complex of finitely

generated free $\mathbf{Z}[t,t^{-1}]$-modules. Since $\mathbf{Z}[t,t^{-1}]$ is noetherian, the homology groups

of X are finitely generated $\mathbf{Z}[t,t^{-1}]$-modules.

For fixed n , let $A = H_n(X;\mathbf{Z})$. By the universal coefficient theorem,

$$H_n(X;\mathbf{Q}) \cong A \otimes_{\mathbf{Z}} \mathbf{Q} \oplus \text{Tor}_{\mathbf{Z}}(H_{n-1}(X);\mathbf{Q}) = A \otimes_{\mathbf{Z}} \mathbf{Q} .$$

Since $\mathbf{Q}[t,t^{-1}]$ is a principal ideal domain, $A \otimes_{\mathbf{Z}} \mathbf{Q}$ is a direct sum of finitely many

cyclic $\mathbf{Q}[t,t^{-1}]$-modules.

If $A \otimes_{\mathbf{Z}} \mathbf{Q}$ has infinite dimension, then at least one of its cyclic summands

must be free. So there is a non-zero $\mathbf{Q}[t,t^{-1}]$-homomorphism $A \otimes_{\mathbf{Z}} \mathbf{Q} \to \mathbf{Q}[t,t^{-1}]$;

restriction to $A \subset A \otimes_{\mathbf{Z}} \mathbf{Q}$ gives a non-zero $\mathbf{Z}[t,t^{-1}]$-homomorphism $f : A \to \mathbf{Q}[t,t^{-1}]$.

Since A is finitely generated over $\mathbf{Z}[t,t^{-1}]$, there is a non-zero integer r such

that $(rf)(A) \subseteq \mathbf{Z}[t,t^{-1}]$. Now $(rf)(A)$ is a non-zero ideal in $\mathbf{Z}[t,t^{-1}]$, so its

additive group is free abelian of infinite rank. This implies that $A \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is infinite.

On the other hand, $A \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is isomorphic to a subgroup of $H_n(X;\mathbf{Z}_p)$. By

hypothesis, $H_n(X;\mathbb{Z}_p)$ is finite, so we have a contradiction. This proves that $A \otimes_\mathbb{Z} \mathbb{Q}$ has finite dimension, as required.

Proof of lemma 4. Following Milnor [8], we first show that $H_*(X;\mathbb{Z}_p)$ is finite. Let $t : H_*(X;\mathbb{Z}_p) \to H_*(X;\mathbb{Z}_p)$ be induced by a generator of the group of covering translations. From the exact sequence :

$$H_{n+1}(Y;\mathbb{Z}_p) \to H_n(X;\mathbb{Z}_p) \xrightarrow{t-1} H_n(X;\mathbb{Z}_p) \to H_n(Y;\mathbb{Z}_p) ,$$

we see that $t-1$ is an automorphism of $H_n(X;\mathbb{Z}_p)$ for all $n \geq 1$. $H_n(X;\mathbb{Z}_p)$ is a finitely generated $\mathbb{Z}_p[t,t^{-1}]$-module ; by induction on the number of generators, it follows that $H_n(X;\mathbb{Z}_p)$ is finite.

First suppose that $\widetilde{X}$ is a p-fold cyclic covering of X . By lemma 5, $H_*(\widetilde{X};\mathbb{Z}_p)$ is finite. The result will follow from lemma 6, if we show that $\widetilde{X}$ is an infinite cyclic covering of a finite complex.

Let $h : X \to X$ generate the group of covering translations of X . Let G be the image of $H_1(\widetilde{X};\mathbb{Z}_p)$ in $H_1(X;\mathbb{Z}_p)$. Since $H_1(X;\mathbb{Z}_p)$ is finite, there is a non-zero integer r such that $(h_*)^r(G) = G$. It follows that $h^r : X \to X$ lifts to a homeomorphism $\widetilde{h}$ of $\widetilde{X}$, and $\widetilde{X}$ is an infinite cyclic covering of the finite complex $\widetilde{X}/\widetilde{h}$.

Finally, observe that a regular p^r-fold covering map $\widetilde{X} \to X$ can be factorised into p-fold cyclic covering maps. The result follows or p^r-fold coverings by repeating the argument given above.

We have shown (lemma 2) that the double branched covering of any smooth 2-disc in B^4 is a rational homology ball. The proof extends to show the same branched cyclic coverings of prime-power order. Similarly, the branched covering L_n of $K \subset S^3$ has the rational homology of S^3 i n is a prime-power. There is then a linking form $\lambda : H_1(L_n) \times H_1(L_n) \to \mathbb{Q}/\mathbb{Z}$ which is symmetric and non-singular.

THEOREM 2. Let the knot K have n-fold branched cyclic covering L_n , with n a prime-power. If K is slice, then there is a subgroup G of $H_1(L_n)$ such that

$\lambda(G \times G) = 0$ and $\tau(K,\chi) = 0$ for every character χ of prime-power order with $\chi(G) = 0$.

Proof. Suppose $K = \partial D$ with $D \subset B^4$, and let U be a regular neighbourhood of D . Let V_r be the r-fold cyclic covering of $\overline{B^4-U}$, and let W_r be the r-fold cyclic covering of B^4 branched over D . Observe that $M_r = \partial V_r$ and $L_r = \partial W_r$. Let G be the kernel of the map $i_* : H_1(L_n) \to H_1(W_n)$ induced by inclusion. Since $\widetilde{H}_*(W_n;\mathbb{Q}) = 0$, λ vanishes on $G \times G$.

Let $\chi : H_1(L_n) \to C^*$ be a character of order m with $\chi(G) = 0$. There is a character $\chi' : H_1(W_n) \to C^*$ of order m^ℓ (for some ℓ) with $\chi = \chi' i_*$. We have commutative diagram :

$$
\begin{array}{ccc}
H_1(M_n) \longrightarrow & H_1(L_n) \xrightarrow{\quad \chi \quad} & C^* \\
\downarrow i_* & \downarrow i_* & \nearrow \chi' \\
H_1(V_n) \longrightarrow & H_1(W_n) &
\end{array}
$$

Let $\widetilde{V}_n$, $\widetilde{W}_n$ be the m-fold coverings induced by χ' ; we can also form the induced covering $\widetilde{V}_\infty$ of V_∞ .

We claim that V_n can be used to compute $\tau(K,\chi)$, even though $\partial \widetilde{V}_n$ is a disconnected covering of M_n . To justify this, suppose M_n bounds V_n' over $C_m \times C_\infty$ and let $\widetilde{V}_n'$, $\widetilde{V}_\infty'$ be the m^ℓ-fold coverings induced by the composite $H_1(V_n') \to C_m \subset C_{m\ell}$. Then :

$$t(V_n) - t(V_n') = t(V_n \cup (-V_n')) = t_o(V_n \cup (-V_n'))$$
$$= t_o(V_n) - t_o(V_n') ;$$

so

$$t(V_n) - t_o(V_n) = t(V_n') - t_o(V_n') .$$

This equation takes place in $L_o(\overline{k}(t))$ where $\overline{k}$ is the cyclotomic field $\mathbb{Q}(C_{m^\ell}) \subset C$. Thus $\overline{k}(t)$ is an extension field of $k(t)$ of degree $m^{\ell-1}$. We have a map $L_o(k(t)) \xrightarrow{p} L_o(\overline{k}(t))$ induced by tensoring with $\overline{k}(t)$ over $k(t)$. This map is also given ty taking a matrix with coefficients in $k(t)$ for a representative class in $L_o(k(t))$ and viewing it as a matrix with coefficients in $\overline{k}(t)$. Now $\widetilde{V}_\infty'$ is a disjoint union of m-fold coverings of V_∞' permuted cyclically by $C_{m^{\ell-1}}$. Thus $t(V_n') - t_o(V_n')$ is the image $\tau(K,\chi)$ under this map (after tensoring with $\mathbb{Q}$). Thus we need to see that

$\rho \otimes id_{\mathbb{Q}}$ is injective. To see this one defines a map $L_0(\bar{k}(t)) \xrightarrow{T} L_0(k(t))$ as follows. Let A be a vector space over $(\bar{k}(t)$ and $\langle\ ,\ \rangle : A \times A \to \bar{k}(t)$ a hermitian form, then we may view A as a vector space over $k(t)$ (the dimension is multiplied by $m^{\ell-1}$) and define a $k(t)$ hermitian form on A by the formula $\langle x,y \rangle ' = \frac{1}{m^{\ell-1}} \operatorname{Tr} \langle x,y \rangle$, where Tr is the trace homomorphism $\bar{k}(t) \to k(t)$. This map is well defined on the Witt level. Since these Witt groups are generated by one dimensional forms, it is not difficult to see that the composition $T \circ \rho$ is multiplication by $m^{\ell-1}$. Therefore $\rho \otimes id_{\mathbb{Q}}$ is injective.

Now suppose m (the order of χ) is prime-power. $\widetilde{V}_\infty$ is an m^ℓ-fold cyclic covering of V_∞, which is an infinite cyclic covering of $\overline{B^4-U}$. By lemma 4 , $H_2(\widetilde{V}_\infty; \mathbb{Q})$ has finite dimension over $\mathbb{Q}$, so it is a torsion module over $\mathbb{Z}[t,t^{-1}]$. It follows that :

$$H_2(V_n; k(t)) \cong H_2(\widetilde{V}_\infty; \mathbb{Z}) \otimes_{\mathbb{Z}[C_m \times C_\infty]} k(t) = 0 .$$

This proves that $t(V_n) = 0$; it only remains to observe that $H_2(V_n; \mathbb{Q}) = 0$ (by the extension of lemma 2) so $t_0(V_n) = 0$. Therefore $\tau(K,\chi) = 0$.

We now explain a relation between $\sigma(K,\chi)$ and $\tau(K,\chi)$ leading to a partial calculation of τ for 2-bridge knots. Let V be a vector space over $k(t)$ and let $\varphi : V \times V \to k(t)$ be non-singular and hermitian with respect to the involution $p(t).q(t) \longleftrightarrow \bar{p}(t^{-1})/\bar{q}(t^{-1})$. We can define a signature $\sigma_\alpha(\varphi)$ for each complex number α of modulus 1 , as follows.

Let A be a matrix for φ over $k(t)$; we write $A = A(t)$ to indicate that the entries of A are rational functions in t . If $|\alpha| = 1$ and all entries of $A(\alpha)$ are finite, then $A(\alpha)$ is hermitian and has a signature $\sigma_\alpha(A)$. Observe that $\sigma_\alpha(A)$ is constant in a neighbourhood of α unless $\det(A(\alpha)) = 0$ or some entry of $A(\alpha)$ is infinite. So $\sigma_\alpha(A)$ is a step-function with finitely many discontinuities. At each discontinuity α , we re-define $\sigma_\alpha(A)$ to be the average of the one-sided limits of $\sigma_\beta(A)$ as β tends to α .

If B is another matrix for φ , then $B = \bar{P}'AP$ for some non-singular matrix

P over $k(t)$. It follows that $\sigma_\alpha(A) = \sigma_\alpha(B)$ except at finitely many values of α , and this implies that $\sigma_\alpha(A) = \sigma_\alpha(B)$ for all α . We define $\sigma_\alpha(\varphi) = \sigma_\alpha(A)$, and note that σ_α induces a homomorphism from $L_0(k(t)) \otimes_{\mathbb{Z}} \mathbb{Q}$ into $\mathbb{Q}$.

THEOREM 3. Let $K \subset S^3$ have n-fold branched cyclic covering L_n . Let χ be a non-constant character on $H_1(L_n)$, inducing a covering $\widetilde{L}_n$ of L_n . If $H_1(\widetilde{L}_n; \mathbb{Q}) = 0$, then $|\sigma(K,\chi) - \sigma_1\tau(K,\chi)| \le 1$.

Proof. Let M_n be constructed as above and suppose rM_n bounds a compact 4-manifold V_n over $C_m \times C_\infty$. We may do surgery on V_n until $\pi_1(V_n) \cong C_m \times C_\infty$, making $\widetilde{V}_\infty$ simply connected. Recall that $H_*(\widetilde{V}_\infty)$ is a $\mathbb{Z}[C_m][t,t^{-1}]$-module, where t represents a covering translation of $\widetilde{V}_\infty$ over $\widetilde{V}_n$. From the exact sequence of this covering; :

$$H_2(\widetilde{V}_n) \cong \operatorname{coker}(t-1 : H_2(\widetilde{V}_\infty) \to H_2(\widetilde{V}_\infty))$$
$$\cong H_2(\widetilde{V}_\infty) \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z} .$$

By tensoring over $\mathbb{Z}[C_m]$ with k , we obtain $H_2(V_n;k) \cong H_2(V_\infty;k) \otimes_{k[t,t^{-1}]} k$.

The intersection form on $H_2(V_n;k(t))$ is obtained from the pairing $H_2(\widetilde{V}_\infty) \times H_2(\widetilde{V}_\infty) \to \mathbb{Z}[C_m][t,t^{-1}]$ by tensoring over $\mathbb{Z}[C_m][t,t^{-1}]$ with $k(t)$. It follows that the intersection form on $H_2(V_n;k(t))$ has a matrix $A(t)$ all of whose entries lie in $k[t,t^{-1}]$. Observe that $A(1)$ is a matrix for the intersection form on $H_2(V_n; k)$.

Let $A(1)$ have rank ρ and nullity ν , and let $B(t)$ be the $\rho \times \rho$ submatrix in the top left-hand corner of $A(t)$. We may transform $A(t)$ by a permutation matrix to make $B(1)$ non-singular. Then $B(\alpha)$ is non-singular for all α close to 1 , so $\sigma_\alpha(B)$ is constant in some neighbourhood of 1 .

Observe that $|\sigma_\alpha(A) - \sigma_\alpha(B)| \le \nu$ for all α close to 1 . Since $B(1)$ has the same rank as $A(1)$, $\sigma_1(B) = \sigma_1(A)$ = signature of $H_2(V_n;k) = s(V_n)$. Let σ_1 be the average of the one-sided limits of $\sigma_\alpha(A)$ as α approaches 1 ; it follows that $|\sigma_1 - s(V_n)| \le \nu$. By definition, $\sigma_1\tau(K,\chi) = \frac{1}{r}(\sigma_1 - s_0(V_n))$, so $|\sigma_1\tau(K,\chi) - \frac{1}{r}(s(V_n) - s_0(V_n))| \le \frac{\nu}{r}$. By the proof of invariance of

$\sigma(K,\chi), \frac{1}{r}(s(V_n) - s_o(V_n))$ depends only on M_n and χ ; the same is true of $\frac{\nu}{r}$.

Suppose rL_n bounds a compact 4-manifold W_n over C_m . Since M_n can be obtained from L_n by a single surgery, rM_n bounds a 4-manifold V_n obtained from W_n by attaching r 2-handles. Although V_n is not the manifold which was called V_n above, the values of $\frac{1}{r}(s(V_n) - s_o(V_n))$ and ν agree.

By hypothesis, $\widetilde{L}_n$ is a rational homology sphere. Observe that $H_2(V_n;k)$ has a subspace of codimension r isometric with $H_2(W_n;k)$. The intersection form on $H_2(W_n;k)$ is non-singular, and the form on $H_2(V_n;k)$ has nullity ν . It follows that $|s(V_n) - s(W_n)| \leq r - \nu$. A similar argument shows that $s_o(V_n) = s_o(W_n)$, so

$$|\sigma_1 \tau(K;\chi) - \sigma(K;\chi)| \leq \frac{\nu}{r} + (1 - \frac{\nu}{r}) = 1 .$$

Let K be the 2-bridge knot $\frac{q}{p}$, with double branched covering $L = L_{p,q}$. If K is slice, then p is a square (lemma 1) , say $p = \ell^2$. By theorem 2, $\tau(K,\chi) = 0$ for every character χ on $H_1(L)$ of prime-power order dividing ℓ . By theorem 3, $|\sigma(K,\chi)| \leq 1$ for every such character.

Suppose $\ell^2 = mn$ with m a prime-power dividing ℓ . We showed that (for some character χ of order m) :

$$\sigma(K,\chi^{qr}) = 4 (\text{area } \Delta(nr, \frac{qr}{m}) - \text{int } \Delta(nr, \frac{qr}{m})) .$$

For K to be slice, this expression (which is always an odd integer if $r \not\equiv 0(m)$) must be ± 1 for $r = 1, 2, \ldots, m-1$. Consider the example $K = 2/\ell^2$, whis has null-cobordant Seifert form. We obtain, for $2r < m$, $\sigma(K,\chi^{2r}) = 4\frac{nr^2}{m} - 2nr + 1$. In particular (since $m > 2$) :

$$\sigma(K,\chi^2) = 4\frac{n}{m} - 2n + 1 = 1 - 2n(1 - \frac{2}{m}) .$$

If $\ell > 3$, then $\ell \geq 5$ and ℓ has a prime-power factor $m \geq 5$. Since $\ell^2 = mn$ and $m | \ell$, we have $n \geq 5$, so :

$$\sigma(K,\chi^2) \leq 1 - 10(1 - \frac{2}{5}) = -5 < -1 ,$$

so K cannot be slice. This proves the result announced at the beginning, that $2/\ell^2$ is slice if and only if $\ell = 3$ (stevedore's knot) or $\ell = 1$ (unknot).

Observe that, in contrast to theorem 1, theorem 2 is restricted to characters χ of prime power order. As a consequence, there are knots (for example, the 2-bridge knot $94/225$) which cannot bound ribbons but which could conceivably be slice.

Finally, we remark that the condition that K be slice could be replaced throughout by the condition that K bound a smooth 2-disc in a homology 4-ball.

REFERENCES

[1] M.F. ATIYAH and I.M. SINGER, The index of elliptic operators III, Ann. of Math. 87 (1968), 546-604.

[2] J.H. CONWAY, An enumeration of knots and links and some of their algebraic properties, Computational Problems in Abstract Algebra, ed. John Leech, Pergamon Press, Oxford and New York, 1969, 329-358.

[3] R.H. FOX, A quick trip through knot theory, Topology of 3-manifolds and Related Topics, Prentice Hall Inc., Englewood Cliffs, N.J. 1962, 120-167.

[4] F. HIRZEBRUCH, Free involutions on manifolds and some elementary number theory, Symposia Mathematica,(Instituto nazionale di alta matematica, Roma), vol. V, Academic Press, 1971, 411-419.

[5] M. KERVAIRE, Les noeuds de dimensions supérieures, Bull. Soc. Math. France 93 (1965), 225-271.

[6] J. LEVINE, Knot cobordism groups in codimension two, Comm. Math. Helv. 44 (1969), 229-244.

[7] W.S. MASSEY, Proof of a conjecture of Whitney, Pacific J. Math. 31 (1969), 143-156.

[8] J. MILNOR, Infinite cyclic coverings, Conference on the Topology of Manifolds, Prindle, Weber and Schmidt, Boston, Mass., 1968, 115-133.

[9] H. SCHUBERT, Knoten mit zwei Brücken, Math. Zeit. 65 (1956), 133-170.

[10] A.G. TRISTAM, Some cobordism invariants for links, Proc. Camb. Phil. Soc. 66 (1969), 251-264.

[11] C.T.C. WALL, Surgery on Compact Manifolds, Academic Press, London and New York, 1970.

Appendix to Cobordism of Classical Knots by A. Casson & C. Gordon

REMARKS ON SUBSEQUENT DEVELOPMENTS

P.M. GILMER

Further progress has been made in several directions since this seminal paper first became available (1975).

The most recent and most dramatic follows from Donaldson's result that certain quadratic forms are not the intersection pairings for simply connected 4-manifolds. Previously Casson had noted that if every knot with Alexander polynomial one is slice, then every quadratic form can be realized as above. Thus, it follows that some knots with trivial Alexander polynomial are not slice. In fact, Casson's argument is constructive. So one can find such knots explicity. The branched cyclic covers of Alexander polynomial one knots are homology spheres. Thus, there are no nontrivial Casson-Gordon invariants. Such a knot also has a null-cobordant Seifert matrix.

Following more directly in the footsteps of "Cobordism of Classical Knots", much work has been done. Gilmer [1] showed these methods could be used to get lower bounds on the slice genus of a knot (the minimal genus for a smooth proper surface in D^4 with boundary K) .

Litherland [2] has a formula for $\tau(K,\chi)$ (for χ associated to L_2) in terms of data derived from a Seifert surface for K . This generalizes a formula for $\tau(K,\chi)$ in the genus-one case and an estimate of $\sigma_1(\tau(K,\chi))$ due to Gilmer [3] .

Gilmer also showed that the characters χ which must extend across W_2 could be more precisely identified, and related to the Seifert form [3] . In this way the work of Levine and that of Casson and Gordon could be combined in a nontrivial way yielding further examples of non-slice knots.

The most interesting are examples, due to Livingston [4], of knots which cannot be cobordant to their reverses. The reverse of a knot is obtained by simply reversing the string orientation of a knot. Thus, a knot and its reverse have identical exteriors and branched cyclic covers (viewing these objects as oriented manifolds. Thus the Casson-Gordon invariants must fail to distinguish them. Their Seifert pairings are cobordant as well.

There is also an interesting preprint by Daryl Cooper [5] which considers a particular family of knots and obtains very strong results. His method is related to that of Casson and Gordon but is definitely different. Casson-Gordon invariants have also found other related uses in topology. For instance in his recent Ph. D. thesis , Ruberman [6] found new obstructions to a knot being the slice of an unknot in every dimension [6] .

[1] P.M. GILMER, On the slice genus of knots, Invent. Math. 66 (1982), 191-197.

[2] R.A. LITHERLAND, A formula for the Casson-Gordon invariants of a knot,
        Preprint.

[3] P.M. GILMER, Slice Knots in S^3 , to appear in Quarterly Jour. Math.

[4] C. LIVINGSTON, Knots which are not cobordant to their reverses,
        to appear in Quarterly Jour. of Math.

[5] D. COOPER, Detecting Knots which are not slice I, preprint, University of
        Warwick (1980).

[6] D. RUBERMAN, Doubly slice knots and the Casson-Gordon invariants,
        Ph. D. Thesis, University of California at Berkeley, CA (1982).