I.1

Deloopings in Algebraic *K*-Theory*

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* Supported in part by NSF DMS 0104162

1.1 Introduction

A crucial observation in Quillen's definition of higher algebraic K-theory was that the right way to proceed is to define the higher K-groups as the homotopy groups of a space ([23]). Quillen gave two different space level models, one via the plus construction and the other via the Q-construction. The Q-construction version allowed Quillen to prove a number of important formal properties of the K-theory construction, namely localization, devissage, reduction by resolution, and the homotopy property. It was quickly realized that although the theory initially revolved around a functor K from the category of rings (or schemes) to the category $\underline{Top}$ of topological spaces, K in fact took its values in the category of infinite loop spaces and infinite loop maps ([1]). In fact, K is best thought of as a functor not to topological spaces, but to the category of *spectra* ([2,11]). Recall that a spectrum is a family of based topological spaces $\{X_i\}_{i\geq 0}$, together with bonding maps $\sigma_i : X_i \to \Omega X_{i+1}$, which can be taken to be homeomorphisms. There is a great deal of value to this refinement of the functor K. Here are some reasons.

- Homotopy colimits in the category of spectra play a crucial role in applications of algebraic K-theory. For example, the assembly map for the algebraic K-theory of group rings, which is the central object of study in work on the Novikov conjecture ([13,24]), is defined on a spectrum obtained as a homotopy colimit of the trivial group action on the K-theory spectrum of the coefficient ring. This spectrum homotopy colimit is definitely not the same thing as the homotopy colimit computed in the category $\underline{Top}$, and indeed it is clear that no construction defined purely on the space level would give this construction.
- The lower K-groups of Bass [5] can only be defined as homotopy groups in the category of spectra, since there are no negative homotopy groups defined on the category $\underline{Top}$. These groups play a key role in geometric topology [3,4], and to define them in a way which is consistent with the definition of the higher groups (i.e. as homotopy groups) is very desirable.
- When computing with topological spaces, there is a great deal of value in being able to study the homology (or generalized homology) of a space, rather than just its homotopy groups. A reason for this is that homology is relatively easy to compute, when compared with homotopy. One has the notion of *spectrum homology* , which can only be defined as a construction on spectra, and which is also often a relatively simple object to study. To simply study the homology of the zero-th space of a spectrum is not a useful thing to do, since the homology of these spaces can be extremely complicated.
- The category of spectra has the convenient property that given a map f of spectra, the fibre of f is equivalent to the loop spectrum of the cofibre. This linearity property is quite useful, and simplifies many constructions.

In this paper, we will give an overview of a number of different constructions of spectra attached to rings. Constructing spectra amounts to constructing "deloopings" of the K-theory space of a ring. We will begin with a "generic" construction, which applies to any category with an appropriate notion of direct sum. Because it is so generic, this construction does not permit one to prove the special formal properties mentioned above for the K-theory construction. We will then outline Quillen's Q-construction, as well as iterations of it defined by Waldhausen [32], Gillet–Grayson [15], Jardine [17], and Shimakawa [27]. We then describe Waldhausen's $S_.$-construction, which is a kind of mix of the generic construction with the Q-construction, and which has been very useful in extending the range of applicability of the K-theoretic methods beyond categories of modules over rings or schemes to categories of spectra, which has been the central tool in studying pseudo-isotopy theory ([33]). Finally, we will discuss three distinct constructions of non-connective deloopings due to Gersten–Wagoner, M. Karoubi, and Pedersen–Weibel. These constructions give interpretations of Bass's lower K-groups as homotopy groups. The Pedersen–Weibel construction can be extended beyond just a delooping construction to a construction, for any metric space, of a K-theory spectrum which is intimately related to the *locally finite homology* of the metric space. This last extension has been very useful in work on the Novikov conjecture (see [19]).

We will assume the reader is familiar with the technology of simplicial sets and multisimplicial sets, the properties of the nerve construction on categories, and the definition of algebraic K-theory via the plus construction ([16]). We will also refer him/her to [2] or [11] for material on the category of spectra.

1.2 Generic Deloopings Using Infinite Loop Space Machines

To motivate this discussion, we recall how to construct Eilenberg–MacLane spaces for abelian groups. Let A be a group. We construct a simplicial set $B_.A$ by setting $B_kA = A^k$, with face maps given by

$$d_0\left(a_0, a_1, \dots, a_{k-1}\right) = \left(a_1, a_2, \dots a_{k-1}\right)$$

$$d_i\left(a_0, a_1, \dots, a_{k-1}\right) = \left(a_0, a_1, \dots, a_{i-2}, a_{i-1} + a_i, a_{i+1}, \dots a_{k-1}\right) \text{ for } 0 < i < k$$

$$d_k\left(a_0, a_1, \dots, a_{k-1}\right) = \left(a_0, a_1, \dots, a_{k-2}\right) \ .$$

We note that due to the fact that A is abelian, the multiplication map $A \times A \to A$ is a homomorphism of abelian groups, so $B_.A$ is actually a simplicial abelian group. Moreover, the construction is functorial for homomorphisms of abelian groups, and so we may apply the construction to a simplicial abelian group to obtain a bisimplicial abelian group. Proceeding in this way, we may start with an abelian group A, and obtain a collection of multisimplicial sets $B^n_.A$, where $B^n_.A$ is an n-simplicial set. Each n-simplicial abelian group can be viewed as a simplicial abelian group by restricting to the diagonal $\Delta^{op} \subseteq (\Delta^{op})^n$, and we obtain a family of simplicial sets, which we also denote by $B^n_{\cdot}A$. It is easy to see that we have exact sequences of simplicial abelian groups

$$B^{n-1}_{\cdot}A \to E^n_{\cdot}A \to B^n_{\cdot}A$$

where $E^n_{\cdot}A$ is a contractible simplicial group. Exact sequences of simplicial abelian groups realize to Serre fibrations, which shows that

$$\left|B^{n-1}_{\cdot}A\right| \cong \Omega\left|B^n_{\cdot}A\right|$$

and that therefore the family $\{B^n_{\cdot}A\}_n$ forms a spectrum. The idea of the infinite loop space machine construction is now to generalize this construction a bit, so that we can use combinatorial data to produce spectra.

We first describe Segal's notion of Γ-spaces, as presented in [26]. We define a category Γ as having objects the finite sets, and where a morphism from X to Y is given by a function $\theta : X \to \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ denotes the power set of Y, such that if $x, x' \in X$, $x \neq x'$, then $\theta(x) \cap \theta(x') = \emptyset$. Composition of $\theta : X \to \mathcal{P}(Y)$ and $\eta : Y \to \mathcal{P}(Z)$ is given by $x \to \bigcup_{y\in\theta(x)} \eta(y)$. There is a functor from the category Δ of finite totally ordered sets and order preserving maps to Γ, given on objects by sending a totally ordered set X to its set of non-minimal elements X^-, and sending an order-preserving map from $f : X \to Y$ to the function θ_f, defined by letting $\theta_f(x)$ be the intersection of the "half-open interval" $(f(x-1), f(x)]$ with Y^-. ($x-1$ denotes the immediate predecessor of x in the total ordering on X if there is one, and if there is not, the interval $[x-1, x)$ will mean the empty set.) There is an obvious identification of the category Γ^{op} with the category of finite based sets, which sends an object X in Γ^{op} to X_+, X "with a disjoint base point added", and which sends a morphism $\theta : X \to \mathcal{P}(Y)$ to the morphism $f_\theta : Y_+ \to X_+$ given by $f_\theta(y) = x$ if $y \in \theta(x)$ and $f_\theta(y) = *$ if $y \notin \bigcup_x \theta(x)$. Let $\underline{n}$ denote the based set $\{1, 2, \ldots, n\}_+$. We have the morphism $p_i : \underline{n} \to \underline{1}$ given by $p_i(i) = 1$ and $p_i(j) = *$ when $i \neq j$.

1 **Definition 1** A *Γ-space* is a functor from Γ^{op} to the category of simplicial sets, so that

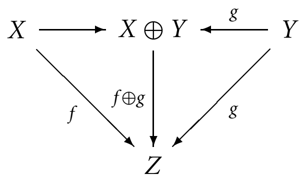
- $F(\emptyset_+)$ is weakly contractible.
- $\Pi_1^n F(p_i) : F(\underline{n}) \to \Pi_1^n F(\underline{1})$ is a weak equivalence of simplicial sets.

Note that we have a functor $\Delta^{op} \to \Gamma^{op}$, and therefore every Γ-space can be viewed as a simplicial simplicial set, i.e. a bisimplicial set.

We will now show how to use category theoretic data to construct Γ-spaces. Suppose that C denotes a category which contains a zero objects, i.e. an object which is both initial and terminal, in which every pair of objects $X, Y \in C$ admits a *categorical sum*, i.e an object $X \oplus Y \in C$, together with a diagram

$$X \to X \oplus Y \leftarrow Y$$

so that given any pair of morphisms $f : X \to Z$ and $g : Y \to Z$, there is a unique morphism $f \oplus g : X \oplus Y \to Z$ making the diagram



commute. We will now define a functor F_C from Γ^{op} to simplicial sets. For each finite based set X, we define $\Pi(X)$ to be the category of finite based subsets of X and inclusions of sets. Consider any functor $\varphi(X) : \Pi(X) \to C$. For any pair of based subsets $S, T \subseteq X$, we obtain morphisms $\varphi(S) \to \varphi(S \cup T)$ and $\varphi(T) \to \varphi(S \cup T)$, and therefore a well defined morphism $\varphi(S) \oplus \varphi(T) \to \varphi(S \cup T)$ for any choice of sum $\varphi(S) \oplus \varphi(T)$. We say the functor $\varphi : \Pi(X) \to C$ is *summing* if it satisfies two conditions.

- $\varphi(\emptyset)$ is a zero object in C
- For any based subsets $S, T \subseteq X$, with $S \cap T = \{*\}$, we have that the natural morphism $\varphi(S) \oplus \varphi(T) \to \varphi(S \cup T)$ is an isomorphism.

Let $Sum_C(X)$ denote the category whose objects are all summing functors from $\Pi(X)$ to C, and whose morphisms are all natural transformations which are isomorphisms at all objects of $\Pi(X)$.

We next observe that if we have a morphism $f : X \to Y$ of based sets, we may define a functor $Sum_C(X) \stackrel{Sum_C(f)}{\to} Sum_C(Y)$ by

$$Sum_C(f)(\varphi)(S) = \varphi(f^{-1}(S))$$

for any based subset $S \subseteq Y$. One verifies that this makes $Sum_C(-)$ into a functor from Γ^{op} to the category $\underline{CAT}$ of small categories. By composing with the nerve functor $N.$, we obtain a functor $Sp_1(C) : \Gamma^{op} \to s.sets$. Segal [26] now proves

Proposition 2 The functor $Sp_1(C)$ is a Γ-space. 2

The category $Sum_C(\emptyset)$ is just the subcategory of zero objects in C, which has contractible nerve since it has an initial object. The map $\prod_{i=1}^n Sp_1(p_i) : Sp_1(\underline{n}) \to \Pi_{i=1}^n Sp_1(C)(\underline{1})$ is obtained by applying the nerve functor to the functor $\prod_{i=1}^n Sum_C(p_i) : Sum_C(\underline{n}) \to \prod_{i=1}^n Sum_C(\underline{1})$. But this functor is an equivalence of categories, since we may define a functor $\theta : \prod_{i=1}^n Sum_C(\underline{1}) \to Sum_C(\underline{n})$ by

$$\theta\left(\varphi_1, \varphi_2, \dots, \varphi_n\right)\left(\{i_1, i_2, \dots, i_s\}\right) = \varphi_{i_1}\left(\underline{1}\right) \oplus \varphi_{i_2}\left(\underline{1}\right) \oplus \cdots \oplus \varphi_{i_s}\left(\underline{1}\right)$$

Here the sum denotes any choice of categorical sum for the objects in question. Any choices will produce a functor, any two of which are isomorphic, and it is easy to verify that $\prod_{i=1}^{n} Sum_C(p_i) \circ \theta$ is equal to the identity, and that $\theta \circ \prod_{i=1}^{n} Sum_C(p_i)$ is canonically isomorphic to the identity functor.

We observe that this construction is also functorial in C, for functors which preserve zero objects and categorical sums. Moreover, when C possesses zero objects and categorical sums, the categories $Sum_C(X)$ are themselves easily verified to possess zero objects and categorical sums, and the functors $Sum_C(f)$ preserve them. This means that we can iterate the construction to obtain functors $Sp_n(C)$ from $(\Gamma^{op})^n$ to the category of $(n+1)$-fold simplicial sets, and by restricting to the diagonal to the category of simplicial sets we obtain a family of simplicial sets we also denote by $Sp_n(C)$. We also note that the category $Sum_C(\underline{1})$ is canonically equivalent to the category C itself, and therefore that we have a canonical map from $N.C$ to $N.Sum_C(\underline{1})$. Since $Sum_C(\underline{1})$ occurs in dimension 1 of $Sp_1(C)$, and since $Sum_C(\emptyset)$ has contractible nerve, we obtain a map from $\Sigma N.C$ to $Sp_1(C)$. Iterating the Sp_1-construction, we obtain maps $\Sigma Sp_n(C) \to Sp_{n+1}(C)$, and hence adjoints $\sigma_n : Sp_n(C) \to \Omega Sp_{n+1}(C)$. Segal proves

3 **Theorem 3** The maps σ_n are weak equivalences for $n > 1$, and for $n = 0$, σ_0 can be described as a group completion. Taken together, the functors Sp_n yield a functor Sp from the category whose objects are categories containing zero objects and admitting categorical sums and whose morphisms are functors preserving zero objects and categorical sums to the category of spectra.

Example 4. For C the category of finite sets, $Sp(C)$ is the sphere spectrum.

Example 5. For the category of finitely generated projective modules over a ring A, this spectrum is the K-theory spectrum of A.

6 **Remark 6** The relationship between this construction and the iterated delooping for abelian groups discussed above is as follows. When C admits zero objects and categorical sums, we obtain a functor $C \times C$ by choosing a categorical sum $a \oplus b$ for every pair of objects a and b in C. Applying the nerve functor yields a simplicial map $\mu : N.C \times N.C \to N.C$. The map μ behaves like the multiplication map in a simplicial monoid, except that the identities are only identities up to simplicial homotopy, and the associativity conditions only hold up to homotopy. Moreover, μ has a form of homotopy commutativity, in that the maps μT and μ are simplicially homotopic, where T denotes the evident twist map on $N.C \times N.C$. So $N.C$ behaves like a commutative monoid up to homotopy. On the other hand, in verifying the Γ-space properties for $Sp_1(C)$, we showed that $Sp_1(C)(\underline{n})$ is weakly equivalent to $\prod_{i=1}^{n} Sp_1(C)(\underline{1})$. By definition of the classifying spaces for abelian groups, the set in the n-th level is the product of n copies of G, which is the set in the first level. So, the construction $Sp_1(C)$ also behaves *up to homotopy equivalence* like the classifying space construction for abelian groups.

The construction we have given is restricted to categories with categorical sums, and functors which preserve those. This turns out to be unnecessarily restrictive. For example, an abelian group A can be regarded as a category $Cat(A)$ whose objects are the elements of the A, and whose morphisms consist only of identity morphisms. The multiplication in A gives a functor $Cat(A) \times Cat(A) \to Cat(A)$, and hence a map $N.Cat(A) \times N.Cat(A) \to N.Cat(A)$, which is in fact associative and commutative. One can apply the classifying space construction to $N.Cat(A)$ to obtain the Eilenberg–MacLane spectrum for A. However, this operation is not induced from a categorical sum. It is desirable to have the flexibility to include examples such as this one into the families of categories to which one can apply the construction Sp. This kind of extension has been carried out by *May* [20] and *Thomason* [28]. We will give a description of the kind of categories to which the construction can be extended. See Thomason [28] for a complete treatment.

Definition 7 A *symmetric monoidal category* is a small category $\mathbf{S}$ together with a functor $\oplus : \mathbf{S} \times \mathbf{S} \to \mathbf{S}$ and an object 0, together with three natural isomorphisms of functors 7

$$\alpha : (S_1 \oplus S_2) \oplus S_3 \xrightarrow{\sim} S_1 \oplus (S_2 \oplus S_3)$$

$$\lambda : 0 \oplus S_1 \xrightarrow{\sim} S_1$$

and

$$\gamma : S_1 \oplus S_2 \xrightarrow{\sim} S_2 \oplus S_1$$

satisfying the condition that $\gamma^2 = Id$ and so that the following three diagrams commute.

$$\begin{array}{ccc}
((S_1 \oplus S_2) \oplus S_3) \oplus S_4 & \xrightarrow{\alpha \oplus S_4} & (S_1 \oplus (S_2 \oplus S_3)) \oplus S_4 \\
\downarrow \alpha & & \\
(S_1 \oplus S_2) \oplus (S_3 \oplus S_4) & & \downarrow \alpha \\
\downarrow \alpha & & \\
S_1 \oplus (S_2 \oplus (S_3 \oplus S_4)) & \xleftarrow{S_1 \oplus \alpha} & S_1 \oplus ((S_2 \oplus S_3) \oplus S_4)
\end{array}$$

$$\begin{array}{ccccc}
(S_1 \oplus S_2) \oplus S_3 & \xrightarrow{\alpha} & S_1 \oplus (S_2 \oplus S_3) & \xrightarrow{\gamma} & (S_2 \oplus S_3) \oplus S_1 \\
\downarrow \gamma \oplus S_3 & & & & \downarrow \alpha \\
(S_2 \oplus S_1) \oplus S_3 & \xrightarrow{\alpha} & S_2 \oplus (S_1 \oplus S_3) & \xrightarrow{S_2 \oplus \gamma} & S_2 \oplus (S_3 \oplus S_1)
\end{array}$$

$$\begin{array}{ccccc} (0 \oplus S_1) \oplus S_2 & \xrightarrow{\gamma \oplus S_2} & (S_1 \oplus 0) \oplus S_2 & \xrightarrow{\alpha} & S_1 \oplus (0 \oplus S_2) \\ & \searrow^{\lambda \oplus S_2} & & \swarrow_{S_1 \oplus \lambda} & \\ & & S_1 \oplus S_2 & & \end{array}$$

Given two symmetric monoidal categories **S** and **T**, a *symmetric monoidal functor* from **S** to **T** is a triple $(F, f, \bar{f})$, where $F : \mathbf{S} \to \mathbf{T}$ is a functor, and where

$$f : FS_1 \oplus FS_2 \to F(S_1 \oplus S_2) \text{ and } \bar{f} : 0 \to F0$$

are natural transformations of functors so that the diagrams

$$\begin{array}{ccccc} (FS_1 \oplus FS_2) \oplus FS_3 & \xrightarrow{f \oplus FS_2} & F(S_1 \oplus S_2) \oplus FS_3 & \xrightarrow{f} & F((S_1 \oplus S_2) \oplus S_3) \\ \downarrow \alpha & & & & \downarrow F\alpha \\ FS_1 \oplus (FS_2 \oplus FS_3) & \xrightarrow{FS_3 \oplus f} & FS_1 \oplus F(S_2 \oplus S_3) & \xrightarrow{f} & F(S_1 \oplus (S_2 \oplus S_3)) \end{array}$$

and

$$\begin{array}{ccc} FS_1 \oplus FS_2 & \xrightarrow{f} & F(S_1 \oplus S_2) \\ \downarrow \gamma & & \downarrow F\gamma \\ FS_2 \oplus FS_1 & \xrightarrow{f} & F(S_2 \oplus S_1) \end{array} \qquad \begin{array}{ccc} F0 \oplus FS & \xrightarrow{f} & F(0 \oplus S) \\ \uparrow \bar{f} \oplus FS & & \downarrow F\lambda \\ 0 \oplus FS & \xrightarrow{\lambda} & FS \end{array}$$

all commute.

It is easy to see that if we are given a category C with zero objects and which admits categorical sums, then one can produce the isomorphisms in question by making arbitrary choices of zero objects and categorical sums for each pair of objects of C, making C into a symmetric monoidal category. In [28], Thomason now shows that it is possible by a construction based on the one given by Segal to produce a Γ-space $Sp_1(\mathbf{S})$ for any symmetric monoidal category, and more generally Γ^n-spaces, i.e functors $(\Gamma^{op})^n \to s.sets$ which fit together into a spectrum, and that these constructions agree with those given by Segal in the case where the symmetric monoidal sum is given by a categorical sum. May [20] has also given a construction for *permutative categories*, i.e. symmetric monoidal categories where the associativity isomorphism α is actually the identity. He uses his theory of *operads* instead of Segal's Γ-spaces. It should be pointed out that the restriction to permutative categories is no real restriction, since every symmetric monoidal category is symmetric monoidally equivalent to a permutative category.

The Q-Construction and Its Higher Dimensional Generalization 1.3

It was Quillen's crucial insight that the higher algebraic K-groups of a ring A could be defined as the homotopy groups of the nerve of a certain category Q constructed from the category of finitely generated projective modules over A. He had previously defined the K-groups as the homotopy groups of the space $BGL^+(A) \times K_0(A)$. The homotopy groups and the K-groups are related via a dimension shift of one, i.e.

$$K_i(A) \cong \pi_{i+1} N.Q$$

This suggests that the loop space $\Omega N.Q$ should be viewed as the "right" space for K-theory, and indeed Quillen ([16, 23]) showed that $\Omega N.Q$ could be identified as a group completion of the nerve of the category of finitely generated projective A-modules and their isomorphisms. From this point of view, the space $N.Q$ can be viewed as a delooping of $BGL^+(A) \times K_0(A)$, and it suggests that one should look for ways to construct higher deloopings which would agree with $N.Q$ in the case $n = 1$. This was carried out by Waldhausen in [32], and developed in various forms by Gillet [15], Jardine [17], and Shimakawa [27]. We will outline Shimakawa's version of the construction.

We must first review Quillen's Q-construction. We first recall that its input is considerably more general than the category of finitely generated projective modules over a ring. In fact, the input is an *exact category*, a concept which we now recall.

Definition 8 A category is *additive* if it admits sums and products and if every *Hom*-set is given an abelian group structure, such that the composition pairings are bilinear. An *exact category* is an additive category **C** equipped with a family **E** of diagrams of the form 8

$$C' \xrightarrow{i} C \xrightarrow{p} C''$$

which we call the *exact sequences*, satisfying certain conditions to be specified below. Morphisms which occur as "i" in an exact sequence are called *admissible monomorphisms* and morphisms which occur as "p" are called *admissible epimorphisms*. The family **E** is now required to satisfy the following five conditions.

- Any diagram in **C** which is isomorphic to one in **E** is itself in **E**.
- The set of admissible monomorphisms is closed under composition, and the cobase change exists for an arbitrary morphism. The last statement says that the pushout of any diagram of the form

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle i}\downarrow & & \vdots\,{\scriptstyle \hat{\imath}} \\ C' & \cdots\cdots\rightarrow & C' \times_C D \end{array}$$

exists, and the morphism $\hat{\imath}$ is also an admissible monomorphism.

- The set of admissible epimorphisms is closed under composition, and the base change exists for an arbitrary morphism. This corresponds to the evident dual diagram to the preceding condition.
- Any sequence of the form

$$C \to C \oplus C' \to C'$$

is in **E**.

- In any element of **E** as above, i is a kernel for p and p is a cokernel for i.

We also define an *exact functor* as a functor between exact categories which preserves the class of exact sequences, in an obvious sense, as well as base and cobase changes.

Exact categories of course include the categories of finitely generated projective modules over rings, but they also contain many other categories. For example, any abelian category is an exact category. For any exact category $(\mathbf{C}, \mathbf{E})$, Quillen now constructs a new category $Q(\mathbf{C}, \mathbf{E})$ as follows. Objects of $Q(\mathbf{C}, \mathbf{E})$ are the same as the objects of $\mathbf{C}$, and a morphism from C to C' in $Q(\mathbf{C}, \mathbf{E})$ is a diagram of the form

$$\begin{array}{ccc} D & \xrightarrow{i} & C' \\ {\scriptstyle p}\downarrow & & \\ C & & \end{array}$$

where i is an admissible monomorphism and p is an admissible epimorphism. The diagrams are composed using a pullback construction, so the composition of the two diagrams

$$\begin{array}{ccc} D & \xrightarrow{i} & C' \\ {\scriptstyle p}\downarrow & & \\ C & & \end{array} \qquad \begin{array}{ccc} D' & \xrightarrow{i'} & C'' \\ {\scriptstyle p'}\downarrow & & \\ C' & & \end{array}$$

is the diagram

$$\begin{array}{ccc} D \times_{C'} D' & \longrightarrow & C'' \\ \downarrow & & \\ C & & \end{array}$$

Quillen now defines the higher K-groups for the exact category $(\mathbf{C}, \mathbf{E})$ by $K_{i-1}(\mathbf{C}, \mathbf{E}) = \pi_i N.Q(\mathbf{C}, \mathbf{E})$. The problem before us is now how to construct higher deloopings, i.e. spaces X_n so that $K_{i-n}(\mathbf{C}, \mathbf{E}) = \pi_i X_n$. Shimakawa [27] proceeds as follows.

We will first need the definition of a *multicategory*. To make this definition, we must first observe that a category $\mathbf{C}$ is uniquely determined by

- The set $A_{\mathbf{C}}$ of all the morphisms in $\mathbf{C}$, between any pairs of objects.
- A subset $O_{\mathbf{C}}$ of $A_{\mathbf{C}}$, called the objects, identified with the set of identity morphisms in $A_{\mathbf{C}}$.
- The source and target maps $S : A_{\mathbf{C}} \to O_{\mathbf{C}}$ and $T : A_{\mathbf{C}} \to O_{\mathbf{C}}$.
- The composition pairing is a map $\circ$ from the pullback

$$\begin{array}{ccc} A_{\mathbf{C}} \times_{O_{\mathbf{C}}} A_{\mathbf{C}} & \longrightarrow & A_{\mathbf{C}} \\ \downarrow & & \downarrow{\scriptstyle T} \\ A_{\mathbf{C}} & \xrightarrow{S} & O_{\mathbf{C}} \end{array}$$

to $A_{\mathbf{C}}$.

Definition 9 An *n-multicategory* is a set A equipped with n different category structures $(S_j, T_j, \circ_j)$ for $j = 1, \ldots, n$ satisfying the following compatibility conditions for all pairs of distinct integers j and k, with $1 \le j, k \le n$. 9

- $S_j S_k x = S_k S_j x$, $S_j T_k x = T_k S_j x$, and $T_j T_k x = T_k T_j x$
- $S_j(x \circ_k y) = S_j x \circ_k S_j y$ and $T_j(x \circ_k y) = T_j x \circ_k T_j y$
- $(x \circ_k y) \circ_j (z \circ_k w) = (x \circ_j z) \circ_k (y \circ_j w)$

The notion of an n-multifunctor is the obvious one.

It is clear that one can define the notion of an n-multicategory object in any category which admits finite limits (although the only limits which are actually needed are the pullbacks $A \times_{O_j} A$). In particular, one may speak of an n-multicategory object in the category <u>*CAT*</u>, and it is readily verified that such objects can be identified with $(n + 1)$-multicategories.

There is a particularly useful way to construct n-multicategories from ordinary categories. Let I denote the category associated with the totally ordered set $\{0, 1\}$,

$0 < 1$, and let λ equal the unique morphism $0 \to 1$ in I. For any category $\mathbf{C}$, define an n-multicategory structure on the set of all functors $I^n \to \mathbf{C}$ as follows. The j-th source and target functions are given on any functor f and any vector $u = (u_1, \dots, u_n) \in I^n$ by

$$\left(S_j f\right)(u) = f\left(u_1, \dots, u_{j-1}, 0, u_{j+1}, \dots, u_n\right)$$

and

$$\left(T_j f\right)(u) = f\left(u_1, \dots, u_{j-1}, 1, u_{j+1}, \dots, u_n\right)$$

The j-th composition pairing is defined by

$$\left(g \circ_j f\right) = \begin{cases} fu & : \quad \text{if } u_j = 0 \\ gu & : \quad \text{if } u_j = 1 \\ gu \circ fu & : \quad \text{if } u_j = \lambda \text{ and } u_k \in \{0,1\} \text{ for all } k \neq j \end{cases}$$

We will write $\mathbf{C}^{[n]}$ for this n-multicategory, for any $\mathbf{C}$.

We will now define an analogue of the usual nerve construction on categories. The construction applied to an n-multicategory will yield an n-multisimplicial set. To see how to proceed, we note that for an ordinary category $\mathbf{C}$, regarded as a set A with S, T, and $\circ$ operators, and O the set of objects, the set $N_k\mathbf{C}$ can be identified with the pullback

$$\underbrace{A \times_O A \times_O A \cdots \times_O A}_{k \text{ factors}}$$

i.e the set of vectors $(a_1, a_2, \dots a_k)$ so that $S(a_j) = T(a_{j-1})$ for $2 \leq j \leq k$. Note that $N_0\mathbf{C}$ conventionally denotes O. In the case of an n-multicategory $\mathbf{C}$, we can therefore construct this pullback for any one of the n category structures. Moreover, because of the commutation relations among the operators S_j, T_j, and $\circ_j$ for the various values of j, the nerve construction in one of the directions respects the operators in the other directions. This means that if we let N_{s,k_s} denote the k-dimensional nerve operator attached to the s-th category structure, we may define an n-multisimplicial set $N\mathbf{C} : (\Delta^{op})^n \to \mathit{Sets}$ by the formula

$$N\mathbf{C}\left(i_1, i_2, \dots, i_n\right) = N_{1,i_1} N_{2,i_2} \cdots N_{n,i_n} \mathbf{C}$$

The idea for constructing deloopings of exact categories is to define a notion of an n-multiexact category, and to note that it admits a Q-construction which is an n-multicategory. whose nerve will become the n-th delooping.

10 **Definition 10** A functor $F : \mathbf{C} \to \mathbf{D}$ of small categories is called *strongly good* if for any object $x \in \mathbf{C}$ and any isomorphism $f : Fx \to y$ in $\mathbf{D}$, there is a unique isomorphism $f' : x \to y'$ in $\mathbf{C}$ such that $Ff' = f$.

Definition 11 Let $P \subseteq \{1, \dots, n\}$. Then by P-exact category, we mean an n-fold category $\mathbf{C}$ so that every $\mathbf{C}_p, p \in P$, is equipped with the structure of an exact category, so that the following conditions hold for every pair p, j, with $p \in P$ and $j \neq p$. 11

- $o_j\mathbf{C}_p$ is an exact subcategory of $\mathbf{C}_p$.
- $(S_j, T_j) : \mathbf{C}_p \to o_j\mathbf{C}_p \times o_j\mathbf{C}_p$ is strongly good and exact.
- $\circ_j : \mathbf{C}_p \times_{o_j\mathbf{C}_p} \mathbf{C}_p \to \mathbf{C}_p$ is exact.
- If j also belongs to P, the class E_j of exact sequences of $\mathbf{C}_j$ becomes an exact subcategory of $\mathbf{C}_p \times_{o_j\mathbf{C}_p} \mathbf{C}_p$

One direct consequence of the definition is that if we regard a P-exact category as an $(n-1)$-multicategory object in $\underline{CAT}$, with the arguments in the $(n-1)$-multicategory taking their values in $\mathbf{C}_p$, with $p \in P$, we find that we actually obtain an $(n-1)$-multicategory object in the category $\underline{EXCAT}$ of exact categories and exact functors. The usual Q-construction gives a functor from $\underline{EXCAT}$ to $\underline{CAT}$, which preserves the limits used to define n-multicategory objects, so we may apply Q in the p-th coordinate to obtain an $(n-1)$-multicategory object in $\underline{CAT}$, which we will denote by $Q_p(\mathbf{C})$. We note that $Q_p(\mathbf{C})$ is now an n-multicategory, and Shimakawa shows that there is a natural structure of a $(P - \{p\})$-exact multicategory on $Q_p(\mathbf{C})$. One can therefore begin with an $\{1, \dots, n\}$-exact multicategory $\mathbf{C}$, and construct an n-multicategory $Q_n Q_{n-1} \cdots Q_1 \mathbf{C}$. It can further be shown that the result is independent of the order in which one applies the operators Q_i. The nerves of these constructions provide us with n-multisimplicial sets, and these can be proved to yield a compatible system of deloopings and therefore of spectra.

Waldhausen's $S_\bullet$-Construction 1.4

In this section, we describe a family of deloopings constructed by F. Waldhausen in [33] which combine the best features of the generic deloopings with the important special properties of Quillen's Q-construction delooping. The input to the construction is a *category with cofibrations and weak equivalences*, a notion defined by Waldhausen, and which is much more general than Quillen's exact categories. For example, it will include categories of spaces with various special conditions, or spaces over a fixed space as input. These cannot be regarded as exact categories in any way, since they are not additive. On the other hand, the construction takes into account a notion of exact sequence, which permits one to prove versions of the localization and additivity theorems for it. It has permitted Waldhausen to construct spectra $A(X)$ for spaces X, so that for X a manifold, $A(X)$ contains the stable pseudo-isotopy space as a factor. See [33] for details.

We begin with a definition.

Definition 12 A category $\mathcal{C}$ is said to be *pointed* if it is equipped with a distinguished object $*$ which is both an initial and terminal object. A *category with cofibrations* 12 is a pointed category $\mathcal{C}$ together with a subcategory $co\mathcal{C}$. The morphisms of $co\mathcal{C}$ are called the cofibrations. The subcategory $co\mathcal{C}$ satisfies the following properties.

1. Any isomorphism in $\mathcal{C}$ is in $co\mathcal{C}$. In particular, any object of $\mathcal{C}$ is in $co\mathcal{C}$.
2. For every object $X \in \mathcal{C}$, the unique arrow $* \to X$ is a cofibration.
3. For any cofibration $i : X \hookrightarrow Y$ and any morphism $f : X \to Z$ in $\mathcal{C}$, there is a pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \hat{\imath}} \\ Y & \longrightarrow & W \end{array}$$

in $\mathcal{C}$, and the natural map $\hat{\imath}$ is also a cofibration.

13 **Definition 13** Let $\mathcal{C}$ be a category with cofibrations. A *category of weak equivalences* in $\mathcal{C}$ is a subcategory $w\mathcal{C}$ of $\mathcal{C}$, called the *weak equivalences*, which satisfy two axioms.

1. All isomorphisms in $\mathcal{C}$ are in $w\mathcal{C}$.
2. For any commutative diagram

$$\begin{array}{ccccc} Y & \xleftarrow{i} & X & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \xleftarrow{i'} & X' & \longrightarrow & Z' \end{array}$$

in $\mathcal{C}$, where i and i' are cofibrations, and all the vertical arrows are weak equivalences, the induced map on pushouts is also a weak equivalence.

If $\mathcal{C}$ and $\mathcal{D}$ are both categories with cofibrations and weak equivalences, we say a functor $f : \mathcal{C} \to \mathcal{D}$ is *exact* if it preserves pushouts, $co\mathcal{C}$, and $w\mathcal{C}$, and $f* = *$.

Here are some examples.

Example 14. The category of based finite sets, with $*$ a single point space, the cofibrations the based inclusions, and the weak equivalences being the bijections.

Example 15. The category of based simplicial sets with finitely many cells (i.e. non-degenerate simplices), the one point based set as $*$, the level-wise inclusions as the cofibrations, and the usual weak equivalences as $w\mathcal{C}$.

Example 16. Any exact category $\mathcal{E}$ in the sense of Quillen [23] can be regarded as a category with cofibrations and weak equivalences as follows. $*$ is chosen to be any zero object, the cofibrations are the admissible monomorphisms, and the weak equivalences are the isomorphisms.

Example 17. Let A be any ring, and let $\mathcal{C}$ denote the category of chain complexes of finitely generated projective A-modules, which are bounded above and below. The zero complex is $*$, the cofibrations will be the levelwise split monomorphisms, and the weak equivalences are the chain equivalences, i.e maps inducing isomorphisms on homology. A variant would be to consider the homologically finite complexes, i.e. complexes which are not necessarily bounded above but which have the property that there exists an N so that $H_n = 0$ for $n > N$.

We will now outline how Waldhausen constructs a spectrum out of a category with cofibrations and weak equivalences. For each n, we define a new category $\mathcal{S}_n\mathcal{C}$ as follows. Let $\underline{n}$ denote the totally ordered set $\{0, 1, \ldots, n\}$ with the usual ordering. Let $Ar[n] \subseteq \underline{n} \times \underline{n}$ be the subset of all (i, j) such that $i \leq j$. The category $Ar[n]$ is a partially ordered set, and as such may be regarded as a category. We define the objects of $S_n\mathcal{C}$ as the collection of all functors $\theta : Ar[n] \to \mathcal{C}$ satisfying the following conditions.

- $\theta(i, i) = *$ for all $0 \leq i \leq n$.
- $\theta((i, j) \leq (i, j'))$ is a cofibration.
- For all triples i, j, k, with $i \leq j \leq k$, the diagram

$$\begin{array}{ccc} \theta(i,j) & \longrightarrow & \theta(i,k) \\ \downarrow & & \downarrow \\ \theta(j,j) = * & \longrightarrow & \theta(j,k) \end{array}$$

  is a pushout diagram.

Remark 18 Note that each object in $\mathcal{S}_n\mathcal{C}$ consists of a composable sequence of cofibrations 18

$$* = \theta(0, 0) \hookrightarrow \theta(0, 1) \hookrightarrow \theta(0, 2) \hookrightarrow \cdots \theta(0, n-1) \hookrightarrow \theta(0, n)$$

together with choices of quotients for each cofibration $\theta((0, i) \leq (0, j))$, when $i \leq j$.

$\mathcal{S}_n\mathcal{C}$ becomes a category by letting the morphisms be the natural transformations of functors. We can define a category of cofibrations on $\mathcal{S}_n\mathcal{C}$ as follows. A morphism $\Phi : \theta \to \theta'$ determines morphisms $\Phi_{ij} : \theta(ij) \to \theta'(ij)$. In order for Φ to be a cofibration in $\mathcal{S}_n\mathcal{C}$, we must first require that Φ_{ij} is a cofibration in $\mathcal{C}$ for every

i and j. In addition, we require that for every triple i, j, k, with $i \le j \le k$, the commutative diagram

$$\begin{array}{ccc} \theta(i,j) & \longrightarrow & \theta(i,k) \\ \downarrow & & \downarrow \\ \theta'(i,j) & \longrightarrow & \theta'(j,k) \end{array}$$

is a pushout diagram in $\mathcal{C}$. We further define a category $w\mathcal{S}_n\mathcal{C}$ of weak equivalences on $\mathcal{S}_n\mathcal{C}$ by $\Phi \in w\mathcal{S}_n\mathcal{C}$ if and only if $\Phi_{ij} \in w\mathcal{C}$ for all $i \le j$. One can now check that $\mathcal{S}_n\mathcal{C}$ is a category with cofibrations and weak equivalences.

We now further observe that we actually have a functor from $\Delta \to \underline{CAT}$ given by $\underline{n} \to Ar[n]$, where Δ as usual denotes the category of finite totally ordered sets and order preserving maps of such. Consequently, if we denote by $\mathcal{F}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$ (the morphisms are natural transformations), we obtain a simplicial category $n \to \mathcal{F}(Ar[n], \mathcal{C})$ for any category $\mathcal{C}$. One checks that if $\mathcal{C}$ is a category with cofibrations and weak equivalences, then the subcategories $\mathcal{S}_n\mathcal{C} \subseteq \mathcal{F}(Ar[n], \mathcal{C})$ are preserved under the face and degeneracy maps, so that we actually have a functor $\mathcal{S}_.$ from the category of categories with cofibrations and weak equivalences, and exact functors, to the category of simplicial categories with cofibrations and weak equivalences and levelwise exact functors. This construction can now be iterated to obtain functors $\mathcal{S}^k_.$ which assign to a category with cofibrations and weak equivalences a k-simplicial category with cofibrations and weak equivalences. To obtain the desired simplicial sets, we first apply the w levelwise, to obtain a simplicial category, and then apply the nerve construction levelwise, to obtain a $(k+1)$-simplicial set $N.w\mathcal{S}^k_.\mathcal{C}$. We can restrict to the diagonal simplicial set $\Delta N.w\mathcal{S}^k_.\mathcal{C}$ to obtain a family of simplicial sets, which by abuse of notation we also write as $\mathcal{S}^k\mathcal{C}$

Waldhausen's next observation is that for any category with cofibrations and weak equivalences $\mathcal{C}$, there is a natural inclusion

$$\Sigma N.w\mathcal{C} \to \mathcal{S}_.\mathcal{C}$$

The suspension is the reduced suspension using $*$ as the base point in $N.w\mathcal{C}$. This map exists because by definition, $\mathcal{S}_0\mathcal{C} = *$, and $\mathcal{S}_1\mathcal{C} = N.w\mathcal{C}$, so we obtain a map $\sigma : \Delta[1] \times N.w\mathcal{C} \to \mathcal{S}_.\mathcal{C}$, and it is easy to see that the subspace

$$\partial\Delta[1] \times N.w\mathcal{C} \cup \Delta[1] \times *$$

maps to $*$ under σ, inducing the desired map which we also denote by σ. The map σ is natural for exact functors, and we therefore obtain maps $\mathcal{S}^k_.(\sigma) : \Sigma\mathcal{S}^k_.\mathcal{C} \to \mathcal{S}^{k+1}_.$ for each k. Waldhausen now proves

Theorem 19 The adjoint to $\mathcal{S}^k_{\cdot}(\sigma)$ is a weak equivalence of simplicial sets from $\mathcal{S}^k_{\cdot}\mathcal{C}$ 19
to $\Omega\mathcal{S}^{k+1}_{\cdot}\mathcal{C}$ for $k \geq 1$, so the spaces $|\mathcal{S}^k_{\cdot}\mathcal{C}|$ form a spectrum except in dimension $k = 0$, when it can be described as a homotopy theoretic group completion. These deloopings agree with Segal's generic deloopings when $\mathcal{C}$ is a category with sums and zero object, and the cofibrations are chosen to be only the sums of the form $X \to X \vee T \to Y$.

We will write $\mathcal{S}\mathcal{C}$ for this spectrum. The point of Waldhausen's construction is that it produces a spectrum from the category $\mathcal{C}$ in such a way that many of the useful formal properties of Quillen's construction hold in this much more general context. We will discuss the analogues of the localization and additivity theorems, from the work of Quillen [23].

We will first consider localization. Recall that Quillen proved a localization theorem in the context of quotients of abelian categories. The context in which Waldhausen proves his localization result is the following. Given a category with cofibrations and weak equivalences $\mathcal{C}$, we define $Ar\mathcal{C}$ to be the category whose objects are morphisms $f : X \to Y$ in $\mathcal{C}$, and where a morphism from $f : X \to Y$ to $f' : X' \to Y'$ is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

It is easy to check that $Ar\mathcal{C}$ becomes a category with cofibrations and weak equivalences if we declare that the cofibrations (respectively weak equivalences) are diagrams such as the ones above in which both vertical arrows are cofibrations (respectively weak equivalences). If $\mathcal{C}$ is the category of based topological spaces, then the *mapping cylinder* construction can be viewed as a functor from $Ar\mathcal{C}$ to spaces, satisfying certain conditions. In order to construct a localization sequence, Waldhausen requires an analogue of the mapping cylinder construction in the category $\mathcal{C}$.

Definition 20 A *cylinder functor* on a category with cofibrations and weak equivalences $\mathcal{C}$ is a functor T which takes objects $f : X \to Y$ in $Ar\mathcal{C}$ to diagrams of shape 20

$$\begin{array}{ccccc} X & \xrightarrow{j} & T(f) & \xleftarrow{k} & Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle id} & \\ & & Y & & \end{array}$$

satisfying the following two conditions.

- For any two objects X and Y in $\mathcal{C}$, we denote by $X \vee Y$ pushout of the diagram

$$X \leftarrow * \rightarrow Y$$

  We require that the canonical map $X \vee Y \rightarrow T(f)$ coming from the diagram above be a cofibration, and further that if we have any morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

  in $Ar\mathcal{C}$, then the associated diagram

$$\begin{array}{ccc} X \vee Y & \longrightarrow & T(f) \\ \downarrow & & \downarrow \\ X' \vee Y' & \longrightarrow & T(f') \end{array}$$

  is a pushout.
- $T(* \rightarrow X) = X$ for every $X \in \mathcal{C}$, and k and p are the identity map in this case.

(The collection of diagrams of this shape form a category with natural transformations as morphisms, and T should be a functor to this category.) We say the cylinder functor satisfies the *cylinder condition* if p is in $w\mathcal{C}$ for every object in $Ar\mathcal{C}$.

We will also need two axioms which apply to categories with cofibrations and weak equivalences.

21 **Axiom 21:** *Saturation axiom:* If f and g are composable morphisms in $\mathcal{C}$, and if two of f, g, and gf are in $w\mathcal{C}$, then so is the third.

22 **Axiom 22:** *Extension Axiom:* If we have a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow{i'} & Y' & \longrightarrow & Z' \end{array}$$

where i and i' are cofibrations, and Z and Z' are pushouts of the diagrams $* \leftarrow X \rightarrow Y$ and $* \leftarrow X' \rightarrow Y'$ respectively, and if the arrows $X \rightarrow X'$ and $Z \rightarrow Z'$ are in $w\mathcal{C}$, then it follows that $Y \rightarrow Y'$ is in $w\mathcal{C}$ also.

The setup for the localization theorem is now as follows. We let $\mathcal{C}$ be a category equipped with a category of cofibrations, and two different categories of weak equivalences v and w, so that $v\mathcal{C} \subseteq w\mathcal{C}$. Let $\mathcal{C}^w$ denote the subcategory with cofibrations on $\mathcal{C}$ given by the objects $X \in \mathcal{C}$ having the property that the map $* \rightarrow X$ is in $w\mathcal{C}$. It will inherit categories of weak equivalences $v\mathcal{C}^w = \mathcal{C}^w \cap v\mathcal{C}$ and $w\mathcal{C}^w = \mathcal{C}^w \cap w\mathcal{C}$. Waldhausen's theorem is now as follows.

Theorem 23 If $\mathcal{C}$ has a cylinder functor, and the category of weak equivalences $w\mathcal{C}$ satisfies the cylinder axiom, saturation axiom, and extension axiom, then the square 23

$$\begin{array}{ccc} v\mathcal{S}.\mathcal{C}^w & \longrightarrow & w\mathcal{S}.\mathcal{C}^w \\ \downarrow & & \downarrow \\ v\mathcal{S}.\mathcal{C} & \longrightarrow & w\mathcal{S}.\mathcal{C} \end{array}$$

is homotopy Cartesian, and $w\mathcal{S}.\mathcal{C}^w$ is contractible. In other words, we have up to homotopy a fibration sequence

$$v\mathcal{S}.\mathcal{C}^w \rightarrow v\mathcal{S}.\mathcal{C} \rightarrow w\mathcal{S}.\mathcal{C}$$

The theorem extends to the deloopings by applying $\mathcal{S}.$ levelwise, and we obtain a fibration sequence of spectra.

Remark 24 The reader may wonder what the relationship between this sequence and Quillen's localization sequence is. One can see that the category of finitely generated projective modules over a Noetherian commutative ring A, although it is a category with cofibrations and weak equivalences, does not admit a cylinder functor with the cylinder axiom, and it seems that this theorem does not apply. However, one can consider the category of chain complexes $Comp(A)$ of finitely generated projective chain complexes over A, which are bounded above and below. The category $Comp(A)$ does admit a cylinder functor satisfying the cylinder axiom (just use the usual algebraic mapping cylinder construction). It is shown in [29] that $Proj(A) \hookrightarrow Comp(A)$ of categories with cofibrations and weak equivalences induces an equivalences on spectra $\mathcal{S}(Proj(A)) \rightarrow \mathcal{S}(Comp(A))$ for any ring. Moreover, if S is a mulitiplicative subset in the regular ring A, then the category $Comp(A)^S$ of objects $C_* \in Comp(A)$ which have the property that $S^{-1}C_*$ is acyclic, i.e. has vanishing homology, has the property that $\mathcal{S}.Comp(A)^S$ is weakly equivalent to the $\mathcal{S}.$ spectrum of the exact category $Mod(A)^S$ of finitely generated 24

A-modules M for which $S^{-1}M = 0$. In this case, the localization theorem above applies, with v being the usual category of weak equivalences, and where w is the class of chain maps whose algebraic mapping cone has homology annihilated by S^{-1}. Finally, if we let $\mathcal{D}$ denote the category with cofibrations and weak equivalences consisting with $\mathcal{C}$ as underlying category, and with w as the weak equivalences, then $\mathcal{S}.\mathcal{D} \cong \mathcal{S}.Comp(S^{-1}A)$. Putting these results together shows that we obtain Quillen's localization sequence in this case.

The key result in proving 23 is Waldhausen's version of the Additivity Theorem. Suppose we have a category with cofibrations and weak equivalences $\mathcal{C}$, and two subcategories with cofibrations and weak equivalences $\mathcal{A}$ and $\mathcal{B}$. This means that $\mathcal{A}$ and $\mathcal{B}$ are subcategories of $\mathcal{C}$, each given a structure of a category with cofibrations and weak equivalences, so that the inclusions are exact. Then we define a new category $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ to have objects cofibration sequences

$$A \hookrightarrow C \to B$$

with $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $C \in \mathcal{C}$. This means that we are given a specific isomorphism from a pushout of the diagram $* \leftarrow A \to C$ to B. The morphisms in $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ are maps of diagrams. We define a category of cofibrations on $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ to consist of those maps of diagrams which are cofibrations at each point in the diagram. We similarly define the weak equivalences to be the pointwise weak equivalences. We have an exact functor $(s, q) : E(\mathcal{A}, \mathcal{C}, \mathcal{B}) \to \mathcal{A} \times \mathcal{B}$, given by $s(A \hookrightarrow C \to B) = A$ and $q(A \hookrightarrow C \to B) = B$.

25 **Theorem 25** The exact functor (s, q) induces a homotopy equivalence from $\mathcal{S}.E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ to $\mathcal{S}.\mathcal{A} \times \mathcal{S}.\mathcal{B}$.

Finally, Waldhausen proves a comparison result between his delooping and the nerve of Quillen's Q-construction.

26 **Theorem 26** There is a natural weak equivalence of spaces from $|w\mathcal{S}_{.}E|$ to $|N_{.}Q(E)|$ for any exact category E. (Recall that E can be viewed as a category with cofibrations and weak equivalences, and hence $w\mathcal{S}_{.}$ can be evaluated on it).

1.5 The Gersten–Wagoner Delooping

All the constructions we have seen so far have constructed simplicial and category theoretic models for deloopings of algebraic K-theory spaces. It turns out that there is a way to construct the deloopings directly on the level of rings, i.e. for any ring R there is a ring μR whose K-theory space actually deloops the K-theory space of R. Two different versions of this idea were developed by S. Gersten [14] and J. Wagoner [30]. The model we will describe was motivated by problems in high dimensional geometric topology [12], and by the observation that the space of Fredholm operators on an infinite dimensional complex Hilbert space provides a delooping of the infinite unitary group U. This delooping, and the Pedersen–Weibel delooping which follows in the last section, are *non-connective*, i.e. we can have $\pi_i X_n \neq 0$ for $i < n$, where X_n denotes the n-th delooping in the spectrum. The homotopy group $\pi_i X_{i+n}$ are equal to Bass's lower K-group $K_{-n}(R)$ ([5]), and these lower K-groups have played a significant role in geometric topology, notably in the study of stratified spaces ([3, 4]).

Definition 27 Let R be a ring, and let lR denote the ring of infinite matrices over R in which each row and column contains only finitely many non-zero elements. The subring $mR \subseteq lR$ will be the set of all matrices with only finitely many non-zero entries; mR is a two-sided ideal in lR, and we define $\mu R = lR/mR$. 27

Remark 28 μR is a ring of a somewhat unfamiliar character. For example, it does not admit a rank function on projective modules. 28

Wagoner shows that $BGL^+(\mu R)$ is a delooping of $BGL^+(R)$. He first observes that the construction of the $n \times n$ matrices $M_n(R)$ for a ring R does not require that R has a unit. Of course, if R doesn't have a unit, then neither will $M_n(R)$. Next, for a ring R (possibly without unit), he defines $GL_n(R)$ be the set of $n \times n$ matrices P so that there is an $n \times n$ matrix Q with $P + Q + PQ = 0$, and equips $GL_n(R)$ with the multiplication $P \circ Q = P + Q + PQ$. (Note that for a ring with unit, this corresponds to the usual definition via the correspondence $P \to I + P$.) GL is now defined as the union of the groups GL_n under the usual inclusions. We similarly define $E_n(R)$ to be the subgroup generated by the elementary matrices $e_{ij}(r)$, for $i \neq j$, whose ij-th entry is r and for which the kl-th entry is zero for $(k, l) \neq (i, j)$. The group $E(R)$ is defined as the union of the groups $E_n(R)$. By definition, $GL(R)/E(R) \cong K_1 R$. The group $E(R)$ is a perfect group, so we may perform the plus construction to the classifying space $BE(R)$. Wagoner proves that there is a fibration sequence up to homotopy

$$BGL^+(R) \times K_0(R) \to E \to BGL^+(\mu R) \tag{1.1}$$

where E is a contractible space. This clearly shows that $BGL^+(\mu R)$ deloops $BGL^+(R) \times K_0(R)$. The steps in Wagoner's argument are as follows.

- There is an equivalence $BGL^+(R) \cong BGL^+(mR)$, coming from a straightforward isomorphism of rings (without unit) $M_\infty(mR) \cong mR$, where $M_\infty(R)$ denotes the union of the rings $M_n(R)$ under the evident inclusions.

- There is an exact sequence of groups $GL(mR) \to E(lR) \to E(\mu R)$. This follows directly from the definition of the rings mR, lR, and μR, together with the fact that $E(lR) = GL(lR)$. It yields a fibration sequence of classifying spaces

$$BGL(mR) \to BE(lR) \to BE(\mu R) \tag{1.2}$$

- The space $BE(lR)$ has trivial homology, and therefore the space $BE^+(lR)$ is contractible.
- The action of $E(\mu(R)) = \pi_1 BE^+(\mu R)$ on the homology of the fiber $BGL^+(mR)$ in the fibration 1.2 above is trivial. Wagoner makes a technical argument which shows that this implies that the sequence

$$BGL^+(mR) \to BGL^+(lR) \to BGL^+(\mu R)$$

is a fibration up to homotopy.
- $K_1(\mu(R)) \cong K_0(R)$.

Wagoner assembles these facts into a proof that we have a fibration of the form 1.1. Iterating the μ construction and applying $BGL^+(-)$ now yields the required family of deloopings.

1.6 Deloopings Based on Karoubi's Derived Functors

Max Karoubi ([18]) developed a method for defining the lower algebraic K-groups which resembles the construction of derived functors in algebra. The method permits the definition of these lower K-groups in a very general setting. As we have seen, the lower K-groups can be defined as the homotopy groups of non-connective deloopings of the the zeroth space of the K-theory spectrum. Karoubi observed that his techniques could be refined to produce deloopings rather than just lower K-groups, and this was carried out by Pedersen and Weibel in [22].

Karoubi considers an additive category $\mathcal{A}$, i.e. a category so that every morphism set is equipped with the structure of an abelian group, so that the composition pairings are bilinear, and so that every finite set of objects admits a sum which is simultaneously a product. He supposes further that $\mathcal{A}$ is embedded as a full subcategory of another additive category $\mathcal{U}$. He then makes the following definition.

29 **Definition 29** $\mathcal{U}$ is said to be $\mathcal{A}$-*filtered* if every object $U \in \mathcal{U}$ is equipped with a family of direct sum decompositions $\varphi_i\colon U \xrightarrow{\sim} E_i \oplus U_i$, $i \in I_U$, where I_U is an indexing set depending on U, with each $E_i \in \mathcal{A}$, satisfying the following axioms.

- For each U, the collection of decompositions form a filtered poset, when we equip it with the partial order $\{\varphi_i\colon U \xrightarrow{\sim} E_i \oplus U_i\} \leq \{\varphi_j\colon U \xrightarrow{\sim} E_j \oplus U_j\}$ if and only if the composite $U_j \hookrightarrow E_j \oplus U_j \stackrel{\varphi_j}{\to} U$ factors as $U_j \to U_i \hookrightarrow E_i \oplus U_i \stackrel{\varphi_i}{\to} U$ and the composite $E_i \hookrightarrow E_i \oplus U_i \stackrel{\varphi_i}{\to} U$ factors as $E_i \to E_j \hookrightarrow E_j \oplus U_j \stackrel{\varphi_j}{\to} U$.

- For any objects $A \in \mathcal{A}$ and $U \in \mathcal{U}$, and any morphism $f\colon A \to U$ in $\mathcal{U}$, f factors as $A \to E_i \hookrightarrow E_i \oplus U_i \stackrel{\varphi_i}{\to} U$ for some i.
- For any objects $A \in \mathcal{A}$ and $U \in \mathcal{U}$, and any morphism $f\colon U \to A$ in $\mathcal{U}$, f factors as $U \stackrel{\varphi_i^{-1}}{\to} E_i \oplus U_i \stackrel{\pi}{\to} E_i \to A$ for some i.
- For each $U, V \in \mathcal{U}$, the given partially ordered set of filtrations on $U \oplus V$ is equivalent to the product of the partially ordered sets of filtrations on U and V. That is to say, if the decompositions for U, V, and $U \oplus V$ are given by $\{E_i \oplus U_i\}_{i \in I_U}$, $\{E_j \oplus V_j\}_{j \in I_V}$, and $\{E_k \oplus W_k\}_{k \in I_{U \oplus V}}$, then the union of the collections of decompositions $\{(E_i \oplus E_j) \oplus (U_i \oplus V_j)\}_{(i,j) \in I_U \times I_V}$ and $\{E_k \oplus W_k\}_{k \in I_{U \oplus V}}$ also form a filtered partially ordered set under the partial ordering specified above.
- If $\varphi_i\colon U \to E_i \oplus U_i$ is one of the decompositions for U, and E_i can be decomposed as $E_i \cong A \oplus B$ in $\mathcal{A}$, then the decomposition $U \cong A \oplus (B \oplus U_i)$ is also one of the given family of decompositions for U.

Karoubi also defines an additive category $\mathcal{U}$ to be *flasque* if there a functor $e : \mathcal{U} \to \mathcal{U}$ and a natural isomorphism from e to $e \oplus id_{\mathcal{U}}$. Given an inclusion $\mathcal{A} \to \mathcal{U}$ as above, he also defines the quotient category $\mathcal{U}/\mathcal{A}$ to be the category with the same objects as $\mathcal{U}$, but with $Hom_{\mathcal{U}/\mathcal{A}}(U, V) \cong Hom_{\mathcal{U}}(U, V)/K$, where K is the subgroup of all morphisms from U to V which factor through an object of $\mathcal{A}$. The quotient category $\mathcal{U}/\mathcal{A}$ is also additive.

In [22], the following results are shown.

- Any additive category $\mathcal{A}$ admits an embedding in an $\mathcal{A}$-filtered flasque additive category.
- For any flasque additive category $\mathcal{U}$, $K\mathcal{U}$ is contractible, where $K\mathcal{U}$ denotes the Quillen K-theory space of $\mathcal{U}$.
- For any semisimple, idempotent complete additive category $\mathcal{A}$ and any embedding of $\mathcal{A}$ into an $\mathcal{A}$-filtered additive category $\mathcal{U}$, we obtain a homotopy fibration sequence

$$K\mathcal{A} \to K\mathcal{U} \to K\mathcal{U}/\mathcal{A}$$

It now follows that if we have an embedding $\mathcal{A} \to \mathcal{U}$ of additive categories, where $\mathcal{A}$ is semisimple and idempotent complete, and $\mathcal{U}$ is flasque, then $K\mathcal{U}/\mathcal{A}$ is a delooping of the K-theory space $K\mathcal{A}$. By applying the idempotent completion construction to $\mathcal{U}/\mathcal{A}$, one can iterate this construction to obtain a non-connective family of deloopings and therefore a non-connective spectrum. This delooping is equivalent to the Gersten–Wagoner deloopings of the last section and to the Pedersen–Weibel deloopings to be described in the next section. Finally, we note that M. Schlichting (see [25]) has constructed a version of the deloopings discussed in this section which applies to any idempotent complete exact category.

1.7
The Pedersen–Weibel Delooping and Bounded *K*-Theory

In this final section we will discuss a family of deloopings which were constructed by Pedersen and Weibel in [22] using the ideas of "bounded topology". This work is based on much earlier work in high-dimensional geometric topology, notably by E. Connell [10]. The idea is to consider categories of possibly infinitely generated free modules over a ring A, equipped with a basis, and to suppose further that elements in the basis lie in a metric space X. One puts restrictions on both the objects and the morphisms, i.e. the modules have only finitely many basis elements in any given ball, and morphisms have the property that they send basis elements to linear combinations of "nearby elements". When one applies this construction to the metric spaces $\mathbb{R}^n$, one obtains a family of deloopings of the K-theory spectrum of A. The construction has seen application in other problems as well, when applied to other metric spaces, such as the universal cover of a $K(\pi,1)$-manifold, or a finitely generated group Γ with word length metric attached to a generating set for Γ. In that context, the method has been applied to prove the so-called Novikov conjecture and its algebraic K-theoretic analogue in a number of interesting cases ([6,7], and [8]). See [19] for a complete account of the status of this conjecture. This family of deloopings is in general non-connective, like the Gersten–Wagoner delooping, and produces a homotopy equivalent spectrum.

We begin with the construction of the categories in question.

30 **Definition 30** Let A denote a ring, and let X be a metric space. We define a category $\mathcal{C}_X(A)$ as follows.

- The objects of $\mathcal{C}_X(A)$ are triples (F, B, φ), where F is a free left A-module (not necessarily finitely generated), B is a basis for F, and $\varphi : B \to X$ is a function so that for every $x \in X$ and $R \in [0, +\infty)$, the set $\varphi^{-1}(B_R(x))$ is finite, where $B_R(x)$ denotes the ball of radius R centered at x.
- Let $d \in [0, +\infty)$, and let (F, B, φ) and (F', B', φ') denote objects of $\mathcal{C}_X(A)$. Let $f : F \to F'$ be a homomorphism of A-modules. We say f is bounded with bound d if for every $\beta \in B$, $f\beta$ lies in the span of $\varphi^{-1}B_d(\varphi(x)) = \{\beta' | d(\varphi(\beta), \varphi'(\beta')) \leq d\}$. The morphisms in $\mathcal{C}_X(A)$ from (F, B, φ) to (F', B', φ') are the A-linear homomorphisms which are bounded with some bound d.

It is now easy to observe that $i\mathcal{C}_X(A)$, the category of isomorphisms in $\mathcal{C}_X(A)$, is a symmetric monoidal category, and so the construction of section 1.2 allows us to construct a spectrum $Sp(i\mathcal{C}_X(A))$. Another observation is that for metric spaces X and Y, we obtain a tensor product pairing $i\mathcal{C}_X(A) \times i\mathcal{C}_Y(B) \to i\mathcal{C}_{X\times Y}(A \otimes B)$, and a corresponding pairing of spectra $Sp(i\mathcal{C}_X(A)) \wedge Sp(i\mathcal{C}_Y(B)) \to Sp(i\mathcal{C}_{X\times Y}(A \otimes B))$ (see [21]). We recall from section 1.2 that for any symmetric monoidal category $\mathbf{C}$, there is a canonical map

$$N.\mathbf{C} \to Sp(\mathbf{C})_0$$

where $Sp(\mathbf{C})_0$ denotes the zero-th space of the spectrum $Sp(\mathbf{C})$. In particular, if f is an endomorphism of any object in $\mathbf{C}$, f determines an element in $\pi_1(Sp(\mathbf{C}))$. Now consider the case $X = \mathbb{R}$. For any ring A, let M_A denote the object $(F_A(\mathbb{Z}), \mathbb{Z}, i)$, where $i \hookrightarrow \mathbb{R}$ is the inclusion. Let σ_A denote the automorphism of M_A given on basis elements by $\sigma_A([n]) = [n+1]$. σ_A determines an element in $\pi_1 Sp(i\mathcal{C}_{\mathbb{R}}(A))$. Therefore we have maps of spectra

$$\Sigma Sp\left(i\mathcal{C}_{\mathbb{R}^n}(A)\right) \cong S^1 \wedge Sp\left(i\mathcal{C}_{\mathbb{R}^n}(A)\right) \to Sp\left(i\mathcal{C}_{\mathbb{R}}(\mathbb{Z})\right) \wedge Sp\left(i\mathcal{C}_{\mathbb{R}^n}(A)\right) \to$$

$$\to Sp\left(i\mathcal{C}_{\mathbb{R}\times\mathbb{R}^n}(\mathbb{Z}\otimes A)\right) \to Sp\left(i\mathcal{C}_{\mathbb{R}^{n+1}}(\mathbb{Z}\otimes A)\right) \cong Sp\left(i\mathcal{C}_{\mathbb{R}^{n+1}}(A)\right)$$

and therefore adjoint maps of spaces

$$Sp\left(i\mathcal{C}_{\mathbb{R}^n}(A)\right)_0 \to \Omega\left(Sp\left(i\mathcal{C}_{\mathbb{R}^{n+1}}(A)\right)\right)_0$$

Assembling these maps together gives the Pedersen–Weibel spectrum attached to the ring A, which we denote by $\mathcal{K}(A)$. Note that we may also include a metric space X as a factor, we obtain similar maps

$$Sp\left(i\mathcal{C}_{X\times\mathbb{R}^n}(A)\right)_0 \to \Omega\left(Sp\left(i\mathcal{C}_{X\times\mathbb{R}^{n+1}}(A)\right)\right)_0$$

We will denote this spectrum by $\mathcal{K}(X;A)$, and refer to it as the *bounded K-theory spectrum of X with coefficients in the ring A*. A key result concerning $\mathcal{K}(X;A)$ is the following excision result (see [8]).

Proposition 31 Suppose that the metric space X is decomposed as a union $X = Y \cup Z$. For any subset $U \subseteq X$, and any $r \in [0,+\infty)$, we let N_rU denote r-neighborhood of U in X. We consider the diagram of spectra 31

$$\underset{r}{colim}\, Sp\left(i\mathcal{C}_{N_rY}(A)\right) \leftarrow \underset{r}{colim}\, Sp\left(i\mathcal{C}_{N_rY\cap N_rZ}(A)\right) \to \underset{r}{colim}\, Sp\left(i\mathcal{C}_{N_rZ}(A)\right)$$

and let $\mathbf{P}$ denote its pushout. Then the evident map $\mathbf{P} \to Sp(i\mathcal{C}_X(A))$ induces an isomorphism on π_i for $i > 0$. It now follows that if we denote by $\mathcal{P}$ the pushout of the diagram of spectra

$$\underset{r}{colim}\, \mathcal{K}\left(N_rY;A\right) \leftarrow \underset{r}{colim}\, \mathcal{K}\left(N_rY\cap N_rZ;A\right) \to \underset{r}{colim}\, \mathcal{K}\left(N_rZ;A\right)$$

then the evident map $\mathcal{P} \to \mathcal{K}(X;A)$ is an equivalence of spectra.

Remark 32 The spectra $colim_r\, \mathcal{K}(N_rY;A)$ and $colim_r\, \mathcal{K}(N_rZ;A)$ are in fact equivalent to the spectra $\mathcal{K}(Y;A)$ and $\mathcal{K}(Z;A)$ as a consequence of the coarse invariance property for the functor $\mathcal{K}(-;A)$ described below. 32

Using the first part of 31, Pedersen and Weibel now prove the following properties of their construction.

- The homotopy groups $\pi_i\mathcal{K}(A)$ agree with Quillen's groups for $i \geq 0$.
- For $i < 0$ the groups $\pi_i\mathcal{K}(A)$ agree with Bass's lower K-groups. In particular, they vanish for regular rings.
- $\mathcal{K}(A)$ is equivalent to the Gersten–Wagoner spectrum.

The Pedersen–Weibel spectrum is particularly interesting because of the existence of the spectra $\mathcal{K}(X;A)$ for metric spaces X other than $\mathbb{R}^n$. This construction is quite useful for studying problems in high dimensional geometric topology. The spectrum $\mathcal{K}(X;A)$ has the following properties.

- **(Functoriality)** $\mathcal{K}(-;A)$ is functorial for *proper eventually continuous* map of metric spaces. A map of $f : X \to Y$ is said to be proper if for any bounded set $U \in Y$, $f^{-1}U$ is a bounded set in X. The map f is said to be eventually continuous if for every $R \in [0,+\infty)$, there is a number $\delta(R)$ so that $d_X(x_1,x_2) \leq R \Rightarrow d_Y(fx_1,fx_2) \leq \delta(R)$.
- **(Homotopy Invariance)** If $f,g : X \to Y$ are proper eventually continuous maps between metric spaces, and so that $d(f(x),g(x))$ is bounded for all x, then the maps $\mathcal{K}(f;A)$ and $\mathcal{K}(g;A)$ are homotopic.
- **(Coarse invariance)** $\mathcal{K}(X;A)$ depends only on the *coarse type* of X, i.e. if $Z \subseteq X$ is such that there is an $R \in [0,+\infty)$ so that $N_RZ = X$, then the map $\mathcal{K}(Z;A) \to \mathcal{K}(X;A)$ is an equivalence of spectra. For example, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ induces an equivalence on $\mathcal{K}(-;A)$. The spectrum $\mathcal{K}(-;A)$ does not "see" any local topology, only "topology at infinity".
- **(Triviality on bounded spaces)** If X is a bounded metric space, then $\mathcal{K}(X;A) \cong \mathcal{K}(A)$.

To show the reader how this $\mathcal{K}(X;A)$ behaves, we first remind him/her about locally finite homology. Recall that the singular homology of a space X is defined to be the homology of the singular complex, i.e. the chain complex C_*X, with C_kX denoting the free abelian group on the set of singular k-simplices, i.e. continuous maps from the standard k-simplex $\Delta[k]$ into X. This means that we are considering *finite* formal linear combinations of singular k-simplices.

33 **Definition 33** Let X denote a locally compact topological space. We define $C_k^{lf}X$ to be the infinite formal linear combinations of singular k-simplices $\sum_\sigma n_\sigma \sigma$, which have the property that for any compact set K in X, there are only finitely many σ with $im(\sigma) \cap K \neq \emptyset$, and $n_\sigma \neq 0$. The groups $C_k^{lf}X$ fit together into a chain complex, whose homology is denoted by $H_*^{lf}X$. The construction H_*^{lf} is functorial with respect to proper continuous maps, and is proper homotopy invariant.

34 **Remark 34** H_*^{lf} is formally dual to cohomology with compact supports.

Example 35. $H_*^{lf}\mathbb{R}^n$ vanishes for $* \neq n$, and $H_n^{lf}\mathbb{R}^n \cong \mathbb{Z}$.

Example 36. If X is compact, $H_*^{lf}X \cong H_*X$.

Example 37. Suppose that X is the universal cover of a bouquet of two circles, so it is an infinite tree. It is possible compactify X by adding a Cantor set onto X. The Cantor set can be viewed as an inverse system of spaces $\underline{C} = \dots C_n \to C_{n-1} \to \dots$, and we have $H_*^{lf}X \cong 0$ for $* \neq 1$, and $H_1^{lf}X \cong \underleftarrow{lim}\, \mathbb{Z}[C_n]$.

Example 38. For any manifold with boundary $(X, \partial X)$, $H_*^{lf}(X) \cong H_*(X, \partial X)$.

A variant of this construction occurs when X is a metric space.

Definition 39 Suppose that X is a *proper metric space*, i.e. that all closed balls are compact. We now define a subcomplex ${}^sC_*^{lf}X \subseteq C_*^{lf}X$ by letting ${}^sC_k^{lf}X$ denote the infinite linear combinations $\sum_\sigma n_\sigma \sigma \in C_k^{lf}X$ so that the set $\{diam(im(\sigma)) | n_\sigma \neq 0\}$ is bounded above. Informally, it consists of linear combinations of singular simplices which have images of uniformly bounded diameter. We denote the corresponding homology theory by ${}^sH_*^{lf}X$. There is an evident map ${}^sH_*^{lf}X \to H_*^{lf}X$, which is an isomorphism in this situation, i.e. when X is proper. 39

In order to describe the relationship between locally finite homology and bounded K-theory, we recall that spectra give rise to *generalize homology theories* as follows. For any spectrum $\mathcal{S}$ and any based space X, one can construct a new spectrum $X \wedge \mathcal{S}$, which we write as $h(X, \mathcal{S})$. Applying homotopy groups, we define the *generalized homology groups* of the space X with coefficients in $\mathcal{S}$, $h_i(X, \mathcal{S}) = \pi_i h(X, \mathcal{S})$. The graded group $h_*(X, \mathcal{S})$ is a generalized homology theory in X, in that it satisfies all of the Eilenberg–Steenrod axioms for a homology theory except the dimension hypothesis, which asserts that $h_i(S^0, \mathcal{S}) = 0$ for $i \neq 0$ and $h_0(X, \mathcal{S}) = \mathbb{Z}$. In this situation, when we take coefficients in the Eilenberg–MacLane spectrum for an abelian group A, we obtain ordinary singular homology with coefficients in A. It is possible to adapt this idea for the theories H_*^{lf} and ${}^sH_*^{lf}$.

Proposition 40 (See [8]) Let $\mathcal{S}$ be any spectrum. Then there are spectrum valued functors $h^{lf}(-, \mathcal{S})$ and ${}^sh^{lf}(-, \mathcal{S})$, so that the graded abelian group valued functors $\pi_* h^{lf}(-, \mathcal{S})$ and $\pi_*^s h^{lf}(-, \mathcal{S})$ agree with the functors $H_*^{lf}(-, A)$ and ${}^sH_*^{lf}(-, A)$ defined above in the case where $\mathcal{S}$ denotes the Eilenberg–MacLane spectrum for A. 40

The relationship with bounded K-theory is now given as follows.

41 **Proposition 41** There is a natural transformation of spectrum valued functors

$$\alpha^R(-) :^s h^{lf}(-, \mathcal{K}(R)) \to \mathcal{K}(-; R)$$

which is an equivalence for discrete metric spaces. (The constructions above extend to metric spaces where the distance function is allowed to take the value $+\infty$. A metric space X is said to be *discrete* if $x_1 \neq x_2 \Rightarrow d(x_1, x_2) = +\infty$.)

The value of this construction is in its relationship to the K-theoretic form of the Novikov conjecture. We recall ([8]) that for any group Γ, we have the assembly map $A^R_\Gamma : h(B\Gamma_+, \mathcal{K}(R)) \to \mathcal{K}(R[\Gamma])$, and the following conjecture.

42 **Conjecture 42:** *(Integral K-theoretic Novikov conjecture for Γ)* A_Γ induces a split injection on homotopy groups.

43 **Remark 43** This conjecture has attracted a great deal of interest due to its relationship with the original Novikov conjecture, which makes the same assertion after tensoring with the rational numbers, and using the analogous statement for L-theory. Recall that L-theory is a quadratic analogue of K-theory, made periodic, which represents the obstruction to completing non simply connected surgery. The L-theoretic version is also closely related to the Borel conjecture, which asserts that two homotopy equivalent closed $K(\Gamma, 1)$-manifolds are homeomorphic. This geometric consequence would require that we prove an isomorphism statement for A_Γ rather than just an injectivity statement.

We now describe the relationship between the locally finite homology, bounded K-theory, and Conjecture 42. We recall that if X is any metric space, and $d \geq 0$, then the *Rips complex for X with parameter d*, $R[d](X)$, is the simplicial complex whose vertex set is the underlying set of X, and where $\{x_0, x_1, \ldots, x_k\}$ spans a k-simplex if and only if $d(x_i, x_j) \leq d$ for all $0 \leq i, j \leq k$. Note that we obtain a directed system of simplicial complexes, since $R[d] \subseteq R[d']$ when $d \leq d'$. We say that a metric space is *uniformly finite* if for every $R \geq 0$, there is an N so that for every $x \in X$, $\#B_R(x) \leq N$. We note that if X is uniformly finite, then each of the complexes $R[d](X)$ is locally finite and finite dimensional. If X is a finitely generated discrete group, with word length metric associated to a finite generating set, then X is uniformly finite. Also, again if $X = \Gamma$, with Γ finitely generated, Γ acts on the right of $R[d](X)$, and the orbit space is homeomorphic to a finite simplicial complex. It may be necessary to subdivide $R[d](X)$ for the orbit space to be a simplicial complex. This Γ-action is free if Γ is torsion free. Further, $R[\infty](\Gamma) = \bigcup_d R[d](\Gamma)$ is contractible, so when Γ is torsion free, $R[\infty](\Gamma)/\Gamma$ is a model for the classifying space $B\Gamma$. The complex $R[d](X)$ is itself equipped with a metric, namely the path length metric. For a uniformly finite metric space X and spectrum $\mathcal{S}$, we now define a new spectrum valued functor $\mathcal{E}(X, \mathcal{S})$ by

$$\mathcal{E}(X, \mathcal{S}) = \underset{d}{colim}\, {}^{s}h^{lf}\left(R[d](X), \mathcal{S}\right)$$

Note that without the uniform finiteness hypothesis, the spaces $R[d](X)$ would not be locally compact. We may now apply the assembly map $\alpha^R(-)$ to obtain a commutative diagram

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
{}^{s}h^{lf}\left(R[d](X), \mathcal{K}(R)\right) & \xrightarrow{\alpha^R(R[d](X)/} & \mathcal{K}\left(R[d](X), R\right) \\
\downarrow & & \downarrow \\
{}^{s}h^{lf}(R[d+1](X), \mathcal{K}(R)) & \xrightarrow{\alpha^R(R[d+1](X)/} & \mathcal{K}\left(R[d+1](X), R\right) \\
\vdots & & \vdots \\
\downarrow & & \downarrow
\end{array}$$

which yields a natural transformation

$$\alpha^R_{\mathcal{E}}(X) : \mathcal{E}\left(X, \mathcal{K}(R)\right) \to \underset{d}{colim}\, \mathcal{K}\left(R[d](X), R\right)$$

It follows directly from the coarse invariance property of $\mathcal{K}(-, R)$ that the natural inclusion $\mathcal{K}(X, R) \to \underset{d}{colim}\, \mathcal{K}(R[d](X), R)$ is an equivalence of spectra, and by abuse of notation we regard $\alpha^R_{\mathcal{E}}(X)$ as a natural transformation from $\mathcal{E}(X, \mathcal{K}(R))$ to $\mathcal{K}(X, R)$.

Theorem 44 ([8]) Let Γ be a finitely generated group, with finite classifying space, and let Γ be regarded as a metric space via the word length metric associated to any finite generating set. If $\alpha^R_{\mathcal{E}}(\Gamma)$ is an equivalence, then the K-theoretic Novikov conjecture holds for the group Γ and the coefficient ring R. **44**

The value of a theorem of this type is that $\pi^s_* h^{lf}(X, \mathcal{K}(R))$ has many good properties, including an excision property. It does not involved the intricacies present in the (complicated) group ring $R[\Gamma]$, which makes it difficult to deal with the algebraic K-theory of this group ring directly. Two important advantages of this method are as follows.

- Experience shows that it generally works equally well for the case of L-theory, which is the case with direct geometric consequences.
- This method produces integral results. As such, it has the potential to contribute directly to the solution of the Borel conjecture.

We give an outline of the proof. A first observation is that there is an equivariant version of the construction $\mathcal{K}(-,R)$, which when applied to the metric space Γ with left action by the group Γ produces a spectrum $\mathcal{K}^{\Gamma}(\Gamma,R)$ with Γ-action, which is equivalent to the usual (non-equivariant) spectrum $\mathcal{K}(\Gamma,R)$, and whose fixed point set is the K-theory spectrum $\mathcal{K}(R[\Gamma])$. Next recall that for any space (or spectrum) X which is acted on by a group Γ, we may define the homotopy fixed point space (or spectrum) $X^{h\Gamma}$ to be the space (or spectrum) $F^{\Gamma}(E\Gamma,X)$ of equivariant maps from $E\Gamma$ to X, where $E\Gamma$ denotes a contractible space on which Γ acts trivially. The homotopy fixed set $X^{h\Gamma}$ has the following properties.

- The construction $X \to X^{h\Gamma}$ is functorial for maps of Γ-spaces (spectra).
- There is a map $X^{\Gamma} \to X^{h\Gamma}$, which is natural for Γ-equivariant maps, where X^{Γ} denotes the fixed point space (spectrum).
- Suppose that $f : X \to Y$ is an equivariant map of Γ-spaces (spectra), which is a weak equivalence as a non-equivariant map. Then the natural map $X^{h\Gamma} \to Y^{h\Gamma}$ is also a weak equivalence.
- For groups with finite classifying spaces, the functor $(-)^{h\Gamma}$ commutes with arbitrary filtering colimits.

In order to apply these facts, we will also need to construct an equivariant version of $\mathcal{E}(X,\mathcal{S})$. The facts concerning this construction are as follows.

- It is possible to construct an equivariant version of the functor on proper metric spaces $X \to {}^{s}h^{lf}(X,\mathcal{S})$, whose fixed point spectrum is equivalent to ${}^{s}h^{lf}(X/\Gamma,\mathcal{S})$. Such a construction yields naturally an equivariant version of the functor $\mathcal{E}(X,\mathcal{S})$.
- When X is a locally finite, finite dimensional simplicial complex with free simplicial Γ action, then the fixed point spectrum of the action of Γ on the equivariant model is ${}^{s}h^{lf}(X/\Gamma,\mathcal{S})$. In particular, we find that for a uniformly finite metric space X, $\mathcal{E}(X,\mathcal{S})^{\Gamma}$ is equivalent to

  $$\operatorname*{colim}_{d} h^{lf}\left(R[d](X)/\Gamma,\mathcal{S}\right)$$

  When $X = \Gamma$, equipped with a word length metric, and Γ is torsion free, we find that since $R[d](\Gamma)/\Gamma$ is a finite simplicial complex, we have

  $$\mathcal{E}(\Gamma,\mathcal{S})^{\Gamma} \cong \operatorname*{colim}_{d} h^{lf}\left(R[d](\Gamma)/\Gamma,\mathcal{S}\right) \cong \operatorname*{colim}_{d} h\left(R[d](X)/\Gamma,\mathcal{S}\right)$$

  $$\cong h\left(\operatorname*{colim}_{d} R[d](X)/\Gamma,\mathcal{S}\right) \cong h(B\Gamma,\mathcal{S})$$

- The assembly map A^{R}_{Γ} is the map obtained by restricting $\alpha^{R}_{\mathcal{E}}(\Gamma)$ to fixed point sets.
- Suppose Γ is a discrete group, and X is a finite dimensional simplicial complex equipped with a free simplicial Γ-action, with only a finite number of orbits in each simplicial dimension. Then $h^{lf}(X,\mathcal{S})^{\Gamma} \simeq h^{lf}(X,\mathcal{S})^{h\Gamma} \cong h(X/\Gamma,\mathcal{S})$, and

similarly for ${}^s h^{lf}$. Therefore, when Γ is torsion free, so that $R[d](\Gamma)$ is a filtering direct system of such complexes, we have

$$\mathcal{E}(\Gamma, \mathcal{S})^{\Gamma} \cong \mathcal{E}(\Gamma, \mathcal{S})^{h\Gamma}$$

In order to prove Theorem 44 from these facts, we consider the following diagram.

$$\begin{array}{ccc} h(B\Gamma, \mathcal{K}R) \cong \mathcal{E}(\Gamma, \mathcal{K}R)^{\Gamma} & \xrightarrow{(\alpha^R_{\mathcal{E}})^{\Gamma} = A^R_{\Gamma}} & \mathcal{K}\left(R[\Gamma]\right) \cong \mathcal{K}(\Gamma, R)^{\Gamma} \\ \downarrow & & \downarrow \\ \mathcal{E}(\Gamma, \mathcal{K}R)^{h\Gamma} & \longrightarrow & \mathcal{K}(\Gamma, R)^{h\Gamma} \end{array}$$

We wish to prove that the upper horizontal arrow is the inclusion of a spectrum summand. To prove this, it will suffice that the composite to the lower right hand corner is an equivalence. From the discussion above, it follows that the left hand vertical arrow is an equivalence. the hypothesis of Theorem 44 shows that the map $\mathcal{E}(\Gamma, \mathcal{K}R) \to \mathcal{K}(\Gamma, R)$ is a weak equivalence. It now follow from the properties of homotopy fixed points enumerated above the composite $h(B\Gamma, \mathcal{K}R) \to \mathcal{K}(\Gamma, R)^{h\Gamma}$ is a weak equivalence. It now follows that the map $h(B\Gamma, \mathcal{K}R) \to \mathcal{K}(R[\Gamma])$ is the inclusion on a wedge product of spectra.

Finally, we wish to give an indication about how one can prove that the hypothesis of Theorem 44 holds for some particular groups. In order to do this, we need to formulate a reasonable excision property for bounded K-theory. By a covering of a metric space X, we will mean a family of subsets $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of X so that $X = \bigcup_\alpha U_\alpha$. We will say that the covering has covering dimension $\leq d$ if whenever $\alpha_0, \alpha_1, \ldots, \alpha_k$ are distinct elements of A, with $k > d$, then $U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_k} = \emptyset$. For $R \geq 0$, we say that a covering $\mathcal{U}$ is *R-lax d-dimensional* if the covering $N_R\mathcal{U} = \{N_R U\}_{U \in \mathcal{U}}$ is d-dimensional. We also say that a covering $\mathcal{V}$ *refines* $\mathcal{U}$ if and only if every element of $\mathcal{V}$ is contained in an element of $\mathcal{U}$.

Definition 45 An *asymptotic covering* of dimension $\leq d$ of a metric space X is a family of coverings $\mathcal{U}_n$ of X satisfying the following properties. 45
- $\mathcal{U}_i$ refines $\mathcal{U}_{i+1}$ for all i.
- $\mathcal{U}_i$ is R_i-lax d-dimensional, where $R_i \to +\infty$.

Bounded K-theory has an excision property for one-dimensional asymptotic coverings. We first recall that the bounded K-theory construction can accept as input metric spaces in which the value $+\infty$ is an allowed value of the metric, where points x, y so that $d(x, y) = +\infty$ are understood to be "infinitely far apart". Suppose that we have a family of metric spaces X_α. Then we define $\coprod_\alpha X_\alpha$ to be the metric space whose underlying set is the disjoint union of the X_α's, and where the metric is given by $d(x,y) = d_\alpha(x,y)$ when $x, y \in X_\alpha$, and where $d(x,y) = +\infty$ when $x \in X_\alpha$, $y \in X_\beta$, and $\alpha \neq \beta$. Consider a one-dimensional asymptotic covering of X, and for each i and each set $U \in \mathcal{U}_i$, select a set $\Theta(U) \in \mathcal{U}_{i+1}$ so that $U \subseteq \Theta(U)$. For each i, we now construct the two metric spaces

$$N_0(i) = \coprod_{U \in \mathcal{U}_i} U$$

and

$$N_1(i) = \coprod_{(U,V) \in \mathcal{U}_i \times \mathcal{U}_i, U \cap V \neq \emptyset} U \cap V$$

There are now two maps of metric spaces $d_0^i, d_1^i : N_1(i) \to N_0(i)$, one induced by the inclusions $U \cap V \hookrightarrow U$ and the other induced by the inclusions $U \cap V \hookrightarrow V$. Recall that for any pair of maps $f, g : X \to Y$, we may construct the *double mapping cylinder* $Dcyl(f,g)$ as the quotient

$$X \times [0,1] \coprod Y / \simeq$$

where $\simeq$ is generated by the relations $(x,0) \simeq f(x)$ and $(x,1) \simeq g(x)$. This construction has an obvious extension to a spectrum level construction, and so we can construct $Dcyl(d_0^i, d_1^i)$ for each i. Moreover, the choices $\Theta(U)$ give us maps

$$Dcyl\left(d_0^i, d_1^i\right) \to Dcyl\left(d_0^{i+1}, d_1^{i+1}\right)$$

for each i. Furthermore, for each i, we obtain a map

$$\lambda_i : Dcyl\left(d_0^i, d_1^i\right) \to \mathcal{K}(X,R)$$

which on the metric spaces $N_0(i)$ and $N_1(i)$ is given by inclusions on the factors U and $U \cap V$ respectively. The excision result for bounded K-theory which we require is now the following.

46 **Theorem 46** The maps λ_i determine a map of spectra

$$\Lambda : \operatorname*{colim}_i Dcyl\left(d_0^i, d_1^i\right) \to \mathcal{K}(X,R)$$

which is a weak equivalence of spectra.

Instead of applying $\mathcal{K}(-,R)$ to the diagram

$$\begin{array}{ccccccc}
\longrightarrow & N_1(i) & \longrightarrow & N_1(i+1) & \longrightarrow \\
 & \downarrow{\scriptstyle d_0^i, d_1^i} & & \downarrow{\scriptstyle d_0^{i+1}, d_1^{i+1}} & \\
\longrightarrow & N_0(i) & \longrightarrow & N_0(i+1) & \longrightarrow
\end{array}$$

we apply $\mathcal{E}(,\mathcal{K}R)$ to it. We obtain double mapping cylinders $Dcyl_{\mathcal{E}}(d_0^i, d_1^i)$, maps $\lambda_{i,\mathcal{E}} : Dcyl_{\mathcal{E}}(d_0^i, d_1^i) \to \mathcal{E}(X, \mathcal{K}R)$, and finally a map

$$\Lambda_{\mathcal{E}} : \operatorname*{colim}_i Dcyl_{\mathcal{E}}\left(d_0^i, d_1^i\right) \to \mathcal{E}(X, \mathcal{K}R) .$$

Proposition 47 The map $\Lambda_{\mathcal{E}}$ is a weak equivalence of spectra. 47

Due to the naturality of the constructions, we now conclude the following.

Corollary 48 Suppose that we have an asymptotic covering of a metric space X of dimension d, and suppose that the maps 48

$$\alpha^R_{\mathcal{E}}\left(N_0(i)\right) \text{ and } \alpha^R_{\mathcal{E}}\left(N_i(i)\right)$$

are weak equivalences of spectra for all i. Then $\alpha^R_{\mathcal{E}}(X)$ is a weak equivalence of spectra.

This result is a useful induction result. We need an absolute statement for some family of metric spaces, though. We say that a metric space X is *almost discrete* if there is a number R so that for all $x, y \in X$, $d(x,y) \geq R \Rightarrow d(x,y) = +\infty$.

Theorem 49 If X is almost discrete, then $\alpha^R_{\mathcal{E}}(X)$ is a weak equivalence of spectra. 49

The proof of this theorem relies on an analysis of the K-theory of infinite products of categories with cofibrations and weak equivalences, which is given in [9]. An iterated application of Corollary 48 now gives the following result.

Theorem 50 Suppose that X is a metric space, and that we have a finite family of asymptotic coverings $\mathcal{U}^j = \{\mathcal{U}^j_k\}_{k\in A^j_i}$ of dimension 1, with $1 \leq j \leq N$. Suppose further that for each i, there is an R_i so that for any family of elements $V_j \in \mathcal{U}^j_i$, the intersection $\cap_j V_j$ has diameter bounded by R_i. Then $\alpha^R_{\mathcal{E}}(X)$ is a weak equivalence of spectra. 50

Remark 51 The existence of asymptotic coverings of this form can be verified in many cases. For instance, in [8], it is shown that such coverings exist for the homogeneous space G/K, where G is a Lie group, and K is a maximal compact subgroup. This gives the result for torsion free, cocompact subgroups of Lie groups. In the case of the real line, such a family of coverings can be given by the family of coverings $\mathcal{U}_i = \{[2^i k, 2^i(k+1)]\}_{k\in\mathbb{Z}}$. By taking product hypercubes, one obtains similar asymptotic coverings for Euclidean space. It can be shown that similar coverings exist for trees, and therefore it follows that one can construct a finite 51 family of asymptotic coverings satisfying the hypotheses of Theorem 50 for any finite product of trees, and therefore for subspaces of products of trees.

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