CONTROLLED ALGEBRAIC K-THEORY, A SURVEY

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0. Introduction

The purpose of this paper is to give the reader an impression of various techniques used in controlled algebra. Controlled algebra was introduced by Connell-Hollingsworth and developed by Quinn [4, 10] in connection with foundational studies in topological manifold theory. Suppose given a ring R and a space X, possibly with some extra structure such as a metric. The objects studied in controlled algebra are free based R-modules together with a map π from the basis to the metric space X. One usually requires that the image of π is nowhere dense in X, and that $\pi^{-1}(x)$ is finite for all x in X. For every point x in X we denote the free R-module generated by $\pi^{-1}(x)$ by A_x. A morphism φ between objects A and B can now be given a matrix decomposition as $\varphi = \{\varphi_x^y\}$ where φ_x^y is the composition $A_x \to A \xrightarrow{\varphi} B \to B_y$. Control is then a requirement that $\varphi_x^y = 0$ if x is "too far away" from y. In this paper we shall consider X exhibited with a metric and require the existence of a k such that $\varphi_x^y = 0$ if the distance from x to y is bigger than k. These are the bounded categories introduced in [7]. Notice that even in the case when X is compact, this category is a little more that just the category of R-modules; indeed the hom-sets come exhibited with a filtration.

Controlled algebra is used to guide geometric constructions. Suppose given a manifold or CW-complex with a reference map to a metric space. The cellular chains of the CW-complex or the cellular chains of a handle-decomposition of the manifold, can be interpreted as being the chain complex in a controlled category, by associating each generator, which comes from a cell in the space, to the image of the barycenter of the cell. By subdividing we can obtain that the boundary maps are in as low filtration degrees as we want.

Geometrically there are two basic moves. The first one is handle addition, consisting of sliding one cell (or handle in the manifold case) across another cell in a cell-decomposition. Algebraically this corresponds to changing the boundary map in the cellular chain complex by an elementary matrix. Sliding a cell across another cell will typically increase the diameter of the cell. The purpose of controlled algebra is to keep track of how much this diameter increases. The second basic geometric move is to introduce a pair of canceling cells at an appropriate spot, and then use these extra cells in the cell-slidings. These extra cells are used in the cell-slidings. This can sometimes help avoid increase the diameters of the cells too much. Algebraically, introducing a pair of extra canceling cells or canceling handles corresponds to stabilization – introducing an extra generator in adjacent

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dimensions letting the boundary map send generator to generator. In the cell-complex situation this is obtained simply by wedging on a disc at an appropriate point.

We can perform many handle additions simultaneously as long as we do it locally finitely. Since every handle addition increases the size of the handle, we also have to make sure that we do not reuse handles that have already been increased in size in a given step. This means that we need to divide the basis in two groups, and only slide handles corresponding to elements in the first group across handles in the second group. These moves are then expressed algebraically by a 2×2 elementary block matrix $e = \left(\begin{smallmatrix} 1 & A \\ 0 & 1 \end{smallmatrix}\right)$. We define the size of e as the size of A, the size of A being an expression of how far away a generator is sent. This size is clearly a measure of the increase in the size of the cells. Repeating this procedure a finite number of times will preserve some control, so in general we define a deformation E to be a composition of such elementary deformations e and we define the size of E to be the sum of the size of e_i when E is the product of the e_i's. We say that α can be δ-deformed to β if $\alpha \cdot E = \beta$ and the size of E is less than δ. A very useful algebraic result due to Quinn is that once an automorphism α is sufficiently small, then we may stabilize α, and deform in a small way to an arbitrarily small automorphism i. e. there exists an ε_X only depending on the metric space such that for every $\varepsilon < \varepsilon_X$ there is a δ so that if α is δ-controlled, then after stabilization, α can be ε-deformed to an automorphism of arbitrarily small control. The crucial point is that δ only depends on ε and X, not on how small we want the automorphism to be. We present a new proof of this result below.

In a final section, we give a new proof of topological invariance of Whitehead torsion. Controlled algebra comes into the picture as follows: If $h : K \to L$ is a homeomorphism between finite polyhedra, we can use simplicial approximation to approximate h by a simplicial homotopy equivalence with simplicial homotopy inverse g. If f and g are chosen close enough to h and h^{-1}, the tracks of the homotopies $f \circ g \simeq \mathrm{id}$ and $g \circ f \simeq \mathrm{id}$ will be small. Following the usual Whitehead prescription represents $\tau(h)$ by a controlled automorphism. The point here is that since polyhedra are locally simply connected, it is unnecessary to specify paths between the barycenters of simplices, provided that these simplices are close together. Thus, the usual passage to $\mathbb{Z}\pi_1 L$-modules is replaced by a passage to geometric algebra. We now use a standard algebraic trick to pass from a K_1-problem in ε-δ controlled algebra to an equivalent K_2 problem in Pedersen-Weibel's bounded algebra. It turns out we can show the Whitehead torsion is in the image from a certain K_2-group. We compute this K_2-group to be $\pm\pi$ which is precisely what is divided out in $K_1(\mathbb{Z}\pi)$ in the definition of the Whitehead group. Basically we prove that the Whitehead torsion of a homeomorphism is in the image of the K-theory assembly map, and hence trivial in the Whitehead group.

Other algebraic approaches to the topological invariance of Whitehead torsion are given by Quinn in [9], and Ranicki and Yamasaki in [11]. They use an ε controlled K_1-group, K_1^ε, where we use an ordinary K_2 of an additive category in the sense of Bass [1].

1. A Category of metric spaces and eventual Lipschitz maps

We shall work in the category of metric spaces and eventual Lipschitz maps. There are several similar, slightly conflicting concepts in the literature, such as the uniformly bornologous maps in [12] and eventual Lipschitz maps in [8] and in [13]. The following definitions seem appropriate for the purposes of this paper:

Definition 1.1. The category $\mathcal{M}$ of metric spaces and eventual Lipschitz maps has as objects arbitrary metric spaces. An *eventual Lipschitz map* $f : M \to N$ is a not necessarily continuous map satisfying:

(i) The inverse image of every bounded subset of N is bounded in M.
(ii) There exist r and k, depending only on f, so that for all $x, y \in M$ $d(f(x), f(y)) < k \cdot d(x, y) + r$.

Subspaces are given the induced metric and product spaces are given the max metric. A *homotopy* of morphisms $f_0, f_1 : M \to N$ is a morphism $F : M \times I \to N$ which restricts to f_0 and f_1 on the ends. It follows immediately that maps f_0 and f_1 are homotopic if and only if $d(f_0(x), f_1(x))$ is uniformly bounded.

To relate this to similar notions, we recall Gromov's notion of quasi-isometry [5]:

Definition 1.2. Let X and Y be metric spaces and let $F : X \to Y$ be a function. F is a (K, c)-quasi-isometry ($c \geq 0$ and $K \geq 1$) provided that for all $x, y \in X$

$$(1/K)d(x, y) - c \leq d(F(x), F(y)) \leq Kd(x, y) + c$$

Metric spaces X and Y are quasi-isometric if there are quasi-isometries in both directions. This is clearly equivalent to saying that there is a quasi-isometry F from X to Y and a constant C such that every point of Y is within C of the image of X.

A homotopy equivalence in the category $\mathcal{M}$ is therefore a quasi-isometry in the sense of Gromov and homotopy equivalent spaces are quasi-isometric.

Remark 1.3. All notions of this section make equally good sense if we replace the metric spaces by pseudo-metric spaces, since the condition $d(x, y) = 0$ implies $x = y$ is not really needed. This is useful, for instance, in case we have a map $p : M \to X$ and we want to use the pseudo-metric $\rho(m_1, m_2) = d(p(m_1), p(m_2))$ to measure distances in M.

2. Bounded algebraic categories

Given a ring R and a pseudo-metric space M, we define a category $\mathcal{C}_M(R)$. The model case of our definition is the case in which M is the infinite open cone $O(K) = \{t \cdot x \in \mathbb{R}^{n+1} | t \in [0, \infty), x \in K\}$ on a complex $K \subset S^n \subset \mathbb{R}^{n+1}$ and $R = \mathbb{Z}\pi$, with π a finitely presented group. The metric on $O(K)$ is inherited from the surrounding Euclidean space.

Definition 2.1. An object A of $\mathcal{C}_M(R)$ is a collection of finitely generated free based right R-modules A_x, one for each $x \in M$, such that for each ball $C \subset M$ of finite radius, only finitely many A_x, $x \in C$, are nonzero. A morphism $\varphi : A \to B$ is a collection of morphisms $\varphi^x_y : A_x \to B_y$ such that there exists $k = k(\varphi)$ such that $\varphi^x_y = 0$ for $d(x, y) > k$. The bound of φ i denoted $\mathrm{bd}(\varphi)$ is the minimal such k.

The composition of $\varphi : A \to B$ and $\psi : B \to C$ is given by $(\psi \circ \varphi)^x_y = \sum_{z \in M} \psi^z_y \varphi^x_z$. The composition $(\psi \circ \varphi)$ satisfies the local finiteness and boundedness conditions whenever ψ and φ do.

This makes $\mathcal{C}_M(R)$ into a filtered additive category, where $F_d \hom(A_1, A_2)$ consists of morphisms having bound $\leq d$.

Definition 2.2. The *idempotent completion* $\hat{\mathcal{C}_M}(R)$ of $\mathcal{C}_M(R)$ is the category whose objects are pairs (A, p) where A is an object of $\mathcal{C}_M(R)$ and $p : A \to A$ is idempotent. A morphism $\varphi : (A_1, p_1) \to (A_2, p_2)$ is a morphism $\varphi : A_1 \to A_2$ in $\mathcal{C}_M(R)$ such that $\varphi = p_2\varphi p_1$. The filtration degree of φ is the smallest d such that $\varphi = p_2 f p_1$ for some $f \in F_d \hom(A_1, A_2)$ with $fp_1 = p_2 f$.

Given a ring R we denote the Quillen K-theory spectrum by $K(R)$. The non-connective Bass-Quillen spectrum which includes the negative K-groups is denoted $K^{-\infty}(R)$.

Theorem 2.3. (Pedersen-Weibel [8]) *If P is a finite polyhedron and $* \geq 0$, then*

$$\pi_{*-1}(P_+ \wedge K^{-\infty}(R)) = H_{*-1}(P; K^{-\infty}(R)) \cong K_*(\hat{\mathcal{C}_{O(P)}}(R)).$$

It is easy to see that an additive category is cofinal in its idempotent completion. It thus follows that the idempotent completion only affects the K-groups in degree 0, so we have

$$H_{*-1}(P; K^{-\infty}(R)) \cong K_*(\mathcal{C}_{O(P)}(R))$$

for $* \geq 1$. In particular, we have

$$K_1(\mathcal{C}_{O(P)}(R)) \cong H_0(P; K^{-\infty}(R)).$$

The group $K_1(\mathcal{C}_M(R))$ has a "classical" description: it consists of equivalence classes of pairs (A, α), where A is an object of $\mathcal{C}_M(R)$ and $\alpha : A \to A$ is an automorphism. If B is another object in $\mathcal{C}_M(R)$, then α is equivalent to $\alpha \oplus \mathrm{id} : A \oplus B \to A \oplus B$. In addition, id is declared to be equivalent to α if α is an elementary automorphism as in Definition 3.1 below. We shall also be discussing K_2 classically as the limit of the endomorphism ring of an object under inclusion of direct summands.

One purpose of this paper is to relate the bounded K-theory of $O(P)$ to the controlled or "ε-δ" K-theory on P in the sense of Chapman and Quinn[3, 10]. We will begin by showing that for each finite polyhedron P there is a critical size ε_P so that automorphisms in $\mathcal{C}_P(R)$ which have bounds less than ε_P can be "deformed by small moves" to automorphisms with arbitrarily small bounds.

3. Squeezing K_1 and K_2

Consider a finite simplicial complex K. As an additive category, $\mathcal{C}_K(R)$ is equivalent to the category of finitely generated free R-modules, since K is eventual Lipschitz equivalent to a point. However, as a filtered category the situation is different. We consider the filtration of the hom-sets given by the bounds on the morphisms. It is the aim of this section to prove that given K there is an ε so that if an automorphism and its inverse are both bounded by ε, then the automorphism may be given a small deformation to an automorphism with arbitrarily small bounds. We introduce some definitions:

Definition 3.1. Let X be a metric space.

(i) If A is an object in $\mathcal{C}_X(R)$, a morphism $\eta : A \to A$ is *strictly triangular* with respect to an internal direct sum decomposition $A = A_1 \oplus A_2$ if η factors as $A \to A_1 \xrightarrow{\eta'} A_2 \to A$. We will denote the *elementary automorphism* $\mathrm{id} + \eta : A \to A$ by e_η when η is strictly triangular.

(ii) If $\eta : B \to C$ is a morphism in $\mathcal{C}_X(R)$, $\eta_{\leq k}$ will denote the morphism such that:
$$(\eta_{\leq k})^x_y = \begin{cases} \eta^x_y & \text{if} \quad ||x||, ||y|| \leq k \\ 0 & \text{if} \quad ||x|| > k \text{ or } ||y|| > k. \end{cases}$$
(iii) If $e_\eta : A \to A$ is an elementary automorphism in $\mathcal{C}_X(R)$, then $e^{\leq k}_\eta$ will denote the elementary automorphism $\mathrm{id} + \eta_{\leq k}$.

(iv) We will say that an object A in $\mathcal{C}_X(R)$ has *support* in $Y \subset X$ if $A_x = 0$ for all $x \notin Y$. We will say that a morphism $\alpha : A \to B$ in $\mathcal{C}_X(R)$ has support in Y if $\alpha^x_y = 0$ whenever x or $y \notin Y$. We will say an elementary automorphism e_η has *support* in Y if η has support in Y.

Remark 3.2. In the definition above, since the direct sum decomposition is required to be internal, and A_i are objects in the category we have an internal direct sum decomposition $A_x = (A_1)_x \oplus (A_2)_x$ for each $x \in X$. Notice that there are two different kinds of "cutting off" in the definition above, one by morphisms and one by objects.

Remark 3.3. If $\{e_{\eta_i}\}$ is a collection of elementary automorphisms on $\mathcal{C}_{O(K)}(R)$ and $\prod_{i=1}^n e_{\eta_i} = 1$, then $\prod_{i=1}^n e^{\leq k}_{\eta_i}$, k large, will be equal to 1 for $||x||$ sufficiently large or sufficiently small, but may be nontrivial in the band $k - \sum \mathrm{bd}(e_{\eta_i}) \leq ||x|| \leq k + \sum \mathrm{bd}(e_{\eta_i})$. Compressing this band onto a single copy of K, say $||x|| = k$, we obtain an object in $\mathcal{C}_K(R)$ and an automorphism of this object. By choosing k to be large, we can force this automorphism to have bound as small as we like, in the original metric on K, since it is bounded in the metric which has been multiplied by k. This construction will be used in the proof of Theorem 3.7, which is our main squeezing theorem.

Lemma 3.4. *If $\alpha : A \to B$ and $\beta : B \to C$ are morphisms in $\mathcal{C}_X(R)$,*

(i) $\mathrm{bd}(\beta \circ \alpha) \leq \mathrm{bd}(\beta) + \mathrm{bd}(\alpha)$.

(ii) $\mathrm{bd}(\alpha_{\leq k}) \leq \mathrm{bd}(\alpha)$.

(iii) *If* $e = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$, *then* $\mathrm{bd}(e) = \mathrm{bd}(\eta)$.

(iv) *If $\alpha, \bar{\alpha} : A \to B$ and $\beta, \bar{\beta} : B \to C$ are morphisms with $\alpha^x_y = \bar{\alpha}^x_y$ and $\beta^x_y = \bar{\beta}^x_y$ for $x, y \in S \subset O(K)$, then $\alpha \circ \beta = \bar{\alpha} \circ \bar{\beta}$ away from a* $(\mathrm{bd}(\alpha) + \mathrm{bd}(\beta))$-*neighborhood of boundary$S$.*

For the rest of this section, we will assume that $\mathbb{R}^n$ has the max metric so that the unit ball is $\prod_{i=1}^n [-1, 1]$. By a *cubical subcomplex* of the boundary of the unit ball in $\mathbb{R}^n$, we will mean a complex consisting entirely of faces of $\prod_{i=1}^n [-1, 1]$. To see that every finite polyhedron is PL homeomorphic to such a cubical complex, take the standard n-simplex Δ^n to be the convex hull of unit vectors $e_1, \ldots, e_{n+1}$ in $\mathbb{R}^{n+1}$. A PL homeomorphism from Δ^n to a cubical subcomplex of the cube $\prod_{i=1}^{n+1} [0, 1]$ in $\mathbb{R}^{n+1}$ is given by sending each barycenter $\langle \widehat{e_{i_0}, \ldots, e_{i_j}} \rangle$ to $e_{i_0} + \cdots + e_{i_j}$ and extending linearly. This PL homeomorphism takes each simplex of Δ^n onto a cubical subcomplex of the cube. Since every finite simplicial complex K is isomorphic to a subcomplex of some Δ^n, composing with the homeomorphism above produces a cubical subcomplex of the cube which is PL homeomorphic to K. Closer inspection shows that the cubes are obtained by amalgamating simplexes of the first barycentric subdivision of K.

Recall the following basic matrix identity

Lemma 3.5. (6 term identity)

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

Proof. This is of course well known. A graphical presentation is given as follows where all the horizontal arrows are identities and matrix multiplication is given by adding all ways through the diagram:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[dr, "\alpha"] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[dr, "\alpha"] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[dr, "-1"] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[ur, "-\alpha^{-1}"'] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[ur, "1"'] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[ur, "1"'] & \bullet
\end{tikzcd}$$

□

Here is the main technical result leading to the K_1-squeezing theorem.

Theorem 3.6. (K_1-vanishing) *Let K be a cubical subcomplex of the boundary of the unit ball in $\mathbb{R}^n$. If A is supported* outside *of the unit ball in $O(K)$ (which is $\prod_1^n[-1,1] \cap O(K)$), and $\alpha : A \to A$ is an automorphism in $\mathcal{C}_{O(K)}(R)$ such that α and α^{-1} are bounded by $\delta < \frac{2}{3} \cdot 6^{-(\dim K)}$ in the max metric, then there exist an object B in $\mathcal{C}_{O(K)}(R)$, and a product of elementary matrices (= a deformation) of bound at most $\delta \cdot 6^{\dim K+1}$, $\prod e_{\eta_i}$ on $A \oplus B$ in the category $\mathcal{C}_{O(K)}(R)$ such that :*

(i) *B is supported outside of the unit ball in $O(K)$.*
(ii) $(\alpha \oplus \mathrm{id}_B) \cdot \prod e_{\eta_i} = \mathrm{id}_{A \oplus B}$.

Before proving Theorem 3.6, we will show how it leads to the squeezing theorem.

Theorem 3.7. (K_1-Squeezing Theorem) *Let K be a cubical subcomplex of the boundary of the unit ball in $\mathbb{R}^n$. If $\varepsilon < 6^{-\dim K}$ and $\alpha : A \to A$ is an automorphism in $\mathcal{C}_K(R)$ such that α and α^{-1} are bounded by ε in the max metric, then for each $\mu > 0$ there is an object C in $\mathcal{C}_K(R)$ and an automorphism $\beta : A \oplus C \to A \oplus C$ such that*

(i) *β and β^{-1} are bounded by μ.*
(ii) *There is a deformation of size $6^{\dim K+1} \cdot \varepsilon$ of $\alpha \oplus \mathrm{id}$ to β*

Proof. We have K a subset of the boundary of the unit cube and a subset of $O(K)$. Let $\alpha : A \to A$ be an automorphism as in the statement of the theorem. If ε is small, Theorem 3.6 guarantees that we can find B and a deformation $\alpha \oplus \mathrm{id}_B \cdot \prod e_{\eta_i} = \mathrm{id}_{A \oplus B}$.

Let $L > 0$ be large and consider $(\alpha \oplus \mathrm{id})^{\leq L} \prod e_{\eta_i}^{\leq L}$. For L sufficiently large, $(\alpha \oplus \mathrm{id})^{\leq L}$ is simply $\alpha \oplus \mathrm{id}_C$ where C denotes the modules of B sitting inside a region a little larger than L. Since each $e_{\eta_i}^{\leq L}$ is an elementary matrix, we will write $e_{\eta_i}^{\leq L} = e_{\bar{\eta}_i}$. Obviously the product $(\alpha \oplus \mathrm{id}_C) \cdot \prod e_{\bar{\eta}_i}$ is no longer the identity. In a region around radius L it will be an automorphism, and we do of course have to be careful to make C so big that it supports this automorphism. Now restrict the attention to a band from radius 1 out to a radius which is L plus the bound of the deformation. This region is being preserved, and since the automorphism is bounded, independently of L, and only different from the identity in a band around radius L. This automorphism, when measured only in K, forgetting the radial direction by using the radial projection $\mathbf{x} \to \frac{\mathbf{x}}{|\mathbf{x}|}$ can now be made as small as wanted by choosing L large. The domain of this radial projection is the region between a sphere of radius 1 and a sphere of radius a little larger than L intersected with $O(K)$. The target is $K = O(K) \cap S^n$. (Remember we are using the max metric, so the unit sphere is the boundary of the unit cube). This radial projection is a map of compact spaces so we do get an induced map at the category level. Under this map $L \cdot K$ is sent to K by a map decreasing distances by the factor L, so we get the desired result. □

Proof of Theorem 3.6. We begin by considering the case $K = *$ even though this case does not exhibit the typical behavior. In this case, $O(K)$ is a ray and we have the picture below:

$$\begin{tikzcd} A \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & {} \end{tikzcd}$$

A is supported outside the unit ball. In the picture we illustrate the case where A is supported on an embedded copy of K, but it is obvious that the arguments to follow only need A supported outside the unit ball. We now take B to be an infinite direct sum of copies of A, one at each "integral" point $n \in O(K)$ for $n \geq 2$. We represent $\alpha \oplus \mathrm{id}$ schematically in the picture below:

$$\begin{tikzcd} A \arrow[loop above, "\alpha"] \arrow[r, dash] & A \arrow[loop above, "\mathrm{id}"] \arrow[r, dash] & A \arrow[loop above, "\mathrm{id}"] \arrow[r, dash] & A \arrow[loop above, "\mathrm{id}"] \arrow[r, dash] & A \arrow[loop above, "\mathrm{id}"] \arrow[r, dash] & {} \end{tikzcd}$$

The 6 term identity allows us to multiply by a product $\prod e_{\eta_i}$ of elementary matrices to obtain the picture:

$$\begin{tikzcd} A \arrow[loop above, "\alpha"] \arrow[r, dash] & A \arrow[loop above, "\alpha^{-1}"] \arrow[r, dash] & A \arrow[loop above, "\alpha"] \arrow[r, dash] & A \arrow[loop above, "\alpha^{-1}"] \arrow[r, dash] & A \arrow[loop above, "\alpha"] \arrow[r, dash] & {} \end{tikzcd}$$

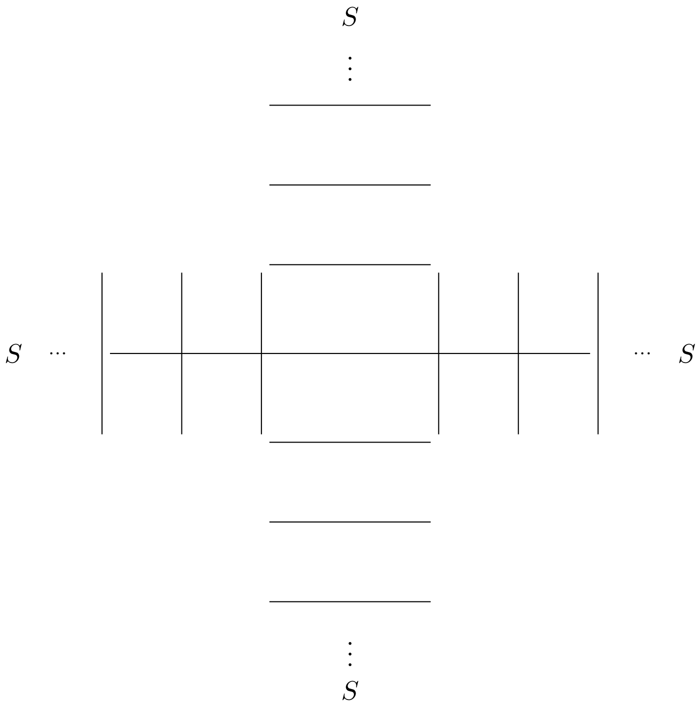
Shifting and using the 6 term identity again, we see that multiplication by a suitable $\prod e'_{\eta_i}$ gives us:

$$\begin{tikzcd} A \arrow[loop above, "\mathrm{id}"] \arrow[r, dash] & A \arrow[loop above, "\mathrm{id}"] \arrow[r, dash] & A \arrow[loop above, "\mathrm{id}"] \arrow[r, dash] & A \arrow[loop above, "\mathrm{id}"] \arrow[r, dash] & A \arrow[loop above, "\mathrm{id}"] \arrow[r, dash] & {} \end{tikzcd}$$

We have therefore obtained that $(\alpha \oplus \mathrm{id}) \cdot (\prod e_{\eta_i})^{-1} \cdot \prod e'_{\eta_i})^{-1} = \mathrm{id}$, as desired. In this too simple case α is bounded by 0, but evidently we have used 12 elementary operations, six of which have $\pm\alpha$ and $\pm\alpha^{-1}$ and six of which have ± 1 off the diagonal. Each elementary operation is bounded by 1 because we placed the modules at the integral lattice points, so the whole deformation is bounded by 12. Had we instead placed the modules at all half integral points, the resulting bound would have been 6, and we can obviously get the bound as small as we want. Since six of the elementary operations When we do this kind of argument in higher dimensions, the α's are not bounded by 0, so this is the reason we only lose control by a factor of 6.

The notion Eilenberg swindle refers to any argument where one exploits the fact that one may get $P \oplus X$ isomorphic to X by choosing X to be an infinite sum of P's. The argument above is a kind of Eilenberg swindle. We now move to the case $K = S^1$, which is more representative of the general case. We would like to perform the same sort of Eilenberg swindle, but now pushing the module A to infinity in all directions at once increases the bound unacceptably. i

The solution to this difficulty is to apply a variation of the swindle rigidly to the top-dimensional faces and induct on dimension. Quinn points out that it is the interplay between Cartesian and polar coordinates that is being exploited. Let $\{F_i\}$ be the collection of top-dimensional faces of K. We stabilize using infinitely many copies of $A|F_i$ with the identity automorphism. Doing this to all of the top-dimensional faces at once, we have the picture below



Let $\alpha|(A|F_i)$ be the composition $(A|F_i) \to A \xrightarrow{\alpha} A \to (A|F_i)$ and define $\alpha^{-1}|(A|F_i)$ similarly. Note that $\mathrm{bd}(\alpha|F_i) \leq \mathrm{bd}(\alpha)$. Since $\alpha|(A|F_i)$ need not be invertible, we cannot use the earlier identities directly. We can, however, consider a similar product of elementary matrices:

$$\prod e_{\eta_i} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} (\begin{pmatrix} 1 & 0 \\ \alpha|F_i & 1 \end{pmatrix} \begin{pmatrix} 1 & -(\alpha^{-1}|F_i) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha|F_i & 1 \end{pmatrix}$$

We can choose the copies of $A|F_i$ close together so that $\sum \mathrm{bd}(e_{\eta_i}) < 3\delta$ and so that $\prod e_{\eta_i}$ is equal to $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ outside of a 3δ-neighborhood of $\cup_i \partial F_i$. Applying the swindle as before, we have $\prod e_{\eta_i} \cdot \prod e'_{\eta_i} \cdot (\alpha \oplus \mathrm{id}) = 1$ outside of a 6δ-neighborhood of $\cup_i \partial F_i$, and therefore in a neighborhood of the rays denoted S, provided that $\delta < \frac{1}{6}$. The $\frac{2}{3}$-factor ensures that we actually have the identity in the middle third of the band around the ray S. The operations on the separate faces are disjoint, so corresponding operations can be combined into single elementary matrices and the bounds need not be summed over the faces. Destabilizing, we excise the area where $\prod e_{\eta_i} \cdot \prod e'_{\eta_i} \cdot (\alpha \oplus \mathrm{id}) = 1$, obtaining modules in four separate quadrants and an automorphism that is forced to preserves each quadrant by the control condition, at least after a further application of an elementary operation that does not increase control.

To be precise, let B be the union of the two axis in $\mathbb{R}^2$, and write $\mathbb{R}^2 = B \cup \bigcup_{i=1}^4 Q_i$ where Q_i are the four quadrants. On $A|B$ the automorphism is now the identity, actually it is the identity in a little neighborhood, and the control conditions ensure that the automorphism preserves $A|Q_i$. We can eliminate $A|Q_i$ by performing the infinite repetition trick in the four directions of 45 degree in each quadrant. That is, we stabilize by making infinitely many identical copies of the nonzero modules and then we apply 3.5 twice to write $\alpha \oplus \mathrm{id}$ as a product of elementary matrices. This completes the case $K = S^1$. Since the last set of elementary matrices is constructed using modified α's, which lead to a loss of control by factor of 6, they are bounded by $6^2 \cdot \delta$. Indeed in each dimension we lost a factor of 6.

Suppose, now, that K is a cubical subcomplex of S^2. We begin by performing the infinite repetition trick along a ray perpendicular to each top-dimensional face. Destabilizing, this eliminates all of the modules inside a distance of $\frac{1}{3}$ of the rays perpendicular to the top-dimensional faces.

Note that we have eliminated all modules A_x such that x has coordinates (x_1, x_2, x_3) with $|x_i| < 1 - 6\delta$ for two values of i. Next, we perform the infinite repetition trick along the 45 degree angled arrows to eliminate all modules A_x such that x has coordinates with $|x_i| < 1 - 36\delta$ for some value of i. After this operation, the nonzero modules are concentrated into separate octants provided, of course, that $\delta < \frac{2}{3} \cdot \frac{1}{36}$ and can be eliminated by doing swindles in directions $(\delta_1, \delta_2, \delta_3)$, where $|\delta_i| = 1$.

The general induction proceeds in an entirely similar fashion. We begin by eliminating all modules A_x where x has $n-1$ coordinates with absolute value $< 1 - 6\delta$. This is done by swindling in directions with one coordinate equal to ± 1 and the rest equal to 0. Next, we eliminate all A_x so that x has $n-2$ coordinates with absolute value $< 1 - 36\delta$. This is done by pushing in directions with *two* coordinates equal to ± 1 and with the rest equal to 0. Continuing, we eventually have no remaining nonzero modules A_x with any coordinates of x having absolute values $< 1 - 6^{(n-1)}\delta$. At this point, the remaining nonzero modules are isolated in the 2^n "octants" and can be eliminated by swindling in directions with all coordinates equal to ± 1. Notice we may do this process to all of S^n, not just the cubical subcomplex of S^n. In the end the modules will have support in $O(K)$ since we are stabilizing with the zero modules on the faces of S^n that are not in K. A good example to think of is the 1-skeleton of S^2 where the first step in the argument is an empty step. □

Addendum 3.8. As mentioned before, the argument we have given above actually works for any automorphism $\alpha : A \to A$ in $\mathcal{C}_{O(K)}(R)$ such that α and α^{-1} are bounded by δ and such that $A_x = 0$ for x inside of the unit ball. The point is that the induction given above naturally starts with any α such that $A_x = 0$ for all x with all coordinates < 1.

We want to prove that the squeezing is unique. For this, we will need a K_2 analog of Theorem 3.6. We begin by recalling Bass' definition of K_2 of an additive category. [1]

Definition 3.9. (Bass) Let $\mathcal{A}$ be an additive category in which the isomorphism classes of objects form a set. Assume $A = A_1 \oplus A_2$, then we get a ring homomorphism $\mathrm{End}(A_1) \to \mathrm{End}(A)$ sending η to $\eta \oplus 0$.

$$K_2(\mathcal{A}) = \lim_{A} K_2(\mathrm{End}(A))$$

where the limit is taken over objects in $\mathcal{A}$ and inclusions as described above.

Conjugating by an automorphism of A induces the identity on $K_2(\mathrm{End}(A))$ so there is no trouble in defining the above limit. C. Weibel proved in [14] that Bass' definition agrees with Quillen's definition of K_2 of the symmetric monoidal category obtained by restricting morphisms to isomorphisms. Since endomorphisms in a sum of r copies of A can be written as an $r \times r$ matrix with entries in $\mathrm{End}(A)$,

$$\mathrm{End}(A^{\oplus r}) = M(r, \mathrm{End}(A)),$$

we can think of an element of $K_2(\mathcal{A})$ as a product of Steinberg symbols in $\mathrm{End}(A)$, for some object A in $\mathcal{A}$ which when evaluated as a product of elementary matrices gives the identity. More commonly we will be given a product of automorphisms of $A = A_1 \oplus A_2 \oplus \ldots \oplus A_r$ which are the identity except for a component sending A_i to A_j, $i \neq j$ i. e. elementary automorphisms with product equal to the identity, and we want to think of this as an element in $K_2(\mathcal{A})$, but this may be done by stabilizing each A_i by $\oplus_{j \neq i} A_j$ to get an element in $K_2(\mathrm{End}(A))$. Whenever we have a product of elementary matrices equal to the identity, we thus get an element of K_2 of the category by replacing the elementary automorphisms by Steinberg symbols in the obvious way.

Theorem 3.10. *(K_2-vanishing) Let K be a cubical subcomplex of the boundary of the unit ball in $\mathbb{R}^n$. If $\{e_{\eta_i}\}$ are elementary automorphisms of a module A in $\mathcal{C}_{O(K)}(R)$ with $A^{<1} = 0$, and such that $\prod_{i=1}^{k} e_{\eta_i} = 1$ and $\sum_{i=1}^{k} \mathrm{bd}(e_{\eta_i}) < \delta < 6^{-(\dim K - 1)}$ in the max metric, then the corresponding element in $K_2(\mathcal{C}_{O(K)}(R))$ is trivial.*

Proof. The proof is very similar to the proof of the squeezing theorem. We shall illustrate the proof by the case $K = S^1 = \partial([-1,1] \times [-1,1])$.

Consider the four regions $B_1 = [1, \infty) \times [-1, 1]$, $B_2 = [-1, 1] \times [1, \infty)$, $B_3 = (-\infty, -1] \times [-1, 1]$ and $B_4 = [-1, 1] \times (-\infty, -1]$. The strategy of the proof will be to write the element in K_2 as an element with support in B_i multiplied by an element with support in the complement and such that the control conditions ensure that the elements in the four quadrants C_i whose union is the complement are independent. This will then ensure the element is trivial since

$$K_*(\mathcal{C}_{B_i}(R)) = K_*(\mathcal{C}_{C_i}(R)) = 0$$

by Eilenberg swindles.

Consider a product of elementary matrices $e_{\eta_1} \cdot e_{\eta_2} \cdot \ldots \cdot e_{\eta_k} = 1$ and let $x_{\eta_1} \cdot x_{\eta_2} \cdot \ldots \cdot x_{\eta_k}$ be the corresponding element in the Steinberg group giving a K_2 element in $\mathcal{C}_{O(K)}(R)$. Write $O(K) = B_1 \cup D_1$ as a disjoint union, and write the modules on which the $e'_{\eta_i}s$ are realized as $A = (A|B_1) \oplus (A|D_1)$. In this direct sum decomposition η_k may be written as $\eta_k = \left\{ \begin{smallmatrix} \eta_{11}^k & \eta_{12}^k \\ \eta_{21}^k & \eta_{22}^k \end{smallmatrix} \right\}$, where η_{11}^k preserves $(A|B_1)$

and η_{22}^k preserves $(A|D_1)$. Clearly

$$\eta_k = \begin{pmatrix} \eta_{11}^k & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \eta_{12}^k \\ \eta_{21}^k & \eta_{22}^k \end{pmatrix}$$

so a Steinberg relation allows us to replace x_{η_k} by

$$x_{\left\{\begin{smallmatrix} \eta_{11}^k & 0 \\ 0 & 0 \end{smallmatrix}\right\}} \cdot x_{\left\{\begin{smallmatrix} 0 & \eta_{12}^k \\ \eta_{21}^k & \eta_{22}^k \end{smallmatrix}\right\}}$$

The morphism $e_{\left\{\begin{smallmatrix} 0 & \eta_{12}^k \\ \eta_{21}^k & \eta_{22}^k \end{smallmatrix}\right\}}$ will only reach a tiny amount into the B_1-region, and we choose B_1' to be $[1,\infty) \times [-1+\mu, 1-\mu]$ where μ is chosen as small as possible so that $e_{\left\{\begin{smallmatrix} 0 & \eta_{12}^k \\ \eta_{21}^k & \eta_{22}^k \end{smallmatrix}\right\}}$ does not reach into the region B_1'. We now decompose using a Steinberg relation

$$x_{\eta_{k-1}} = x_{\left\{\begin{smallmatrix} 0 & \eta_{12}^{k-1} \\ \eta_{21}^{k-1} & \eta_{22}^{k-1} \end{smallmatrix}\right\}} \cdot x_{\left\{\begin{smallmatrix} \eta_{11}^{k-1} & 0 \\ 0 & 0 \end{smallmatrix}\right\}}$$

based on B_1' and its complement, but since η_{11}^{k-1} preserves B_1' and $\left\{\begin{smallmatrix} 0 & \eta_{12}^k \\ \eta_{21}^k & \eta_{22}^k \end{smallmatrix}\right\}$ does not reach into B_1' we may commute $x_{\left\{\begin{smallmatrix} \eta_{11}^{k-1} & 0 \\ 0 & 0 \end{smallmatrix}\right\}}$ and $x_{\left\{\begin{smallmatrix} 0 & \eta_{11}^k \\ \eta_{21}^k & \eta_{22}^k \end{smallmatrix}\right\}}$ using a Steinberg relation. Continuing this process we get

$$x_{\eta_1} \cdot x_{\eta_2} \cdots x_{\eta_k} =$$

$$x_{\left\{\begin{smallmatrix} 0 & \eta_{12}^1 \\ \eta_{21}^1 & \eta_{22}^1 \end{smallmatrix}\right\}} \cdots x_{\left\{\begin{smallmatrix} 0 & \eta_{12}^k \\ \eta_{21}^k & \eta_{22}^k \end{smallmatrix}\right\}} \cdot x_{\left\{\begin{smallmatrix} \eta_{11}^1 & 0 \\ 0 & 0 \end{smallmatrix}\right\}} \cdots x_{\left\{\begin{smallmatrix} \eta_{11}^k & 0 \\ 0 & 0 \end{smallmatrix}\right\}}$$

in the Steinberg group. The product of the corresponding elementary matrices is of course still the identity, but

$$e_{\left\{\begin{smallmatrix} \eta_{11}^1 & 0 \\ 0 & 0 \end{smallmatrix}\right\}} \cdots e_{\left\{\begin{smallmatrix} \eta_{11}^k & 0 \\ 0 & 0 \end{smallmatrix}\right\}}$$

is an automorphism, say α satisfying the same control as the original deformation, with support in a small band around $[1,\infty) \times \{-1, 1\}$. Such an automorphism may be deformed to the identity using Lemma 3.5 along the ray $[1,\infty)$, and putting these deformations in the middle leads to a description of the given K_2-element as a product of 2 elements, one with support in B_1 and one with support in a small neighborhood of the complement. We loose a factor of 6 in the control as usual when we deform α to the identity. The above process may be done simultaneously on $B_1, \ldots B_4$ in the end writing the given K_2-element as a product of 8 elements each with support in B_i or C_i and hence all 0.

The general case proceeds as in the squeezing theorem generalizing this idea, starting with the top-dimensional simplices, noticing that on the faces not belonging to K nothing is done precisely as in the squeezing theorem.

□

4. Bounded algebra and geometric groups

Definition 4.1. ([4] and [9]) Let K be a finite polyhedron. A *geometric* $\mathbb{Z}\pi$ *-module on* K is an object A in $\mathcal{C}_K(\mathbb{Z}\pi)$. A *deformation* is a composable string $e_1, \ldots, e_n$ of elementary isomorphisms. The *bound* of a deformation is the sum $\sum \mathrm{bd}(e_{\eta_i})$. A *$\delta$-isomorphism* is an isomorphism $\alpha : A \to B$ such that $\mathrm{bd}(\alpha)$ and

$\mathrm{bd}(\alpha^{-1})$ are both less than δ. A δ-isomorphism $\alpha : A \to B$ is *geometric* if α is given by a bijection of basis sets. We will identify α with $\alpha \oplus \mathrm{id} : A \oplus C \to B \oplus C$.

Here is Quinn's Stability Theorem. For clarity, we state the theorem for finite polyhedra. The generalization to locally compact ANR's is not difficult.

Theorem 4.2. ([10, p. 381]) *Suppose that K is a finite polyhedron represented as a cubical subcomplex of the boundary of the unit cube in $\mathbb{R}^n$. Then there is an $\varepsilon_K > 0$ such that for every $\delta < \varepsilon_K$, and automorphism α in the category $\mathcal{C}_K(R)$, bounded by δ, there is a $6^{\dim K+1} \cdot \delta$-deformation to the identity if and only if an invariant $\sigma(\alpha) \in H^1(K; K^{-\infty}(R))$ vanishes. Any element in $H_1(K; K^{-\infty}(R))$ may be realized by such an α with arbitrarily small prescribed control. We may take $\varepsilon_K = \frac{2}{3} \cdot 6^{-\dim K}$.*

Quinn's proof is a torus argument, and extracting explicit bounds would be painful. Working directly with bounded topology avoids the torus and makes the argument remarkably concrete.

Proof. Let K_+ be the disjoint union of K and a basepoint. Given α as above, we will now describe an element $\sigma(\alpha) \in K_2(\mathcal{C}_{O(K_+)}(R))$ for δ sufficiently small. Recall that Theorem 2.3 tells us that $K_2(\mathcal{C}_{O(K_+)}(R) = H_1(K_+; K^{-\infty}(R))$. By Theorem 3.6, $\alpha \oplus \mathrm{id} = \prod e_{\eta_i}$, where the e_{η_i}'s are elementary automorphisms in $\mathcal{C}_{O(K)}(R)$ which are supported outside of $B_1 \cap O(K)$, B_1 being the unit ball in $\mathbb{R}^n$. On the other hand, the usual swindle through the origin and "out the tail" $O(+)$, writes $\alpha \oplus \mathrm{id} = \prod e_{\xi_i}$, where the e_{ξ_i} are supported on $O(+) \cup (O(K) \cap B_1)$. Thus, $(\prod e_{\xi_i})^{-1} \cdot \prod e_{\eta_i} = \mathrm{id}$. We define

$$\sigma(\alpha) = (\prod x_{\xi_i})^{-1} (\prod_{i=1}^{12} x_{\eta_i}) \in K_2(C_{O(K^+)}(R)).$$

We need to see this is well defined, so suppose given two such deformations. The difference will then be supported on $O(K)$ outside K, and the difference is 0 by Theorem 3.10. To see that the vanishing of $\sigma(\alpha)$ ensures a small deformation to the identity, suppose that $\sigma(\alpha)$ represents $0 \in K_2(\mathcal{C}_{O(K_+)}(R))$. This means that using Steinberg relations, $\sigma(\alpha)$ can be written as a product of Steinberg relations. We do not a priori know how many of these relations are needed. But starting with the original representative $\sigma(\alpha)$, we may cut down to a band from 1 to L, L large, to obtain a small deformation from α to some much smaller automorphism α'. At this point we have not used any Steinberg relations, so we have precise control of the size of the deformation. Operating on $\sigma(\alpha)$ by Steinberg relations and cutting at L gives deformations of α', but we know how many Steinberg relations are needed to trivialize $\sigma(\alpha)$, so we may choose L so large that the deformation from α' to a trivial automorphism can be made as small as we wish. This two-stage process is necessary to keep control since there is no known upper bound on the number of Steinberg relations needed to trivialize a given element in K_2, and we only know the elementary automorphisms corresponding to the Steinberg relations are bounded not a priory what the bound is. The key point here is that cutting a Steinberg relation produces an elementary automorphism, and we only have to choose L after we know the reason for the K_2-element being 0.

We thus obtain the deformation to the identity by first using the deformation of α to α', where α' is obtained by cutting at a very large L, and then use the fact that Steinberg relations produce elementary automorphisms to show that α' can be written as a product of elementary automorphisms of controlled size. To see that all obstructions are realized, assume an element $\beta \in K_2(\mathcal{C}_{O(K_+)}(R))$ is given. Cutting at a large L produces an automorphism α as small as we wish hence we get an element of $K_1^\delta(\mathcal{C}_K(R))$ by choosing L sufficiently large. We also get a product of elementary matrices with support outside L whose product is the identity except in a region around L where we get α. We want to show that $\sigma(\alpha) = \beta$. The difference however can be written as a product of an element with support inside L and a sufficiently bounded element with support outside L and these are both 0, so $\sigma([\alpha]) = \beta$. (see the final section for more details of this type of argument). $\square$

5. An algebraic proof of topological invariance of Whitehead torsion

In this section we give a quick proof that a homeomorphism of finite PL-complexes has trivial Whitehead torsion, using the techniques developed in this paper. Let K and L be finite complexes $\pi = \pi_1(K)$, and let $h : L \to K$ be a homeomorphism. The Whitehead torsion is given as follows: approximate h by a PL map f and consider the induced map on universal covers $f_\# : C_\#(\tilde{L}) \to C_\#(\tilde{K})$, a chain homotopy equivalence of based $\mathbb{Z}\pi$-chain complexes. As usual, there are of courses choices in getting such a basis. Let $C_\#$ denote the mapping cylinder of $f_\#$, then $C_\#$ is a contractible chain complex of based $\mathbb{Z}\pi$-modules so $\alpha = s + \partial : \oplus C_{\text{even}} \to \oplus C_{\text{odd}}$ is an isomorphism of based modules. The modules must have the same rank, so we can identify the modules by identifying basis. This way we obtain an automorphism, whose torsion in $K_1(\mathbb{Z}\pi)$ represents the Whitehead torsion of h. Since there are choices involved this is only well defined in $\mathrm{Wh}(\pi) = K_1(\mathbb{Z}\pi)/\pm\pi$, where the $\pm$ should be interpreted as $K_1(\mathbb{Z})$. (We obviously have to divide out by all permutations of the generators, but these permutations generate $K_1(\mathbb{Z})$).

We want to make a couple of inessential reinterpretations of this. The category $\mathcal{C}_K(\mathbb{Z}\pi)$ is equivalent to the category of finitely generated free $\mathbb{Z}\pi$-modules, and associating the barycenter of each simplex to the generator the simplex represents, the chain complexes $C_\#(K)$ and $C_\#(L)$ can be thought of as chain complexes in $\mathcal{C}_K(\mathbb{Z}\pi)$ and we can compute the torsion in K_1 of this category. Refining a bit further consider the universal cover of K, $\widetilde{K}$. Consider the category $\mathcal{C}^\pi_{\widetilde{K}}(\mathbb{Z})$ the subcategory of π-invariant $\mathbb{Z}$-modules parameterized by $\widetilde{K}$. This is actually the most obvious way to think of the chain complexes $C_\#(K)$ and $C_\#(L)$. We are using the chain complexes of the universal cover anyway, and we associate a generator given by a simplex in $\widetilde{K}$ to the barycenter of the simplex.

The method we intend to use is to refine the torsion of a homeomorphism to the point where it lives in a group which algebraically maps to zero in the Whitehead group. First we want to refine the element to a K_2-element in an associated category. Consider $\mathcal{C}_+(\mathcal{C}^\pi_{\widetilde{K}}(\mathbb{Z})^\infty)$. Objects are π-invariant $\mathbb{Z}$-modules parameterized equivariantly by $\widetilde{K} \times \mathbb{Z}_+$, and morphisms are germs at infinity of bounded morphisms invariant under the π-action. This means we identify morphisms if they agree on objects with $\mathbb{Z}_+$-coordinate sufficiently large. Given an automorphism in $\mathcal{C}^\pi_{\widetilde{K}}(\mathbb{Z})$, (A, α), consider the object $(A, A, A, \ldots)$ with the i'th copy of A placed at the i'th coordinate in $\mathbb{Z}_+$. The automorphism

$$(\alpha, 1, 1, \ldots)$$

represents the identity in $\mathcal{C}_+\mathcal{C}^{\pi}_{\widetilde{K}}(\mathbb{Z})^\infty$ because the germ at infinity is the identity. There is an obvious deformation of $(\alpha, 1, 1, 1, \ldots)$ (which represents the identity in the category) to the identity through $(\alpha, \alpha^{-1}, \alpha, \alpha^{-1}, \ldots)$ and further to $(1, 1, 1 \ldots)$, using the six term identity, setting the parenthesis two different ways. This defines an element in $K_2(\mathcal{C}_+\mathcal{C}^{\pi}_{\widetilde{K}}(\mathbb{Z})^\infty)$ which we shall denote by $[\alpha]$.

Lemma 5.1. *There is an epimorphism $K_2(\mathcal{C}_+\mathcal{C}^{\pi}_{\widetilde{K}}(\mathbb{Z})^\infty) \to K_1(\mathcal{C}^{\pi}_{\widetilde{K}}(\mathbb{Z}))$ sending $[\alpha]$ to $[(A, \alpha)]$.*

Proof. The map we define is actually an isomorphism, but we have no need for that in the present argument. An element in K_2 is represented by a product of elementary automorphisms in the category with product the identity. Choosing representatives we get a product of elementary automorphisms with product the identity except on modules with $\mathbb{Z}_+$-coordinate smaller than some number l. Cutting the product of elementary automorphisms at some k larger than l plus the bound r of the deformation produces an automorphism which is the identity on modules with $\mathbb{Z}_+$ coordinate less than $k - r$ and bigger than $k + r$, so summing the modules in the region $k - r$ to $k + r$ produces an object in $\mathcal{C}^{\pi}_{\widetilde{K}}(\mathbb{Z})$ together with an automorphism. It is easy to see that a Steinberg relation gives rise to an elementary automorphism, thus the 0-element in K_1, and that products are sent to products, direct sums to direct sums, so we have a well defined map from $K_2(\mathcal{C}_+\mathcal{C}^{\pi}_{\widetilde{K}}(\mathbb{Z})^\infty) \to K_1(\mathcal{C}^{\pi}_{\widetilde{K}}(\mathbb{Z}))$. Inspection shows that $[\alpha]$ is sent to $[(A, \alpha)]$. More precisely consider the diagram

$$\begin{tikzcd}
A \arrow[d, "1"'] \arrow[dr, "\alpha"] & A \arrow[d, "1"] & & A \arrow[d, "1"'] \arrow[dr, "\alpha"] & A \arrow[d, "1"] & & A \arrow[d, "1"'] \arrow[dr, "\alpha"] & A \arrow[d, "1"] \\
A \arrow[d, "1"'] & A \arrow[d, "1"] \arrow[dl, "-\alpha^{-1}"'] & & A \arrow[d, "1"'] & A \arrow[d, "1"] \arrow[dl, "-\alpha^{-1}"'] & & A \arrow[d, "1"'] & A \arrow[d, "1"] \arrow[dl, "-\alpha^{-1}"'] \\
A \arrow[d, "1"'] \arrow[dr, "\alpha"] & A \arrow[d, "1"] & & A \arrow[d, "1"'] \arrow[dr, "\alpha"] & A \arrow[d, "1"] & & A \arrow[d, "1"'] \arrow[dr, "\alpha"] & A \arrow[d, "1"] \\
A \arrow[d, "1"'] & A \arrow[d, "1"] \arrow[dl, "1"'] & & A \arrow[d, "1"'] & A \arrow[d, "1"] \arrow[dl, "1"'] & & A \arrow[d, "1"'] & A \arrow[d, "1"] \arrow[dl, "1"'] \\
A \arrow[d, "1"'] \arrow[dr, "-1"] & A \arrow[d, "1"] & & A \arrow[d, "1"'] \arrow[dr, "-1"] & A \arrow[d, "1"] & & A \arrow[d, "1"'] \arrow[dr, "-1"] & A \arrow[d, "1"] \\
A \arrow[d, "1"'] & A \arrow[d, "1"] \arrow[dl, "1"'] & & A \arrow[d, "1"'] & A \arrow[d, "1"] \arrow[dl, "1"'] & & A \arrow[d, "1"'] & A \arrow[d, "1"] \arrow[dl, "1"'] \\
A & A & & A & A & & A & A
\end{tikzcd}$$

This is the deformation of the identity to $(\alpha, \alpha^{-1}, \ldots)$ (the matrix multiplication is computed by adding all the ways one can get from one point in the diagram to another). Cutting at a certain level consists of replacing all diagonal arrows by 0 to the left of a certain point. Consider the identity as the composite

$$\begin{array}{cccccc}
A & A & A & A & A & A \\
{\scriptstyle\alpha}\big\downarrow & {\scriptstyle\alpha^{-1}}\big\downarrow & {\scriptstyle\alpha}\big\downarrow & {\scriptstyle\alpha^{-1}}\big\downarrow & {\scriptstyle\alpha}\big\downarrow & {\scriptstyle\alpha^{-1}}\big\downarrow \\
A & A & A & A & A & A \\
{\scriptstyle\alpha^{-1}}\big\downarrow & {\scriptstyle\alpha}\big\downarrow & {\scriptstyle\alpha^{-1}}\big\downarrow & {\scriptstyle\alpha}\big\downarrow & {\scriptstyle\alpha^{-1}}\big\downarrow & {\scriptstyle\alpha}\big\downarrow \\
A & A & A & A & A & A
\end{array}$$

and replace the upper arrows by the diagram above, and the lower arrows by the diagram above shifted. We see that cutting in the middle we get

$$\begin{array}{cccccc}
A & A & A & A & A & A \\
{\scriptstyle 1}\big\downarrow & {\scriptstyle 1}\big\downarrow & {\scriptstyle 1}\big\downarrow & {\scriptstyle 1}\big\downarrow & {\scriptstyle\alpha}\big\downarrow & {\scriptstyle\alpha^{-1}}\big\downarrow \\
A & A & A & A & A & A \\
{\scriptstyle 1}\big\downarrow & {\scriptstyle 1}\big\downarrow & {\scriptstyle 1}\big\downarrow & {\scriptstyle\alpha}\big\downarrow & {\scriptstyle\alpha^{-1}}\big\downarrow & {\scriptstyle\alpha}\big\downarrow \\
A & A & A & A & A & A
\end{array}$$

which is the identity except on the fourth module where it is α as claimed. □

Since we started with a homeomorphism h and we can triangulate as finely as we need, and PL-approximate as closely as needed, we have

Lemma 5.2. *We can find representatives of the Whitehead torsion of h, (A_n, α_n) satisfying the following conditions:*

(i) *α_n and α_n^{-1} are bounded by $\frac{1}{n}$.*

(ii) *There is a deformation of $(A_n \oplus A_{n+1}, \alpha_n \oplus \alpha_{n+1}^{-1})$, and hence also of $(A_n \oplus A_{n+1}, \alpha_n^{-1} \oplus \alpha_{n+1})$ bounded by $\frac{1}{2n}$.*

Proof. By triangulating and PL-approximating finely we may obtain an isomorphism $\beta_n : A_n \to B_n$ smaller than any given r, so choose $r < \frac{1}{n}$. Here A_n is the sum of the odd dimensional cellular chains, and B_n the sum of the even-dimensional cellular chains in the mapping cylinder of the chain complex of the cellular chains K and L. Let $\delta = \frac{1}{n} - r$. We may construct an isomorphism γ_n from B_n to A_n with bound smaller than δ, sending generators to generators at least after stabilizing, but stabilizing is no problem, we just replace A_n and B_n by the stabilized modules. To see this consider a generator g_b in B_n and a generator g_a in A_n. Assume g_a is at the point $x \in \widetilde{K}$ and g_b at the point $y \in \widetilde{K}$. We may now choose a finite sequence of points in $\widetilde{K}$, $x = x_0, x_1, \dots, x_m = y$ such that the distance from x_{n-1} to x_n is less than δ. Stabilize A_n and B_n by generators g_i at x_i , $i = 1, \dots, m-1$ and the isomorphism β_n by the identity, and define $\gamma_n(g_i) = g_{i-1}$. At this point we only need to define g_n on a free module with one less generator than before, and the proof is completed by induction. We may now define $\alpha_n = \gamma_n \beta_n$, an automorphism that clearly represent the torsion of h. To see the second part, notice that if we have two fine PL-approximations they are homotopic by a small homotopy. We may change the homotopy to be PL relative to endpoints. Denote the PL homotopy by H. We may now use $C_\#(H)$ to compute the torsion, but the torsion of $C_\#(H)$ is connected to α_n and α_{n+1} by a sequence of collapses. It is however easy to see that the inverse of a collapse algebraically consists of stabilizing and multiplying by an elementary matrix. Since the collapses can be made as small as we wish we are done. □

This means we can find another K_2 element in $\mathcal{C}_+\mathcal{C}^{\pi}_{\widetilde{K}}(\mathbb{Z})^{\infty}$ representing the torsion of h namely given by the deformation first from $(\alpha_1, 1, 1, \ldots)$ to $(\alpha_1, \alpha_2^{-1}, \alpha_3, \ldots)$, and then to $(1, 1, 1, \ldots)$. Once again we see that if we cut this deformation at some large k we get (A_n, α_n) hence a representative for α. The advantage of this representation is that it remains a bounded deformation if we replace the metric on the n'th copy of K by a metric n times larger, and require the morphisms to be bounded with respect to this metric.

Thinking of K as embedded in $\{1\} \times \mathbb{R}^m$ for some big m, and $O(K)$ as the set of rays in $\mathbb{R}^{m+1}$ from 0 going through K. The category $\mathcal{C}_{O(K)}(\mathbb{Z})$ is equivalent to the full subcategory where the objects are 0 except when the first coordinate is an integer, and the level of $O(K)$ with first coordinate equal to n is a copy of K with a metric which is precisely n times the metric in K. It is now easy to see that the subcategory of $\mathcal{C}_+\mathcal{C}^{\pi}_{\widetilde{K}}(\mathbb{Z})^{\infty}$ with morphisms bounded in the metric where the n'th copy of K has the metric multiplied by n, is isomorphic to the category $\mathcal{C}_{O(K)}(\mathbb{Z})^{\infty}$, where the upper index ∞ indicates that we take germs at infinity. The point is that we may divide out the π-action, and since morphisms near infinity are allowed to move very little there is a canonical choice of morphisms near infinity. We have thus shown that the torsion of the homeomorphism h lies in the image of a map from $K_2(\mathcal{C}_{O(K)}(\mathbb{Z})^{\infty})$ to $K_1(\mathbb{Z}\pi)$, but according to [8] we have

$$K_2(\mathcal{C}_{O(K)}(\mathbb{Z})^{\infty}) = H_1(K_+; \mathrm{Alg}K(\mathbb{Z}))$$

which by the Atiyah-Hirzebruch spectral sequence is

$$H_1(K, K_0(\mathbb{Z})) \oplus H_0(K; K_1(\mathbb{Z})) = H_1(K, \mathbb{Z}) \oplus H_0(K; \mathbb{Z}_2) = \pi/[\pi, \pi] \oplus \pm 1.$$

But this is precisely what we divide out from $K_1(\mathbb{Z}\pi)$ to get the Whitehead group, so the image is trivial in $\mathrm{Wh}(\pi)$ as claimed.

Remark 5.3. We could obviously give a different proof by representing the torsion by just *one* sufficiently small automorphism, and then use the squeezing results of the previous sections to produce the element in K_2. Since a homeomorphism is squeezed as much as possible it seems more natural to utilize that directly.

6. Pedestrian Algebraic K-theory

The proof of topological invariance of Whitehead torsion in the previous section does depend on the computation in [8] of $K_2(\mathcal{C}_{O(K_+)}(R))$, and thus of the whole machinery of higher algebraic K-theory. This however is not necessary, it is perfectly possible to compute this group by elementary (pedestrian) methods which we shall proceed to indicate in this section.

Suppose M is a metric space, M_1 and M_2 metric subspaces. We shall consider the category $\mathcal{C}_M(R)$ and various subcategories. Abusing notation we denote the full subcategories on objects with support in a bounded neighborhood of M_i by $\mathcal{C}_{M_i}(R)$. This is not too bad because these subcategories are obviously equivalent to $\mathcal{C}_{M_i}(R)$. We shall denote the full subcategory on objects with support in a bounded neighborhood of M_1 intersected with a bounded neighborhood of M_2 by $\mathcal{C}_{M_1,M_2}(R)$. This could be quite different from $\mathcal{C}_{M_1\cap M_2}(R)$, in particular $M_1 \cap M_2$ could be empty, but it does represent the intersection in the "bounded" sense, and in many favorable cases it is true that this category is equivalent to $\mathcal{C}_{M_1\cap M_2}(R)$ in particular this will be the case if we take finite complexes $K = K_1 \cup K_2$ and put $M_i = O(K_i)$ which is the case we have on mind.

Theorem 6.1. *With assumptions as above, denoting the maps induced by inclusions by i_i and j_i, there is an exact sequence*

$$K_2(\mathcal{C}_{M_1,M_2}(R)) \xrightarrow{(i_1,i_2)} K_2(\mathcal{C}_{M_1}(R)) \oplus K_2(\mathcal{C}_{M_2}(R)) \xrightarrow{j_1-j_2} K_2(\mathcal{C}_M(R)) \xrightarrow{\partial}$$
$$K_1(\mathcal{C}_{M_1,M_2}(R)) \xrightarrow{(i_1,i_2)} K_1(\mathcal{C}_{M_1}(R)) \oplus K_1(\mathcal{C}_{M_2}(R)) \xrightarrow{j_1-j_1} K_1(\mathcal{C}_M(R)) \xrightarrow{\partial}$$
$$K_0(\mathcal{C}_{M_1,M_2}(R)) \xrightarrow{(i_1,i_2)} K_0(\mathcal{C}_{M_1}(R)) \oplus K_0(\mathcal{C}_{M_2}(R)) \xrightarrow{j_1-j_2} K_0(\mathcal{C}_M(R)) \xrightarrow{\partial}$$
$$K_{-1}(\mathcal{C}_{M_1,M_2}(R)) \xrightarrow{(i_1,i_2)} K_{-1}(\mathcal{C}_{M_1}(R)) \oplus K_{-1}(\mathcal{C}_{M_2}(R)) \xrightarrow{j_1-j_2} K_{-1}(\mathcal{C}_M(R)) \xrightarrow{\partial} \dots$$

where K_0 should be interpreted as K_0 of the idempotent completion, or K_1 of the category where M and M_i have been crossed with $\mathbb{R}$, K_{-1} similarly as K_1 after crossing with $\mathbb{R}^2$ etc.

Proof. This follows immediately from [8], see also [2] for an easier proof, but our aim here is to avoid higher Algebraic K-theory. The lower part of the sequence is proved by a slight generalization of the methods in [6] so we shall concentrate on the higher part. The boundary map is defined as follows: Given an element in $K_2(\mathcal{C}_M(R))$ i. e. $\prod_{i=1}^k e_{\eta_i}$ each bounded by some r. We cut the elementary matrices in a bounded neighborhood of M_1 the following way: We write $\eta_i = \mu_i + \nu_i$ where μ_i has support in a $(k-i+1)\cdot r$ neighborhood of M_1 and ν_i has support outside a $(k-i)\cdot r$ neighborhood of M_1. This means that, using Steinberg relations e_{ν_i} commutes with e_{μ_j} when $i<j$, so using Steinberg relations we get

$$e_{\eta_1}\cdot \dots \cdot e_{\eta_k} = e_{\mu_1}\cdot e_{\nu_1}\cdot \dots \cdot e_{\mu_k}\cdot e_{\nu_k} = e_{\mu_1}\cdot \dots \cdot e_{\mu_k}\cdot e_{\nu_1}\cdot \dots \cdot e_{\mu_k}$$

The product $\prod e_{\mu_i}$ will no longer be the identity, but it will be an automorphisms which is the identity except for a bounded neighborhood of M_1 intersected with a bounded neighborhood of M_2 thus defining an element in $K_1(\mathcal{C}_{M_1,M_2}(R))$. It is easy to see this is well defined. If we cut differently, the difference will be given by an elementary automorphism with support in a bounded neighborhood of $M_1 \cap M_2$. If the element is 0 in $K_1(\mathcal{C}_{M_1,M_2}(R))$ we may write it as a product of elementary matrices $\prod e_{\lambda_j}$, and we see that we may write the original element in K_2

$$\prod e_{\eta_i} = \prod e_{\mu_i}\cdot \prod e_{\nu_i} = \prod e_{\mu_i}\cdot (\prod e_{\lambda_j})^{-1}\cdot \prod e_{\lambda_j}\cdot \prod e_{\nu_i}$$

thus as the difference of an element $\prod e_{\mu_i}\prod (e_{\lambda_j})^{-1}$ coming from $K_2(\mathcal{C}_{M_1}(R))$ and an element $(\prod e_{\lambda_j}\cdot \prod e_{\nu_i})^{-1}$ coming from $K_2(\mathcal{C}_{M_2}(R))$. While it is tedious to work out all the details the rest of the proof is elementary. □

Writing $\mathbb{R}^{k+2}$ as a union of two halfspaces along $\mathbb{R}^{k+1}$ and using an Eilenberg swindle on the two half spaces it follows from this that $K_2(\mathcal{C}_{\mathbb{R}^{k+2}}(R)) \cong K_1(\mathcal{C}_{\mathbb{R}^{k+1}}(R))$ which in turn is isomorphic to $K_{-k}(R)$ by [6]. As remarked above when K is the union of two PL-subcomplexes K_1 and K_2 we have $\mathcal{C}_{O(K_1),O(K_2)}(R) \simeq \mathcal{C}_{O(K_1\cap K_2)}(R)$ and this thus provides sufficient information to compute $K_2(\mathcal{C}_{O(K)}(R))$ by the usual Mayer-Vietoris arguments, attaching one cell at a time.

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