ANNALS OF MATHEMATICS
Vol. 75, No. 3, May, 1962
Printed in Japan

COHOMOLOGY THEORIES*

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(Received December 20, 1960)
(Revised August 14, 1961)

Introduction

Suppose that C is a category of topological spaces with base point and continuous maps preserving base points, $\mathcal{S}$ is the category of sets with a distinguished element and set maps preserving distinguished elements, and $H: C \rightarrow \mathcal{S}$ is a contravariant functor. The main result of this paper is that, if H satisfies certain axioms, there is a space Y, unique up to homotopy type, such that H is naturally equivalent to the functor which assigns to each $X \in C$ the set of homotopy classes of maps of X into Y. This result is stated in § 1 and proved in §§ 1, 2, and 3.

The main application of this result is a representation theorem for cohomology theories which satisfy all the Eilenberg-Steenrod axioms except the dimension axiom. Suppose that for each integer q, H^q is a contravariant functor from the category of pairs of finite CW-complexes to the category of abelian groups and $\delta^q: H^q(A) \rightarrow H^{q+1}(X, A)$ are a collection of natural transformations. Furthermore, suppose H^q and δ^q satisfy all the Eilenberg-Steenrod axioms except the dimension axiom which is replaced by the condition that H^q on a point be countable. In § 4 we show that there is an Ω-spectrum Y[4], i.e., a sequence of spaces Y_q and homotopy equivalences $h_q: Y_q \rightarrow \Omega Y_{q+1}$, such that $H^q(X)$ is naturally equivalent to the group of homotopy classes of maps of X into Y_q. The hypothesis that H^q of a point is countable is admittedly unfortunate, but the author seems unable to remove it without making some more drastic assumptions about H (see § 4, Theorem II.)

In § 5 we apply our main result to prove the existence of a universal bundle for a topological group, and also to characterize singular cohomology theory on the category of all CW-complexes.

Finally, in an appendix, we briefly indicate how this theory may be dualized so as to give similar representation theories for covariant functors.

The author acknowledges with pleasure his indebtedness to Arnold Shapiro for many helpful suggestions in the development of this theory.

Part of this research was supported by a National Science Foundation Research Grant No. G-8207.

1. Statement of the main theorem

A category C will be called a *category of spaces* if its objects are pathwise connected topological spaces with base point which admit the structure of a CW-complex, and its maps are all continuous maps of X into Y carrying the base point of X into the base point of Y for each pair $X, Y \in C$. Furthermore it will be assumed that, if $X \in C$ and X' is a subcomplex of X with respect to some CW-complex structure on X, then $X' \in C$. C_1 and C_0 will denote the category of spaces which have as objects all spaces admitting a CW-complex structure and all spaces admitting the structure of a finite CW-complex, respectively. S will denote the category of sets with a distinguished element and set maps preserving distinguished elements.

$(X_1 \cup X_2, X_1, X_2)$ will be called a *proper triad* of C if X_1, X_2, $X_1 \cup X_2$ and $X_1 \cap X_2$ are in C, all have the same base point, and $(X_1, X_1 \cap X_2)$ and $(X_2, X_1 \cap X_2)$ each have the homotopy extension property.

If X and $Y \in C$, $[X, Y]$ will denote the set of homotopy classes of maps of X into Y with respect to homotopies which leave the base point of X fixed. $[\ \ , Y]$ will denote the functor from C to S which assigns to each $X \in C$ the set $[X, Y]$ with the class of the constant map as distinguished element, and assigns to each map $f\colon X \to X'$ the map $f^*\colon [X', Y] \to [X, Y]$ defined as follows:

$$f^*[g] = [gf]$$

where $[g]$ denotes the homotopy class of g. The notation $[\ \ , Y]$ is slightly ambiguous in that it does not indicate the domain category C, but this category will be clear from the context.

Let $H\colon C \to S$ be a contravariant functor. Below we give a set of axioms of which one combination will be used in dealing with H when $C = C_0$, and another combination will be used when $C = C_1$.

Axiom h. If $f, g\colon X \to Y$ are homotopic, $H(f) = H(g)$.

Axiom e. (e1) If p is a point, $H(p)$ contains only one element.

(e2) Suppose (X, X_1, X_2) is a proper triad, $A = X_1 \cap X_2$ and $j_i\colon A \to X_i$ and $k_i\colon X_i \to X$ are the inclusion maps. If $u_1 \in H(X_1)$ and $u_2 \in H(X_2)$ such that $H(j_1)u_1 = H(j_2)u_2$, then there is a $v \in H(X)$ such that $H(k_1)v = u_1$ and $H(k_2)v = u_2$. Furthermore, if A is a point, v is unique.

Axiom c. If S^n is the n-sphere, $H(S^n)$ is countable for all $n > 0$.

Axiom w. Suppose S^n_α is a collection of n-spheres whose wedge product $\bigvee S^n_\alpha$ is in C. Let $i_\beta\colon S^n_\beta \to \bigvee S^n_\alpha$ be the inclusion map.

$$\prod H(i_\alpha)\colon\ H(\bigvee S^n_\alpha) \to \prod H(S^n_\alpha)$$

is an isomorphism.

Axiom l. Suppose $X_1X_2\cdots X_n\cdots$ are a collection of subcomplexes of $X=\bigcup X_n\in\mathcal{C}$ with respect to some CW-complex structure on X such that $X_n^n=X^n$. Let i_n: $X_n\to X$ be the inclusion map. Let inv lim $H(X_n)$ be the inverse limit of $H(X_n)$ with respect to the maps induced by the inclusions of X_n into X_m.

$$\text{inv lim } H(i_n)\colon H(X)\to \text{inv lim } H(X_n)$$

is an epimorphism.

Note that if $\mathcal{C}=\mathcal{C}_0$, axiom l automatically holds. Also axiom e implies that $H(X\vee Y)\approx H(X)\times H(Y)$, i.e., when A in axiom e is a point. Thus,

1.1. LEMMA. *If $\mathcal{C}=\mathcal{C}_0$ and H satisfies axiom* c, *H satisfies axioms* w *and* l.

One may easily check that $[\ ,Y]$ satisfies all of these axioms except c.

THEOREM I. *If H: $\mathcal{C}\to\mathcal{S}$ is a contravariant functor, $\mathcal{C}=\mathcal{C}_0$, and H satisfies axioms* h, e *and* c; *or $\mathcal{C}=\mathcal{C}_1$, and H satisfies* h, e, w *and* l, *then there is a space $Y\in\mathcal{C}_1$, unique up to homotopy type, such that $[\ ,Y]$ on $\mathcal{C}$ and H are naturally equivalent.*

In the remainder of this section we outline the proof of Theorem I by reducing it to a sequence of propositions which are then proved in §§ 2 and 3.

Suppose H: $\mathcal{C}\to\mathcal{S}$ is a contravariant functor satisfying axiom h. Given $Y\in\mathcal{C}$ and $u\in H(Y)$, we construct a natural transformation $T(u)$: $[\ ,Y]\to H$ as follows: If $[g]\in[X,Y]$, let $T(u)[g]=H(g)u$. Since H satisfies h, $T(u)[g]$ is independent of the choice of g. If f: $X'\to X$,

$$\begin{aligned} H(f)T(u)[g] &= H(f)H(g)u \\ &= H(fg)u \\ &= T(u)f^*[g]\,. \end{aligned}$$

Hence $T(u)$ is natural. When $\mathcal{C}=\mathcal{C}_1$, we use this construction to give an equivalence between $[\ ,Y]$ and H for an appropriately chosen Y; but when $\mathcal{C}=\mathcal{C}_0$, this construction will not suffice because in general the appropriate Y will not lie in $\mathcal{C}_0$. To circumvent this difficulty we extend H to a larger category. The following is an auxiliary axiom which describes the properties this extension must have in order that Y be constructed.

Let λ be an infinite cardinal number greater than or equal to the cardinality of $H(S^n)$ for all $n>0$. If f: $A\to X$, let Z_f denote the identification space formed by attaching the cone over A to X by the map f (see § 2.)

Axiom e′. $\mathcal{C}$ contains all spaces which admit a CW-complex structure with λ or less cells. If $A=\bigvee S^n$, X and Z_f, where f: $A\to X$, are in $\mathcal{C}$; then the following sequence is exact:

$$H(Z_f) \xrightarrow{H(i)} H(X) \xrightarrow{H(f)} H(A) ,$$

where i is the inclusion map.

If $\mathcal{C} = \mathcal{C}_1$, $A \in \mathcal{C}$. In § 2 we show that the exactness of the above sequence follows from axiom e. Thus,

1.2 LEMMA. *If* $H: \mathcal{C}_1 \to \mathcal{S}$ *satisfies* e, *it satisfies* e′.

Let $\mathcal{C}_\omega$ be the category of spaces which admit a countable CW-complex structure. In § 3 we prove:

1.3 LEMMA. *If* $H: \mathcal{C}_0 \to \mathcal{S}$ *satisfies* h, e *and* c, *then* H *can be extended to a functor* $\bar{H}: \mathcal{C}_\omega \to \mathcal{S}$ *satisfying* h, w, l *and* e′.

It is exactly in the proof of 1.3 that we need axiom c. Possibly this lemma can be generalized to larger cardinals by some transfinite argument, but the author has not succeeded in doing this.

Combining 1.1, 1.2 and 1.3 we see that if H satisfies the hypothesis of Theorem I it can be extended so as to satisfy the hypothesis of the following lemma.

1.4 LEMMA. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be categories of spaces such that* $\mathcal{C}$ *is a subcategory of* $\mathcal{C}'$ *and let* $H: \mathcal{C}' \to \mathcal{S}$ *be a contravariant functor such that*:

(i) H *satisfies* h, e′, w *and* l.

(ii) $H|\mathcal{C}$ *satisfies* h, e, w *and* l.

Then there is a $Y \in \mathcal{C}'$ *such that* $[\ \ , Y]$ *and* $H | \mathcal{C}$ *are equivalent.*

We now reduce 1.4 to some further lemmas.

1.5 LEMMA. *If* H_1 *and* H_2: $\mathcal{C} \to \mathcal{S}$ *both satisfy* h, e *and* w, *and* $T: H_1 \to H_2$ *is a natural transformation such that* $T: H_1(S^n) \approx H_2(S^n)$ *for all* $n > 0$, *then* $T: H_1(X) \approx H_2(X)$ *for all* $X \in \mathcal{C}$ *such that* $\dim X < \infty$.

Lemma 1.5 is proved in § 2.

To eliminate the finite dimensional restriction when $\mathcal{C} = \mathcal{C}_1$, we show in § 2:

1.6 LEMMA. *Suppose* $H: \mathcal{C}_1 \to \mathcal{S}$ *satisfies* h, e, w *and* l, *and* $T: [\ \ , Y] \to H$ *is a natural transformation such that* $T: [X, Y] \approx H(X)$ *for all* $X \in \mathcal{C}$ *such that* $\dim X < \infty$. *Then* $T: [X, Y] \approx H(X)$ *for all* $X \in \mathcal{C}_1$.

The proof of 1.5 is motivated by the usual five lemma argument, namely: Suppose $X \in \mathcal{C}$ and $\dim X = n + 1$. Then $X = Z_f$ where $f: \bigvee S^n_\alpha \to X'$ and $\dim X' \leqq n$. Axiom e gives an exact sequence:

$$H(SX') \to H(SA) \to H(Z_f) \to H(X') \to H(A)$$

where S is the suspension. Thus if H took its values in the category of abelian groups, induction and the five lemma would give the desired result. This is essentially the proof we use except that a more delicate five lemma, which is applicable when some of the objects are not groups, must be used. One might suppose that the proof of Theorem I could be simplified by assuming H took its values in the category of abelian groups. Surprisingly, this hypothesis can not be used because one is unable to prove that the space Y, constructed from H, is an H-space until one knows that $[\ ,Y]$ and H are equivalent. Hence, one cannot prove that $[\ ,Y]$ is a group before applying 1.5. A simplifying assumption which does work is to suppose that $H(X) = H'(SX)$ for some functor H' satisfying the axioms.

Suppose $H: \mathcal{C} \to \mathcal{S}$ satisfies the hypotheses of 1.4. We next show how to construct a space Y and a $u \in H(Y)$ such that $T(u): [S^n, Y] \approx H(S^n)$ for all $n > 0$. We construct spaces $Y_n \in \mathcal{C}'$ and $u_n \in H(Y_n)$ by induction on n such that:

(i) $Y_{n-1} \subset Y_n$.

(ii) $H(i_n)u_n = u_{n-1}$ where $i_n: Y_{n-1} \to Y_n$ is the inclusion map.

(iii) $T(u_n): [S^m, Y_n] \to H(S^m)$ is onto for all $m > 0$ and $1-1$ for $m \leqq n$.

(iv) The cardinality of the number of cells in $Y_n \leqq \lambda$ (see e′ for a definition of λ.)

Choose generators g^n_α for $H(S^n)$. Let S^n_α be a copy of S^n for each g^n_α, $Y_0 = \bigvee S^n_\alpha$ over all α and $n > 0$, and let $h^n_\alpha: S^n \to Y_0$ be a homeomorphism of S^n onto S^n_α followed by the inclusion map. By axiom w, $H(Y_0) \approx \prod H(S^n_\alpha)$. Hence there is an element $u_0 \in H(Y_0)$ such that $H(h^n_\alpha)u_0 = g^n_\alpha$. Therefore $T(u_0)[h^n_\alpha] = g^n_\alpha$.

Suppose Y_{n-1} and u_{n-1} have been defined and satisfy (i)–(iv). Let $[f_\beta]$ generate the kernel of $T(u_{n-1}): [S^n, Y_{n-1}] \to H(S^n)$. Let S^n_β be a copy of S^n for each β, $A = \bigvee S^n_\beta$ and $f = \bigvee f_\beta$. One can easily convince oneself that cardinality of $\{f_\beta\} < \lambda$ so that by e′, A and $Z_f \in \mathcal{C}'$. Again by axiom w, $H(A) \approx \prod H(S^n_\beta)$. Let $k_\beta: S^n_f \to A$ be the inclusion map.

$$\begin{aligned} H(k_\beta)H(f)u_{n-1} &= H(f_\beta)u_{n-1} \\ &= T(u_{n-1})[f_\beta] \\ &= 0 \end{aligned}$$

Therefore $H(f)u_{n-1} = 0$. Let $Y_n = Z_f$. Axiom e′ yields $u_n \in H(Y_n)$ such that $H(i_n)u_n = u_{n-1}$. Consider the commutative diagram:

$$\begin{tikzcd} {[S^m, Y_{n-1}]} \arrow[dd, "i_{n*}"'] \arrow[dr, "T(u_{n-1})"] & \\ & H(S^m) \\ {[S^m, Y_n]} \arrow[ur, "T(u_n)"'] & \end{tikzcd}$$

$T(u_{n-1})$ is onto for all m, hence $T(u_n)$ is. Y_n was formed from Y_{n-1} by adding $n+1$ cells. Therefore i_{n*} is onto for $m < n$. $T(u_{n-1})$ is $1-1$ for $m < n$, hence $T(u_n)$ is. For $m = n$, the kernel of $T(u_{n-1})$ is contained in the kernel of i_{n*} and hence $T(u_n)$ is $1-1$.

Let $Y = \bigcup Y_n$. By axiom 1 there is a $u \in H(Y)$ which restricts to u_n for each n. Hence $T(u)$: $[S^n, Y] \approx H(S^n)$ for all $n > 0$. We have therefore shown, modulo the proofs of 1.3, 1.5, and 1.6, that if H satisfies the hypotheses of Theorem I, there is a $Y \in C$, such that $[\ , Y]$ and H are equivalent. It thus remains to show that the homotopy type of Y is unique. When $C = C_1$ this follows from the obvious fact that, if $T: [X, Y] \to [X, Y']$ is a natural transformation defined for all $X \in C_1$, then there is a map $f: Y \to Y'$, unique up to homotopy, such that $T = f_*$. Namely, $f \in T(\mathrm{id})$ where id: $Y \to Y$ is the identity map.

In § 3 we prove:

1.7 LEMMA. *If Y and $Y' \in C_\omega$, and $T: [X, Y] \to [X, Y']$ is a natural transformation defined for all $X \in C_0$, then there is a map f: $Y \to Y'$ such that $T = f_*$.*

In general f will not be unique up to homotopy type. Nevertheless if $T = f_*$ is an equivalence, f_*: $[S^n, Y] \approx [S^n, Y']$ for all $n > 0$. Therefore by J.H.C.Whitehead's theorem, f is a homotopy equivalence.

2. Proof of Lemma 1.5

We begin by developing the machinery to be used in place of exact sequences and the five lemma. Let A, X_0, X_1, and f_i: $A \to X_i$ be in C. Z_{f_0, f_1} will denote the identification space formed from $X_0 \cup X_1 \cup (A \times I)$ by the following identifications. $*$ denotes the base point.

$$(a, i) \sim f_i(a) \qquad\qquad i = 0, 1, a \in A$$
$$(*, t) \sim * \qquad\qquad t \in I$$

(We will always assume A, X_0, X_1 have been made disjoint.)

Let c_i: $* \to X_i$ and c: $A \to *$. The following are notational conventions:

$$X_0 \vee X_1 = Z_{c_0, c_1}\,,$$
$$Z_f = Z_{c, f}\,,$$
$$SA = Z_{c, c}\,,$$
$$S^n = SS^{n-1}\,, \qquad S^0 = \text{two points.}$$

We assume throughout this section that H is a contravariant functor satisfying h and e. Let f_i: $A \to X_i$, $i = 0, 1$, and let j_i: $X_i \to Z_{f_0, f_1}$ be the inclusion maps.

2.1 LEMMA. *If $u_i \in H(X_i)$ are such that $H(f_0)u_0 = H(f_1)u_1$, then there is a $v \in H(Z_{f_0, f_1})$ such that $H(j_0)v = u_0$ and $H(j_1)v = u_1$. Furthermore if A is a point, v is unique.*

PROOF. Let $\bar{X}_0$, $\bar{X}_1$ and $\bar{A}$ be the image in Z_{f_0,f_1} of $X_0 \cup (A \times [0, 1/2])$, $X_1 \cup (A \times [1/2, 1])$ and $A \times 1/2$. Then $(Z_{f_0,f_1}, \bar{X}_0, \bar{X}_1)$ is a proper triad, $\bar{A} = \bar{X}_0 \cap \bar{X}_1$ and $\bar{X}_0$ and $\bar{X}_1$ are deformation retracts of X_0 and X_1. Lemma 2.1 then follows from e2 applied to $(Z_{f_0,f_1}, \bar{X}_0, \bar{X}_1)$, and h applied to the various maps involved.

Let j_i: $X_i \to X_0 \vee X_1$ be the inclusion maps.

2.2 LEMMA. $H(j_0) \times H(j_1)$: $H(X_0 \vee X_1) \approx H(X_0) \times H(X_1)$.

PROOF. By e1, $H(c_0)u_0 = H(c_1)u_1$ for any pair of elements $u_i \in H(X_i)$. Lemma 2.2 then follows from 2.1 when A is a point.

Let t: $SX \to SX \vee SX$ be the map which pinches $X \times 1/2$ to a point. By 2.2, t defines a map t^*: $H(SX) \times H(SX) \to H(SX)$. By standard arguments we have:

2.3 LEMMA. *t^* defines a natural group structure on $H(SX)$ and $H(S(SX))$ is abelian.*

Let r: $Z_f \to SA \vee Z_f$ be the map which pinches $A \times 1/2$ to a point. Applying 2.2 we obtain a map r^*: $H(SA) \times H(Z_f) \to H(Z_f)$.

Again by standard arguments:

2.4 LEMMA. *r^* defines an action of the group $H(SA)$ on the set $H(Z_f)$.*

Let i: $X \to Z_f$ be the inclusion map.

2.5 LEMMA. *$H(i)$ induces an isomorphism of the orbits of $H(Z_f)$ onto the kernel of $H(f)$.*

Note that if we were dealing with groups instead of sets, 2.5 would simply say that the following sequence is exact.

$$H(SA) \to H(Z_f) \to H(X) \to H(A) .$$

PROOF. By e1 and 2.1, image $H(i) =$ kernel $H(f)$. Let j_1 and j_2 be the inclusion maps of SA and Z_f into $SA \vee Z_f$. Then $ri = j_2 i$. Let $w \in H(SA)$ and $u \in H(Z_f)$.

$$\begin{aligned} H(i)(wu) &= H(i)H(r)(H(j_1) \times H(j_2))^{-1}(w, u) \\ &= H(i)H(j_2)\big(H(j_1) \times H(j_2)\big)^{-1}(w, u) \\ &= H(i)u . \end{aligned}$$

Hence $H(i)$ induces a map from the orbits of $H(Z_f)$ onto the kernel of $H(f)$.

Suppose u_0 and $u_1 \in H(Z_f)$ and $H(i)u_0 = H(i)u_1$. Let k_0, k_1: $Z_f \to Z_{i,i}$ be the two inclusion maps. By 2.1, there is an element $w^1 \in H(Z_{i,i})$ such that $u_1 = H(k_i)w^1$. Let h: $SA \to Z_{i,i}$ be defined as follows:

$$\begin{aligned} h(a, t) &= k_0(a, 3t) & 0 < t < 1/3, a \in A\ , \\ &= (f(a), 3t - 1) & 1/3 < t < 2/3\ , \\ &= k_1(a, 1 - (3t - 2)) & 2/3 < t < 1\ . \end{aligned}$$

Let $w = H(h)w^1$. One can easily check by drawing a picture that $(h \vee k_0)r$ and k_1 are homotopic.

$$\begin{aligned} u_1 &= H(k_1)w^1 \\ &= H(r)H(h \vee k_0)w^1 \\ &= (H(h)w^1)(H(k_0)w^1) \\ &= wu_0\ . \end{aligned}$$

Hence u_0 and u_1 are in the same orbit and therefore $H(i)$ is $1-1$ on orbits.

Lemma 2.5 enables us to prove the onto part of the five lemma. Lemma 2.6 below describes a gadget we use to prove the $1-1$ part.

Suppose $A = SB$ so that $H(SA)$ is abelian. Let $v \in H(X)$ be in the kernel of $H(f)$. By 2.5, there is a $u \in H(Z_f)$ such that $H(i)u = v$. Let $\varphi(v)$ be the isotropy group of u under the action of $H(SA)$. Since $H(SA)$ is abelian $\varphi(v)$ does not depend on u.

2.6 LEMMA. *There is a space W_f and maps $h\colon X \to W_f$ and $k\colon SA \to W_f$ which depend only on f such that $\varphi(v) = H(k)H(h)^{-1}v$ for all $v \in \ker H(f)$.*

The construction of W_f, is accompanied by a proof that

$$\varphi(v) \subset H(k)H(h)^{-1}v\ .$$

Suppose $v \in \ker H(f)$, $u \in H(Z_f)$, $w \in H(SA)$, $H(i)u = v$ and $wu = u$. There is a $z_1 \in H(SA \vee Z_f)$ such that $H(j_1)z_1 = w$ and $H(j_2)z_1 = u$, where j_1 and j_2 are the inclusion maps of SA and Z_f into $S(A) \vee Z_f$, respectively.

$$H(r)z_1 = wu = u = H(j)u$$

where $j\colon Z_f \to Z_f$ is the identity map. Therefore there is an element $z_2 \in H(Z_{r,j})$ such that $H(i_1)z_2 = z_1$ and $H(i_2)z_2 = u$, where i_1 and i_2 are the inclusion maps of $SA \vee Z_f$ and Z_f into $Z_{r,j}$. Roughly speaking, each element of $H(Z_{r,j})$ gives elements $u, u' \in H(Z_f)$, and $w \in H(SA)$ such that $wu = u'$. We now add a homotopy to $Z_{r,j}$ to make $u = u'$.

$$\begin{aligned} H(i_1 j_2)z_2 &= H(j_2)H(i_1)z_2 \\ &= H(j_2)z_1 \\ &= u\ . \end{aligned}$$

Hence $H(i_2 \vee i_1 j_2)z_2 \in H(Z_f \vee Z_f)$ corresponds to the element

$$(u, u) \in H(Z_f) \times H(Z_f) \ .$$

Let $m_1 = i_2 \vee i_1 j_2$ and let m_2 be the folding map of $Z_f \vee Z_f$ onto Z_f. Under $H(m_2)$, u also corresponds to (u, u). Let $W_f = Z_{m_1, m_2}$. There is an element $z_3 \in H(W_f)$ such that $H(s_1)z_3 = z_2$ and $H(s_2)z_3 = u$, where s_1 and s_2 are the inclusion maps of $Z_{r,j}$ and Z_f into W_f.

Let $h = s_2 i$ and $k = s_1 i_1 j_1$. Recall $i: X \to Z_f$.

$$H(k)z_3 = H(j_1)H(i_1)H(s_1)z_3 = w$$
$$H(h)z_3 = H(i)H(s_2)z_3 = v \ ,$$

therefore $w \in H(k)H(h)^{-1}v$.

Suppose $z \in H(W_f)$ and $H(h)z = v$. Let $u = H(s_2)z$.

$$\begin{aligned} H(i)u &= H(i)H(s_2)z \\ &= H(h)z \\ &= v \ . \end{aligned}$$

Note $s_2 m_2 = s_1 m_1$. Therefore,

$$\begin{aligned} H(m_2)u &= H(m_2)H(s_2)z \\ &= H(m_1)H(s_1)z \ . \end{aligned}$$

But $m_1 = i_2 \vee i_1 j_2$. Therefore, since m_2 is the folding map, $H(i_2)H(s_1)z = u$ and $H(j_2)H(i_1)H(s_1)z = u$.

$$\begin{aligned} u &= H(i_2)H(s_1)z \\ &= H(j)H(i_2)H(s_1)z \\ &= H(r)H(i_1)H(s_1)z \\ &= \big(H(j_1)H(i_1)H(s_1)z\big)\big(H(j_2)H(i_1)H(s_1)z\big) \\ &= \big(H(k)z\big)u \ . \end{aligned}$$

Therefore $H(k)z \in \varphi(v)$ and the proof of 2.6 is complete.

PROOF OF LEMMA 1.5. Let H_1 and H_2: $\mathcal{C} \to \mathcal{S}$ be functors satisfying h, e and w, and let T: $H_1 \to H_2$ be a natural transformation such that $T: H_1(S^n) \approx H_2(S^n)$ for all $n > 0$.

If $X \in \mathcal{C}$ and $\dim X < \infty$, to within homotopy type, X may be given a CW-complex structure in the following fashion: Let $X_n \in \mathcal{C}$ be a sequence of spaces, $n = 1, 2, \cdots, \dim X$, and $f_n: \{\bigvee S^n_\alpha \mid \alpha \in \Lambda_n\} \to X_n$ such that $X_{n+1} = Z_{f_n}$ and $X_1 = \bigvee S^1_\alpha$. Then we may assume $X = \bigcup X_n$.

We first show that T: $H_1(X) \to H_2(X)$ is onto for all $X \in \mathcal{C}$ such that $\dim X < \infty$. Let X have the structure described above. Let $A_n = \bigvee S^n_\alpha$. We show by induction on n that T: $H_1(X_n) \to H_2(X_n)$ is onto. Note first that

$H_i(A_n) = H_i(\bigvee S^n_\alpha) \approx \prod H_i(S^n_\alpha)$ by axiom w. Hence T: $H_1(X_1) \approx H_2(X_1)$. The inductive step then follows from 2.5 by moving elements around a diagram as in the five lemma.

We next show that T is an isomorphism. Again we apply induction to T: $H_1(X_n) \to H_2(X_n)$. We have already noted that this is an isomorphism for $n = 1$. Suppose it is an isomorphism for $n - 1$. For $v_i \in H_i(X_{n-1})$, let $\varphi_i(v_i) \in H_i(SA_{n-1})$, $i = 1, 2$, be the isotropy groups described in 2.6. The fact that T is onto and 2.6 then yield: $T\varphi_1(v) = \varphi_2(Tv)$ for each $v \in$ kernel $H_1(f_{n-1})$. The inductive step then follows by moving elements around a diagram.

PROOF OF LEMMA 1.6. Suppose H: $\mathcal{C}_1 \to \mathcal{S}$ satisfies h, e, w and l, and T: $[\ \ , Y] \to H$ is a natural transformation such that T: $[X, Y] \approx H(X)$ for all $X \in \mathcal{C}_1$ such that dim $X < \infty$. We wish to show that T: $[X, Y] \approx H(X)$ for all $X \in \mathcal{C}_1$.

Suppose $X \in \mathcal{C}_1$, X^n is the n-skeleton of X with respect to some CW-complex decomposition, and i_n: $X^n \to X$ is the inclusion map.

2.7 LEMMA. *If $u \in H(X)$, there is an $[f] \in [X, Y]$ such that $T[fi_n] = H(i_n)u$.*

PROOF. For each n there is a unique element $[f_n] \in [X^n, Y]$ such that $T[f_n] = H(i_n)u$. Clearly $f_n \mid X^{n-1}$ and f_{n-1} are homotopic, and hence there is an f: $X \to Y$ such that $f \mid X^n$ and f_n are homotopic.

2.8 LEMMA. *If $[f]$ and $[g] \in [X, Y]$ and $T[f] = T[g]$, there is a homotopy equivalence k: $Y \to Y$ such that f and kg are homotopic.*

PROOF. Let $u = T(\iota) \in H(Y)$ where $\iota \in [Y, Y]$ is the class of the identity map. $H(f)u = T[f] = T[g] = H(g)u$. Therefore there is a $w \in H(Z_{f,g})$ such that $u = H(j_1)w = H(j_2)w$ where j_i: $Y \to Z_{f,g}$ are the two inclusions. Let t: $Z_{f,g} \to Y$ be the map corresponding to w as described in 2.7. Let i_n: $Y^n \to Y$ be the inclusion map ($Y^n = n$-skeleton). $T[tj_1 i_n] = H(j_1 i_n)w = H(i_n)H(j_1)w = H(i_n)u = T[i_n]$. But on $[Y^n, Y]$, T is an isomorphism. Therefore $tj_1 i_n$ and i_n are homotopic. This implies that $(tj_i)_*$: $[S^n, Y] \approx [S^n, Y]$ for all n and hence that tj_1 is a homotopy equivalence. Similarly, tj_2 is a homotopy equivalence. Let h be a homotopy inverse of tj_1 and let $k = htj_2$. Let η: $X \times I \to Z_{f,g}$ be given by $\eta(z, t) = \{z, t\}$ and let $\bar{h} = h\eta$. Then $\bar{h}$ provides a homotopy between f and kg.

We now show that T is $1-1$. Suppose $T[f] = T[g]$. Then $T[f \vee f] = T[f \vee g] \in H(X \vee X)$. Therefore there is a map k: $Y \to Y$ such that $[f \vee g] = [(f \vee f)k] = [fk \vee fk]$ and hence $[g] = [fk] = [f]$.

Finally we show that T is onto. In § 3 (Lemma 3.1) we show that there is a $u \in H(Y)$ such that $T = T(u)$. Let $v \in H(X)$. Let $w \in H(X \vee Y)$ be the element corresponding to (v, u) and let $j\colon Y \to X \vee Y$ be the inclusion map. Consider the commutative diagram:

$$\begin{tikzcd}
{[S^n, Y]} \arrow[d, "j_*"'] \arrow[dr, "T(u)"] & \\
{[S^n, X \vee Y]} \arrow[r, "T(w)"'] & H(S^n)\,.
\end{tikzcd}$$

Since $T(u)$ is an isomorphism, $T(w)$ is onto. Just as in the construction of Y at the end of § 1, we may adjoin cells to $X \vee Y$ to obtain a space Z and an element $z \in H(Z)$ such that $T(z)\colon [S^n, Z] \approx H(S^n)$, and such that z restricted to $X \vee Y$ is w. ($X \vee Y$ and w play the role of Y_0 and u_0.) If $k\colon X \vee Y \to Z$ is the inclusion map, $T(z)(kj)_* = T(u)$ and hence $(kj)_*\colon [S^n, Y] \approx [S^n, Z]$. Therefore kj has a homotopy inverse $h\colon Z \to Y$. If $i\colon X \to Z$ is the inclusion map,

$$\begin{aligned}
T[hi] &= T(u)[hi] \\
&= H(i)H(h)u \\
&= H(i)H(kj)^{-1}u \\
&= H(i)w \\
&= v\,.
\end{aligned}$$

This completes the proof of 1.6.

3. Natural transformations

Let $H\colon \mathcal{C} \to \mathcal{S}$ be a contravariant functor satisfying axiom h. In this section we investigate natural transformations $T\colon [\ \ , Y] \to H$. Suppose $Y \in \mathcal{C}$. Let $\iota \in [Y, Y]$ be the homotopy class of the identity map and let $u(T) = T(\iota) \in H(Y)$. Given an element $u \in H(Y)$, we may define a natural transformation $T(u)\colon [\ \ , Y] \to H$ as follows: If $[f] \in [X, Y]$, let $T(u)[f] = H(f)u$. It is easily checked that $T(u)$ is in fact a natural transformation and that:

3.1 LEMMA. *$u \to T(u)$ is a $1-1$ function from $H(Y)$ onto the natural transformations of $[\ \ , Y]$ into H. Furthermore, $T \to u(T)$ is its inverse.*

Suppose $\mathcal{C} = \mathcal{C}_0$. We wish to extend H to $\mathcal{C}_1$. For each $X \in \mathcal{C}_1$, let $\bar{H}(X) =$ set of natural transformations of $[\ \ , X]$ into H with the trivial transformation as distinguished element. If $f\colon X \to X'$, let $\bar{H}(f)T = Tf_*$, $T \in \bar{H}(X')$ and $f_*\colon [\ \ , X] \to [\ \ , X']$ the map induced by f. Note that if f and f' are homotopic $\bar{H}(f) = \bar{H}(f')$. Hence by 3.1:

3.2 LEMMA. *$\bar{H}$ is a contravariant functor from $\mathcal{C}_1$ to $\mathcal{S}$ satisfying axiom* h *which agrees with H on $\mathcal{C}_0$.*

We next wish to give another characterization of $\bar{H}$. Let $X \in \mathcal{C}$, and let $\{X_\alpha\}$ be its finite subcomplexes with respect to some CW-complex decomposition. If $X_\alpha \subset X_\beta$, let $i_{\alpha\beta}$, be the inclusion map. Let i_α: $X_\alpha \to X$ be the inclusion map. Then $\{H(X_\alpha), H(i_{\alpha\beta})\}$ forms an inverse system and $\bar{H}(i_\alpha)$: $\bar{H}(X) \to \bar{H}(X_\alpha) = H(X_\alpha)$ defines a map

$$\lambda: \bar{H}(X) \to \operatorname{inv}\lim H(X_\alpha)$$

3.3 LEMMA. *λ is an isomorphism.*

PROOF. Suppose T_1 and $T_2 \in \bar{H}(X)$ and $\lambda T_1 = \lambda T_2$. Let $Y \in \mathcal{C}_0$ and $[f] \in [Y, X]$. There is an X_α and f_α: $Y \to X_\alpha$ such that $i_\alpha f_\alpha = f$. Therefore $T_1[f] = \bar{H}(i_\alpha)T_1[f_\alpha] = H(i_\alpha)T_2[f_\alpha] = T_2[f]$. Therefore $T_1 = T_2$. Suppose $u \in \operatorname{inv}\lim H(X_\alpha)$. Let $T_u \in H(X)$ be defined as follows: Let Y, f, f_α and i_α be as above. Let $T_u[f] = H(f_\alpha)u_\alpha$, where u_α is the projection of u into $H(X_\alpha)$. It is easily checked that $T_u \in \bar{H}(X)$ and $\bar{H}(i_\alpha)T_u = u$.

PROOF OF LEMMA 1.7. Let $[\ \ , Y]_0$, $Y \in \mathcal{C}_1$, denote the restriction of $[\ \ , Y]$ to $\mathcal{C}_0$ and let $\overline{[\ \ , Y]}$: $\mathcal{C}_1 \to \mathcal{S}$ denote the extension of $[\ \ , Y]_0$ described in 3.2, i.e., $\overline{[Y', Y]}$ = natural transformations of $[\ \ , Y']_0$ into $[\ \ , Y]_0$. Let μ: $[Y', Y] \to \overline{[Y', Y]}$ be given by $\mu[f] = f_*$. We must show that μ is onto if Y, $Y' \in \mathcal{C}_\omega$. By 3.3,

$$\lambda: \overline{[Y', Y]} \approx \operatorname{inv}\lim [Y'_\alpha, Y]$$

where Y'_α are the finite subcomplexes of Y' with respect to some CW-complex decomposition. Also $\lambda\mu[f] = [fi_\alpha]$ where i_α: $Y'_\alpha \to Y'$ is the inclusion map. But when $Y' \in \mathcal{C}_\omega$, the homotopy extension theorem and the fact that there are finite subcomplexes $Y'_1, Y'_2, \cdots$ such that $Y = \bigcup Y_n$, imply that $\lambda\mu$ is onto. Therefore μ is onto.

PROOF OF LEMMA 1.3. Suppose H: $\mathcal{C}_0 \to \mathcal{S}$ satisfies axioms h, e and c. Let $\bar{H}$: $\mathcal{C} \to \mathcal{S}$ be the extension of H described in 3.2 restricted to $\mathcal{C}_\omega$. we wish to show that $\bar{H}$ satisfies h, w, l and e′. Axiom h follows from 3.2, w and l follow from 3.3.

Let S^n_i, $i = 1, 2, \cdots$ be a finite or countable collection of n-spheres, $A = \bigvee S^n_i$, $X \in \mathcal{C}$ and f: $A \to X$. We must show that

$$\bar{H}(Z_f) \xrightarrow{\bar{H}(i)} H(X) \xrightarrow{\bar{H}(f)} \bar{H}(A)$$

is exact, where i: $X \to Z_f$ is the inclusion map.

We first prove this when $A = S^n$. Suppose X has been given a countable CW-complex structure. There is a finite subcomplex $X_0 \subset X$ and a map f_0: $S^n \to X_0$ such that f is f_0 followed by the inclusion map. Since X is a countable CW-complex, there is a sequence of finite subcomplexes $X_0 \subset X_1 \subset X_2 \subset \cdots$ such that $X = \bigcup X_i$. Let $f_i = f_0$ followed by the inclusion of X_0 into X_i, $Z_i = Z_{f_i}$ and j_i: $X_i \to Z_i$ and k_i: $Z_i \to Z_{i+1}$ be the inclusion maps. $\bar{H}(f)\bar{H}(j) = \bar{H}(jf)$ is trivial because jf is homotopic to the constant map, and $\bar{H}$ of a point contains only one element. Therefore image $\bar{H}(j) \subset$ kernel $\bar{H}(f)$. Suppose $v \in$ kernel $\bar{H}(f)$. Let v_i be the projection of v into $H(X_i)$. By 3.3, it is sufficient to find $u_i \in H(Z_i)$ such that $H(j_i)u_i = v_i$ and $H(k_i)u_{i+1} = u_i$. Since $H(f)v = 0$, $H(f_i)v_i = 0$. Therefore, by 2.5, there are elements $\bar{u}_i \in H(Z_i)$ such that $H(j_i)\bar{u}_i = v_i$. Let m_i: $X_i \to X_{i+1}$ be the inclusion map. Then $k_i j_i = j_{i+1} m_i$ and $H(m_i)v_{i+1} = v_i$ since the v_i's come from a $v \in H(X)$. Therefore $H(j_i)H(k_i)\bar{u}_{i+1} = H(j_i)u_i$. Hence, by 3.5, there is a $w_i \in H(S^{n+1})$ such that $w_i\bar{u}_i = H(k_i)\bar{u}_{i+1}$. Let $y_i = (w_0 w_1 \cdots w_{i-1})^{-1}$ and $u_i = y_i\bar{u}_i$.

$$\begin{aligned} H(j_i)u_i &= v_i \ , \\ H(k_i)u_{i+1} &= y_{i+1}H(k_i)\bar{u}_{i+1} \\ &= y_{i+1}w_i\bar{u}_i \\ &= y_i\bar{u}_i \\ &= u_i \ . \end{aligned}$$

We next consider the case when $A = \bigvee S^n_j$. Let $A_i = S^n_1 \vee S^n_2 \vee \cdots \vee S^n_i$ and $f_i = f \mid A_i$. Again, since jf is homotopically trivial, image $H(j) \subset$ kernel $H(f)$. Suppose $v \in$ kernel $H(f)$. By induction, and as a result of what we have proved above, we can find $u_i \in H(Z_{f_i})$ such that u_i restricted to X is v; and restricted to $Z_{f_{i-1}}$, is u_{i-1}. Lemma 3.3 will then yield an element $u \in H(Z_f)$ which restricts to v.

4. Cohomology theories

Let $\mathcal{A}$ be a category of pairs of topological spaces which admit the structure of a CW-complex and subcomplex and all continuous maps from pairs to pairs. The pair $(X, \varnothing)$, where $\varnothing$ is the null set, is denoted by X. Let $\mathcal{G}$ be the category of abelian groups and homomorphisms.

A cohomology theory on $\mathcal{A}$ with values in $\mathcal{G}$ is defined to be a collection of contravariant functors H^q: $\mathcal{A} \to \mathcal{G}$, $-\infty < q < \infty$, and a collection of natural homomorphisms δ^q: $H^q(A) \to H^{q+1}(X, A)$ defined for each pair $(X, A) \in \mathcal{A}$ which satisfy the following axioms:

(4.1) If f, g: $(X, A) \to (Y, B)$ are homotopic, $H^q(f) = H^q(g)$ for all q.

(4.2) Let i: $(X, \varnothing) \to (X, A)$ and j: $A \to X$ be the inclusion maps. Then the following sequence is exact.

$$\cdots \to H^{q-1}(A) \xrightarrow{\delta^{q-1}} H^q(X, A) \xrightarrow{H^q(i)} H^q(X) \xrightarrow{H^q(j)} H^q(A) \to \cdots$$

(4.3) If $(X_1, X_1 \cap X_2)$ and $(X_2, X_1 \cap X_2) \in \mathcal{A}$ the map $H^q(X_1 \cup X_2, X_2) \to H^q(X_1, X_1 \cap X_2)$ induced by the inclusion map is an isomorphism for all q.

For each pair $(X, A) \in \mathcal{A}$, let $Z(X, A)$ be the space with base point p formed from $X \cup (A \times I) \cup \{p\}$, where p is not in X, by identifying a and $(a, 1)$ for each $a \in A$ and $A \times \{0\}$ with p. Note that $Z(X) = Z(X, \varnothing) = X \cup \{p\}$. Let $SZ(A)$ be the suspension of $Z(A)$ and let S: $Z(X, A) \to SZ(A)$ be the map which identifies X to p. Let $Y = \{Y_q, h_q\}$ be an Ω-spectrum; that is, a sequence of spaces Y_q with base point and homotopy equivalences h_q: $Y_q \to \Omega Y_{q+1}$ where ΩY_q is the space of loops based at the base point and h_q takes the base point into the constant loop. Then $[Z(X, A), Y_q]$, with the group structure induced by h_q: $Y_q \to \Omega Y_{q+1}$, is a functor from $\mathcal{A}$ to $\mathcal{G}$. Let η: $[Z(A), \Omega Y_{q+1}] \to [SZ(A), Y_{q+1}]$ be the usual isomorphism, and let $\bar{\delta}^q$: $[Z(A), Y_q] \to [Z(X, A), Y_{q+1}]$ be the homomorphism defined by $\bar{\delta}^q = S^* \eta h_{q*}[f]$. Standard arguments then yield:

4.4 LEMMA. $\{[Z(\ \ ), Y_q], \bar{\delta}^q\}$ *is a cohomology theory.*

We next formulate axioms w and l in terms of cohomology theories. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ denote the categories of pairs of topological spaces which admit, respectively, the structure of a finite CW-complex and subcomplex and the structure of a CW-complex and subcomplex.

(4.5) $\mathcal{A} = \mathcal{A}_0$ and, if p is a point, $H^q(p)$ is countable for all q.

(4.6) $\mathcal{A} = \mathcal{A}_1$. If U is a topological space with the discrete topology,

$$\prod H^q(i_x): H^q(U) \approx \prod H^q(x) ,$$

where i_x: $\{x\} \to U$ is the inclusion map. If $X_1 \subset X_2 \subset \cdots$ is a sequence of subcomplexes of $X = \bigcup X_n$ with respect to some CW-complex structure on X, then the inclusion maps i_n: $X_n \to X$ induce an epimorphism:

$$H^q(X) \to \operatorname{inv\,lim} H^q(X_n) .$$

THEOREM II. *If* $\{H^q, \delta^q\}$ *is a cohomology theory satisfying* (4.5) *or* (4.6), *there is an* Ω*-spectrum* $Y = \{Y_q, h_q\}$ *such that* $Y_q \in C$ *and natural equivalences* T^q: $[Z(\ \ ), Y_q] \to H^q$ *such that* $\delta^q T^q = T^{q+1}\bar{\delta}^q$.

We next describe the extent to which the spectrum Y in Theorem II is unique. If Y and Y' are spectra, a map F: $Y \to Y$ is a sequence of maps f_q: $Y_q \to Y'_q$ such that $h'_q f_q$ and $\Omega f_{q+1} h_q$ are homotopic. F is a homotopy equivalence if there is a map G: $Y' \to Y$ such that $g_q f_q$ and $f_q g_q$ are homotopic to the identity maps for each q.

THEOREM III. *If* $\{H^q, \delta^q\}$ *satisfies* (4.6) *the spectrum given by Theorem* II *is unique up to homotopy type. If* $\{H^q, \delta^q\}$ *satisfies* (4.5) *and* Y *and* Y' *are two spectra giving cohomology theories equivalent to* $\{H^q, \delta^q\}$, *then* *there is a collection of homotopy equivalences* f_q: $Y_q \to Y'_q$ *such that* $h'_q f_q$ *and* $\Omega f_{q+1} h_q$ *are homotopic on each finite subcomplex of* Y_q.

The complicated form of Theorem III for the condition (4.5) arises because of the following fact. Suppose X is a CW-complex, and X_α are its finite subcomplexes. In general the map of $[X, Y] \to \operatorname{inv} \lim [X_\alpha, Y]$ is not $1-1$. Thus we can have two maps f and f': $Y \to Y$ which induce the identity transformation on $[\ \ , Y]$ restricted to $\mathcal{C}_0$ but are not homotopic.

The proof of the first part of Theorem III is obvious, and the second part follows from 1.7.

PROOF OF THEOREM II. We only give the proof for the case in which $\{H^q, \delta^q\}$ satisfies (4.5). The proof when it satisfies (4.6) is analogous.

Let J^q: $\mathcal{C}_0 \to \mathcal{S}$ be the contravariant functor defined as follows: If $X \in \mathcal{C}_0$ and $x_0 \in X$ is its base point, $J^q(X)$ is the underlying set of $H^q(X, x_0)$ with the identity element of this group as distinguished element. If f: $X \to Y$ is a map in $\mathcal{C}_0$, $J^q(f) = H^q(f')$ where f': $(X, x_0) \to (Y, y_0)$ is the map in $\mathcal{A}$ defined by f. We first show that J^q satisfies axioms h, e and c.

Axiom h follows immediately from (4.1). If p is a point, $J^q(p) = H^q(p, p) = 0$ by the exact sequence for the pair (p, p). Let $x_0 \in X$ be the base point of X, k: $\{x_0\} \to X$ the inclusion map, and C: $X \to x_0$ the constant map. Then $H^q(k)H^q(C) = H^q(Ck)$ is the identity map on $H^q(x_0)$ and hence:

(4.7) $H(k)$: $H^q(X) \to H^q(x_0)$ is onto.

By the exact sequence of the pair (X, x_0),

(4.8) $J^q(X) = H^q(X, x_0)$ is isomorphic to the kernel of $H(k)$.

Let (X, X_1, X_2) be a proper triad, $A = X_1 \cap X_2$, and let j_i: $X_i \to X$ and and k_i: $A \to X_i$, $i = 1, 2$, be the inclusion maps. Consider the Mayer-Vietoris sequence [1],

$$H^{q-1}(A) \xrightarrow{\Delta} H^q(X) \xrightarrow{a} \bar{H}^q(X_1) + \bar{H}^q(X_2) \xrightarrow{b} H^q(A)$$

where $a = H^q(j_1) + H^q(j_2)$ and $b = H^q(k_1) - H^q(k_2)$. In [1] it is shown that this sequence is exact without using the dimension axiom, hence it is exact in the present context. Using this sequence, (4.7) and (4.8) one can easily check that J^q satisfies axiom e.

4.9 LEMMA. *There is a natural isomorphism* E: $H^q(X, x_0) \to H^{q+1}(SX, x_0)$.

PROOF. Recall that the exact sequence of a triple follows from the exact sequence of a pair. Let X^+ and X^- be the upper and lower cones of SX. $H^q(X, x_0) \approx H^{q+1}(X^+, X)$ by the exact sequence of the triple (X^+, X, x_0) and the fact that X^+ is contractible. By excision $H^{q+1}(X^+, X) \approx$

$H^{q+1}(SX, X^-)$ and, by the exact sequence of the triple (SX, X^-, x), $H^{q+1}(SX, X^-) \approx H^{q+1}(SX, x_0)$. Clearly all these isomorphisms are natural.

Let p be the base point of S^n. By excision, $H^q(p) \approx H^q(S^0, p)$ and by 4.9, $H^q(S^n, p) \approx H^{q-n}(S^0, p)$. Hence by (4.5), $J^q(S^n) = H^{q-n}(p)$ is countable, and thus J^q satisfies axiom c.

Applying Theorem I to J^q, we obtain a CW-complex $\bar{Y}_q$ and a natural equivalence T: $[\ ,\bar{Y}_q] \to J^q$. Furthermore, $\bar{Y}_q$ can be chosen to be a countable CW-complex. Lemma 4.9 yields a natural equivalence v: $[X, \bar{Y}_q] \to [SX, \bar{Y}_{q+1}] = [X, \Omega \bar{Y}_{q+1}]$ which, by 1.7, is induced by a map h_q: $Y_q \to \Omega \bar{Y}_{q+1}$. Milnor has shown in [2] that the loops on a CW-complex have the same homotopy type as a CW-complex. Therefore by J.H.C. Whitehead's theorem, $\bar{h}_q$ defines a homotopy equivalence between $\bar{Y}_q$ and the path component of the constant map of $\Omega \bar{Y}_{q+1}$. Choose a loop α from each path component of $\Omega \bar{Y}_{q+1}$ and let $Y_q = \bar{Y}_q \times \{\alpha\}$. Let h_q: $Y_q \to \Omega Y_{q+1} = \Omega \bar{Y}_{q+1}$ be given by $h_q(y, \alpha) = \bar{h}_q(y) \cdot \alpha$ where $y \in \bar{Y}_q$ and "$\cdot$" denotes path multiplication. Then h_q is a homotopy equivalence and hence $\{Y_q, h_q\}$ is an Ω-spectrum. Also Y_q admits a CW-complex structure. The reason for passing from $\bar{Y}_q$ to Y_q, besides the fact that this is necessary to obtain an Ω-spectrum, is that the spaces in $\mathcal{A}$ are not required to be pathwise connected.

By the usual arguments, the map $H^q(SX, x_0) \times H^q(SX, x_0) \to H^q(SX, x_0)$ induced by the folding map is given by $(u, v) \to u + v$. Hence the group structure on $H^q(SX, x_0)$ is determined by the folding map. Therefore the following are all group isomorphisms:

$$\begin{aligned} H^q(X, x_0) &\approx H^{q+1}(SX, x_0) \\ &\approx [SX, Y_{q+1}] \\ &\approx [X, \Omega Y_{q+1}] \\ &\approx [X, Y_q] . \end{aligned}$$

Furthermore, these isomorphisms are all natural.

Let $\hat{A} = Z(A, A)$. By excision, $H^q(X, A) \approx H^q(Z(X, A), \hat{A})$ and by the exact sequence of the triple $(Z(X, A), \hat{A}, p)$, $H^q(Z(X, A), A) \approx H^q(Z(X, A), p)$. Combining these isomorphisms with the isomorphisms described in the previous paragraph, we obtain a natural equivalence T^q: $[Z(X,A), Y^q] \approx H^q(X, A)$. Finally, the fact that $\delta^q T^q = T^{q+1} \bar{\delta}^q$ can be verified by moving around a few diagrams.

5. Furthermore applications of Theorem I

1. *Classification of principal bundles.* Let G be a topological group. We define a *principal G-bundle with base points* to be a principal bundle $\{B, X, p, G\}$ as defined by Steenrod in [3], together with base points $b_0 \in B$

and $x_0 \in X$ such that $p(b_0)=x_0$. Two such bundles will be called equivalent if there is a bundle equivalence mapping one into the other, and which carries the base point into the base point. For each $X \in \mathcal{C}_1$, let $B(X; G) \in \mathcal{S}$ be the set of equivalence classes of principal G bundles with base points, with the class of the trivial bundle as distinguished element. If $f\colon X' \to X$, let $B(f)\colon B(X; G) \to B(X'; G)$ be the map which is induced by assigning to each bundle over X' the bundle over X induced by f. Elementary bundle arguments show that B satisfies axioms h, e, w and l. (Bundles with base points are needed in order to show that

$$B(X \vee Y; G) \approx B(X; G) \times B(Y; G).$$

We thus have a variation of the well known theorem concerning the classification of principal bundles, namely:

5.1 THEOREM. *There is $Y_G \in \mathcal{C}_1$ unique up to homotopy type, and an element $\alpha \in B(Y_G; G)$ such that $T(\alpha)\colon [X, Y_G] \approx B(X; G)$ for all $X \in \mathcal{C}$.*

Note that α is the class of a universal bundle for G, and $T(\alpha)$ is the usual transformation from homotopy classes of maps into the classifying space into equivalence classes of bundles. If $\{X, Y_G\}$ and $\bar{B}(X; G)$ denote, respectively, homotopy classes of maps without base points and equivalence classes of bundles without base points, 5.1 together with fairly simple arguments show that $T(\alpha)$ induces an isomorphism

$$\{X, Y_G\} \approx \bar{B}(X; G) .$$

2. *Ordinary cohomology theory on* CW-*complexes*. Suppose $\{H^q, \delta^p\}$ is a cohomology theory on $\mathcal{A}_1$ satisfying (4.1), (4.2), (4.3) and the following condition: If S is a topological space with the discrete topology,

$$\begin{aligned} &H^0(S) \approx \textstyle\prod_{p \in S} H^0(p) , \\ &H^q(S) = 0 \qquad\qquad q > 0 . \end{aligned} \tag{5.3}$$

Let $G = H^0(p)$, where p is a point, and let $K(G) = \{K(G, q\}, h_q\}$ be the Ω-spectrum obtained from Eilenberg-MacLane spaces $K(G, q)$. Let $Z(X, A)$ and $\bar{\delta}^q$ be as described in § 4.

5.2 THEOREM. *There are natural equivalences $T^q\colon [Z(X, A), K(G, q)] \approx H^q(X, A)$ defined for all $(X, A) \in \mathcal{A}_1$ such that $\delta^q T^q = T^{q+1}\bar{\delta}^q$.*

PROOF. $\{H^q, \delta^q\}$ satisfies (4.6) because of (5.3) and the fact that $H^q(X) \approx H^q(X^n)$ for $q<n$ and X^n, the n skeleton of X. Hence by Theorem II, there is an Ω-spectrum $\{Y_q, h_q\}$ such that $[Z(\ ), Y_q]$ and H^q are equivalent. $[S^n, Y_q] \approx H^q(S^n) \approx H^{q-n}(p)$ and hence Y_q is an Eilenberg-MacLane space of type (G, q).

Appendix

Most of the machinery developed in the previous sections can be dualized so as to deal with the functor $[X, \ ]$, but unfortunately the dual of the proof of the dual of Theorem I breaks down in at least two places. First, it is difficult to recover $\pi_1(X)$ from $[X, \ ]$; and second, annihilating cohomology classes is more difficult than annihilating homotopy classes. Therefore the construction of X is more difficult. In this appendix, we briefly describe how the first part of § 1 may be dualized, and we state a theorem analogous to Theorem I with sufficiently stringent hypotheses so that the obvious duals of the arguments given in §§ 1, 2 and 3 provide a proof for it.

Suppose $f_i\colon Y_i \to B$, $i = 0, 1$. Let $P(B)$ be the space of paths on B with the compact open topology, and let $E_{f_0, f_1} \subset Y_0 \times P(B) \times Y_1$ be the space defined as follows:

$$E_{f_0, f_1} = \{(y_0, \alpha, y_1) \mid \alpha(0) = f_0(y_0),\ \alpha(1) = f_1(y_1)\}\ .$$

Let K_n be an Eilenberg-MacLane space of type (Z, n). Let $\mathcal{P}_1$ be the category of all pathwise connected spaces with base point which have the same homotopy as a CW-complex and all continuous maps. Let $\mathcal{P}_0$ be the subcategory of $\mathcal{P}_1$ of all space with only a finite number of non-zero homotopy groups and for which π_1 operates trivially on π_n for all $n > 0$. E_{f_0, f_1}, K_n, $\mathcal{P}_0$ and $\mathcal{P}_1$ play the role of Z_{f_1, f_2}, S^n, $\mathcal{C}_0$ and $\mathcal{C}_1$.

Suppose $\pi\colon \mathcal{P}_0 \to \mathcal{S}$ is a covariant functor.

Axiom h*. If f and $g\colon X \to Y$ are homotopic, $\pi(f) = \pi(g)$.

Axiom e*. Let $f_i\colon Y_i \to B$ and let $p_i\colon E_{f_0, f_1} \to Y_i$, $i = 0, 1$, be the projections. If $\alpha_i \in \pi(Y_i)$ and $\pi(f_0)\alpha_0 = \pi(f_1)\alpha_1$, there is a $\beta \in \pi(E_{f_0, f_1})$ such that $\pi(p_0)\beta = \alpha_0$ and $\pi(p_1)\beta = \alpha_1$. Furthermore, if B is a point, β is unique. If p is a point, $\pi(p)$ contains only one element.

Axiom c*. $\pi(K_1) = 0$, and $\pi(K_n)$ is finitely generated for all n.

THEOREM. If π satisfies h*, e* and c*, then there is an $X \in \mathcal{P}$, unique up to homotopy type, and a natural equivalence $T\colon [X, \ ] \approx \pi$.

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ANNALS OF MATHEMATICS
Vol. 78, No. 1, July, 1963
Printed in Japan

Correction to

COHOMOLOGY THEORIES

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These Annals, Vol. 75 (1962), 467-484
(Received January 29, 1963)

Theorem 5.2 is incorrect. A counter-example map can be found in [1]. The first sentence in the proof is false, but fortunately this does not affect any of the other theorems in the paper. As Milnor has shown in [2], the theorem may be corrected by replacing (5.3) by a different additivity condition, i.e., if X is a disjoint union of a collection of spaces $X_\alpha \in \mathcal{A}_1$, then the inclusion maps of X_α into X induce an isomorphism,

$$H^q(X) \approx \prod H^q(X_\alpha) .$$

Using this notion of additivity, Theorem I may be simplified as follows:

AXIOM a. If $X = \bigvee X_\alpha$, $X_\alpha \in C_1$ and $i_\alpha: X_\alpha \to X$ are the inclusion maps, then

$$\prod H(i_\alpha): H(X) \approx \prod H(X_\alpha) .$$

Using the techniques of [2] one may easily prove:

LEMMA. *If $C = C_1$, and H satisfies axioms* h, e *and* a, *then H satisfies axioms* w *and* l.

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