Homotopy type of differentiable manifolds

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It is our aim to give some homotopical conditions on a space which are necessary and sufficient under certain circumstances for it to be the homotopy type of a differentiable manifold.

We first mention some necessary conditions:

Let M^n be a closed differentiable manifold.

(1) M satisfies the Poincaré Duality Theorem, i.e. $H_n(M) = \mathbb{Z}$, with generator g, and $\cap g : H^i(M) \to H_{n-i}(M)$ is an isomorphism.

By a theorem of Whitney, M may be differentiably embedded in a sphere, say S^{n+k}. Let ν be the normal bundle which is oriented if M is oriented. A tubular neighborhood of M in S^{n+k} is diffeomorphic to a neighborhood of the zero cross section in $E(\nu)$, the total space of ν. If we collapse the exterior of this neighborhood to a point, we get the Thom complex of ν, $T(\nu)$. The collapsing map $c : S^{n+k} \to T(\nu)$ has the property that $c_*(\iota) = \Phi(g)$, where ι generates $H_{n+k}(S^{n+k})$ and Φ is the Thom isomorphism $\Phi : H_j(M) \to H_{j+k}(T(\nu))$. Hence we get:

(2) There exists an oriented vector bundle ν over M, such that $\Phi(g)$ is spherical in $H_{n+k}(T(\nu))$.

It turns out that in some circumstances the conditions (1) and (2) may be sufficient.

Theorem 1. *Let X be a connected finite polyhedron, with $\pi_1(X) = 0$. Suppose the following two conditions are satisfied:*

(1) *X satisfies the Poincaré Duality Theorem, i.e. for some n $H_n(X) = \mathbb{Z}$, and if g is a generator, $\cap g : H^i(X) \to H_{n-i}(X)$ is an isomorphism for all i.*

(2) *There exists an oriented vector bundle ξ^k over X, such that $\Phi(g) \in H_{n+k}(T(\xi))$ is spherical. Then, if n is odd, X is the homotopy type of an n-dimensional closed differentiable manifold, whose stable normal bundle is induced from ξ by the homotopy equivalence.*

In case M^n is a closed differentiable manifold of dimension $n = 4k$, the Hirzebruch Index Theorem is another property. In case $n = 4k$, this

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condition together with (1) and (2) is sufficient.

Theorem 2. *Let X be as in Theorem 1, except that $n = 4q \neq 4$. In addition, suppose:*

(3) $I(X) = \langle L_q(\overline{p}_1, \ldots, \overline{p}_q), g\rangle$, *where $I(X)$= index (signature) of the cup product bilinear form $H^{2q}(X;\mathbb{R}) \times H^{2q}(X;\mathbb{R}) \to H^{4q}(X;\mathbb{R}) = \mathbb{R}$, L_q is the Hirzebruch polynomial, and $\overline{p}_1, \ldots, \overline{p}_q$ are the dual classes to the Pontrjagin classes of ξ.*

Then X is the homotopy type of an n-dimensional closed differentiable manifold with normal bundle induced from ξ.

In case ξ is the trivial bundle, condition (2) is equivalent to:

(2′) $\Sigma^k(g) \in H_{n+k}(\Sigma^k X)$ is spherical, where Σ denotes suspension. Then the conclusions to Theorems 1 and 2 in this case, give that X is the homotopy type of a π-manifold.

M. Kervaire [1] has given an example of a 10-dimensional combinatorial manifold K which is not the homotopy type of a differentiable manifold. K has properties (1) and (2′). One may deduce from Theorems 1 and 2 then, that $K \times S^q$ *is* the homotopy type of a π-manifold if $4 \nmid q$, $q > 1$. It follows from a theorem of S. Cairns and M. Hirsch that $K \times S^q$ is not itself a differentiable manifold. Similarly $K \times K$ is the homotopy type of a π-manifold.

(It also follows from Theorems 1 and 2 that if X is a connected polyhedral H-space, $\pi_1(X) = 0$, and $H_n(X) \neq 0$, $H_i(X) = 0$, $i > n$, $n \not\equiv 2 (\mathrm{mod}\, 4)$, then X is the homotopy type of a π-manifold).

We will sketch the proof of Theorem 1.

By embedding X in an Euclidean space and taking a small neighborhood of it, we may replace X by a homotopically equivalent space which is an open manifold. We assume then that X is an open manifold. Let $f : S^{n+k} \to T(\xi)$ be a map, such that $f_*(\iota) = \Phi(g)$. By a theorem of Thom [4], f is homotopic to a map transverse regular on $X \subset T(\xi)$, so we may assume that f is transverse regular on X. Then $N = f^{-1}(X)$ is a closed n-submanifold of S^{n+k} with normal bundle induced from ξ.

Lemma 1. *$f : N \to X$ is of degree 1, i.e. $f_*(\iota) = g$, where $\iota \in H_n(N)$ is the canonical generator given by the orientation induced from the orientation of the normal bundle.*

This follows easily from the naturality of the Thom isomorphism with respect to bundle maps.

Lemma 2. *If $f : H_*(A) \to H_*(B)$ is a map of degree 1, where A, B are spaces satisfying* (1), *then* Kernel(f_*) *is a direct summand of $H_*(A)$.*

Proof. $\alpha = P_A f^*(P_B)^{-1}$ is a map $\alpha : H_*(B) \to H_*(A)$ such that $f_*\alpha = 1$, where P_A, P_B are the isomorphisms given by Poincaré Duality.

Note also that Kernel(f_*) is orthogonal to $\alpha(H_*(B))$ under the intersection pairing.

We wish that N were connected and simply connected and that Kernel(f_*) $= 0$, so that we might apply J. H. C. Whitehead's Theorem to conclude that f is a homotopy equivalence. These things are not true, but we may make them true by changing N by surgery in certain cases. This is where the extra assumption that n is odd (or the assumption on the index) will come in.

Surgery is the following process: We have a differentiable embedding $\phi : S^p \times D^{q+1} \to M^n$, $n = p + q + 1$. We remove interior $\phi(S^p \times D^{q+1})$ and replace it by $D^{p+1} \times S^q$, which has the same boundary $S^p \times S^q$. This is again a differentiable manifold.

We should like to do surgery to make N connected, then simply connected, and then kill the Kernel(f_*). At each stage we produce a new manifold, and we must verify that we still have a map of degree 1 of the new manifold $N' = \chi(N, \phi)$ into X which induces the normal bundle of N' from ξ.

Lemma 3. *Let $\phi : S^p \times D^{q+1} \to M^n$ be an embedding, and let $\phi | S^p \times 0 = \phi'$. Suppose that $f \circ \phi'$ is homotopic to a constant map. Then there is a map $f'' : N' \to X$, $N' = \chi(N, \phi)$ of degree 1.*

Proof. $S^p \times 0$ is a deformation retract of $S^p \times D^{q+1}$, so that since $f \circ \phi' \simeq *$, then $f \circ \phi \simeq *$, $f'(\phi(S^p \times D^{q+1})) = *$. Then we set $f'' = f'$ outside $D^{p+1} \times S^q$ and $f''(D^{p+1} \times S^q) = *$, and $f'' : N' \to X$ is clearly of degree 1 if f was.

It can be shown that by choosing the product structure carefully on $S^p \times D^{q+1}$, the new manifold N' still has its normal bundle induced from ξ, cf. [2,3].

Then by surgery we may make N into a connected manifold, ($p = 0$ in the surgery) and simply connected, ($p = 1$). It follows from the relative Hurewicz Theorem that Kernel(f_*) in the lowest non-zero dimension consists of spherical classes.

To make sure we can do surgery to kill a spherical class we must first find an embedding of a sphere representing the class. This we can do by Whitney's Theorem in dimensions $\leq \frac{1}{2}n$, if $n \neq 4$. We must also have the normal bundle to the embedded sphere trivial.

Lemma 4. *Let $\phi : S^p \to N^n$ with $f \circ \phi \simeq *$, $p < \frac{1}{2}n$. Then the normal bundle γ to S^p in N^n is trivial.*

Proof. $\phi^*(\tau(N)) = \tau(S^p) \oplus \gamma$, and $f^*(\xi) = \nu$, the normal bundle to N in S^{n+k}. Then $\tau(N) \oplus \nu$ is trivial. Hence $\phi^*(\tau(N) \oplus \nu) = \tau(S^p) \oplus \gamma \oplus \phi^*(\nu)$ is trivial. Since $\nu = f^*(\xi)$, $\phi^*(\nu) = \phi^* f^* \xi = (f \circ \phi)^*(\xi)$ is trivial. But $\tau(S^p) \oplus \epsilon$ is trivial if ϵ is trivial. Hence $\gamma \oplus \epsilon'$ is trivial, where ϵ' is trivial, $\epsilon' = \tau(S^p) \oplus \phi^*(\nu)$. Since γ is an $(n-p)$-dimensional bundle over S^p, and

$n - p > p$, this implies that γ is trivial.

If n is odd, we only have to kill Kernel(f_*) in dimensions $< \frac{1}{2}n$, and by Poincaré Duality we have killed the whole kernel. (If n is even, we have also dimension $\frac{1}{2}n$ to consider, and here the extra hypothesis on the index is necessary to take care of questions about triviality of γ).

It remains to show that the surgery can be used to kill all of Kernel(f_*), particularly around the middle dimension. With the aid of Lemma 2, however, the techniques of [1], [2], and [3] show that this can be done in the various cases.

References

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