Vector fields on $\pi$-manifolds

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1. Introduction

Let $M^n$ be an $n$-dimensional differentiable manifold. As in [8], we denote by $\sigma(M)$ (the "span" of $M$) the maximal number of linearly independent vector fields on $M$, and we also put $\sigma_n = \sigma(S^n)$.

The number $\sigma_n$ has been determined by J. F. Adams in [1]. In this paper we show how to determine the span of an arbitrary $\pi$-manifold. This is done by two theorems, below, the first of which states that only two values are possible for the span of a $\pi$-manifold of given dimension, and the second provides an easy way to decide which value is the case.

**Theorem 1.** If $M^n$ is a $\pi$-manifold, then either $\sigma(M^n) = \sigma_n$ or $\sigma(M^n) = n$.

The invariant which separates these cases is the semi-characteristic $\chi^*$ defined as follows: if $n$ is even, then $\chi^*(M^n) = \frac{1}{2} \chi(M^n)$ where $\chi$ denotes the Euler characteristic; if $n = 2r + 1$, then $\chi^*(M^n)$ is the mod 2 congruence class of $\sum_{i=0}^{r} \text{rank } H_i(M; \mathbb{Z}_2)$. We also define the reduced semi-characteristic $\hat{\chi}$ by $\hat{\chi}(M) = 1 - \chi^*(M)$, and note that it is additive with respect to connected sum of manifolds.

**Theorem 2.** $\hat{\chi}$ is a homomorphism of the semi-group of connected oriented $n$-dimensional $\pi$-manifolds onto $\mathbb{Z}$ for $n$ even, and onto $\mathbb{Z}_2$ for $n$ odd, such that if $n \neq 1, 3, 7$, then $\sigma(M^n) = n$ if and only if $\hat{\chi}(M^n) = 1$.

It is clear that Theorem 2 reduces to the theorems of Kervaire [4], [5]. The proof given here differs from Kervaire's. Theorem 1 was first proved for $n \neq 4k - 1$ by E. Thomas. Later the authors and Thomas produced independent proofs of this for general $n$. Thomas' treatment of the subject is substantially different from that of the present paper and is presented in [8].

We wish to thank E. Thomas for bringing this problem to our attention, and for many fruitful discussions concerning it.

An interesting example of applications of Theorems 1 and 2 is provided by Stiefel manifolds (real, complex, or quaternionic). It is known that they are $\pi$-manifolds [2]. We prove

**Corollary.** Stiefel manifolds $V_{n,k}$ are parallelizable if $k > 1$.

**Proof.** Compare [7]. One could of course apply Theorem 2. The following

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method simplifies computations and applies in many situations. Since $V_{n,k}$ is fibered over $V_{n,k-1}$, it follows that $\sigma(V_{n,k}) \geq \sigma(V_{n,k-1})$. Using the elementary fact that $\dim V_{n,k-1} > (2/3) \dim V_{n,k}$ for $k > 2$, together with the inequality $\sigma_m < 2m/3$ (which follows easily from [1]) for $m = 1, 3, 7$, we deduce from Theorem 1 that, if $V_{n,2}$ is parallelizable, then so are all the $V_{n,k}$ for $k > 1$. To check that $V_{n,2}$ is parallelizable, we use Theorem 2.

Later on in this paper we shall need a sharper inequality for $\sigma_n$, namely: $2\sigma_n < n - 1$ for $n \neq 1, 3, 7, 15$. This is also easily checked from the explicit formula for $\sigma_n$ in [1].

2. The Gauss map

We recall that a $\pi$-manifold is a manifold which can be embedded in some euclidean space with trivial normal bundle. It is known [6] that $M$ is a $\pi$-manifold if and only if $M$ is stably parallelizable, that is, if and only if $\tau(M) + \varepsilon$ is trivial, where $\tau(M)$ is the tangent bundle of $M$, and $\varepsilon$ is a trivial line bundle on $M$.

Let $M^\ast$ be a closed oriented $\pi$-manifold, and let $F$ be a framing of the stable tangent bundle $\tau + \varepsilon$ of $M$ compatible with the given orientation on $\tau + \varepsilon$. Referring a vector $v \in \tau + \varepsilon$ to the coordinate system based on the frame $F$ (at the base point of $v$), we obtain a point $F^\ast(v) \in R^{n+1}$. If $x \mapsto \varepsilon(x)$ denotes the canonical cross-section of the trivial line bundle $\varepsilon$, then $x \mapsto F^\ast(\varepsilon(x))$ is the "Gauss map"

$$\nu_F: M^\ast \longrightarrow S^n.$$ Clearly $\nu_F$ is covered by a bundle map of $\tau(M^\ast)$ into $\tau(S^n)$. Thus $\nu_F^\ast(\tau(S^n)) = \tau(M^\ast)$ which implies $\sigma(M^\ast) \geq \sigma_n$.

The degree of the map $\nu_F$ will be denoted by $d(M, F)$. If $-F$ is the framing (of $\tau(-M) + \varepsilon$) obtained from $F$ by reversing the first vector, then

$$(2.1) \quad d(-M, -F) = d(M, F),$$

since $\nu_{-F}$ differs from $\nu_F$ by a reflection through a hyperplane, but the orientation of $M$ has also been changed.

We note that $\sigma(M) \geq k$ if and only if for each framing $F$ there is a map $f_k: M^\ast \to V_{n+1,k+1}$ such that $\nu_F = \pi \circ f_k$ where $\pi: V_{n+1,k+1} \to V_{n+1,1} = S^n$ is the canonical projection.

If $F$ and $G$ are two framings of $\tau + \varepsilon$, then there is a map $g: M^\ast \to SO(n+1)$ such that, for each $x \in M$, $F_x = g(x) \cdot G$. Thus $\nu_{g}(x) = g(x)(\nu_F(x))$. Let $\pi: SO(n+1) \to S^n$ be the canonical projection, and note that $\nu_F$ is homotopic to a map $\nu'_F$ taking the complement of some $n$-cell $U$ onto the base point $\pi(e) \in S^n$, while $g$ is homotopic to a map taking $U$ into $e$. Thus $\nu_{g}$ is homotopic to the composition

$$M^\ast \xrightarrow{g' \times \nu'_F} SO(n+1) \vee S^n \xrightarrow{\pi \cup 1} S^n \vee S^n \longrightarrow S^n,$$
the last map being of degree (1,1). It follows that
(2.2) \[ d(M, G) = \deg (\pi \circ g) + d(M, F) \].

Notice that for \( n \) odd, \( \deg (\pi \circ g) \) can be any even integer (at least), since there is a map \( f: S^n \to \text{SO}(n + 1) \) with \( \deg (\pi \circ f) = 2 \).

3. Two lemmas

3.1. Lemma. Let \( W^n \) be a \( \pi \)-manifold with boundary \( S^{n-1} \) \( (n \) odd\). Let \( K \) be a \( (k - 1) \)-connected complex with \( 2k \geq n \). Then, for any map \( f: W \to K \), \( f|\partial W \) is null-homotopic.

Proof. By [6; Th. 6.6] \( W \) can be submitted to a sequence of spherical modifications (away from \( \partial W \)) ending with a contractible manifold. We must only check that at each stage there is a map into \( K \) extending the given map \( f \) on \( \partial W \). But, by the proof of [6; Th. 6.6] surgery need only be performed on embedded \( m \)-spheres for \( m \leq (n - 1)/2 < k \). Thus, at any stage, the map into \( K \) may be assumed to be constant in the neighborhood of this embedded \( m \)-sphere, and hence the map into \( K \) can be defined in the obvious way on the manifold resulting from this spherical modification.

3.2. Lemma. Let \( M^n \) be a \( \pi \)-manifold with \( n \) odd. Let \( \nu: M^n \to S^n \) be any map of degree one. Then \( \nu \) can be lifted to \( f: M^n \to V_{n+1,k+1} \) with \( \nu = \pi \circ f \) if and only if \( k \leq \sigma_n \).

Proof. The lemma is trivial for \( n = 1, 3, 7 \). Assume for the moment that \( n \neq 1, 3, 7, 15 \). Suppose that \( k = \sigma_n + 1 \), and that \( f: M^n \to V_{n+1,k+1} \) exists such that \( \pi \circ f = \nu \) is of degree one. Let \( W \) be the complement of an open cell in \( M \). We may assume that \( \nu \) is one-one on \( M - W \) and maps \( W \) to a point. Then \( f \) maps \( W \) into a fibre \( V_{n,k} \) of \( \pi \). Since \( V_{n,k} \) is \( (n - k - 1) \)-connected and (for \( n \neq 1, 3, 7, 15 \)) \( 2(n - k) = 2(n - \sigma_n - 1) > n - 1 \) (that is, \( 2\sigma_n < n - 1 \), see [1]), it follows from 3.1 that \( f|\partial W: \partial W \to V_{n,k} \) is null-homotopic. However, \( f|\partial W \) is just the characteristic map of the fiber \( \pi: V_{n+1,k+1} \to S^n \) since \( \nu \) is one-one on \( M - W \). Thus \( f|\partial W \) is not null-homotopic in \( V_{n,k} \), for otherwise \( \pi: V_{n+1,k+1} \to S^n \) would have a cross-section, and \( S^n \) would have a field of \( k \)-frames, \( k = \sigma_n + 1 \).

It remains to consider the case \( n = 15 \). More generally, let \( \sigma'_n \) be the largest integer \( k \) for which there exists a \( \pi \)-manifold \( M^n \), and a map \( f: M^n \to V_{n+1,k+1} \) with \( \pi \circ f \) of degree one. We have shown that \( \sigma'_n = \sigma_n \) for \( n \neq 15 \), and we wish to show this for \( n = 15 \). We claim that, if \( k \leq \sigma'_n, \sigma'_m \), then \( k \leq \sigma'_{n+m+1} \). To see this, let \( f_i: M^i_1 \to V_{n+1,k+1} \) and \( f_2: M^2_1 \to V_{m+1,k+1} \) be as above. Let \( g: M_1 \times M_2 \times S^1 \to M_1 \ast M_2 \) (the join of \( M_1 \) and \( M_2 \)) be a map of degree one (that is, inducing isomorphism on homology in dimension \( n + m + 1 \)). (\( g \) can

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be taken to be the natural map into the reduced join of $M_i$ and $M_j$ followed by a homotopy equivalence with $M_i * M_j$.) Consider the diagram

$$
\begin{array}{c}
M_i \times M_j \times S^1 \longrightarrow M_i * M_j \\
\downarrow \\
S^* \uparrow \quad \cong \\
\downarrow \\
S^{*+1}
\end{array}
$$

where the map $V_{n+1,k+1} * V_{m+1,k+1} \to V_{n+m+2,k+1}$ is the map defined by James [3]. The bottom row of this diagram consists of maps of degree one, so that $k \leq \sigma_{n+m+1}$ by definition.

Now we have, by induction, $\sigma_n' \leq \sigma_{p+1}^n$. For $n = 15$, $p = 3$ this yields $\sigma_7' \leq \sigma_47 = \sigma_47 = 8 = \sigma_15$.

Consequently $\sigma_n' = \sigma_n$ for all $n$, and the lemma follows.

4. Proof of Theorem 1

4.1. Suppose that either $n$ is even and, for some $F'$, $\text{d}(M^*, F') = 0$; or $n$ is odd, and $\text{d}(M^*, F') = 0 \mod 2$. Then $\sigma(M^*) = n$.

**Proof.** The assumptions imply, by 2.2 and the remark following it, that in either case there is a framing $F'$ such that $\text{d}(M, F') = 0$. But then $\nu_F$ is null-homotopic and can be lifted to $f_n : M \to SO(n+1) = V_{n+1,n+1}$. This gives the desired framing of $\tau(M)$.

4.2. Let $n$ be even, and suppose that for some $F$, $\text{d}(M, F') \neq 0$. Then $\sigma(M) = \sigma_n$.

**Proof.** For $n$ even, $\sigma_n = 0$. Suppose $\sigma(M) \neq 0$. Then there is, for each $F$, a map $f_1 : M \to V_{n+1,2}$ such that $\nu_F = \pi \circ f_1$. Since $H_n(V_{n+1,2}) = \mathbb{Z}_2$, it follows that $\text{d}(M, F') = \deg(\pi \circ f_1) = 0$, which proves 4.2.

4.3. Let $n$ be odd $\neq 1, 3, 7$ and suppose that, for some $F$, $\text{d}(M, F) = 1 \mod 2$. Then $\sigma(M) = \sigma_n$.

**Proof.** By 2.2 we can find a framing $F$ such that $\text{d}(M, F') = 1$. Therefore 4.3 follows from 3.2.

Now, 4.1, 4.2 and 4.3 imply Theorem 1.

5. Proof of Theorem 2

5.1. If $n$ is even, then $\text{d}(M, F')$ does not depend on $F$. If $n$ is odd $\neq 1, 3, 7$, then the mod 2 congruence class of $\text{d}(M, F')$ does not depend on $F$.

**Proof.** Suppose first that $n$ is even. Then, in 2.2, we have $\deg(\pi \circ g) = 0$, for $\pi$ factors through $V_{n+1,2}$, and $H_n(V_{n+1,2}) = \mathbb{Z}_2$ in this case. Therefore $\text{d}(M, F')$ does not depend on $F$. 
Suppose now that $n$ is odd. By 4.1 and 4.3, the mod 2 congruence class of $d(M, F)$ is determined by the span of $M$, and therefore does not depend on $F$.

We now define $d(M)$ to be the mod 2 congruence class of $d(M, F)$ if $n$ is odd, and to be $d(M, F)$ if $n$ is even.

To prove Theorem 2, it is now sufficient to prove the following theorem:

**THEOREM 3.** If $M^n$ is a $\pi$-manifold and $n \neq 1, 3, 7$, then $d(M^n) = \chi^*(M^n)$.

The proof is preceded by a lemma.

**5.2. Lemma.** If $M^n$ is the boundary of an oriented $\pi$-manifold $W^{n+1}$, and if $F$ is a framing of $\tau(W)$, then $d(M, F) = \chi(W)$.

**Proof.** Here we have used $F$ to denote also the restriction of $F$ to a framing of $\tau(W)|M = \tau(M) + \varepsilon$. Also, we select our orientation conventions so that the canonical section (orientation) of $\varepsilon$ is the outward normal vector. Let $W$ be riemannian and let $f$ be a real-valued differentiable function on $W$, which has only a finite number of non-degenerate critical points, and which is constant on $M$. We may assume that $(\text{grad } f)|M$ is the canonical section of $\varepsilon$ (the outward normal to $M$ in $W$) and that $||\text{grad } f|| \leq 1$ everywhere on $W$.

Let $\mu_\nu$ denote the map $W^{n+1} \to D^{n+1}$ defined by $x \to F^*(\text{grad } f)_x$, where $D^{n+1}$ is the unit disk in $R^{n+1}$. Clearly $\nu_\nu = \mu_\nu | M$, and thus $\deg \nu_\nu = \deg \mu_\nu$ (considering $\mu_\nu$ as a map of pairs $(W, M) \to (D^{n+1}, S^n)$), as follows easily from the homology sequences of the pairs $(W, M)$ and $(D^{n+1}, S^n)$. Since $f$ is non-degenerate, 0 is a regular value of $\mu_\nu$, and thus $\deg \mu_\nu$ is the sum of the “local degrees” at the critical points of $f$; that is, $\deg \mu_\nu$ is the sum of the indices of the vector field $\text{grad } f$. However, it is well-known that the latter is also the Euler characteristic of $W$.

**Proof of Theorem 3.** Suppose first that $n$ is even, and consider $M \times I$. By 2.1, 2.4, and 5.1, we have

$$\chi(M) = \chi(M \times I) = d(M) + d(-M) = 2d(M),$$

which proves the theorem in this case.

Assume now that $n$ is odd and different from 1, 3, and 7. By [6, Th. 6.6], $M^n$ is frame-cobordant to a homotopy sphere $-\Sigma^n$, that is, there exists an $(n + 1)$-manifold $W_{n+1}$, and a framing $F$ of $\tau(W)$, such that $\partial W = M \cup \Sigma$, and $F|\partial M$ is the given framing of $\tau(M) + \varepsilon$.

Recall that for any even dimensional $\pi$-manifold $V^n$, $\chi(V) = \chi^*(\partial V) + \rho$ where $\rho$ is the rank of the intersection pairing $H_*(V, \mathbb{Z}) \otimes H_*(V, \mathbb{Z}) \to \mathbb{Z}$ [6, Lem. 5.9], and the fact that
$\rho$ is even since $V$ is a $\pi$-manifold [6, p. 525].

Thus, in the present situation,

$$\chi^*(M) + \chi^*(\Sigma) = \chi^*(\partial W) = \chi(W) = d(\partial W, F) \mid \partial W = d(M) + d(\Sigma) \pmod{2}.$$  

Now $\chi^*(\Sigma) = 1$, and $d(\Sigma)$ must be odd; for if $d(\Sigma)$ were even, then 4.1 would imply that $\Sigma$ is parallelizable contrary to the assumption that $n \neq 1, 3, 7$. This concludes the proof of Theorem 3, and hence also of Theorem 2.

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References


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