ON HYPERBOLIC GROUPS WITH SPHERES AS BOUNDARY

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Dedicated to Steve Ferry on the occasion of his 60th birthday

Abstract

Let $G$ be a torsion-free hyperbolic group and let $n \geq 6$ be an integer. We prove that $G$ is the fundamental group of a closed aspherical manifold if the boundary of $G$ is homeomorphic to an $(n-1)$-dimensional sphere.

Introduction

If $G$ is the fundamental group of an $n$-dimensional closed Riemannian manifold with negative sectional curvature, then $G$ is a hyperbolic group in the sense of Gromov (see for instance [6, 7, 21, 22]). Moreover, such a group is torsion-free and its boundary $\partial G$ is homeomorphic to a sphere. This leads to the natural question whether a torsion-free hyperbolic group with a sphere as boundary occurs as a fundamental group of a closed aspherical manifold (see Gromov [23, page 192]). We settle this question if the dimension of the sphere is at least 5.

Theorem A. Let $G$ be a torsion-free hyperbolic group and let $n$ be an integer $\geq 6$. The following statements are equivalent:

(i) The boundary $\partial G$ is homeomorphic to $S^{n-1}$.
(ii) There is a closed aspherical topological manifold $M$ such that $G \cong \pi_1(M)$, its universal covering $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\tilde{M}$ by $\partial G$ is homeomorphic to $D^n$.

The aspherical manifold $M$ appearing in our result is unique up to homeomorphism. This is a consequence of the validity of the Borel Conjecture for hyperbolic groups [2]; see also Section 3.

The proof depends on the surgery theory for homology ANR-manifolds due to Bryant, Ferry, Mio, and Weinberger [9] and the validity of the $K$- and $L$-theoretic Farrell-Jones Conjecture for hyperbolic groups due to Bartels, Reich, and Lück [4] and Bartels-Lück [2]. It seems likely that this result holds also if $n = 5$. Our methods can be extended to this case if the surgery theory from [9] can be extended to the case of

Received 1/15/2010.
5-dimensional homology ANR-manifolds—such an extension has been announced by Ferry and Johnston. We also hope to give a treatment elsewhere by more algebraic methods.

We do not get information in dimensions \( n \leq 4 \) for the usual problems about surgery. For instance, our methods give no information in the case where the boundary is homeomorphic to \( S^3 \), since virtually cyclic groups are the only hyperbolic groups which are known to be good in the sense of Friedman \([19]\). In the case \( n = 3 \) there is the conjecture of Cannon \([11]\) that a group \( G \) acts properly, isometrically, and cocompactly on the 3-dimensional hyperbolic plane \( \mathbb{H}^3 \) if and only if it is a hyperbolic group whose boundary is homeomorphic to \( S^2 \). Provided that the infinite hyperbolic group \( G \) occurs as the fundamental group of a closed irreducible 3-manifold, Bestvina and Mess \([5, Theorem 4.1]\) have shown that its universal cover is homeomorphic to \( \mathbb{R}^3 \) and its compactification by \( \partial G \) is homeomorphic to \( D^3 \), and the Geometrization Conjecture of Thurston implies that \( M \) is hyperbolic and \( G \) satisfies Cannon’s conjecture. The problem is solved in the case \( n = 2 \), essentially as a consequence of Eckmann’s theorem that 2-dimensional Poincare duality groups are surface groups (see \([16]\)). Namely, for a hyperbolic group \( G \), its boundary \( \partial G \) is homeomorphic to \( S^1 \) if and only if \( G \) is a Fuchsian group (see \([12, 18, 20]\)).

In general, the boundary of a hyperbolic group is not locally a Euclidean space but has a fractal behavior. If the boundary \( \partial G \) of an infinite hyperbolic group \( G \) contains an open subset homeomorphic to Euclidean \( n \)-space, then it is homeomorphic to \( S^n \). This is proved in \([25, Theorem 4.4]\), where more information about the boundaries of hyperbolic groups can be found.

We also prove the following result.

**Theorem B.** Let \( G \) and \( H \) be torsion-free hyperbolic groups such that \( \partial G \cong \partial H \). Then \( G \) can be realized as the fundamental group of a closed aspherical manifold of dimension at least 6 if and only if \( H \) can be realized as the fundamental group of such a manifold.

Moreover, even in case that neither can be realized by a closed aspherical manifold, they can both be realized by closed aspherical homology ANR-manifolds, which both have the same Quinn obstruction \([30]\) (see Theorem 1.3 for a review of this notion) provided that \( \partial G \) has the integral Čech cohomology of \( S^{n-1} \) for \( n \geq 6 \).

In particular, if \( G \) is hyperbolic and realized as the fundamental group of a closed aspherical manifold of dimension at least 6, then any torsion-free group \( H \) that is quasi-isometric to \( G \) can also be realized as the fundamental group of such a manifold. This follows from Theorem B, because the homeomorphism type of the boundary of a hyperbolic group is invariant under quasi-isometry (and so is the property of being hyperbolic). The attentive reader will realize that most of the content
of Theorem A can also be deduced from Theorem B, as every sphere appears as the boundary of the fundamental group of some closed hyperbolic manifold.

Acknowledgments. This paper was financially supported by the Sonderforschungsbereich 478 (Geometrische Strukturen in der Mathematik), the Max-Planck-Forschungspreis and the Leibniz-Preis of the second author, and NSF grant 0852227 of the third author.

The techniques and ideas of this paper are very closely related to the work of Steve Ferry; indeed, his unpublished work could have been used to simplify some parts of this work. It is a pleasure to dedicate this paper to him on the occasion of his 60th birthday.

1. Homology manifolds

A topological space $X$ is called an absolute neighborhood retract, or ANR, if it is normal and for every normal space $Z$, every closed subset $Y \subseteq Z$ and every (continuous) map $f: Y \to X$ there exists an open neighborhood $U$ of $Y$ in $Z$ together with an extension $F: U \to X$ of $f$ to $U$.

Definition 1.1 (Homology ANR-manifold). An $n$-dimensional homology ANR-manifold $X$ is an absolute neighborhood retract satisfying:

- $X$ has a countable base for its topology;
- the topological dimension of $X$ is finite;
- $X$ is locally compact;
- for every $x \in X$ the $i$th singular homology group $H_i(X, X - \{x\})$ is trivial for $i \neq n$ and infinite cyclic for $i = n$.

Notice that a normal space with a countable basis for its topology is metrizable by the Urysohn Metrization Theorem (see [29, Theorem 4.1 in Chapter 4-4 on page 217]) and is separable, i.e., contains a countable dense subset [29, Theorem 4.1]. Notice furthermore that every metric space is normal (see [29, Theorem 2.3 in Chapter 4-4 on page 198]), and has a countable basis for its topology if and only if it is separable (see [29, Theorem 1.3 in Chapter 4-1 on page 191 and Exercise 7 in Chapter 4-1 on page 194]). Hence a homology ANR-manifold in the sense of Definition 1.1 is the same as a generalized manifold in the sense of Daverman [14, page 191]. A closed $n$-dimensional topological manifold is an example of a closed $n$-dimensional homology ANR-manifold (see [14, Corollary 1A in V.26, page 191]). A homology ANR-manifold $M$ is said to have the disjoint disk property (DDP), if for any $\varepsilon > 0$ and maps $f, g: D^2 \to M$, there are maps $f', g': D^2 \to M$ so that $f'$ is $\varepsilon$-close to $f$, $g'$ is $\varepsilon$-close to $g$, and $f'(D^2) \cap g'(D^2) = \emptyset$; see for example [9, page 435]. We recall that a Poincaré duality group $G$ is a finitely presented group satisfying the following two conditions: first,
the $\mathbb{Z}G$-module $\mathbb{Z}$ (with the trivial $G$-action) admits a resolution of finite length by finitely generated projective $\mathbb{Z}G$-modules; second, there is $n$ such that $H^i(G; \mathbb{Z}G) = 0$ for $n \neq i$ and $H^n(G; \mathbb{Z}G) \cong \mathbb{Z}$. In this case $n$ is the formal dimension of the Poincaré duality group $G$.

**Theorem 1.2.** Let $G$ be a torsion-free group.

(i) Assume that

- the (non-connective) $K$-theory assembly map
  $$H_i(BG; K_{\mathbb{Z}}) \to K_i(\mathbb{Z}G)$$
  is an isomorphism for $i \leq 0$ and surjective for $i = 1$;
- the (non-connective) $L$-theory assembly map
  $$H_i(BG; w L_{\mathbb{Z}}^{(-\infty)}) \to L_i^{(-\infty)}(\mathbb{Z}G, w)$$
  is bijective for every $i \in \mathbb{Z}$ and every orientation homomorphism $w : G \to \{\pm 1\}$.

Then for $n \geq 6$ the following are equivalent:

- $G$ is a Poincaré duality group of formal dimension $n$;
- there exists a closed ANR-homology manifold $M$ homotopy equivalent to $BG$. In particular, $M$ is aspherical and $\pi_1(M) \cong G$;

(ii) If the statements in assertion (i) hold, then the homology ANR-manifold $M$ appearing there can be arranged to have the DDP.

(iii) If the statements in assertion (i) hold, then the homology ANR-manifold $M$ appearing there is unique up to $s$-cobordism of ANR-homology manifolds.

**Proof.** (i) The assumption on the $K$-theory assembly map implies that $\text{Wh}(G) = 0$, $\tilde{K}_0(\mathbb{Z}G) = 0$, and $K_i(\mathbb{Z}G) = 0$ for $i < 0$; compare [27, Conjecture 1.3 on page 653 and Remark 2.5 on page 679]. This implies that we can change the decoration in the above $L$-theory assembly map from $(-\infty)$ to $s$ (see [27, Proposition 1.5 on page 664]). Thus the assembly map $A$ in the algebraic surgery exact sequence [31, Definition 14.6] (for $R = \mathbb{Z}$ and $K = BG$) is an isomorphism. This implies in particular that the quadratic structure groups $S_i(\mathbb{Z}, BG)$ are trivial for all $i \in \mathbb{Z}$.

Assume now that $G$ is a Poincaré duality group of dimension $n \geq 3$. We conclude from Johnson and Wall [24, Theorem 1] that $BG$ is a finitely dominated $n$-dimensional Poincaré complex in the sense of Wall [35]. Because $\tilde{K}_0(\mathbb{Z}G) = 0$, the finiteness obstruction vanishes and hence $BG$ can be realized as a finite $n$-dimensional simplicial complex (see [34, Theorem F]). We will now use Ranicki’s (4-periodic) total surgery obstruction $\overline{s}(BG) \in \overline{S}_n(BG)$ of the Poincaré complex $BG$; see [31, Definition 25.6]. The main result of [9] asserts that this obstruction vanishes if and only if there is a closed $n$-dimensional homology ANR-manifold $M$ homotopy equivalent to $BG$. The groups $\overline{s}_k(BG)$ arise in a 0-connected version of the algebraic surgery sequence [31, Definition 15.10]. It is a consequence of [31, Proposition 15.11(iii)] (and the
fact that \( L_{-1}(\mathbb{Z}) = 0 \) that \( \mathbb{S}_n(BG) = S_n(\mathbb{Z}, BG) \). Since \( S_n(\mathbb{Z}, BG) = 0 \), we conclude \( \pi(BG) = 0 \). This shows that (i)a implies (i)b. (In this argument we ignored that the orientation homomorphism \( w: G \to \{\pm 1\} \) may be non-trivial. The argument however extends to this case; compare [31, Appendix A].) Homology manifolds satisfy Poincaré duality and therefore (i)b implies (i)a.

(ii) It is explained in [9, Section 8] that this homology manifold \( M \) appearing above can be arranged to have the DDP. (Alternatively, we could appeal to [10] and resolve \( M \) by an \( n \)-dimensional homology ANR-manifold with the DDP.)

(iii) The uniqueness statement follows from Theorem 3.1(ii). q.e.d.

In order to replace homology ANR-manifolds by topological manifolds we will later use the following result that combines work of Edwards and Quinn; see [14, Theorems 3 and 4 on page 288, [30]).

**Theorem 1.3.** There is an invariant \( \iota(M) \in 1 + 8\mathbb{Z} \) (known as the Quinn obstruction) for connected homology ANR-manifolds with the following properties:

(i) If \( U \subset M \) is an open subset, then \( \iota(U) = \iota(M) \).

(ii) Let \( M \) be a homology ANR-manifold of dimension \( \geq 5 \). Then the following are equivalent:
   - \( M \) has the DDP and \( \iota(M) = 1 \);
   - \( M \) is a topological manifold.

**Definition 1.4.** An \( n \)-dimensional homology ANR-manifold \( M \) with boundary \( \partial M \) is an absolute neighborhood retract which is a disjoint union \( M = \text{int} M \cup \partial M \), where

- \( \text{int} M \) is an \( n \)-dimensional homology ANR-manifold;
- \( \partial M \) is an \((n-1)\)-dimensional homology ANR-manifold;
- for every \( z \in \partial M \) the singular homology group \( H_i(M, M \setminus \{z\}) \) vanishes for all \( i \).

**Lemma 1.5.** If \( M \) is an \( n \)-dimensional homology ANR-manifold with boundary, then \( \bar{M} := M \cup_{\partial M} \partial M \times [0, 1) \) is an \( n \)-dimensional homology ANR-manifold.

**Proof.** Suppose that \( Y \) is the union of two closed subsets \( Y_1 \) and \( Y_2 \) and set \( Y_0 := Y_1 \cap Y_2 \). If \( Y_0, Y_1, \) and \( Y_2 \) are ANRs, then \( Y \) is an ANR; see [14, Theorem 7 on page 117]. If \( Y_1 \) and \( Y_2 \) have countable bases \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) of the topology, then sets \( U_1 \setminus Y_2 \) with \( U_1 \in \mathcal{U}_1 \), \( U_2 \setminus Y_1 \) with \( U_2 \in \mathcal{U}_2 \) and \( (U_1 \cup U_2)^c \) with \( U_i \in \mathcal{U}_i \) form a countable basis of the topology of \( Y \). (Here \( (\cdot)^c \) is the operation of taking the interior in \( Y \).) If \( Y_1 \) and \( Y_2 \) are both finite dimensional, then \( Y \) is finite dimensional [29, Theorem 9.2 on page 303]. If \( Y_1 \) and \( Y_2 \) are both locally compact, then \( Y \) is locally compact.
Thus the only non-trivial requirement is that for \( x = (z, 0) \in \hat{M} \) with \( z \in \partial M \), we have \( H_i(\hat{M}, \hat{M} \setminus \{x\}) = 0 \) if \( i \neq n \) and \( \cong \mathbb{Z} \) if \( i = n \). Let \( I_z := \{z\} \times [0, 1/2) \). Because of homotopy invariance we can replace \( \{x\} \) by \( I_z \). Let \( U_1 := M \cup \partial M \times (0, 1/2) \subset \hat{M} \) and \( U_2 := \partial M \times (0, 1) \subset \hat{M} \). Then \( H_i(U_1, U_1 \setminus I_z) \cong H_i(M, M \setminus \{z\}) = 0 \) and \( H_i(U_2, U_2 \setminus I_z) = 0 \). Because \( U_1 \) and \( U_2 \) are both open, we can use a Mayer-Vietoris sequence to deduce

\[
H_i(\hat{M}, \hat{M} \setminus I_z) \cong H_{i-1}(U_1 \cup U_2, U_1 \cap U_2 \setminus I_z) \cong H_{i-1}(\partial M, \partial M \setminus \{z\}).
\]

The result follows as \( \partial M \) is an \((n-1)\)-dimensional homology ANR-manifold. q.e.d.

**Corollary 1.6.** Let \( M \) be a homology ANR-manifold with boundary \( \partial M \). If \( \partial M \) is a manifold, then \( \iota(\text{int } M) = 1 \).

**Proof.** We use \( \hat{M} \) from Lemma 1.5. If \( \partial M \) is a manifold, then so is \( \partial M \times (0,1) \). The result follows now from Theorem 1.3. q.e.d.

### 2. Hyperbolic groups and aspherical manifolds

For a hyperbolic group we write \( \overline{G} := G \cup \partial G \) for the compactification of \( G \) by its boundary; compare [7, III.H.3.12], [5]. Left multiplication of \( G \) on \( G \) extends to a natural action of \( G \) on \( \overline{G} \). We will use the following properties of the topology on \( \overline{G} \).

**Proposition 2.1.** Let \( G \) be a hyperbolic group. Then

(i) \( \overline{G} \) is compact;

(ii) \( \overline{G} \) is finite dimensional;

(iii) \( \partial G \) has empty interior in \( \overline{G} \);

(iv) the action of \( G \) on \( \overline{G} \) is small at infinity: if \( z \in \partial G \), \( K \subset G \) is finite and \( U \subset \overline{G} \) is a neighborhood of \( z \), then there exists a neighborhood \( V \subset \overline{G} \) of \( z \) with \( V \subseteq U \) such that for any \( g \in G \) with \( gK \cap V \neq \emptyset \) we have \( gK \subseteq U \);

(v) if \( z \in \partial G \) and \( U \) is an open neighborhood of \( z \) in \( \overline{G} \), then for every finite subset \( K \subset G \) there is an open neighborhood \( V \) of \( z \) in \( \overline{G} \) such that \( V \subseteq U \) and \( (V \cap G) \cdot K \subseteq U \cap G \).

**Proof.**

(i) see for instance [7, III.H.3.7(4)].

(ii) see for instance [3, 9.3.(ii)].

(iii) is obvious from the definition of the topology in [5].

(iv) see for instance [32, page 531].

(v) follows from (iv): We may assume \( 1_G \in K \). Pick \( V \) as in (iv). If \( g \in V \cap G \) and \( k \in K \), then \( g \in gK \cap V \). Thus \( gK \subseteq U \). Therefore \( gK \in U \cap G \). q.e.d.

Let \( X \) be a locally compact space with a cocompact and proper action of a hyperbolic group \( G \). Then we equip \( \overline{X} := X \cup \partial G \) with the topology
for which a typical open neighborhood of \( x \in X \) is an open subset of \( X \) and a typical (not necessarily open) neighborhood of \( z \in \partial G \) is of the form

\[
(U \cap \partial G) \cup (U \cap G) \cdot K
\]

where \( U \) is an open neighborhood of \( z \) in \( \overline{G} \) and \( K \) is a compact subset of \( X \) such that \( G \cdot K = X \). We observe that we could fix the choice of \( K \) in the definition of \( O_X \): Let \( U, z, \) and \( K \) be as above and let \( K' \) be a further compact subset of \( X \) such that \( G \cdot K' = X \). Because the \( G \)-action is proper, there is a finite subset \( L \) of \( G \) such that \( K' \subseteq L \cdot K \).

By Proposition 2.1(v) there is an open neighborhood \( V \subseteq U \) of \( z \) in \( G \) such that \( (V \cap G) \cdot L \subseteq U \cap G \). Thus

\[
(V \cap \partial G) \cup (V \cap G) \cdot K' \subseteq (U \cap \partial G) \cup (V \cap G) \cdot L \subseteq (U \cap \partial G) \cup (U \cap G) \cdot K.
\]

If \( f : X \to Y \) is a \( G \)-equivariant continuous map where \( Y \) is also a locally compact space with a cocompact proper \( G \)-action, then we define \( \overline{f} : X \to \overline{Y} \) by \( \overline{f}|_X := f \) and \( \overline{f}|_{\partial G} := \text{id}_{\partial G} \).

**Lemma 2.2.** Let \( G \) be a hyperbolic group and \( X \) be a locally compact space with a cocompact and proper \( G \)-action.

(i) \( \overline{X} \) is compact;
(ii) \( \partial G \) is closed in \( \overline{X} \) and its interior in \( \overline{X} \) is empty;
(iii) if \( \dim X \) is finite, then \( \dim \overline{X} \) is also finite;
(iv) if \( f : X \to Y \) is a \( G \)-equivariant continuous map where \( Y \) is also a locally compact space with a cocompact proper \( G \)-action, then \( \overline{f} \) is continuous.

**Proof.** These claims are easily deduced from the observation following the definition of the topology \( O_{\overline{X}} \) and Proposition 2.1. q.e.d.

We recall that for a hyperbolic group \( G \) equipped with a (left invariant) word-metric \( d_G \) and a number \( d > 0 \), the Rips complex \( P_d(G) \) is the simplicial complex whose vertices are the elements of \( G \), and a collection \( g_1, \ldots, g_k \in G \) spans a simplex if \( d_G(g_i, g_j) \leq d \) for all \( i, j \). The action of \( G \) on itself by left translation induces an action of \( G \) on \( P_d(G) \). Recall that a closed subset \( Z \) in a compact ANR \( Y \) is a \( Z \)-set if for every open set \( U \) in \( Y \) the inclusion \( U \setminus Z \to U \) is a homotopy equivalence. An important result of Bestvina and Mess [5] asserts that (for sufficiently large \( d \)) \( \overline{P_d(G)} \) is an ANR such that \( \partial G \subset \overline{P_d(G)} \) is \( \partial \)-set. The proof uses the following criterion [5, Proposition 2.1]:

**Proposition 2.3.** Let \( Z \) be a closed subspace of the compact space \( Y \) such that

(i) the interior of \( Z \) in \( Y \) is empty;
(ii) \( \dim Y < \infty \);
(iii) for every $k = 0, \ldots, \dim Y$, every $z \in Z$ and every neighborhood $U$ of $z$, there is a neighborhood $V$ of $z$ such that every map $\alpha : S^k \to V \setminus Z$ extends to $\tilde{\alpha} : D^{k+1} \to U \setminus Z$;
(iv) $Y \setminus Z$ is an ANR.

Then $Y$ is an ANR and $Z \subset Y$ is a $Z$-set.

Condition (iii) is sometimes abbreviated by saying that $Z$ is $k$-LCC in $Y$, where $k = \dim Y$.

**Theorem 2.4.** Let $X$ be a locally compact ANR with a cocompact and proper action of a hyperbolic group $G$. Assume that there is a $G$-equivariant homotopy equivalence $X \to P_d(G)$. If $d$ is sufficiently large, then $X$ is an ANR, $\partial G$ is $Z$-set in $X$, and $Z$ is $k$-LCC in $X$ for all $k$.

**Proof.** Bestvina and Mess [5, page 473] show that (for sufficiently large $d$) $P_d(G)$ satisfies the assumptions of Proposition 2.3. Moreover, they show that $Z$ is $k$-LCC in $X$ for all $k$. Using this, it is not hard to show, that $X$ satisfies these assumptions as well: Assumptions (i) and (ii) hold because of Lemma 2.2. Assumption (iv) holds because $X$ is an ANR. Because $f \mapsto \overline{f}$ is clearly functorial, the homotopy equivalence $X \to P_d(G)$ induces a homotopy equivalence $\overline{X} \to P_d(G)$ that fixes $\partial G$. Using this homotopy equivalence, it is easy to check that $\partial G$ is $k$-LCC in $\overline{X}$, because it is $k$-LCC in $P_d(G)$. Thus Assumption (iii) holds. q.e.d.

**Proposition 2.5.** Let $M$ be a finite-dimensional locally compact ANR which is the disjoint union of an $n$-dimensional ANR-homology manifold $\text{int} M$ and an $(n-1)$-dimensional ANR-homology manifold $\partial M$ such that $\partial M$ is a $Z$-set in $M$. Then $M$ is an ANR-homology manifold with boundary $\partial M$.

**Proof.** The $Z$-set condition implies that there exists a homotopy $H_t : M \to M, t \in [0,1]$ such that $H_0 = \text{id}_M$ and $H_t(M) \subseteq \text{int} M$ for all $t > 0$, see [5, page 470].

Let $z \in \partial M$. Then the restriction of $H_1$ to $M \setminus \{z\}$ is a homotopy inverse for the inclusion $M \setminus \{z\} \to M$. Thus $H_i(M, M \setminus \{z\}) = 0$ for all $i$.

q.e.d.

There is the following (harder) manifold version of Proposition 2.5 due to Ferry and Seebeck [17, Theorem 5 on page 579].

**Theorem 2.6.** Let $M$ be a locally compact with a countable basis of the topology. Assume that $M$ is the disjoint union of an $n$-dimensional manifold $\text{int} M$ and an $(n-1)$-dimensional manifold $\partial M$ such that $\text{int} M$ is dense in $M$ and $\partial M$ is $(n-1)$-LCC in $M$. Then $M$ is an $n$-manifold with boundary $\partial M$.

**Theorem 2.7.** Let $G$ be a torsion-free word-hyperbolic group. Let $n \geq 6$. 

(i) The following statements are equivalent:
   a) The boundary $\partial G$ has the integral $\check{\text{C}}$ech cohomology of $S^{n-1}$.
   b) $G$ is a Poincaré duality group of formal dimension $n$.
   c) There exists a closed ANR-homology manifold $M$ homotopy equivalent to $BG$. In particular, $M$ is aspherical and $\pi_1(M) \cong G$.

(ii) If the statements in assertion (i) hold, then the homology ANR-manifold $M$ appearing there can be arranged to have the DDP;

(iii) If the statements in assertion (i) hold, then the homology ANR-manifold $M$ appearing there is unique up to $s$-cobordism of ANR-homology manifolds.

Proof. By [21, page 73], torsion-free hyperbolic groups admit a finite CW-model for $BG$. Thus the $\mathbb{Z}G$-module $\mathbb{Z}$ admits a resolution of finite length of finitely generated free $\mathbb{Z}G$ modules. By [5, Corollary 1.3], the $(n-1)$-th $\check{\text{C}}$ech cohomology of the boundary $\partial G$ agrees with $H^n(G; \mathbb{Z}G)$. This shows that the statements (i)a and (i)b in assertion (i) are equivalent.

The Farrell-Jones Conjecture in $K$- and $L$-theory holds by [2, 4]. This implies that the assumptions of Theorem 1.2 are satisfied; compare [27, Proposition 2.2 on page 685]. This finishes the proof of Theorem 2.7.

q.e.d.

Proof of Theorem A. (i) Let $G$ be a torsion-free hyperbolic group. Assume that $\partial G \cong S^{n-1}$ and $n \geq 6$. Theorem 2.7 implies that there is a closed $n$-dimensional homology ANR-manifold $N$ homotopy equivalent to $BG$. Moreover, we can assume that $N$ has the DDP. The universal cover $M$ of $N$ is an $n$-dimensional ANR-homology manifold with a proper and cocompact action of $G$. The homotopy equivalence $N \to BG$ lifts to a $G$-homotopy equivalence $M \to EG$. For sufficiently large $d$, $P_d(G)$ is a model for $EG$ (see [21, page 73]). Thus there is a $G$-homotopy equivalence $M \to P_d(G)$. Theorem 2.4 implies that $\overline{M}$ is an ANR and $\partial G$ is a $Z$-set in $\overline{M}$. We conclude from Lemma 2.2 that $\overline{M}$ is compact and has finite dimension. Thus we can apply Proposition 2.5 and deduce that $\overline{M}$ is a homology ANR-manifold with boundary. Its boundary is a sphere and in particular a manifold. Corollary 1.6 implies that $\iota(M) = 1$. By Theorem 1.3(i) this implies $\iota(N) = 1$. Using Theorem 1.3(ii) we deduce that $N$ is a topological manifold. By Theorem 2.4 the boundary $\partial G \cong S^{n-1}$ is $k$-LCC in $M$ for all $k$. Therefore we can apply Theorem 2.6 and deduce that $\overline{M}$ is a manifold with boundary $S^{n-1}$. The $Z$-condition implies that $\overline{M}$ is contractible, because $M$ is contractible as the universal cover of the aspherical manifold $N$. The $h$-cobordism theorem for topological manifolds implies that $\overline{M} \cong D^n$. In particular, $M \cong \mathbb{R}^n$. This shows that (i) implies (ii). The converse is obvious.

q.e.d.
3. Rigidity

The uniqueness question for the manifold appearing in our result from the introduction is a special case of the Borel Conjecture that asserts that aspherical manifolds are topological rigid: any isomorphism of fundamental groups of two closed aspherical manifolds should be realized (up to inner automorphism) by a homeomorphism. The connection of this rigidity question to assembly maps is well known and one of the main motivations for the Farrell-Jones Conjecture. For homology ANR-manifolds, the corresponding rigidity statement is (because of the lack of an s-cobordism theorem) somewhat weaker.

Theorem 3.1. Let $G$ be a torsion-free group. Assume that

- the (non-connective) $K$-theory assembly map
  
  $H_i(BG; K\mathbb{Z}) \to K_i(\mathbb{Z}G)$

  is an isomorphism for $i \leq 0$ and surjective for $i = 1$;

- the (non-connective) $L$-theory assembly map
  
  $H_i(BG; wL^{(-\infty)}\mathbb{Z}) \to L^{(-\infty)}_i(\mathbb{Z}G, w)$

  is bijective for every $i \in \mathbb{Z}$ and every orientation homomorphism $w: G \to \{\pm 1\}$.

Then the following holds:

(i) Let $M$ and $N$ be two aspherical closed $n$-dimensional manifolds together with isomorphisms $\phi_M: \pi_1(M) \xrightarrow{\cong} G$ and $\phi_N: \pi_1(N) \xrightarrow{\cong} G$. Suppose $n \geq 5$.

Then there exists a homeomorphism $f: M \to N$ such that $\pi_1(f)$ agrees with $\phi_N \circ \phi_M^{-1}$ (up to inner automorphism);

(ii) Let $M$ and $N$ be two aspherical closed $n$-dimensional homology ANR-manifolds together with isomorphisms $\phi_M: \pi_1(M) \xrightarrow{\cong} G$ and $\phi_N: \pi_1(N) \xrightarrow{\cong} G$. Suppose $n \geq 6$.

Then there exists an s-cobordism of homology ANR-manifolds $W = (W, \partial_0W, \partial_1W)$, homeomorphisms $u_0: M_\partial \to \partial_0W$, and $u_1: M_\partial \to \partial_1W$ and an isomorphism $\phi_W: \pi_1(W) \to G$ such that $\phi_W \circ \pi_1(i_0 \circ u_0)$ and $\phi_W \circ \pi_1(i_1 \circ u_1)$ agree (up to inner automorphism), where $i_k: \partial_kW \to W$ is the inclusion for $k = 0,1$.

Proof. (i) As discussed in the proof of Theorem 1.2, the assumptions imply that $Wh(G) = 0$. Therefore it suffices to show that the structure set $S^{TOP}(M)$ (see [31, Definition 18.1]) in the Sullivan-Wall geometric surgery exact sequence consists of precisely one element. This structure set is identified with the quadratic structure group $S_{n+1}(M) = S_{n+1}(BG)$ in [31, Theorem 18.5]. A discussion similar to the one in the proof of Theorem 1.2 shows that our assumptions imply that the quadratic structure group is trivial.
(ii) This follows from a similar argument that uses the surgery exact sequences for homology ANR-manifolds due to Bryant, Ferry, Mio, and Weinberger [9, Main Theorem on page 439]. q.e.d.

4. The Quinn obstruction depends only on the boundary

Let $G$ be a torsion-free hyperbolic group. Assume that $\partial G$ has the integral Čech cohomology of a sphere $S^{n-1}$ with $n \geq 6$. By Theorem 2.7 there is a closed aspherical ANR-homology manifold $N$ whose fundamental group is $G$.

**Proposition 4.1.** In the above situation, the Quinn obstruction (see Theorem 1.3) $\iota(N)$ depends only on $\partial G$.

**Proof.** Let $H$ be a further torsion-free hyperbolic group such that $\partial H \cong \partial G$. Let $N'$ be a closed aspherical ANR-homology manifold whose fundamental group is $H$. Then both the universal covers $M$ of $N$ and $M'$ of $N'$ can be compactified to $\overline{M}$ and $\overline{M}'$ such that $\partial G \cong \partial H$ is a Z-set in both; see Theorem 2.4. Now set $X := \overline{M} \cup_{\partial G} \overline{M}'$. We claim that $X$ is a connected ANR-homology manifold. Thus

$$\iota(N) = \iota(M) = \iota(X) = \iota(M') = \iota(N')$$

by Theorem 1.3(i). To prove the claim we refer to [1], in particular pages 1270–1271. Both, $M$ and $M'$ are homology manifolds in the sense of this reference. By fact 6 of this reference, $X$ is also a homology manifold. It remains to show that $X$ is an ANR. This follows from an argument given during the proof of Theorem 9 of this reference. q.e.d.

**Proof of Theorem B.** Let $G$ and $H$ be torsion-free hyperbolic groups, such that $\partial G \cong \partial H$. Assume that $G$ is the fundamental group of a closed aspherical manifold of dimension at least 6. Theorem 2.7(i) implies that $\partial G \cong \partial H$ has the integral Čech cohomology of a sphere $S^{n-1}$ with $n \geq 6$ and that $H$ is the fundamental group of a closed aspherical ANR-homology manifold $M$ of dimension $n$. Because of Theorem 2.7(ii) this ANR-homology manifold can be arranged to have the DDP. Now by Proposition 4.1 (and Theorem 1.3(ii)) we have $\iota(M) = 1$. Using Theorem 1.3(ii) again, it follows that $M$ is a manifold.

A similar argument works if $G$ is the fundamental group of a closed aspherical homology ANR-manifold that is not necessarily a closed manifold. q.e.d.

5. Exotic examples

In light of the results of this paper one might be tempted to wonder if for a torsion-free hyperbolic group $G$, the condition $\partial G \cong S^n$ is equivalent to the existence of a closed aspherical manifold whose fundamental group is $G$. This is however not correct: Davis and Januszkiewicz, and
Charney and Davis constructed closed aspherical manifolds whose fundamental group is hyperbolic with boundary not homeomorphic to a sphere. We review these examples below.

**Example 5.1.**

(i) For every \( n \geq 5 \) there exists an example of an aspherical closed topological manifold \( M \) of dimension \( n \) which is a piecewise flat, non-positively curved polyhedron such that the universal covering \( \tilde{M} \) is not homeomorphic to Euclidean space (see [15, Theorem 5b.1 on page 383]). There is a variation of this construction that uses the strict hyperbolization of Charney and Davis [13] and produces closed aspherical manifolds whose universal cover is not homeomorphic to Euclidean space and whose fundamental group is hyperbolic.

(ii) For every \( n \geq 5 \) there exists a strictly negative curved polyhedron of dimension \( n \) whose fundamental group \( G \) is hyperbolic, which is homeomorphic to a closed aspherical smooth manifold and whose universal covering is homeomorphic to \( \mathbb{R}^n \), but the boundary \( \partial G \) is not homeomorphic to \( S^{n-1} \); see [15, Theorem 5c.1 on page 384 and Remark on page 386].

On the other hand, one might wonder if assertion (ii) in Theorem A can be strengthened to the existence of more structure on the aspherical manifold. Strict hyperbolization [13] can be used to show that in general there may be no smooth closed aspherical manifold in this situation.

**Example 5.2.** Let \( M \) be a closed oriented triangulated PL-manifold. It follows from [13, Theorem 7.6] that there is a hyperbolization \( \mathcal{H}(M) \) of \( M \) has the following properties:

(i) \( \mathcal{H}(M) \) is a closed oriented PL-manifold. (This uses properties (2) and (4) from [13, page 333].)

(ii) There is a degree 1-map \( \mathcal{H}(M) \to M \) under which the rational Pontrjagin classes of \( M \) pull back to those of \( \mathcal{H}(M) \). In particular, the Pontrjagin numbers of \( M \) and \( \mathcal{H}(M) \) coincide. (See properties (5) and (6′) from [13, page 333].)

(iii) \( \mathcal{H}(M) \) is a negatively curved piece-wise hyperbolic polyhedra. In particular, \( G := \pi_1(\mathcal{H}(M)) \) is hyperbolic. Moreover, by [15, page 348] the boundary of \( \partial G \) is a sphere.

Suppose that some Pontrjagin number of \( \mathcal{H}(M) \) is not an integer. Then the same is true for \( \mathcal{H}(M) \). In particular, \( \mathcal{H}(M) \) does not carry the structure of a smooth manifold. If in addition \( \dim \mathcal{H}(M) = \dim M \geq 5 \), then by Theorem 3.1 (i) any other closed aspherical manifold \( N \) with \( \pi_1(N) = G \) is homeomorphic to \( \mathcal{H}(M) \) and does not carry a smooth structure either. Such manifolds \( M \) exist in all dimensions \( 4k, k \geq 2 \); see Lemma 5.3. This shows that there are for all \( k \geq 2 \) torsion-free hyperbolic groups \( G \) with \( \partial G \cong S^{4k-1} \) that are not fundamental groups of smooth closed aspherical manifolds. In particular, such a \( G \) is not
the fundamental group of a closed Riemannian manifolds of non-positive curvature.

In the previous example we needed $PL$-manifolds that do not carry a smooth structure. Such manifolds are classically contructed using Hirzebruch’s Signature Theorem.

**Lemma 5.3.** Let $k \geq 2$. There is an oriented closed $4k$-dimensional $PL$-manifold $M^{4k}$ whose top Pontrjagin number $\langle p_k(M^{4k}) \mid [M^{4k}] \rangle$ is not an integer.

**Proof.** For all $k \geq 2$ there are smooth framed compact manifolds $N^{4k}$ whose signature is 8 and whose boundary is a $(4k - 1)$-homotopy sphere; see [8] and [26, Theorem 3.4]. By [33], this homotopy sphere is $PL$-isomorphic to a sphere. We can now cone off the boundary and obtain a $PL$-manifold $M^{4k}$ (often called the Milnor manifold) whose only non-trivial Pontrjagin class is $p_k$ and whose signature $\sigma(M^{2k})$ is 8. Hirzebruch’s Signature Theorem implies that

$$8 = \sigma(M^{4k}) = \frac{2^{2k}(2^{2k-1} - 1)B_k}{2k!}\langle p_k(M^{4k}) \mid [M^{4k}] \rangle$$

where $B_k$ is the $k$th Bernoulli number; see [26, page 75]. For $k = 2, 3$ we have then

$$8 = \frac{7}{45}\langle p_2(M^8) \mid [M^8] \rangle = \frac{62}{945}\langle p_3(M^{12}) \mid [M^{12}] \rangle;$$

compare [28, page 225]. This yields examples for $k = 2, 3$. Taking products of these examples, we obtain examples for all $k \geq 2$. q.e.d.

### 6. Open questions

We conclude this paper with two open questions.

(i) Can the boundary of a hyperbolic group be an ANR-homology sphere that is not a sphere?

(ii) Can one give an example of a hyperbolic group (with torsion) whose boundary is a sphere, such that the group does not act properly discontinuously on some contractible manifold?

### References


