The signature of fibre-bundles*

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1. Introduction

For a compact oriented differentiable manifold X of dimension $4k$ the signature (or index) of X is defined as the signature of the quadratic form in $H^{2k}(X; \mathbf{R})$ given by the cup product. Thus

$$\mathrm{Sign}\,(X) = p - q$$

where p is the number of $+$ signs in a diagonalization of the quadratic form and q is the number of $-$ signs. If $\dim X$ is not divisible by 4 one defines $\mathrm{Sign}\,(X)$ to be zero. Then one has the multiplicative formula

$$\mathrm{Sign}\,(X \times Y) = \mathrm{Sign}\,(X) \cdot \mathrm{Sign}\,(Y)\ .$$

In [5] it was proved that this multiplicative formula continues to hold when $X \times Y$ is replaced by a fibre bundle with base X and fibre Y *provided that the fundamental group of X acts trivially on the cohomology of Y.*

In this paper we exhibit examples which show that this restriction on the action of $\pi_1(X)$ is necessary, and that the signature *is not multiplicative in general fibre-bundles.* Our examples are actually in the lowest possible dimension namely when $\dim X = \dim Y = 2$. The total space Z of the bundle has dimension 4 and non-zero signature, whereas $\mathrm{Sign}\,(X) = \mathrm{Sign}\,(Y) = 0$ (because their dimensions are not divisible by 4). Of course if one wants an example in which base and fibre have dimensions divisible by 4 it suffices to take Z^2, which is fibered over X^2 with fibre Y^2; we have

$$\mathrm{Sign}\,(Z^2) = \mathrm{Sign}\,(Z)^2 \neq 0$$
$$\mathrm{Sign}\,(X^2) = \mathrm{Sign}\,(X)^2 = 0$$
$$\mathrm{Sign}\,(Y^2) = \mathrm{Sign}\,(Y)^2 = 0\ .$$

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Since this paper was submitted, my attention has been drawn to a very similar paper of Kodaira J. Anal. Math. **19** (1967), 207-215. Although this decreases the originality of the present paper, it enhances the appropriateness of the dedication.

Our 4-manifold Z will actually arise as a complex algebraic surface and the projection $\pi: Z \to X$ will be holomorphic (for some complex structure on X). The fibres $Y_x = \pi^{-1}(x)$ will therefore be algebraic curves but the complex structure will vary with x, so that Z is not a holomorphic fibre bundle. This is an essential feature of the example as we shall explain in § 3.

If T, E denote the Todd genus and Euler characteristic respectively, one has the following simple relations (for curves and surfaces)

$$2T(X) = E(X)\,, \qquad 2T(Y) = E(Y)$$
$$4T(Z) = \mathrm{Sign}\,(Z) + E(Z)\,.$$

Since E is always multiplicative for fibre bundles these relations imply

$$T(Z) = T(X)T(Y) + \frac{1}{4}\,\mathrm{Sign}\,(Z)\,.$$

Thus the non-vanishing of Sign (Z) is equivalent to the non-multiplicativity

$$T(Z) \neq T(X)T(Y) \tag{1.1}$$

of the Todd genus.

If $d \in H^2(Z)$ denotes the first Chern class of the tangent bundle T_π along the fibres of Z, the total Pontrjagin class $p(Z)$ is given by

$$\begin{aligned} p(Z) &= p(T_\pi)\cdot\pi^*p(X) \\ &= 1 + d^2\,. \end{aligned}$$

The Hirzebruch formula for the signature therefore gives

$$\mathrm{Sign}\,(Z) = \frac{d^2}{3}[Z]\,. \tag{1.2}$$

Thus the crucial property of our examples will be that

$$d^2 \neq 0\,. \tag{1.3}$$

In the next section we shall construct the surface Z and show that (1.3) holds. In § 3 we shall explain the connection with moduli of algebraic curves. Finally in § 4 we shall investigate in general the effect of the fundamental group on the signature of a fibre-bundle. We shall see that this is closely related to the homomorphism

$$H^*(B_G; \mathbf{Q}) \longrightarrow H^*(B_\Gamma; \mathbf{Q})$$

induced by a homomorphism of the discrete group Γ into the real Lie group G.

2. Construction of the surface Z

We first choose a curve C with a fixed-point-free involution τ. In other words C is a double covering of a curve C': these exist as soon as the genus $g' \geqq 1$, but we shall take $g' \geqq 2$. Note that the genus g of C is $2g' - 1$ and so $g \geqq 3$. We now take X to be the covering of C given by the homomorphism

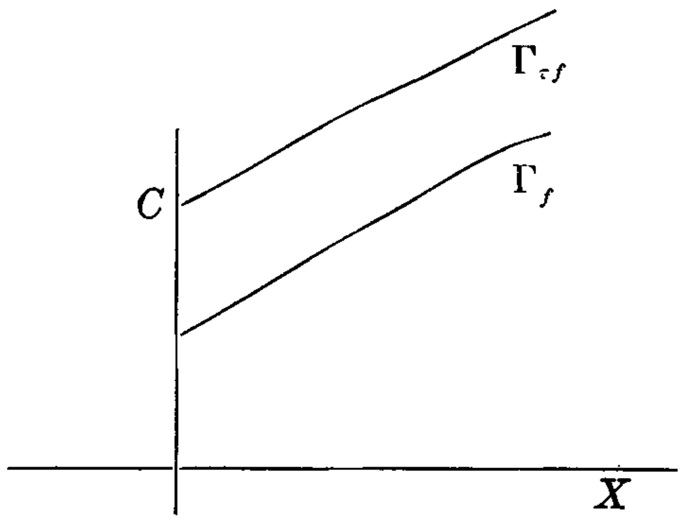
$$\pi_1(C) \longrightarrow H_1(C; \mathbf{Z}) \longrightarrow H_1(C; Z_2) \cong Z_2^{2g} ,$$

where g is the genus of C. It has the property that any double covering of C becomes trivial when lifted to X. Thus, if $f: X \rightarrow C$ is the covering map, the induced homomorphism with mod 2 coefficients

$$f_1^*: H^1(C; Z_2) \longrightarrow H^1(X; Z_2)$$

is zero.

Consider now, in $X \times C$, the graphs Γ_f, $\Gamma_{\tau f}$



Our choice of f was to ensure the following property of these graphs:

LEMMA 2.1. *The homology class in $H^2(X \times C; \mathbf{Z})$ defined by the curve $\Gamma_f + \Gamma_{\tau f}$ is even.*

PROOF. Let $\gamma_f \in H^2(X \times C; Z_2)$ be the mod 2 reduction of the class of Γ_f. We have to show

$$\gamma_f + \gamma_{\tau f} = 0 .$$

Now, if we use the Künneth formula and Poincaré duality (for mod 2 coefficients) to identify

$$H^*(X \times C) \cong \operatorname{Hom}\big(H^*(C), H^*(X)\big)$$

it is well known that

$$\gamma_f = \textstyle\sum_p f_p^*$$

where f_p^* is the homomorphism induced by f in $H^p(\ ; Z_2)$. By our choice of f we have

$$\begin{aligned} f_1^* &= 0 \\ f_2^* &= \deg f (\mathrm{mod}\ 2) = 0 \\ f_0^* &= 1 \qquad \text{(identifying } H^0(C) \text{ and } H^0(X) \text{ with } Z_2) . \end{aligned}$$

Similarly

$$\begin{aligned} (\tau f)_1^* &= f_1^* \tau_1^* = 0 \\ (\tau f)_2^* &= 0 \\ (\tau f)_0^* &= 1 . \end{aligned}$$

Putting these together we get

$$\gamma_f + \gamma_{\tau f} = \textstyle\sum_p \big(f_p^* + (\tau f)_p^*\big) = 0 .$$

Let us next recall that, if A is a non-singular curve on an algebraic surface M, and if the homology class of A in $H^2(M; \mathbf{Z})$ is even, then we can construct a double covering $\tilde{M}$ of M *ramified along* A. To see this[1] let L be the holomorphic line-bundle defined by A. We shall first show that we can find a holomorphic line-bundle $\tilde{L}$ with $\tilde{L}^2 \cong L$. Consider the exact cohomology sequence

$$\longrightarrow H^1(M; \mathcal{O}) \longrightarrow H^1(M; \mathcal{O}^*) \xrightarrow{\ \delta\ } H^2(M; \mathbf{Z}) \longrightarrow H^2(M; \mathcal{O}) \longrightarrow$$

arising from the exact sequence of sheaves

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O} \xrightarrow{\ \exp 2\pi i\ } \mathcal{O}^* \longrightarrow 0$$

($\mathcal{O}$ the sheaf of germs of holomorphic functions, $\mathcal{O}^*$ the multiplicative sheaf of germs of non-zero holomorphic functions). Since $H^1(M; \mathcal{O})$ and $H^2(M; \mathcal{O})$ are both complex vector spaces, it follows easily from this exact sequence that, for $k \in \mathbf{Z}$,

$$\delta(l) \text{ divisible by } k \text{ in } H^2(M; \mathbf{Z}) \Longrightarrow l \text{ divisible by } k \text{ in } H^1(M; \mathcal{O}^*) .$$

Taking l to be the class of L and $k = 2$ we see that A being even

[1] The argument which follows applies quite generally to any complex manifold M (not necessarily compact) with a non-singular divisor A defining an even class in $H^2(M; \mathbf{Z})$.

implies L is even, i.e. there exists a holomorphic line-bundle $\tilde{L}$ with $\tilde{L}^2 \cong L$. If s is now a section of L vanishing at A the ramified covering $\tilde{M}$ is just the inverse image of $s(M) \subset L$ under the squaring map $\tilde{L} \to L$. It is clear that $\tilde{M}$ is algebraic.

If $\tilde{A} \subset \tilde{M}$ is the branch curve (mapped isomorphically onto A by $\tilde{M} \to M$) we have

$$\tilde{A}^2 = A^2/2 \tag{2.2}$$

for the self-intersection numbers. This follows from the fact that the normal bundles of $A, \tilde{A}$ in $M, \tilde{M}$ are just the restrictions $L \mid A$, $\tilde{L} \mid \tilde{A}$. If B is any curve on M having no component in common with A, and if $\tilde{B}$ is its inverse image in $\tilde{M}$, we have

$$A \cdot B = \tilde{A} \cdot \tilde{B} . \tag{2.3}$$

If a, b denote the corresponding cohomology classes (2.3) follows from the formulas

$$\begin{aligned} \{\pi^*(a)\pi^*(b)\}[\tilde{M}] &= 2(a \cdot b)[M] \\ \pi^*(b) &= \tilde{b} \\ \pi^*(a) &= 2\tilde{a} . \end{aligned}$$

Returning now to the curve $A = \Gamma_f + \Gamma_{\tau f}$ on the surface $M = X \times C$ we see from Lemma (2.1) that a ramified double cover $\tilde{M}$ exists. This is our surface Z referred to in § 1. It is clear that the composite projection

$$Z \longrightarrow X \times C \longrightarrow X$$

has as fibres the double coverings of C ramified at pairs of points of our involution τ. Thus the genus h of Y is given by

$$2 - 2h = 2(2 - 2g) - 2$$

or

$$h = 2g .$$

It remains now to calculate the first Chern class d of the tangent bundle T_π along the fibres of Z and to show that $d^2 \neq 0$.

Let ω be a holomorphic differential on C, and let Ω, $\tilde{\Omega}$ be its lift to $X \times C$, Z respectively. Then $\tilde{\Omega}$ is a holomorphic section of the dual T_π^* of T_π so that

$$d = -(\tilde{\Omega})$$

where $(\tilde{\Omega})$ denotes the class of the zeros of $\tilde{\Omega}$. But clearly

$$(\tilde{\Omega}) = \tilde{p}^*(w) + (\tilde{\Gamma}_f) + (\tilde{\Gamma}_{\tau f})$$

where $\tilde{p}: Z \to C$ is the projection. Hence, computing self-intersection numbers, and using (2.2) and (2.3) we get

$$\begin{aligned} d^2[Z] &= 2\tilde{p}^*(\omega)\cdot(\tilde{\Gamma}_f) + 2\tilde{p}^*(\omega)\cdot(\tilde{\Gamma}_{\tau f}) + (\tilde{\Gamma}_f)^2 + (\tilde{\Gamma}_{\tau f})^2 \\ &= 2p^*(\omega)\cdot(\Gamma_f) + 2p^*(\omega)\cdot(\Gamma_{\tau f}) + \frac{1}{2}(\Gamma_f)^2 + \frac{1}{2}(\Gamma_{\tau f})^2 \\ &= 2(2g-2)\deg f + 2(2g-2)\deg \tau f + \frac{1}{2}(\Gamma_f)^2 + \frac{1}{2}(\Gamma_{\tau f})^2 \\ &= 8(g-1)\deg f + (\Gamma_f)^2 . \end{aligned}$$

Here p denotes the projection $X \times C \to C$ and we used the involution $1 \times \tau$ on $X \times C$ to take Γ_f into $\Gamma_{\tau f}$ and so derive the equality $(\Gamma_f)^2 = (\Gamma_{\tau f})^2$. Finally, if I denotes the identity map of C, we have

$$\begin{aligned} (\Gamma_f)^2 &= \deg f\cdot(\Gamma_I)^2 \\ &= \deg f\cdot(2-2g) . \end{aligned}$$

Putting this in the formula above for $d^2[Z]$ we therefore obtain

$$\begin{aligned} d^2[Z] &= 8(g-1)\deg f - 2(g-1)\deg f \\ &= 6(g-1)\deg f \\ &= 6(g-1)\cdot 2^{2g} . \end{aligned}$$

Thus the signature of our surface Z is given by

$$\text{Sign}(Z) = (g-1)2^{2g+1} = -E(X) \tag{2.4}$$

and (since $g \geq 3$) this is non-zero as required.

Remarks. (1) An alternative method of calculating Sign (Z) has been pointed out to me by F. Hirzebruch. This is to use the following general formula for the signature of a ramified double covering $\tilde{M}$ of M:

$$\text{Sign}(\tilde{M}) = 2\,\text{Sign}(M) - \text{Sign}(\tilde{S}\cdot\tilde{S}) .$$

Here $\tilde{S} \subset \tilde{M}$ is the ramification submanifold (of codimension 2) and $\tilde{S}\cdot\tilde{S}$ denotes a "self-intersection manifold" of $\tilde{S}$ in $\tilde{M}$. This formula is an easy consequence of the general G-signature theorem of [1] with G of order two (see [1; (6.15)]) and, applied in our case, it gives

$$\begin{aligned} \text{Sign}\, Z &= 0 - (\tilde{\Gamma}_f^2 + \tilde{\Gamma}_{\tau f}^{2}) \\ &= -\Gamma_f^2 = -E(X) . \end{aligned}$$

(2) The formula (2.4) shows that there exist algebraic

surfaces with arbitrarily large signature, contrary to a conjecture of Zappa. In fact Borel in [3] produced counter-examples to this conjecture which are somewhat similar to our examples. In both cases the surfaces are classifying spaces for a discrete group Γ, that is the universal covering surface is contractible. In Borel's example the universal covering is the unit ball B^2 in C^2, but in our examples the universal covering is definitely not B^2. One way to distinguish the two cases is to observe that the Borel surfaces are rigid [4] whereas our surface Z has moduli. In fact it is easy to see (using the footnote in § 2) that, as we vary the original curve C' in a local family C'_t, we obtain a family Z_t and that the moduli of C'_t (i.e., the holomorphic periods) give rise to non-trivial moduli for Z_t. We may also note that the universal covering of Z cannot be the product $B^1 \times B^1$ of two unit discs because, if it were, the argument of Borel [3] would imply Sign $(Z)=0$. Since B^2 and $B^1 \times B^1$ are the only homogeneous bounded domains in C^2 it follows that the universal covering of Z is not a homogeneous bounded complex domain. Note finally that the fundamental group $\pi_1(Z)$ is a split extension with $\pi_1(Y)$ as subgroup and $\pi_1(X)$ as quotient.

(3) The smallest non-zero value of Sign (Z) in (2.4) occurs when $g = 3$. This gives Sign $(Z) = 2^8$.

(4) Since the fibre Y_x of Z is, by construction, a double covering of C ramified at x and $\tau(x)$ it is fairly clear that the "moduli" of Y_x vary non-trivially with $x \in X$. In the next section we shall in fact see that this is always the case for any family of curves $Z \to X$ for which the total space Z has non-vanishing signature. Conversely, appealing to facts about the space of moduli of curves, one could deduce the existence of families $Z \to X$ with Sign $(Z) \neq 0$. This was my original approach to the problem. The specific construction by ramified double coverings was suggested to me by I. R. Šafarevic.

(5) Despite the algebro-geometric aspect of the preceding remark it should be emphasized that the complex structure of the fibering $Z \to X$ is purely auxiliary. We could have constructed Z as a ramified covering of $X \times C$ without using the complex structure.

3. Relation with moduli

If we apply the Grothendieck Riemann-Roch theorem to the map $Z \xrightarrow{\pi} X$ we get

$$\mathrm{ch}\,(1 - \mathcal{H}) = \pi_*\left(1 + \frac{d}{2} + \frac{d^2}{12}\right) \tag{3.1}$$

where $\mathcal{H}$ is the vector bundle over X whose fibre $\mathcal{H}_x$ is $H^1(Y_x, \mathcal{O}_{Y_x})$: its dual $\mathcal{H}^*$ is the bundle whose fibre $\mathcal{H}_x^*$ is the space of holomorphic differentials on Y_x. Equating the two-dimensional terms in (3.1) we get

$$c_1(\mathcal{H}^*) = -c_1(\mathcal{H}) = \pi_*\left(\frac{d^2}{12}\right). \tag{3.2}$$

Thus the essential feature of our example is that $c_1(\mathcal{H}^*) \neq 0$.

The cohomology class $c_1(\mathcal{H}^*)$ can also be interpreted as follows. In the differentiable fibre bundle $Z \to X$ the fundamental group $\pi_1(X)$ acts on the cohomology $H^1(Y; \mathbf{Z})$ preserving the symplectic form given by the cup product. Thus we get a homomorphism

$$\alpha: \pi_1(X) \longrightarrow \mathrm{Sp}\,(2h; \mathbf{Z})$$

and hence a homomorphism

$$\beta: \pi_1(X) \longrightarrow \mathrm{Sp}\,(2h; \mathbf{R})\ .$$

This induces a homomorphism in the cohomology of the classifying spaces

$$\beta^*: H^*(B_G; \mathbf{Q}) \longrightarrow H^*(X; \mathbf{Q})$$

where $G = \mathrm{Sp}\,(2h; \mathbf{R})$. Now the maximal compact subgroup of G is isomorphic to the unitary group $U(h)$ and so there is a universal complex vector bundle V of dimension h over B_G. One can verify that the bundle $\beta^*(V)$ over X induced by β is isomorphic to $\mathcal{H}^*$ (or $\mathcal{H}$, depending on how we choose V) and so

$$c_1(\mathcal{H}^*) = \beta^*(c_1(V))\ .$$

Since $c_1(\mathcal{H}^*) \neq 0$ and since β factors through α it follows that *the inclusion*

$$\mathrm{Sp}\,(2h; \mathbf{Z}) \longrightarrow \mathrm{Sp}\,(2h; \mathbf{R})$$

induces a non-zero homomorphism in the rational cohomology of the classifying spaces (in dimension 2). Note that in our example $h = 2(2g' - 1)$ where $g' \geqq 2$, so that the smallest value is 6.

Remark. For a discrete subgroup Γ of a real Lie group G, one might ask under what circumstances the homomorphism

$$H^*(B_G; \mathbf{Q}) \longrightarrow H^*(B_\Gamma; \mathbf{Q})$$

is non-trivial. When G/Γ is compact one can use the theory of harmonic forms to investigate this question. Our example here, with $G = \mathrm{Sp}(2h; \mathbf{R})$, $\Gamma = \mathrm{Sp}(2h; \mathbf{Z})$ is of a different type since G/Γ has finite volume but is not compact.

Let D denote the Siegel upper half-plane $\mathrm{Sp}(2h; \mathbf{R})/U(h)$ and let $\Gamma\backslash D$ denote the space obtained by dividing by the action of the discrete group $\Gamma = \mathrm{Sp}(2h; \mathbf{Z})$. Then the holomorphic family of curves $Z \to X$ defines a holomorphic map

$$\varphi: X \longrightarrow \Gamma\backslash D .$$

Except for the presence of fixed points of Γ on D the space $\Gamma\backslash D$ would be a classifying space for Γ. In any case, for real cohomology, the fixed points cause no trouble[2] and we can identify

$$\varphi^*: H^*(\Gamma\backslash D; \mathbf{Q}) \longrightarrow H^*(X; \mathbf{Q})$$

with the homomorphism

$$\alpha^*: H^*(B_\Gamma; \mathbf{Q}) \longrightarrow H^*(X; \mathbf{Q})$$

induced by $\alpha: \pi_1(X) \to \Gamma$. This shows that we cannot get examples of families Z with $c_1(\mathcal{H}^*) \neq 0$ unless φ is non-trivial, that is unless the holomorphic structure of the fibres varies.

4. General remarks

We shall now discuss in general the effect of the fundamental group of the base on the signature of a fibre bundle. When our manifolds are not complex we cannot use holomorphic differentials as in § 3 but, choosing a riemannian metric, we can use harmonic forms. With this modification we shall see that the discussion of § 3 can be paralleled in the general case.

Let $\pi: Z \to X$ be a differentiable fibre bundle with fibre Y. We assume X, Y, Z are compact oriented and that X, Y have even[3] dimension. We fix riemannian metrics on the fibres, for example by taking a riemannian metric on Z and giving Y_x the induced metric.

Let $\dim Y = 2k$ and consider the bundle H^k over X given by the real cohomology of the fibres in dimension k. This is the bundle associated to the action of $\Gamma = \pi_1(X)$ on $H^k(Y; \mathbf{R})$. This action

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[2] We can replace Γ by a subgroup Γ_0 of finite index acting freely on D.

[3] When the dimensions are odd one always has

$$\mathrm{Sign}(X) \cdot \mathrm{Sign}(Y) - \mathrm{Sign}(Z) = 0 .$$

preserves the bilinear form given by the cup product. Thus for k odd we have a homomorphism

$$\beta\colon \Gamma \longrightarrow \mathrm{Sp}(2n; \mathbf{R})$$

and for k even we have

$$\beta\colon \Gamma \longrightarrow O(p, q; \mathbf{R}) .$$

Here $2n = \dim H^k(Y; \mathbf{R})$ for k odd, and $\sum_1^p x_i^2 - \sum_1^q x_j^2$ is the diagonalization of the quadratic form on $H^k(Y; R)$ for k even.

If we identify $H^k(Y_x; \mathbf{R})$ with the space of harmonic k-forms on Y_x we get an inner product in the bundle H^k and so a reduction of its structure group to a maximal compact subgroup. For k odd this is $U(n)$ and for k even it is $O(p) \times O(q)$. For k odd we thus have an associated n-dimensional complex vector bundle V over X, and for k even we have two real vector bundles W^+, W^- of dimensions p, q respectively. These can be defined directly in terms of the inner product on $H^k(Y_x; \mathbf{R})$ as follows. The $*$-operator on the harmonic k-forms satisfies $*^2 = (-1)^k$. Thus for k odd it defines a complex structure on the bundle H^k — this is V — and for k even it defines a decomposition $H^k = H_+^k \oplus H_-^k$ into the ± 1-eigenspaces — these are W^+, W^-. We now define the signature of the map π to be

$$\begin{aligned} \mathrm{Sign}(\pi) &= V^* - V \in K(X) && (k \text{ odd}) \\ &= W^+ - W^- \in KO(X) && (k \text{ even}) . \end{aligned}$$

From the classifying space description of the bundles V, W^+, W^- it is clear that[4] $\mathrm{ch}(\mathrm{Sign}(\pi))$ is induced from a universal characteristic class

$$\mathrm{ch}(\mathrm{Sign}) \in H^*(B_G; \mathbf{Q}) = H^*(B_K; \mathbf{Q})$$

by the composite map

$$X \longrightarrow B_\Gamma \longrightarrow B_G$$

where $G = \mathrm{Sp}(2n; \mathbf{R})$ or $O(p, q; \mathbf{R})$ according as k is odd or even and K is a maximal compact subgroup. The map $X \to B_\Gamma$ is the classifying map of the universal covering of X and $B_\Gamma \to B_G$ is induced by the homomorphism $\beta\colon \Gamma \to G$.

On the other hand from the harmonic form description of V, W^+, W^- one can show that

[4] For a real bundle ch is defined as the Chern character of the complexification.

$$(4.1)\qquad \begin{aligned} V^* - V &= \operatorname{index} D^+ \in K(X) && k \text{ odd} \\ W^+ - W^- &= \operatorname{index} D^+ \in KO(X) && k \text{ even} \end{aligned}$$

where D^+ is a family of elliptic operators along the fibres of π. For $x \in X$ the operator D_x^+ is the signature operator defined in [1; § 6]. The vector space $\operatorname{Ker} D_x^+$ has a constant dimension and is the fibre of a bundle $\operatorname{Ker} D^+$ over X (complex for k odd, real for k even). Similarly for $\operatorname{Coker} D^+$ and the index of the family D^+ is defined to be the element of $K(X)$ or $KO(X)$ (as k is odd or even) given by

$$\operatorname{index} D^+ = \operatorname{Ker} D^+ - \operatorname{Coker} D^+ .$$

Note that (for k even say) $\operatorname{Ker} D^+$ contains W^+ as a sub-bundle and similarly $\operatorname{Coker} D^+$ contains W^-. The equality (4.1) holds because the remaining parts cancel, that is

$$\operatorname{Ker} D^+/W^+ \cong \operatorname{Coker} D^+/W^- .$$

The index theorem for families of elliptic operators [2] applied to D^+ gives

$$(4.2)\qquad \operatorname{ch}(\operatorname{index} D^+) = \pi_*(\widetilde{\mathfrak{L}}T_\pi) \in H^*(X; \mathbf{Q})$$

where, for any real vector bundle E of dimension $2k$, $\widetilde{\mathfrak{L}}(E)$ is that function of the Pontrjagin classes of E defined by

$$\widetilde{\mathfrak{L}}(E) = \prod_{i=1}^{k} \frac{x_i}{\tanh x_i/2} .$$

(As usual the Pontrjagin classes of E are interpreted as the elementary symmetric functions in $x_1^2, \cdots, x_k^2$.) Combined with (4.1) we therefore obtain

$$(4.3)\qquad \operatorname{ch}(\operatorname{Sign}(\pi)) = \pi_*(\widetilde{\mathfrak{L}}(E)) .$$

For the special case considered in § 3 formula (4.3) coincides essentially with (3.2) which was derived from the Grothendieck Riemann-Roch theorem.

The Hirzebruch index theorem gives the signature of Z in terms of $\widetilde{\mathfrak{L}}(Z)$; i.e., $\widetilde{\mathfrak{L}}$ of the tangent bundle of Z,

$$\begin{aligned} \operatorname{Sign}(Z) &= \widetilde{\mathfrak{L}}(Z)[Z] \\ &= \{\widetilde{\mathfrak{L}}(T_\pi)\cdot\pi^*\widetilde{\mathfrak{L}}(X)\}[Z] \\ &= \{\pi_*\widetilde{\mathfrak{L}}(T_\pi)\cdot\widetilde{\mathfrak{L}}(X)\}[X] \\ &= \{\operatorname{ch}(\operatorname{Sign}(\pi))\cdot\widetilde{\mathfrak{L}}(X)\}[X] && \text{by (4.3)} . \end{aligned}$$

This last expression for Sign (Z) shows clearly why the signature is not multiplicative in general. The deviation from multiplicativity is in a sense measured by the (positive-dimensional components of) the cohomology class ch (Sign (π)). As we have seen this is induced from a universal characteristic class in B_G *via* the fundamental group of X.

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